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Instability of Electroweak Homogeneous Vacua in Strong Magnetic Fields

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To Nicholas Ercolani and Gian Michele Graf, scientists and friends.

Abstract. We consider the classical vacua of the Weinberg–Salam (WS) model of electroweak forces. These are no-particle, static solutions to the WS equations minimizing the WS energy locally. We study the WS vacuum solutions exhibiting a non-vanishing average magnetic field of strength b and prove that (i) there is a magnetic field threshold b_* such that for $b < b_*$, the vacua are translationally invariant (and the magnetic field is constant), while, for $b > b_*$, they are not, (ii) for $b > b_*$, there are non-translationally invariant solutions with lower energy per unit volume and with the discrete translational symmetry of a 2D lattice in the plane transversal to b, and (iii) the lattice minimizing the energy per unit volume approaches the hexagonal one as the magnetic field strength approaches the threshold b_*. In the absence of particles, the Weinberg–Salam model reduces to the Yang–Mills–Higgs (YMH) equations for the gauge group $U(2)$. Thus, our results can be rephrased as the corresponding statements about the $U(2)$-YMH equations.

Mathematics Subject Classification. 81T13 (primary), 35Q40, 70S15 (secondary).

1. Introduction

The Weinberg–Salam (WS) model of electroweak interactions [22, 35, 47] was the first triumph of the program to unify the four fundamental forces of nature. It is a key part of the standard model of elementary particles. It unifies electromagnetic and weak interactions, two of the three forces dealt with in the standard model. It involves particle, gauge and the Higgs fields.

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While the gauge fields describe the electroweak interactions, the role of the Higgs field is to convert the original massless fields (zero masses are required by the relativistic invariance) to massive ones. This phenomenon is called the Higgs mechanism. This mechanism, together with the Goldstone theorem, leads to all gauge particles but one acquiring mass, resulting in two massive bosons—denoted W and Z—and a massless one—the photon. The W and Z particles where discovered experimentally 16 years after their theoretical prediction.

In this paper, we consider the vacuum solutions of the classical WS model with a non-vanishing *average magnetic field* $\langle\vec{b}\rangle$. These are static, no-particle solutions minimizing the WS energy locally for a fixed $\vec{b}$. They are also no-particle solutions of the entire standard model.[1]

We prove that (i) there is a magnetic field threshold b_* such that for $|\vec{b}| < b_*$, the vacua are translationally invariant, while, for $|\vec{b}| > b_*$, they are not, (ii) for $|\vec{b}| > b_*$, there are non-translationally invariant solutions with lower energy per unit volume and with the discrete translational symmetry of a 2D lattice in the plane transversal to the magnetic field, and (iii) the lattice minimizing the energy of the latter solutions per unit volume approaches the hexagonal one as the magnetic field strength approaches the threshold b_*. We expect that these solutions are stable under field fluctuations and, in fact, minimize the energy locally.

The phenomenon above was investigated extensively in the physics literature (see, e.g. [5–12,19,26–29,32,41,42] and the references therein). It is similar to the one occurring in superconductivity and the solutions whose existence we establish are analogous to the superconducting Abrikosov vortex lattices ([1], see, e.g. [37], for a review). It is estimated in [26] that the spontaneous symmetry breaking takes place at the critical average magnetic field of approximately 10^{24} Gauss $= 10^{20}$ Tesla. By comparison, the strongest magnetic field produced on Earth is 10^{14} Tesla.

Note that, in the absence of particles, the WS system reduces to the Yang–Mills–Higgs (YMH) one with the gauge group $U(2)$. So ultimately, these are the equations we deal with.

The only rigorous result [43,44] on the classical WS model deals with the vortices in the self-dual regime, where the WS (or corresponding YMH) equations are equivalent to the first-order equations, and it uses this equivalence in an essential way. (The self-dual regime in this context was discovered in [6–8], see also [41,42].)

Open problems and further directions:

(a) Stability of the emerging solutions.

(b) Existence of vortex lattices at $|\vec{b}| \gg b_*$.

[1]The no-particle sector of the standard model splits into the $U(2)$-YMH (electroweak) and $U(3)$-YM (strong, or QCD) parts. Correspondingly, the vacuum of the standard model is the product of the electroweak and strong vacua and the vacuum energy is the sum of the corresponding energies.)

(c) Quantum corrections to the values of the classical critical magnetic field b_* and the optimal lattice shape parameter τ_*.

For the stability and existence problems, (a) and (b), see, e.g. [38,40] and [39], respectively. The last problem brings up the regime of 'sparse' vortex lattices as opposite to the case of $|\vec{b}|$ close to (and $>$) b_* resulting in densely packed vortices: the lattice step $\to 0$ as $|\vec{b}| \to b_*$ and $\to \infty$ as $|\vec{b}| \to \infty$. Hence, the existence of vortex lattices at $|\vec{b}| \gg b_*$ is closely related to the problem of *existence of vortices* (elementary excitations).

For existence of Abrikosov lattices in the Ginzburg–Landau and Chern–Simons equations, see [33,37].

For the quantum corrections, problem (c), it would be natural to start with a BCS-type, or quasi-free, version of the WS model and a Bogoliubov-type expansion of a regularized (say, lattice) WS model around it, see, e.g. [14,15].

The paper is organized as follows: In Sect. 2, we formulate the problem and describe results. In Sects. 3–4, we fix the gauge and pass from the original Yang–Mills fields to the W and Z (massive boson) and A (photon) fields and rescale the resulting equations. The proofs of the main results are given in Sect. 5 (Theorem 2.1), Sects. 6–10 (Theorem 2.2) and Sect. 11 (Theorem 2.3). In "Appendix A", we discuss various covariant derivatives used in the main text, and in "Appendix B", we review the time-dependent YMH equations and derive the expression for the conserved energy as well as the YMH equations used in the main text. Furthermore, there we write the YMH equations in coordinate form and derive a convenient expression for the energy functional. In "Appendices D.1–D.2", we derive the WS equations in 3D and 2D, respectively, in terms of the fields W, Z, A and φ. In the remaining appendices, we carry out technical computations.

Throughout the paper, we use the Einstein convention of *summing over repeated indices.*

2. No-particle and Vacuum Sectors of the Weinberg–Salam Model

The no-particle sector of the Weinberg–Salam (WS) model involves the interacting Higgs and $SU(2)$ and $U(1)$ gauge fields, Φ and V and X, while the particle fields are set to zero. The field Φ is a vector function defined on the Minkowski space-time $\mathbb{R}^{3+1}$ with values in $\mathbb{C}^2$, and the fields V and X are one-forms on $\mathbb{R}^{3+1}$ with values in the algebras $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$, respectively. We write

$$Q = gV + g'X,$$

where g and g' are coupling constants, which is a one-form with values in $\mathfrak{u}(2)$. We consider $SU(2)$ as a matrix group and $U(1)$ as multiples of the identity matrix $\mathbf{1}$ acting on $\mathbb{C}^2$.

These fields satisfy the WS equations, which are the Euler–Lagrange equations for the action functional

$$\mathcal{S}(Q,\Phi)=\int_M\big(\langle\nabla_Q\Phi,\nabla_Q\Phi\rangle^{\eta}_{\Omega^1_V}-\frac{1}{2}\lambda(\|\Phi\|^2_{\mathbb{C}^2}-\varphi_0^2)^2+\langle F_Q,F_Q\rangle^{\eta}_{\Omega^2_{\mathfrak{g}}}\big),\qquad(2.1)$$

where M is a bounded domain in spacetime $\mathbb{R}^{3+1}$ equipped with the Minkowski metric η of signature $(-,+,+,+)$, λ and φ_0 are positive parameters, and the remaining symbols are defined as follows:

∇_Q is the covariant derivative mapping $\mathbb{C}^2$-valued functions (sections) into $\mathbb{C}^2$-valued one-forms defined as

$$\nabla_Q=d+Q,\qquad(2.2)$$

with d, the exterior derivative; F_Q is the curvature 2-form of the connection one-form Q, given by

$$F_Q=dQ+\frac{1}{2g}[Q,Q],\qquad(2.3)$$

where $[A,B]$ is defined in local coordinates $\{x^i\}$ as

$$[A,B]:=[A_i,B_j]dx^i\wedge dx^j=[B,A],\qquad(2.4)$$

with $A=A_idx^i$ and $B=B_idx^i$;

$\Omega^p_U\equiv U\otimes\Omega^p$ denotes the space of U-valued p-forms with the Minkowski, indefinite inner product,

$$\langle A,B\rangle^{\eta}_{\Omega^p_U}:=\langle A_\alpha(x),B^\alpha(x)\rangle_U,\qquad(2.5)$$

where $A=A_\alpha(x)dx^\alpha$ and $B=B_\alpha(x)dx^\alpha$ are U-valued p-forms, α is a p-form index and $\langle\cdot,\cdot\rangle_U$ is the standard, positive definite inner product on U with the indices raised and lowered with help of the Minkowski metric η on M. For instance, for $U=\mathfrak{su}(2)$, the inner product is given by

$$\langle A,B\rangle^{\eta}_{\Omega^p_{\mathfrak{su}(2)}}:=2\,\mathrm{Tr}(A_\alpha(x)^*B^\alpha(x))=-2\,\mathrm{Tr}(A_\alpha(x)B^\alpha(x)).\qquad(2.6)$$

Solutions of the no-particle WS equations solve also the full WS system as well as that for the standard model of the particle physics.

The vacuum sector of the Weinberg–Salam (WS) model consists of static, no-particle solutions.

The static Higgs and $SU(2)$ and $U(1)$ gauge fields Φ, V and X are now defined on the physical space $\mathbb{R}^3$ with the same respective values as in the time-dependent case. Geometrically, V,X and Q can be thought of as connection one-forms on the trivial bundles $\mathbb{R}^3\times SU(2),\mathbb{R}^3\times U(1)$ and $\mathbb{R}^3\times U(2)$.

The fields Φ, V and X satisfy the static no-particle WS equations, which are the Euler–Lagrange equations for the static WS energy functional originating in (2.1)[2]

$$E_N(Q,\Phi):=\int_N\big(\|\nabla_Q\Phi\|^2_{\Omega^1_{\mathbb{C}^2}}+\frac{1}{2}\lambda(\|\Phi\|^2_{\mathbb{C}^2}-\varphi_0^2)^2+\frac{1}{2}\|F_Q\|^2_{\Omega^2_{\mathfrak{u}(2)}}\big),\qquad(2.7)$$

[2]For a discussion of the time-dependent theory and a derivation of the energy functional (2.7), see [24,28,34,36] and "Appendix B".

where N is a bounded domain in $\mathbb{R}^3$ with appropriate boundary conditions (specified in (2.17) below) and $\|\cdot\|_{\Omega^p_U}$ is the standard norm on the space $\Omega^p_U := U \otimes \Omega^p$ of U-valued p-forms at $x \in N$ (e.g. for $B = B_i(x)dx^i \in \Omega^1_U$, we have $\|B\|_{\Omega^1_U} := (\sum_i \|B_i(x)\|^2_U)^{1/2}$ with the usual Euclidean metric and with the indices running through $1,2,3$), while now, (2.5) (and (2.6)) become the usual inner products. The symbols ∇_Q and F_Q are as defined above but without the time component.

Since $Q = gV + g'X$ and X has the values in the centre, $u(1)$, of the algebra $u(2)$, we have $F_Q = gF_V + g'F_X$, where

$$F_V := dV + \frac{g}{2}[V,V] \;\text{ and }\; F_X := dX \tag{2.8}$$

are the curvatures of the connections V and X[3] and $\|F_Q\|^2_{\Omega^2_{\mathfrak{u}(2)}} = \|F_V\|^2_{\Omega^2_{\mathfrak{u}(2)}} + \|F_X\|^2_{\Omega^2_{\mathfrak{u}(1)}}$.

We introduce the covariant derivative d_Q mapping $\mathfrak{u}(2)$-valued k-forms into $\mathfrak{u}(2)$-valued $(k+1)$-forms, $k \geq 1$, as

$$d_Q B := dB + [Q,B] = d_V B := dB + g[V,B]. \tag{2.9}$$

This formula originates in the equation $(\delta_Q F_Q)(B) = d_Q B$, where δ_Q is the Gâteaux derivative with respect to Q. For 0-forms, we set $d_Q = \nabla_Q$.

The Euler–Lagrange equations for energy functional (2.7) are given by (see "Appendix B"[4])

$$\nabla_Q^* \nabla_Q \Phi = \lambda(\varphi_0^2 - \|\Phi\|^2)\Phi, \tag{2.10}$$

$$d_Q^* F_Q = J(Q,\Phi), \tag{2.11}$$

where ∇_Q^* is the adjoint of ∇_Q and maps $\mathbb{C}^2$-valued one-forms into $\mathbb{C}^2$-valued functions, d_Q^* is the adjoint of d_Q and maps $\mathfrak{u}(2)$-valued two-forms into $\mathfrak{u}(2)$-valued one-forms, and $J(Q,\Phi)$ is the electroweak current, which is the $\mathfrak{u}(2)$-valued one-form given by

$$J(Q,\Phi) := -\frac{ig}{2}\tau_a \operatorname{Im}\langle \tau_a \Phi, \nabla_Q \Phi\rangle - \frac{ig'}{2}\tau_0 \operatorname{Im}\langle \tau_0 \Phi, \nabla_Q \Phi\rangle, \tag{2.12}$$

where summing over repeated indices is understood, $\tau_0 := \mathbf{1}$ and $\tau_a, a = 1,2,3$, are the Pauli matrices,

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \tau_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

(The Pauli matrices, multiplied by $-i/2$, form an orthonormal basis in $\mathfrak{su}(2)$ with the inner product $\langle g,h\rangle_{\mathfrak{su}(2)} := 2\operatorname{Tr}(g^*h) = -2\operatorname{Tr}(gh)$.) We call system (2.10)–(2.11) the (static) WS equations.

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[3] For more discussion of covariant derivatives and their curvatures, see "Appendices A" for the general case, or "Appendix C", for the case of the gauge group $G = U(2)$.

[4] These equations could be converted formally back into the time-dependent ones by taking the adjoints in the Minkowski metric instead of the Euclidian one, see (B.4)–(B.5), "Appendix B".

The energy functional (2.7) and Euler–Lagrange equations (2.10)–(2.11) are invariant under the group of rigid motions and the gauge transformations (gauge symmetry)

$$(V(x), X(x), \Phi(x)) \mapsto (V_\gamma(x), X_\gamma(x), \Phi_\gamma(x)), \tag{2.14}$$

where $\gamma = \gamma(x) = h_1(x)h_2(x)$, with $h_1(x) \in SU(2)$, $h_2(x) \in U(1)$, and

$$\begin{cases} V(x) \mapsto h_1(x)V(x)h_1^{-1}(x) - i\frac{2}{g}h_1(x)dh_1^{-1}(x), \\ X(x) \to X(x) - i\frac{2}{g'}h_2(x)dh_2^{-1}(x), \\ \Phi(x) \to h_1(x)h_2(x)\Phi(x). \end{cases}$$

The physical quantities here are (a) the Higgs field density $\|\Phi\|$, (b) the magnetic field $\operatorname{Tr} F_Q$ and (c) the YM current $J(Q,\Phi)$. It is easy to check that these quantities are gauge invariant. We say that a solution (Q,Φ) to (2.10)–(2.11) is *homogeneous* if $\|\Phi\|$, $\operatorname{Tr} F_Q$ and $J(Q,\Phi)$ are independent of x. (We say that $\operatorname{Tr} F_Q$ is independent of x, if it is a multiple of a constant 2-form, see (2.16).) Otherwise, we say that (Q,Φ) is *inhomogeneous*.

Furthermore, we say that a solution (Q,Φ) is *gauge-translation invariant* if it is invariant under translations up to gauge transformations.

Clearly, a solution (Q,Φ) which is gauge-translation invariant is also homogeneous. The converse in general might not be true.

We are interested in the vacuum solutions of the WS equations with a non-vanishing average magnetic field,

$$\lim_{R\to\mathbb{R}^3} \frac{1}{|R|}\int_R \operatorname{Tr} F_Q = -ie\sum_{(ijk)} b_i dx^j \wedge dx^k,$$

i.e. solutions minimizing the WS energy locally under the constraint above. In physical field theories, one expects the vacua to have the maximal available symmetry. Consequently, we first consider gauge-translation invariant solutions with a fixed (constant) magnetic field.

For $\vec{b} = (b_1, b_2, b_3) \neq 0$, Eqs. (2.10)–(2.11) have the gauge-translation invariant solution given (up to a gauge symmetry) by

$$U_*^{\vec{b}} := (Q^{\vec{b}}, \Phi^{\vec{b}}), \tag{2.15}$$

where $\Phi^{\vec{b}}$ is a constant field and $Q^{\vec{b}}$ is a connection with a constant magnetic field

$$\operatorname{Tr} F_{Q^{\vec{b}}} = -ie\omega_{\vec{b}}, \quad \text{where } \omega_{\vec{b}} := \sum_{(ijk)} b_i dx^j \wedge dx^k, \tag{2.16}$$

with the sum taken over all cyclic permutations of $(1,2,3)$, and $e := \frac{gg'}{\sqrt{g^2+g'^2}}$. ($e$ turns out to be the electron charge.) We specify this solution at the end of this section in Eqs. (2.23) and (2.24). (For it, $Q^{\vec{b}}$ solves the YM equation $d_Q^* F_Q = 0$.)

Fixing the average magnetic field breaks the full special Euclidean symmetry (i.e. translations and rotations but not reflections) but maintains the special Euclidean symmetry in the plane orthogonal to $\vec{b}$ and the translational symmetry along $\vec{b}$. Looking for the simplest non-trivial solutions, we consider solutions which *do not depend on the coordinate along* $\vec{b}$ and look for solutions spontaneously breaking the transversal translational symmetry.

With the notation $b = |\vec{b}|$, we show that for appropriate perturbations:

(i) (2.15) is linearly stable for $b < b_*$ and unstable for $b > b_*$, where $b_* := g^2\varphi_0^2/2e$;
(ii) At $b = b_*$, a new inhomogeneous solution (breaking the gauge-translational invariance) bifurcates, and this solution has the discrete translational symmetry of a lattice in the plane orthogonal to $\vec{b}$ and has lower energy per unit area;
(iii) The lattice shape minimizing the energy per unit area approaches the hexagonal lattice as b approaches b_*.

To formulate these results precisely, we introduce some definitions. Since we consider solutions which *do not depend on the coordinate along* $\vec{b}$, we can restrict our analysis to the plane $\perp \vec{b}$. We choose the x^3-axis along $\vec{b}$ and identify the plane $\perp \vec{b}$ with $\mathbb{R}^2$.

We fix a lattice $\mathcal{L}$ in $\mathbb{R}^2$ and say a triple $(\Phi(x), V(x), X(x))$ is $\mathcal{L}$-*gauge-periodic*, or, $\mathcal{L}$-*equivariant*, if and only if it satisfies the equation

$$(T^{\text{gauge}}_{\gamma_s})^{-1} T^{\text{trans}}_s (V, X, \Phi) = (V, X, \Phi), \qquad \forall s \in \mathcal{L}, \tag{2.17}$$

for some $\gamma_s \in C^1(\mathbb{R}^2, SU(2) \times U(1))$. Here T^{gauge}_γ is given by (2.14) and T^{trans}_s is the group of translations, $T^{\text{trans}}_s f(x) = f(x+s)$. (When $\mathcal{L}$ is clear, we omit it from the definition above.)

We denote by $\mathcal{H}^s_{\mathcal{L}}$, $s \in \mathbb{N}$, the Sobolev space of $\mathcal{L}$-equivariant triples $U \equiv (V, X, \Phi)$ on $\mathbb{R}^2$, with the norm

$$\|U\|_{\mathcal{H}^s_{\mathcal{L}}} := \Big(\frac{1}{|\Omega|}\sum_{k=0}^{s}\int_\Omega \|d_Q^k U\|^2\Big)^{\frac{1}{2}}, \tag{2.18}$$

where Ω is an arbitrary fundamental domain of $\mathcal{L}$, d_Q^k is the k-th iterate of the covariant derivative d_Q and $\|\cdot\|$ is the (fibre) norm in the space $\Omega^{k+1}_{\mathfrak{su}(2)} \times \Omega^{k+1}_{\mathfrak{u}(1)} \times \Omega^k_{\mathbb{C}^2}$, see (2.5) (and (2.6)), and with corresponding the inner product. Note that $L^2_{\mathcal{L}} = \mathcal{H}^0_{\mathcal{L}}$.

The resulting Sobolev spaces $\mathcal{H}^s_{\mathcal{L}}$ are independent (up to isomorphism) of the choice of the fundamental domain, Ω. All Sobolev embedding theorems are valid for $\mathcal{H}^s_{\mathcal{L}}$. They can be proven by passing to a vector bundle over the torus $\mathbb{R}^2/\mathcal{L}$ and then to the local charts and then using standard Sobolev embedding theorems. By the Sobolev embedding $\mathcal{H}^1_{\mathcal{L}} \subset L^p_{\mathcal{L}}$, $p < \infty$, and the definitions 2.7 and 2.18,

$$E_\Omega(Q, \Phi) < \infty \quad \text{on} \quad \mathcal{H}^1_{\mathcal{L}} \tag{2.19}$$

(recall that $\Omega \subset \mathbb{R}^2$) and is independent of a choice of Ω.

We say a solution $U_* := (V_*, X_*, \Phi_*)$ of the WS system (2.10)–(2.11) is *energetically stable* if and only if it is a local minimum of the WS energy E_N, in the sense that the spectrum of the L^2-*Hessian* of E_N at U_* on $L^2_{\mathcal{L}}$ (which is real) is non-negative. U_* is said to be *unstable* if it is a saddle point of E_N (so that the spectrum of its Hessian has a negative part).

For an $\mathcal{L}$-equivariant triple U and a fundamental domain Ω of $\mathcal{L}$, we define the energy per fundamental cell by

$$E^{\mathcal{L}}(U) := \frac{1}{|\Omega|} E_\Omega(U), \tag{2.20}$$

where $|\Omega|$ denotes the area of Ω. This energy is independent of the choice of Ω.

In what follows, Ω denotes an arbitrary (but fixed throughout) fundamental domain of $\mathcal{L}$, and $|\mathcal{L}|$, the area of a fundamental cell of $\mathcal{L}$, which is independent of the choice of the cell Ω and is called the covolume of $\mathcal{L}$.

Let $M_W := \frac{1}{\sqrt{2}} g\varphi_0$, $M_Z := \frac{1}{\sqrt{2}\cos\theta} g\varphi_0$ and $M_H := \sqrt{2}\lambda\varphi_0$, where θ is the Weinberg angle defined by $\cos\theta = \frac{g}{\sqrt{g^2+g'^2}}$. These are the masses of the W, Z and Higgs bosons, respectively (this nomenclature will be explained in the discussion following Eq. (3.10)). Finally, let

$$b_* := \frac{g^2\varphi_0^2}{2e} = \frac{M_W^2}{e}, \qquad e := g\sin\theta. \tag{2.21}$$

With the above definitions, we will prove the following:

Theorem 2.1. *The gauge-translational invariant solution* (2.15) *is energetically stable for $b < b_*$ and unstable for $b > b_*$.*

Theorem 2.2. *Let $\mathcal{L}$ be a lattice satisfying $0 < 1 - \frac{M_W^2}{2\pi}|\mathcal{L}| \ll 1$ and assume that $M_Z < M_H$.*[5] *Then there exist $\delta > 0$ such that the following holds:*

(a) Equations (2.10)–(2.11) *have an inhomogeneous solution $U_{\mathcal{L}} \in \mathcal{H}^2_{\mathcal{L}}$ in the δ-ball $B_{\mathcal{H}^2_{\mathcal{L}}}(U_*^{\vec{b}};\delta)$ in $\mathcal{H}^2_{\mathcal{L}}$ around the homogeneous solution* (2.15)*;*

(b) $U_{\mathcal{L}}$ is the unique, up to gauge symmetry transformation, inhomogeneous solution in the δ-ball $B_{\mathcal{H}^2_{\mathcal{L}}}(U_*^{\vec{b}};\delta)$;

(c) $U_{\mathcal{L}}$ has energy per unit area less than vacuum solution (2.15)*: $E^{\mathcal{L}}(U_{\mathcal{L}}) < E^{\mathcal{L}}(U_*^b)$.*

The solutions described in this theorem can be reinterpreted geometrically as representing sections $(\Phi(x))$ and connections $((V(x), X(x)))$ on a $U(2)$ vector bundle over a torus (cf. [20]). However, a vector bundle over a torus is topologically equivalent to a direct sum of line bundles. In our case, this equivalence follows from Eqs. (3.5)–(3.7).

For the next result, we use the topology on the space of (normalized) lattices induced by the standard parameterization of lattices defined as follows. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $(x_1, x_2) \leftrightarrow x_1 + ix_2$ and viewing a lattice $\mathcal{L} \subset \mathbb{R}^2$ as a subset of $\mathbb{C}$ and using a translation and a rotation, any lattice $\mathcal{L} \subset \mathbb{C}$ can be reduced to the form $\mathcal{L} = r\mathcal{L}_\tau$, where $r > 0$, $\mathcal{L}_\tau := \mathbb{Z} + \tau\mathbb{Z}$ and $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$. Furthermore, any two τ's produce the same lattice iff they

[5] This assumption is justified experimentally since $M_Z = 91.1876 \pm 0.0021 GeV/c^2$ [17] and $M_H = 125.09 \pm 0.31$ GeV/c^2 [13]

are related by an element the modular group $SL(2,\mathbb{Z})$ acting on the Poincaré half-plane $\mathbb{H}$ (see, e.g. [4]). Hence, it suffices to restrict τ to the fundamental domain of $SL(2,\mathbb{Z})$,

$$\mathbb{H}/SL(2,\mathbb{Z}) = \left\{\tau \in \mathbb{H} : |\tau| \geq 1,\ -\frac{1}{2} < \operatorname{Re}\tau \leq \frac{1}{2}\right\}. \tag{2.22}$$

Theorem 2.3. *For $M_Z < M_H$, the lattice $\mathcal{L}_*$ minimizing the average energy, $E^{\mathcal{L}}(U_{\mathcal{L}})$, approaches the hexagonal lattice $\mathcal{L}_{\text{hex}}$ as $b \to b_*$ in the sense that the shape parameter τ_* of the lattice $\mathcal{L}_*$ approaches $\tau_{\text{hex}} = e^{i\pi/3}$ in $\mathbb{C}$.*

Now, we construct explicitly the solution (2.15). We define

$$Q^{\vec{b}} = \left(-\frac{i}{2}\tau_3 A^{\vec{b}}\sin\theta, -\frac{i}{2}\tau_0 A^{\vec{b}}\cos\theta\right) \quad \text{and} \quad \Phi^{\vec{b}} \equiv \Phi_0 := (0,\varphi_0), \tag{2.23}$$

where $A^{\vec{b}}(x)$ be a ($U(1)$-) magnetic potential of the constant magnetic field $dA^{\vec{b}} = \omega_{\vec{b}}$ and θ is *Weinberg's angle*, given by $\tan\theta = g'/g$. We have

Lemma 2.4. *The pair $(Q^{\vec{b}}, \Phi_0)$ satisfies (2.10)–(2.11). Moreover, the connection $Q^{\vec{b}}$ has the constant curvature*

$$F_{Q^{\vec{b}}} = -\frac{i}{2}e(\tau_3+\tau_0)\omega_{\vec{b}}, \quad \textit{(with the magnetic field } \operatorname{Tr} F_{Q^{\vec{b}}} = -ie\omega_{\vec{b}}). \tag{2.24}$$

Proof. (2.24) follows easily from $dA^{\vec{b}} = \omega_{\vec{b}}$. To check that $(Q^{\vec{b}}, \Phi_0)$ satisfies (2.10)–(2.11), we observe that $d_{Q^{\vec{b}}}\Phi_0 = (gV^{\vec{b}} + g'X^{\vec{b}})\Phi_0 = (gA^{\vec{b}}\sin\theta\tau_3 + g'A^{\vec{b}}\cos\theta\tau_0)\Phi_0 = eA^{\vec{b}}(\tau_3+\tau_0)\Phi_0$. Since $(\tau_3+\tau_0)\Phi_0 = 0$, this implies $\nabla_Q\Phi_0 = 0$. This gives (2.10) and reduces (2.11) to $d^*_{Q^{\vec{b}}}F_{Q^{\vec{b}}} = 0$, which follows easily from (2.24). □

Our approach is based on a careful examination of the linearization of the WS equations on the homogeneous vacuum. The spectrum of the linearized problem determines the domains of the linear, or energetic, stability and the transition threshold. In the instability domain, we apply an equivariant bifurcation theory. This gives Theorem 2.2(a) and (b). For Theorems 2.2(c) and 2.3, we carefully study the asymptotic behaviour of the energy functions for small values of the bifurcation parameter.

3. Gauge Fixing and W and Z Bosons

In this section, we choose a particular gauge and pass from the fields (one-forms) V and X to more suitable gauge fields. We eliminate a part of the gauge freedom by assuming that the Higgs field Φ is of the form

$$\Phi = (0,\varphi), \tag{3.1}$$

with φ real. (This can be done using only the $SU(2)$ part of the gauge group.) Then

$$\tau_a\Phi \neq 0,\ \ a = 0,1,2,3, \tag{3.2}$$

where, recall, $\tau_a,\ a = 1,2,3$, are the Pauli matrices generating the Lie algebra $su(2)$, and $\tau_0 = \mathbf{1}$. However, there is one linear combination of τ_a's (unique up to a scalar multiple) which annihilates Φ:

$$(\tau_3 + \tau_0)\Phi = 0. \tag{3.3}$$

Thus, for the gauge $\Phi = (0, \varphi)$ the symmetries generated by $\tau_1, \tau_2, \tau_3 - \tau_0$ are broken and the $U(1)$ symmetry generated by $\tau_3 + \tau_0$ remains unbroken. The unbroken gauge symmetry is given by transformations (2.14) with

$$h_1(x) := e^{-\frac{i}{2}\gamma(x)\tau_3} \in SU(2),\ \ h_2(x) := e^{-\frac{i}{2}\gamma(x)\tau_0} \in U(1), \tag{3.4}$$

where $\gamma \in C^1(\mathbb{R}^3, \mathbb{R})$.

Continuing in the gauge $\Phi = (0, \varphi)$ and writing $V = -\frac{i}{2}\tau_a V^a$[6] and $X = -\frac{i}{2}\tau_0 X^0$, where X^0 and $V^a, a = 1,2,3$, are real fields (since V takes values in $su(2)$ and therefore $V^* = -V$), we pass to the new fields corresponding to the broken and unbroken generators, $\tau_3 - \tau_0$ and $\tau_3 + \tau_0$, respectively:

$$Z = V^3 \cos\theta - X^0 \sin\theta \quad \text{and} \quad A = V^3 \sin\theta + X^0 \cos\theta, \tag{3.5}$$

where recall, θ is Weinberg's angle, defined by $\tan\theta = g'/g$. Note that Z and A are real fields. Moreover, it is convenient to pass from the remaining two components, V^1, V^2, of V to a single complex field

$$W = \frac{1}{\sqrt{2}}(V^1 - iV^2). \tag{3.6}$$

The gauge invariance of the original field equations with the unbroken gauge symmetry given by transformations (2.14) with (3.4) leads to the invariance under following gauge transformations:

$$\tilde{T}^{\text{gauge}}_{\gamma} : (W, A, Z, \varphi) \mapsto (e^{i\gamma}W, A - \frac{1}{e}d\gamma, Z, \varphi), \tag{3.7}$$

for $\gamma \in C^1(\mathbb{R}^3, \mathbb{R})$, where $e^{i\gamma}W = \sum e^{i\gamma}W_i dx^i$ for $W = \sum W_i dx^i$, e is the electron charge. Here, we replaced $\Phi := (0, \varphi)$ by φ.

The WS energy in terms of W, Z, A and φ fields in 3D is given in (D.1), "Appendix D". The WS equations in terms of W, Z, A and φ in 3D can be found by taking variational derivatives of this energy w.r.t different fields.

In terms of W, A, Z and φ fields, the vacua (2.15) of the Weinberg–Salam model become (up to a gauge symmetry):

$$(0, A^b(x), 0, \varphi_0), \tag{3.8}$$

where recall, $A^b(x)$ is a magnetic potential for the constant magnetic field of strength b in the x^3-direction, $dA^b(x) = b dx_1 \wedge dx_2$, and φ_0 is a positive constant from (2.7). We choose the gauge so that $A^b(x)$ is of the form

$$A^b(x) = \frac{b}{2}(-x_2 dx_1 + x_1 dx_2). \tag{3.9}$$

[6]Note that the lower indices i, j, k, as in $A = A_i dx^i$, refer to vectorial components and run through $1, 2$, while the upper indices a, b, c, as in $V = -\frac{i}{2}\tau_a V^a$, refer to $U(2)$-algebra components.

We will show that for a large magnetic field b, these homogeneous vacua become unstable and new, inhomogeneous vacua emerge from them. This is a bifurcation problem from the branch of gauge-translationally invariant (homogeneous) solutions, (3.8).

Since we consider the WS system with the fields independent of the third dimension x^3, i.e. in $\mathbb{R}^2$, we can choose the gauge with $V_3 = X_3 = 0$ (and hence $W_3 = A_3 = Z_3 = 0$).

Also, we will work in a fixed coordinate system, $\{x^i\}_{i=1}^2$, and write the fields as $W = W_i dx^i$, $Z = Z_i dx^i$ and $A = A_i dx^i$. For ease of comparing our arguments with earlier results, and given that we use the standard Euclidean metric in $\mathbb{R}^2$, we identify (complex) one-forms W, Z and A with the (complex) vector fields $(W_1, W_2), (Z_1, Z_2)$ and (A_1, A_2). With this, we show in "Appendix D.2" that in this case, WS energy functional (2.7) can be written as

$$\begin{aligned} E_\Omega(W, A, Z, \varphi) = \int_\Omega \big[&|\operatorname{curl}_{gV^3} W|^2 + \frac{1}{2}|\operatorname{curl} Z|^2 + \frac{1}{2}|\operatorname{curl} A|^2 \\ &+ \frac{1}{2}g^2\varphi^2|W|^2 + \frac{1}{2}\kappa g^2\varphi^2|Z|^2 + \frac{g^2}{2}|\overline{W} \times W|^2 \\ &+ ig(\operatorname{curl} V^3)\overline{W} \times W + |\nabla\varphi|^2 + \frac{1}{2}\lambda(\varphi^2 - \varphi_0^2)^2 \big], \end{aligned} \quad (3.10)$$

where $\kappa := \frac{g^2}{2\cos^2\theta}$, $\operatorname{curl}_U W := (\nabla_U)_1 W_2 - (\nabla_U)_2 W_1$, $(\nabla_U)_i := \partial_i - iU_i$, $\partial_i \equiv \partial_{x^i}$ (for a $\mathfrak{u}(1)-$valued vector field U), $\xi \times \eta := \xi_1\eta_2 - \xi_2\eta_1$ and $\operatorname{curl} V^3 := \partial_1 V_2^3 - \partial_2 V_1^3$. It follows from (3.5) that $V^3 = Z\cos\theta + A\sin\theta$.

Expanding (3.10) in φ around φ_0, we see that the W, Z and ϕ (Higgs) fields have the masses $M_W := \frac{1}{\sqrt{2}}g\varphi_0$, $M_Z := \frac{1}{\sqrt{2}\cos\theta}g\varphi_0$ and $M_H = \sqrt{2}\lambda\varphi_0$, respectively.

Using the relation $\xi \times \eta = J\xi \cdot \eta$, where $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^2$ and J is the symplectic matrix,

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (3.11)$$

we find the Euler–Lagrange equations for (3.10), which give the WS system (2.10)–(2.11) in 2D in terms of the fields W, A, Z and φ

$$[\operatorname{curl}^*_{gV^3} \operatorname{curl}_{gV^3} + \frac{g^2}{2}\varphi^2 - ig(\operatorname{curl} V^3)J + g^2(\overline{W} \times W)J]W = 0, \quad (3.12)$$

$$\operatorname{curl}^* \operatorname{curl} A + 2e\operatorname{Im}[(\operatorname{curl}_{gV^3} W)J\overline{W} - \operatorname{curl}^*(\overline{W}_1 W_2)] = 0, \quad (3.13)$$

$$[\operatorname{curl}^* \operatorname{curl} + \kappa\varphi^2]Z + 2g\cos\theta\operatorname{Im}[(\operatorname{curl}_{gV^3} W)J\overline{W} - \operatorname{curl}^*(\overline{W}_1 W_2)] = 0, \quad (3.14)$$

$$[-\Delta + \lambda(\varphi^2 - \varphi_0^2) + \frac{g^2}{2}|W|^2 + \frac{1}{2}\kappa|Z|^2]\varphi = 0, \quad (3.15)$$

where, recall, $\kappa = \frac{g^2}{2\cos^2\theta}$, $V^3 = Z\cos\theta + A\sin\theta$ and Δ is the standard Laplacian. (For a derivation of (3.12)–(3.15) from (3.10), see "Appendix D.2" and also [26,43].) Of course, (3.12)–(3.15) can also be derived directly from WS system (2.10)–(2.11).

In terms of the (W, A, Z, φ) fields, the lattice gauge—periodicity (2.17) is expressed as:

$$(\tilde{T}^{\mathrm{gauge}}_{\gamma_s})^{-1} T^{\mathrm{trans}}_s (W, A, Z, \varphi) = (W, A, Z, \varphi), \tag{3.16}$$

for all $s \in \mathcal{L}$, where $\gamma_s \in C^1(\mathbb{R}^2, \mathbb{R})$ for all $s \in \mathcal{L}$, $\tilde{T}^{\mathrm{gauge}}_{\gamma}$ given in (3.7) and T^{trans}_s is the group of translations, $T^{\mathrm{trans}}_s f(x) = f(x+s)$. We say that (W, A, Z, φ) satisfying (3.16) is an $\mathcal{L}$-*equivariant* state. By evaluating the effect of translation by $s+t$ in two different ways, we see that the family of functions γ_s has the co-cycle property[7]

$$\gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}, \qquad \forall s, t \in \mathcal{L}. \tag{3.17}$$

Since T^{trans}_s is an Abelian group, the co-cycle condition (3.17) implies that, for any basis $\{j_1, j_2\}$ in $\mathcal{L}$, the quantity

$$c(\gamma_s) = \frac{1}{2\pi}(\gamma_{j_2}(x+j_1) + \gamma_{j_1}(x) - \gamma_{j_1}(x+j_2) - \gamma_{j_2}(x)) \tag{3.18}$$

is independent of x and of the choice of the basis $\{j_1, j_2\}$, and is an integer. This topological invariant is equal to the degree of the corresponding line bundle.

Using Stokes' theorem, one can show, for any A satisfying (3.16)–(3.18), that the magnetic flux through any fundamental domain Ω of the lattice $\mathcal{L}$ is quantized:

$$\frac{e}{2\pi}\int_{\Omega} dA = n, \tag{3.19}$$

where e is defined after (3.7) and $n = c(\gamma_s) \in \mathbb{Z}$ defined in (3.18). The left-hand side of (3.19) is called the *Chern number* of the line bundle corresponding to γ_s. (We note that n is independent of the choice of Ω.)

The vacuum state (3.8) is $\mathcal{L}$-equivariant if and only if the magnetic field b is given by the relation

$$b = \frac{2\pi}{e|\mathcal{L}|} n, \tag{3.20}$$

where, by definition, $|\mathcal{L}| = |\Omega|$ for any fundamental cell Ω. In particular, b is quantized. For such b, the vector field $\frac{1}{e}A^b$ satisfies (3.19).

Furthermore, due to the reflection symmetry of the problem, we may assume that $b \geq 0$. Clearly, we have:

Lemma 3.1. *Equations* (2.10)–(2.11) *for* $\mathcal{L}$-equivariant *fields* (2.17) *in the gauge* $\Phi = (0, \varphi)$ *are equivalent to Equations* (3.12)–(3.15) *for* $\mathcal{L}$-equivariant *fields* (3.16)*, with the equivalence realized by the transformation* (3.5)–(3.6)*.*

Finally, we use the invariance of (3.12)–(3.15) under the gauge transformation (3.7) to choose a convenient gauge for the fields $W(x)$ and $A(x)$. We say that the fields (W, A, Z, φ) and (W', A', Z', φ') are *gauge-equivalent* if there is $\gamma \in C^1(\mathbb{R}^2, \mathbb{R})$ such that

$$(W', A', Z', \varphi') = \tilde{T}^{\mathrm{gauge}}_{\gamma}(W, A, Z, \varphi).$$

---

[7] A function $\gamma_s : \mathcal{L} \times \mathbb{R}^2 \to G$ satisfying the co-cycle property (3.17) is called the automorphy exponent and $e^{i\gamma_s}$, the automorphy factor.

Clearly, if (W, A, Z, φ) and (W', A', Z', φ') are gauge-equivalent, then (W, A, Z, φ) solves (3.12)–(3.15) if and only if (W', A', Z', φ') solves (3.12)–(3.15). The following proposition was first used in [31] and proven in [45] (an alternate proof is given in "Appendix A" of [46]):

Proposition 3.2. *Let (W', A', Z', φ') be an $\mathcal{L}$-equivariant state and let b be given by (3.20). Then, there is a $\mathcal{L}$-equivariant state (W, A, Z, φ), gauge-equivalent to (W', A', Z', φ'), which satisfies (3.16), with $\chi_s(x) = \frac{eb}{2}\, s \wedge x + k_s$, i.e. such that, $\forall s \in \mathcal{L}$,*

$$W(x+s) = e^{i(\frac{eb}{2} s\wedge x + k_s)} W(x), \tag{3.21}$$

$$A(x+s) = A(x) + \frac{b}{2} Js, \tag{3.22}$$

$$\operatorname{div} A = 0, \tag{3.23}$$

$$Z(x+s) = Z(x), \quad \varphi(x+s) = \varphi(x). \tag{3.24}$$

Here k_s satisfies the condition $k_{s+t} - k_s - k_t - \frac{eb}{2}\, s \wedge t \in 2\pi\mathbb{Z}$, for all $s, t \in \mathcal{L}$, the matrix J is given in (3.11).

Note that with the gauge (3.23), the homogeneous vacua (3.8) satisfy (3.21)–(3.24).

Our goal is to prove the instability of the vacuum state (3.8) and the existence of $\mathcal{L}-$equivariant (in the sense of (3.16)) solutions to transformed WS system (3.12)–(3.15) having the properties described in Theorems 2.2 and 2.3.

4. Rescaling

In this section, we rescale transformed WS system (3.12)–(3.15) to keep the lattice size fixed. Specifically, we define the rescaled fields (w, a, z, ϕ) to be

$$(w(x), a(x), z(x), \phi(x)) := (rW(rx), rA(rx), rZ(rx), r\varphi(rx)), \tag{4.1}$$

$$r := \sqrt{\frac{n}{eb}} = \sqrt{\frac{|\Omega|}{2\pi}}. \tag{4.2}$$

where in the second equality (4.2), we used (3.20). Clearly, $(W(x), A(x), Z(x), \varphi(x))$ is $\mathcal{L}$-equivariant if and only if $(w(x), a(x), z(x), \phi(x))$ is $\mathcal{L}'$-equivariant, where

$$\mathcal{L}' := \frac{1}{r}\mathcal{L}.$$

Now, the rescaled lattice $\mathcal{L}'$ is independent of b and the size of a fundamental domain, Ω', of $\mathcal{L}'$ is fixed as $|\Omega'| = 2\pi$.

Plugging the rescaled fields into (3.12)–(3.15) gives the rescaled Weinberg–Salem equations:

$$[\operatorname{curl}_\nu^* \operatorname{curl}_\nu + \frac{g^2}{2}\phi^2 - i(\operatorname{curl}\nu)J + g^2(\overline{w} \times w)J]w = 0, \tag{4.3}$$

$$\operatorname{curl}^* \operatorname{curl} a + 2e\,\operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)] = 0, \tag{4.4}$$

$$[\operatorname{curl}^* \operatorname{curl} + \kappa\phi^2]z + 2g\cos\theta \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)] = 0, \tag{4.5}$$

$$[-\Delta + \lambda(\phi^2 - \xi^2) + \frac{g^2}{2}|w|^2 + \frac{1}{2}\kappa|z|^2]\phi = 0, \tag{4.6}$$

where $\xi := r\varphi_0$ (with r given in (4.2)), $\nu := g(a\sin\theta + z\cos\theta)$ and, recall, $\operatorname{curl}_q w = \nabla_1 w_2 - \nabla_2 w_1$, $\nabla_i := \partial_i - iq_i$, $\partial_i \equiv \partial_{x^i}$ (for a $\mathfrak{u}(1)-$valued vector-field iq) and, recall, $\overline{w} \times w := \overline{w}_1 w_2 - \overline{w}_2 w_1$. We define the rescaled energy by:

$$\mathcal{E}_{\Omega'}(w, a, z, \phi; r) := r^2 E_\Omega(W, A, Z, \varphi). \tag{4.7}$$

with (W, A, Z, φ) related to (w, a, z, ϕ) by (4.1) and $E_\Omega(W, A, Z, \varphi)$ given in (3.10). Explicitly, we have

$$\begin{aligned}\mathcal{E}_{\Omega'}(w, a, z, \phi; r) = \int_{\Omega'} \big(&|\operatorname{curl}_\nu w|^2 + \frac{1}{2}|\operatorname{curl} a|^2 + \frac{1}{2}|\operatorname{curl} z|^2 \\ &+ \frac{1}{2}g^2\phi^2|w|^2 + \frac{1}{2}\kappa\phi^2|z|^2 + \frac{g^2}{2}|\overline{w} \times w|^2 \\ &+ i(\operatorname{curl}\nu)\overline{w} \times w + |\nabla\phi|^2 + \frac{1}{2}\lambda(\phi^2 - \xi^2)^2\big).\end{aligned} \tag{4.8}$$

We note that after rescaling, the average magnetic flux per fundamental domain becomes n/e and the vacuum solution (3.8),

$$m^{n,r} := \left(0, \frac{1}{e}a^n, 0, \xi\right), \tag{4.9}$$

where $a^n(x) \equiv A^n(x) = \frac{n}{2}Jx$,. Furthermore, (3.16) and Proposition 3.2 imply that (w, a, z, ϕ) satisfy

$$w(x+s) = e^{i\left(\frac{n}{2}s \times x + c_s\right)}w(x) \text{ for all } s \in \mathcal{L}', \tag{4.10}$$

$$a(x+s) = a(x) + \frac{n}{2e}Js \text{ for all } s \in \mathcal{L}', \tag{4.11}$$

$$\operatorname{div} a = 0, \tag{4.12}$$

$$z(x+s) = z(x), \quad \phi(x+s) = \phi(x) \text{ for all } s \in \mathcal{L}', \tag{4.13}$$

where c_s satisfies the condition $c_{s+t} - c_s - c_t - \frac{n}{2}s \times t \in 2\pi\mathbb{Z}$, for all $s, t \in \mathcal{L}'$.

Finally, the Sobolev spaces here, denoted again by $\mathcal{H}^s_{\mathcal{L}'}$, can be obtained by rescaling the Sobolev spaces defined above or defined directly, again as above, see (2.18) and the text around it. Similarly to (2.19), by a Sobolev embedding theorem, the rescaled energy is finite,

$$E_{\Omega'}(w, a, z, \phi; r) < \infty \quad \text{on} \quad \mathcal{H}^1_{\mathcal{L}'} \tag{4.14}$$

and is independent of a choice of Ω'.

5. The Linearized Problem

In this section, we prove Theorem 2.1, describing the stability properties of the vacuum (3.8). Equivalently, we will investigate the energetic stability of the rescaled vacuum solution (4.9) of the rescaled WS equations (4.3)–(4.6).

Let $m := (w, a, z, \phi)$ and denote by $G(b, m) \equiv G(m)$ the map $G : \mathcal{H}^2_{\mathcal{L}'} \to \mathbb{C}^7$ given by the left-hand side of (4.3)–(4.6), written explicitly as

$$G(b, m) \equiv G(m) = (G_1(m), \dots, G_4(m)), \tag{5.1}$$

$$G_1(m) := \left[\operatorname{curl}^*_\nu \operatorname{curl}_\nu + \frac{g^2}{2}\phi^2 - i(\operatorname{curl}\nu)J + g^2(\overline{w} \times w)J\right] w, \tag{5.2}$$

$$G_2(m) := \operatorname{curl}^* \operatorname{curl} a + 2e \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)], \tag{5.3}$$

$$G_3(m) := [\operatorname{curl}^* \operatorname{curl} + \kappa\phi^2]z + 2g\cos\theta \operatorname{Im}[(\operatorname{curl}_\nu w)J\overline{w} - \operatorname{curl}^*(\overline{w}_1 w_2)], \tag{5.4}$$

$$G_4(m) := \left[-\Delta + \lambda(\phi^2 - \xi^2) + \frac{g^2}{2}|w|^2 + \frac{1}{2}\kappa|z|^2\right]\phi, \tag{5.5}$$

where, recall, J is the symplectic matrix given in (3.11), $\xi := r\varphi_0$ (with r given in (4.2)), $\nu := g(a\sin\theta + z\cos\theta)$, Δ is the standard Laplacian and the parameter b enters through periodicity conditions (4.10)–(4.13). Now, the WS system can be written as:

$$G(m) = 0. \tag{5.6}$$

Recall the definition of stability given above Eq. (2.20). To apply it to the rescaled WS Eqs. (4.3)–(4.6), we observe that the map G is the L^2-gradient, $\operatorname{grad}_{L^2} \mathcal{E}_{\Omega'}$, of the energy $\mathcal{E}_{\Omega'}$, see (4.8), considered as a functional of $u = (w, a, z, \phi)$. Namely, $\langle G(m), \xi\rangle_{L^2} = \delta\mathcal{E}_{\Omega'}(m)\xi$, where $\delta\mathcal{E}_{\Omega'}(m)$ is the Gâteau derivative

$$\delta\mathcal{E}_{\Omega'}(u)\xi \equiv \frac{d}{d\tau}\mathcal{E}_{\Omega'}(u + \tau\xi)|_{\tau=0}, \tag{5.7}$$

of $E_{\Omega'}$ at m, defined on the space of variations $\mathcal{Y}$ tangent to the space of L^2_{loc} functions of the form (w, a, z, ϕ) satisfying the gauge—periodicity conditions (4.10)–(4.13):

$$\mathcal{Y} := L^2_n \times L^2_0 \times L^2_0 \times L^2. \tag{5.8}$$

Here L^2_n, L^2_0 and L^2 are given by

$$L^2_n := \left\{w \in L^2_{loc}(\mathbb{R}^2, \mathbb{C}^2) : w(x + s) = e^{i\left(\frac{n}{2}s\times x + c_s\right)}w(x)\ \forall s \in \mathcal{L}'\right\}, \tag{5.9}$$

$$L^2_0 := \left\{\alpha \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}^2) : \alpha(x + s) = \alpha(x)\ \forall s \in \mathcal{L}',\ \operatorname{div}\alpha = 0\right\}, \tag{5.10}$$

$$L^2 := \{\psi \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}) : \psi(x + s) = \psi(x)\ \forall s \in \mathcal{L}'\} \tag{5.11}$$

(see (4.10)–(4.12)).

Since $G(m) = \operatorname{grad}_{L^2} \mathcal{E}_{\Omega'}(m)$, the L^2-Hessian for $\mathcal{E}_{\Omega'}$ and m is the formally symmetric operator

$$\mathcal{E}''_{\Omega'}(m) := \delta \operatorname{grad}_{L^2} \mathcal{E}_{\Omega'}(m) = \delta G(m),$$

Denote the L^2-Hessian at the vacuum solution $m^{n,r}$ (see (4.9)) by

$$L_{n,\mu} := \delta G(m^{n,r}).$$

As seen from its explicit form given below, the operator $L_{n,\mu}$, acting on the space $\mathcal{Y}$, is self-adjoint, and therefore, its spectrum is real.

Thus, applied to the rescaled WS equations (4.3)–(4.6), the definition of stability can be rephrased as:

the vacuum solution $m^{n,r}$ is *energetically stable* (respectively, *unstable*) if and only if $\inf \operatorname{spec}(L_{n,\mu}) \geq 0$ (respectively, $\inf \operatorname{spec}(L_{n,\mu}) < 0$).

We consider the operator $L_{n,\mu}$ on the space $\mathcal{Y}$, with the domain

$$\mathcal{X} := \mathcal{H}_n^2 \times \mathcal{H}_0^2 \times \mathcal{H}_0^2 \times \mathcal{H}^2, \tag{5.12}$$

where $\mathcal{H}_n^s$, $\mathcal{H}_0^s$ and $\mathcal{H}^s$ are the respective Sobolev spaces for the L^2-spaces (5.9)–(5.11), with inner products given (for $s \in \mathbb{Z}_{\geq 0}$) by

$$\langle w, w' \rangle_{\mathcal{H}_n^s} := \frac{1}{|\Omega'|} \sum_{i=1}^{2} \sum_{|\gamma| \leq s} \int_{\Omega'} \overline{(\nabla_{a^n})^\gamma w_i} (\nabla_{a^n})^\gamma w_i', \tag{5.13}$$

$$\langle a, a' \rangle_{\mathcal{H}_0^s} := \frac{1}{|\Omega'|} \sum_{i=1}^{2} \sum_{|\gamma| \leq s} \int_{\Omega'} \partial^\gamma a_i \partial^\gamma a_i', \tag{5.14}$$

$$\langle \psi, \psi' \rangle_{\mathcal{H}^s} := \frac{1}{|\Omega'|} \sum_{|\gamma| \leq s} \int_{\Omega'} \partial^\gamma \psi \partial^\gamma \psi', \tag{5.15}$$

where $w^\# = (w_1^\#, w_2^\#)$, $a^\# = (a_1^\#, a_2^\#)$, Ω' is an arbitrary fundamental domain of the lattice $\mathcal{L}'$ and γ is a multi-index. The $\mathcal{L}'$-equivariance of the above functions implies that these inner products do not depend on the choice of fundamental domain Ω'.

We compute the linear operator $L_{n,\mu}$ explicitly. In what follows we use the notation $\oplus_j A_j$ for diagonal operator-matrices with the operators A_j on the diagonal.

Passing from the parameter $\xi = r\varphi_0$, or r, to the parameter $\mu := g^2\xi^2/2$ and using that $\nu\big|_{a=a^n/e, z=0} = \frac{1}{e} a^n g \sin\theta = a^n$, we find

$$L_{n,\mu} = \oplus_{j=1}^4 H_j, \tag{5.16}$$

$$H_1(\mu) := \operatorname{curl}_{a^n}^* \operatorname{curl}_{a^n} + \mu - niJ, \tag{5.17}$$

$$H_2(\mu) := \operatorname{curl}^* \operatorname{curl}, \tag{5.18}$$

$$H_3(\mu) := \operatorname{curl}^* \operatorname{curl} + \frac{\mu}{\cos^2\theta}, \tag{5.19}$$

$$H_4(\mu) := -\Delta + \frac{4\lambda\mu}{g^2}, \tag{5.20}$$

where, recall, $\operatorname{curl}_q w = (\nabla_q)_1 w_2 - (\nabla_q)_2 w_1$, $(\nabla_q)_i := \partial_i - iq_i$, $\partial_i \equiv \partial_{x^i}$. (Note that the matrix iJ is self-adjoint.)

The gauge invariance of Eq. (5.6) and the partial symmetry breaking of vacuum solution (4.9) imply that $L_{n,\mu=n}$ has the gauge zero mode:

$$L_{n,\mu=n}\Gamma_f = 0, \;\; \Gamma_f := (0, \nabla f, 0, 0). \tag{5.21}$$

For a null vector Γ_f defined in (5.21) to be in $\mathcal{X}$, f must satisfy $\operatorname{div}(\nabla f) = -\Delta f = 0$. This implies that f is a linear function, $f(x) = c \cdot x + d$ for some $c \in \mathbb{R}^2$ and $d \in \mathbb{R}$, and so

$$\Gamma_f \in \mathcal{X} \Longrightarrow \Gamma_f = (0, c, 0, 0). \tag{5.22}$$

In this section, we shall prove the following result implying Theorem 2.1:

Theorem 5.1. *The operator $L_{n,\mu}$ on the space $\mathcal{X}$ has purely discrete spectrum. For $\mu \neq n$, $L_{\mu,n}$ has the multiplicity 2 eigenvalue 0 with the eigenfuctions $(0, e_i, 0, 0)$, $i = 1, 2, e_1 = (1, 0), e_2 = (0, 1)$ (see (5.22)).*

Furthermore, the smallest non-zero eigenvalue given by $\mu - n$, having multiplicity n. For $\mu = n$, the eigenvalue 0 has the multiplicity $n + 2$.

Theorem 5.1 follows from Propositions 5.2 and 5.3 given below. □

Proposition 5.2. *The operators $H_2(\mu)$, $H_3(\mu)$ and $H_4(\mu)$ have purely discrete spectra. Furthermore, $H_3(\mu)$ and $H_4(\mu)$ are strictly positive and $H_2(\mu)$ is non-negative and has the null space $\{(0, c, 0, 0) : c \in \mathbb{R}^2\}$ of dimension 2.*

Proof. The strict positivity of $H_3(\mu)$ and $H_4(\mu)$ and the non-negativity of $H_2(\mu)$ are obvious. The discreteness of the spectra and the form of the null space of $H_2(\mu)$ follow from the discreteness of the spectrum of the Laplacian on compact domains and the identity $\operatorname{curl}^* \operatorname{curl} v = -\Delta v$ when $\operatorname{div}(v) = 0$. To compute the null space of $H_2(\mu)$, we observe that the solutions of the equations $\Delta v = 0$ and $\operatorname{div}(v) = 0$ are constant vectors in $\mathbb{R}^2$. □

Let $\nabla_q := \nabla - iq = ((\nabla_q)_1, (\nabla_q)_2)$, $(\nabla_q)_j := \partial_j - iq_j$, and $\Delta_q := \nabla_q^2 = -\nabla_q^* \nabla_q$. We also introduce the complexified covariant derivative $\bar{\partial}_q := (\nabla_q)_1 + i(\nabla_q)_2$.

We have

Proposition 5.3. *(i) $H_1(\mu)$ is a self-adjoint operator on $\mathcal{H}_n^2$ and its spectrum is given by*

$$\sigma(H_1(\mu)) = \{(m - 1)n + \mu : m \in \mathbb{Z}_{\geq 0}\} \cup \{\mu\}, \tag{5.23}$$

where $n := eb|\mathcal{L}|/2\pi$.

(ii) The eigenspace of the eigenvalue $-n + \mu$ is n-dimensional and is spanned by functions of the form[8]

$$\chi = (\beta, i\beta), \qquad \operatorname{curl}_{a^n} \chi = i\bar{\partial}_{a^n} \beta = 0, \tag{5.24}$$

and the eigenspace of the eigenvalue μ is of the form

$$\operatorname{Null}(H_1(\mu) - \mu) = \{\nabla_{a^n} f : f \in \mathcal{H}_n^3\}. \tag{5.25}$$

In the proof of this proposition, we use the following standard result whose proof, for reader's convenience, is given in "Appendix G":

Proposition 5.4. *The operator $-\Delta_{a^n}$ is self-adjoint on its natural domain and its spectrum is given by:*

$$\sigma(-\Delta_{a^n}) = \{\, (m + 1)n : m \in \mathbb{Z}_{\geq 0} \,\}, \tag{5.26}$$

with each eigenvalue is of the multiplicity n. Moreover,

$$\operatorname{Null}(-\Delta_{a^n} - n) = \operatorname{Null} \bar{\partial}_{a^n}. \tag{5.27}$$

[8] β can be expressed in terms of the Jacobi theta function, see Proposition 5.4 and "Appendix G".

In more detail, with $z = (x^1 + ix^2)/\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}$ *and* τ *coming from* $\mathcal{L}' = \mathbb{Z} + \tau\mathbb{Z}$, *we have*

$$\operatorname{Null}(-\Delta_{a^n} - n) = e^{\frac{in}{2}x^2(x^1+ix^2)}V_n, \tag{5.28}$$

where V_n *is spanned by functions of the form*

$$\theta(z,\tau) := \sum_{m=-\infty}^{\infty} c_m e^{i2\pi m z},\ c_{m+n} = e^{-in\pi z}e^{i2m\pi\tau}c_m. \tag{5.29}$$

Remark 5.5. Functions of the form (5.29) are determined entirely by the values of $c_0, \dots, c_{n-1}$ and therefore form an n-dimensional vector space..

Proof of Proposition 5.3. First, we will show that $\mathcal{H}^2_n = \mathcal{Y} \oplus \mathcal{Z}$ (the Hodge decomposition), where

$$\mathcal{Y} :=\{w \in \mathcal{H}^2_n : \operatorname{div}_{a^n} w = 0\}, \tag{5.30}$$

$$\mathcal{Z} :=\{w \in \mathcal{H}^2_n : w = \nabla_{a^n} f \text{ for some } f \in \mathcal{H}^3_n\}, \tag{5.31}$$

with $\operatorname{div}_{a^n} w := (\nabla_{a^n})_1 w_1 + (\nabla_{a^n})_2 w_2 = -\nabla^*_{a^n}$. We write any $w \in \mathcal{H}^2_n$ as $w = w_0 + \nabla_{a^n} f$, where f solves the equation $\Delta_{a^n} f = \operatorname{div}_{a^n} w$ and w_0 is defined by this relation. By Proposition 5.4, 0 is not in the spectrum of Δ_{a^n} and therefore the equation $\Delta_{a^n} f = \operatorname{div}_{a^n} w$ has the unique solution $f \in \mathcal{H}^3_n$. Then, since $\Delta_{a^n} := \operatorname{div}_{a^n} \nabla_{a^n}$, we have $\operatorname{div}_{a^n} w_0 = 0$. This proves $\mathcal{H}^2_n = \mathcal{Y} \oplus \mathcal{Z}$.

Now, recall that the operator $H_1(\mu)$ acts on complex vectors $w = (w_1, w_2)$. The definition $H_1(\mu) := \operatorname{curl}^*_{a^n} \operatorname{curl}_{a^n} - niJ + \mu$ and the relations $\operatorname{curl}^*_{a^n} = -J\nabla_{a^n}$ and

$$\operatorname{curl}_{a^n} \nabla_{a^n} = [(\nabla_{a^n})_1, (\nabla_{a^n})_2] = -in$$

yield that $(H_1(\mu) - \mu)\nabla_{a^n} f = 0$, which proves that the μ-eigenspace of $H_1(\mu)$ is of the form (5.25) giving the second part of (ii).

By the above the subspace $\mathcal{Y}$ is invariant under $H_1(\mu)$. To compute the spectrum of the operator $H_1(\mu)$ on the subspace $\mathcal{Y}$, we use the definitions of $\operatorname{curl}_{a^n}$ and $\operatorname{curl}^*_{a^n}$ and recall the relation $[(\nabla_{a^n})_1, (\nabla_{a^n})_2] = -in$ to compute

$$\operatorname{curl}^*_{a^n} \operatorname{curl}_{a^n} = -\Delta_{a^n} - niJ + \nabla_{a^n} \operatorname{div}_{a^n}.$$

By above, we have $H_1(\mu)w_0 = (-\Delta_{a^n} - 2niJ - \mu)w_0$, for any $w_0 \in \mathcal{Y}$.

(We check using $\operatorname{div}_{a^n}(-\Delta_{a^n} - 2niJ)w_0 = (-\Delta_{a^n}) \operatorname{div}_{a^n} w_0 = 0$, that $H_1(\mu)$ sends $\mathcal{Y}$ to $\mathcal{Y}$ and hence, $\mathcal{Y}$ is invariant under $H_1(\mu)$.) Thus, we conclude that

$$H_1(\mu)(w_0 \oplus 0) = (h_1 - \mu)w_0 \oplus 0, \tag{5.32}$$

$$h_1 := -\Delta_{a^n} - 2niJ. \tag{5.33}$$

Identifying one-forms with vector fields, we compute

$$U^*(iJ)U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ U := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{5.34}$$

which gives

$$U^* h_1 U = \begin{pmatrix} -\Delta_{a^n} + 2n & 0 \\ 0 & -\Delta_{a^n} - 2n \end{pmatrix}. \tag{5.35}$$

By Proposition 5.4, we know that

$$\sigma(-\Delta_{a^n}) = \{\, (m+1)n : m \in \mathbb{Z}_{\geq 0}\} \tag{5.36}$$

and so the spectrum of $H_1(\mu)$ on $\mathcal{Y}$ is given by the first set on the r.h.s. of (5.23). Hence, by $\mathcal{H}_n^2 = \mathcal{Y} \oplus \mathcal{Z}$, (5.23) follows, giving (i).

Furthermore, by (5.35) and (5.36), any eigenvector χ of h_1 corresponding to the eigenvalue $-n$ must be of the form

$$\chi = U(0,\beta) = \frac{1}{\sqrt{2}}(\beta, i\beta), \tag{5.37}$$

where β satisfies

$$-\Delta_{a^n}\beta = n\omega. \tag{5.38}$$

This relation, together with the equation $\mathrm{Null}(-\Delta_{a^n} - n) = \mathrm{Null}\,\bar{\partial}_{a^n}$ (see (5.27)), implies $\overline{\partial}_{a^n}\beta = 0$. Since $\mathrm{curl}_{a^n}\chi = i\overline{\partial}_{a^n}\beta$, this gives

$$\mathrm{curl}_{a^n}\chi = i\overline{\partial}_{a^n}\omega = 0. \tag{5.39}$$

Furthermore, by Proposition 5.4, the space of such functions is n-dimensional. Thus (after rescaling ω by a factor of $\sqrt{2}$) χ is of the form (5.24). This gives also the first part of (ii) completing the proof of the proposition. □

We see that the operator $H_1(\mu)$ is non-negative for the magnetic fields satisfying $b < b_* := g^2\varphi_0^2/2e = M_W^2/e$ and acquires a negative eigenvalue $\mu - n = (b_*/b - 1)\,n$ of multiplicity n as the magnetic field increases to $b > b_*$. Theorem 2.1 follows by undoing the rescaling (4.1)–(4.2).

6. Setup of the Bifurcation Problem

We substitute $a = \frac{1}{e}a^n + \alpha$ (with $\mathrm{div}(\alpha) = 0$), $\phi = \xi + \psi$, $\nu = a^n + \tilde{\nu}$ and $\xi = \sqrt{2\mu}/g$ into (4.3)–(4.6) and relabel the unknowns w, α, z, ψ as u_1, u_2, u_3, u_4 to obtain the system

$$H_i u_i = -J_i(\mu, u),\ i = 1, \ldots, 4, \tag{6.1}$$

where $u = (u_1, u_2, u_3, u_4) \equiv (w, \alpha, z, \psi)$, the operators H_i on the left-hand side are defined in (5.17) - (5.20), and

$$J_1(\mu, u) := Mw + \frac{g^2}{2}\psi^2 w + g\sqrt{2\mu}\psi w - i(\mathrm{curl}\,\tilde{\nu})Jw + g^2(\overline{w} \times w)Jw, \tag{6.2}$$

$$J_2(\mu, u) := 2e\,\mathrm{Im}[(\mathrm{curl}_\nu w)J\overline{w} - \mathrm{curl}^*(\overline{w}_1 w_2)], \tag{6.3}$$

$$J_3(\mu, u) := 2g\cos\theta\,\mathrm{Im}[(\mathrm{curl}_\nu w)J\overline{w} - \mathrm{curl}^*(\overline{w}_1 w_2)] + \kappa\frac{2\sqrt{2\mu}}{g}\psi z + \kappa\psi^2 z, \tag{6.4}$$

$$J_4(\mu, u) := 3\lambda\frac{\sqrt{2\mu}}{g}\psi^2 + \lambda\psi^3 + \frac{g^2}{2}|w|^2\left(\frac{\sqrt{2\mu}}{g} + \psi\right) + \frac{1}{2}\kappa|z|^2\left(\frac{\sqrt{2\mu}}{g} + \psi\right), \tag{6.5}$$

with $\tilde{\nu} := g(\alpha \sin\theta + z\cos\theta)$, $\xi \times \eta := \xi_1\eta_2 - \xi_2\eta_1$, recall, $\mathrm{curl}_q\, w = \nabla_1 w_2 - \nabla_2 w_1$, $\nabla_i := \partial_i - iq_i$ and, recalling that $w : \mathbb{R}^2 \to \mathbb{C}^2$,

$$M := \mathrm{curl}_\nu^* \,\mathrm{curl}_\nu - \mathrm{curl}_{a^n}^* \,\mathrm{curl}_{a^n} = \begin{pmatrix} M_{22} & -M_{21} \\ -M_{12} & M_{11} \end{pmatrix}, \tag{6.6}$$

with $M_{ij} := i\tilde{\nu}_i(\nabla_{a^n})_j + i\tilde{\nu}_j(\nabla_{a^n})_i + i\partial_i\tilde{\nu}_j + \tilde{\nu}_i\nu_j$.

Note that system (6.1) can be also written as $G(m^{n,r} + u)|_{\xi=\sqrt{2\mu}/g} = 0$, where G is defined in (5.1) and $m^{n,r} := (0, \frac{1}{e}a^n, 0, \xi)$.

Applying div to the second equation in (6.1), we find that a solution (μ, u) should satisfy $\mathrm{div}\, J_2(\mu, u) = 0$. To prove that a solution (μ, u) satisfies this constraint, we consider the following auxiliary problem

$$F(\mu, u) = 0, \quad \text{where} \quad F(\mu, u) := L_{n,\mu} u + P' J(\mu, u), \tag{6.7}$$

where $P' = \mathbf{1} \otimes P_0 \otimes \mathbf{1} \otimes \mathbf{1}$, with P_0 the orthogonal projection onto the divergence-free vector fields ($P_0 = \frac{1}{-\Delta}\mathrm{curl}^*\,\mathrm{curl}$), and, recall, $L_{n,\mu} = \oplus H_i$ and $J(\mu, u)$ given in (5.16) and

$$J(\mu, u) := (J_1(\mu, u), \ldots, J_4(\mu, u)). \tag{6.8}$$

We consider $F(\mu, u)$ as a map from the space $\mathbb{R}_{>0} \times \mathcal{X}$, where $\mathcal{X} := \mathcal{H}_n^2 \oplus \mathcal{H}_0^2 \oplus \mathcal{H}_0^2 \oplus \mathcal{H}^2$, to the space $\mathcal{Y} := L_n^2 \oplus L_0^2 \oplus L_0^2 \oplus L^2$, and let $F = (F_1, \ldots, F_4)$, where

$$F_i(\mu, u) = H_i u + \delta_{i,2} P_0 J_i(\mu, u), \quad i = 1, \ldots, 4. \tag{6.9}$$

In what follows, we denote the partial (real) Gâteaux derivatives with respect to $\#$ by $\delta_\#$.

Proposition 6.1. *Assume (μ, u) is a solution of the system (6.7) satisfying the gauge—periodicity conditions (4.10)–(4.13). Then $\mathrm{div}\, J(\mu, u) = 0$ and therefore (μ, u) solves the original system (6.1).*

Proof. We follow [46]. Assume $\chi \in H^1_{\mathrm{loc}}$ and is $\mathcal{L}-$periodic (we say, $\chi \in H^1_{\mathrm{per}}$). The gauge invariance implies that

$$E_{\Omega'}(e^{is\chi}w, a + s\nabla\chi, z, \phi) = E_{\Omega'}(w, a, z, \phi), \tag{6.10}$$

where $E_{\Omega'}(w, a, z, \phi)$ is given in (4.8). Differentiating this equation with respect to s at $s = 0$ gives $\delta_w E_{\Omega'}(w, a, z, \phi)(i\chi w) + \delta_a E_{\Omega'}(w, a, z, \phi)(\nabla\chi) = 0$. Now, we use the fact that the partial Gâteaux derivative with respect to w vanishes, $\delta_w E_{\Omega'}(w, a, z, \phi) = 0$, and that $\mathrm{curl}\,\nabla\chi = 0$, and integrate by parts, to obtain

$$\langle J(\mu, u), \nabla\chi\rangle = 0. \tag{6.11}$$

(Due to conditions (4.10)–(4.13) and the $\mathcal{L}-$periodicity of χ, there are no boundary terms.) Since the last equation holds for any $\chi \in H^1_{\mathrm{per}}$, we conclude that $\mathrm{div}\, J(\mu, u) = 0$. □

In Sects. 7–8, we solve Eq. (6.7), subject to conditions (4.10)–(4.13).

In conclusion of this section, we investigate properties of the map $F(\mu, u)$. For $f = (f_1, f_2, f_3, f_4)$ and $\delta \in \mathbb{R}$, define the global transformation

$$T_\delta f = (e^{i\delta} f_1, f_2, f_3, f_4). \tag{6.12}$$

Proposition 6.2. *The map $F(\mu, u)$ defined in (6.7) has the following properties:*

(i) $F : \mathbb{R}_{>0} \times \mathcal{X} \to \mathcal{Y}$ *is continuously Gâteau differentiable of all orders;*
(ii) $F(\mu, 0) = 0$ *for all* $\mu \in \mathbb{R}_{>0}$*;*
(iii) $\delta_u F(\mu, 0) = L_{n,\mu}$ *for all* $\mu \in \mathbb{R}_{>0}$*;*
(iv) $F(\mu, T_\delta u) = T_\delta F(\mu, u)$ *for all* $\delta \in \mathbb{R}$*;*
(v) $\langle u, F(\mu, u)\rangle_{\mathcal{Y}} \in \mathbb{R}$ *(respectively* $\langle w, F_1(\mu, u)\rangle_{L^2_n} \in \mathbb{R}$*) for all* $u \in \mathcal{X}$ *(respectively* $w \in \mathcal{H}^2_n$*).*

Proof. (i) follows because F is a polynomial in the components of u and their first- and second-order (covariant) derivatives. (ii), (iii) and (iv) follow from an easy calculation (in fact, u and $L_{n,\mu}$ were defined so that (ii) and (iii) hold). For (v), it suffices to show that $\langle w, F_1(\mu, u)\rangle_{L^2_n} \in \mathbb{R}$. To simplify notation, we return to the coordinates $(w, a, z, \phi) = (w, \frac{1}{e}a^n + \alpha, z, \frac{\sqrt{2\mu}}{g} + \psi)$. Then

$$\begin{aligned}\langle w, F_1(\mu, u)\rangle_{L^2_n} &= \frac{1}{|\Omega'|}\int_{\Omega'} |\operatorname{curl}_\nu w|^2 + \frac{1}{|\Omega'|}\int_{\Omega'} \frac{g^2}{2}\phi^2 |w^2| \\ &\quad + \frac{1}{|\Omega'|}\int_{\Omega'} i(\operatorname{curl}\nu)(\overline{w} \times w) + \frac{1}{|\Omega'|}\int_{\Omega'} g^2 |\overline{w} \times w|^2. \end{aligned} \tag{6.13}$$

The first, second and fourth terms are clearly real, while the third term is real because ν is real and $\overline{w} \times w$ is imaginary. □

7. Reduction to a Finite-Dimensional Problem

In this section, we shall reduce solving Eq. (6.7), i.e. $F(\mu, u) = 0$, with $u = (u_1, u_2, u_3, u_4) \equiv (w, \alpha, z, \psi)$ and $F : \mathbb{R}_{>0} \times \mathcal{X} \to \mathcal{Y}$ defined in (6.7)–(6.8)), to a finite-dimensional problem. To this end, we use the Lyapunov–Schmidt reduction.

Recall that $L_{n,\mu}$ is defined in (5.16). Let P be the orthogonal projection onto $\mathcal{K} := \operatorname{Null}(L_{n,\mu=n})$, which can be written explicitly as

$$P = P_1 \oplus P_2 \oplus 0 \oplus 0, \tag{7.1}$$

$$P_1 w := -\frac{1}{2\pi i}\oint_{\gamma_n} (H_1(n) - z)^{-1} w \, \mathrm{d}z, \tag{7.2}$$

$$P_2 \alpha := \langle \alpha \rangle, \tag{7.3}$$

where $H_1(n)$ is defined in (5.17), γ_n is any simple closed curve in $\mathbb{C}$ containing the eigenvalue 0 and no other eigenvalues of $H_1(n)$ (see Proposition 5.3), and $\langle\alpha\rangle$ is the mean value of α in Ω', $\langle\alpha\rangle := \frac{1}{|\Omega'|}\int_{\Omega'} \alpha$. P_1 is a projection onto $\operatorname{Null}(H_1(n))$ (spanned by vectors of the form (5.24)). Since $H_1(n)$ is self-adjoint, P_1 is an orthogonal projection (relative to the inner product of L^2_n). By Theorem 5.1, $\mathcal{K} := \operatorname{Null}(L_{n,\mu=n})$ is $(n+1)$-dimensional.

Let $P^\perp = 1 - P$ be the projection onto the orthogonal complement of $\mathcal{K}$. Applying P and $P^\perp$ to the equation $F(\mu, u) = 0$ (see (6.7)), we split it into two equations for two unknowns as

$$PF(\mu, v + u') = 0, \tag{7.4}$$

$$P^{\perp}F(\mu, v+u')=0, \tag{7.5}$$

where $v:=Pu,\ u':=P^{\perp}u$.

Our next goal is to solve (7.5) for u' in terms of μ and v. For $n=1$, $\mathcal{K}$ is two-dimensional and we write $v=(v_1,v_2,v_3,v_4)\equiv(v_1,v_2,0,0)\in\mathcal{K}$. Let $\mathcal{X}^{\perp}:=P^{\perp}\mathcal{X}=\mathcal{X}\ominus\mathcal{K}$ and $\mathcal{Y}^{\perp}:=P^{\perp}\mathcal{Y}=\mathcal{Y}\ominus\mathcal{K}$, and let $\partial_i\equiv\partial_{x_i}$.

Proposition 7.1. *There is a neighbourhood $U\subset\mathbb{R}_{>0}\times\mathcal{K}$ of $(n,0)$ such that for every $(\mu,v)\in U$, Eq. (7.5) for u' has a unique solution $u'=u'(\mu,v)=(u'_1,u'_2,u'_3,u'_4)$. Furthermore, this solution has the following properties:*

$$u':\mathbb{R}_{>0}\times\mathcal{K}\to\mathcal{X}^{\perp}\ \textit{is continuously differentiable of all orders;} \tag{7.6}$$

$$\|(\nabla_{a^n})_j^m u'_1\|_{\mathcal{H}^2_n}=\mathcal{O}(\|v\|^2_{\mathcal{X}}), \tag{7.7}$$

$$\|\partial_j^m u'_k\|_{\mathcal{H}^2_k}=\mathcal{O}(\|v\|^2_{\mathcal{X}}), \tag{7.8}$$

$$||\partial_{v_i}(\nabla_{a^n})_j^m u'_1(\mu,v^i)\|_{\mathcal{H}^2_n}\lesssim\|v^i\|_{\mathcal{X}}, \tag{7.9}$$

$$\|\partial_{v_i}\partial_j^m u'_k(\mu,v^i)\|_{\mathcal{H}^2_0}\lesssim\|v^i\|_{\mathcal{X}}, \tag{7.10}$$

$$\|\partial_\mu u'(\mu,v)||_{\mathcal{X}}\lesssim\|v\|^2_{\mathcal{X}}; \tag{7.11}$$

where $i=1,\dots4,\ m=0,1,\ j=1,2,\ k=2,3,4,\ v=(v_1,v_2,v_3,v_4),\ v^i\equiv v|_{v_i=0}$ and $\mathcal{H}^2_k=\mathcal{H}^2_0,\ \mathcal{H}^2_0,\ \mathcal{H}^2$ for $k=2,3,4$.

Proof. Define $F^{\perp}:\mathbb{R}_{>0}\times\mathcal{K}\times\mathcal{X}^{\perp}\to\mathcal{Y}^{\perp}$ by

$$F^{\perp}(\mu,v,u'):=P^{\perp}F(\mu,v+u'). \tag{7.12}$$

By Proposition 6.2 (i) and (ii), $F^{\perp}$ is continuously differentiable of all orders as a map between Banach spaces and $F^{\perp}(\mu,0,0)=0$ for all $\mu\in\mathbb{R}_{>0}$. Furthermore,

$$\delta_{u'}F^{\perp}(\mu,0,0)=P^{\perp}L_{n,\mu}P^{\perp}|_{\mathcal{X}^{\perp}}, \tag{7.13}$$

which is invertible for $\mu=n$ because $P^{\perp}$ is the projection onto the orthogonal complement of $\mathcal{K}=\mathrm{Null}(L_{n,\mu=n})$. By the Implicit Function Theorem (see, e.g. [16]), there exists a function $u'(\mu,v)$ with continuous derivatives of all orders such that for (μ,v) in a sufficiently small neighbourhood $U\subset\mathbb{R}_{>0}\times\mathcal{K}$ of $(n,0)$, (μ,v,u') solves (7.5) if and only if $u'=u'(\mu,v)$. This proves the first statement and property (7.6).

We define the operator

$$L^{\perp}_{n,\mu}:=P^{\perp}L_{n,\mu}P^{\perp}|_{\mathcal{X}^{\perp}}:\mathcal{X}^{\perp}\to\mathcal{Y}^{\perp}. \tag{7.14}$$

Then by (6.7) and (7.13), we can write equation (7.5) as $L^{\perp}_{n,\mu}u'=-P^{\perp}P'J(\mu,u)$. By Theorem 5.1 and the relation $\mathcal{K}:=\mathrm{Null}(L_{n,\mu=n})=\mathrm{Null}(L_{n,\mu}-\mu+n)$, for μ in a neighbourhood of n, the operator $L^{\perp}_{n,\mu}$ has a uniformly bounded inverse $(L^{\perp}_{n,\mu})^{-1}:\mathcal{Y}^{\perp}\to\mathcal{X}^{\perp}$. Hence, equation $L^{\perp}_{n,\mu}u'=-P^{\perp}P'J(\mu,u)$, with $(\mu,v)\in U$ (replacing U with a smaller neighbourhood if necessary), is equivalent to

$$u'=-(L^{\perp}_{n,\mu})^{-1}P^{\perp}P'J(\mu,u); \tag{7.15}$$

hence

$$\|u'\|_{\mathcal{X}} \lesssim \|J(\mu,u)\|_{\mathcal{Y}}, \tag{7.16}$$

uniformly in μ. Recall that $\mathcal{X} = \mathcal{H}^2_n \oplus \mathcal{H}^2_0 \oplus \mathcal{H}^2_0 \oplus \mathcal{H}^2$ and $\mathcal{Y} = L^2_n \oplus L^2_0 \oplus L^2_0 \oplus L^2$. $J(\mu,u)$ is a polynomial in the components of u and their first-order (covariant) derivatives consisting of terms of degree at least 2, so the left-hand side of (7.16) can be bounded above by a sum of products of one $\mathcal{L}^2$-norm and at least one $\mathcal{L}^\infty$-norm of these terms. $\mathcal{H}^1$ is trivially continuously embedded in $\mathcal{L}^2$, and by the Sobolev embedding theorem, $\mathcal{H}^1$ is continuously embedded in $\mathcal{L}^\infty$. Therefore,

$$\|J(\mu,u)\|_{\mathcal{Y}} \lesssim \|u\|^2_{\mathcal{X}}. \tag{7.17}$$

Recalling that $u = v + u'$, this proves (7.7) and (7.8) when $m = 0$. The other case is proven similarly.

For $v = (v_1,\ldots,v_4)$, we let $v_{\hat{i}} \equiv v|_{v_i=0}$, $i = 1,\ldots,4$. By the Taylor theorem for Banach spaces (see, e.g. [16]), we have

$$u'(\mu,v) = u'(\mu,v_{\hat{i}}) + \delta_{v_i}u'(\mu,v_{\hat{i}})v_i + R_2(\mu,v_{\hat{i}})(v_i), \tag{7.18}$$

$$R_2(\mu,v_{\hat{i}})(v_i) := \int_0^1 (1-t)\delta^2_{v_i}u'(\mu,v_{\hat{i}}+tv_i)(v_i,v_i)dt. \tag{7.19}$$

Let $(\mu,v) \in U$ with $\|v_{\hat{i}}\| = \|v_i\| = 1$, and let $\epsilon > 0$. Then

$$\begin{aligned}\|\delta_{v_i}u'(\mu,\epsilon v_{\hat{i}})\epsilon v_i\|_{\mathcal{X}} &= \|u'(\mu,\epsilon v) - u'(\mu,\epsilon v_{\hat{i}}) - R_2(\mu,\epsilon v_{\hat{i}})(\epsilon v_i)\|_{\mathcal{X}}\\ &\le \|u'(\mu,\epsilon v)\|_{\mathcal{X}} + ||u'(\mu,\epsilon v_{\hat{i}})\|_{\mathcal{X}}\\ &\quad + \epsilon^2\|v_i\|^2 \sup_{0\le t\le 1}(1-t)\|\delta^2_{v_i}u'(\mu,\epsilon v_{\hat{i}}+t\epsilon v_i)\|^2_{\mathcal{X}^*\otimes\mathcal{X}^*\otimes\mathcal{X}}\\ &\lesssim \epsilon^2.\end{aligned} \tag{7.20}$$

with the norm taken in the appropriate space for v_i. Taking the supremum over all v_i with $||v_i|| = 1$ gives

$$\|\delta_{v_i}u'(\mu,\epsilon v_{\hat{i}})\|_{\mathcal{X}} \lesssim \epsilon,\ \ \|v_{\hat{i}}\|_{\mathcal{X}} = 1, \tag{7.21}$$

proving (7.9)–(7.10) for $m = 0$. The other cases are proven in exactly the same way.

Again by Taylor's theorem,

$$\partial_\mu u'(\mu,v) = \partial_\mu u'(\mu,0) + \partial_\mu\delta_v u'(\mu,0)v + \tilde{R}_2(\mu,0)(v), \tag{7.22}$$

$$\tilde{R}_2(\mu,0)(v) := \int_0^1 (1-t)\partial_\mu\delta^2_v u'(\mu,tv)(v,v)\mathrm{d}t. \tag{7.23}$$

By Eqs. (7.8) and (7.9)–(7.10) with $m = 0$, we have $u'(\mu,0) = 0$ and $\delta_v u'(\mu,0) = 0$, so

$$\|\partial_\mu u'(\mu,v)\|_{\mathcal{X}} = \|\tilde{R}_2(\mu,0)(v)\|_{\mathcal{X}} \tag{7.24}$$

$$\le \|v\|^2_{\mathcal{X}} \sup_{0\le t\le 1}(1-t)\|\partial_\mu\delta^2_v u'(\mu,tv)\|^2_{\mathcal{X}^*\otimes\mathcal{X}^*\otimes\mathcal{X}} \tag{7.25}$$

$$\lesssim \|v\|^2_{\mathcal{X}}, \tag{7.26}$$

proving (7.11). □

We plug the solution $u' = u'(\mu, v)$ into Eq. (7.4) to get the *bifurcation equation*

$$\gamma(\mu, v) := PF(\mu, v + u'(\mu, v)) = 0. \tag{7.27}$$

Corollary 7.2. *In a neighbourhood of* $(n, 0)$ *in* $\mathbb{R}_{>0} \times \mathcal{X}$*, the pair* (μ, u) *solves* (6.7) *if and only if* (μ, v) *solves the finite-dimensional Eq.* (7.27)*. Moreover, a solution of* (6.7) *can be constructed from a solution* (μ, v) *of* (7.27) *by setting* $u = v + u'(\mu, v)$*, where* $u'(\mu, v)$ *is given by Proposition* 7.1*.*

Since $F : \mathbb{R}_{>0} \times \mathcal{X} \to \mathcal{Y}$ and $u' : \mathbb{R}_{>0} \times \mathcal{K} \to \mathcal{Y}^{\perp}$ have been shown to be continuously differentiable of all orders, we conclude:

Corollary 7.3. $\gamma : \mathbb{R} \times \mathcal{K} \to \mathcal{K}$ *is continuously Gâteau differentiable of all orders.*

Furthermore, $\gamma(\mu, v)$ inherits the following symmetry of $F(\mu, u)$, which we will use to find a solution of (7.27):

Lemma 7.4. *Let* T_δ *be given by* (6.12)*. For every* $\delta \in \mathbb{R}$ *and* (μ, v) *in a neighbourhood of* $(n, 0)$*, we have*

$$u'(\mu, T_\delta v) = T_\delta u'(\mu, v), \tag{7.28}$$

$$\gamma(\mu, T_\delta v) = T_\delta \gamma(\mu, v). \tag{7.29}$$

Proof. For Eq. (7.28), we note that by Proposition 6.2 (iv)

$$\begin{aligned} P^{\perp} F(\mu, T_\delta v + T_\delta u'(\mu, v)) &= P^{\perp} T_\delta F(\mu, v + u'(\mu, v)) \\ &= T_\delta P^{\perp} F(\mu, v + u'(\mu, v)) = 0. \end{aligned} \tag{7.30}$$

(Here we used $P^{\perp} T_\delta = T_\delta P^{\perp}$, which follows because $T_\delta = e^{i\delta} \oplus 1 \oplus 1 \oplus 1$ and $P^{\perp} = 1 - P$ where P is defined in (7.1).) Since $u' = u'(\mu, T_\delta v)$ is the unique solution to $P^{\perp} F(\mu, T_\delta v + u') = 0$ for (μ, v) in a neighbourhood $U \subset \mathbb{R} \times \mathcal{K}$ of $(n, 0)$, we conlcude that $u'(\mu, T_\delta v) = T_\delta u'(\mu, v)$.

For Eq. (7.29), we note that by (7.28) and Proposition 6.2 (iv),

$$\begin{aligned} \gamma(\mu, T_\delta v) &= PF(\mu, T_\delta v + u'(\mu, T_\delta v)) = PF(\mu, T_\delta(v + u'(\mu, v))) \\ &= T_\delta PF(\mu, v + u'(\mu, v)) = T_\delta \gamma(\mu, v) \end{aligned}$$

(where again we used $PT_\delta = T_\delta P$). $\square$

8. The Bifurcation Result When $n = 1$

Theorem 8.1. *Assume that* $n = 1$ *and* $|1 - b_*/b| \ll 1$, $b_* := M_W^2/e$*. Then there exists* $\epsilon > 0$ *and a branch* $(\mu_s, u_s) := (\mu_s, w_s, \alpha_s, z_s, \psi_s)$*, with* $s \in [0, \sqrt{\epsilon})$*, of non-trivial solutions of Eq.* (6.1)*, unique modulo a gauge symmetry in a* *sufficiently small neighbourhood of the rescaled vacuum solution* (4.9) *in* $\mathbb{R}_{>0} \times \mathcal{X}$*, such that*

$$\begin{cases} w_s = s\chi + sg_1(s^2), \\ \alpha_s = g_2(s^2), \\ z_s = g_3(s^2), \\ \alpha\alpha_s = g_4(s^2), \\ \mu_s = n + g_5(s^2), \end{cases} \tag{8.1}$$

where χ *solves the eigenvalue problem* $H_1(n)\chi = 0$ *(it is defined in* (5.24)*, see Proposition* 5.3*),* $\mu := g^2\xi^2/2 = g^2r^2\varphi_0^2/2$*,* $g_1 : [0, \epsilon) \to \mathcal{H}_n^2$ *and is orthogonal to* $\mathrm{Null}(H_1(n))$*,* $g_2 : [0, \epsilon) \to \mathcal{H}_0^2$*,* $g_3 : [0, \epsilon) \to \mathcal{H}_0^2$*,* $g_4 : [0, \epsilon) \to \mathcal{H}^2$*,* $g_5 : [0, \epsilon) \to \mathbb{R}_{>0}$*, and* g_j *for* $j = 1, \cdots, 5$ *are functions, continuously differentiable of all orders in* s*, such that* $g_j(0) = 0$*.*

Proof of Theorem 8.1. For the proof below, recall that we denote the partial (real) Gâteaux derivatives with respect to # by $\delta_\#$, and let $\partial_i \equiv \partial_{x_i}$.

By Proposition 6.1, solving Eq. (6.1) is equivalent to solving (6.7). By Corollary 7.2, solving (6.7) is equivalent to solving the bifurcation equation (7.27). Hence, we address the latter equation.

Recall that P is the projection onto $\mathcal{K} = \mathrm{Null}\, L_{n,\mu=n} = \mathrm{Null}(H_1(n)) \times \{\text{constants}\} \times \{0\} \times \{0\}$. The projection onto constant vector fields in $\mathcal{H}_0^2$ can be written as the mean value $\langle\alpha\rangle := \frac{1}{|\Omega'|}\int_{\Omega'} \alpha$. Since $\dim \mathrm{Null}(H_1(n)) = 1$ for $n = 1$, we may choose $\chi \in \mathrm{Null}(H_1(n))$ such that

$$P(w, \alpha, z, \psi) = (s\chi, c, 0, 0), \tag{8.2}$$

$$s := \langle\chi, w\rangle_{L_n^2} \in \mathbb{C}, \;\; c := \langle\alpha\rangle \in \mathbb{R}^2, \tag{8.3}$$

and χ satisfies $\|\chi\|_{L_n^2}^2 = \langle|\chi|^2\rangle = 1$ (see (5.13)), where, recall, χ is described in (5.24). Hence, we may write the γ from the bifurcation equation (7.27) as $\gamma = (\tilde{\gamma}_1\chi, \tilde{\gamma}_2, 0, 0)$, where $\tilde{\gamma}_1, \tilde{\gamma}_2 : \mathbb{R}_{>0} \times \mathbb{C} \times \mathbb{R}^2 \to \mathbb{C}$ are given by

$$\tilde{\gamma}_1(\mu, s, c) := \langle\chi, F_1(\mu, v(s, c) + u'(\mu, v(s, c))\rangle_{L_n^2}, \tag{8.4}$$

$$\tilde{\gamma}_2(\mu, s, c) := \langle F_2(\mu, v(s, c) + u'(\mu, v(s, c))\rangle, \tag{8.5}$$

where, recall, F_j, $j = 1, \ldots, 4$ are defined by (6.9), $s \in \mathbb{C}$, $c \in \mathbb{R}^2$ and (see (8.2))

$$v(s, c) := (s\chi, c, 0, 0). \tag{8.6}$$

Note that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are continuously differentiable of all orders in μ, s and c by Corollary 7.3. ($\tilde{\gamma}_2$ is independent of μ.) The bifurcation equation (7.27) is then equivalent to the equations

$$\tilde{\gamma}_1(\mu, s, c) = 0, \tag{8.7}$$

$$\tilde{\gamma}_2(\mu, s, c) = 0. \tag{8.8}$$

Lemma 8.2. *There exists a neighbourhood* $U \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ *of* $(n, 0)$ *and a unique function* $c : U \to \mathbb{R}^2$ *with continuous derivatives of all orders such that*

$$\tilde{\gamma}_2(\mu, s, c(\mu, s^2)) = 0 \tag{8.9}$$

and

$$\|\partial_\mu^l c(\mu, s^2)\|_{\mathbb{R}^2} = \mathcal{O}(|s|^2),\ l = 0, 1. \tag{8.10}$$

Proof. Recall that $F_2(\mu, u) = H_2(\mu)\alpha + P_0 J_2(\mu, u)$ (see Equation (6.7)), with P_0 the projection onto the divergence-free vector fields and

$$u = (w, \alpha, z, \psi) = v + u', \tag{8.11}$$

where $v = v(s, c)$ and $u' = u'(\mu, v)$ solves (7.5). By definition, $(\mathbf{1} - P_0)f = \Delta^{-1}\nabla \operatorname{div} f$ and therefore $\langle(\mathbf{1} - P_0)f\rangle = 0$. Hence $\langle P_0 f\rangle = \langle f\rangle$. This and the relation $\langle H_2(\mu)\alpha\rangle = \frac{1}{|\Omega'|}\int_{\Omega'} \operatorname{curl}^* \operatorname{curl}\alpha = 0$ give

$$\tilde{\gamma}_2(\mu, s, c) = \langle J_2(\mu, v(s, c) + u'(\mu, v(s, c)))\rangle. \tag{8.12}$$

Using (6.3), $\nu = a^n + \tilde{\nu}$, $\operatorname{curl}_{a^n} w = \operatorname{curl}_{a^n} w - i\tilde{\nu} \times w$ and that the final term in (6.3) vanishes after taking the mean, we find

$$\langle J_2(\mu, u)\rangle = 2e \operatorname{Im}\langle(\operatorname{curl}_{a^n} w - i\tilde{\nu} \times w)J\overline{w}\rangle. \tag{8.13}$$

Recall $u' = (w', \alpha', z', \psi')$. Then (8.6) and (8.11) give $w = s\chi + w'$ and (using that $e = g\sin\theta$) $\tilde{\nu} = ec + \nu'$. Using these relations and $\operatorname{curl}_{a^n}\chi = 0$ (by (5.24)) and (8.12) and (8.13), we find for $\overline{\gamma}_2(\mu, s, c) := (2e)^{-1}|s|^{-2}\tilde{\gamma}_2(\mu, s, c)$

$$\overline{\gamma}_2(\mu, s, c) := -e\langle\operatorname{Re}[(c \times \chi)J\overline{\chi}]\rangle + \operatorname{Im} s^{-1}\langle(\operatorname{curl}_{a^n} w')J\overline{\chi}\rangle \tag{8.14}$$

$$+ \operatorname{Im}\langle\bar{R}_2(\mu, s, c)\rangle, \tag{8.15}$$

$$\bar{R}_2(\mu, s, c) := |s|^{-2}[-i(ec \times s\chi)J\overline{w'} - i(ec \times w')J\overline{w'} \tag{8.16}$$

$$- i(ec \times w')J\overline{s\chi} - i(\nu' \times w')J\overline{s\chi} - i(\nu' \times s\chi)J\overline{w'} \tag{8.17}$$

$$- i(\nu' \times s\chi)J\overline{s\chi} - i(\nu' \times w')J\overline{w'} + (\operatorname{curl}_{a^n} w')J\overline{w'}]. \tag{8.18}$$

Note that we expect (8.14) $= \mathcal{O}(|s|^2)$ and (8.15) $= \mathcal{O}(|s|^4)$. We now simplify (8.14). For the first term on the right-hand side, we use (5.24) and the condition $\langle|\chi|^2\rangle = 1$ to compute

$$\langle\operatorname{Re}[(c \times \chi)J\overline{\chi}]\rangle = -\frac{1}{2}c. \tag{8.19}$$

For the second term on the right-hand side of (8.14), we use $\langle fJ\overline{\chi}\rangle = \langle f(i\overline{\eta}, \overline{\eta})\rangle = \langle f\overline{\eta}\rangle(i, 1) = \langle\eta, f\rangle(i, 1)$ and integrate by parts to compute

$$\langle(\operatorname{curl}_{a^n} w')J\overline{\chi}\rangle = \langle\eta, \operatorname{curl}_{a^n} w'\rangle(i, 1) = \langle\operatorname{curl}^*_{a^n}\eta, w'\rangle(i, 1). \tag{8.20}$$

Abusing notation, we write in what follows $w(\mu, s, c) \equiv w(\mu, v(s, c))$. Then (8.14) becomes

$$\overline{\gamma}_2(\mu, s, c) = \frac{1}{2}ec + \operatorname{Im} s^{-1}\langle\operatorname{curl}^*_{a^n}\eta, w'(\mu, s, c)\rangle\rangle(i, 1) + \operatorname{Im}\langle\tilde{R}_2(\mu, s, c)\rangle. \tag{8.21}$$

Now, Equation (7.7), with $m = 0$, implies that

$$|\operatorname{Im}\langle\operatorname{curl}^*_{a^n}\eta, w'(\mu, s, c)\rangle\rangle| = \mathcal{O}(|s|^2). \tag{8.22}$$

Furthermore, we show below the following estimate on the remainder:

$$||\operatorname{Im}\langle\partial_c^l\tilde{R}_2(\mu, s, c)\rangle||_{\mathbb{R}^2} = \mathcal{O}(|s|^{2-l}),\ l = 0, 1. \tag{8.23}$$

Hence $\overline{\gamma}_2(\mu, 0, 0) = 0$.

To apply the implicit function theorem to solve for c as a function of μ and s, we have to estimate the derivative

$$\partial_c \overline{\gamma}_2(\mu, s, c) = \frac{1}{2} e\mathbf{1} + \operatorname{Im} s^{-1}\langle \operatorname{curl}^*_{a^n} \eta, \partial_c w'(\mu, s, c)\rangle (i, 1) + \operatorname{Im}\langle \partial_c \tilde{R}_2(\mu, s, c)\rangle. \tag{8.24}$$

at $(n, s, 0)$. At the first step, we use the following

Lemma 8.3. *Using Dirac's bra-ket notation, we have*

$$(\partial_c w')(n, s, 0) = -n^{-1} es |\operatorname{curl}^*_{a^n} \eta\rangle\langle(1, i)| + \mathcal{O}(|s|^2). \tag{8.25}$$

Proof of Lemma 8.3. By definition (7.2), $P_1^{\perp}$ projects onto the orthogonal complement of the eigenspace of $H_1(n)$ corresponding to the eigenvalue 0 and therefore the operator $H_1^{\perp}(n)$ is invertible on $\operatorname{Ran} P_1^{\perp}$.

Hence (6.1) with $i = 1$ can be rewritten as $w' = -(H_1^{\perp}(n))^{-1} P_1^{\perp} J_1(n, u)$ (which is the first component of (7.15)), which gives

$$\partial_c w' = -(H_1^{\perp}(n))^{-1} P_1^{\perp} \partial_c J_1(n, u), \tag{8.26}$$

where $u \equiv u(s, c) := v(s, c) + u'(\mu, v(s, c))$. By (6.2) and (6.6), we have

$$\partial_c J_1(n, u) = \partial_c[\operatorname{curl}^*_{\nu} \operatorname{curl}_{\nu} w]. \tag{8.27}$$

Using $w = s\chi + w'$, $\nu = a^n + ec + \nu'$ and $\operatorname{curl}_{\nu} = \operatorname{curl}_{a^n} + iJ(ec + \nu')\cdot$, $\operatorname{curl}^*_{\nu} = \operatorname{curl}^*_{a^n} - iJ(ec + \nu')$ and that $\nu' = \mathcal{O}(|s|^2)$, we compute

$$\begin{aligned} \partial_c J_1(n, u)c' &= s\partial_c[\operatorname{curl}^*_{\nu} \operatorname{curl}_{\nu}]\chi c' + \mathcal{O}(|s|^2) \\ &= sie[-Jc' \operatorname{curl}_{\nu} + \operatorname{curl}^*_{\nu} Jc'\cdot]\chi + \mathcal{O}(|s|^2) &(8.28)\\ &= sie[-Jc' \operatorname{curl}_{a^n+ec} + \operatorname{curl}^*_{a^n+ec} Jc'\cdot]\chi + \mathcal{O}(|s|^2). &(8.29)\end{aligned}$$

Since $\operatorname{curl}_{a^n} \chi = \nabla_1 i\beta - \nabla_2 \beta = i\bar{\partial}_{a^n}\beta = 0$ and $Jc' \cdot \chi = (-c_2', c_1') \cdot (\beta, i\beta) = -c_2'\beta + c_1' i\beta = i(c_1' + ic_2')\beta$ and therefore $\operatorname{curl}^*_{a^n} Jc' \cdot \chi = i \operatorname{curl}^*_{a^n} \beta(c_1' + ic_2')$, this yields

$$\partial_c J_1(n, u)c'\big|_{c=0} = -se \operatorname{curl}^*_{a^n} \beta(c_1' + ic_2') + \mathcal{O}(|s|^2). \tag{8.30}$$

By Proposition 5.3(ii), $\operatorname{Null}(H_1(\mu) - \mu + n) = \{\chi = (\beta, i\beta) : \operatorname{curl}_{a^n} \chi = i\bar{\partial}_{a^n}\eta = 0\}$. The relation $\operatorname{curl}_{a^n} \chi = 0$ implies also $\langle \chi, \operatorname{curl}^*_{a^n} \chi\rangle = \langle \operatorname{curl}_{a^n} \chi, \chi\rangle = 0$, which, for $n = 1$, gives that $P_1^{\perp}\partial_c J_1(n, u)c' = \partial_c J_1(n, u)c'$ and therefore

$$P_1^{\perp}\partial_c J_1(n, u)c' = -se \operatorname{curl}^*_{a^n} \beta(c_1' + ic_2') + \mathcal{O}(|s|^2). \tag{8.31}$$

By (5.24), we have $\operatorname{curl}^*_{a^n} \eta = i\nabla_{a^n}\beta$, and by (5.25), we have $H_1(n)\nabla_{a^n}\beta = n\nabla_{a^n}\beta$; hence, $(H_1^{\perp}(n))^{-1} \operatorname{curl}^*_{a^n} \beta = n^{-1} \operatorname{curl}^*_{a^n} \beta$. This relation, together with (8.26) and (8.31), yields

$$\partial_c w' c' = sen^{-1} \operatorname{curl}^*_{a^n} \beta(c_1' + ic_2') + \mathcal{O}(|s|^2), \tag{8.32}$$

which gives (8.25). □

Using Eq. (8.25), we calculate the second term on the right-hand side of (8.24) at $(n, s, 0)$:

$$\begin{aligned}\operatorname{Im} s^{-1}\langle \operatorname{curl}_{a^n}^* \beta, \partial_c w'(\mu, s, c)c'\rangle(i, 1)\\ = en^{-1} \operatorname{Im}\langle \operatorname{curl}_{a^n}^* \beta, \operatorname{curl}_{a^n}^* \beta\rangle(c_1' + ic_2')(i, 1).\end{aligned} \tag{8.33}$$

The inner product term is real. Integrating it by parts and using that, by Equation (5.38), β satisfies $\operatorname{curl}_{a^n} \operatorname{curl}_{a^n}^* \beta = -\Delta_{a^n}\beta = n\beta$ and using that $\|\beta\|_{L_n^2}^2 = \frac{1}{2}\|\chi\|_{L_n^2}^2 = \frac{1}{2}$, gives

$$\langle \operatorname{curl}_{a^n}^* \beta, \operatorname{curl}_{a^n}^* \beta\rangle = \langle \beta, -\Delta_{a^n}\beta\rangle_{L_n^2} = \frac{1}{2}n. \tag{8.34}$$

The last two equations and the relation $\operatorname{Im}(c_1'+ic_2')(i,1) = \operatorname{Im}\begin{pmatrix} i & -1 \\ 1 & i\end{pmatrix} c' = \mathbf{1}c'$ imply

$$\operatorname{Im} s^{-1}\langle \operatorname{curl}_{a^n}^* \beta, \partial_c w'(\mu, s, c)\rangle(i, 1) = \frac{1}{2}e\mathbf{1}. \tag{8.35}$$

This, together with (8.24), gives

$$\partial_c\overline{\gamma}_2(n, s, 0) = \frac{1}{2}e\mathbf{1} + \frac{1}{2}e\mathbf{1} + \operatorname{Im}\langle\partial_c\tilde{R}_2(n, s, 0)\rangle. \tag{8.36}$$

Therefore, (8.36) and (8.23) (with $l = 1$) imply

$$\partial_c\overline{\gamma}_2(n, 0, 0) = e\mathbf{1}, \tag{8.37}$$

proving that $\partial_c\overline{\gamma}_2(n, 0, 0)$ is invertible, as required.

Recall that, by (8.21), (8.22) and (8.23) (with $l = 0$), we have

$$\overline{\gamma}_2(n, 0, 0) = 0. \tag{8.38}$$

Since $\partial_c\overline{\gamma}_2(n, 0, 0)$ is invertible, by the implicit function theorem there exists a unique function $\tilde{c} : \mathbb{R}_{>0}\times\mathbb{C} \to \mathbb{R}^2$ with continuous derivatives of all orders such that $\overline{\gamma}_2(\mu, s, \tilde{c}(\mu, s)) = 0$ for (μ, s) in a sufficiently small neighbourhood of $(n, 0)$. Furthermore, the symmetry (7.29) implies that $\overline{\gamma}_2(\mu, |s|, \tilde{c}(\mu, s)) = \overline{\gamma}_2(\mu, e^{i \arg s}|s|, \tilde{c}(\mu, s)) = \overline{\gamma}_2(\mu, s, \tilde{c}(\mu, s)) = 0$, so by the uniqueness of the branch $\tilde{c}(\mu, s)$ we have

$$\tilde{c}(\mu, s) = \tilde{c}(\mu, |s|). \tag{8.39}$$

In particular, $\partial_\mu^l\tilde{c}(\mu, s)$, $l = 0, 1$, restricted to $s \in \mathbb{R}$ are even functions with continuous derivatives of all orders; thus $\partial_s\partial_\mu^l\tilde{c}(\mu, 0) = 0$ and hence $\partial_\mu^l\tilde{c}(\mu, s) = \mathcal{O}(|s|^2)$, since the first two terms of the Taylor expansion are 0. We define $c : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}^2$ by $c(\mu, s) := \tilde{c}(\mu, \sqrt{s})$, which is a function with continuous derivatives of all orders satisfying $||\partial_\mu^l c(\mu, s^2)||_{\mathbb{R}^2} = \mathcal{O}(|s|^2)$, $l = 0, 1$, and $\tilde{\gamma}_2(\mu, s, c(\mu, s^2)) = |s|^2\overline{\gamma}_2(\mu, s, c(\mu, s^2)) = 0$, as required. $\square$

Lemma 8.4. *For $\epsilon > 0$ sufficiently small, there exists a unique function $\mu : [0, \epsilon) \to \mathbb{R}_{>0}$ with continuous derivatives of all orders such that*

$$\tilde{\gamma}_1(\mu(s^2), s, c(\mu(s^2), s^2)) = 0. \tag{8.40}$$

Proof. To simplify notation for this lemma, we set $u = v_s + u'_s$, with $v \equiv v_s \equiv (s\chi, c(\mu, s^2), 0, 0)$, $u' \equiv u'_s \equiv u'(\mu, v_s)$, $c \equiv c(\mu, s^2)$.

We first show that $\tilde{\gamma}_1(\mu, s, c) \in \mathbb{R}$ for $s \in \mathbb{R}$. Since u' by definition solves $P_1^{\perp} F_1(\mu, v + u') = 0$, where $P_1^{\perp} w' = w'$ and $P_1^{\perp}$ is self-adjoint, we have

$$\langle w', F_1(\mu, v + u') \rangle_{L^2_n} = \langle w', P_1^{\perp} F_1(\mu, v + u') \rangle_{L^2_n} = 0. \tag{8.41}$$

Therefore, for $s \neq 0$, we find

$$\begin{aligned} \tilde{\gamma}_1(\mu, s, c) &= s^{-1} \langle s\chi, F_1(\mu, v + u' \rangle_{L^2_n} \\ &= s^{-1} \langle s\chi + w', F_1(\mu, v + u') \rangle_{L^2_n}, \end{aligned} \tag{8.42}$$

which is real by Proposition 6.2 (v). Furthermore, by equations (7.29) and (8.39), we have $\tilde{\gamma}_1(\mu, s, c(\mu, s^2)) = e^{i\arg(s)} \tilde{\gamma}_1(\mu, |s|, c(\mu, |s|^2))$, so we may restrict s to be real.

Next, we show that

$$\tilde{\gamma}_1(n, s, c(n, s^2)) = \mathcal{O}(|s|^2) \tag{8.43}$$

Indeed,

$$\begin{aligned} |\tilde{\gamma}_1(n, s, c(n, s^2))| &\leq \|\chi\|_{L^2_n} \|F_1(n, v + u')\|_{L^2_n} \\ &\leq \|\chi\| \|_{L^2_n} \big[\|H_1(n)(s\chi + w')\|_{L^2_n} \\ &\quad + \|J_1(n, v + u')\|_{L^2_n}) \big]. \end{aligned} \tag{8.44}$$

Recall that $H_1(n)\chi = 0$, so that

$$\begin{aligned} |\tilde{\gamma}_1(n, s, c(n, s^2))| \leq \|\chi\|_{L^2_n} \big[&\|H_1(n)\|_{L^2_n \otimes (L^2_n)^*} \|w'\|_{L^2_n} \\ &+ \|J_1(n, v + u')\| \end{aligned} \tag{8.45}$$

By the definition $v \equiv v_s \equiv (s\chi, c(\mu, s^2), 0, 0)$ and equation (8.10), $\|v\|_{\mathcal{X}} = \mathcal{O}(|s|)$; hence, by Proposition 7.1,

$$\|w'\|_{L^2_n} \leq \|w'\|_{\mathcal{H}^2_n} = \mathcal{O}(|s|^2). \tag{8.46}$$

Furthermore, by equation (7.17) and recalling that $H_1(n)\chi = 0$,

$$\|J_1(n, v + u')\|_{L^2_n} \leq \|J_1(n, v + u')\|_{\mathcal{H}^2_n} \lesssim \|v + u'\|^2_{\mathcal{X}} = \mathcal{O}(|s|^2). \tag{8.47}$$

This proves that $\tilde{\gamma}_1(n, s, c(n, s^2))$ is $\mathcal{O}(|s|^2)$, as required.

In light of equation (8.43), we can define a function $\overline{\gamma}_1 : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ with continuous derivatives of all orders by

$$\overline{\gamma}_1(\mu, s) \equiv \begin{cases} s^{-1} \tilde{\gamma}_1(\mu, s, c(\mu, s^2)), & s \neq 0, \\ 0, \quad s = 0. \end{cases} \tag{8.48}$$

We now find a non-trivial branch of solutions $(\mu, s) = (\tilde{\mu}(s), s)$ by applying the implicit function theorem to $\overline{\gamma}_1$. First, we prove the following proposition to bound the polynomials of functions appearing below:

Lemma 8.5. *Let X be one of the spaces $\mathcal{H}^2_n$, $\mathcal{H}_0$ or $\mathcal{H}^2$ defined before equation (5.9). Let $p(x_1, ..., x_n)$ be a polynomial with positive coefficients and let $f_1, ..., f_n \in X$. Then $\|p(f_1, ..., f_n)\|_X \lesssim p(\|f_1\|_X, ..., \|f_n\|_X)$.*

Proof. Write $p(x_1,...,x_n)=\sum_{|\alpha|\le N} p_\alpha x^\alpha$, where $\alpha=(\alpha_1,...,\alpha_n)$ is a multi-index, $x^\alpha=\prod_{i=1}^n x_i^{\alpha_i}$ and $p_\alpha \ge 0$. Since by the Sobolev Embedding Theorem (see, e.g. [2]), X is a Banach algebra, we have

$$\begin{aligned}\|p(f_1,...,f_n)\|_X &\le \sum_{|\alpha|\le N} p_\alpha \|f^\alpha\|_X\\ &\lesssim \sum_{|\alpha|\le N} p_\alpha \prod_{i=1}^n \|f_i\|_X^{\alpha_i}\\ &= p(\|f_1\|_X,...,\|f_n\|_X),\end{aligned}$$

which implies the desired result. □

Lemma 8.6. *There exists $\epsilon>0$ and a unique function $\tilde\mu:(-\sqrt{\epsilon},\sqrt{\epsilon})\to\mathbb{R}_{>0}$ with continuous derivatives of all orders such that $\tilde\mu(0)=n$ and $\mu=\tilde\mu(s)$ solves $\overline{\gamma}_1(\mu,s)=0$ for $s\in(-\sqrt{\epsilon},\sqrt{\epsilon})$. Moreover, $\tilde\mu$ is an even function: $\tilde\mu(s)=\tilde\mu(-s)$.*

Proof. Recall that $F_1(\mu,u)=H_1(\mu)w+J_1(\mu,u)$ (where $H_1(\mu)$ and $J_1(\mu,u)$ are defined in (5.17) and (6.2)). Using that $\partial_\mu F_1(\mu,u)=(1+\frac{g}{2\sqrt{2\mu}}\psi)w$ and setting $u=v_s+u_s'$, with $v\equiv v_s\equiv(s\chi,c(\mu,s^2),0,0),\ u'\equiv u_s'\equiv u'(\mu,v_s),\ c=c(\mu,s^2)$, we compute

$$\begin{aligned}\partial_\mu[s^{-1}F_1(\mu,v+u')] &= s^{-1}\left(1+\frac{g}{2\sqrt{2\mu}}\psi'\right)(s\chi+w')\\ &\quad+s^{-1}\sum_{i=1}^4 \delta_{u_i}F_1 w(\mu,v+u')(\partial_\mu v_i+\partial_\mu u_i')\\ &= s^{-1}\left(1+\frac{g}{2\sqrt{2\mu}}\psi'\right)(s\chi+w')+s^{-1}\delta_\alpha F_1(\mu,v+u')\partial_\mu c\\ &\quad+s^{-1}\sum_{i=1}^5 \delta_{u_i}F_1(\mu,v+u')\partial_\mu u_i'.\end{aligned} \tag{8.49}$$

By Lemma 8.2, $\|\partial_\mu^l c\|_{\mathbb{R}^2}=\mathcal{O}(|s|^2)$, $l=0,1$. Since $||v||_{\mathcal{X}}$ is $\mathcal{O}(|s|)$, by Proposition 7.1 the terms $\|\partial_\mu^l u_i'\|$ ($l=0,1,\ i=1,\dots,4$, with the norms taken in the appropriate spaces), are $\mathcal{O}(|s|^2)$. By Lemma 8.5, this implies that all terms in (8.49) containing c,w',α',z',ψ' or their μ-derivatives vanish at $(\mu,s)=(n,0)$. Therefore,

$$\partial_\mu[s^{-1}F_1(\mu,v+u')]|_{(\mu,s)=(n,0)}=\chi \tag{8.50}$$

and hence

$$\partial_\mu\overline{\gamma}_1(n,0)=\langle\chi,\partial_\mu[s^{-1}F_1(\mu,s)]\,|_{(\mu,s)=(n,0)}\rangle_{L_n^2}=\|\chi\|_{L_n^2}^2\ne 0. \tag{8.51}$$

Since $\overline{\gamma}_1(\mu,s)$ is continuously differentiable of all orders in μ and s, by the implicit function theorem, we obtain the first statement of the lemma.

By the symmetry $\overline{\gamma}_1(\mu,-s)=-\overline{\gamma}_1(\mu,s)$ of $\overline{\gamma}_1$ and the uniqueness of the branch $\tilde\mu(s)$, we have $\tilde\mu(s)=\tilde\mu(-s)$, which gives the second statement. □

We define $\mu(s) \equiv \tilde{\mu}(\sqrt{s})$, which is a function with continuous derivatives of all orders for $s \in [0, \epsilon)$ for the same reasons that $c(\mu, s) := \tilde{c}(\mu, \sqrt{s})$ was shown to be continuously differentiable of all orders in Lemma 8.2. Furthermore, μ satisfies $\tilde{\gamma}_1(\mu(s^2), s, c(\mu(s^2), s^2)) = s\overline{\gamma}_1(\mu(s^2), s, c(\mu(s^2), s^2)) = 0$, as required. $\square$

We will now use the branch of solutions to (8.7)–(8.8), provided by Lemmas 8.2 and 8.4, and Corollary 7.2 to obtain the corresponding unique branch, (μ_s, u_s), of solutions to (6.7), with

$$\mu_s \equiv \mu(s^2), \quad u_s \equiv v_s + u'_s, \tag{8.52}$$

$$v_s \equiv (s\chi, c_s, 0, 0), \quad c_s \equiv c(\mu_s, s^2), \tag{8.53}$$

$$u'_s \equiv u'(\mu, v_s). \tag{8.54}$$

(8.52)–(8.54) have continuous s-derivatives of all orders because each component function has continuous derivatives of all orders. Symmetry (7.28) with $\delta = \pi$ and the relation $T_\pi(f_1, f_2, f_3, f_4) = (-f_1, f_2, f_3, f_4)$ imply that $(u'_s)_1$ is an odd function of s and $(u'_s)_2$, $(u'_s)_3$ and $(u'_s)_4$ are even functions of s. Arguing as in the case of Lemma 8.2 shows that the functions:

$$g_1(s) := \begin{cases} \frac{1}{\sqrt{s}}(u'_{\sqrt{s}})_1, & s \neq 0, \\ 0, & s = 0, \end{cases} \quad g_2(s) := c_{\sqrt{s}} + (u'_{\sqrt{s}})_2, \tag{8.55}$$

$$g_3(s) := (u'_{\sqrt{s}})_3, \quad g_4(s) := (u'_{\sqrt{s}})_4, \quad g_5(s) := \mu_{\sqrt{s}} - n, \tag{8.56}$$

are well-defined for $s \geq 0$ and have continuous derivatives of all orders. By Proposition 7.1, these functions have the properties listed in Theorem 8.1. The above definitions and equations (8.52)–(8.54) imply $u_s = (s\chi, \frac{1}{e}a^n, 0, 0) + (g_1(s), \ldots, g_4(s))$. Hence, this solution is of the form (8.1). Now, by Proposition 6.1, this also solves system (4.3)–(4.6), completing the proof. $\square$

9. Proof of Theorem 2.2(a), (b)

Recall that M_W, M_Z, M_H are the masses of the W, Z and Higgs bosons, respectively, and that τ is the shape parameter of the lattice $\mathcal{L}$ (see the paragraph before Theorem 2.3 of Sect. 2). We introduce the notation

$$\langle f \rangle := \frac{1}{|\Omega'|} \int_{\Omega'} f, \tag{9.1}$$

the average of f over fundamental domain $\Omega' = \sqrt{\frac{2\pi}{|\Omega|}}\Omega$. Furthermore, we introduce the function (cf. [26])

$$\eta \equiv \eta_{m_z, m_h}(\tau) := [m_w^2 \alpha_{m_z, m_h}(\tau) + \sin^2 \theta]^{-1}, \tag{9.2}$$

with, recall, $m_w := \sqrt{n}$, $m_z := \frac{\sqrt{n}}{\cos\theta}$ and $m_h := \frac{\sqrt{4\lambda n}}{g}$ the masses of the rescaled W, Z and Higgs boson fields, w, z and ϕ, respectively, and

$$\alpha_{m_z, m_h}(\tau) := \langle |\chi|^2 G_{m_z, m_h}(|\chi|^2) \rangle / (\langle |\chi|^2 \rangle)^2. \tag{9.3}$$

Here χ is defined in (5.24) and $G_{m,m'}$ is the operator-family on the space (5.11) given by

$$G_{m,m'} := G_{m'} - G_m, \qquad \text{where} \qquad G_m := (-\Delta + m^2)^{-1}. \tag{9.4}$$

Note that $G_{m,m'} > 0$ for $m' < m$. Recall $M_W := \frac{1}{\sqrt{2}} g\varphi_0$, $M_Z := \frac{1}{\sqrt{2}\cos\theta} g\varphi_0$ and $M_H := \sqrt{2}\lambda\varphi_0$.

Proposition 9.1. *If $M_Z < M_H$, the parameter s of the branch (8.1) is related to the magnetic field strength by*

$$s^2 = \frac{n}{g^2\langle|\chi|^2\rangle}\eta_{m_z,m_h}(\tau)\omega + R_s(\omega), \qquad \omega := 1 - \frac{M_W^2}{eb}, \tag{9.5}$$

where $R_s(\omega)$ is a real, smooth function of ω satisfying

$$R_s(\omega) = \mathcal{O}(|\omega|^2). \tag{9.6}$$

Before proving Proposition 9.1, we shall see how it implies statements (a) and (b) of Theorem 2.2.

Proof of Theorem 2.2(a), (b). Since the operator G_{m_z,m_h} is positivity preserving, the function $G_{m_z,m_h}(|\chi|^2)$ is positive for $M_Z < M_H$, and hence $\alpha_{m_z,m_h}(\tau)$ and $\eta_{m_z,m_h}(\tau)$ are positive. Furthermore, when the right-hand side of (9.5) is positive, we solve (9.5) for s as a function of b, $s = s(b)$, having continuous derivatives of all orders. When $|1 - \frac{M_W^2}{eb}| \ll 1$, the right-hand side of (9.5) is positive if and only if $1 - \frac{M_W^2}{eb} > 0$.[9] Plugging $s = s(b)$ into (8.1) (i.e. passing from the bifurcation parameter s to the physical parameter b), undoing the rescaling (4.1), and recalling that $b_* = \frac{M_W^2}{e}$, we arrive at the branch, $U_{\mathcal{L}} \equiv (W_b, A_b, Z_b, \varphi_b)$, of solutions of (3.12)–(3.15), which has the properties listed in statements (a) and (b) of Theorem 2.2. □

The following statement follows from the proof above:

Lemma 9.2. *$U_{\mathcal{L}}$ is continuously differentiable of all orders in b for b in an open right half-interval of b_**

Proof of Proposition 9.1. Consider the solution branch (μ_s, w_s, a_s, z_s) given in equation (8.1) and described in Theorem 8.1. Using Taylor's theorem for Banach space-valued finctions (see, e.g. [16]) and recalling the relation $\xi = \sqrt{2\mu}/g$, we may expand this branch in s as follows:

$$\begin{cases} w_s = s\chi + s^3 w' + \mathcal{O}(|s|^5), \\ a_s = \frac{1}{e}a^n + s^2 a' + s^4 a'' + \mathcal{O}(|s|^6), \\ z_s = s^2 z' + \mathcal{O}(|s|^4), \\ \psi_s := \phi_s - \xi_s = s^2\psi' + \mathcal{O}(|s|^4), \\ \xi_s := \sqrt{2\mu_s}/g = \sqrt{2n}/g + s^2\xi' + \mathcal{O}(|s|^4), \end{cases} \tag{9.7}$$

[9] The condition $0 < 1 - \frac{M_W^2}{eb} \ll 1$ is equivalent to the condition $0 < 1 - \frac{M_W^2}{2\pi}|\mathcal{L}| \ll 1$ of Theorem 2.2.

where w', a', z', ψ', ξ' and a'' are the coefficients of s^2 and s^4, respectively, in the Taylor expansion of $g_j(s^2)$, $j = 0, ..., 5$, in (8.1), and χ is defined in (5.24). Here $\mathcal{O}(|s|^p)$ stand for various error terms which, together with their (covariant) derivatives, have norms of order $\mathcal{O}(|s|^p)$ when taken in the appropriate spaces.

To rewrite the asymptotics in terms of the parameter b, we analyse how s depends on b. For this, we use the definitions $\xi_s = \sqrt{2\mu_s}/g$ and $\mu := \frac{1}{2}(g\xi)^2 = \frac{1}{2}(gr\varphi_0)^2$, with $r := \sqrt{\frac{n}{eb}}$ (see (4.2)) to find the following equation for s^2:

$$\xi_s = \sqrt{\frac{n}{eb}}\varphi_0. \tag{9.8}$$

To solve this equation for s^2, we use the Implicit Function Theorem. By (9.7), we can write $\xi_s = \sqrt{2n}/g + g_\xi(s^2)$, where recall, $g_\xi(0) = 0$ and $g'_\xi(0) = \xi'$. Hence, we have to show that $\xi' \neq 0$.

Lemma 9.3. *We have* $\xi' \neq 0$.

Proof. We find relations between ψ', a' and z' entering (9.7). Plugging (9.7) into Equations (4.4)–(4.6), we obtain at order s^4

$$\begin{cases} -\Delta a' - e\,\mathrm{curl}^*\,|\chi|^2 = 0 \\ \left(-\Delta + \frac{n}{\cos^2\theta}\right) z' - g\cos\theta\,\mathrm{curl}^*\,|\chi|^2 = 0 \\ \left(-\Delta + \frac{4\lambda n}{g^2}\right)\psi' + \frac{g}{2}\sqrt{2n}|\chi|^2 = 0. \end{cases} \tag{9.9}$$

We solve these equations, using that $\mathrm{curl}^*\,|\chi|^2 = \mathrm{curl}^*(|\chi|^2 - \langle|\chi|^2\rangle')$ and $|\chi|^2 - \langle|\chi|^2\rangle' \in \mathrm{Ran}(\Delta)$, for the first one, to find [10]

$$\begin{cases} a' = e\,\mathrm{curl}^*\,G_0(|\chi|^2 - \langle|\chi|^2\rangle') \\ z' = g\cos\theta\,\mathrm{curl}^*\,G_{m_z}(|\chi|^2) \\ \psi' = -\frac{g}{2}\sqrt{2n}\,G_{m_h}(|\chi|^2), \end{cases} \tag{9.10}$$

where $G_m := (-\Delta + m^2)^{-1}$ acting on the space (5.11) (cf. (9.4)), and $m_z := \frac{\sqrt{n}}{\cos\theta}$ and $m_h := \frac{\sqrt{4\lambda n}}{g}$ are the masses of the rescaled Z and Higgs boson (Φ) fields, z and ϕ, respectively. Next, we use the following relation proven in "Appendix E":

$$\langle g\sqrt{2n}\xi'|\chi|^2\rangle = \langle -g\sqrt{2n}\psi'|\chi|^2 + \mathrm{curl}\,\nu'|\chi|^2 - g^2|\chi|^4\rangle, \tag{9.11}$$

where, recall, $\nu' := g(a'\sin\theta + z'\cos\theta)$. First, we evaluate $\mathrm{curl}\,\nu'$. The relations $(-\Delta + m^2)G_m = \mathbf{1}$ and $\mathrm{curl}\,\mathrm{curl}^* = -\Delta$ imply $\mathrm{curl}\,a' = e(|\chi|^2 - \langle|\chi|^2\rangle)$. Next, the second relation in (9.10) and the relation $\mathrm{curl}\,\mathrm{curl}^* = -\Delta$ yield $\mathrm{curl}\,z' = g\cos\theta(-\Delta)(-\Delta + m_z^2)^{-1}|\chi|^2$, which, together with $m_z := \frac{\sqrt{n}}{\cos\theta}$, gives $\mathrm{curl}\,z' = g\cos\theta|\chi|^2 - g\frac{n}{\cos\theta}G_{m_z}|\chi|^2$. Finally, using that $e := g\sin\theta$, we conclude that

$$\mathrm{curl}\,\nu' = g^2|\chi|^2 - e^2\langle|\chi|^2\rangle - g^2 n G_{m_z}(|\chi|^2). \tag{9.12}$$

Plugging the last relation and equation (9.10) into the relation (9.11), gives

$$g\sqrt{2n}\xi'\langle|\chi|^2\rangle = -g^2[m_w^2\langle G_{m_z,m_h}(|\chi|^2)|\chi|^2\rangle + \sin^2\theta(\langle|\chi|^2\rangle)^2], \tag{9.13}$$

[10] To check the solutions, one may use that $\mathrm{curl}\,\mathrm{curl}^* = -\Delta$.

where $m_w := \sqrt{n}$ is the mass of the rescaled W boson field w and the operator-family $G_{m,m'}$ is defined by (9.4). We solve for ξ' and write the solution as

$$\xi' = -\frac{g}{\sqrt{2n}}\langle|\chi|^2\rangle\eta^{-1}, \tag{9.14}$$

where $\eta \equiv \eta_{m_z,m_h}(\tau)$ is defined in (9.2)–(9.3). The operator G_{m_z,m_h} in (9.3) is positivity preserving and therefore the function $\alpha_{m_z,m_h}(\tau)$ (and hence $\eta_{m_z,m_h}(\tau)$) is positive, if and only if $m_z < m_h$ (equivalently, $M_Z < M_H$), in which case $\xi' < 0$. □

We now derive the estimate (9.5)–(9.6) for s^2. Equations (9.7) and (9.8) give ξ_s as a function of s and b respectively, yielding

$$\xi_s^2 = \left[\frac{\sqrt{2n}}{g} + g_\xi(s^2)\right]^2 = \frac{n}{eb}\varphi_0^2, \tag{9.15}$$

which can be rearranged to give

$$\frac{2\sqrt{2n}}{g}g_\xi(s^2) + g_\xi(s^2)^2 = \frac{2n}{g^2}\omega, \tag{9.16}$$

where, recall, $\omega = 1 - \frac{M_W^2}{eb}$, with $M_W = \frac{1}{\sqrt{2}}g\varphi_0$. Recall that $g_\xi(0) = 0$ and $g_\xi'(0) = \xi'$. We have

$$\frac{d}{ds^2}\Big|_{s^2=0}\left[\frac{2\sqrt{2n}}{g}g_\xi(s^2) + g_\xi(s^2)^2\right] = \frac{2\sqrt{2n}}{g}\xi'. \tag{9.17}$$

Since $\xi' \neq 0$ and $g_\xi(s^2)$ is continuously differentiable of all orders (see Theorem 8.1), by the implicit function theorem, we may solve (9.16) for s^2, with the solution, $s^2 = s^2(\omega)$, with continuous derivatives of all orders in ω. Explicitly, (9.16)–(9.17) give

$$s^2 = \frac{g}{2\sqrt{2n}}\xi'^{-1}\frac{2n}{g^2}\omega + \mathcal{O}(|\omega|^2). \tag{9.18}$$

Plugging (9.14) into (9.18) gives

$$s^2 = \frac{n}{g^2}\frac{\omega}{\langle|\chi|^2\rangle}\eta + R_s(\omega), \tag{9.19}$$

which is (9.5), with $R_s(\omega)$ satisfying $R_s(\omega) = \mathcal{O}(|\omega|^2)$. Furthermore, since the solution $s^2 = s^2(\omega)$ is continuously differentiable of all orders in ω, so is the remainder term $R_s(\omega)$. □

10. Asymptotics of the Weinberg–Salam Energy Near $b = M_W^2/e$

Recall $\omega = 1 - \frac{M_W^2}{eb}$, with $M_W = \frac{1}{\sqrt{2}}g\varphi_0$, and $\eta_{m_z,m_h}(\tau)$ is defined in (9.2). The main result of this section is the following:

Theorem 10.1. *If $M_Z < M_H$, then the WS energy (3.10) of the branch of solutions (8.1) has the following expansion:*

$$\frac{1}{|\Omega|}E_\Omega(W_b, A_b, Z_b, \varphi_b) = \frac{1}{2}b^2 - \frac{1}{2}b^2 \sin^2\theta\ \eta_{m_z,m_h}(\tau)\omega^2 + R_E(\omega), \tag{10.1}$$

where $R_E(\omega)$ is a real function with continuous derivatives of all orders satisfying

$$R_E(\omega) = \mathcal{O}(|\omega|^3). \tag{10.2}$$

Before proving Theorem 10.1, we derive from it Theorem 2.2 (c).

Proof of Theorem 2.2 (c). Since $\eta_{m_z,m_h}(\tau)$ is positive,[11] the second term in Equation (10.1) is negative, and so for $0 < 1 - \frac{M_W^2}{eb} \ll 1$, E_Ω^{WS} is less than the vacuum energy $\frac{1}{2}b^2|\Omega|$. This proves Theorem 2.2 (c). □

Proof of Theorem 10.1. Let $\mathcal{E}'(w_s, a_s, z_s, \psi_s + \xi_s; r) := \frac{1}{|\Omega'|}\mathcal{E}_{\Omega'}(w_s, a_s, z_s, \psi_s + \xi_s; r)$, where $\mathcal{E}_{\Omega'}$ is the rescaled WS energy given in (4.8). In "Appendix F", we derive the following expansion (to order s^4) of $\mathcal{E}'$ evaluated at family (9.7) of solutions:

$$\begin{aligned}\mathcal{E}'(w_s, a_s, z_s, \psi_s + \xi_s; r) =& \frac{1}{2}\frac{n^2}{e^2} + s^4\langle\frac{1}{2}|\operatorname{curl} z'|^2 + \frac{1}{2}|\operatorname{curl} a'|^2 \\ &+\ g\sqrt{2n}(\psi' + \xi')|\chi|^2 + \frac{n}{2\cos^2\theta}|z'|^2 + |\nabla\psi'|^2 \\ &+ \frac{4\lambda n}{g^2}\psi'^2 - |\chi|^2 \operatorname{curl}\nu' + \frac{g^2}{2}|\chi|^4\rangle \\ &+ R_\varepsilon(s),\end{aligned} \tag{10.3}$$

where $R_\varepsilon(s) = \mathcal{O}(|s|^6)$ and has continuous derivatives of all orders, $\nu' := g(a'\sin\theta + z'\cos\theta)$ and, recall, $\xi_s = \sqrt{2\mu_s}/g$.

To simplify notation, in what follows, we shall suppress the argument $(w_s, a_s, z_s, \psi_s + \xi_s; r)$ of $\mathcal{E}'$.

We claim the following relation:

$$\mathcal{E}' = \frac{1}{2}\frac{n^2}{e^2} - s^4\frac{g^2}{2}\langle|\chi|^2\rangle^2\eta^{-1} + R_\varepsilon(s). \tag{10.4}$$

Proof of (10.4). We simplify the integral at order s^4 in (10.3) by applying eq. (9.9) for a', z' and ψ' to convenient groupings of terms.

First, we address the z' terms in (10.3). Integrating by parts and factoring out z' gives

$$\frac{1}{2}\left\langle|\operatorname{curl} z'|^2 + \frac{n}{\cos^2\theta}|z'|^2\right\rangle = \frac{1}{2}\left\langle z'\cdot\left(-\Delta + \frac{n}{\cos^2\theta}\right)z'\right\rangle. \tag{10.5}$$

Applying (9.9) for z' gives

$$\frac{1}{2}\left\langle|\operatorname{curl} z'|^2 + \frac{n}{\cos^2\theta}|z'|^2\right\rangle = \frac{1}{2}\langle z'\cdot g\cos\theta \operatorname{curl}^*|\chi|^2\rangle. \tag{10.6}$$

[11] See the discussion following Proposition 9.1 for details.

Integrating by parts again gives

$$\frac{1}{2}\left\langle |\operatorname{curl} z'|^2 + \frac{n}{\cos^2\theta}|z'|^2 \right\rangle = \frac{1}{2}\langle g\cos\theta(\operatorname{curl} z')|\chi|^2\rangle. \tag{10.7}$$

Next, we address the a' term in (10.3). Integrating by parts gives

$$\left\langle \frac{1}{2}|\operatorname{curl} a'|^2 \right\rangle = \left\langle \frac{1}{2}a'\cdot(-\Delta)a' \right\rangle. \tag{10.8}$$

Inserting into this expression (9.9) for a' gives

$$\left\langle \frac{1}{2}|\operatorname{curl} a'|^2 \right\rangle = \left\langle \frac{1}{2}a'\cdot e\operatorname{curl}^* |\chi|^2 \right\rangle. \tag{10.9}$$

Integrating by parts again gives

$$\left\langle \frac{1}{2}|\operatorname{curl} a'|^2 \right\rangle = \left\langle \frac{1}{2}g\sin\theta(\operatorname{curl} a')|\chi|^2 \right\rangle. \tag{10.10}$$

Next, we address the ψ' terms. Integrating by parts and factoring out ψ' gives

$$\begin{aligned}&\left\langle |\nabla\psi'|^2 + \frac{4\lambda n}{g^2}\psi'^2 + g\sqrt{2n}\psi'|\chi|^2 \right\rangle\\ &\quad= \left\langle \psi'\left(-\Delta + \frac{4\lambda n}{g^2} + g\sqrt{2n}|\chi|^2\right)\psi' \right\rangle.\end{aligned} \tag{10.11}$$

Inserting into this expression (9.9) for ψ' gives

$$\left\langle |\nabla\psi'|^2 + \frac{4\lambda n}{g^2}\psi'^2 + g\sqrt{2n}\psi'|\chi|^2 \right\rangle = \left\langle \frac{g}{2}\sqrt{2n}\psi'|\chi|^2 \right\rangle. \tag{10.12}$$

For the ξ' term in (10.3), we have by (9.11) and (9.14),

$$\begin{aligned}\langle g\sqrt{2n}\xi'|\chi|^2\rangle &= \frac{1}{2}\langle g\sqrt{2n}\xi'|\chi|^2\rangle + \frac{1}{2}\langle g\sqrt{2n}\xi'|\chi|^2\rangle\\ &= \frac{1}{2}\langle -g\sqrt{2n}\psi'|\chi|^2 + \operatorname{curl}\nu'|\chi|^2 - g^2|\chi|^4\rangle\\ &\quad - \frac{1}{2}g^2\langle|\chi|^2\rangle^2\eta^{-1},\end{aligned} \tag{10.13}$$

where, recall, $\nu' := g(a'\sin\theta + z'\cos\theta)$. Finally, there are two remaining terms of the integral at order s^4 in (10.3),

$$\langle -|\chi|^2\operatorname{curl}\nu' + \frac{g^2}{2}|\chi|^4\rangle, \tag{10.14}$$

which we will not presently simplify.

Adding Eqs. (10.7), (10.10), (10.12), (10.13) and (10.14) and remembering (10.3) gives Eq. (10.4), as required. □

Plugging (9.19) into (10.4) gives

$$\mathcal{E}' = \frac{1}{2}\frac{n^2}{e^2} - \frac{1}{2}\frac{n^2}{g^2}\omega^2\eta_{m_z,m_h}(\tau,r) + \tilde{R}_\varepsilon(\omega), \tag{10.15}$$

where $\tilde{R}_\varepsilon(\omega)$ has continuous derivatives of all orders and satisfies $\tilde{R}_\varepsilon(\omega) = \mathcal{O}(|\omega|^3)$.

For the WS energy (3.10), evaluated at $(W_b, A_b, Z_b, \varphi_b)$, we recall that $E_\Omega^{WS} = \frac{1}{r^2}\mathcal{E}_{\Omega'} = \frac{1}{r^2}|\Omega'|\mathcal{E}'$, which implies

$$\frac{1}{|\Omega|}E_\Omega = \frac{|\Omega'|}{r^2|\Omega|}\mathcal{E}' = \frac{e^2b^2}{n^2}\mathcal{E}', \quad r = \sqrt{\frac{|\Omega|}{|\Omega'|}} = \sqrt{\frac{n}{eb}}. \tag{10.16}$$

Eq. (10.1) follows by plugging (10.15) into (10.16). Since the remainder term $\tilde{R}_\varepsilon$ of (10.15) has continuous derivatives of all orders, so does the remainder term R_E of (10.1). $\square$

11. Shape of Lattice Solutions

In this section, we shall prove Theorem 2.3. Recall the shape parameter τ described in the paragraph preceding (2.22). We return briefly to working with the rescaled fields to prove that $\mathcal{E}_{\Omega'}(u;r)$, $u = (w, \alpha, z, \psi)$, given in (4.8) (and hence $E_\Omega(U)$) is continuously Gâteau differentiable of all orders in the shape parameter τ (restricted to domain (2.22)), which enters through Ω' (and Ω), as well as the spaces containing u (and U). Below, we write

$$u_{\tau,b}(x) \equiv (w_{\tau,b}(x), a_{\tau,b}(x), z_{\tau,b}(x), \phi_{\tau,b}(x)), \tag{11.1}$$

$$\mathcal{E}(\tau, b, u) \equiv \mathcal{E}_{\Omega'}(u;r) =: \int_{\Omega'} e(u;r), \tag{11.2}$$

$$U_{\tau,b}(x) \equiv (W_{\tau,b}(x), A_{\tau,b}(x), Z_{\tau,b}(x), \varphi_{\tau,b}(x)), \tag{11.3}$$

$$E(\tau, b, U) \equiv E_\Omega(U), \tag{11.4}$$

$$\mathcal{X}_\tau \equiv \mathcal{X}, \tag{11.5}$$

to emphasize the dependence of the family of solutions (9.7), the corresponding energy (4.8) (respectively (3.10)) and the space (5.12) containing these solutions on the shape parameter τ, the magnetic field strength b and the position in space $x \in \mathbb{R}^2$. Also, recall the notation $r := \sqrt{n/eb}$.

To get rid of the dependency of the space $\mathcal{X}_\tau$ containing $u_{\tau,b}$, on the shape parameter τ, we make the change of coordinates

$$M_\tau : \mathcal{X}_\tau \to \mathcal{X}_1, (M_\tau u)(x) = u(m_\tau x),\ m_\tau = \frac{1}{\sqrt{\mathrm{Im}(\tau)}}\begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ 0 & \mathrm{Im}(\tau) \end{pmatrix}, \tag{11.6}$$

mapping Ω' into a square of area 2π. This allows us to define the functions $G' : \mathbb{C} \times \mathbb{R} \times \mathcal{X}_1 \to \mathbb{C} \times \mathbb{R} \times \mathcal{Y}_1$ and $\varepsilon' : \mathbb{C} \times \mathbb{R} \times \mathcal{X}_1 \to \mathbb{C} \times \mathbb{R} \times \mathcal{Y}_1$ on the fixed space $\mathcal{X}_1$:

$$G'(\tau, b, v) = M_\tau G(b, M_\tau^{-1}v), \tag{11.7}$$

$$\varepsilon'(\tau, b, v) = M_\tau \varepsilon(b, M_\tau^{-1}v), \tag{11.8}$$

where, recall, $G(b, v)$ is the map given by the left-hand side of (4.3)–(4.6), given explicitly in (5.1), and $\varepsilon_{WS}(b, u) := \varepsilon(u;r)$ is the rescaled energy density given by the integrand in (4.8), see (11.2) (ε depends on the magnetic field strength b but does not directly depend on the shape parameter τ).

Lemma 11.1. *$G'(\tau, b, v)$ and $\varepsilon'(\tau, b, v)$ are continuously Gâteau differentiable of all orders in $\mathrm{Re}(\tau)$, $\mathrm{Im}(\tau)$, b and v.*

Proof. Since $G(b, u)$ and $\varepsilon(b, u)$ have continuous b and u derivatives of all orders, and M_τ is a linear map independent of b and v, it follows that $G'(\tau, b, v)$ and $\varepsilon'(\tau, b, v)$ have continuous b- and v-derivatives of all orders.

For the τ-derivatives, note that

$$M_\tau \circ \partial_{x_1} \circ M_\tau^{-1}(v_j)(x) = \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \partial_{x_1} v_j(x), \quad j = 1, ..., 4, \tag{11.9}$$

$$\begin{aligned} &M_\tau \circ \partial_{x_2} \circ M_\tau^{-1}(v_j)(x) \\ &\quad = \frac{1}{\sqrt{\mathrm{Im}(\tau)}} (\mathrm{Re}(\tau)\partial_{x_1} v_j(x) + \mathrm{Im}(\tau)\partial_{x_2} v_j(x)), \quad j = 1, ..., 4, \end{aligned} \tag{11.10}$$

are continuously differentiable of all orders in $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$. Since $G(b, u)$ and $\varepsilon_{WS}(b, u)$ are polynomials in the components of u and their (covariant) derivatives, G' and Σ are simply G and ε_{WS} with the coefficients of the derivative-containing terms multiplied by smooth functions of $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$. Therefore $G'(\tau, b, v)$ and $\Sigma(\tau, b, v)$ have continuous $\mathrm{Re}(\tau)$- and $\mathrm{Im}(\tau)$-derivatives of all orders. □

Lemma 11.2. *$v_{\tau,b} := M_\tau u_{\tau,b}$ is continuously differentiable of all orders in $Re(\tau)$ and $Im(\tau)$.*

Proof. Let τ_0 be an arbitrary shape parameter, and recall that $\delta_\#$ denotes the partial (real) Gâteaux derivative with respect to $\#$. Then $G'(\tau_0, b, v_{\tau_0,b}) = M_{\tau_0} G(b, u_{\tau_0,b}) = 0$, $\delta_v G(\tau_0, b, v_{\tau_0,b}) = M_{\tau_0} \circ \delta_u G(b, u_{\tau_0,b}) \circ M_{\tau_0}^{-1}$ is invertible, and by Lemma 11.1, G' is continuously Gâteau differentiable of all orders in τ, b and v. Therefore, by the Implicit Function Theorem, the unique solution $v_{\tau,b}$ to the equation $G(\tau, b, v) = 0$ is continuously differentiable of all orders in $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ near $(\mathrm{Re}(\tau), \mathrm{Im}(\tau)) = (\mathrm{Re}(\tau_0), \mathrm{Im}(\tau_0))$. Since τ_0 was arbitrary, this proves the result. □

Proposition 11.3. *$E(\tau, b, U_{\tau,b})$ is continuously differentiable of all orders in $Re(\tau)$ and $Im(\tau)$.*

Proof. To get rid of the dependency of $\mathcal{E}(\tau, b, u_{\tau,b})$ on the domain of integration Ω', we again make the change of coordinates $y = m_\tau^{-1} x$. Then

$$\begin{aligned} \mathcal{E}(\tau, b, u_{\tau,b}) &= \int_{\Omega'} \varepsilon(b, u_{\tau,b})(x)\, d^2x \\ &= \int_0^{\sqrt{2\pi}} \int_0^{\sqrt{2\pi}} \varepsilon'(\tau, b, v_{\tau,b})(y) d^2y. \end{aligned} \tag{11.11}$$

By Lemma 11.2, $v_{\tau,b}$ has continuous $\mathrm{Re}(\tau)$- and $\mathrm{Im}(\tau)$-derivatives of all orders, and by Lemma 11.1, Σ has continuous derivatives of all orders mapping $\mathbb{C} \times \mathbb{R} \times \mathcal{X}_1$ to $\mathbb{C} \times \mathbb{R} \times \mathcal{Y}_1$. In particular, the $\mathrm{Re}(\tau)$- and $\mathrm{Im}(\tau)$-derivatives of $\varepsilon'(\tau, b, v_{\tau,b})$ remain integrable, so we conclude that $\mathcal{E}(\tau, b, u_{\tau,b})$ (and hence $E(\tau, b, U_{\tau,b})$) is continuously differentiable of all orders in $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$. $\square$

Theorem 11.4. *When $M_Z < M_H$, the minimizers τ_b of $E(\tau, b, U_{\tau,b})$ are related to the maximizers τ_* of $\eta_{m_z,m_h}(\tau)$ as $\tau_b - \tau_* = \mathcal{O}(|1 - \frac{M_W^2}{eb}|^{\frac{1}{2}})$. In particular, $\tau_b \to \tau_*$ as $b \to b_* = M_W^2/e$.*

Proof. The minimizers (τ_b) of $E(\tau, b, U_{\tau,b})$ are equivalent to the minimizers of the energy functional $\tilde{E}(\tau, U_{\tau,b}) := \omega^{-2}(E(\tau, b, U_{\tau,b}) - \frac{1}{2}b^2)$. By Theorem 10.1, we have

$$\tilde{E}(\tau, U_{\tau,b}) = -\frac{1}{2}b^2 \sin^2\theta\ \eta_{m_z,m_h}(\tau) + \mathcal{O}(|\omega|),$$

where, recall, $\omega = 1 - \frac{M_W^2}{eb}$. Since $\partial_\tau \tilde{E}(\tau, U_{\tau,b})|_{\tau=\tau_b} = 0$, we have the expansion

$$\begin{aligned}\tilde{E}(\tau_*, U_{\tau_*,b}) - \tilde{E}(\tau_b, U_{\tau_b,b}) &= \frac{1}{2}\partial_\tau^2 \tilde{E}(\tau_b, U_{\tau_b,b})[\tau_* - \tau_b]^2 + \mathcal{O}([\tau_* - \tau_b]^3)\\ &= -\frac{1}{4}b^2 \sin^2\theta\ \partial_\tau^2 \eta_{m_z,m_h}(\tau_b)[\tau_* - \tau_b]^2\\ &\quad + \mathcal{O}([\tau_* - \tau_b]^3) + \mathcal{O}(|\omega|).\end{aligned} \tag{11.12}$$

For both expansions to hold, we must have $\tau_b - \tau_* = \mathcal{O}(|\omega|^{\frac{1}{2}})$, as required. $\square$

The maximizer of $\eta_{m_z,m_h}(\tau)$, defined in (9.2), was found numerically in [26] with some analytical results in [27]:

Theorem 11.5. *([26]) For $M_Z < M_H$, the function $\eta_{m_z,m_h}(\tau)$ has a maximum at $\tau_* = e^{i\pi/3}$.*

Theorem 2.3 follows from Theorems 11.4 and 11.5.

Remark 11.6. Using symmetries of $\eta_{m_z,m_h}(\tau)$, one might be able to prove that its only critical points are $\tau = e^{i\pi/3}$ and $\tau = e^{i\pi/2}$, i.e. the hexagonal and square lattices, cf. [3, 30].

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Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose.

Appendix A: Covariant Derivatives and Curvature

In this appendix, we briefly review some basic definitions from gauge theory. For some geometrical background, see [21,25,34]. Recall that we use the Einstein convention of *summing over repeated indices*.

Let V be an inner product vector space, G a Lie group acting transitively on V via a unitary representation $\rho : g \mapsto \rho_g$, and let $\mathfrak{g}$ be the Lie algebra of G acting on V via the representation $\tilde{\rho} : A \mapsto \tilde{\rho}_A$ induced by ρ.

To simplify notation below, we take $V = \mathbb{C}^m$ and G a matrix group, acting on V by matrix rules (and similarly for $\mathfrak{g}$) and write $\rho_g \Psi = g\Psi$ and $\tilde{\rho}_A \Psi = A\Psi$. Moreover, we assume that G is either $U(m)$ or a Lie subgroup of $U(m)$.

Let M be an open subset in a finite-dimensional vector space, with a metric h and local coordinates $\{x^i\}$, and let $\partial_i \equiv \partial_{x^i}$.

For a $\mathfrak{g}$-valued connection (one-form) $A \equiv A_i dx^i$ on M, we define the covariant derivatives:

- ∇_A, mapping functions (sections), $\Psi : M \to V$, into $\mathfrak{g}$-valued one-forms, as

$$\nabla_A \Psi := d\Psi + A\Psi \equiv (\partial_i \Psi + A_i \Psi) dx^i; \tag{A.1}$$

- d_A, mapping $\mathfrak{g}$-valued functions (0-forms) f into $\mathfrak{g}$-valued one-forms

$$d_A f := df + [A, f] \equiv (\partial_i f + [A_i, f]) dx^i; \tag{A.2}$$

- d_A, mapping $\mathfrak{g}$-valued one-forms into $\mathfrak{g}$-valued two-forms

$$d_A B := dB + [A, B], \tag{A.3}$$

with $[A, B]$ defined in local coordinates $\{x^i\}$ as

$$[A, B] := [A_i, B_j] dx^i \wedge dx^j \equiv [B, A], \tag{A.4}$$

for $A = A_i dx^i$ and $B = B_j dx^j$.[12]

[12]More generally, if A is a $\mathfrak{g}-$valued $p-$form and B is a $\mathfrak{g}-$valued $q-$form, written as $A = A^a \otimes \gamma_a$ and $B = B^b \otimes \gamma_b$, where A^a and B^b are $p-$ and $q-$forms and $\{\gamma_a\}$ is a basis

The curvature form of the connection A is the $\mathfrak{g}$-valued two-form given by the formula

$$F_A = dA + \frac{1}{2}[A, A]. \tag{A.6}$$

It is related to the curvature operator (denoted by the same symbol) $F_A := d_A \circ d_A$. As a simple computation shows, this operator is a matrix-multiplication operator given by the matrix-valued 2-form (A.6).

Let U be a vector space (V or $\mathfrak{g}$ in our case) and let $\Omega^p_U \equiv U \otimes \Omega^p$ denote the space of U-valued p-forms. On Ω^p_U, one defined the inner product, $\langle\cdot,\cdot\rangle_{\Omega^p_U} \equiv \langle\cdot,\cdot\rangle^h_{\Omega^p_U}$ as

$$\langle A, B\rangle_{\Omega^p_U} \equiv \langle A, B\rangle^h_{\Omega^p_U} := \langle A_\alpha, B^\alpha\rangle_U, \tag{A.7}$$

where $A = A_\alpha dx^\alpha$ and $B = B_\alpha dx^\alpha$ are U -valued p-forms, α is a p-form index and $\langle\cdot,\cdot\rangle_U$ is the inner product on U. Here the indices are raised and lowered with help of the inner product h on M.

Above, we did not display the coupling constants. Doing so would change the covariant derivative to $d_A\Psi = (d + gA)\Psi$, if G is simple. If G is not simple, then each simple component of G gets its own coupling constant, as was done in the main text for $G = SU(2) \times U(1)$ (see also (C.2)–(C.6) below).

Appendix B: The Time-Dependent Yang–Mills–Higgs System

In this appendix, we briefly review the Yang–Mills–Higgs theory, including the derivation of the energy functional (2.7). In what follows, we use the convention of raising or lowering an index by contracting a tensor T with the metric tensor:

$$T^\alpha_{i,\beta} = \eta_{ij} T^{j,\alpha}_\beta \tag{B.1}$$

where η is the Minkowski metric of signature $(+, -, ..., -)$ on $M \subset \mathbb{R}^{d+1}$ and α, β are multi-indices. The same equations could be *reinterpreted as stationary* equations by taking the *Euclidean metric* δ_{ij}, *instead of* η_{ij}, *and letting the indices range over* $1, \ldots, d$, *rather than* $1, \ldots, d+1$. In this case, $T^\alpha_{i,\beta} = T^{i,\alpha}_\beta$.

Lagrangian. Let Ω be a bounded domain in $\mathbb{R}^d$ and $M = \Omega \times [0, T] \subset \mathbb{R}^{d+1}$ be spacetime equipped with the Minkowski metric η of signature $(+, -, ..., -)$ and V and G be as in "Appendix A". The theory involves a Higgs field $\Psi : M \to V$ interacting with the gauge field A, a connection (one-form) on M with values in the algebra $\mathfrak{g}$. The dynamics are given by the Lagrangian

$$\mathcal{L}(\Psi, A) := \int_\Omega \left(\langle\nabla_A\Psi, \nabla_A\Psi\rangle^\eta_{\Omega^1_V} - U(\Psi) + \langle F_A, F_A\rangle^\eta_{\Omega^2_{\mathfrak{g}}}\right), \tag{B.2}$$

---

in $\mathfrak{g}$, then

$$[A, B] := (A^a \wedge B^b) \otimes [\gamma_a, \gamma_b] = (-1)^{pq+1}[B, A]. \tag{A.5}$$

with corresponding action $\mathcal{S} := \int_0^T \mathcal{L}(\Psi, A)dt$, $T > 0$, given explicitly by

$$\mathcal{S}(\Psi, A) = \int_M \big(\langle \nabla_A \Psi, \nabla_A \Psi\rangle^\eta_{\Omega^1_V} - U(\Psi) + \langle F_A, F_A\rangle^\eta_{\Omega^2_{\mathfrak{g}}}\big), \tag{B.3}$$

where $U : V \to \mathbb{R}^+$ is a self-interaction potential, which is assumed to be gauge invariant: $U(\rho_g\Psi) = U(\Psi)$. Typical examples of G, V and $U(\Psi)$ are $U(m), \mathbb{C}^m$ and $U(\Psi) = \frac{1}{2}\lambda(1 - \|\Psi\|^2_V)^2$.

Euler–Lagrange equations. The Euler–Lagrange equations (called Yang–Mills–Higgs equations) for the fields Ψ and A are

$$\nabla^{*\eta}_A \nabla_A \Psi = U'(\Psi), \tag{B.4}$$

$$d^{*\eta}_A F_A = J(\Psi, A), \tag{B.5}$$

where $\nabla^{*\eta}_A$ and $d^{*\eta}_A$ are the adjoints of ∇_A and d_A in the appropriate inner products *involving the metric* η and $J(\Psi, A)$ is the YMH current given by

$$J(\Psi, A) := \mathrm{Re}\langle \gamma_a \Psi, \nabla_A \Psi\rangle_V \gamma_a \tag{B.6}$$

$$= \mathrm{Re}\langle \gamma_a \Psi, \nabla_i \Psi\rangle_V \gamma_a \otimes dx^i, \tag{B.7}$$

where γ_a is an orthonormal basis of $\mathfrak{g}$ and $\nabla_i := \partial_i + A_i$, with $\partial_i \equiv \partial_{x^i}$, so that $\nabla_A \Psi = \nabla_i \Psi dx^i$. (B.5) is the Yang–Mills equation.

Proof of (B.4) - (B.5). For convenience, we *assume periodic or Dirichlet boundary conditions and that* Ψ *and* A *are* T*-periodic in* t and calculate the Gâteaux derivatives *formally*.

Recall that $\delta_\#$ denotes the partial (real) Gâteaux derivative with respect to $\#$. First we calculate the (complex) Gâteaux derivative of (B.3) in the Ψ-direction. Define $\partial_z \equiv \frac{1}{2}(\partial_{\mathrm{Re}\, z} - i\partial_{\mathrm{Im}\, z})$ and $\delta_\Psi \equiv \frac{1}{2}(\delta_{\mathrm{Re}\,\Psi} - i\delta_{\mathrm{Im}\,\Psi})$. Then $\delta_\Psi \mathcal{S}(\Psi, A)\Psi' = \partial_z \mathcal{S}(\Psi_z, A)|_{z=0}$, where $\Psi_z = \Psi + z\Psi'$, $z \in \mathbb{C}$. Using this, we find

$$\delta_\Psi \mathcal{S}(\Psi, A)\Psi' = \int_M \big(\langle \nabla_A \Psi, \nabla_A \Psi'\rangle_{\Omega^1_V} - \langle U'(\Psi), \Psi'\rangle_V\big). \tag{B.8}$$

Integrating the first term by parts and factoring out Ψ' gives

$$\delta_\Psi \mathcal{S}(\Psi, A)\Psi' = \int_M \langle \nabla^*_A \nabla_A \Psi - U'(\Psi), \Psi'\rangle_V. \tag{B.9}$$

For this derivative to be zero for every variation Ψ', (B.4) must hold.

Next we calculate the Gâteaux derivative of (B.3) in the A-direction. Using the definition $\delta_A f(A)B = \partial_s f(A_s)|_{s=0}$, where $A_s = A + sA'$, $s \in \mathbb{R}$, we find

$$\delta_A \mathcal{S}(\Psi, A)B = \int_M \big(\langle B\Psi, \nabla_A \Psi\rangle_{\Omega^1_V} + c.c. + 2\langle d_A B, F_A\rangle_{\Omega^2_{\mathfrak{g}}}\big) \tag{B.10}$$

$$= I + II. \tag{B.11}$$

Writing $B = B^a \gamma_a = B^a_i dx^i \otimes \gamma_a$ (with B^a_i real) and $\nabla_A \Psi = \nabla_i \Psi dx^i$, so that

$$\langle B\Psi, \nabla_A \Psi\rangle_{\Omega^1_V} = \langle B^a, \langle \gamma_a \Psi, \nabla_i \Psi\rangle_V dx^i\rangle_{\Omega^1}, \tag{B.12}$$

and using that $B^a C^a = -\operatorname{Tr}[(B^c\gamma_c)(C^a\gamma_a)]$ (since $\operatorname{Tr}(\gamma_c^*\gamma_a) = -\operatorname{Tr}(\gamma_c\gamma_a) = \delta_{ca}$), gives

$$I = -\int_M \langle B, \langle\gamma_a\Psi, \nabla_i\Psi\rangle_V \gamma_a \otimes dx^i\rangle_{\Omega^1_{\mathfrak{g}}} + c.c.. \tag{B.13}$$

which gives $I = \int_M \langle B, J(\Psi, A)\rangle_{\Omega^1_{\mathfrak{g}}}$. For the second term on the r.h.s. of (B.10), integrating by parts yields $II = \int_M \langle B, d_A^* F_A\rangle_{\Omega^1_{\mathfrak{g}}}$. Collecting the last two equations gives

$$\delta_A \mathcal{S}(\Psi, A)B = 2\int_M \langle B, -J(\Psi, A) + d_A^* F_A\rangle_{\Omega^1_{\mathfrak{g}}}. \tag{B.14}$$

For this derivative to be zero for every variation B, (B.5) must hold. $\square$

Conserved energy. Again, the Gâteaux derivative calculations in the following subsection are formal. Recall that $M := \Omega \times [0, T] \subset \mathbb{R}^{d+1}$.

To find the expression for the energy, we use, as in classical mechanics, the (partial, i.e. without passing to the momentum fields) Legendre transform of (B.2) is given by

$$\begin{aligned} E(\Psi, A) = \partial_{\nabla_0\Psi}\mathcal{L}(\Psi, A)\nabla_0\Psi + \partial_{\overline{\nabla_0\Psi}}\mathcal{L}(\Psi, A)\overline{\nabla_0\Psi} \\ + \sum_{i=1}^{d} \partial_{F_{0i}}\mathcal{L}(\Psi, A)F_{0i} - \mathcal{L}(\Psi, A). \end{aligned} \tag{B.15}$$

Proposition B.1. *The (partial) Legendre transform* (B.15) *of Lagrangian* (B.2) *yields the conserved energy*

$$E(\Psi, A) := \int_\Omega \left(\|\nabla_A\Psi\|^2_{\Omega^1_V} + U(\Psi) + \|F_A\|^2_{\Omega^2_{\mathfrak{g}}}\right), \tag{B.16}$$

where the norms are taken using the Euclidean metric on $\mathbb{R}^{d+1}$ *(rather than the Minkowski metric).*

Note that for static (time-independent) fields, $E(\Psi, A) = -\mathcal{L}(\Psi, A)$.

Proof. Let $\partial_\#$ denote the partial derivative with respect to the symbol #, and recall that $\delta_\#$ denotes the partial (real) Gâteaux derivative with respect to #. We calculate

$$\partial_{\nabla_0\Psi}\mathcal{L}(\Psi, A)\nabla_0\Psi = \int_\Omega \|\nabla_0\Psi\|^2_V = \partial_{\overline{\nabla_0\Psi}}\mathcal{L}(\Psi, A)\overline{\nabla_0\Psi} \tag{B.17}$$

and

$$\sum_{i=1}^{d} \partial_{F_{0i}}\mathcal{L}(\Psi, A)F_{0i} = \int_\Omega 2\sum_{i=1}^{d} |F_{0i}|^2. \tag{B.18}$$

(B.16) results.

It remains to show that (B.16) is conserved by the YMH equations (B.4)–(B.5). This can be done by using the (partial) Legendre transform (B.15) as in classical mechanics, or by a direct computation. We proceed in the second way. Applying the chain rule gives

$$\frac{d}{dt}E(\Psi, A) = \delta_\Psi E(\Psi, A)\partial_0 \Psi + \delta_{\overline{\Psi}} E(\Psi, A)\partial_0 \overline{\Psi} + \delta_A E(\Psi, A)\partial_0 A, \tag{B.19}$$

where, recall, $\partial_i \equiv \partial_{x^i}$. We now calculate the first term using (B.4).

$$\begin{aligned}\delta_\Psi E(\Psi, A)\partial_0 \Psi = \int_\Omega \big(&\langle \nabla_0 \Psi, \nabla_0 \partial_0 \Psi\rangle_V + \sum_{k=1}^{d} \langle \nabla_k \Psi, \nabla_k \partial_0 \Psi\rangle_V \\ &+ \langle U'(\Psi), \partial_0 \Psi\rangle_V\big).\end{aligned} \tag{B.20}$$

Integrating the second term by parts gives

$$\begin{aligned}\delta_\Psi E(\Psi, A)\partial_0 \Psi = \int_\Omega \big(&\langle \nabla_0 \Psi, \nabla_0 \partial_0 \Psi\rangle_V + \sum_{k=1}^{d} \langle \nabla_k^* \nabla_k \Psi, \partial_0 \Psi\rangle_V \\ &+ \langle U'(\Psi), \partial_0 \Psi\rangle_V\big).\end{aligned} \tag{B.21}$$

By (B.4), we have

$$\nabla_0^* \nabla_0 \Psi - \sum_{k=1}^{d} \nabla_k^* \nabla_k \Psi = U'(\Psi), \tag{B.22}$$

so (B.21) becomes

$$\delta_\Psi E(\Psi, A)\partial_0 \Psi = \int_\Omega \big(\langle \nabla_0 \Psi, \nabla_0 \partial_0 \Psi\rangle_V + \langle \nabla_0^* \nabla_0 \Psi, \partial_0 \Psi\rangle_V\big). \tag{B.23}$$

Here $\nabla_0^* = -\partial_0 + A_0^\dagger = -\partial_0 - A_0$, where the second equality follows because the representation of $\mathfrak{g}$ is unitary. Therefore,

$$\begin{aligned}\delta_\Psi E(\Psi, A)\partial_0 \Psi &= \int_\Omega \big(\langle (\partial_0 + A_0)\Psi, (\partial_0 + A_0)\partial_0 \Psi\rangle_V \\ &\qquad + \langle (-\partial_0 - A_0)(\partial_0 + A_0)\Psi, \partial_0 \Psi\rangle_V\big) \\ &= \int_\Omega \partial_0 \langle \Psi, A_0 \partial_0 \Psi\rangle_V.\end{aligned} \tag{B.24}$$

Similarly,

$$\delta_{\overline{\Psi}} E(\Psi, A)\partial_0 \overline{\Psi} = \int_\Omega \partial_0 \langle A_0 \partial_0 \Psi, \Psi\rangle_V, \tag{B.25}$$

and so

$$\delta_\Psi E(\Psi, A)\partial_0 \Psi + \delta_{\overline{\Psi}} E(\Psi, A)\partial_0 \overline{\Psi} = \int_\Omega \partial_0 J_0(\Psi, A), \tag{B.26}$$

where $J_0(\Psi, A)$ is the time component of the YMH current (B.6).

One may show using (B.5) that

$$\delta_A E(\Psi, A)\partial_0 A = -\int_\Omega \partial_0 J_0(\Psi, A). \tag{B.27}$$

Hence, by (B.19) we have $\frac{d}{dt}E(\Psi, A) = 0$, as required. $\square$

Gauge symmetries. We define the local action, $\rho_g A$,[13] of the group G on A, by the equation $d_{\rho_g A} = g d_A g^{-1}$, for all $g \in C^1(N, G)$, where N is either M or Ω. We compute

$$\rho_g A = gAg^{-1} + gdg^{-1}. \tag{B.28}$$

Proposition B.2. *The Lagrangian* (B.2) *is invariant under the Poincaré group and the gauge transformations*

$$T_g^{\text{gauge}} : (\Psi, A) \mapsto (g\Psi, \rho_g A), \qquad \forall g \in C^1(M, G). \tag{B.29}$$

Proof. The invariance under the Poincaré group follows from the definition of this group and the choice of the Minkowski metric on $M \subset \mathbb{R}^{d+1}$.

For the gauge invariance, recall that $U(\Psi)$ is $\mathfrak{g}$-invariant, and that the representations $g \mapsto \rho_g$ (on V) and the adjoint representation $g \mapsto \mathrm{ad}_g$ (on $\mathfrak{g}$) are unitary. Therefore, to prove invariance under the gauge transformation (B.29), it suffices to show that

$$\nabla_{\rho_g A} g\Psi = g\nabla_A \Psi, \tag{B.30}$$

$$F_{\rho_g A} = gF_A g^{-1}. \tag{B.31}$$

We shall use the equation

$$hdh^{-1} = -dhh^{-1}, \quad \forall h \in G \tag{B.32}$$

which follows from $d(hh^{-1}) = 0$. For (B.30) we compute

$$\begin{aligned} \nabla_{\rho_g A} g\Psi &= d(g\Psi) + (gAg^{-1} + gdg^{-1})(g\Psi) && \text{(B.33)}\\ &= (dg)\Psi + gd\Psi + gA\Psi + gdg^{-1}g\Psi. && \text{(B.34)} \end{aligned}$$

Since $gdg^{-1}g = -gg^{-1}dg = -dg$, this gives $\nabla_{\rho_g A} g\Psi = g\nabla_A \Psi$.

For (B.31), computing in coordinates $\{x^i\}$ and writting $F_{\rho_g A} := (F_{\rho_g A})_{ij}$ $dx^i \wedge dx^j$ and $F_A := (F_A)_{ij} dx^i \wedge dx^j$, we find

$$\begin{aligned} (F_{\rho_g A})_{ij} = &\frac{1}{2}[\partial_i(gA_j g^{-1} + g\partial_j g^{-1}) - \partial_j(gA_i g^{-1} + g\partial_i g^{-1})]\\ &+ \frac{1}{2}[gA_i g^{-1} + g\partial_i g^{-1}, gA_j g^{-1} + g\partial_j g^{-1}], \end{aligned} \tag{B.35}$$

where, recall, $\partial_i \equiv \partial_{x^i}$. Expanding the partial derivative and commutators gives

$$\begin{aligned} (F_{\rho_g A})_{ij} =&\frac{1}{2}[\partial_i g A_j g^{-1} + g\partial_i A g^{-1} + gA_j \partial_i g^{-1} + \partial_i g \partial_j g^{-1} + g\partial_i \partial_j g^{-1}\\ &+ (gA_i g^{-1} + \partial_i g g^{-1})(gA_j g^{-1} + g\partial_j g^{-1})\\ &- (i \leftrightarrow j)]. \end{aligned} \tag{B.36}$$

Expanding the product on the second line gives

$$(F_{\rho_g A})_{ij} =\frac{1}{2}[\partial_i g A_j g^{-1} + g\partial_i A g^{-1} + gA_j \partial_i g^{-1} + \partial_i g \partial_j g^{-1} + g\partial_i \partial_j g^{-1}$$

[13] Compared with the notation of "Appendix A", to simplify the notation we omit the tilde over ρ_g in action of the Lie algebra $\mathfrak{g}$ on V.

$$+ gA_iA_jg^{-1} + \partial_i gA_jg^{-1} + gA_i\partial_j g^{-1} + \partial_i g\partial_j g^{-1} - (i \leftrightarrow j)]. \tag{B.37}$$

Cancelling terms symmetrical in i and j and simplifying gives

$$(F_{\rho_g A})_{ij} = g\left(\frac{1}{2}[\partial_i A_j - \partial_j A_i] + \frac{1}{2}[A_iA_j - A_jA_i]\right)g^{-1} \tag{B.38}$$

$$= g(F_A)_{ij}g^{-1}, \tag{B.39}$$

as required. □

Specifying (B.16) to the WS model gives (2.7).

The YMH equations in coordinate form. In coordinate form, the differential form (gauge field) entering the YMH Lagrangian (B.2) is written as $A = A_i dx^i$. The local coordinate expression for the curvature is $F_A = F_{ij}dx^i \wedge dx^j$, where $F_{ij} := \frac{1}{2}(\partial_i A_j - \partial_j A_i) + \frac{1}{2}[A_i, A_j]$. Furthermore, for the covariant derivatives ∇_A and d_A, we have $\nabla_A \Psi = \nabla_i \Psi dx^i$ and $d_A^* F_A = -\nabla^i F_{ij} dx^j$, where $\nabla_i \Psi := (\partial_i + A_i)\Psi$ and $\nabla^i F_{ij} := \partial^i F_{ij} + [A^i, F_{ij}]$.

For an arbitrary $\mathfrak{g}$-valued one-form $B = B_i dx^i$, we have $d_A B = \nabla_i B_j dx^i \wedge dx^j$ and $d_A^* B = -\nabla^i B_i$, where

$$\nabla^i B_j := \partial^i B_j + [A^i, B_j]. \tag{B.40}$$

We write $F_{ij} = F_{ij}^a \gamma_a$ for an orthonormal basis γ_a of $\mathfrak{g}$ and the lower case roman indices run over the spatial components $1, 2, \ldots, d$. Note that $F_{ij} = [\nabla_i, \nabla_j]$, but $F_{ij} \neq \frac{1}{2}(\nabla_i A_j - \nabla_j A_i)$.

Let Ω be either a bounded domain in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$. In the former case, we assume either periodic or Dirichlet boundary conditions.

Proposition B.3. *The Lagrangian and energy for the YMH model are given in coordinates by*

$$\mathcal{L}(\Psi, A) = \int_\Omega \langle \nabla_k \Psi, \nabla^k \Psi \rangle_V - U(\Psi) + \frac{1}{2}F_{ij}^a F^{a,ij}, \tag{B.41}$$

$$E_\Omega(\Psi, A) = \int_\Omega \langle \nabla_k \Psi, \nabla_k \Psi \rangle_V + U(\Psi) + \frac{1}{2}F_{ij}^a F_{ij}^a \tag{B.42}$$

(with different ranges of indices as mentioned above). The YMH equations are given in coordinates by:

$$-\nabla^i \nabla_i \Psi = U'(\Psi), \tag{B.43}$$

$$-\nabla^i F_{ij} = \mathrm{Re}\langle \gamma_a \Psi, \nabla_j \Psi \rangle_V \gamma_a. \tag{B.44}$$

Proof. Equations (B.41) and (B.42) follow from the coordinate expressions $d_A \Psi = \nabla_k \Psi dx^k$ and $F_A = F_{ij}^a \gamma_a \otimes dx^i \wedge dx^j$, together with the fact that dx^k and $\gamma_a \otimes dx^i \wedge dx^j$ form orthonormal bases for Ω^1 and $\Omega^2_{\mathfrak{g}}$, respectively.

Equations (B.43)–(B.44) follow from Eqs. (B.4)–(B.6) and the coordinate expressions for d_A and d_A^* above. □

Appendix C: The WS Equations in Coordinate Form

For the gauge group $G = U(2) = SU(2) \times U(1)$, we choose the standard inner product

$$\langle \gamma, \delta \rangle_{\mathfrak{u}(2)} := 2\,\mathrm{Tr}\,\gamma^*\delta = -2\,\mathrm{Tr}\,\gamma\delta \tag{C.1}$$

on $\mathfrak{u}(2)$, for which $-\frac{i}{2}\tau_a$, $a = 0, 1, 2, 3$ (where τ_a, $a = 1, 2, 3$, are the Pauli matrices together with $\tau_0 := \mathbf{1}$) form an orthonormal basis. It is customary to factor out the coefficient of $-\frac{i}{2}$. In coordinates, we write

$$\nabla_Q \Phi = \nabla_i \Phi dx^i, \quad Q = -\frac{i}{2} Q_i dx^i \ \text{ and } \ F_Q = -\frac{i}{2} Q_{ij} dx^i \wedge dx^j, \tag{C.2}$$

with $Q_i(x)$, $Q_{ij}(x) \in i\mathfrak{u}(2)$. Using Eq. (2.3), we compute $Q_{ij} = \frac{1}{2}(\partial_i Q_j - \partial_j Q_i) - \frac{i}{4}[Q_i, Q_j]$. Furthermore, we write $Q = V + X$ and

$$V = -\frac{i}{2} V_i dx^i \ \text{ and } \ X = -\frac{i}{2} X_i dx^i, \tag{C.3}$$

with $V_i(x) \in i\mathfrak{su}(2)$ and $X_i(x) \in i\mathfrak{u}(1)$. Then $Q_{ij} = V_{ij} + X_{ij}$ and

$$\nabla_i \Phi := (\partial_i - \frac{ig}{2} V_i - \frac{ig'}{2} X_i)\Phi, \tag{C.4}$$

$$V_{ij} := \frac{1}{2}(\partial_i V_j - \partial_j V_i) - \frac{ig}{4}[V_i, V_j], \tag{C.5}$$

$$X_{ij} := \frac{1}{2}(\partial_i X_j - \partial_j X_i). \tag{C.6}$$

We specify Eqs. (B.41)–(B.44) for to the Weinberg–Salam (WS) model, which has the gauge group $G = U(2) = SU(2) \times U(1)$. As was mentioned in "Appendix A", in this case, there is a slight discrepancy in the definition of the covariant derivative due to the fact that $U(2)$ is not simple, but a (semi-)direct product of the simple group $SU(2)$ and $U(1)$, with each component having a coupling constant, see (C.2)–(C.6).

Using Eqs (C.2)–(C.6), we express the Lagrangian and the energy in coordinates as

$$\mathcal{L}(\Phi, Q) := \int_\Omega \langle \nabla_i \Phi, \nabla^i \Phi \rangle_{\mathbb{C}^2} - U(\Phi) + \frac{1}{2}\,\mathrm{Tr}\, Q_{ij} Q^{ij}, \tag{C.7}$$

$$E(\Phi, Q) := \int_\Omega \langle \nabla_i \Phi, \nabla_i \Phi \rangle_{\mathbb{C}^2} + U(\Phi) + \frac{1}{2}\,\mathrm{Tr}\, Q_{ij} Q_{ij}, \tag{C.8}$$

(with indices ranging from 0 to d and 1 to d, respectively, as mentioned above), and the Euler–Lagrange equations are written in coordinates as

$$-\nabla^i \nabla_i \Phi = U'(\Phi), \tag{C.9}$$

$$\nabla^i Q_{ij} = \frac{1}{2} g\,\mathrm{Im}\langle \tau_a \Phi, \nabla_j \Phi \rangle_{\mathbb{C}^2} \tau_a + \frac{1}{2} g'\,\mathrm{Im}\langle \tau_0 \Phi, \nabla_j \Phi \rangle_{\mathbb{C}^2} \tau_0. \tag{C.10}$$

Equations (C.8)–(C.10) can be expressed in terms of the W, Z, Higgs and electromagnetic fields resulting in 2D Eqs. (3.10)–(3.15), see "Appendix D.2".

Appendix D: The Weinberg–Salam Energy in Terms of the Fields W, A, Z and φ

Appendix D.1: Dimension 3

We work in a fixed coordinate system, $\{x^i\}_{i=1}^3$ and write the fields as $W = W_i dx^i$, $Z = -\frac{i}{2} Z_i dx^i$ and $A = -\frac{i}{2} A_i dx^i$. We show

Proposition D.1. *Energy* (2.7)*, written in terms of the fields W, A, Z and φ and coordinates $\{x^i\}_{i=1}^3$, is given by (see also* [43]*):*

$$\begin{aligned} E_\Omega(W, A, Z, \varphi) := \int_\Omega \Big[\sum_{ij} \big(\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2 \big) \\ + \frac{1}{2} g^2 \varphi^2 |W|^2 + \frac{1}{4\cos^2\theta} g^2 \varphi^2 |Z|^2 + T(W, A, Z) \\ + |\nabla\varphi|^2 + \frac{1}{2}\lambda(\varphi^2 - \varphi_0^2)^2 \Big], \end{aligned} \tag{D.1}$$

where $W_{ij} := \nabla_i W_j - \nabla_j W_i$, with $\nabla_k := \partial_k - igV_k^3, \partial_k \equiv \partial_{x^k}$, $Z_{ij} := \partial_i Z_j - \partial_j Z_i$, $A_{ij} := \partial_i A_j - \partial_j A_i$ and $T(W, A, Z)$ is the sum of super-quadratic terms,

$$T(W, A, Z) := \frac{g^2}{2} \sum_{ij} (|W_i W_j|^2 - W_i^2 \overline{W}_j^2) - ig \sum_{ij} V_{ij}^3 W_i \overline{W}_j, \tag{D.2}$$

where $V^3 := Z\cos\theta + A\sin\theta$ and $V_{ij}^3 := \partial_i V_j - \partial_j V_i$, with the important property that $T(W, A, Z)$ is invariant under the gauge transformation (3.7)*.*

Proof of (D.1). We proceed by rewriting the terms in the coordinate expression of the WS energy (C.8),

in terms of the fields $W = W_i dx^i$, $Z = -\frac{i}{2} Z_i dx^i$, $A = -\frac{i}{2} A_i dx^i$ and φ.

For the first term, first we calculate $\nabla_i \Phi$. Recall the definition $\nabla_i \Phi := (\partial_i - \frac{ig}{2} V_i - \frac{ig'}{2} X_i)\Phi$. We simplify the matrix representing the connection's action on Φ:

$$\begin{aligned} -\frac{ig}{2} V_i - \frac{ig'}{2} X_i &= -\frac{ig}{2} V_i^a \tau_a - \frac{ig'}{2} X_i \tau_0 \\ &= -\frac{ig}{2} \begin{pmatrix} 0 & V_i^1 \\ V_i^1 & 0 \end{pmatrix} - \frac{ig}{2} \begin{pmatrix} 0 & -iV_i^2 \\ iV_i^2 & 0 \end{pmatrix} \\ &\quad - \frac{ig}{2} \begin{pmatrix} V_i^3 & 0 \\ 0 & -V_i^3 \end{pmatrix} - \frac{ig}{2} \tan\theta \begin{pmatrix} X_i & 0 \\ 0 & X_i \end{pmatrix} \\ &= -\frac{ig}{2\cos\theta} \begin{pmatrix} V_i^3 \cos\theta + X_i \sin\theta & V_i^1 \cos\theta - iV_i^2 \cos\theta \\ V_i^1 \cos\theta + iV_i^2 \cos\theta & -V_i^3 \cos\theta + X_i \sin\theta \end{pmatrix}. \end{aligned} \tag{D.3}$$

In terms of the fields Z, A and W (see Eqs. (3.5)–(3.6) for the definitions of these fields), (D.3) becomes

$$-\frac{ig}{2} V_i - \frac{ig'}{2} X_i = -\frac{ig}{2\cos\theta} \begin{pmatrix} Z_i \cos 2\theta + A_i \sin 2\theta & \sqrt{2}\, W_i \cos\theta \\ \sqrt{2}\, \overline{W}_i \cos\theta & -Z_i \end{pmatrix}. \tag{D.4}$$

Hence, for $\Phi = (0, \varphi)$,

$$\nabla_i \Phi = \begin{pmatrix} -\frac{ig}{\sqrt{2}} W_i \varphi \\ \partial_i \varphi + \frac{ig}{2\cos\theta} Z_i \varphi \end{pmatrix}. \tag{D.5}$$

Therefore, the first term of (C.8), written in terms of the fields W, A, Z and φ, becomes

$$\begin{aligned}\langle \nabla_i \Phi, \nabla^i \Phi \rangle_{\mathbb{C}^2} &= \overline{\frac{ig}{\sqrt{2}} W_i} \frac{ig}{\sqrt{2}} W^i \\ &\quad + \overline{\left(\partial_i \varphi + \frac{ig}{2\cos\theta} Z_i \varphi\right)} \left(\partial^i \varphi + \frac{ig}{2\cos\theta} Z^i \varphi\right) \\ &= \frac{g^2}{2} \varphi^2 |W|^2 + |\nabla \varphi|^2 + \frac{g^2}{4\cos^2\theta} \varphi^2 |Z|^2. \end{aligned} \tag{D.6}$$

The second term of (C.8) becomes

$$U(\Phi) = \frac{1}{2}\lambda(\|\Phi\|^2 - \varphi_0^2)^2 = \frac{1}{2}\lambda(\varphi^2 - \varphi_0^2)^2. \tag{D.7}$$

For the third term of (C.8), we will use the fact that $\operatorname{Tr} Q_{ij} Q^{ij} = \operatorname{Tr} V_{ij} V^{ij} + \operatorname{Tr} X_{ij} X^{ij}$, where V_{ij} and X_{ij} are defined in (C.5) and (C.6). Furthermore, we have

$$V_i := V_i^a \tau_a = \begin{pmatrix} V_i^3 & \sqrt{2}\, W_i \\ \sqrt{2}\, \overline{W}_i & -V_i^3 \end{pmatrix}. \tag{D.8}$$

We recall $V_{ij}^3 = \partial_i V_j^3 - \partial_j V_i^3$ and $W_{ij}^0 = \partial_i W_j - \partial_j W_i$ and calculate

$$\frac{1}{2}(\partial_i V_j - \partial_j V_i) = \frac{1}{2} \begin{pmatrix} V_{ij}^3 & \sqrt{2}\, W_{ij} \\ \sqrt{2}\, \overline{W}^0_{ji} & -V_{ij}^3 \end{pmatrix}, \tag{D.9}$$

and, with $K_{ij} := V_i^3 W_i - V_j^3 W_i$,

$$\begin{aligned} -\frac{ig}{4}[V_i, V_j] &= -\frac{ig}{4} \begin{pmatrix} V_i^3 & \sqrt{2}\, W_i \\ \sqrt{2}\, \overline{W}_i & -V_i^3 \end{pmatrix} \begin{pmatrix} V_j^3 & \sqrt{2}\, W_j \\ \sqrt{2}\, \overline{W}_j & -V_j^3 \end{pmatrix} - (i \leftrightarrow j) \\ &= -\frac{ig}{4} \begin{pmatrix} V_i^3 V_j^3 + 2 W_i \overline{W}_j & \sqrt{2}\, K_{ij} \\ \sqrt{2}\, \overline{K}_{ij} & -V_i^3 V_j^3 - 2 W_i \overline{W}_j \end{pmatrix} - (i \leftrightarrow j) \\ &= -\frac{ig}{2} \begin{pmatrix} W_i \overline{W}_j - \overline{W}_i W_j & \sqrt{2}\, K_{ij} \\ \sqrt{2}\, \overline{K}_{ji} & -W_i \overline{W}_j + \overline{W}_i W_j \end{pmatrix}. \end{aligned} \tag{D.10}$$

Adding (D.9) and (D.10), using that $W_{ij} = W_{ij}^0 + K_{ij}$ and denoting $L_{ij} := V_{ij}^3 - ig(W_i \overline{W}_j - \overline{W}_i W_j)$ gives

$$V_{ij} = \frac{1}{2} \begin{pmatrix} L_{ij} & \sqrt{2}\, W_{ij} \\ -\sqrt{2}\, \overline{W}_{ij} & -L_{ij} \end{pmatrix}. \tag{D.11}$$

Since V_{ij} and X_{ij} are Hermitian, $\operatorname{Tr} V_{ij} V^{ij}$ and $\operatorname{Tr} X_{ij} X^{ij}$ are the sum of the squared absolute values of the matrix coefficients of V_{ij} and X_{ij}, respectively. Thus

$$\begin{aligned} \frac{1}{2} \operatorname{Tr} Q_{ij} Q^{ij} &= \frac{1}{2} \operatorname{Tr} V_{ij} V^{ij} + \frac{1}{2} \operatorname{Tr} X_{ij} X^{ij} \\ &= \frac{1}{8} \sum_{ij} 2|L_{ij}|^2 + 4|W_{ij}|^2 + 2|X_{ij}|^2. \end{aligned} \tag{D.12}$$

Using $L_{ij} = V^3_{ij} - ig(W_i\overline{W}_j - \overline{W}_iW_j)$ and expanding the first term gives

$$\frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} = \sum_{ij} \frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|V^3_{ij}|^2 + \frac{1}{4}|X_{ij}|^2$$
$$+ \frac{g^2}{4}\sum_{ij}|W_i\overline{W}_j - \overline{W}_iW_j|^2 - \frac{ig}{4}\sum_{ij} 2V^3_{ij}(W_i\overline{W}_j - \overline{W}_iW_j). \tag{D.13}$$

Recall that $A_{ij} = V^3_{ij}\sin\theta + X_{ij}\cos\theta$ and $Z_{ij} = V^3_{ij}\cos\theta - X_{ij}\sin\theta$. Writing the first line of (D.13) in terms of these fields gives

$$\frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} = \sum_{ij} \frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2$$
$$+ \frac{g^2}{4}\sum_{ij}|W_i\overline{W}_j - \overline{W}_iW_j|^2 - \frac{ig}{2}\sum_{ij} V^3_{ij}(W_i\overline{W}_j - \overline{W}_iW_j). \tag{D.14}$$

Expanding the first term of the second line, and using $V^3_{ij} = -V^3_{ij}$ in the second term, (D.14) becomes

$$\frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} = \sum_{ij} \frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2$$
$$+ \frac{g^2}{4}\sum_{ij}(|W_i|^2|\overline{W}_j|^2 - W_i^2\overline{W}_j^2 + (i \leftrightarrow j))$$
$$- \frac{ig}{2}\sum_{ij}(V^3_{ij}W_i\overline{W}_j + (i \leftrightarrow j)). \tag{D.15}$$

Recalling the definition (D.2) of $T(W, A, Z)$ gives

$$\frac{1}{2}\operatorname{Tr} Q_{ij}Q^{ij} = \sum_{ij} \frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|A_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + T(W, A, Z). \tag{D.16}$$

Adding (D.6), (D.7) and (D.16) gives (D.1). □

Appendix D.2: Dimension 2: Proof of (3.10)

Proof of (3.10). Now, we consider the Weinberg–Salam (WS) model in $\mathbb{R}^2$ with fields independent of the third dimension x_3, and correspondingly choose the gauge with $V_3 = X_3 = 0$ (and hence $W_3 = A_3 = Z_3 = 0$). In this case the summation in (D.1) contains only two terms, $(ij) = (12)$ and $(ij) = (21)$, and we use this to simplify (D.1).

We proceed by simplifying the terms of (D.2) and the first line of (D.1); the remaining terms are unchanged.

$$\sum_{ij}\left(\frac{1}{2}|W_{ij}|^2 + \frac{1}{4}|Z_{ij}|^2 + \frac{1}{4}|A_{ij}|^2\right) = \sum_{i<j}\left(|W_{ij}|^2 + \frac{1}{2}|Z_{ij}|^2 + \frac{1}{2}|A_{ij}|^2\right)$$
$$= |\operatorname{curl}_{gV^3} W|^2 + \frac{1}{2}|\operatorname{curl} Z|^2 + \frac{1}{2}|\operatorname{curl} A|^2; \tag{D.17}$$

$$\sum_{ij}(|W_iW_j|^2 - W_i^2\overline{W}_j^2)$$

$$
\begin{aligned}
&= W_1 W_2 \overline{W}_1 \overline{W}_2 - W_1^2 \overline{W}_2^2 + W_2 W_1 \overline{W}_2 \overline{W}_1 - W_2^2 \overline{W}_1^2 \\
&= \overline{(\overline{W}_1 W_2 - W_1 \overline{W}_2)}(\overline{W}_1 W_2 - W_1 \overline{W}_2) \\
&= |\overline{W} \times W|^2; \qquad \text{(D.18)}
\end{aligned}
$$

$$
\begin{aligned}
-\sum_{ij} V^3_{ij} W_i \overline{W}_j &= \sum_{i<j} V^3_{ij}(-W_i \overline{W}_j + W_j \overline{W}_i) \\
&= (\operatorname{curl} V^3)\overline{W} \times W. \qquad \text{(D.19)}
\end{aligned}
$$

Replacing corresponding terms in (D.1)–(D.2) with (D.17)–(D.19) proves (3.10). □

Proof of (3.12) - (3.15). We proceed by calculating the (complex) Gâteaux derivatives of (3.10).

Let $\delta_{\#}$ denote the partial (real) Gâteaux derivative with respect to #. Let $W_z = W + zW'$, $z \in \mathbb{C}$, and define $\partial_{\overline{z}} \equiv \frac{1}{2}(\partial_{\operatorname{Re} z} + i\partial_{\operatorname{Im} z})$ and $\delta_{\overline{W}} \equiv \frac{1}{2}(\delta_{\operatorname{Re} W} + i\delta_{\operatorname{Im} W})$. Then

$$
\begin{aligned}
\delta_{\overline{W}} E_\Omega(W, A, Z, \varphi)\overline{W'} &= \partial_{\overline{z}} E^{WS}_\Omega(W_z, A, Z, \varphi)|_{z=0} \\
&= \int_\Omega \operatorname{curl}_{gV^3} W \cdot \overline{\operatorname{curl}_{gV^3} W'} + \frac{1}{2} g^2 \varphi^2 W \cdot \overline{W'} \\
&\quad - ig(\operatorname{curl} V^3) JW \cdot \overline{W'} + g^2(\overline{W} \times W) JW \cdot \overline{W'}. \qquad \text{(D.20)}
\end{aligned}
$$

Integrating the first term by parts and factoring out W and $\overline{W}'$ gives

$$
\begin{aligned}
\delta_{\overline{W}} E_\Omega(W, A, Z, \varphi)\overline{W'} = \int_\Omega &[\operatorname{curl}^*_{gV^3} \operatorname{curl}_{gV^3} + \frac{g^2}{2}\varphi^2 - ig(\operatorname{curl} V^3) J \\
&+ g^2(\overline{W} \times W) J] W \cdot \overline{W'}. \qquad \text{(D.21)}
\end{aligned}
$$

For the derivative to be zero for every variation W', (3.12) must hold.

Let $A_s = A + sA'$, $s \in \mathbb{R}$. Then

$$
\begin{aligned}
\delta_A E_\Omega(W, A, Z, \varphi) A' &= \partial_s E^{WS}_\Omega(W, A_s, Z, \varphi)|_{s=0} \\
&= \int_\Omega \operatorname{curl}_{gV^3} W \overline{(-ieA' \times W)} + \overline{\operatorname{curl}_{gV^3} W}(-ieA' \times W) \\
&\quad + (\operatorname{curl} A)(\operatorname{curl} A') + ie(\operatorname{curl} A')\overline{W} \times W. \qquad \text{(D.22)}
\end{aligned}
$$

Using $A' \times W = -JW \cdot A'$ in the first two terms, and integrating the last two terms by parts, gives

$$
\begin{aligned}
\delta_A E_\Omega(W, A, Z, \varphi) A' = \int_\Omega &[-ie(\operatorname{curl}_{gV^3} W) J\overline{W} + ie\overline{(\operatorname{curl}_{gV^3} W) J\overline{W}} \\
&+ \operatorname{curl}^* \operatorname{curl} A + ie \operatorname{curl}^*(\overline{W} \times W)] \cdot A', \qquad \text{(D.23)}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\delta_A E_\Omega(W, A, Z, \varphi) A' = \int_\Omega &[\operatorname{curl}^* \operatorname{curl} A + 2e \operatorname{Im}[(\operatorname{curl}_{gV^3} W) J\overline{W} \\
&- \operatorname{curl}^*(\overline{W}_1 W_2)]] \cdot A'. \qquad \text{(D.24)}
\end{aligned}
$$

For the derivative to be zero for every variation A', (3.13) must hold.

The proof of (3.14) is essentially the same as the proof of (3.13), so we omit it.

Let $\varphi_s = \varphi + s\varphi'$, $s \in \mathbb{R}$. Then

$$\begin{aligned}\delta_\varphi E_\Omega(W, A, Z, \varphi)\varphi' &= \partial_s E_\Omega^{WS}(W, A, Z, \varphi_s)|_{s=0} \\ &\int_\Omega g^2\varphi\varphi'|W|^2 + \frac{g^2}{2\cos^2\theta}\varphi\varphi'|Z|^2 \\ &+ 2\nabla\varphi' \cdot \nabla\varphi + 2\lambda(\varphi^2 - \varphi_0^2)\varphi\varphi' \end{aligned} \tag{D.25}$$

Integrating the third term by parts and factoring out $2\varphi'$ gives

$$\begin{aligned} &= \int_\Omega \left[\frac{g^2}{2}|W|^2 + \frac{1}{2}\kappa|Z|^2 \right. \\ &\left. -\Delta + \lambda(\varphi^2 - \varphi_0^2)\right]\varphi \cdot 2\varphi'. \end{aligned} \tag{D.26}$$

For the derivative to be zero for every variation φ', (3.15) must hold. □

Appendix E: Proof of (9.11)

In the proof below, we will use the following result:

Lemma E.1. *Let L^2_{per} denote any of the spaces (5.9)–(5.11), and let $\mathcal{H}^2_{per}$ denote the corresponding Sobolev space. Suppose that $f_s, g_s : \mathbb{R} \to \mathcal{H}^2_{per}$ satisfy $||f_s||_{\mathcal{H}^2_{per}} = \mathcal{O}(|s|^k)$ and $||g_s||_{\mathcal{H}^2_{per}} = \mathcal{O}(|s|^l)$ for some $k, l \in \mathbb{Z}$. Then for $i, j = 1, 2$ and $p, q = 0, 1$,*

$$\left|\int_{\Omega'} \partial_i^p f_s \partial_j^q g_s\right| = \mathcal{O}(|s|^{k+l}). \tag{E.1}$$

Furthermore, if f_s and g_s have continuous derivatives of all orders in s, then so does the above integral.

Proof. Equation (E.1) follows from the following chain of inequalities:

$$\begin{aligned}\left|\int_{\Omega'} \partial_i^p f_s \partial_j^q g_s\right| &\lesssim ||\partial_i^p f_s||_{\mathcal{L}^2_{per}} ||\partial_j^q g_s||_{\mathcal{L}^2_{per}} \\ &\lesssim ||f_s||_{\mathcal{H}^2_{per}} ||g_s||_{\mathcal{H}^2_{per}} = \mathcal{O}(|s|^{k+l}). \end{aligned} \tag{E.2}$$

If f_s and g_s have continuous derivatives of all orders in s, then their s-derivatives of all orders are in $\mathcal{H}^2_{per}$. In particular, this means that $\partial_s^k(f_s g_s)$, $k \in \mathbb{Z}_{\geq 0}$, remains integrable, so the $s-$derivatives of the above integral (obtained by differentiation under the integral sign) are well-defined. □

Proof of (9.11). To prove (9.11), we use the w-field Eq. (4.3), and $\nu_s := g(a_s \sin\theta + z_s \cos\theta)$, to get

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot &\left[\operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} + \frac{g^2}{2}(\psi_s + \xi_s)^2 \right. \\ &\left. -i(\operatorname{curl}\nu_s)J + g^2(\overline{w_s} \times w_s)J\right] w_s = 0. \end{aligned} \tag{E.3}$$

We shall calculate each term of the integral (E.3) up to order s^3 using Lemma E.1 and the Taylor expansions (9.7).

Integrating the first term of (E.3) by parts gives

$$\int_{\Omega'} \overline{\chi} \cdot \operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} w_s = \int_{\Omega'} \overline{\operatorname{curl}_{\nu_s} \chi} \cdot \operatorname{curl}_{\nu_s} w. \tag{E.4}$$

Plugging in the Taylor expansions (9.7) gives

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot \operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} w_s = \int_{\Omega'} &[\overline{\operatorname{curl}_{a^n} \chi} + \mathcal{O}(|s|^2)] \\ &\cdot [s \operatorname{curl}_{a^n} \chi - s^3 i\nu' w' + \mathcal{O}(|s|^5)],\end{aligned} \tag{E.5}$$

where, recall, $\nu' := g(a' \sin\theta + z' \cos\theta)$. Recall from Equation (5.24) that $\operatorname{curl}_{a^n} \chi = 0$. Therefore, applying Lemma E.1 gives

$$\int_{\Omega'} \overline{\chi} \cdot \operatorname{curl}^*_{\nu_s} \operatorname{curl}_{\nu_s} w_s = \mathcal{O}(|s|^5). \tag{E.6}$$

Plugging the Taylor expansions (9.7) into the second term of (E.3) gives

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\psi_s + \xi_s)^2 w_s = \int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2} &\left(\frac{\sqrt{2n}}{g} + s^2(\psi' + \xi') + \mathcal{O}(|s|^4)\right)^2 \\ &\times (s\chi + \mathcal{O}(|s|^5)).\end{aligned} \tag{E.7}$$

Expanding this product and applying Lemma E.1 gives

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\psi_s + \xi_s)^2 w_s = s \int_{\Omega'} n|\chi|^2 &+ s^3 \int_{\Omega'} g\sqrt{2n}(\psi' + \xi')|\chi|^2 \\ &+ s^3 \int_{\Omega'} n\overline{\chi} \cdot w' + \mathcal{O}(|s|^5).\end{aligned} \tag{E.8}$$

Recall that $\chi \in \operatorname{Null}(H_1(n))$ and that w' is orthogonal to $\operatorname{Null} H_1(n)$. Therefore the third term vanishes:

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot \frac{g^2}{2}(\psi_s + \xi_s)^2 w_s = s \int_{\Omega'} n|\chi|^2 &+ s^3 \int_{\Omega'} g\sqrt{2n}(\psi' + \xi')|\chi|^2 \\ &+ \mathcal{O}(|s|^5).\end{aligned} \tag{E.9}$$

Plugging the Taylor expansions (9.7) into the third term of (E.3) gives

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s) J w_s) = \int_{\Omega'} &\overline{\chi} \cdot (-in - s^2 i(\operatorname{curl} \nu') + \mathcal{O}(|s|^4)) \\ &\times (sJ\chi + s^3 J w' + \mathcal{O}(|s|^5)).\end{aligned} \tag{E.10}$$

Recall from Eq. (5.24) that χ is of the form $\chi = (\omega, i\omega)^T$, so $\overline{\chi} \cdot J\chi = -i|\chi|^2$ and $\overline{\chi} \cdot Jw' = -i\overline{\chi} \cdot w'$. Therefore (E.10) simplifies to

$$\begin{aligned}\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s) J w_s) = \int_{\Omega'} &(-in - s^2 i(\operatorname{curl} \nu') + \mathcal{O}(|s|^4)) \\ &\times (-si|\chi|^2 - s^3 i\overline{\chi} \cdot w' + \mathcal{O}(|s|^5)).\end{aligned} \tag{E.11}$$

Expanding this product and applying Lemma E.1 gives

$$\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s) J w_s) = -s \int_{\Omega'} n|\chi|^2 - s^3 \int_{\Omega'} (\operatorname{curl} \nu')|\chi|^2 \\ - s^3 \int_{\Omega'} n\overline{\chi} \cdot w' + \mathcal{O}(|s|^5). \tag{E.12}$$

Recall that $\chi \in \operatorname{Null}(H_1(n))$ and that w' is orthogonal to Null $H_1(n)$. Therefore the third term vanishes:

$$\int_{\Omega'} \overline{\chi} \cdot (-i(\operatorname{curl} \nu_s) J w_s) = -s \int_{\Omega'} n|\chi|^2 - s^3 \int_{\Omega'} (\operatorname{curl} \nu')|\chi|^2 \\ + \mathcal{O}(|s|^5). \tag{E.13}$$

Using $\overline{\chi} \cdot J w_s = -\overline{\chi} \times w_s$, the fourth term of (E.3) becomes

$$\int_{\Omega'} \overline{\chi} \cdot (g^2 \overline{w}_s \times w_s) J w_s = \int_{\Omega'} -g^2 (\overline{\chi} \times w_s) \times (\overline{w}_s \times w_s). \tag{E.14}$$

Plugging in the Taylor expansions (9.7) gives

$$\int_{\Omega'} \overline{\chi} \cdot (g^2 \overline{w}_s \times w_s) J w_s = \int_{\Omega'} -g^2 (s\overline{\chi} \times \chi + \mathcal{O}(|s|^3)) \\ \times (s^2 \overline{\chi} \times \chi + \mathcal{O}(|s|^4)). \tag{E.15}$$

Recall from Eq. (5.24) that χ is of the form $\chi = (\omega, i\omega)$, so $\overline{\chi} \times \chi = i|\chi|^2$. This fact and Lemma E.1 gives

$$\int_{\Omega'} \overline{\chi} \cdot (g^2 \overline{w}_s \times w_s) J w_s = s^3 \int_{\Omega'} g^2 |\chi|^4 + \mathcal{O}(|s|^5). \tag{E.16}$$

The s^3 terms of (E.6), (E.9), (E.13) and (E.16) must sum to 0, and so (9.11) results. □

Appendix F: Proof of (10.3)

Proof of (10.3). We shall calculate each term in the integral (4.8) up to order s^6 using Lemma E.1 and the Taylor expansions (9.7).

Plugging the Taylor expansions (9.7) into the first term of (4.8) gives

$$\int_{\Omega'} |\operatorname{curl}_\nu w_s|^2 = \int_{\Omega'} |s \operatorname{curl}_{a^n} \chi + \mathcal{O}(|s|^3)|^2. \tag{F.1}$$

Recall from Eq. (5.24) that $\operatorname{curl}_{a^n} \chi = 0$. Therefore, applying Lemma E.1 gives

$$\int_{\Omega'} |\operatorname{curl}_\nu w_s|^2 = \mathcal{O}(|s|^6). \tag{F.2}$$

Plugging the Taylor expansions (9.7) into the second term of (4.8) gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} z_s|^2 = \int_{\Omega'} \frac{1}{2} |s^2 \operatorname{curl} z' + \mathcal{O}(|s|^4)|^2. \tag{F.3}$$

Expanding the square and applying Lemma E.1 gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} z_s|^2 = s^4 \int_{\Omega'} \frac{1}{2} |\operatorname{curl} z'|^2 + \mathcal{O}(|s|^6). \tag{F.4}$$

Plugging the Taylor expansions (9.7) into the third term of (4.8) gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} a_s|^2 = \int_{\Omega'} \frac{1}{2} \left| \operatorname{curl} \frac{1}{e} a^n + s^2 \operatorname{curl} a' + s^4 \operatorname{curl} a'' + \mathcal{O}(|s|^6) \right|^2 . \quad \text{(F.5)}$$

Recall that $\operatorname{curl} a^n = n$. Expanding the square gives

$$\begin{aligned} \int_{\Omega'} \frac{1}{2} |\operatorname{curl} a_s|^2 = \int_{\Omega'} \bigg[& \frac{1}{2} \frac{n^2}{e^2} + s^2 \frac{n}{e} \operatorname{curl} a' + s^4 \frac{n}{e} \operatorname{curl} a'' \\ & + s^4 \frac{1}{2} |\operatorname{curl} a'|^2 + \mathcal{O}(|s|^6) \bigg] . \end{aligned} \quad \text{(F.6)}$$

The second and third terms vanish because a' and a'' are $\mathcal{L}'$-periodic. Therefore, applying Lemma E.1 gives

$$\int_{\Omega'} \frac{1}{2} |\operatorname{curl} a_s|^2 = \frac{1}{2} \frac{n^2}{e^2} |\Omega'| + s^4 \int_{\Omega'} \frac{1}{2} |\operatorname{curl} a'|^2 + \mathcal{O}(|s|^6). \quad \text{(F.7)}$$

Plugging the Taylor expansions (9.7) into the fourth term of (4.8) gives

$$\begin{aligned} \int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = \int_{\Omega'} \frac{1}{2} g^2 & \left[\frac{\sqrt{2n}}{g} + s^2 (\xi' + \psi') + \mathcal{O}(|s|^4) \right]^2 \\ \times & |s\chi + s^3 w' + \mathcal{O}(|s|^6)|^2 . \end{aligned} \quad \text{(F.8)}$$

Expanding the square terms gives

$$\begin{aligned} \int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = \int_{\Omega'} \frac{1}{2} g^2 & \left[\frac{2n}{g^2} + s^2 2 \frac{\sqrt{2n}}{g} (\xi' + \psi') + \mathcal{O}(|s|^4) \right] \\ \times & [s^2 |\chi|^2 + s^4 2 \operatorname{Re}(\overline{\chi} \cdot w') + \mathcal{O}(|s|^6)] . \end{aligned} \quad \text{(F.9)}$$

Expanding this product and applying Lemma E.1 gives

$$\begin{aligned} \int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = & s^2 \int_{\Omega'} n |\chi|^2 \\ & + s^4 \int_{\Omega'} [g\sqrt{2n} (\xi' + \psi') |\chi|^2 + 2n \operatorname{Re}(\overline{\chi} \cdot w')] + \mathcal{O}(|s|^6). \end{aligned} \quad \text{(F.10)}$$

Recall that $\chi \in \operatorname{Null}(H_1(n))$ and that w' is orthogonal to $\operatorname{Null}(H_1(n))$. Therefore the third term vanishes:

$$\int_{\Omega'} \frac{1}{2} g^2 \phi_s^2 |w_s|^2 = s^2 \int_{\Omega'} n |\chi|^2 + s^4 \int_{\Omega'} g\sqrt{2n} (\xi' + \psi) |\chi|^2 + \mathcal{O}(|s|^6). \quad \text{(F.11)}$$

Plugging the Taylor expansions (9.7) into the fifth term of (4.8) and expanding the square terms gives

$$\begin{aligned} \int_{\Omega'} \frac{1}{4 \cos^2 \theta} g^2 \phi_s^2 |z_s|^2 = & \int_{\Omega'} \frac{1}{4 \cos^2 \theta} g^2 \\ \times & \left[\frac{2n}{g^2} + s^2 2 \frac{\sqrt{2n}}{g} (\xi' + \psi') + \mathcal{O}(|s|^4) \right] [s^4 |z'|^2 + \mathcal{O}(|s|^6)] . \end{aligned} \quad \text{(F.12)}$$

Expanding this product and applying Lemma E.1 gives

$$\int_{\Omega'} \frac{1}{4\cos^2\theta} g^2\phi_s^2|z_s|^2 = s^4 \int_{\Omega'} \frac{n}{2\cos^2\theta}|z'|^2 + \mathcal{O}(|s|^6). \tag{F.13}$$

Plugging the Taylor expansions (9.7) into the sixth term of (4.8) gives

$$\int_{\Omega'} |\overline{w}_s \times w_s|^2 = \int_{\Omega'} |s^2\overline{\chi} \times \chi + \mathcal{O}(|s|^4)|^2, \tag{F.14}$$

Recall from Eq. (5.24) that χ is of the form $\chi = (\omega, i\omega)$, so $\overline{\chi} \times \chi = i|\chi|^2$. Therefore, applying Lemma E.1 gives

$$\int_{\Omega'} |\overline{w}_s \times w_s|^2 = s^4 \int_{\Omega'} |\chi|^4 + \mathcal{O}(|s|^6). \tag{F.15}$$

Plugging the Taylor expansions (9.7) into the seventh term of (4.8) gives

$$\begin{aligned}\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s &= \int_{\Omega'} i\left[g\sin\theta \operatorname{curl}\frac{1}{e}a^n + s^2\operatorname{curl}\nu' + \mathcal{O}(|s|^4)\right]\\ &\quad\times [s\overline{\chi} + s^3\overline{w'} + \mathcal{O}(|s|^5)] \times [s\chi + s^3 w' + \mathcal{O}(|s|^5)].\end{aligned} \tag{F.16}$$

where, recall, $\nu' := g(a'\sin\theta + z'\cos\theta)$. Recall that $\operatorname{curl} a^n = n$ and $e = g\sin\theta$. Expanding the wedge product of the second and third terms gives

$$\begin{aligned}\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s &= \int_{\Omega'} i\left[\frac{n}{g}^2 + s^2\operatorname{curl}\nu' + \mathcal{O}(|s|^4)\right]\\ &\quad\times\left[s^2\overline{\chi}\times\chi + s^4(\overline{\chi}\times w' + \overline{w'}\times\chi) + \mathcal{O}(|s|^6)\right].\end{aligned} \tag{F.17}$$

Recall from Eq. (5.24) that χ is of the form $\chi = (\omega, i\omega)$, so $\overline{\chi} \times \chi = i|\chi|^2$ and $\overline{\chi} \times w' = i\overline{\chi}\cdot w'$. Therefore

$$\begin{aligned}\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s &= \int_{\Omega'} [in + s^2 i\operatorname{curl}\nu' + \mathcal{O}(|s|^4)]\\ &\quad\times [s^2 i|\chi|^2 + s^4 2\operatorname{Re}(i\overline{\chi}\cdot w') + \mathcal{O}(|s|^6)].\end{aligned} \tag{F.18}$$

Expanding this product and using Lemma E.1 gives

$$\begin{aligned}\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s &= -s^2\int_{\Omega'} n|\chi|^2 - s^4\int_{\Omega'} [2in\operatorname{Im}(\overline{\chi}\cdot w')\\ &\quad - s^4\int_{\Omega'} (\operatorname{curl}\nu')|\chi|^2 + \mathcal{O}(|s|^6).\end{aligned} \tag{F.19}$$

Recall that $\chi \in \operatorname{Null}(H_1(n))$ and w' is orthogonal to $\operatorname{Null}(H_1(n))$. Therefore the second term vanishes:

$$\begin{aligned}\int_{\Omega'} i(\operatorname{curl}\nu_s)\overline{w}_s \times w_s &= -s^2\int_{\Omega'} n|\chi|^2 - s^4\int_{\Omega'} (\operatorname{curl}\nu')|\chi|^2\\ &\quad + \mathcal{O}(|s|^6).\end{aligned} \tag{F.20}$$

Plugging the Taylor expansions (9.7) into the eigth term of (4.8) gives

$$\int_{\Omega'} |\nabla\phi_s|^2 = \int_{\Omega'} |s^2\nabla\psi' + \mathcal{O}(|s|^4)|^2. \tag{F.21}$$

Expanding the square and using Lemma E.1 gives

$$\int_{\Omega'} |\nabla\phi_s|^2 = s^4 \int_{\Omega'} |\nabla\psi'|^2 + \mathcal{O}(|s|^6). \tag{F.22}$$

Plugging the Taylor expansions (9.7) into the ninth term of (4.8) and expanding the inner squares gives

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{2}\lambda(\phi_s^2 - \xi_s^2) \\
&\quad = \int_{\Omega'} \frac{1}{2}\lambda\left[\frac{2n}{g^2} + s^2 2\frac{\sqrt{2n}}{g}(\xi' + \psi') - \frac{2n}{g^2} - s^2 2\frac{\sqrt{2n}}{g}\xi' + \mathcal{O}(|s|^4)\right]^2 \\
&\quad = \int_{\Omega'} \frac{1}{2}\lambda\left[s^2 2\frac{\sqrt{2n}}{g}\psi' + \mathcal{O}(|s|^4)\right]^2 .
\end{aligned} \tag{F.23}$$

Expanding the outer square gives and using Lemma E.1 gives

$$\int_{\Omega'} \frac{1}{2}\lambda(\phi_s^2 - \xi_s^2) = s^4 \int_{\Omega'} \frac{4\lambda n}{g^2}\psi'^2 + \mathcal{O}(|s|^6). \tag{F.24}$$

Adding (F.2)–(F.24) and dividing by $|\Omega'|$ gives (10.3), where R_ε collects the $\mathcal{O}(|s|^6)$ remainder terms. R_ε has continuous derivatives of all orders because it is a sum of integrals of the form (E.1) with f_s and g_s coming from the continuously differentiable remainder terms $\mathcal{O}(|s|^p)$ of (9.7). $\square$

Appendix G: Spectral Analysis of the Operator $-\Delta_{a^n}$

Recall from the main text, but in vector notation, that $a^n := \frac{n}{2}x^\perp$, where $(x^1, x^2)^\perp = (-x^2, x^1)$, $\nabla_q := \nabla - iq = (\nabla_1, \nabla_2)$, $\nabla_j := \partial_j - iq_j, \partial_j \equiv \partial_{x^j}$, and $\Delta_q := \nabla_q^2 = -\nabla_q^*\nabla_q$. The next proof follows Section 5 of [18].

Proof of Proposition 5.4. The self-adjointness of the operator $-\Delta_{a^n}$ is well-known. To find its spectrum, we introduce the complexified covariant derivatives (harmonic oscillator annihilation and creation operators), $\bar\partial_{a^n}$ and $\bar\partial_{a^n}^* = -\partial_{a^n}$, with

$$\bar\partial_{a^n} := (\nabla_{a^n})_1 + i(\nabla_{a^n})_2 = \partial_{x^1} + i\partial_{x^2} + \frac{1}{2}n(x^1 + ix^2). \tag{G.1}$$

One can redily verify that these operators satisfy the following relations:

$$[\bar\partial_{a^n}, (\bar\partial_{a^n})^*] = \operatorname{curl} a^n = n; \tag{G.2}$$

$$-\Delta_{a^n} - n = (\bar\partial_{a^n})^*\bar\partial_{a^n}. \tag{G.3}$$

As for the harmonic oscillator (see, e.g. [23]), this gives explicit information about the spectrum of $-\Delta_{a^n}$, namely (5.26), with each eigenvalue is of the same multiplicity. Furthermore, the above properties imply (5.27).

We find $\operatorname{Null}\bar\partial_{a^n}$. A simple calculation gives the following operator equation

$$e^{-\frac{n}{2}(ix^1x^2-(x^2)^2)}\bar\partial_{a^n}e^{\frac{n}{2}(ix^1x^2-(x^2)^2)} = \partial_{x^1} + i\partial_{x^2}.$$

(The transformation on the left-hand side is highly non-unique.) This immediately proves that

$$\bar{\partial}_{a^n}\psi = 0, \tag{G.4}$$

if and only if $\theta = e^{-\frac{n}{2}(ix^1x^2-(x^2)^2)}\psi$ satisfies $(\partial_{x^1}+i\partial_{x^2})\theta = 0$. We now identify $x \in \mathbb{R}^2$ with $z = x^1 + ix^2 \in \mathbb{C}$ and see that this means that θ is analytic and

$$\psi(x) = e^{-\frac{\pi n}{2\,\mathrm{Im}\,\tau}(|z|^2-z^2)}\theta(z,\tau),\ \ z = (x^1+ix^2)/\sqrt{\frac{2\pi}{\mathrm{Im}\,\tau}}. \tag{G.5}$$

where we display the dependence of θ on τ. The quasiperiodicity of ψ transfers to θ as follows:

$$\theta(z+1,\tau) = \theta(z,\tau), \qquad \theta(z+\tau,\tau) = e^{-2\pi i n z}e^{-in\pi\tau}\theta(z,\tau).$$

The first relation ensures that θ have a absolutely convergent Fourier expansion of the form $\theta(z,\tau) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi m i z}$. The second relation, on the other hand, leads to relation for the coefficients of the expansion: $c_{m+n} = e^{-in\pi z}e^{i2m\pi\tau}c_m$, which together with the previous statement implies (5.29). □

Next, we claim that the solution (G.5) satisfies

$$\psi(x) = \psi(-x). \tag{G.6}$$

By (G.5), it suffices to show that $\theta(z) = \theta(-z)$. We show this for $n = 1$. Denote the corresponding θ by $\theta(z,\tau)$. Iterating the recursive relation for the coefficients in (5.29), we obtain the following standard representation for the theta function

$$\theta(z,\tau) = \sum_{m=-\infty}^{\infty} e^{2\pi i\left(\frac{1}{2}m^2\tau+mz\right)}. \tag{G.7}$$

We observe that $\theta(-z,\tau) = \theta(z,\tau)$ and therefore $\psi_0(-x) = \psi_0(x)$. Indeed, using the expression (G.7), we find, after changing m to $-m'$, we find

$$\theta(-z,\tau) = \sum_{m=-\infty}^{\infty} e^{2\pi i\left(\frac{1}{2}m^2\tau-mz\right)} = \sum_{m'=-\infty}^{\infty} e^{2\pi i\left(\frac{1}{2}m'^2\tau+m'z\right)} = \theta(z,\tau). \tag{G.8}$$

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