Ann. Henri Poincaré *Online First*

https://doi.org/10.1007/s00023-023-01381-3

Annales Henri Poincaré

Entanglement Entropy of Ground States of the Three-Dimensional Ideal Fermi Gas in a Magnetic Field

Paul Pfeiffer and Wolfgang Spitzer

Abstract. We study the asymptotic growth of the entanglement entropy of ground states of non-interacting (spinless) fermions in $\mathbb{R}^3$ subject to a constant magnetic field perpendicular to a plane. As for the case with no magnetic field we find, to leading order $L^2 \ln(L)$, a logarithmically enhanced area law of this entropy for a bounded, piecewise Lipschitz region $L\Lambda \subset \mathbb{R}^3$ as the scaling parameter L tends to infinity. This is in contrast to the two-dimensional case since particles can now move freely in the direction of the magnetic field, which causes the extra $\ln(L)$ factor. The explicit expression for the coefficient of the leading order contains a surface integral similar to the Widom–Sobolev formula in the non-magnetic case. It differs, however, in the sense that the dependence on the boundary, $\partial\Lambda$, is not solely on its area but on the "surface perpendicular to the direction of the magnetic field". We utilize a two-term asymptotic expansion by Widom (up to an error term of order one) of certain traces of one-dimensional Wiener–Hopf operators with a discontinuous symbol. This leads to an improved error term of the order L^2 of the relevant trace for piecewise $\mathsf{C}^{1,\alpha}$ smooth surfaces $\partial\Lambda$.

Mathematics Subject Classification. Primary 47G30, 35S05; Secondary 45M05, 47B10, 47B35.

Contents

Published online: 07 November 2023

Birkhäuser

1. Introduction

In recent years, entanglement entropy (EE) has become an important and intensively studied quantity of states of many-particle quantum systems. For an introduction to this topic, we refer to [3,6,15]. In this paper, we study the EE of ground states of the ideal Fermi gas in a magnetic field in three-dimensional Euclidean space, $\mathbb{R}^3$, see [18]. The two-dimensional Fermi gas in a constant magnetic field was recently analyzed in [5,21], starting from the earlier work in [31]. Here, a strict area-law holds, while for the free Fermi gas in any dimension d a logarithmically enhanced area-law is valid, see [13,20]. Stability of these area-laws has been proved in [25,26] for $d \geq 2$ and in [29] in the sense that adding a "small" electric or magnetic potential to the Hamiltonian does not change the leading asymptotics of the entropy. The one-dimensional case seems to be still open (for the γ-Rényi entropy with $\gamma \leq 1$).

There is an extensive literature on EE by now with many fascinating connections and implications to related fields. Here, we only mention and refer to a small fraction of mathematical results. In [30], an enhanced area-law was proved for the one-dimensional free Fermi gas in a periodic potential; the higher-dimensional case remains an open problem. By the work in [8,24,28], we understand EE in Anderson-type models on the lattice. An extension to the EE of positive temperature equilibrium states (of the ideal Fermi gas) was presented in [22,23,35]. Finally, we mention results on the XY and XXZ quantum spin chain [1,4,7,11,12,16].

By a (strict) area-law for a ground state of the infinitely extended Fermi gas, say in $\mathbb{R}^d$ with spatial dimension $d \in \mathbb{N}$, we mean that the entanglement (or local) entropy of this state reduced to the scaled (bounded) region $L\Lambda$ grows to leading order like $L^{d-1}\mathcal{H}^2(\partial\Lambda)$ as the dimensionless real parameter L tends to infinity. Here, $\mathcal{H}^2(\partial\Lambda)$ is the (Hausdorff) surface area of the boundary $\partial\Lambda$. If there is an extra $\ln(L)$ factor in this leading asymptotics, then we call it a logarithmically enhanced area-law.

Whether one should expect a strict area-law or an enhanced area-law is related to the spectral properties of the one-particle Hamiltonian of the non-interacting many-particle Fermi gas. If the off-diagonal part of the integral kernel of the corresponding spectral (Fermi) projection has a fast decay (e.g., exponential), then we expect a strict area-law to hold. It is not difficult to argue for that (see [28]) but to compute and finally prove the precise leading coefficient has only been accomplished in special cases. On the other hand, if the decay of the off-diagonal part of the integral kernel is weak (e.g., inverse linear), then we can expect an enhanced area-law. In the present model, we have a mixture. Namely, we have an exponential decay in the planar coordinate (orthogonal to the magnetic field) and a $1/|\cdot|$ decay in the longitudinal coordinate along the magnetic field. The latter prevails and leads to a logarithmically enhanced area-law. Our main result is formulated in Theorem 4.1.

As in previous proofs there are two parts to proving such a result. Firstly, we prove a two-term asymptotic expansion for polynomials (see Theorem 2.3). Due to the product structure of the ground state, see (2.7), we can dimensionally reduce the asymptotics of a three-dimensional problem to an asymptotic expansion of a one-dimensional problem with localizing sets $L\Lambda_{x^\perp} \subset \mathbb{R}$ and with the spectral projection of the one-dimensional Laplacian, see Lemma 3.2. The corresponding asymptotic expansion was already proved by Landau and Widom [19] and then improved by Widom [36]. But here we need to take care of the error term which depends on the planar coordinate $x^\perp \in \mathbb{R}^2$ and integrate over $x^\perp$. To this end, we show that the error term is of order one and is integrable as a function of $x^\perp$ under some assumptions on Λ. We believe that the precise description of the error term for the one-dimensional free case in terms of the finite collection of intervals $\Lambda_{x^\perp}$ is of independent interest and we provide a proof in Appendix C. This dimensional reduction is also the strategy of Widom [37] and of Sobolev in the proof of the Widom conjecture in [32]. In fact, due to the fast (exponential) decay in the planar direction error estimates are simpler to obtain than in the case with no magnetic field. This and the improved Landau–Widom (or Widom) asymptotics allows us to prove for $\mathsf{C}^{1,\alpha}$ (smooth) regions Λ an error term (for polynomials as in Theorem 2.3) of the order L^2 rather than merely of lower order than $L^2\ln(L)$ in [32, Theorem 2.9].

Secondly, in Sect. 4 we make the transition in the asymptotic expansion from polynomials to the entropy function. This requires certain Schatten–von Neumann quasi-norm bounds presented in Sect. 5, which in turn are based on bounds obtained in previous papers [20,21] and notably by Sobolev [34].

The smoothness conditions on the region Λ to prove our two-term asymptotic result with error term $o(L^2\ln(L))$ are rather weak; namely, we require Λ to be only piecewise Lipschitz smooth. For a smooth region Λ, one would expect the next lower order term to be of the order L^2. This is indeed true if the boundary $\partial\Lambda$ is piecewise $\mathsf{C}^{1,\alpha}$ smooth. We also present regions with weaker regularity on the boundary for which the error term (for a quadratic polynomial) can be arbitrarily close to the leading $L^2\ln(L)$-term. This may also be of independent interest and is the content of Sect. 6.

A note on our notation: As L, $L \geq 1$, is our scaling parameter that tends to infinity, we use the "big-O" and "small-o" notation in the sense that for two functions f and g on $\mathbb{R}^+$, $f = O(g)$ if $\limsup_L f(L)/g(L) < \infty$ and $f = o(g)$ if $\limsup_L f(L)/g(L) = 0$. By C with or without indices, we denote various positive, finite constants, whose precise values is of no importance, and may even change from line to line.

2. Setup

We consider a nonzero constant magnetic field in $\mathbb{R}^3$ of strength B which is perpendicular to a plane. We assume without loss of generality that this constant magnetic field points in the positive z-direction with $B > 0$.

We denote the Euclidean norm in $\mathbb{R}^d$, $d \in \mathbb{N}$, or the norm in the Hilbert-space $\mathrm{L}^2(\mathbb{R}^d)$ of complex-valued, square-integrable functions on $\mathbb{R}^d$ by the same symbol $\|\cdot\|$. For $x \in \mathbb{R}$, let $\langle x \rangle := \sqrt{1+x^2}$ denote the Japanese bracket. For a Borel set $\Omega \subset \mathbb{R}^d$ and $k < d$, let $\mathcal{H}^k(\Omega)$ be the k-dimensional Hausdorff measure of Ω, $\#\Omega = \mathcal{H}^0(\Omega)$ its counting measure, and let $|\Omega|$ be its d-dimensional Lebesgue measure/volume. By $\mathbb{1}_\Omega$ we denote the multiplication operator on $\mathsf{L}^2(\mathbb{R}^d)$ by the indicator function 1_Ω of the set Ω. As usual, we write for the complement $\Omega^\complement := \mathbb{R}^d \backslash \Omega$.

For $r > 0$, $x \in \mathbb{R}^d$, and a set $X \subset \mathbb{R}^d$ we denote by

$$B_r(x) := \{y \in \mathbb{R}^d : \|y - x\| < r\}, \quad B_r(X) := X + B_r(0) := \{x + y : x \in X, \|y\| < r\} \tag{2.1}$$

the open ball of radius r with center x and the (open) r-*neighborhood* of the set $X \subset \mathbb{R}^d$ of width r, respectively. In most cases, the dimension, d, is clear from the context and we omit it in the definition; if not, we write $B_r^{(d)}(x)$. We denote the closed ball of radius r with center x by $\overline{B}_r^{(d)}(x)$.

For a point $x \in \mathbb{R}^3$, we write $x = (x^\perp, x^\parallel)$ with (planar coordinate) $x^\perp \in \mathbb{R}^2$ and (longitudinal coordinate) $x^\parallel \in \mathbb{R}$, and $\nabla = (\nabla^\perp, \nabla^\parallel)$, where $\nabla^\perp$ and $\nabla^\parallel$ are the gradients in the respective Cartesian coordinates.

By our assumption, the magnetic field is equal to $B \cdot e_3$ with $e_3 := (0, 0, 1)$. We use the symmetric gauge $a : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $a(x^\perp) := B/2\,(-x_2^\perp, x_1^\perp)$ so that the curl

$$\nabla \times (a, 0) = B \cdot e_3. \tag{2.2}$$

The one-particle Hamiltonian of the ideal Fermi gas in three-dimensional Euclidean space $\mathbb{R}^3$ subject to the magnetic field $B \cdot e_3$ is informally given by

$$\mathrm{H}_B := (-\mathrm{i}\nabla^\perp + a)^2 + (-\mathrm{i}\nabla^\parallel)^2. \tag{2.3}$$

We use physical units such that Planck's constant $\hbar = 1$, the mass is equal to $1/2$ and the charge of the particles is equal to one. H_B is well defined as a self-adjoint operator on a suitable domain in the one-particle Hilbert space $\mathsf{L}^2(\mathbb{R}^3)$.

The ground state of free fermions with one-particle Hamiltonian H_B is described by the spectral projection (or Fermi projection) $\mathrm{D}_\mu := \mathbb{1}(\mathrm{H}_B \le \mu) := 1_{(-\infty,\mu]}(\mathrm{H}_B)$ of H_B below some so-called Fermi energy (or chemical potential) $\mu \in \mathbb{R}$. As is well-known, we have [10,18]

$$(-\mathrm{i}\nabla^\perp + a)^2 = B \sum_{\ell=0}^{\infty} (2\ell + 1) \mathrm{P}_\ell \tag{2.4}$$

with explicitly known (infinite-dimensional) eigenprojections P_ℓ on $\mathsf{L}^2(\mathbb{R}^2)$. In order to write down these projections, let us introduce the Laguerre polynomials, $\mathcal{L}_\ell(t) := \sum_{j=0}^{\ell} \frac{(-1)^j}{j!} \binom{\ell}{\ell-j} t^j$, $t \geq 0$, of degree $\ell \in \mathbb{N}_0$. Then, the integral kernel of P_ℓ is given by the function

$$p_\ell(x^\perp, y^\perp) := \frac{B}{2\pi} \mathcal{L}_\ell\big(B\|x^\perp - y^\perp\|^2/2\big) \exp\big(-B\|x^\perp - y^\perp\|^2/4 + \mathrm{i}\tfrac{B}{2} x^\perp \wedge y^\perp\big),$$
$$x^\perp, y^\perp \in \mathbb{R}^2. \tag{2.5}$$

Here, $\wedge$ refers to the exterior or wedge product on $\mathbb{R}^2$. The explicit description of this kernel is not relevant for this paper. We only use the exponential decay in $\|x^\perp - y^\perp\|^2$ and $p_\ell(x^\perp, x^\perp) = B/(2\pi)$. In the z-direction, we meet the spectral projection $\mathbb{1}((-\nabla^\|)^2 \leq \mu)$ with (sine) integral kernel, $\mathbb{1}((-\nabla^\|)^2 \leq \mu)(z, z') = k_\mu(z - z')$,

$$k_\mu(z) := \begin{cases} \frac{\sin(\sqrt{\mu} z)}{\pi z} & \text{for } z \in \mathbb{R}\backslash\{0\} \\ \lim_{z\to 0} k_\mu(z) = \frac{\sqrt{\mu}}{\pi} & \text{for } z = 0 \end{cases} \qquad \mu > 0. \tag{2.6}$$

The following factorization of spectral projections is crucial, which stems from the fact that the magnetic field is pointing in the z-direction. We work with the identification $\mathsf{L}^2(\mathbb{R}^2) \otimes \mathsf{L}^2(\mathbb{R}) = \mathsf{L}^2(\mathbb{R}^3)$. Since the spectrum of H_B is the set $[B, \infty)$, we may always consider $\mu > B$ since for smaller values of μ the ground state is zero. If $B < \mu \leq 3B$ then $\mathrm{D}_\mu = \mathrm{P}_0 \otimes \mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \leq \mu - B]$. For higher values of μ, let $\nu := \lceil \frac{1}{2}(\mu/B - 1) \rceil \in \mathbb{N}$ be the smallest integer larger or equal to $\frac{1}{2}(\mu/B - 1)$, and let us set $\mu(\ell) := \mu - B(2\ell + 1)$. Then,

$$\mathrm{D}_\mu = \mathbb{1}(\mathrm{H}_B \leq \mu) = \sum_{\ell=0}^{\nu-1} \mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \leq \mu(\ell)] \tag{2.7}$$

with integral kernel $(x = (x^\perp, x^\|), y = (y^\perp, y^\|))$

$$\mathrm{D}_\mu(x, y) = \sum_{\ell=0}^{\nu-1} p_\ell(x^\perp, y^\perp) k_{\mu(\ell)}(x^\| - y^\|). \tag{2.8}$$

For any Borel subset $\Lambda \subset \mathbb{R}^3$ we define the spatial reduction (or truncation) of D_μ to Λ by

$$\mathrm{D}_\mu(\Lambda) := \mathbb{1}_\Lambda \mathrm{D}_\mu \mathbb{1}_\Lambda. \tag{2.9}$$

Before we define the main object in this paper, we introduce for any $\gamma > 0$ the γ-Rényi entropy function, $h_\gamma : [0, 1] \to [0, \ln(2)]$,

$$h_\gamma(t) := \frac{1}{1-\gamma} \ln\big(t^\gamma + (1-t)^\gamma\big), \ \gamma \neq 1, \tag{2.10}$$
$$h_1(t) := -\,t\ln(t) - (1-t)\ln(1-t) \text{ if } t \notin \{0, 1\} \text{ and } h_1(0) := h_1(1) := 0. \tag{2.11}$$

Now, for a ground state described by the projection $\mathrm{D}_\mu = \mathbb{1}(\mathrm{H}_B \leq \mu)$ as above, a Borel subset $\Lambda \subset \mathbb{R}^3$, and localized ground-state projection, $\mathrm{D}_\mu(\Lambda)$, we define the γ-Rényi entanglement entropy of the ground state at Fermi energy μ localized (in space) to Λ by

$$\mathrm{S}_\gamma(\Lambda) := \operatorname{tr} h_\gamma(\mathrm{D}_\mu(\Lambda)). \tag{2.12}$$

Here, tr refers to the (usual Hilbert space) trace on $\mathsf{L}^2(\mathbb{R}^d)$. For bounded Λ, $h_\gamma(\mathrm{D}_\mu(\Lambda))$ is trace-class by the same arguments as in the proof of Lemma 7 in [21]; thus, the entanglement entropy $\mathrm{S}_\gamma(\Lambda)$ is trivially a positive number. This entropy is a rather complicated function of Λ, but there is a chance to describe it for large regions. To this end, we scale a fixed set Λ by $L, L \geq 1$, and we determine the leading growth (scaling) of the entropy $\mathrm{S}_\gamma(L\Lambda)$ as $L \to \infty$.

As there does not seem to be a common definition for regions with piecewise differentiable boundary, we will now provide the one used in this paper.

Definition 2.1. Let $0 < \alpha < 1, d \in \mathbb{N}$. A *region* $\Lambda \subset \mathbb{R}^{d+1}$ is a finite union of bounded, open, connected sets in $\mathbb{R}^{d+1}$ such that their closures (denoted by $\bar{\cdot}$) are disjoint. The boundary $\partial\Lambda$ is the set $\bar{\Lambda}\backslash\Lambda$. We assume that the closures $\overline{\Lambda}$ and $\Lambda^\complement$ are topological manifolds with boundary $\partial\Lambda$.

We call a bi-Lipschitz[1] map $\Psi\colon [0,1]^d \to \partial\Lambda$ a Lipschitz chart of $\partial\Lambda$ if $\Psi((0,1)^d) \subset \partial\Lambda$ is relatively open. If in addition $\Psi \in \mathsf{C}^1((0,1)^d)$ and its differential $D\Psi$ satisfies the Hölder condition

$$\|D\Psi(x) - D\Psi(y)\| \leq C\|x-y\|^\alpha, \quad x, y \in (0,1)^d, \tag{2.13}$$

for some constant C, we say that Ψ is a $\mathsf{C}^{1,\alpha}$ chart. A finite set of charts $(\Psi_i)_{i\in I}$ is called a piecewise atlas of $\partial\Lambda$ if $\partial\Lambda = \bigcup_{i\in I}\Psi_i([0,1]^d)$, and a global atlas of $\partial\Lambda$ if $\partial\Lambda = \bigcup_{i\in I}\Psi_i((0,1)^d)$. We say an atlas is a Lipschitz atlas (resp. $\mathsf{C}^{1,\alpha}$) if it consists of Lipschitz (resp. $\mathsf{C}^{1,\alpha}$) charts.

We say that Λ is a *piecewise Lipschitz region* (resp. *global Lipschitz region*) if $\partial\Lambda$ admits a piecewise Lipschitz atlas $(\Psi_{\mathrm{pL},i})_{i\in I}$ (resp. global Lipschitz atlas $(\Psi_{\mathrm{gL},i})_{i\in I}$). We call Λ a *piecewise* $\mathsf{C}^{1,\alpha}$ *region* if it admits both a global Lipschitz atlas $(\Psi_{\mathrm{gL},j})_{j\in J}$ and a piecewise $\mathsf{C}^{1,\alpha}$ atlas $(\Psi_{\mathrm{pC},i})_{i\in I}$.

For a *piecewise* $\mathsf{C}^{1,\alpha}$ *region* Λ, we fix a piecewise $\mathsf{C}^{1,\alpha}$ atlas $(\Psi_{\mathrm{pC},i})_{i\in I}$ and define the set of all edges, Γ by

$$\Gamma := \bigcup_{i\in I}\Psi_{\mathrm{pC},i}(\partial([0,1]^d)). \tag{2.14}$$

Remarks 2.2. (i) Any *global* Lipschitz region is obviously a *piecewise* Lipschitz region.

(ii) Our definition of a *global Lipschitz region* is a bit more general than the usual notion of a *strong Lipschitz region* (see [2, Pages 66–67]), where every $v \in \partial\Lambda$ has a neighborhood $U_v \subset \mathbb{R}^{d+1}$ such that, after an affine-linear transformation, the set $\Lambda\cap U_v$ looks like the graph below a Lipschitz function $\Psi_v\colon (0,1)^d \to \mathbb{R}$. To get to our definition from this, one can choose the graph function $x \mapsto (x, \Psi_v(x))$ on $(0,1)^d$ as the bi-Lipschitz

[1] A function f is bi-Lipschitz if there is a constant $C_{\mathrm{lip}} \in \mathbb{R}^+$ such that $C_{\mathrm{lip}}^{-1}\|x-y\| \leq \|f(x)-f(y)\| \leq C_{\mathrm{lip}}\|x-y\|$. Such a function f is (obviously) invertible on its image and satisfies $C_{\mathrm{lip}}^{-1}\|x-y\| \leq \|f^{-1}(x)-f^{-1}(y)\| \leq C_{\mathrm{lip}}\|x-y\|$.

function needed in our definition. (As a Lipschitz function, it naturally extends to all of $[0,1]^d$.)

(iii) For a piecewise Lipschitz region $\Lambda \subset \mathbb{R}^{d+1}$ and for $v \in \partial\Lambda$, let $n(v)$ be the unit outward normal vector at v. This is only well defined up to null sets with respect to the d-dimensional Hausdorff (surface) measure $\mathcal{H}^d$ on $\partial\Lambda$, see Lemma A.6.

(iv) As the set of edges, Γ, depends on the piecewise $\mathsf{C}^{1,\alpha}$ atlas $\Psi_{\mathrm{pC},i}$ it may be a different set depending on the atlas.

For a continuous function $f : [0,1] \to \mathbb{C}$ with $f(0) = 0$ and being Hölder continuous at the two endpoints 0 and 1, we introduce the linear functional

$$f \mapsto \mathsf{I}(f) := \frac{1}{4\pi^2} \int_0^1 \mathrm{d}t \, \frac{f(t) - tf(1)}{t(1-t)}. \tag{2.15}$$

By our assumption, $|\mathsf{I}(f)| < \infty$. We note for later use two special cases. Namely, $\mathsf{I}(m) := \mathsf{I}((\cdot)^m) = -1/(4\pi^2) \sum_{r=1}^{m-1} r^{-1}$; as usual we interpret the sum on the right-hand side as zero if $m = 1$, which coincides with the vanishing of I on affine linear functions. The second example concerns the γ-Rényi entropy function h_γ defined in (2.10). Here, $\mathsf{I}(h_\gamma) = (1+\gamma)/(24\gamma)$, see [20].

Our first main result is the following theorem, which we prove in the next section.

Theorem 2.3. *Let $f : [0,1] \to \mathbb{C}$ be a polynomial with $f(0) = 0$, let $\Lambda \subset \mathbb{R}^3, \mu > B > 0, \nu := \lceil \frac{1}{2}(\mu/B - 1) \rceil \in \mathbb{N}$, the smallest integer larger or equal to $\frac{1}{2}(\mu/B - 1)$, and $\mu(\ell) := \mu - (2\ell+1)B$. Let $\mathrm{D}_\mu(L\Lambda)$ be the operator defined in (2.9).*

(i) *If Λ is a piecewise Lipschitz region (see Definition 2.1), then we have the asymptotic expansion of the trace on $\mathsf{L}^2(\mathbb{R}^3)$,*

$$\begin{aligned} \operatorname{tr} f(\mathrm{D}_\mu(L\Lambda)) &= L^3 \frac{B}{2\pi^2} \sum_{\ell=0}^{\nu-1} \sqrt{\mu(\ell)} f(1) |\Lambda| \\ &\quad + L^2 \ln(L) \nu B \, \mathsf{I}(f) \, \frac{1}{\pi} \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v) \, |n(v) \cdot e_3| + o(L^2 \ln(L)), \end{aligned} \tag{2.16}$$

as $L \to \infty$. Here, $n(v)$ is the unit normal outward vector at $v \in \partial\Lambda$, which is well defined for almost every $v \in \partial\Lambda$, and $\mathcal{H}^2$ is the two-dimensional (surface) Hausdorff measure on $\partial\Lambda$.

(ii) *If Λ is a piecewise $\mathsf{C}^{1,\alpha}$ region (see Definition 2.1), then the error term is $O(L^2)$ instead of $o(L^2 \ln(L))$.*

Remarks 2.4. (i) The condition $f(0) = 0$ is no restriction in the sense that in general the operator on the left-hand side has to be replaced by $f(\mathrm{D}_\mu(L\Lambda)) - f(0)\mathrm{D}_\mu(L\Lambda)$ and $\mathsf{I}(f)$ on the right-hand side by $\mathsf{I}(\tilde{f})$ with $\tilde{f}(t) := f(t) - (1-t)f(0)$.

(ii) For the ideal Fermi gas with one-particle Hamiltonian $\mathrm{H}_0 = -\Delta$ on $\mathsf{L}^2(\mathbb{R}^3)$, Fermi energy $\mu > 0$, ground state Fermi projection $\mathrm{D}_\mu = \mathbb{1}(-\Delta \leq \mu)$ and Fermi sea $\Gamma := \{p \in \mathbb{R}^3 : p^2 \leq \mu\}$ it was proved in [20] that

$$\operatorname{tr} f(\mathrm{D}_\mu(L\Lambda)) = L^3 f(1)|\Gamma/(2\pi)||\Lambda| + L^2\ln(L)\,\frac{\mu}{2\pi}\,\mathsf{I}(f)\,\mathcal{H}^2(\partial\Lambda) + o(L^2\ln(L)) \tag{2.17}$$

as $L \to \infty$. To this end, note that $|\Gamma| = \frac{4\pi}{3}\mu^{3/2}$ and that our functional I here is the same as the functional I in [20]. The double-surface integral $J(\partial\Gamma, \partial\Lambda)$ [20, (2)] equals $\frac{\mu}{2\pi}\mathcal{H}^2(\partial\Lambda)$.

Letting B tend to zero in (2.16) but keeping the Fermi energy μ fixed, the prefactor νB tends to $\mu/2$. The remaining integral over $\partial\Lambda$ is independent of the strength B and remains fixed. For the volume term, we have in this limit

$$\begin{aligned}\frac{B}{2\pi^2}\sum_{\ell=0}^{\nu-1}\sqrt{\mu-(2\ell+1)B} &\sim \frac{\mu^{3/2}}{4\pi^2\nu}\\ \sum_{\ell=0}^{\nu}\sqrt{1-\ell/\nu} &\sim \frac{\mu^{3/2}}{4\pi^2}\int_0^1 \mathrm{d}x\,\sqrt{x} = \frac{\mu^{3/2}}{6\pi^2}.\end{aligned}$$

In this limit the volume term equals the above volume term at $B = 0$ as in (2.17). To summarize, we obtain

$$\begin{aligned}\lim_{B\downarrow 0}\text{rhs of (2.16)} = L^3 f(1)\frac{\mu^{3/2}}{6\pi^2}|\Lambda| + L^2\ln(L)\frac{\mu}{2\pi}\,\mathsf{I}(f)\int_{\partial\Lambda}\mathrm{d}\mathcal{H}^2(v)\,|n(v)\cdot e_3|\\ + o(L^2\ln(L)),\end{aligned}$$

which is identical to the right-hand side (rhs) of (2.17) except for the prefactor depending on $\partial\Lambda$.

(iii) There is no 'level mixing' at the order in $L^2\ln(L)$ in the sense that each Landau level enters individually in the numerical coefficient. In [21], we proved that level mixing occurs in the two-dimensional setting at the next-to-leading order, namely at the order L. We expect level mixing to occur in the present case at the order L^2. This is certainly possible to prove, say for a cylindrical region, but it requires a three-term expansion in the $x^{\|}$-coordinate and the by now proved two-term expansion in the $x^{\perp}$-coordinate [21]. The caveat for us to proceed with this question is that the mentioned three-term expansion has not been proved so far for the entropy function. This is an interesting open problem.

(iv) For (2.16) to hold we require only weak regularity of the boundary $\partial\Lambda$ like in the proof in [20] for the ideal Fermi gas. In contrast, the proof of the corresponding two-term asymptotics for the two-dimensional model in [21] required C^3 smooth regions. This smoothness was a technical condition and may not be necessary. On the other hand and more importantly, only the leading contribution of the two-dimensional Hamiltonian enters and the extra logarithm stems from an expansion in the longitudinal direction, where weaker conditions suffice.

3. Proof of Theorem 2.3

We split the proof into two steps. The first one is the lemma below, which reduces the computation of the trace to an integral of the trace of the projection operator $\mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]$ localized to the sets $L\Lambda_{x^\perp} \subset \mathbb{R}$ with respect to $x^\perp \in \mathbb{R}^2$. The second step starts from there, proves an asymptotic expansion of this trace, and finishes the proof of Theorem 2.3.

Definition 3.1. For any Borel set $E \subset \mathbb{R}^3$ and any $x^\perp \in \mathbb{R}^2$ we define $E_{x^\perp} := \{x^\| \in \mathbb{R} : (x^\perp, x^\|) \in E\}$ to collect the third components of the intersection $E \cap (\{x^\perp\} \times \mathbb{R})$.

Lemma 3.2. *Let $m \in \mathbb{N}$ with $m \ge 2$. Then, under the same conditions as in Theorem 2.3(i), there is a constant C depending only on B, m and μ such that*

$$\Big| tr\,(D_\mu(L\Lambda))^m - L^2 \frac{B}{2\pi} \sum_{\ell=0}^{\nu-1} \int_{\mathbb{R}^2} dx^\perp\, tr\Big(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-i\nabla^\|)^2 \le \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}}\Big)^m \Big|$$
$$\le C\mathcal{K}(\Lambda) L^2, \tag{3.1}$$

where the Λ dependent constant $\mathcal{K}(\Lambda)$ is defined in Lemma A.3; it is positive and finite for any piecewise Lipschitz region Λ. Note that $L\Lambda_{x^\perp} := L\,(\Lambda_{x^\perp})$ is (in general) different from $(L\Lambda)_{x^\perp}$.

Proof. We utilize the same changes of coordinates in the first two components (that is, for the planar $x_0^\perp$-coordinates) as in [21]. For the convenience of the reader we repeat all steps.

As Λ is bounded, the operator $\mathbb{1}_{L\Lambda}\mathrm{D}_\mu$ is Hilbert–Schmidt and therefore $\mathrm{D}_\mu(L\Lambda)$ is trace-class. We may write

$$\mathrm{tr}\, \mathrm{D}_\mu(L\Lambda)^m = \int_{\mathbb{R}^3} \mathrm{d}x_0\, \mathrm{D}_\mu(L\Lambda)^m(x_0, x_0) \tag{3.2}$$

with integral kernel

$$\mathrm{D}_\mu(L\Lambda)(x,y) = \sum_{\ell=0}^{\nu-1} p_\ell(x^\perp, y^\perp) k_{\mu(\ell)}(x^\|, y^\|), \quad x = (x^\perp, x^\|), y = (y^\perp, y^\|), \tag{3.3}$$

as in (2.8). Therefore, the trace is of the form

$$\mathrm{tr}\, \mathrm{D}_\mu(L\Lambda)^m = \int_{L\Lambda} \mathrm{d}x_0 \sum_{\ell_1,\dots,\ell_m=0}^{\nu-1} \int_{\mathbb{R}^{2(m-1)}} \mathrm{d}\, x_1^\perp \cdots \mathrm{d}\, x_{m-1}^\perp\, p_{\ell_1}(x_0^\perp, x_1^\perp)$$
$$p_{\ell_2}(x_1^\perp, x_2^\perp) \cdots p_{\ell_m}(x_{m-1}^\perp, x_0^\perp)$$
$$\times \int_{\mathbb{R}^{m-1}} \mathrm{d}\, x_1^\| \cdots \mathrm{d}\, x_{m-1}^\|\, k_{\mu(\ell_1)}(x_0^\| - x_1^\|) \cdots k_{\mu(\ell_m)}(x_{m-1}^\| - x_0^\|)$$
$$\times 1_{L\Lambda}(x_1) \cdots 1_{L\Lambda}(x_{m-1}).$$

We begin by approximating $1_{L\Lambda}(x_j)$ by $1_{L\Lambda}(x_0^\perp, x_j^\|)$. We call the resulting approximate term $T(L\Lambda)$. This means

$$\begin{aligned} T(L\Lambda) := \int_{L\Lambda} \mathrm{d}x_0 \sum_{\ell_1,\ldots,\ell_m=0}^{\nu-1} \int_{\mathbb{R}^{2(m-1)}} &\mathrm{d}x_1^\perp \cdots \mathrm{d}x_{m-1}^\perp \, p_{\ell_1}(x_0^\perp, x_1^\perp) p_{\ell_2}(x_1^\perp, x_2^\perp) \\ &\cdots p_{\ell_m}(x_{m-1}^\perp, x_0^\perp) \\ &\times \int_{\mathbb{R}^{m-1}} \mathrm{d}x_1^\| \cdots \mathrm{d}x_{m-1}^\| \, k_{\mu(\ell_1)}(x_0^\| - x_1^\|) \cdots k_{\mu(\ell_m)}(x_{m-1}^\| - x_0^\|) \\ &\times 1_{L\Lambda}(x_0^\perp, x_1^\|) \cdots 1_{L\Lambda}(x_0^\perp, x_{m-1}^\|) \, . \end{aligned}$$

As the second line is independent of $x_j^\perp$, the integrals over $x_1^\perp, \ldots, x_{m-1}^\perp$ can be easily resolved and yield the diagonal of the integral kernel of the operator $\mathrm{P}_{\ell_1} \cdots \mathrm{P}_{\ell_m}$ at $x_0^\perp$, which is $B/(2\pi)$, if $\ell_1 = \cdots = \ell_m$ and 0 otherwise. Thus, we have

$$\begin{aligned} T(L\Lambda) = \int_{L\Lambda} \mathrm{d}x_0 \sum_{\ell=0}^{\nu-1} \frac{B}{2\pi} \quad &\int_{\mathbb{R}^{m-1}} \mathrm{d}x_1^\| \cdots \mathrm{d}x_{m-1}^\| \, k_{\mu(\ell)}(x_0^\| - x_1^\|) \cdots k_{\mu(\ell)}(x_{m-1}^\| - x_0^\|) \\ &\times 1_{L\Lambda}(x_0^\perp, x_1^\|) \cdots 1_{L\Lambda}(x_0^\perp, x_{m-1}^\|). \end{aligned}$$

Now, we set $x^\perp := x_0^\perp / L$ and observe $1_{L\Lambda}(x_0^\perp, x_j^\|) = 1_{L\Lambda_{x^\perp}}(x_j^\|)$. Therefore, we have

$$\begin{aligned} T(L\Lambda) &= L^2 \int_{\mathbb{R}^2} \mathrm{d}x^\perp \sum_{\ell=0}^{\nu-1} \frac{B}{2\pi} \int_{L\Lambda_{x^\perp}} \mathrm{d}x_0^\| \int_{\mathbb{R}^{m-1}} \mathrm{d}x_1^\| \cdots \mathrm{d}x_{m-1}^\| \\ &\qquad \times k_{\mu(\ell)}(x_0^\| - x_1^\|) \cdots k_{\mu(\ell)}(x_{m-1}^\| - x_0^\|) \, 1_{L\Lambda}(Lx^\perp, x_1^\|) \cdots 1_{L\Lambda}(Lx^\perp, x_{m-1}^\|) \\ &= L^2 \frac{B}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x^\perp \sum_{\ell=0}^{\nu-1} \mathrm{tr} \left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m , \end{aligned}$$

which is the expression in the claim. Thus, we are left to bound the error term of our approximation. Let us denote by $U \subset \mathbb{R}^{3m}$ the set of all tuples $(x_0, x_1, \ldots, x_{m-1})$ where $1_{L\Lambda}(x_0) 1_{L\Lambda}(x_1) \cdots 1_{L\Lambda}(x_{m-1})$ is not equal to $1_{L\Lambda}(x_0) 1_{L\Lambda}(x_0^\perp, x_1^\|) \cdots 1_{L\Lambda}(x_0^\perp, x_{m-1}^\|)$. Then, using the notation $x_m := x_0$ we trivially have

$$\left| T(L\Lambda) - \mathrm{tr}\, \mathrm{D}_\mu(L\Lambda)^m \right| \leq \int_U \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \prod_{j=0}^{m-1} |\mathrm{D}_\mu(x_j, x_{j+1})|. \tag{3.4}$$

We will now enlarge U until we get a set where the integral can easily be calculated. Let $(x_0, x_1, \ldots, x_{m-1}) \in U$. Then, there is a $j \in \{1, \ldots, m-1\}$ such that $1_{L\Lambda}(x_j) \neq 1_{L\Lambda}(x_0^\perp, x_j^\|)$. Thus, the line between x_j and $(x_0^\perp, x_j^\|)$ has to intersect the boundary $L\partial\Lambda$, which implies $\mathrm{dist}(x_j, L\partial\Lambda) \leq \|x_j^\perp - x_0^\perp\|$. By the triangle and mean inequalities, we observe that

$$\mathrm{dist}(x_j, L\partial\Lambda) \le \|x_j^\perp - x_0^\perp\| \le \sum_{k=1}^{m} \|x_k^\perp - x_{k-1}^\perp\| \le \sqrt{m}\sqrt{\sum_{k=1}^{m} \|x_k^\perp - x_{k-1}^\perp\|^2}. \tag{3.5}$$

For $j \in \{0, \dots, m-1\}$, let $U_j \subset \mathbb{R}^{3m}$ be the set of all $(x_0, x_1, \dots, x_{m-1}) \in \mathbb{R}^{3m}$ satisfying

$$\mathrm{dist}(x_j, L\partial\Lambda) \le \sqrt{m}\sqrt{\sum_{k=1}^{m} \|x_k^\perp - x_{k-1}^\perp\|^2}. \tag{3.6}$$

As $U \subset \bigcup_{j=1}^{m-1} U_j$, we see that

$$\int_U \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \prod_{j=0}^{m-1} |\mathrm{D}_\mu(x_j, x_{j+1})| \le \sum_{k=1}^{m-1} \int_{U_k} \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \prod_{j=0}^{m-1} |\mathrm{D}_\mu(x_j, x_{j+1})| \tag{3.7}$$

$$= (m-1) \int_{U_0} \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \prod_{j=0}^{m-1} |\mathrm{D}_\mu(x_j, x_{j+1})|. \tag{3.8}$$

The cyclic parameter shift $(x_0, x_1, \dots, x_{m-1}) \mapsto (x_1, x_2, \dots, x_0)$ sends U_j to U_{j+1} and does not change the integrand. For $1 \le j \le m$, let $y_j := x_j - x_{j-1}$. We will change variables from $(x_0, x_1, \dots, x_{m-1})$ to $(x_0, y_1, \dots, y_{m-1}) =: (x_0, \mathbf{y})$. Using $y_m^\perp = -\sum_{j=1}^{m-1} y_j^\perp$, similar to (3.5), we observe that

$$m \sum_{k=1}^{m} \|x_k^\perp - x_{k-1}^\perp\|^2 = m(\|\mathbf{y}^\perp\|^2 + \|y_m^\perp\|^2) \le m\|\mathbf{y}^\perp\|^2 + m(m-1)\|\mathbf{y}^\perp\|^2 = m^2\|\mathbf{y}^\perp\|^2. \tag{3.9}$$

Thus, under this change of variables the set U_0 is mapped into the set

$$V := \left\{(x_0, y_1, \dots, y_{m-1}) \in \mathbb{R}^{3m} \colon\ \mathrm{dist}(x_0, L\partial\Lambda) \le m\|\mathbf{y}^\perp\|\right\}. \tag{3.10}$$

Let us first estimate the integrand in terms of the y_j's. With (2.8), (2.5) and (2.6), we get

$$|\mathrm{D}_\mu(x_j, x_{j+1})| \le C_{\mu,B,1} \frac{\exp(-B\|y_{j+1}^\perp\|^2/8)}{\langle y_{j+1}^\parallel \rangle}. \tag{3.11}$$

We recall that $\langle x \rangle = \sqrt{1 + x^2}$ is the Japanese bracket.

For $x_0 \in \mathbb{R}^3$, let $\Omega_{x_0} := \{\mathbf{y}^\perp \in \mathbb{R}^{2(m-1)} \colon\ \mathrm{dist}(x_0, L\partial\Lambda) \le m\|\mathbf{y}^\perp\|\}$, and thus $V = \{(x_0, \mathbf{y}) \in \mathbb{R}^{3m} \colon \mathbf{y}^\perp \in \Omega_{x_0}\}$. We have

$$\int_{U_0} \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \prod_{j=0}^{m-1} |\mathrm{D}_\mu(x_j, x_{j+1})| \tag{3.12}$$

$$\le C_{\mu,B,1}^m \int_V \mathrm{d}x_0 \mathrm{d}\mathbf{y} \prod_{j=1}^{m} \frac{\exp(-B\|y_j^\perp\|^2/8)}{\langle y_j^\parallel \rangle} \tag{3.13}$$

$$= C_{\mu,B,1}^m \left(\int_{\mathbb{R}^{m-1}} \mathrm{d}\mathbf{y}^{\|} \prod_{j=1}^m \frac{1}{\langle y_j^{\|} \rangle} \right) \int_{\mathbb{R}^3} \mathrm{d}x_0 \int_{\Omega_{x_0}} \mathrm{d}\mathbf{y}^{\perp} \exp(-B\|\mathbf{y}^{\perp}\|^2/8). \tag{3.14}$$

We need the estimate

$$\int_{\mathbb{R}^{m-1}} \mathrm{d}\mathbf{y}^{\|} \prod_{j=1}^m \frac{1}{\langle y_j^{\|} \rangle} \leq 2^m m!, \tag{3.15}$$

which is proved in Appendix B. We also have the bound

$$\int_{\Omega_{x_0}} \mathrm{d}\mathbf{y}^{\perp} \exp(-B\|\mathbf{y}^{\perp}\|^2/8) \leq \sup_{\mathbf{y}^{\perp}\in\Omega_{x_0}} \left(\exp(-B\|\mathbf{y}^{\perp}\|^2/9)\right) \int_{\mathbb{R}^{2(m-1)}} \mathrm{d}\mathbf{y}^{\perp} \exp(-B\|\mathbf{y}^{\perp}\|^2/72) \tag{3.16}$$

$$= \exp\left(\frac{-B \operatorname{dist}(x_0, L\partial\Lambda)^2}{9m^2} \right) \sqrt{72\pi/B}^{2(m-1)}. \tag{3.17}$$

Thus, we arrive at

$$\int_{U_0} \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \prod_{j=0}^{m-1} |\mathrm{D}_\mu(x_j, x_{j+1})|$$

$$\leq C_{\mu,B,1}^m C_{B,2}^m m! \int_{\mathbb{R}^3} \mathrm{d}x_0 \exp\left(\frac{-B \operatorname{dist}(x_0, L\partial\Lambda)^2}{9m^2} \right) \tag{3.18}$$

$$\leq C_{\mu,B,1}^m C_{B,2}^m m! \sum_{k=0}^{\infty} |B_{k+1}(L\partial\Lambda)| \exp\left(-\frac{B}{9m^2} k^2 \right). \tag{3.19}$$

Here, we used an $(1, \infty)$ Hölder estimate on the sets $k \leq \operatorname{dist}(x_0, L\partial\Lambda) \leq k+1$ for the integral over $\mathbb{R}^3$. We then enlarged these sets to the $(k+1)$-neighborhood $B_{k+1}(L\partial\Lambda)$, as their measures can be estimated more easily. Thus, using Lemma A.3 with $d = 2$ and $r = k+1$, we arrive at

$$|T(L\Lambda) - \operatorname{tr} \mathrm{D}_\mu(L\Lambda)^m| \tag{3.20}$$

$$\leq (m-1)(C_{\mu,B,1} C_{B,2})^m m! \sum_{k=0}^{\infty} |B_{k+1}(L\partial\Lambda)| \exp\left(-\frac{B}{9m^2} k^2 \right) \tag{3.21}$$

$$\leq (m-1)(C_{\mu,B,1} C_{B,2})^m m! \sum_{k=0}^{\infty} L^3 \left| B_{\frac{k+1}{L}}(\partial\Lambda) \right| \exp\left(-\frac{B}{9m^2} k^2 \right) \tag{3.22}$$

$$\leq (m-1)(C_{\mu,B,1} C_{B,2})^m m! \sum_{k=0}^{\infty} L^3 \mathcal{K}(\Lambda) \left(\frac{k+1}{L} + \frac{(k+1)^3}{L^3} \right) \exp\left(-\frac{B}{9m^2} k^2 \right) \tag{3.23}$$

$$\leq (m-1)(C_{\mu,B,1} C_{B,2})^m m! \mathcal{K}(\Lambda) L^2 \sup_{t>0} \left((t+1)^3 (t+2)^2 \exp\left(-\frac{B}{9m^2} t^2 \right) \right)$$

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \tag{3.24}$$

$$\leq (m-1)(C_{\mu,B,1}C_{B,2})^m m! \mathcal{K}(\Lambda) L^2 C_{B,3} m^5 \leq \mathcal{K}(\Lambda) L^2 C_{\mu,B}^m m!, \tag{3.25}$$

which was our claim. □

In the next step, we accomplish the

Proof of Theorem 2.3. As the expression is linear in f, it suffices to consider monomials $f(t) = t^m$ with integer $m \geq 1$. In the special case $m = 1$, we just use (2.8), (2.5), and (2.6) to see

$$\operatorname{tr} \mathrm{D}_\mu(L\Lambda) = \int_{L\Lambda} \mathrm{d}x_0 \, \mathrm{D}_\mu(x_0, x_0) = \int_{L\Lambda} \mathrm{d}x_0 \sum_{\ell=0}^{\nu-1} k_{\mu(\ell)}(0) p_\ell(x_0^\perp, x_0^\perp) \tag{3.26}$$

$$= \int_{L\Lambda} \mathrm{d}x_0 \sum_{\ell=0}^{\nu-1} \frac{\sqrt{\mu(\ell)}}{\pi} \frac{B}{2\pi} = L^3 \frac{B}{2\pi^2} \sum_{\ell=0}^{\nu-1} \sqrt{\mu(\ell)} 1^1 |\Lambda|. \tag{3.27}$$

As $\mathsf{I}(1) = \mathsf{I}(id) = 0$, this covers the case $m = 1$ and we may from now on assume $m \geq 2$.

Our first aim is to understand the open sets $\Lambda_{x^\perp}$. This is essentially a question about the nature of the sets Λ. There are some results to choose from, so let us take a look. Due to Lemmas A.6 and A.8, for Lebesgue almost every $x^\perp \in \mathbb{R}^2$, the set $\Lambda_{x^\perp}$ is a finite union of disjoint intervals, $\partial(\Lambda_{x^\perp}) = (\partial\Lambda)_{x^\perp}$, and $\#(\partial(\Lambda_{x^\perp}))$ is twice the number of these intervals. Henceforth, we set $\partial\Lambda_{x^\perp} := \partial(\Lambda_{x^\perp})$. The (improved) asymptotic expansion goes back to Landau and Widom [19] and is presented in Appendix C, see Corollary C.3. The coefficient $\mathsf{I}(m) = -1/(4\pi^2) \sum_{r=1}^{m-1} r^{-1}$ is mentioned below (2.15).

For fixed $\Lambda_{x^\perp}$, the error term $\varepsilon(\Lambda_{x^\perp}, L)$ remains bounded as $L \to \infty$. However, we need to know, whether this error term is integrable over $x^\perp$. Thus, the dependency on $\Lambda_{x^\perp}$ is relevant.

To derive the $o(L^2 \ln(L))$ error term, we subtract the volume term, divide by $L^2 \ln(L)$ and use dominated convergence in order to exchange the limit $L \to \infty$ with the integral over $x^\perp$. Thus, instead of an estimate for the error term that is of a lower order in L than $\ln(L)$, we only need an upper bound for the difference to the volume term, which is of order $\ln(L)$. This upper bound is provided by Lemma 6.1. As any interval in $L\Lambda_{x^\perp}$ has length at most CL, we arrive at

$$\left| \operatorname{tr} \left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m - \frac{\sqrt{\mu(\ell)}}{\pi} L |\Lambda_{x^\perp}| \right| \tag{3.28}$$

$$= \left| \operatorname{tr} \left[\left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m - \mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right] \right| \tag{3.29}$$

$$\leq \left\| \left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m - \mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right\|_1 \tag{3.30}$$

$$\leq C \#(\partial\Lambda_{x^\perp}) \ln(L), \tag{3.31}$$

where the constant C depends on m, $\mu(\ell)$ and Λ, but not on $x^\perp$. With this estimate, we apply dominated convergence to get

$$\lim_{L\to\infty} \frac{1}{L^2 \ln(L)} \left(\operatorname{tr} \mathrm{D}_\mu(L\Lambda)^m - BL^3|\Lambda| \sum_{\ell=0}^{\nu-1} \frac{\sqrt{\mu(\ell)}}{2\pi^2} \right) \tag{3.32}$$

$$= \sum_{\ell=0}^{\nu-1} \lim_{L\to\infty} \frac{1}{L^2 \ln(L)} \frac{B}{2\pi} L^2 \Big(\int_{\mathbb{R}^2} \mathrm{d}x^\perp \ \operatorname{tr} \left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m$$
$$- \frac{\sqrt{\mu(\ell)}}{\pi} |L\Lambda_{x^\perp}| \Big) \tag{3.33}$$

$$= \sum_{\ell=0}^{\nu-1} \frac{B}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x^\perp \lim_{L\to\infty} \frac{1}{\ln(L)} \Big(\operatorname{tr} \left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu(\ell)] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m - \frac{\sqrt{\mu(\ell)}}{\pi} |L\Lambda_{x^\perp}| \Big) \tag{3.34}$$

$$= \nu \frac{B}{2\pi} 2\,\mathsf{I}(m) \int_{\mathbb{R}^2} \mathrm{d}x^\perp \, \#(\partial\Lambda_{x^\perp}) = \nu B\,\mathsf{I}(m) \frac{1}{\pi} \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v) \, |n(v)\cdot e_3|. \tag{3.35}$$

We moved the sum over ℓ to the front, as every summand converges as $L \to \infty$. In the second line we used that $\int_{\mathbb{R}^2} \mathrm{d}x^\perp \, |\Lambda_{x^\perp}| = |\Lambda|$. Finally, we inserted (A.44) to obtain the expansion with error term $o(L^2 \ln(L))$ as claimed in the theorem.

For the second part, we need to show that the error term for polynomials can be bounded by CL^2, if Λ is a piecewise $\mathsf{C}^{1,\alpha}$ region for some $0 < \alpha < 1$, as defined in Definition 2.1. This time, we use Corollary C.3 to deal with the trace of the one-dimensional operator. For that, we arrange each $\partial\Lambda_{x^\perp} := (\partial\Lambda)_{x^\perp} = \{w_{x^\perp 1}, \ldots, w_{x^\perp \#(\partial\Lambda_{x^\perp})}\} \subset \mathbb{R}$ in the order of increasing third components and write

$$\left| \operatorname{tr} \left(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu] \mathbb{1}_{L\Lambda_{x^\perp}} \right)^m - \frac{\sqrt{\mu}}{\pi} L|\Lambda_{x^\perp}| - 2\,\mathsf{I}(m)\#(\partial\Lambda_{x^\perp}) \ln(1+L) \right| \tag{3.36}$$

$$\leq C \sum_{i=1}^{\#(\partial\Lambda_{x^\perp})-1} \big(1 + |\ln(|w_{x^\perp i} - w_{x^\perp i+1}|)|\big) \tag{3.37}$$

$$\leq C \sum_{i=1}^{\#(\partial\Lambda_{x^\perp})} \Big(1 + \big|\ln\big(\inf_{v\in\partial\Lambda_{x^\perp}\setminus w_{x^\perp i}} |w_{x^\perp i} - v|\big)\big|\Big). \tag{3.38}$$

In the last step, we used that the distance between any two points in $\partial\Lambda$ is bounded from above, as Λ is bounded to conclude that only short distances $|v - v_i|$ can lead to an error term larger than the $O(\#(\partial\Lambda_{x^\perp}))$-term we have in front. A lower bound for the infimum is provided by Lemma A.1. This bound is zero in some cases, which leads to the logarithm being infinite. This just means that our integrand in the integral over $x^\perp$ attains infinity. The integral can still exist and we will show that it does.

As the terms of order L^3 and $L^2\ln(L)$ work just like in the previous case, we will only consider the error term. Hence, we need to estimate

$$\int_{\mathbb{R}^2} \mathrm{d}x^\perp \sum_{i=1}^{\#(\partial\Lambda_{x^\perp})} \Big(1 + |\ln(\inf_{v\in\partial\Lambda_{x^\perp}\setminus w_{x^\perp i}} |w_{x^\perp i} - v|)|\Big) \tag{3.39}$$

$$\leq C\int_{\mathbb{R}^2} \mathrm{d}x^\perp \sum_{i=1}^{\#(\partial\Lambda_{x^\perp})} \Big(1 + |\ln(\min\{\operatorname{dist}(w_{x^\perp i},\Gamma), |n((x^\perp, w_{x^\perp i}))\cdot e_3|^{\frac{1}{\alpha}})|\}\Big) \tag{3.40}$$

$$\leq C\int_{\mathbb{R}^2} \mathrm{d}x^\perp \sum_{w\in\{x^\perp\}\times\partial\Lambda_{x^\perp}} \Big(1 + |\ln(\operatorname{dist}(w,\Gamma))| + |\ln(|n(w)\cdot e_3|)|\Big). \tag{3.41}$$

In the first step, we applied Lemma A.1 with the vectors $v_1 := (x^\perp, w_{x^\perp i})$ amd $v_2 := (x^\perp, v)$ noting that $\frac{v_1-v_2}{\|v_1-v_2\|} = \pm e_3$. We now want to rewrite this integral as an integral over the boundary $\partial\Lambda$. This is possible by Lemma A.7. Hence, we have (recall that $\mathcal{H}^2$ is the canonical surface measure on $\partial\Lambda$),

$$\int_{\mathbb{R}^2} \mathrm{d}x^\perp \sum_{i=1}^{\#(\partial\Lambda_{x^\perp})} \Big(1 + |\ln(\inf_{v\in\partial\Lambda_{x^\perp}\setminus w_{x^\perp i}} |w_{x^\perp i} - v|)|\Big) \tag{3.42}$$

$$\leq C\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(w)\left[1 + |\ln(\operatorname{dist}(w,\Gamma))| + |\ln(|n(w)\cdot e_3|)|\right]|n(w)\cdot e_3| \tag{3.43}$$

$$\leq C + C\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(w)\,|\ln(\operatorname{dist}(w,\Gamma))| \leq C. \tag{3.44}$$

In the second step, we used that $0 \leq |n(w)\cdot e_3| \leq 1$ and that for $0 \leq t \leq 1$, we have $0 \leq |t\ln(t)| \leq 1/\mathrm{e}$. The last step is a rather lengthy, not particularly insightful calculation, which can be found in Lemma A.9.

Once we put the factor L^2 back in front of this, we arrive at the error term $O(L^2)$ which completes the proof of the second part of this theorem. □

4. Entanglement Entropy

Here is the main result of this paper.

Theorem 4.1. *Suppose that $\Lambda \subset \mathbb{R}^3$ is a piecewise Lipschitz region and let $\mu > B$. Let $\nu := \lceil \frac{1}{2}(\mu/B - 1)\rceil$ and let $h\colon [0,1] \to \mathbb{R}$ be a continuous function, which is β-Hölder continuous at* 0 *and* 1 *for some $1 \geq \beta > 0$, and assume that $h(0) = h(1) = 0$. Then, we have the asymptotic expansion*

$$\operatorname{tr} h(\mathrm{D}_\mu(L\Lambda)) = L^2\ln(L)\nu B\frac{1}{\pi}\,\mathsf{I}(h)\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\,|n(v)\cdot e_3| + o(L^2\ln(L)). \tag{4.1}$$

In particular, as the γ-Rényi entropy function h_γ is β-Hölder continuous for any $\beta < \min(\gamma, 1)$, the γ-Rényi entanglement entropy, $\mathrm{S}_\gamma(L\Lambda)$, of the ground state at Fermi energy μ localized to $L\Lambda$, satisfies the asymptotic expansion

$$\mathrm{S}_\gamma(L\Lambda) = L^2 \ln(L)\nu B \, \frac{1+\gamma}{24\gamma\pi} \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v) \, |n(v) \cdot e_3| + o(L^2 \ln(L)) \tag{4.2}$$

as $L \to \infty$.

Remarks 4.2. (1) Unlike in Theorem 2.3, we cannot improve the $o(L^2 \ln(L))$ error term in (4.2) if we assume stronger regularity conditions on the boundary of Λ. This is a limitation of our method of proof, which relies on the Stone–Weierstrass approximation. Here, we lose control of the error term.

(2) Let us compare (4.2) to the asymptotic expansion of the entanglement entropy of the ground state at Fermi energy $\mu > 0$ in the ideal Fermi gas, as introduced in Remark 2.4(ii). Here, the γ-Rényi entanglement entropy satisfies

$$\mathrm{S}_\gamma(L\Lambda) = L^2 \ln(L) \frac{\mu(1+\gamma)}{48\gamma\pi} \mathcal{H}^2(\partial\Lambda) + o(L^2 \ln(L)).$$

(3) To the best of our knowledge, the result (4.2) is new, even in the physics literature. The factor νB satisfies $\nu B = \mu/2 + (\delta - \frac{1}{2})B$ for some $\delta \in [0, 1)$. We can bound the surface integral in (4.2) by $\mathcal{H}^2(\partial\Lambda)$ due to $|n(v) \cdot e_3| \leq 1$. Let us set $\mu_{\mathrm{ref}} := 2\nu B$. This corresponds to the same number ν of Landau levels as the original μ, and seems to be a suitable reference value for comparing the Landau Hamiltonian with the free Hamiltonian. Thus the entanglement entropy associated to the Landau Hamiltonian is always smaller than the one associated to the free Hamiltonian at the reference value μ_{ref}.

We use certain estimates on traces. To this end, let us denote by $s_n(T), n \in \mathbb{N}$, the singular values of the compact operator T on a (separable) Hilbert space, arranged in decreasing order. The standard notation $\mathfrak{S}_p, 0 < p < \infty$ is used for the class of operators with a finite *Schatten–von Neumann quasi-norm*:

$$\|T\|_p := \left[\sum_{n=1}^{\infty} s_n(T)^p \right]^{\frac{1}{p}} < \infty.$$

If $p \geq 1$, then $\|\cdot\|_p$ defines a norm. For $0 < p < 1$ it is a quasi-norm that satisfies the *p-triangle inequality*

$$\|T_1 + T_2\|_p^p \leq \|T_1\|_p^p + \|T_2\|_p^p. \tag{4.3}$$

The class $\mathfrak{S}_1$ is the standard trace-class. The class $\mathfrak{S}_2$ is the ideal of Hilbert–Schmidt operators. The p-Schatten quasi-norm estimate required for this proof is shown in Theorem 5.5.

Proof of Theorem 4.1. The proof goes along the same line of arguments as presented in [20,21]. We recall that $\mathsf{I}(h_\gamma) = (1+\gamma)/(24\gamma)$ and thus we are left to show the claim for the function h. Let $r = \beta/2$ and $\varepsilon > 0$. We choose a smooth cutoff function ζ_ε such that $0 \le \zeta_\varepsilon \le 1$ and such that ζ_ε vanishes on $[\varepsilon, 1-\varepsilon]$ and equals 1 on $[0, \varepsilon/2] \cup [1-\varepsilon/2, 1]$. As h is continuous and β-Hölder continuous at 0 and 1, there is a constant C such that

$$h(t) \le C t^\beta (1-t)^\beta, \quad t \in [0,1]. \tag{4.4}$$

This implies

$$|(\zeta_\varepsilon h)(t)| \le C\varepsilon^r t^r (1-t)^r \quad t \in [0,1]. \tag{4.5}$$

As the function $t \mapsto \frac{(1-\zeta_\varepsilon(t))h(t)}{t(1-t)}$ is continuous, we can infer from the Stone–Weierstrass approximation theorem that there is a polynomial p and a function $\delta_\varepsilon \colon [0,1] \to \mathbb{R}$ with $\|\delta_\varepsilon\|_{\mathsf{L}^\infty([0,1])} \le \varepsilon^r$ and

$$\frac{(1-\zeta_\varepsilon(t))h(t)}{t(1-t)} = p(t) + \delta_\varepsilon(t), \quad t \in [0,1]. \tag{4.6}$$

Thus, we have

$$h(t) = p(t)t(1-t) + \delta_\varepsilon(t)t(1-t) + \zeta_\varepsilon(t)h(t) =: p(t)t(1-t) + \phi_\varepsilon(t)\,. \tag{4.7}$$

As $t(1-t) \le t^r(1-t)^r$, we observe

$$|\phi_\varepsilon(t)| \le C\varepsilon^r t^r (1-t)^r, \quad t \in [0,1]. \tag{4.8}$$

Thus, using Theorem 5.5, (2.7) and (4.3), we arrive at

$$\begin{aligned}
|\mathrm{tr}\, \phi_\varepsilon(\mathrm{D}_\mu(L\Lambda))| &\le C\varepsilon^r \mathrm{tr}\,(\mathrm{D}_\mu(L\Lambda)^r(1-\mathrm{D}_\mu(L\Lambda))^r) &(4.9)\\
&= C\varepsilon^r \|\mathbb{1}_{L\Lambda}\mathrm{D}_\mu \mathbb{1}_{L\Lambda^\complement}\mathrm{D}_\mu\mathbb{1}_{L\Lambda}\|_r^r &(4.10)\\
&= C\varepsilon^r \|\mathbb{1}_{L\Lambda}\mathrm{D}_\mu \mathbb{1}_{L\Lambda^\complement}\|_{2r}^{2r} &(4.11)\\
&\le C\varepsilon^r \sum_{\ell=0}^{\nu-1} \|\mathbb{1}_{L\Lambda}(\mathrm{P}_\ell \otimes \mathbb{1}((-\mathrm{i}\nabla^\parallel)^2 \le \mu(\ell)))\mathbb{1}_{L\Lambda^\complement}\|_{2r}^{2r} &(4.12)\\
&\le C\varepsilon^r C L^2 \ln(L). &(4.13)
\end{aligned}$$

In (4.10), we used that D_μ is a projection. Let $q(t) := p(t)t(1-t)$. Now, by linearity of I and the estimate (4.8), we have

$$|\mathsf{I}(h) - \mathsf{I}(q)| = |\mathsf{I}(\phi_\varepsilon)| \le C\varepsilon^r\, \mathsf{I}(t \mapsto t^r(1-t)^r) \le C\varepsilon^r. \tag{4.14}$$

Theorem 2.3(i) applied for the polynomial q with $q(0) = q(1) = 0$ yields

$$\mathrm{tr}\, q(\mathrm{D}_\mu(L\Lambda)) = L^2\ln(L) B\nu\, \mathsf{I}(q) \frac{1}{\pi}\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\,|n(v)\cdot e_3| + o(L^2\ln(L)). \tag{4.15}$$

Now, combining (4.13), (4.14) and (4.15), we arrive at

$$\limsup_{L\to\infty} \left| \frac{\mathrm{tr}\, h(\mathrm{D}_\mu(L\Lambda))}{L^2\ln(L)} - \nu B\, \mathsf{I}(h)\, \frac{1}{\pi}\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\,|n(v)\cdot e_3| \right| \le C\varepsilon^r. \tag{4.16}$$

As $\varepsilon > 0$ is arbitrary, we have proved the claim. □

5. Schatten–von Neumann Quasi-Norm Estimates

By a box in $\mathbb{R}^d$, we mean a Cartesian product of d intervals. These intervals do not have to be bounded. We will denote subsets of $\mathbb{R}$ by I, of $\mathbb{R}^2$ by Υ, and of $\mathbb{R}^3$ by Λ. We will combine known estimates for the two-dimensional magnetic Hamiltonian from [21] and for the one-dimensional Hamiltonian [20, 34] without a magnetic field.

Let $\Upsilon, \Upsilon' \subset \mathbb{R}^2$ be Lipschitz regions and let $I, I' \subset \mathbb{R}$ be finite unions of closed intervals. Then, we have

$$\mathbb{1}_{\Upsilon\times I}\big(\mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]\big)\mathbb{1}_{\Upsilon'\times I'} = \big(\mathbb{1}_{\Upsilon}\mathrm{P}_\ell\mathbb{1}_{\Upsilon'}\big) \otimes \big(\mathbb{1}_I\mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]\mathbb{1}_{I'}\big). \tag{5.1}$$

As the singular values of the tensor product of two operators are given by all possible products of pairs of the individual singular values, we have for any $0 < p \le \infty$:

$$\left\|\mathbb{1}_{\Upsilon\times I}\left(\mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]\right)\mathbb{1}_{\Upsilon'\times I'}\right\|_p = \|\mathbb{1}_{\Upsilon}\mathrm{P}_\ell\mathbb{1}_{\Upsilon'}\|_p \left\|\mathbb{1}_I\mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]\mathbb{1}_{I'}\right\|_p. \tag{5.2}$$

The following general properties will be useful:

Lemma 5.1. *For any self-adjoint bounded operators $S, T\colon \mathsf{L}^2(\mathbb{R}^d) \to \mathsf{L}^2(\mathbb{R}^d)$, any measurable sets Ω_1, Ω_2, $\Omega_1', \Omega_2' \subset \mathbb{R}^d$ and any $0 < p \le 1$, we have*

Symmetry	$\|\mathbb{1}_{\Omega_1}T\mathbb{1}_{\Omega_2}\|_p = \|\mathbb{1}_{\Omega_2}T\mathbb{1}_{\Omega_1}\|_p$,
Monotonicity I	$\|\mathbb{1}_{\Omega_1}T\mathbb{1}_{\Omega_2}\|_p \le \left\|\mathbb{1}_{\Omega_1\cup\Omega_1'}T\mathbb{1}_{\Omega_2\cup\Omega_2'}\right\|_p$,
Monotonicity II	*If* $0 \le S \le T$, *then* $\|S\|_p \le \|T\|_p$,
Subadditivity	$\left\|\mathbb{1}_{\Omega_1\cup\Omega_1'}T\mathbb{1}_{\Omega_2}\right\|_p^p \le \|\mathbb{1}_{\Omega_1}T\mathbb{1}_{\Omega_2}\|_p^p + \left\|\mathbb{1}_{\Omega_1'}T\mathbb{1}_{\Omega_2}\right\|_p^p$.

A proof of these properties can be found, for example, in [29].

We assume now that the magnetic-field strength has been "scaled out" so that $B = 1$ for the remainder of this section. The effective scale in the planar coordinates is $L\sqrt{B}$ and in the perpendicular it is $L\sqrt{\mu}$.

Next, we collect some more specific (quasi-)norm estimates for both the one dimensional free Hamiltonian and the constant magnetic field Hamiltonian in two dimensions.

Proposition 5.2. *Let $0 < p \le 1, \ell \in \mathbb{N}_0$ and let $\mu > 0$. Then, there is a constant C such that for any $x \in \mathbb{R}^2, t \in \mathbb{R}, h \ge 2, \delta \ge 1$, any measurable set $\Upsilon \subset \mathbb{R}^2$ such that $[-\delta, 1+\delta]^2 + x \subset \Upsilon$ and any measurable set $I \subset \mathbb{R}$ such that $[t, t+h] \subset I$, we have the estimates*

$$\left\|\mathbb{1}_{[0,1]^2+x}\mathrm{P}_\ell\right\|_p^p \le C, \tag{5.3}$$

$$\left\|\mathbb{1}_{[0,1]^2+x}\mathrm{P}_\ell\mathbb{1}_{\Upsilon^\complement}\right\|_p^p \le C\exp(-p\delta^2/18), \tag{5.4}$$

$$\left\|\mathbb{1}_{[t,t+h]}\mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]\right\|_p^p \le Ch, \tag{5.5}$$

$$\left\|\mathbb{1}_{[t,t+h]}\mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]\mathbb{1}_{I^\complement}\right\|_p^p \le C\ln(h), \tag{5.6}$$

Proof. The first two inequalities follow by [21, Lemma 12], monotonicity I in Lemma 5.1, and the unitary translation invariance of P_ℓ. The 8 in the denominator was increased to 18 in (5.4) as we switched from circles to squares[2]. To prove the last inequality, we first use monotonicity I and the translation invariance, then the standard unitary equivalence, see, for example, [19, (7–10)], and finally [34, Corollary 4.7]. Thus,

$$\left\|\mathbb{1}_{[t,t+h]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\mathbb{1}_{I^\complement}\right\|_p^p \le \left\|\mathbb{1}_{[0,h]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\mathbb{1}_{[0,h]^\complement}\right\|_p^p \tag{5.7}$$
$$= \left\|\mathbb{1}_{[0,1]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le h^2\mu]\mathbb{1}_{[0,1]^\complement}\right\|_p^p \tag{5.8}$$
$$\le C\ln(h). \tag{5.9}$$

For the third inequality, we will reduce to the case $h = 2, t = 0$ by subadditivity, monotonicity I and translation invariance. Let $m := \lceil h/2 \rceil \in \mathbb{N}$ be the smallest integer larger or equal to $h/2$. Thus, as $h \ge 2$, we have $m \le h$. We observe that

$$\left\|\mathbb{1}_{[t,t+h]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\right\|_p^p \le \sum_{k=0}^{m-1} \left\|\mathbb{1}_{[t+2k,t+2k+2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\right\|_p^p \tag{5.10}$$
$$= m\left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\right\|_p^p \tag{5.11}$$
$$\le 2h\left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\right\|_p^p. \tag{5.12}$$

Using subadditivity once more we now estimate

$$\left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\right\|_p^p \tag{5.13}$$
$$\le \left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\mathbb{1}_{[0,2]}\right\|_p^p + \left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\mathbb{1}_{[0,2]^\complement}\right\|_p^p \tag{5.14}$$
$$\le \left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\mathbb{1}_{[0,2]}\right\|_p^p + C \tag{5.15}$$
$$= \left\|\mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]\right\|_{2p}^{2p} + C. \tag{5.16}$$

The second summand was bounded by (5.6), and the last quasi-norm identity is derived by the singular value identity $s_n(A)^2 = s_n(A^*A)$. Define $Q := \mathbb{1}_{[0,2]}\mathbb{1}[(-\mathrm{i}\nabla^\|)^2 \le \mu]$. Our claim is $Q \in \mathfrak{S}_p$ for all $0 < p \le 1$. The last estimate shows that $Q \in \mathfrak{S}_p$, if $Q \in \mathfrak{S}_{2p}$ for $p \le 1$. We now observe

$$\|Q\|_2^2 = \int_0^2 \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t\, k_\mu^2(s-t) = \frac{2\sqrt{\mu}}{\pi}. \tag{5.17}$$

Thus, we have $Q \in \mathfrak{S}_2$ and hence $Q \in \mathfrak{S}_{2^{1-n}}$ for any $n \in \mathbb{N}$. Lastly, as $\mathfrak{S}_p \subset \mathfrak{S}_q$ for $p < q$, we arrive at $Q \in \mathfrak{S}_p$ for any $0 < p \le \infty$, which finishes the proof. $\square$

[2] The choice of the value $18 = 3^2 \times 2$ is convenient for this paper.

After all these preparations we finally state the crucial local estimates that are needed in the proof of Theorem 5.5.

Lemma 5.3. *Let $0 < p \leq 1$. Then, there is a constant C such that for any $x \in \mathbb{R}^2, t \in \mathbb{R}, h \geq 2$, $\delta \geq 1$, any measurable $\Upsilon \subset \mathbb{R}^2$ such that $[-\delta, 1+\delta]^2 + x \subset \Upsilon$ and any interval $I \subset \mathbb{R}$ such that $[t, t+h] \subset I$, we have the estimates*

$$\left\| \mathbb{1}_{([0,1]^2+x)\times[t,t+h]} \left(\mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu] \right) \right\|_p^p \leq Ch, \tag{5.18}$$

$$\left\| \mathbb{1}_{([0,1]^2+x)\times[t,t+h]} \left(\mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^\parallel)^2 \leq \mu] \right) \mathbb{1}_{(\Upsilon\times I)^\complement} \right\|_p^p \leq Ch \exp\left(-p\delta^2/18\right) + C\ln(h). \tag{5.19}$$

Proof. For the first inequality, we use (5.2) and Proposition 5.2. For the second inequality, we first observe

$$(\Upsilon \times I)^\complement = (\Upsilon^\complement \times I) \cup (\mathbb{R}^2 \times I^\complement) \subset (\Upsilon^\complement \times \mathbb{R}) \cup (\mathbb{R}^2 \times I^\complement) \tag{5.20}$$

and then we use the p-triangle inequality, (5.2) and Proposition 5.2. □

We now fix a region $\Lambda \subset \mathbb{R}^3$ and define the signed distance function

$$\mathrm{d}_\Lambda(x) := \begin{cases} +\operatorname{dist}(x, \partial\Lambda) & \text{for } x \notin \Lambda \\ -\operatorname{dist}(x, \partial\Lambda) & \text{for } x \in \Lambda \end{cases}, \tag{5.21}$$

where dist is the Euclidean distance. The signed distance function is Lipschitz-continuous with Lipschitz constant 1.

In order to utilize Lemma 5.3, we need to essentially cover $L\Lambda$ with a lot of very long boxes (of dimensions $1 \times 1 \times O(L)$). This boils down to choosing appropriate intervals that cover most of Λ_x (as defined in Definition 3.1), for any $x \in \mathbb{R}^2$. Let $G(x, \varepsilon)$ be the number of these intervals. The following lemma explicitly constructs such intervals and lists the properties that $G(x, \varepsilon)$ and the intervals satisfy, which we need for our estimates. The basic idea is to collect connected components of Λ_x, which go sufficiently deep inside Λ.

Lemma 5.4. *For any $x \in \mathbb{R}^2$ and $\varepsilon > 0$, there is a finite (possibly empty) set of intervals $A(x, \varepsilon) = \{I_{1,x,\varepsilon}, \ldots, I_{G(x,\varepsilon)),x,\varepsilon}\}$, satisfying the following conditions:*

(1) *We have $\mathrm{d}_\Lambda(I_{k,x,\varepsilon}) \subset (-\infty, -\varepsilon)$ and $\operatorname{dist}(I_{k,x,\varepsilon}, \partial\Lambda) = \varepsilon$.*
(2) *For any $\lambda \in \Lambda_x$, there exists a j with $1 \leq j \leq G(x, \varepsilon)$: $\lambda \in I_{j,x,\varepsilon}$, or $\mathrm{d}_\Lambda((x, \lambda)) > -2\varepsilon$.*
(3) *We have $G(x, \varepsilon) = \#A(x, \varepsilon) \leq \mathcal{H}^1\left(\mathrm{d}_\Lambda^{-1}((-2\varepsilon, -\varepsilon)) \cap (\{x\} \times \mathbb{R})\right)/\varepsilon$.*

The signed distance function d_Λ, dependent on the piecewise Lipschitz region Λ, is defined in (5.21).

We regard the lemma and its proof as the definitions of $A(x, \varepsilon), I_{j,x,\varepsilon}$ and $G(x, \varepsilon)$.

Proof. We consider the set $A_0(x, \varepsilon)$ of all connected components of $\left(\mathrm{d}_\Lambda^{-1}((-\infty, -\varepsilon))\right)_x \subset \mathbb{R}$ (with the convention that the empty set has no connected components). The set $A(x, \varepsilon)$ is defined as the set of all $I \in A_0(x, \varepsilon)$, such that there is a $\lambda \in I$ with $\mathrm{d}_\Lambda((x, \lambda)) \leq -2\varepsilon$. The first point is already satisfied for all $I \in A_0(x,\varepsilon)$ and thus holds for all I in the smaller set $A(x,\varepsilon)$. For the second claim, we observe that if $\lambda \in \Lambda_x$ with $\mathrm{d}_\Lambda((x,\lambda)) \le -2\varepsilon$, then $\lambda \in \left(\mathrm{d}_\Lambda^{-1}((-\infty,-\varepsilon))\right)_x$ and thus there is an $I \in A_0(x,\varepsilon)$ with $\lambda \in I$. By definition of $A(x,\varepsilon)$, this ensures $I \in A(x,\varepsilon)$.

For the third claim, if $A(x,\varepsilon) \neq \emptyset$, let $I = (\lambda_1,\lambda_4) \in A(x,\varepsilon)$ and define $\lambda_2 := \inf\{\lambda \in I\colon \mathrm{d}_\Lambda((x,\lambda) \le -2\varepsilon\}, \lambda_3 := \sup\{\lambda \in I\colon \mathrm{d}_\Lambda((x,\lambda) \le -2\varepsilon\}$. Thus, $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$,

$$\mathrm{d}_\Lambda(\{x\} \times (\lambda_1,\lambda_2)) = \mathrm{d}_\Lambda(\{x\} \times (\lambda_3,\lambda_4)) = (-2\varepsilon,-\varepsilon), \tag{5.22}$$

and, as d_Λ has Lipschitz constant 1, this means that

$$\mathcal{H}^1((\{x\} \times I) \cap \mathrm{d}_\Lambda^{-1}((-2\varepsilon,-\varepsilon))) \ge \mathcal{H}^1(\{x\} \times ((\lambda_1,\lambda_2) \cup (\lambda_3,\lambda_4))) \ge 2\varepsilon. \tag{5.23}$$

As $(\lambda_1,\lambda_2) \subset I$, $(\lambda_3,\lambda_4) \subset I$ and different elements of $A(x,\varepsilon)$ are disjoint (as they are connected components), we can sum the inequality over all elements of $A(x,\varepsilon)$ and arrive at

$$\mathcal{H}^1((\{x\} \times \mathbb{R}) \cap \mathrm{d}_\Lambda^{-1}((-2\varepsilon,-\varepsilon))) \ge \sum_{I \in A(x,\varepsilon)} \mathcal{H}^1((\{x\} \times I) \cap \mathrm{d}_\Lambda^{-1}((-2\varepsilon,\varepsilon))) \ge 2G(x,-\varepsilon)\varepsilon, \tag{5.24}$$

which implies the last claim (with an additional factor 1/2). $\square$

Theorem 5.5. *Let Λ be a piecewise Lipschitz region (see Definition 2.1) and let $0 < p \le 1, \ell \in \mathbb{N}, \mu \in \mathbb{R}^+$. Then, there are constants $L_0 = L_0(\Lambda,p,\ell,\mu) > 3$ and $C = C(\Lambda,p,\ell,\mu)$ such that for all $L > L_0$*

$$\left\| \mathbb{1}_{L\Lambda} \mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^{\parallel})^2 \le \mu] \mathbb{1}_{L\Lambda^\complement} \right\|_p^p \le C(\Lambda,p) L^2 \ln(L). \tag{5.25}$$

Proof. We want to cover most of $L\Lambda$ with translates of cubes $[0,1]^2 \times [0,h]$, where h grows like L and will use Lemma 5.3 on these.[3] We set $\delta := 6p^{-1/2}$ $\sqrt{\ln(L)}$. Let L_0 be large enough to ensure that $\delta \ge 1$ for $L > L_0$. Hence, these cubes need to keep a distance of at least δ from the boundary. Set $\varepsilon := \frac{2(\delta+1)}{L}$. We also define the shorthand

$$\mathrm{P} := \mathrm{P}_\ell \otimes \mathbb{1}[(-\mathrm{i}\nabla^{\parallel})^2 \le \mu]. \tag{5.26}$$

Let h_0 be the length of the longest straight line contained in Λ.

Consider any $x \in \mathbb{R}^2$ with $G(x,\varepsilon) \ge 1$, as defined in Lemma 5.4. For $k \in \{1,\ldots,G(x,\varepsilon)\}$, we define the boxes

$$Q_{x,k} := ([0,1]^2 + Lx) \times (LI_{k,x,\varepsilon}) \qquad \subset L\Lambda, \tag{5.27}$$

$$Q'_{x,k} := ([-\delta,1+\delta]^2 + Lx) \times (LI_{k,x,\varepsilon}) \qquad \subset L\Lambda. \tag{5.28}$$

These inclusions hold because $\sqrt{2} < \sqrt{2}(\delta+1) < L\varepsilon = L\,\mathrm{dist}(\{x\} \times I_{k,x,\varepsilon}, \partial\Lambda) = \mathrm{dist}(\{Lx\} \times (LI_{k,x,\varepsilon}), L\partial\Lambda)$.

[3] The choice of 18 in Proposition 5.2 makes the definition of δ quite nice to work in the estimate (5.31).

We assume $L > h_0$, $Lh_0 > 1$ and $L > 2$. Now we have by monotonicity I in Lemma 5.1 and by Lemma 5.3

$$\left\|\mathbb{1}_{Q_{x,k}}\mathrm{P}\mathbb{1}_{L\Lambda^\complement}\right\|_p^p \leq \left\|\mathbb{1}_{Q_{x,k}}\mathrm{P}\mathbb{1}_{(Q'_{x,k})^\complement}\right\|_p^p \tag{5.29}$$

$$\leq CL|I_{k,x,\varepsilon}|\exp\left(-p\delta^2/18\right) + C\ln(L|I_{k,x,\varepsilon}|) \tag{5.30}$$

$$\leq Ch_0L^{-1} + C\ln(L) + C\ln(h_0) \leq C\ln(L). \tag{5.31}$$

The constant C depends only on p and h_0.

Now we consider some offset parameter $s \in [0,1)^2$, and we define

$$\Lambda_{\varepsilon,s} := \Lambda \setminus \bigcup_{z\in\mathbb{Z}^2} \bigcup_{k=1}^{G\left(\frac{z+s}{L},\varepsilon\right)} \frac{1}{L} Q_{\frac{z+s}{L},k} \subset \mathrm{d}_\Lambda^{-1}\left((-3\varepsilon, 0)\right). \tag{5.32}$$

The inclusion is based on the fact that for each $y \in \mathbb{R}^3$, there is a $\frac{z+s}{L} \in \mathbb{R}^2$ with $z \in \mathbb{Z}^2$ such that $y \in \left(\frac{z+s}{L} + \frac{1}{L}[0,1]^2\right) \times \mathbb{R}$. If $\mathrm{d}_\Lambda(y) \leq -3\varepsilon$, then the point $\left(\frac{z+s}{L}, y_3\right)$ is at most $\frac{\sqrt{2}}{L} < \varepsilon$ away from y and hence at least 2ε away from the boundary. Therefore, there is a k such that $y \in \frac{1}{L}Q_{\frac{z+s}{L},k}$.

We further define

$$Z_\varepsilon := \left\{u \in \mathbb{Z}^3 \colon \left(u + [0,1]^3\right) \cap L\,\mathrm{d}_\Lambda^{-1}\left((-3\varepsilon, 0)\right) \neq \emptyset\right\}, \tag{5.33}$$

so that $L\,\mathrm{d}_\Lambda^{-1}\left((-3\varepsilon,0)\right) \subset \bigcup_{u\in Z_\varepsilon}\left(u + [0,1]^3\right)$. As $\varepsilon > \frac{\sqrt{3}}{L}$, the length of the diagonal in a cube $\frac{1}{L}[0,1]^3$, we have (second inclusion)

$$L\,\mathrm{d}_\Lambda^{-1}\left((-3\varepsilon,0)\right) \subset \bigcup_{u\in Z_\varepsilon}\left(u + [0,1]^3\right) \subset L\,\mathrm{d}_\Lambda^{-1}\left((-4\varepsilon,\varepsilon)\right). \tag{5.34}$$

Hence, the volume of the middle term, which is the cardinality, $\#Z_\varepsilon$, of Z_ε, can be bounded by the volume of the right-hand side. For $\varepsilon < 1$, using Lemma A.3, this is bounded by $L^3C(\Lambda)\varepsilon$. Hence for $L > L_0$:

$$\#Z_\varepsilon \leq L^3 C(\Lambda)\varepsilon \leq C(\Lambda,p)L^2\sqrt{\ln(L)}. \tag{5.35}$$

Using the monotonicity I and subadditivity properties in Lemma 5.1 and the covering (5.32), we can finally estimate,

$$\|\mathbb{1}_{L\Lambda}\mathrm{P}\mathbb{1}_{L\Lambda^\complement}\|_p^p \leq \sum_{z\in\mathbb{Z}^2}\sum_{k=1}^{G\left(\frac{z+s}{L},\varepsilon\right)} \left\|\mathbb{1}_{Q_{\frac{z+s}{L},k}}\mathrm{P}\mathbb{1}_{L\Lambda^\complement}\right\|_p^p + \left\|\mathbb{1}_{L\Lambda_{\varepsilon,s}}\mathrm{P}\mathbb{1}_{L\Lambda^\complement}\right\|_p^p. \tag{5.36}$$

The summands in the first sum can be bounded by $C\ln(L)$ using (5.31). The second term will be bounded using monotonicity I, (5.32) and (5.34) in the first step and using monotonicity II, subadditivity, (5.18) and (5.35) in the second step. Hence, we have

$$\|\mathbb{1}_{L\Lambda}\mathrm{P}\mathbb{1}_{L\Lambda^{\complement}}\|_p^p \leq \sum_{z\in\mathbb{Z}^2} G\left(\frac{z+s}{L},\varepsilon\right) C\ln(L) + \sum_{u\in Z_\varepsilon} \left\|\mathbb{1}_{[0,1]^3+u}\mathrm{P}\mathbb{1}_{L\Lambda^{\complement}}\right\|_p^p \tag{5.37}$$

$$\leq C\sum_{z\in\mathbb{Z}^2} G\left(\frac{z+s}{L},\varepsilon\right)\ln(L) + C(\Lambda,p)L^2\sqrt{\ln(L)}C. \tag{5.38}$$

For any fixed $L > L_0$ and $s \in [0,1)^2$, this is finite. Hence, we can integrate this over $s \in [0,1)^2$ and get a different upper bound. As the volume of $[0,1)^2$ is 1, the left-hand side and the last term do not change, as it is an integral over a constant in both cases.

$$\|\mathbb{1}_{L\Lambda}\mathrm{P}\mathbb{1}_{L\Lambda^{\complement}}\|_p^p \leq C\int_{[0,1)^2}\mathrm{d}s \sum_{z\in\mathbb{Z}^2} G\left(\frac{z+s}{L},\varepsilon\right)\ln(L) + C(\Lambda,p)L^2\sqrt{\ln(L)}. \tag{5.39}$$

Now we can use Fubini on the product $\mathbb{Z}^2 \times [0,1)^2 = \mathbb{R}^2$. Hence we have

$$\|\mathbb{1}_{L\Lambda}\mathrm{P}\mathbb{1}_{L\Lambda^{\complement}}\|_p^p \leq C\ln(L)\int_{\mathbb{R}^2} G\left(\frac{x}{L},\varepsilon\right)\mathrm{d}x + C(\Lambda,p)L^2\sqrt{\ln(L)} \tag{5.40}$$

$$= C\ln(L)L^2\int_{\mathbb{R}^2} G(x,\varepsilon)\,\mathrm{d}x + C(\Lambda,p)L^2\sqrt{\ln(L)} \tag{5.41}$$

$$\leq C\ln(L)L^2\int_{\mathbb{R}^2}\left|\left(\mathrm{d}_\Lambda^{-1}((-2\varepsilon,-\varepsilon))\right)_x\right|/\varepsilon\mathrm{d}x + C(\Lambda,p)L^2\sqrt{\ln(L)} \tag{5.42}$$

$$= C\ln(L)L^2|\mathrm{d}_\Lambda^{-1}((-2\varepsilon,-\varepsilon))|/\varepsilon + C(\Lambda,p)L^2\sqrt{\ln(L)} \leq C(\Lambda,p)L^2\ln(L). \tag{5.43}$$

In the first step, we did a change of variables, in the third step we used Lemma 5.4, in the last but one step Fubini and in in the final step we applied (A.9). □

6. The Error Term can be Large and not Smaller than $o(L^2\ln(L))$

Without loss of generality, we assume throughout this section that $\nu = 1$ and $B = 1$ because the precise values are not relevant now. The non-asymptotic bound in the following lemma is simple and useful in the proof of the main theorem in this section.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}$ be a finite union of intervals of finite lengths $\ell_1,\ldots,\ell_n$ with disjoint closures. Let $m \in \mathbb{N}$ with $m \geq 2$, $\mu > 0$, and $\Delta = \mathrm{d}^2/\mathrm{d}^2x$ the one-dimensional Laplacian. Then, we have the estimate*

$$\|(\mathbb{1}_\Omega\mathbb{1}(-\Delta\leq\mu)\mathbb{1}_\Omega)^m - \mathbb{1}_\Omega\mathbb{1}(-\Delta\leq\mu)\mathbb{1}_\Omega\|_1 \leq \frac{m-1}{\pi^2}\sum_{j=1}^n \ln(1+\sqrt{\mu}\ell_j) + Cmn, \tag{6.1}$$

where C is an entirely independent constant.

For $m = 2$, this estimate is sharp in the sense that the prefactor $1/\pi^2$ equals the coefficient of the leading asymptotic behavior of $\operatorname{tr}(\mathbb{1}_{L\Omega}\mathbb{1}(-\Delta \leq \mu)\mathbb{1}_{L\Omega})^2$ for large L.

Proof. By scaling we can assume $\mu = 1$ since $\mathbb{1}_\Omega \mathbb{1}(-\Delta \leq \mu)$ is unitarily equivalent to $\mathbb{1}_{\sqrt{\mu}\Omega}\mathbb{1}(-\Delta \leq 1)$. In other words, we may set $\mu = 1$ and eventually replace the lengths ℓ_j by $\sqrt{\mu}\ell_j$.

Then, we use the geometric series $a^m - a = a(a-1)(a^{m-2} + \cdots + a + 1)$ with $a := \mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_\Omega$. As a has on operator norm of at most 1, we can estimate

$$\begin{aligned}
\|(\mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_\Omega)^m &- \mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_\Omega\|_1 \\
&\leq (m-1)\|a(a-1)\|_1 \\
&= (m-1)\|\mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_{\Omega^\complement}\mathbb{1}(-\Delta \leq 1)\mathbb{1}_\Omega\|_1 \\
&= (m-1)\|\mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_{\Omega^\complement}\|_2^2 \\
&= (m-1)\int_\Omega \mathrm{d}x \int_{\Omega^\complement} \mathrm{d}y\, k(x-y)^2, \qquad (6.2)
\end{aligned}$$

with the function $k = k_1$ defined in (2.6).

For a fixed $x \in \Omega$, we now enlarge the domain of integration in y by allowing $y \in \Omega$, as long as x and y are in different intervals in Ω. In a formula, with $\pi_0(\Omega)$ denoting the connected components (subintervals) of Ω, the new domain of integration in (6.2) is

$$\bigcup_{I \in \pi_0(\Omega)} \big\{(x,y) \colon x \in I, y \notin I\big\}. \qquad (6.3)$$

As the integrand only depends on $x - y$, we may translate I to be of the form $(0, \ell_j)$. Hence, with $n := \#\pi_0(\Omega)$ the number of connected components of Ω, we have

$$\begin{aligned}
&\|(\mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_\Omega)^m - \mathbb{1}_\Omega \mathbb{1}(-\Delta \leq 1)\mathbb{1}_\Omega\|_1 && (6.4)\\
&\quad\leq (m-1)\sum_{j=1}^n \int_0^{\ell_j} \mathrm{d}x \int_{(0,\ell_j)^\complement} \mathrm{d}y\, k(x-y)^2 && (6.5)\\
&\quad= (m-1)\sum_{j=1}^n \|\mathbb{1}_{(0,\ell_j)}\mathbb{1}(-\Delta \leq 1)\mathbb{1}_{(0,\ell_j)^\complement}\|_2^2 && (6.6)\\
&\quad= (m-1)\sum_{j=1}^n \operatorname{tr}\Big[\mathbb{1}_{(0,\ell_j)}\mathbb{1}(-\Delta \leq 1)\mathbb{1}_{(0,\ell_j)} - \big(\mathbb{1}_{(0,\ell_j)}\mathbb{1}(-\Delta \leq 1)\mathbb{1}_{(0,\ell_j)}\big)^2\Big] && (6.7)\\
&\quad\leq (m-1)\sum_{j=1}^n \frac{1}{\pi^2}\ln(1+\ell_j) + Cmn. && (6.8)
\end{aligned}$$

The last step relies on an improved result of Landau and Widom with $L = 1$, see Corollary C.3. $\square$

In Theorem 2.3, we obtained for a general Lipschitz region $\Lambda \subset \mathbb{R}^3$ an error term $o(L^2 \ln(L))$ and not of the order L^2. Specifically, using $\mathsf{I}(2) = -1/(4\pi^2)$, we have the asymptotic expansion

$$\operatorname{tr}\left(\mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right) = \frac{L^2 \ln(L)}{4\pi^3} \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\, |n(v) \cdot e_3| + o(L^2 \ln(L)). \tag{6.9}$$

This allows us to define the error term $\varepsilon(L, \Lambda)$ by the identity

$$\operatorname{tr}\left(\mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right) = \frac{L^2 \ln(L)}{4\pi^3} \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\, |n(v) \cdot e_3| - L^2 \ln(L)\, \varepsilon(L, \Lambda). \tag{6.10}$$

In this notation, Theorem 2.3 states that $\lim_{L\to\infty} \varepsilon(L, \Lambda) = 0$ for a piecewise Lipschitz region Λ and we have $\sup_{L\geq 2} |\varepsilon(L, \Lambda)| \ln(L) < \infty$, if Λ is a piecewise $\mathsf{C}^{1,\alpha}$ region. The main result of this section, which is the next theorem, shows that the estimate for Lipschitz regions is sharp and the error term can be large and just $o(L^2 \ln(L))$. The negative sign in front of the error term does not necessarily mean that it has a definite sign although in our example it will be. Although our result only deals with the error term for the simplest, non-trivial polynomial, namely $t \mapsto t(1-t)$, we believe that also for the entropy the error term can be as large and only $o(L^2 \ln(L))$ for a Lipschitz region.

Theorem 6.2. *Let $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded function with $\lim_{L\to\infty} \varphi(L) = 0$. Then, there is a piecewise Lipschitz region Λ and an L_0 such that for any $L \geq L_0$, the error term defined in (6.10) satisfies*

$$\varepsilon(L, \Lambda) \geq \varphi(L). \tag{6.11}$$

Remark 6.3. Let $\mathcal{A}$ be the subset of the space of all polynomials vanishing at 0 and 1 such that the error term in Theorem 2.3 for $f \in \mathcal{A}$ is of order $O(L^2)$ for any Lipschitz domain Λ. This is clearly a linear subspace and the theorem tells us that $t \mapsto t(1-t) \notin \mathcal{A}$. Thus, the subspace has at least codimension one, which means that it satisfies (at least) one linear constraint. We conjecture that this constraint might be $f \in \mathcal{A} \implies \mathsf{I}(f) = 0$. That is, the error term can only achieve the order $O(L^2)$, if the leading term of order $L^2 \ln(L)$ vanishes.

Proof of Theorem 6.2. We begin with a non-negative, summable sequence $(a_i)_{i\in\mathbb{N}}$ with $\sum_{i\in\mathbb{N}} a_i = 1$, which we will choose later. Let $g_0 : [0, 1] \to \mathbb{R}^+$ be the zigzag function defined by $g_0(0) = 1$ and for $t > 0$,

$$g_0'(t) = \begin{cases} +1 & \text{if } \exists j \in \mathbb{N}\colon 0 < t - \sum_{i<j} a_i \leq \frac{1}{2} a_j \\ -1 & \text{if } \exists j \in \mathbb{N}\colon \frac{1}{2} a_j < t - \sum_{i<j} a_i \leq a_j \end{cases}. \tag{6.12}$$

If $j = 1$, then we use the convention that $\sum_{j<1} a_j := 0$. Clearly, g_0 is Lipschitz continuous with Lipschitz bound 1. We expand g_0 to $[-1, 2]$ by setting $g_0(t) = t + 1$ for $t < 0$ and $g_0(t) = 2 - t$ for $t > 1$. This extension is still Lipschitz continuous and satisfies $g_0(-1) = g_0(2) = 0$. Now, we can define the region Λ,

$$\Lambda := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1 \in (0, 1), x_3 \in (-1, 2), -g_0(x_3) < x_2 < g_0(x_3)\right\}. \tag{6.13}$$

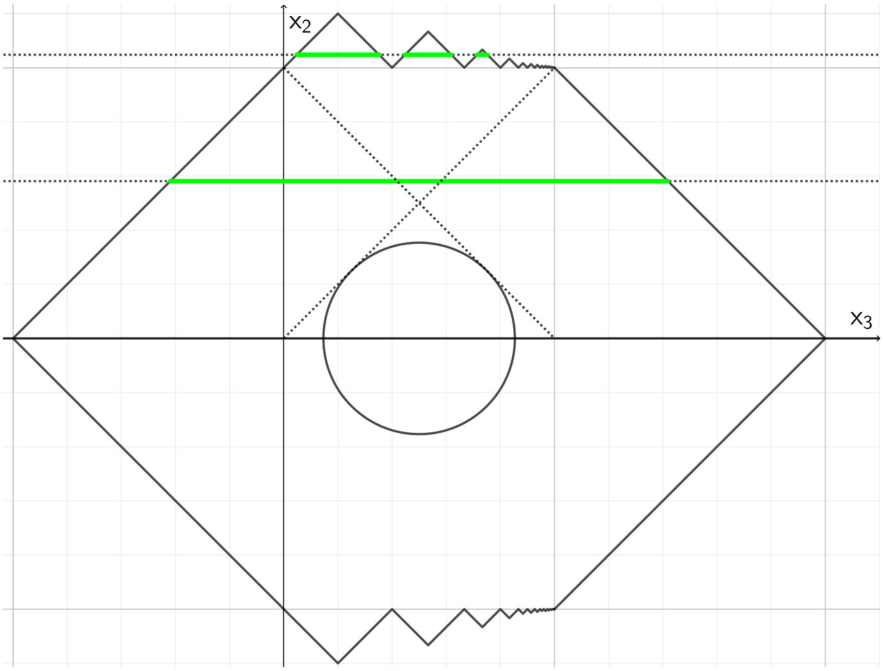

FIGURE 1. Example of a x_2-x_3-plot of the domain Λ for any $x_1 \in (0,1)$ and some sequence $(a_i)_{i\in\mathbb{N}}$. The upper half is the graph of g_0. In green, one can see two sets $\Lambda_{x^\perp}$. In the middle, one can see the ball of all points, with respect to which Λ is star-shaped

This clearly defines a piecewise Lipschitz region. We will now sketch why this is even a strong Lipschitz domain (See [2, Pages 66–67] for the definition.)

For any $x_0 \in B_{1/(2\sqrt{2})}(1/2, 0, 1/2)$, the region Λ is star shaped with respect to x_0. For the definition of a strong Lipschitz domain, we need to choose an open cover of $\partial\Lambda$ and a projection with a certain direction on every set of the cover. For any orthogonal (rank 2) projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$, on the two connected components of the set $\pi^{-1}(\pi(B_{1/(4\sqrt{2})}(1/2, 0, 1/2))) \cap \partial\Lambda$, one can define the chart as the inverse of π, which has a Lipschitz constant less than 10. This leads to an open cover of $\partial\Lambda$ and one can then choose a finite subcover.

The boundary $\partial\Lambda$ can be covered by the sets $\partial_1\Lambda := \big\{x \in \partial\Lambda\colon x_1 \in \{0,1\}\big\}$ and $\partial_2\Lambda := \big\{x \in \partial\Lambda\colon x_1 \in [0,1]\big\}$. These two boundary sets have a non-empty intersection, but $\partial_1\Lambda \cap \partial_2\Lambda$ is a "one-dimensional" set with two-dimensional Hausdorff measure zero, that is, $\mathcal{H}^2(\partial_1\Lambda \cap \partial_2\Lambda) = 0$.

For almost every $x \in \partial_1\Lambda$, the outward normal vector $n(x)$ is given by $\pm e_1$, while for almost every $x \in \partial_2\Lambda$, the outward normal vector is given by $\frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)$; the vectors e_1, e_2, e_3 are the usual unit vectors in the positive

x_1, x_2, x_3 directions. Hence, we observe

$$\mathcal{H}^2(\partial_1\Lambda) = 4\int_{-1}^{2} g_0(t)\,\mathrm{d}t \le 9, \tag{6.14}$$

$$\mathcal{H}^2(\partial_2\Lambda) = 2\int_{-1}^{2} \sqrt{1+(g_0')^2(t)}\,\mathrm{d}t = 6\sqrt{2}, \tag{6.15}$$

$$\int_{\partial\Lambda} |n(x)\cdot e_3|\,\mathrm{d}\mathcal{H}^2(x) = \frac{1}{\sqrt{2}}\mathcal{H}^2(\partial_2\Lambda) = 6. \tag{6.16}$$

It is important that $\mathcal{H}^2(\partial\Lambda) = \mathcal{H}^2(\partial_1\Lambda) + \mathcal{H}^2(\partial_2\Lambda)$ is bounded independently of the sequence $(a_i)_{i\in\mathbb{N}}$ and that the surface integral in (6.16) is completely independent of the sequence.

The leading asymptotic term for the trace on the left-hand side of (6.10) is provided by Theorem 2.3. Here, $\mathsf{I}(2) = -1/(4\pi^2)$ and hence

$$\operatorname{tr}\left(\mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right) = \frac{L^2\ln(L)}{4\pi^3}\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\,|n(v)\cdot e_3| + o(L^2\ln(L)) \tag{6.17}$$

$$= \frac{L^2\ln(L)}{\pi^3}\,\frac{3}{2} + o(L^2\ln(L)), \tag{6.18}$$

where we used (6.16).

We need an upper bound for the constant $\mathcal{K}(\Lambda)$ defined in Lemma A.3, which is independent of the function g. We observe that $\partial_1\Lambda$ is the image of the Lipschitz functions $f_j\colon [0,1]^2 \to \partial_1\Lambda; (x_1,x_2) \mapsto (j, 3x_1-1, g(3x_1-1)x_2)$ for $j=0,1$ and $\partial_2\Lambda$ is in the image of the Lipschitz functions $\tilde{f}_\pm\colon [0,1]^2 \to \partial_2\Lambda; (x_1,x_2)\mapsto (x_1, 3x_2-1, \pm g(3x_2-1))$. Thus, the set $\{f_0, f_1\tilde{f}_+, \tilde{f}_-\}$ defines a piecewise Lipschitz atlas of $\partial\Lambda$. Hence, as $C_{\mathrm{lip}}(f_j) = 3\sqrt{2}$ and $C_{\mathrm{lip}}(\tilde{f}_\pm) = 3$, we observe

$$\mathcal{K}(\Lambda) \le (16\sqrt{2})^2 \times 2 \times (1 + (3\sqrt{2})^2 + 1 + 3^2) < \infty. \tag{6.19}$$

Hence, by Lemma 3.2, we have

$$\operatorname{tr}\mathrm{D}_\mu(L\Lambda)^m = \frac{L^2}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}x^\perp\ \operatorname{tr}\left(1_{L\Lambda_{x^\perp}} 1[(-\mathrm{i}\nabla^\parallel)^2 \le \mu]\, 1_{L\Lambda_{x^\perp}}\right)^m + O(L^2) \tag{6.20}$$

with $x^\perp = (x_1, x_2)$. To get to the polynomial $f(t) = t(1-t)$ we have to subtract this term with $m=2$ from the term with $m=1$. Now, we intend to use Lemma 6.1. To do so, we need to describe the lengths of the (sub)intervals of $\Lambda_{x^\perp}$ depending on $x^\perp = (x_1,x_2)$.

We can ignore the case $x_1 \in \{0,1\}$, as this is a null set with respect to the Lebesgue measure on $\mathbb{R}^2$. If $x_1\in(0,1)$ and $|x_2|\le 1$ then the set $\Lambda_{x^\perp}$ is a single interval of length $\ell_1(x_2) = 3-2|x_2|$. The interesting case is $x_1\in(0,1)$ and $1<|x_2|<2$. Here, for any $i\in\mathbb{N}$ with $a_i > 2(|x_2|-1)$, there is an interval of size $\ell_i(x_2) = a_i - 2(|x_2|-1)$, as illustrated in Equation (1). For any $|x_2|>1$, this will only lead to finitely many intervals, as the sequence $(a_i)_{i\in\mathbb{N}}$ is a null sequence. Now, we apply (6.20) for $m = 1$ and $m = 2$, and then Lemma 6.1 and see that

$$\frac{2\pi}{L^2}\Big|\mathrm{tr}\left(\mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right)\Big| \tag{6.21}$$

$$\leq C + \int_0^1 \mathrm{d}x_1 \int_{\mathbb{R}} \mathrm{d}x_2 \, \|(\mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}(-\Delta \leq \mu)\mathbb{1}_{L\Lambda_{x^\perp}})^2 - \mathbb{1}_{L\Lambda_{x^\perp}} \mathbb{1}(-\Delta \leq \mu)\mathbb{1}_{L\Lambda_{x^\perp}}\|_1 \tag{6.22}$$

$$\leq \frac{2}{\pi^2} \int_0^1 \mathrm{d}x_2 \left(\ln(1 + L\sqrt{\mu}(3 - 2x_2)) + C\right) \tag{6.23}$$

$$+ \frac{2}{\pi^2} \int_1^2 \mathrm{d}x_2 \sum_{i \in \mathbb{N}:\ a_i > 2(x_2 - 1)} \left(\ln(1 + L\sqrt{\mu}(a_i - 2(x_2 - 1)) + C\right) \tag{6.24}$$

$$= \frac{2}{\pi^2} \int_0^1 \mathrm{d}x_2 \Big(\ln(1 + L\sqrt{\mu}(3 - 2x_2)) + C\Big)$$

$$+ \frac{2}{\pi^2} \sum_{i \in \mathbb{N}} \int_0^{\frac{1}{2}a_i} \mathrm{d}t \Big(\ln(1 + L\sqrt{\mu}2t) + C\Big). \tag{6.25}$$

In the second step, we also used that the set $\Lambda_{x^\perp}$ is independent of $x_1 \in (0, 1)$. The third step uses Fubini to exchange the sum and the integral and then transforms the integration variable to $t := \frac{1}{2}a_i + 1 - x_2$. The lower bound 0 in the last integral stems from the condition $a_i > 2(x_2 - 1)$, respectively, from $t > 0$.

We intend to show that this upper bound is significantly smaller than the known asymptotics. The difference between the asymptotics and this upper bound can then be used as a lower bound for the error term. This is why it is very important that the coefficient in front of the upper bound is equal to the asymptotic coefficient and is thus the reason why we can only do this here for the polynomial $f(t) = t(1 - t)$.

We now allow our constants to depend on μ (for general $\nu \geq 1$, they depend on all values of $\mu(\ell)$) and use the trivial inequality $\ln(1 + ab) \leq \ln(1 + a) + \ln(1 + b)$ for $a, b \geq 0$ to arrive at

$$\frac{\pi^3}{L^2}\left|\mathrm{tr}\, \mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right| \tag{6.26}$$

$$\leq \int_0^1 (\ln(1 + L) + C)\, \mathrm{d}x_2 + \sum_{i \in \mathbb{N}} \Big[\int_0^{\frac{1}{2}a_i} \ln(1 + Lt)\, \mathrm{d}t + Ca_i\Big] + C \tag{6.27}$$

$$= \ln(1 + L) + \frac{1}{L} \sum_{i \in \mathbb{N}} \Big(\Big(1 + \frac{1}{2}a_iL\Big)\Big(\ln\Big(1 + \frac{1}{2}a_iL\Big) - 1\Big) - (-1)\Big) + C \tag{6.28}$$

$$= \ln(1 + L) + \sum_{i \in \mathbb{N}} \Big[\frac{1}{2}a_i \ln\Big(1 + \frac{1}{2}a_iL\Big) + \frac{\ln(1 + \frac{1}{2}a_iL)}{L} - \frac{1}{2}a_i\Big] + C \tag{6.29}$$

$$\leq \ln(L) + \sum_{i \in \mathbb{N}} \frac{1}{2}a_i \ln\Big(1 + \frac{1}{2}a_iL\Big) + C, \tag{6.30}$$

or equivalently,

$$\begin{aligned}&\big(0 \leq \big) \operatorname{tr}\left(\mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right)\\ &\quad \leq \frac{L^2 \ln(L)}{\pi^3}\left[1 + \frac{1}{\ln(L)}\sum_{i\in\mathbb{N}}\frac{1}{2}a_i \ln\left(1+\frac{1}{2}a_i L\right) + \frac{C}{\ln(L)}\right].\end{aligned} \tag{6.31}$$

In the first step, we used $1 \leq 3 - x_2 \leq 3$, and in the fourth step, we used $\ln(1+L) \leq \ln(L)+1$ for $L \geq 1$ and $\ln(1+\frac{1}{2}a_i L) \leq \frac{1}{2}a_i L$.

Now we rewrite (6.10) and use (6.16) and (6.31) to obtain

$$2\pi^3\varepsilon(L,\Lambda) = \frac{1}{2}\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\,|n(v)\cdot e_3| - \frac{2\pi^3}{L^2\ln(L)}\operatorname{tr}\left(\mathrm{D}_\mu(L\Lambda) - \mathrm{D}_\mu(L\Lambda)^2\right) \tag{6.32}$$

$$\geq 3 - \left(2 + \frac{1}{\ln(L)}\sum_{i\in\mathbb{N}} a_i \ln\left(1+\frac{1}{2}a_i L\right) + \frac{2C}{\ln(L)}\right) \tag{6.33}$$

$$= 1 - \frac{1}{\ln(L)}\sum_{i\in\mathbb{N}} a_i \ln\left(1+\frac{1}{2}a_i L\right) - \frac{C}{\ln(L)} \tag{6.34}$$

$$= -\frac{1}{\ln(L)}\sum_{i\in\mathbb{N}} a_i \ln\left(\frac{1}{L}+\frac{1}{2}a_i\right) - \frac{C}{\ln(L)} \tag{6.35}$$

$$\geq -\frac{1}{\ln(L)}\sum_{i\in\mathbb{N}, a_i<\frac{1}{L}} a_i \ln\left(\frac{3}{2L}\right) - \frac{C}{\ln(L)} \tag{6.36}$$

$$\geq \sum_{i\in\mathbb{N}, a_i<\frac{1}{L}} a_i - \frac{C}{\ln(L)} =: \varepsilon_0(L). \tag{6.37}$$

The fourth step uses $\sum_i a_i = 1$ and the fifth step relies on $L \geq 2, a_i \leq 1$ to get $\ln(\frac{1}{L}+\frac{1}{2}a_i) \leq 0$. In the last step, C changed. Now, we just need to find a good sequence $(a_i)_{i\in\mathbb{N}}$. To show our claim, it suffices to find a sequence $(a_i)_{i\in\mathbb{N}}$ such that

$$\lim_{L\to\infty} \varphi(L)/\varepsilon_0(L) = 0, \tag{6.38}$$

since then the quotient $\varphi(L)/\varepsilon(L,\Lambda) \leq 2\pi^3\varphi(L)/\varepsilon_0(L) \to 0$ is less than 1 for large L, that is, $\varepsilon(L,\Lambda) \geq \varphi(L) \geq 0$ for $L \geq L_0$, where L_0 is chosen below.

The construction of the sequence a_i relies on Lemma D.1, and we apply this Lemma with f as φ. With the resulting function $\operatorname{Env}(\varphi)$ we define the sequence of real numbers $a_i := \operatorname{Env}(\varphi)(i-1) - \operatorname{Env}(\varphi)(i)$ for $i \in \mathbb{N}$. As $\lim_{L\to\infty}\operatorname{Env}(\varphi)(L) = 0$, we have $\sum_{i\geq L} a_i = \operatorname{Env}(\varphi)(L)$ and in particular $\sum_{i\in\mathbb{N}} a_i = \operatorname{Env}(\varphi)(0) = 1$. As $\operatorname{Env}(\varphi)$ is non-increasing and convex, the a_i are non-negative and non-increasing. As the sequence defined this way is non-increasing and $\sum_{i\in\mathbb{N}} a_i = 1$, we have $a_i \leq \frac{1}{i}\sum_{j\leq i} a_j \leq \frac{1}{i}$. Hence, we know that $i \geq L$ implies $a_i \leq \frac{1}{L}$. Thus, we have the estimate

$$\sum_{i\in\mathbb{N}, a_i\leq\frac{1}{L}} a_i \geq \sum_{i\geq L} a_i = \operatorname{Env}(\varphi)(L). \tag{6.39}$$

Furthermore, as $\mathrm{Env}(\varphi)(L) \geq C/\sqrt{\ln(2+L)}$, for L large enough, we have

$$\mathrm{Env}(\varphi)(L) - \frac{C}{\ln(L)} \geq \frac{1}{2}\mathrm{Env}(\varphi)(L). \tag{6.40}$$

Hence, we conclude that

$$0 \leq \lim_{L\to\infty} \frac{\varphi(L)}{\varepsilon_0(L)} = \lim_{L\to\infty} \frac{\varphi(L)}{\sum_{i\in\mathbb{N},a_i\leq\frac{1}{L}} a_i - \frac{C}{\ln(L)}} \leq 2 \lim_{L\to\infty} \frac{\varphi(L)}{\mathrm{Env}(\varphi)(L)} = 0. \tag{6.41}$$

One choice of L_0 could be that $4\varphi(L) \leq \mathrm{Env}(\varphi)(L)$ for $L > L_0$ is satisfied. Thus, by this and (6.37), there is an $L_0 > 0$ such that for any $L > L_0$, we have

$$\varepsilon(L,\Lambda) \geq \varepsilon_0(L)/(2\pi^3) \geq \varphi(L). \tag{6.42}$$

This finishes the proof. □

Acknowledgements

We are grateful to the referees for a careful reading of the manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL.

Appendix A. Some Geometric Results

Here, we assemble a few geometric statements that we used.

Lemma A.1. *Let $\Lambda \subset \mathbb{R}^{d+1}$ be a piecewise $\mathsf{C}^{1,\alpha}$ region for some $0 < \alpha < 1$. Let $(\Psi_{\mathrm{pC},i})_{i\in I}$ be a piecewise $\mathsf{C}^{1,\alpha}$ atlas of $\partial\Lambda$ and Γ as defined in Definition 2.1. Then, there is a constant C depending only on Λ such that for all unequal v_1 and v_2 in $\partial\Lambda$, we have*

$$\|v_1 - v_2\| \geq C \min\left\{ \left| n(v_1) \cdot \frac{v_1 - v_2}{\|v_1 - v_2\|} \right|^{\frac{1}{\alpha}}, \mathrm{dist}(v_1,\Gamma) \right\}. \tag{A.1}$$

Remark A.2. The normal vector $n(v_1)$ is well defined if $v_1 \notin \Gamma$. In the case $v_1 \in \Gamma$, the minimum on the right-hand side is meant to be $0 = \operatorname{dist}(v_1, \Gamma)$, which turns it into a trivial statement.

Proof. We begin with the case $v_1 \in \Gamma$ or $v_2 \in \Gamma$; $v_1 \in \Gamma$ is explained in the above remark. If $v_2 \in \Gamma$, then we trivially have $\|v_1 - v_2\| \geq \operatorname{dist}(v_1, \Gamma)$ and thus the claim holds for any $C \leq 1$.

Let us now consider the case that there is an $i \in I$ such that both v_1 and v_2 are in $\Psi_{\mathrm{pC},i}((0,1)^d)$. As $\Psi := \Psi_{\mathrm{pC},i}$ is injective, there are unique $x_k \in (0,1)^d$ such that $\Psi(x_k) = v_k \in \mathbb{R}^{d+1}$ for $k = 1, 2$. We observe that $n(v_1) \cdot D\Psi(x_1) = 0 \in \mathbb{R}^d$, as the image of the matrix $D\Psi(x_1)$ is the tangent space to $\partial\Lambda$ at v_1 and hence is orthogonal to the outward normal vector $n(v_1)$. Thus, using (2.13), we see

$$\begin{aligned} \big|n(v_1) \cdot (v_1 - v_2)\big| &= \big|n(v_1) \cdot \big(\Psi(x_1) - \Psi(x_2)\big)\big| && \text{(A.2)} \\ &\leq \|x_1 - x_2\| \sup_{t \in [0,1]} \big|n(v_1) \cdot D\Psi(tx_1 + (1-t)x_2)\big| && \text{(A.3)} \\ &= \|x_1 - x_2\| \sup_{t \in [0,1]} \big|n(v_1) \cdot (D\Psi(tx_1 + (1-t)x_2) - D\Psi(x_1)\big| && \text{(A.4)} \\ &\leq \|x_1 - x_2\|\, C\|x_1 - x_2\|^\alpha = C\|x_1 - x_2\|^{1+\alpha}. && \text{(A.5)} \end{aligned}$$

As Ψ is bi-Lipschitz, we know that $\|v_1 - v_2\| = \|\Psi(x_1) - \Psi(x_2)\| \geq C\|x_1 - x_2\|$. Using this and dividing both sides by $\|v_1 - v_2\|$, we arrive at

$$\left|n(v_1) \cdot \frac{v_1 - v_2}{\|v_1 - v_2\|}\right| \leq C\|v_1 - v_2\|^\alpha. \tag{A.6}$$

We are now in the remaining case that v_1 and v_2 lie in different sets $\Psi_{\mathrm{pC},i}((0,1)^d)$ since $\partial\Lambda = \Gamma \cup \bigcup_{i \in I} \Psi_{\mathrm{pC},i}((0,1)^d)$.

Let $(\Psi_{\mathrm{gL},j})_{j \in J}$ be a global Lipschitz atlas of $\partial\Lambda$. As the boundary $\partial\Lambda = \bigcup_{j \in J} \Psi_{\mathrm{gL},j}((0,1)^d)$ is a cover by (relatively) open sets and $\partial\Lambda$ is a compact metric space, by Lebesgue's number lemma, there is an $\varepsilon > 0$ such that for all $v \in \partial\Lambda$ there is an $j \in J$ with $B_\varepsilon(v) \cap \partial\Lambda \subset \Psi_{\mathrm{gL},j}((0,1)^d)$, where $B_\varepsilon(v) \subset \mathbb{R}^{d+1}$ is the open ball of radius ε at v.

If $\|v_1 - v_2\| \geq \varepsilon$, we can choose $C = \varepsilon$ to get the statement, as the first expression inside the minimum is at most 1.

Hence, we are left with the case $\|v_1 - v_2\| < \varepsilon$. Now, we get an $j \in J$ such that $v_1, v_2 \in \Psi_{\mathrm{gL},j}((0,1)^d)$. Again, we define y_k by $\Psi_{\mathrm{gL},j}(y_k) = v_k$ for $k = 1, 2$. The image γ of the linear path from y_1 to y_2 is at most $C\|y_1 - y_2\| \leq C\|v_1 - v_2\|$ long. As v_1 and v_2 are in the images of two different $\Psi_{\mathrm{pC},i}$'s, the path γ has to intersect some edge $\Psi_{\mathrm{pC},i}(\partial(0,1)^d)$ which implies $\gamma \cap \Gamma \neq \emptyset$. Hence, we have

$$\operatorname{dist}(v_1, \Gamma) \leq \mathcal{H}^1(\gamma) \leq C\|y_1 - y_2\| \leq C\|v_1 - v_2\|. \tag{A.7}$$

The last inequality follows since $\Psi_{\mathrm{gL},j}$ is bi-Lipschitz. This finishes the proof. $\square$

Lemma A.3. *For $d \geq 1$, let $f\colon [0,1]^d \to \mathbb{R}^{d+1}$ be a Lipschitz continuous function with Lipschitz constant $C_{\mathrm{lip}}(f)$ and let $\Lambda \subset \mathbb{R}^{d+1}$ be a piecewise Lipschitz* *region. Then, for any $r > 0$, the $(d+1)$-dimensional Lebesgue volume of the r-neighborhood (see (2.1)) of the set $f([0,1]^d)$ in $\mathbb{R}^{d+1}$ satisfies*

$$|B_r(f([0,1]^d))| \le (16\sqrt{d})^d(C_{\rm lip}(f)^d + 1)(r + r^{d+1}), \tag{A.8}$$

and the set $\partial\Lambda$ satisfies the bounds

$$|B_r(\partial\Lambda)| \le \mathcal{K}(\Lambda)(r + r^{d+1}), \tag{A.9}$$

$$\mathcal{H}^d(\partial\Lambda) \le \mathcal{K}(\Lambda), \tag{A.10}$$

where $\mathcal{K}(\Lambda)$ is described as follows: Let $\mathcal{A}$ be the set of all piecewise Lipschitz atlases of $\partial\Lambda$, as defined in Definition 2.1. Then, we define

$$\mathcal{K}(\Lambda) := \inf_{(\Psi_{{\rm pL},i})_{i\in I}\in\mathcal{A}} \sum_{i\in I} (16\sqrt{d})^d(C_{\rm lip}(\Psi_{{\rm pL},i})^d + 1). \tag{A.11}$$

Proof. We consider the set

$$A_r := \left(\frac{r}{C_{\rm lip}(f)\sqrt{d}}\mathbb{Z}\right)^d \cap [0,1]^d. \tag{A.12}$$

The maximum distance a point in $[0,1]^d$ can have from A_r is less than $\frac{r}{C_{\rm lip}(f)}$. For the cardinality $\#A_r$ of A_r, we observe

$$\begin{aligned}\#A_r &\le \left(1 + \frac{C_{\rm lip}(f)\sqrt{d}}{r}\right)^d \le 2^{d-1}\left(1 + \frac{C_{\rm lip}(f)^d\sqrt{d}^d}{r^d}\right)\\ &\le 2^{d-1}\sqrt{d}^d(1 + C_{\rm lip}(f)^d)(1 + r^{-d}).\end{aligned} \tag{A.13}$$

For any $x \in [0,1]^d$, there is a $z \in A_r$ such that $\|x - z\| \le r/C_{\rm lip}(f)$ and thus $\|f(x) - f(z)\| \le r$. This implies $B_r(f(A_r)) \supset f([0,1]^d)$, which leads to $B_{2r}(f(A_r)) \supset B_r(f([0,1]^d))$. Hence, we get

$$\begin{aligned}|B_r(f([0,1]^d))| &\le |B_{2r}(f(A_r))| \le |B_1(0)|\,\#A_r(2r)^{d+1} \le 4^{d+1}\,\#A_r r^{d+1}\\ &\le (16\sqrt{d})^d(1 + C_{\rm lip}(f)^d)(r + r^{d+1}).\end{aligned}$$

This finishes the proof of the first statement. The second statement is trivially implied by the first one. Furthermore, as $B_r(f(A_r)) \supset f([0,1]^d)$ due to the definition of the Hausdorff measure (see, e.g., [9, Definition 2.1]) we observe

$$\begin{aligned}\mathcal{H}^d(f([0,1]^d) &\le \lim_{r\to 0}|B_1^{(d)}(0)|\#A_r r^d \le \lim_{r\to 0}(4\sqrt{d})^d(1 + C_{\rm lip}(f)^d)(1 + r^{-d})r^d\\ &= (4\sqrt{d})^d(1 + C_{\rm lip}(f)^d)\,.\end{aligned}$$

The final statement is a corollary of this inequality. We want to note that $\mathcal{K}(\Lambda) < \infty$ for any piecewise Lipschitz region Λ, as we require in this paper our atlases to be a finite collection of charts. □

Lemma A.4. *Let $\Lambda \subset \mathbb{R}^3$ be a piecewise Lipschitz region with piecewise Lipschitz atlas $(\Psi_{{\rm pL},i})_{i\in I}$. Let $v_0 \in \partial\Lambda$ satisfy that there are $i_0 \in I$ and $x_0 \in (0,1)^2$ such that $\Psi_{{\rm pL},i_0}(x_0) = v_0$ and the Jacobi matrix $D\Psi_{{\rm pL},i_0}(x_0)$*

exists. Then, the signed distance function d_Λ *is differentiable at* v_0*, the outward unit normal vector* $n(v_0)$ *is well-defined, orthogonal to the image of* $D\Psi_{\mathrm{pL},i_0}(x_0)$*, and* $D\,\mathrm{d}_\Lambda(v_0) = n(v_0)$*.*

To prove this statement, we need the following result from intersection theory.

Lemma A.5. *Let* $R > 0$*,* $f_1\colon [-1,1] \to \overline{B}_R^{(3)}(0), f_2\colon \overline{B}_R^{(2)}(0) \to \overline{B}_R^{(3)}(0)$ *be continuous functions such that* $f_1(\pm 1) = \pm R e_3$ *and* f_2 *restricted to the boundary is the equatorial embedding, that is,* $f_2(x) = (x,0)$ *for* $\|x\| = R$*. Then, the images of* f_1 *and* f_2 *intersect.*

Proof. Without loss of generality, we assume $R = 1$. Assume f_1 and f_2 were two such functions such that their images do not intersect. Let $\eta_1\colon \mathbb{R}^1 \to \mathbb{R}^3, t \mapsto (0,0,t)$ and $\eta_2\colon \mathbb{R}^2 \to \mathbb{R}^3, x \mapsto (x,0)$ be the natural orthogonal inclusions. The assumptions on f_j can now be stated as $f_j(x_j) = \eta_j(x_j)$ for $x_j \in \mathbb{R}^j$ with $\|x_j\| = 1$ for $j \in \{1,2\}$. We extend the maps f_j to the $\mathbb{R}^j$ by setting

$$f_j(x_j) := \begin{cases} f_j(x_j) & \text{if } \|x_j\| \le 1 \\ \eta_j(x_j) & \text{if } \|x_j\| > 1 \end{cases}, \quad x_j \in \mathbb{R}^j,\ j \in \{1,2\}. \tag{A.14}$$

Trivially, these extensions are still continuous and their images still do not intersect. As the images do not intersect and only get close to each other in the compact set $\overline{B}_1^{(3)}(0)$, they have a positive distance. We can now mollify f_j by convolution with an appropriately chosen, compactly supported smooth function to get $\hat{f}_j$ such that the images of $\hat{f}_1$ and $\hat{f}_2$ still have positive distance and $\hat{f}_j(x_j) = \eta_j(x_j)$ for any $x_j \in \mathbb{R}^j$ with $\|x_j\| \ge 2$.

For $d = 1,2,3$, consider the sphere $\mathbb{S}^d = \mathbb{R}^d \cup \{\infty\}$. With the charts $\mathrm{id}\colon \mathbb{R}^d \to \mathbb{S}^d, x \mapsto x$ and $\iota_d\colon \mathbb{R}^d \to \mathbb{S}^d, x \mapsto x/\|x\|^2, 0 \mapsto \infty$, it becomes a differentiable manifold. We now extend $\hat{f}_j$ to a function from $\mathbb{S}^j$ to $\mathbb{S}^3$ by setting $\hat{f}_j(\infty) := \infty$ for $j \in \{1,2\}$. The point ∞ is now an intersection point of $\hat{f}_1$ and $\hat{f}_2$. We want to show that the extended functions are still smooth and that they intersect transversely at ∞, see for instance [14, Page 113]. For $j \in \{1,2\}$ and $x_j \in \mathbb{R}^j$ with $\|x_j\| < \frac{1}{2}$, we observe

$$(\iota_3^{-1} \circ \hat{f}_j \circ \iota_j)(x_j) = (\iota_3^{-1} \circ \eta_j)(x_j \|x_j\|^{-2}) = \eta_j(x_j). \tag{A.15}$$

Thus, in the charts ι_j, ι_3 the maps $\hat{f}_j$ are linear and orthogonal at 0 (which corresponds to $\infty \in \mathbb{S}^j$). The maps $\hat{f}_1, \hat{f}_2$ are therefore smooth and intersect transversely at ∞. In conclusion, we have just constructed two smooth maps $\hat{f}_j\colon \mathbb{S}^j \to \mathbb{S}^3$, which intersect transversely and have a unique intersection point. Thus, their oriented intersection number is equal to the local intersection number at this intersection point, which is $+1$ or -1 (in fact, it is $+1$). However, both maps are contractible (homotopic to a constant map) and thus, as oriented intersection numbers are homotopy invariant (see [14, Page 115]), they should have intersection number 0. This is a contradiction. Hence, the assumption that f_1 and f_2 do not intersect was wrong. $\square$

Proof of Lemma A.4. Let C_{lip} be a bi-Lipschitz constant of Ψ_{pL,i_0}, as in footnote 1. Then, for any $x \in \mathbb{R}^2$ with $\|x\| = 1$, we observe $C_{\text{lip}}^{-1} \leq \|D\Psi_{\text{pL},i_0}(x_0)x\| \leq C_{\text{lip}}$. This means that the Jacobi matrix is invertible. Thus, using affine linear transformations on $\mathbb{R}^2$ and on $\mathbb{R}^3$, we can transform the function Ψ_{pL,i_0} into a function Ψ such that x_0, v_0 are mapped to 0[4] and the Jacobi matrix turns into the standard inclusion $J\colon \mathbb{R}^2 \to \mathbb{R}^3, x \mapsto (x, 0)$. The function Ψ is now defined on some closed parallelogram P containing 0 in its interior P^{int}. Let $C_{\text{lip}} \geq 2$ be a bi-Lipschitz constant for Ψ. Let $0 < \varepsilon < 1/2$. Then, there is an $r > 0$ such that

- For any $x \in B^{(2)}_{2C_{\text{lip}}r}(0)$, we have $\|\Psi(x) - (x, 0)\| \leq \varepsilon\|x\|$, as $D\Psi(0) = J$;
- $B^{(3)}_{3r}(0) \cap \partial\Lambda \subset \Psi(P^{\text{int}})$, as Ψ^{int} is relatively open in $\partial\Lambda$;
- The set $B^{(3)}_r(0) \cap (\partial\Lambda)^\complement$ has exactly two connected components, as $\overline{\Lambda}$ and $\Lambda^\complement$ are topological manifolds with common boundary $\partial\Lambda$.

As Ψ is bi-Lipschitz, we observe

$$B^{(3)}_{2r}(0) \cap \partial\Lambda \subset \Psi\left(B^{(2)}_{2C_{\text{lip}}r}(0)\right) \subset \bigcup_{x \in B^{(2)}_{2C_{\text{lip}}r}(0)} \overline{B}^{(3)}_{\varepsilon\|x\|}((x, 0)) \tag{A.16}$$

$$\subset \bigcup_{x \in \mathbb{R}^2} \overline{B}^{(3)}_{\varepsilon\|x\|}((x, 0)) = \left\{v \in \mathbb{R}^3 \colon |v \cdot e_3| \leq \varepsilon\|v\|\right\}. \tag{A.17}$$

We define

$$U_0 := \left\{v \in \mathbb{R}^3 \colon |v \cdot e_3| \leq \varepsilon\|v\|\right\}, \tag{A.18}$$

$$U_\pm := \left\{v \in \mathbb{R}^3 \colon \pm v \cdot e_3 > \varepsilon\|v\|\right\}. \tag{A.19}$$

The sets $U_\pm \cap B^{(3)}_r(0)$ are open, convex and do not intersect $\partial\Lambda$ due to (A.17). We will now use Lemma A.5 to show that, up to a binary choice, we may assume $U_- \cap B^{(3)}_r(0) \subset \Lambda$ and $U_+ \cap B^{(3)}_r(0) \subset \Lambda^\complement$. As $B^{(3)}_r(0) \cap (\partial\Lambda)^\complement$ has exactly two connected components, it is sufficient to prove that any (continuous) path $p\colon [-1/2, 1/2] \to B^{(3)}_r(0)$ with $p(\pm 1/2) \in U_\pm$ intersects $\partial\Lambda$.

We first use the convexity to extend p by an (affine) linear path at both ends to get a path $f_1\colon [-1, 1] \to \overline{B}^{(3)}_{5r}(0)$ with $f_1(\pm 1) = \pm 5re_3$. Then we define $f_2\colon \overline{B}^{(2)}_{5r}(0) \to \mathbb{R}^3$ by

$$f_2(x) := \begin{cases} \Psi(x) & \text{if } \|x\| < 2r \\ \frac{\|x\| - 3r}{r}(x, 0) + \frac{3r - \|x\|}{r}\Psi(x) & \text{if } 2r \leq \|x\| < 3r\,. \\ (x, 0) & \text{if } 3r \leq \|x\| \leq 5r \end{cases} \tag{A.20}$$

We see that f_2 is Lipschitz continuous. The middle case is just a convex combination between the two other cases. Let $x \in \mathbb{R}^2$ with $\|x\| \leq 3r$. We observe

$$\|f_2(x) - (x, 0)\| \leq \sup_{t \in [0,1]} \|t\Psi(x) + (1 - t)(x, 0) - (x, 0)\| \leq \sup_{t \in [0,1]} t\|\Psi(x) - (x, 0)\| < \varepsilon\|x\|. \tag{A.21}$$

[4] In $\mathbb{R}^2$ resp. $\mathbb{R}^3$.

This implies

$$|f_2(x)\cdot e_3| = |f_2(x)\cdot e_3 - (x,0)\cdot e_3| \le \|f_2(x)-(x,0)\| < \varepsilon\|x\|,$$

that is, $f_2\left(\overline{B}^{(2)}_{3r}(0)\right) \subset U_0$ and thus, by the definition of $f_2(x)$ for $\|x\| \ge 3r$, $f_2\left(\overline{B}^{(2)}_{5r}(0)\right) \subset U_0$. By the triangle inequality and with $\varepsilon < \frac{1}{2}$ we obtain

$$\frac{1}{2}\|x\| < \|f_2(x)\| < \frac{3}{2}\|x\|\,. \tag{A.22}$$

These inequalities yield $f_2^{-1}\left(B^{(3)}_r(0)\right) \subset B^{(2)}_{2r}(0)$ and $f_2\left(\overline{B}^{(2)}_{3r}(0)\right) \subset B^{(3)}_{5r}(0)$. The latter inclusion together with the definition of f_2 outside $B^{(2)}_{3r}(0)$ implies $f_2\left(B^{(2)}_{5r}(0)\right) \subset B^{(3)}_{5r}(0)$. Thus, f_1 and f_2 satisfy the assumptions of Lemma A.5 (with $R := 5r$) and consequently, they have an intersection point $s \in \mathbb{R}^3$. We have $s \in f_2\left(\overline{B}^{(2)}_{5r}(0)\right) \subset U_0$ and $f_1^{-1}(U_0) \subset (-\frac{1}{2},\frac{1}{2})$, which means s is in the image of the original path p. Thus, $s \in B^{(3)}_r(0)$, which implies that $s \in f_2\left(B^{(2)}_{2r}(0)\right) = \Psi\left(B^{(2)}_{2r}(0)\right) \subset \partial\Lambda$. Therefore, the path p intersects $\partial\Lambda$, which was our claim.

As a result, we know that the sets $U_\pm \cap B^{(3)}_r(0)$ lie on opposite sides of $\partial\Lambda$. Without loss of generality, we assume $U_- \cap B^{(3)}_r(0) \subset \Lambda$ and $U_+ \cap B^{(3)}_r(0) \subset \Lambda^\complement$. In terms of the signed distance function d_Λ, this means that $\pm\,\mathrm{d}_\Lambda(v) > 0$ for $v \in U_\pm \cap B^{(3)}_r(0)$.

We are left to estimate $\mathrm{dist}(v,\partial\Lambda)$ for $v \in B^{(3)}_r(0)$. We start with the case $v \in U_0 \cap B^{(3)}_r(0)$. For that, we consider the map

$$\Phi\colon B^{(2)}_r(0)\times[-2\varepsilon,2\varepsilon] \to \mathbb{R}^3, \quad (y,t) \mapsto (\sqrt{1-t^2}y, t\|y\|). \tag{A.23}$$

We see that $\Phi\left(B^{(2)}_r(0)\times[-\varepsilon,\varepsilon]\right) = U_0 \cap B^{(3)}_r(0)$ and $\|\Phi(y,t)\| = \|y\|$. Furthermore, for a fixed y, the map $t \mapsto \Phi(y,t)$ defined on $[-2\varepsilon,2\varepsilon]$ is a path between U_+ and U_- inside $B^{(3)}_r(0)$ and must thus intersect $\partial\Lambda$. This path has a length of $2\|y\|\sin^{-1}(2\varepsilon) \le 2\pi\varepsilon\|y\|$. Hence, as each point $v \in U_0 \cap B^{(3)}_r(0)$ is on such a path for a y with $\|y\| = \|v\|$, we get $|\mathrm{d}_\Lambda(v)| \le 2\pi\varepsilon\|v\|$. Therefore, for $v \in U_0 \cap B^{(3)}_r(0)$, we get

$$|\mathrm{d}_\Lambda(v) - v\cdot e_3| \le |\mathrm{d}_\Lambda(v)| + |v\cdot e_3| \le (2\pi+1)\varepsilon\|v\|. \tag{A.24}$$

For $v \in U_\pm \cap B^{(3)}_r(0)$, we know that $\pm\,\mathrm{d}_\Lambda(v) > 0$ and only need upper and lower bounds for the distance to $\partial\Lambda$. For the lower bound, as $\partial\Lambda \cap B^{(3)}_{2r}(0) \subset U_0$, we have

$$|\mathrm{d}_\Lambda(v)| \ge \mathrm{dist}(v,U_0) = \sqrt{1-\varepsilon^2}|v\cdot e_3| - \varepsilon\|v^\perp\| \ge |v\cdot e_3| - \varepsilon^2|v\cdot e_3| - \varepsilon\|v^\perp\| \ge |v\cdot e_3| - 2\varepsilon\|v\|. \tag{A.25}$$

For the upper bound, we just use

$$|\mathrm{d}_\Lambda(v)| \le |v\cdot e_3| + |\mathrm{d}_\Lambda((v^\perp,0))| \le |v\cdot e_3| + 2\pi\|v^\perp\| \le |v\cdot e_3| + 2\pi\|v\|. \tag{A.26}$$

As the signs align, we finally get

$$|\mathrm{d}_\Lambda(v) - v \cdot e_3| \leq 2\pi\|v\|. \tag{A.27}$$

Thus, (A.24) holds for all $v \in B_r^{(3)}(0)$, which, by definition, says that d_Λ is differentiable at 0 and its differential is e_3. We also see that e_3 is orthogonal to the image of J and points toward $\Lambda^\complement$, which means that it is the outward normal vector to $\partial\Lambda$ at 0. This finishes the proof. □

Lemma A.6. *Let $\Lambda \subset \mathbb{R}^3$ be a piecewise Lipschitz region. Then, the outward normal vector $n(v)$ exists for $\mathcal{H}^2$ almost every $v \in \partial\Lambda$ and the set*

$$\mathcal{N} := \left\{x^\perp \in \mathbb{R}^2 \colon \partial\overline{(\Lambda_{x^\perp})} \neq (\partial\Lambda)_{x^\perp}\right\} \tag{A.28}$$

is a (two-dimensional) Lebesgue null set, where $\Lambda_{x^\perp}$ and $(\partial\Lambda)_{x^\perp}$ are defined in Definition 3.1.

Proof. We observe

$$\partial\overline{(\Lambda_{x^\perp})} \subset \partial\left(\Lambda_{x^\perp}\right) \subset (\partial\Lambda)_{x^\perp}. \tag{A.29}$$

The first inclusion is trivial. The second inclusion can be seen as follows. Let $t \in \partial\left(\Lambda_{x^\perp}\right)$. Then for all $r > 0$, $B_r^{(1)}(t) \cap \Lambda_{x^\perp} \neq \emptyset$ and $B_r^{(1)}(t) \cap (\Lambda_{x^\perp})^\complement \neq \emptyset$. Therefore, $B_r^{(3)}(x^\perp, t) \cap \Lambda \neq \emptyset$ and $B_r^{(3)}(x^\perp, t) \cap \Lambda^\complement \neq \emptyset$. Therefore, $(x^\perp, t) \in \partial\Lambda$ and $t \in (\partial\Lambda)_{x^\perp}$.

Let $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection with $\pi(e_3) = 0$ and let $(\Psi_{\mathrm{pL},i})_{i\in I}$ be a piecewise Lipschitz atlas of $\partial\Lambda$. For $i \in I$, we define the sets

$$\mathcal{N}_i := \partial[0,1]^2 \cup \left\{x \in (0,1)^2 \colon D\Psi_{\mathrm{pL},i}(x) \text{ does not exist}\right\}, \tag{A.30}$$

which are Lebesgue null sets due to Rademacher's theorem.[5] Thus, the set $\bigcup_{i\in I} \Psi_{\mathrm{pL},i}(\mathcal{N}_i)$ is an $\mathcal{H}^2$ null set, see [9, Theorem 2.8(i)]. Combining this with Lemma A.4, we now know that the outward normal vector $n(v)$ is well-defined for $\mathcal{H}^2$ every $v \in \partial\Lambda$. As $\pi \circ \Psi_{\mathrm{pL},i} \colon [0,1]^2 \to \mathbb{R}^2$ is Lipschitz, this implies that $(\pi \circ \Psi_{\mathrm{pL},i})(\mathcal{N}_i)$ is a Lebesgue null set [9, Lemma 3.2(iii)]. Furthermore, we define the sets

$$\mathcal{M}_i := \left\{x \in (0,1)^2 \colon D(\pi \circ \Psi_{\mathrm{pL},i})(x) \text{ exists, but is not invertible}\right\}. \tag{A.31}$$

By [9, Theorem 3.8], we know that $(\pi \circ \Psi_{\mathrm{pL},i})(\mathcal{M}_i)$ is a Lebesgue null set. Let

$$\Omega_i := [0,1]^2 \backslash (\mathcal{N}_i \cup \mathcal{M}_i), \tag{A.32}$$

and

$$\mathcal{M} := \partial\Lambda \backslash \bigcup_{i\in I} \Psi_{\mathrm{pL},i}(\Omega_i). \tag{A.33}$$

Let $v \in \partial\Lambda\backslash\mathcal{M}$. Hence, there is an $i \in I$ and a $y \in \Omega_i$, such that $v = \Psi_{\mathrm{pL},i}(y)$. Thus, $D\Psi_{\mathrm{pL},i}(y)$ exists, has full rank and does not have e_3 in its image. By Lemma A.4, we know that $n(v)$ exists and that $n(v) \cdot e_3 \neq 0$. Thus, the function $p \colon \mathbb{R} \to \mathbb{R}$ given by $p(t) = \mathrm{d}_\Lambda(v + te_3)$ with d_Λ being the signed distance function to the boundary $\partial\Lambda$ has non-vanishing differential at 0 and satisfies $p(0) = 0$. Hence, p changes sign at 0, which means that $v^\parallel \in \partial\overline{(\Lambda_{v^\perp})}$.

[5] See, e.g., [9, Theorem 3.2].

Conversely, this means that for $v \in \partial\Lambda$, the property $v^{\|} \notin \partial\overline{(\Lambda_{v^\perp})}$ implies that $v \in \mathcal{M}$. Thus, we have for the set $\mathcal{N}$ defined in (A.28),

$$\mathcal{N} \subset \pi\left(\mathcal{M}\right). \tag{A.34}$$

Finally, we observe

$$\mathcal{N} \subset \pi\left(\bigcup_{i\in I} \Psi_{\mathrm{pL},i}(\mathcal{N}_i \cup \mathcal{M}_i)\right) = \bigcup_{i\in I} \left((\pi \circ \Psi_{\mathrm{pL},i})(\mathcal{N}_i) \cup (\pi \circ \Psi_{\mathrm{pL},i})(\mathcal{M}_i)\right), \tag{A.35}$$

which shows that $\mathcal{N}$ is a Lebesgue null set. $\square$

Lemma A.7. *Let $\Lambda \subset \mathbb{R}^3$ be a piecewise Lipschitz region, $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the canonical projection and $f\colon \partial\Lambda \to \mathbb{R}^+$ be measurable. Let $n\colon \partial\Lambda \to \mathbb{R}^3$ be the outward normal vector field, which is defined almost everywhere (see Lemma A.6). Then, we have*

$$\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\, f(v)|n(v)\cdot e_3| = \int_{\mathbb{R}^2} \mathrm{d}x \sum_{v\in\pi^{-1}(x)\cap\partial\Lambda} f(v). \tag{A.36}$$

Proof. The proof is based on a quite general form of changing variables. We use the following area-formula (see [9, Theorem 3.9]) with one slight modification. To this end, let $n, m \in \mathbb{N}$ with $n \le m$, $U \subset \mathbb{R}^n$ be open, $\Phi\colon U \to \mathbb{R}^m$ be Lipschitz continuous and let $g\colon U \to \mathbb{R}^+$ be measurable. Then, we have the identity

$$\int_U \mathrm{d}y\, g(y)|D\Phi(y)| = \int_{\mathbb{R}^m} \mathrm{d}\mathcal{H}^n(x) \sum_{y\in\Phi^{-1}(x)} g(y), \tag{A.37}$$

where $|D\Phi(y)|^2 = \det(D\Phi(y)^* D\Phi(y))$. In [9], this is stated for $g \in \mathsf{L}^1(U)$. However, their proof also applies to positive, measurable functions g as an identity in $[0,\infty]$.

We cannot apply this directly to π, as it decreases the dimension and π does not have an inverse. Thus, we have to introduce a new map.

Let $(\Psi_{\mathrm{pL},i})_{i\in I}$ be a piecewise Lipschitz atlas of $\partial\Lambda$ so that $\partial\Lambda = \bigcup_{i\in I} \Psi_{\mathrm{pL},i}([0,1]^2)$. We may assume that $\mathrm{supp}(f) \subset \Psi_{\mathrm{pL},i}((0,1)^2)$ for some $i \in I$. For the remainder of this proof, we write Ψ for $\Psi_{\mathrm{pL},i}$.

Now, we can apply (A.37) with $\Phi := \pi \circ \Psi$ and $g := f \circ \Psi$. Thus, we see

$$\int_{(0,1)^2} \mathrm{d}y\, f(\Psi(y))|D(\pi\circ\Psi)(y)| = \int_{\mathbb{R}^2} \mathrm{d}x \sum_{y\in(\pi\circ\Psi)^{-1}(x)} f(\Psi(y)) \tag{A.38}$$

$$= \int_{\mathbb{R}^2} \mathrm{d}x \sum_{v\in\pi^{-1}(x)\cap\partial\Lambda} f(v). \tag{A.39}$$

We used that Ψ is bijective. So, we already have the right-hand side of the claim. We will apply again (A.37) with the functions $\Phi := \Psi$ and g given by

$$g(y) := f(\Psi(y))\frac{|D(\pi\circ\Psi)(y)|}{|D\Psi(y)|}, \quad y \in (0,1)^2. \tag{A.40}$$

Thus, using that Ψ is bijective and that the measure $\mathcal{H}^2$ on $\partial\Lambda$ is the 2-dimensional Hausdorff measure, we have

$$\int_{(0,1)^2} \mathrm{d}y\, f(\Psi(y))|D(\pi\circ\Psi)(y)| = \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\, f(v)\frac{|D(\pi\circ\Psi)(\Psi^{-1}(v))|}{|D\Psi(\Psi^{-1}(v))|}. \tag{A.41}$$

To conclude the proof, we only need to show that the quotient of the functional determinants is given by $|n(v)\cdot e_3|$ for almost every $v\in\partial\Lambda$. Let $v\in\partial\Lambda$ be such that $B := D\Psi(\Psi^{-1}(v)) \in \mathbb{R}^{3\times 2}$ is well defined. We identify the two column vectors of the 3×2 matrix B as w_1 and w_2. The image of B is the tangent space to $\partial\Lambda$ at v. As Ψ is bi-Lipschitz continuous, the matrix B has full rank. The normal vector $n(v)$ is now orthogonal to the linear independent vectors w_1, w_2. Thus, $n(v)|w_1\times w_2| = \pm w_1\times w_2$.

As π is linear, we have $D(\pi\circ\Psi)(\Psi^{-1}(v)) = \pi B$. For the determinant of this 2×2 matrix, we get $\det(\pi B) = (w_1\times w_2)\cdot e_3$. For the denominator, we observe

$$\det(B^*B) = |w_1|^2|w_2|^2 - (w_1\cdot w_2)^2 = |w_1\times w_2|^2. \tag{A.42}$$

In conclusion, we have

$$\frac{|D(\pi\circ\Psi)(\Psi^{-1}(v))|}{|D\Psi(\Psi^{-1}(v))|} = \frac{|(w_1\times w_2)\cdot e_3|}{|w_1\times w_2|} = |n(v)\cdot e_3|. \tag{A.43}$$

In combination with (A.39) and (A.41), we have proved the statement. □

Corollary A.8. *Let $\Lambda\subset\mathbb{R}^3$ be a piecewise Lipschitz region. Then, for Lebesgue almost every $x^\perp\in\mathbb{R}^2$, the set $\Lambda_{x^\perp}$ is a finite (possibly empty) union of intervals with disjoint closures.*

Proof. As $\mathcal{H}^2(\partial\Lambda)$ is finite, see Lemma A.3, we have by Lemma A.7 (with $f=1$)

$$\int_{\mathbb{R}^2} \mathrm{d}x^\perp\, \#(\partial(\Lambda_{x^\perp})) = \int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(v)\, |n(v)\cdot e_3| \le \mathcal{H}^2(\partial\Lambda) < \infty. \tag{A.44}$$

This implies that the set $\partial(\Lambda_{x^\perp})\subset\mathbb{R}$ is finite for almost every $x^\perp$. Hence, $\Lambda_{x^\perp}$ is almost everywhere a finite union of intervals. If, for some $x_0^\perp\in\mathbb{R}^2$, two different connected components of $\Lambda_{x_0^\perp}$ share a boundary point, t, then $t\in \partial(\Lambda_{x_0^\perp})\backslash\partial\Big(\overline{\Lambda_{x_0^\perp}}\Big)$. Looking at Lemma A.6, we realize that this means $x_0^\perp\in\mathcal{N}$. Thus, we have proved the claim. □

Lemma A.9. *Let $\Lambda\subset\mathbb{R}^3$ be a piecewise $\mathsf{C}^{1,\alpha}$ region with Γ as in Definition 2.1. Then, there is a constant $C<\infty$ such that*

$$\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(w)|\ln(\mathrm{dist}(w,\Gamma))| \le C. \tag{A.45}$$

Proof. We start with

$$\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(w)\,|\ln(\mathrm{dist}(w,\Gamma))| \le C\sum_{k\in\mathbb{Z}}(|k|+1)\cdot\mathcal{H}^2\big(\{w\in\partial\Lambda\colon 2^{k-1}\le \mathrm{dist}(w,\Gamma)\le 2^k\}\big) \tag{A.46}$$

$$\leq C \sum_{k=-\infty}^{k_{\max}} (|k|+1) \cdot \mathcal{H}^2\big(\{w \in \partial\Lambda \colon \operatorname{dist}(w,\Gamma) \leq 2^k\}\big). \tag{A.47}$$

For the first step, we just bound the integrand by a step function from above. As Λ is bounded, the associated set is empty for $k > k_{\max}$ with some finite $k_{\max}$. We are left to estimate the volume of these sets. Specifically, we will show that there is an $r_0 > 0$ and a $C < \infty$ such that for any $r < r_0$, we have

$$\mathcal{H}^2(B_r(\Gamma) \cap \partial\Lambda) \leq Cr. \tag{A.48}$$

We recall that $B_r(\Gamma) \subset \mathbb{R}^3$ is the r-neighborhood of Γ. By assumption, there is a global Lipschitz atlas $(\Psi_{\mathrm{gL},j})_{j\in J}$ of $\partial\Lambda$, as in Definition 2.1. For each $i \in I$, the set $U_j := \Psi_{\mathrm{gL},j}((0,1)^d) \subset \partial\Lambda$ is a (relatively) open subset of the compact metric space $\partial\Lambda$ and we have $\partial\Lambda \subset \bigcup_{j\in J} U_j$. Thus, by Lebesgue's number lemma, there is a constant $r_0 > 0$ such that any $v \in \partial\Lambda$ there is a $j \in J$ such that $B_{2r_0}(v) \cap \partial\Lambda \subset U_j$.

Now, we need to understand the set Γ. We recall its definition

$$\Gamma := \bigcup_{i\in I} \Psi_{\mathrm{pC},i}(\partial[0,1]^2). \tag{A.49}$$

Let C_0 be a Lipschitz constant for all $\Psi_{\mathrm{pC},i}$'s which exists, as I is finite. As $\partial[0,1]^2$ is just the boundary of the unit square, there is a surjective (piecewise linear) function $\vartheta \colon [0,1] \to \partial[0,1]^2$ with Lipschitz constant 4. Let $N \in \mathbb{N}$ with $N > 4C_0/r_0$ and $f_k \colon [0,1] \to [0,1]$ be the functions satisfying $f_k(t) = \frac{k-1+t}{N}$. Now, for any $1 \leq k \leq N$ and $i \in I$, we define $g_{ik} \colon [0,1] \to \Gamma$ by $g_{ik} := \Psi_{\mathrm{pC},i} \circ \vartheta \circ f_k$ and observe

$$C_{\mathrm{Lip}}\,(g_{ik}) \leq 4C_0/N < r_0. \tag{A.50}$$

Furthermore, $\Gamma = \bigcup_{i\in I} \bigcup_{k=1}^N g_{ik}([0,1])$. By (A.50), we know $g_{ik}([0,1]) \subset B_{r_0}(g_{ik}(0))$ and thus

$$B_r(g_{ik}((0,1)) \subset B_{2r_0}(g_{ik}(0)) \tag{A.51}$$

for $r \leq r_0$. Hence, there is an $j = j(i,k) \in J$, such that $B_r(g_{ik}([0,1]) \cap \partial\Lambda \subset U_{j(i,k)}$. For any $r \leq r_0$, we can estimate

$$\mathcal{H}^2(B_r(\Gamma) \cap \partial\Lambda) \leq \sum_{i\in I} \sum_{k=1}^N \mathcal{H}^2\big(B_r(g_{ik}([0,1])) \cap \partial\Lambda\big). \tag{A.52}$$

As $\Psi_{\mathrm{gL},j(i,k)}$ is bi-Lipschitz, there is a constant C such that

$$\mathcal{H}^2\big(B_r(g_{ik}([0,1])) \cap \partial\Lambda\big) \leq C \left| \Psi^{-1}_{\mathrm{gL},j(i,k)}\big(B_r(g_{ik}([0,1])) \cap \partial\Lambda\big) \right|, \tag{A.53}$$

and

$$\Psi^{-1}_{\mathrm{gL},j(i,k)}\,(B_r(g_{ik}([0,1])) \cap \partial\Lambda) \subset B_{Cr}(\Psi^{-1}_{\mathrm{gL},j(i,k)}(g_{ik}([0,1]))). \tag{A.54}$$

We now apply Lemma A.3 with $f = \Psi^{-1}_{\mathrm{gL},j(i,k)} \circ g_{ik}$ and $d = 1$ to obtain

$$|B_{Cr}(\Psi^{-1}_{\mathrm{gL},j(i,k)}(g_{ik}([0,1])))| \leq C(r + r^2) \leq Cr, \tag{A.55}$$

as $r < r_0$.

In conclusion, as I is finite, we have

$$\mathcal{H}^2(B_r(\Gamma) \cap \partial\Lambda) \leq \sum_{i\in I} \sum_{k=1}^{N} \mathcal{H}^2\big(B_r(g_{ik}([0,1])) \cap \partial\Lambda\big) \tag{A.56}$$

$$\leq C \sum_{i\in I} \sum_{k=1}^{N} \Big|\Psi^{-1}_{\mathrm{gL},j(i,k)}\big(B_r(g_{ik}([0,1])) \cap \partial\Lambda\big)\Big| \tag{A.57}$$

$$\leq C \sum_{i\in I} \sum_{k=1}^{N} |B_{Cr}(\Psi^{-1}_{\mathrm{gL},j(i,k)}(g_{ik}([0,1])))| \tag{A.58}$$

$$\leq C \sum_{i\in I} \sum_{k=1}^{N} Cr \leq Cr. \tag{A.59}$$

For $r_0 < r < 2^{k_{\max}}$, we trivially arrive at the same estimate as long as $C \geq \mathcal{H}^2(\partial\Lambda) r_0^{-1}$, that is, $\mathcal{H}^2(B_r(\Gamma) \cap \partial\Lambda) \leq Cr$ also for "large" r.

Now, we are able to finish (A.47) and obtain for some (finite) constant C

$$\int_{\partial\Lambda} \mathrm{d}\mathcal{H}^2(w)\, |\ln(\mathrm{dist}(w,\Gamma))| \leq C \sum_{k=-\infty}^{k_{\max}} (|k|+1)2^k \leq C, \tag{A.60}$$

which was the claim. □

Appendix B. Proof of (3.15)

We observe

$$\int_{\mathbb{R}^{m-1}} \prod_{j=1}^{m} \frac{1}{\langle y_j^{\|}\rangle} \mathrm{d}\mathbf{y}^{\|} = \int_{\mathbb{R}^{m-1}} \prod_{j=1}^{m} \frac{1}{\langle y_j^{\|}\rangle} \prod_{j=1}^{m-1} \mathrm{d}y_j^{\|}$$

$$= \int_{\mathbb{R}^{m-1}} \mathrm{d}x_1^{\|} \cdots \mathrm{d}x_{m-1}^{\|} \prod_{j=1}^{m} \frac{1}{\langle x_j^{\|} - x_{j-1}^{\|}\rangle}, \tag{B.1}$$

where we switched back to the integration variables $x_1, \ldots, x_m$ and set[6] $x_0 := x_m := 0$. As we can see, the last expression is the $(m-1)$-fold convolution of $\langle\,\cdot\,\rangle^{-1}$ with itself evaluated at 0. This is a job for the Fourier transform. We use the convention

$$\mathcal{F}(f)(\xi) := \lim_{R\to\infty} \int_{-R}^{R} \mathrm{d}t\, f(t)\, \mathrm{e}^{-2\pi \mathrm{i}\xi t}, \quad \xi \in \mathbb{R}. \tag{B.2}$$

Thus, we have

$$\int_{\mathbb{R}^{m-1}} \mathrm{d}x_1^{\|} \cdots \mathrm{d}x_{m-1}^{\|} \prod_{j=1}^{m} \frac{1}{\langle x_j^{\|} - x_{j-1}^{\|}\rangle} = \int_{\mathbb{R}} \mathrm{d}\xi\, \mathcal{F}(\langle\,\cdot\,\rangle^{-1})(\xi)^m\,. \tag{B.3}$$

[6]The values x_0 and x_m only matter through $x_0 - x_m = 0$. Thus, we can set both to 0.

The Fourier transform of $\langle\,\cdot\,\rangle^{-1}$ can be expressed in terms of the modified Bessel function of the second kind K_0, see [27, (10.32.6)],[7]

$$\begin{aligned}\mathcal{F}(\langle\,\cdot\,\rangle^{-1})(\xi) &= \lim_{R\to\infty}\int_{-R}^{R}\mathrm{d}t\,\frac{1}{\langle t\rangle}\mathrm{e}^{2\pi\mathrm{i}t\xi} = 2\lim_{R\to\infty}\int_{0}^{R}\mathrm{d}t\,\frac{\cos(2\pi|\xi|t)}{\sqrt{t^2+1}}\\ &= 2K_0(2\pi|\xi|).\end{aligned}\tag{B.4}$$

We observe that

$$\begin{aligned}\mathcal{F}(\langle\,\cdot\,\rangle^{-1})(\xi) &= \lim_{R\to\infty}\int_{-R}^{R}\mathrm{d}t\,\frac{1}{\langle t\rangle}\mathrm{e}^{2\pi\mathrm{i}t\xi} = 2\lim_{R\to\infty}\int_{0}^{R}\mathrm{d}t\,\frac{\cos(2\pi|\xi|t)}{\sqrt{t^2+1}}\\ &= 2K_0(2\pi|\xi|).\end{aligned}\tag{B.5}$$

We need the (known) estimate,

$$0<\ln(2)-\gamma_E<\frac{1}{8},\tag{B.6}$$

where γ_E is Euler's constant (see, e.g., [27, (5.2.3)]). Using this inequality, the series representations [27, (10.31.2),(10.25.2)], the harmonic series $H_n := \sum_{k=1}^{n} k^{-1} \le n!$, the identity $\Gamma(n+1)=n!$ (where Γ is the Gamma function, see, e.g., [27, (5.2.1)(5.4.1)]) and the geometric series, we get for any $t\in(0,1)$

$$K_0(t) = -\left(\ln\left(\frac{1}{2}t\right)+\gamma_E\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}t^2\right)^k}{(k!)^2}+\sum_{k=1}^{\infty}H_k\frac{\left(\frac{1}{4}t^2\right)^k}{(k!)^2}\tag{B.7}$$

$$< -\left(\ln\left(\frac{1}{2}t\right)+\gamma_E\right)\frac{1}{1-\frac{1}{4}t^2}+\frac{t^2}{4-t^2}.\tag{B.8}$$

Using the last two inequalities, we can infer

$$2K_0(1) < 2\left(\frac{1}{8}\frac{4}{3}+\frac{1}{3}\right)=1.\tag{B.9}$$

Thus, as K_0 is decreasing on $\mathbb{R}^+$ (See [27, §10.37]), we have $2K_0(t)<1$ for $t>1$. For $t\in(0,1)$, we estimate using $\ln(t/2)+\gamma_E<0$ and $0<\gamma_E<1$,

$$K_0(t) \le -\left(\ln\left(\frac{1}{2}t\right)+\gamma_E\right)\frac{1}{1-\frac{1}{4}t^2}+\frac{t^2}{4-t^2}\tag{B.10}$$

$$= -\ln\left(\frac{1}{2}t\right)-\gamma_E+\frac{t^2}{4-t^2}\left(-\ln(t)+\ln(2)-\gamma_E+1\right)\tag{B.11}$$

$$< -\ln\left(\frac{1}{2}t\right)-\gamma_E+\frac{1}{3}\Big(\sup_{t\in(0,1)}\left(-t^2\ln(t)+1+\ln(2)-\gamma_E\right)\Big)\tag{B.12}$$

$$\le -\ln\left(\frac{1}{2}t\right)-\gamma_E+\frac{1}{3}(1/(2\mathrm{e})+1+\ln(2)-\gamma_E)) < -\ln\left(\frac{1}{2}t\right).\tag{B.13}$$

[7] See [27, (1.4.22)] to verify their usage of an improper Riemann integral, while this paper uses Lebesgue integrals.

The last inequality relies on a numerical computation to verify that $-\gamma_E + \frac{1}{3}(1/(2\mathrm{e}) + 1 + \ln(2) - \gamma_E))$ is negative. Altogether, this yields

$$2K_0(2\pi\xi) \leq -2\ln(\pi\xi) \tag{B.14}$$

for $0 < 2\pi\xi < 1$. Thus, we are able to estimate

$$\int_{\mathbb{R}} \mathrm{d}\xi\, \mathcal{F}(\langle\,\cdot\,\rangle^{-1})(\xi)^m = 2\int_0^\infty \mathrm{d}\xi\, (2K_0(2\pi\xi))^m \tag{B.15}$$

$$\leq 2^{m+1}\int_0^{\frac{1}{2\pi}} \mathrm{d}\xi\, (-\ln(\pi\xi))^m + 2\int_{\frac{1}{2\pi}}^\infty \mathrm{d}\xi\, 2K_0(2\pi\xi) \tag{B.16}$$

$$\leq \frac{2}{\pi}2^m \int_0^1 \mathrm{d}t\, (-\ln(t))^m + 2\int_0^\infty \mathrm{d}\xi\, 2K_0(2\pi\xi) \tag{B.17}$$

$$= \frac{2}{\pi}2^m m! + 1 < 2^m m!. \tag{B.18}$$

The final estimate relies on $m \geq 2$ and $\pi > 3$, while the last identity is based on (B.5) and

$$\int_0^1 \mathrm{d}\xi\, (-\ln(\xi))^m = \int_0^\infty \mathrm{d}t\, t^m\, \mathrm{e}^{-t} = \Gamma(m+1) = m!. \tag{B.19}$$

Combining (B.1), (B.3) and (B.18), we arrive at

$$\int_{\mathbb{R}^{m-1}} \prod_{j=1}^m \frac{\mathrm{d}y_j^{\|}}{\langle y_j^{\|}\rangle} < 2^m m!, \tag{B.20}$$

which was the claim.

Appendix C. Asymptotic Expansion with Order One Error Term

Our final result, Corollary C.3, in this section deals with the asymptotic expansion for a finite union of bounded intervals. That is, we assume that we have $k \in \mathbb{N}$ open and bounded intervals $I_1, \ldots, I_k$, whose closures are disjoint. More precisely, there exist $d_i > 0$ for $1 \leq i < k$ with $\sup I_i + d_i = \inf I_{i+1}$. Let $\ell_j := |I_j|$ be the length of I_j and let $\Omega := \bigcup_{j=1}^k I_j$. The symbol ℓ for the length of intervals in this section has, of course, nothing to do with the index of a Landau level.

The proof of Corollary C.3 is based upon two lemmata. The first lemma is per se not an asymptotic result but reduces the analysis to a single interval including an error term. The second lemma deals with the asymptotic expansion for a single interval, including an order one error term, and improves a seminal result by Landau and Widom in [19]. This is achieved by improving a certain estimate in their proof which allows for an order one error term instead of $o(\ln(L))$. Later, Widom [36] extended their result and proved that the error term is indeed of order one. This was used by Sobolev in [32, Chapter 8] to obtain concrete error terms. Our error term is somewhat different and fits our purposes. It is important to notice that there is still an undetermined error term of order one which is however independent of the scaling and the lengths of the intervals and depends only on the energy.

The first lemma is the following.

Lemma C.1. *Let $\mu > 0$ and $m \in \mathbb{N}$. Then, under the above assumptions on Ω we have*

$$\begin{aligned} &\left| \operatorname{tr}\left[\Big(\mathbb{1}_\Omega \mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu] \mathbb{1}_\Omega \Big)^m - \sum_{j=1}^{k} \Big(\mathbb{1}_{I_j} \mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu] \mathbb{1}_{I_j} \Big)^m \right] \right| \\ &\qquad \le C \sum_{j=1}^{k-1} \ln\Big(1 + \frac{\ell_j}{1+d_j}\Big), \end{aligned} \tag{C.1}$$

where C is a constant depending on m and μ, but crucially not on k or the intervals themselves.

Proof. Let $Q := \mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \le \mu]$. It is convenient to make a slight generalization by allowing I_k to be any measurable set such that $d_{k-1} := \inf(I_k) - \sup(I_{k-1}) > 0$. We note that ℓ_k is undefined, but it is also not present in our claim. We proceed by induction with respect to the number of intervals, k. Once we have proved the statement for $k = 2$, the statement follows for any k, as we can choose $I'_k := I_k \cup I_{k+1}$.

Hence, we just have to deal with the case $k = 2$. We observe $1_\Omega = 1_{I_1} + 1_{I_2}$. We multiply out the first term and, using $I_1 \cap I_2 = \emptyset$, we have

$$(\mathbb{1}_\Omega Q \mathbb{1}_\Omega)^m = \sum_{j \in \{1,2\}^{\{0,\dots,m\}}} \mathbb{1}_{I_{j_0}} \prod_{i=1}^{m} Q \mathbb{1}_{I_{j_i}}. \tag{C.2}$$

The two summands $j = (1,1,\dots,1)$ and $j = (2,2,\dots,2)$ are the ones we subtract in the statement of the lemma. Hence, we have to estimate all other summands. In the case $j_0 \neq j_m$, we use $\operatorname{tr} AB = \operatorname{tr} BA$ with $A = \mathbb{1}_{I_{j_0}} Q$ and $B = \prod_{i=1}^m Q \mathbb{1}_{I_{j_i}}$ to conclude that the trace vanishes. We are left to estimate the terms where $j_0 = j_m$ and there is an $i \in \{1, 2, \dots, m-1\}$ with $j_i \neq j_0$. In this case, we consider i_- and i_+ as the smallest and largest such i (which can be the same). Now, we write

$$\mathbb{1}_{I_{j_0}} \prod_{i=1}^{m} Q \mathbb{1}_{I_{j_i}} = \big(\mathbb{1}_{I_{j_0}} Q\big)^{i_- - 1} \mathbb{1}_{I_{j_0}} Q \mathbb{1}_{I_{j_{i_-}}} A_{i_-,i_+} \mathbb{1}_{I_{j_{i_+}}} Q \mathbb{1}_{I_{j_0}} \big(Q \mathbb{1}_{I_{j_0}}\big)^{m-i_+-1}, \tag{C.3}$$

where A_{i_-,i_+} is the identity, if $i_- = i_+$ and a product of some operators Q, $\mathbb{1}_{I_1}$, and $\mathbb{1}_{I_2}$ otherwise. As all of the operators are projections, their operator norm can be bounded by 1. As we are interested in the trace, we will bound the trace norm. To do so, it suffices to bound two operators in the Hilbert–Schmidt norm and all others in the operator norm. The operators we will bound in Hilbert–Schmidt norm are $\mathbb{1}_{I_{j_0}} Q \mathbb{1}_{I_{j_{i_-}}}$ and $\mathbb{1}_{I_{j_{i_+}}} Q \mathbb{1}_{I_{j_0}}$. These operators are adjoint and hence have the same Hilbert–Schmidt norm. As $j_{i_-} \neq j_0 \neq j_{i_+}$, we know

$\{j_0, j_{i_-}\} = \{j_0, j_{i_+}\} = \{1, 2\}$. Thus, we are left to estimate $\|\mathbb{1}_{I_1} Q \mathbb{1}_{I_2}\|_2^2$. Since the operator Q has integral kernel $Q(x, y) = k_\mu(x - y) = \frac{\sin(\sqrt{\mu}(x-y))}{\pi(x-y)}, x, y \in \mathbb{R}$, the square of the Hilbert–Schmidt norm can be easily calculated as the square of the integral of this kernel for $x \in I_1$ and $y \in I_2$. By translation invariance we may assume that $I_1 = (0, \ell_1)$. By the definition of d_1, we know $I_2 \subset (\ell_1 + d_1, \infty)$. Hence, using the estimate $|k_\mu(z)| \leq C/(1 + |z|)$ for some constant C we get

$$\begin{aligned} \left|\mathrm{tr}\,(C.3)\right| \leq \|\mathbb{1}_{I_1} Q \mathbb{1}_{I_2}\|_2^2 &\leq C \int_{I_1} \mathrm{d}x \int_{I_2} \mathrm{d}y \, \frac{1}{(1 + y - x)^2} &\quad (C.4)\\ &\leq C \int_0^{\ell_1} \mathrm{d}x \int_{\ell_1 + d_1}^{\infty} \mathrm{d}y \, \frac{1}{(1 + y - x)^2} &\quad (C.5)\\ &= C \int_0^{\ell_1} \mathrm{d}x \, \frac{1}{1 + d_1 + \ell_1 - x} = C \ln\left(\frac{1 + d_1 + \ell_1}{1 + d_1}\right). &\quad (C.6) \end{aligned}$$

The number of such error terms is $2^m - 1$. Thus, the error bound in m is quite bad, but we only need it to be good in k. The proof is now finished. □

Here is our second lemma on the mentioned improved asymptotic expansion for a single interval of Landau and Widom. This agrees with the improvement of Widom in [36]. As the paper of Landau and Widom [19] is freely accessible, but the later paper by Widom [36] is not,[8] we provide this different proof for the reader's convenience. We do not claim any originality.

Lemma C.2. *Let $\Omega \subset \mathbb{R}$ be an open and bounded interval of length $\ell > 0$ and let $\mu > 0$. Then, for any $m \in \mathbb{N}$ and $L > 0$, we have with $\mathsf{I}(m)$ explained after (2.15),*

$$\mathrm{tr}\left(\mathbb{1}_{L\Omega} \mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \leq \mu] \mathbb{1}_{L\Omega}\right)^m = \frac{\sqrt{\mu}}{\pi} L\ell + 4\,\mathsf{I}(m) \ln(1 + L\ell) + O(1), \quad (C.7)$$

where the order one error term is independent of L and ℓ but depends on μ.

Proof. The case $m = 1$ is trivial, as the integral kernel is constant on the diagonal and only the volume term $\frac{\sqrt{\mu}}{\pi} L\ell$ appears. Thus, by linearity, it suffices to show the statement for a basis of the polynomials vanishing at 0 and 1.

As μ is fixed, the result depends only on $L\ell$, which can be small or large. If $L\ell \leq 1$ then the trace on the left-hand side of (C.7) is bounded uniformly for these L, ℓ by continuity as a function of $L\ell \in [0, 1]$. The same is true for the first two terms on the right-hand side of (C.7) and hence the equality holds true with an $O(1)$ error term. In the following, we will assume that $L\ell > 1$.

From now on we use the same notation as in [19], where c takes the role of L. The last equation in the proof of their Theorem 1, where they still carry the order one error term is [19, (18)]. Afterward they allow for a larger $o(\ln(c))$ error term and here we take a different route.

[8] As of August 25, 2022.

They consider the polynomials $(t(1-t))^n$ and $t(t(1-t))^n$ for $n \in \mathbb{N}$, which span all polynomials that vanish at 0 and 1. We proceed with these polynomials instead of t^m as in the statement of our lemma. Their equation [19, (18)] states

$$\operatorname{tr} A_c[A_c(I-A_c)]^n = 2\operatorname{tr} K_c^n + O(1) = \frac{1}{2}\operatorname{tr}[A_c(I-A_c)]^n + O(1), \qquad \text{(C.8)}$$

where $A_c = P(0,c)Q(0,1)P(0,c)$, which is unitarily equivalent to the operator $\mathbb{1}_{[0,c]}\mathbb{1}\big(-\Delta \leq \frac{1}{4}\big)\mathbb{1}_{[0,c]}$ in our notation, and $K_c = P(1,c)Q(-\infty,0)P(-\infty,0)$ $Q(-\infty,0)P(1,c)$, as stated below [19, (17)]. They state below [19, (18)] that the integral kernel of the operator K_c on $\mathsf{L}^2([1,c])$ is given for $1 \leq x, y \leq c$ by

$$f(x,y) := \frac{1}{4\pi^2}\int_0^\infty \frac{\mathrm{d}u}{(u+x)(u+y)} = \frac{1}{4\pi^2}\begin{cases} \frac{\ln(x)-\ln(y)}{x-y} & \text{if } x \neq y \\ \frac{1}{x} & \text{if } x = y \end{cases}. \qquad \text{(C.9)}$$

Let K be the operator on $\mathsf{L}^2(\mathbb{R}^+)$ with integral kernel $f(x,y)$ for $0 < x, y < \infty$. Thus, $K_c = P(1,c)KP(1,c)$ and $K = Q(-\infty,0)P(-\infty,0)Q(-\infty,0)$. Hence, we can conclude

$$\operatorname{tr} K_c^n = \int_{[1,c]^n} \mathrm{d}x \prod_{i=1}^n f_1(x_i, x_{i+1}), \qquad \text{(C.10)}$$

with the convention $x_{n+1} = x_1$. We denote the integrand $f_n(x) := \prod_{i=1}^n$ $f(x_i, x_{i+1})$. It satisfies for $\lambda > 0$ the homogeneity property $f_n(\lambda x) = \lambda^{-n} f_n(x)$, which indicates that we should use spherical coordinates to calculate the integral. The problem is, however, that the integration domain does not look particularly nice in spherical coordinates. Thus, we would like to change the integration domain without changing the integral too much.

The first thing to observe is that as ln is increasing, $f_n(x) \geq 0$ holds for any $n \in \mathbb{N}, x \in (\mathbb{R}^+)^n$. For any (Borel) measurable $X \subset (\mathbb{R}^+)^n$, we define

$$\iota(X) := \int_X \mathrm{d}x\, f_n(x) = \int_X \mathrm{d}x \prod_{i=1}^n f(x_i, x_{i+1}). \qquad \text{(C.11)}$$

As the integrand is non-negative, ι is a measure. We also observe that ι is invariant under the cyclic shift $(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, x_1)$. Assuming $n > 1$, for $i = 1, \ldots, n$, we consider the set

$$U_i := \big\{x \in (\mathbb{R}^+)^n : x_i \leq 1 \leq x_{i+1}\big\}. \qquad \text{(C.12)}$$

We observe $\iota(U_i) = \iota(U_1)$ by the cyclic shift property. We see

$$\iota(U_1) = \operatorname{tr} P(0,1)KP(1,\infty)(KP(0,\infty))^{n-2}KP(0,1). \qquad \text{(C.13)}$$

Since $P(0,1)KP(1,\infty) = P(0,1)Q(-\infty,0)P(-\infty,0)Q(-\infty,0)P(1,\infty)$ is the operator R [19, (9)] with appropriately chosen intervals J, M, K, N, L we see that this is trace class by [19, Lemma, (L2)]. By the homogeneity of f_n, we even have $\iota(cU_1) = c^{n-n}\iota(U_1) = \iota(U_1)$.

Next, we introduce the set

$$V := \big\{x \in (\mathbb{R}^+)^n : \sqrt{n} \leq \|x\| \leq \sqrt{n}\, c\big\},$$

which looks very nice in spherical coordinates. For $n = 1$, we just have $V = [1, c]$. For $n > 1$, we observe the chain

$$[1, c]^n \subset V \subset [1, c]^n \cup \bigcup_{i=1}^{n} (U_i \cup cU_i). \tag{C.14}$$

The first inclusion is trivial. We call $x_1, \ldots, x_n$ the coordinates of x. If x has both a coordinate above λ and one below λ, then it has to be in the set $\bigcup_{i=1}^{n} \lambda U_i$. Any $x \in V$ has at least one coordinate above 1 and a coordinate below c. Thus, if $x \notin [1, c]^n$, it has to have a coordinate above and below 1 or a coordinate above and below c, which proves the second inclusion. These inclusions and the subadditivity and monotonicity of ι imply that there is a constant C_n (depending on n but not on c) such that

$$\iota([1, c]^n) \leq \iota(V) \leq \iota([1, c]^n) + C_n \implies \iota([1, c]^n) = \iota(V) + O(1). \tag{C.15}$$

This holds with $O(1)$ replaced by 0 for $n = 1$. Finally, we introduce $W := \{x \in (\mathbb{R}^+)^n : \|x\| = \sqrt{n}\}$ with Hausdorff measure $\mathcal{H}^{n-1}$ and observe $V = [1, c]W = \{\lambda x : 1 \leq \lambda \leq c, x \in W\}$. Now, we are just left to calculate

$$\iota(V) = \int_V \mathrm{d}x\, f_n(x) = \int_1^c \mathrm{d}r\, r^{n-1} \int_W \mathrm{d}\mathcal{H}^{n-1}(x)\, f_n(rx) \tag{C.16}$$

$$= \int_1^c \mathrm{d}r\, r^{n-1} \int_W \mathrm{d}\mathcal{H}^{n-1}(x)\, r^{-n} f_n(x) = \int_1^c \mathrm{d}r\, r^{-1} \int_W \mathrm{d}\mathcal{H}^{n-1}(x)\, f_n(x) \tag{C.17}$$

$$= \ln(c) \int_W \mathrm{d}\mathcal{H}^{n-1}(x)\, f_n(x) = \tilde{C}(n) \ln(c), \tag{C.18}$$

where $\tilde{C}(n)$ is the result of the surface integral. We did a change to spherical coordinates in the second step. As the integrand is positive, $\tilde{C}(n) \in (0, \infty]$ is well-defined. By (C.15), we conclude (for fixed n and as $c \to \infty$)

$$\operatorname{tr} K_c^n = \tilde{C}(n) \ln(c) + O(1). \tag{C.19}$$

From [19, (19)], we know that $\tilde{C}(n) = \frac{1}{4\pi^2} \int_0^1 \mathrm{d}t\, (t(1-t))^{n-1}$. In conjunction with (C.8), we get the improved error term $O(1)$ with the same leading term for any polynomial, which vanishes at 0 and 1.

To get the claim of our lemma, we just have to replace c by $2\sqrt{\mu} L\ell$ and then use $\ln(2\sqrt{\mu} L\ell) = \ln(2) + \frac{1}{2}\ln(\mu) + \ln(L\ell) = \ln(1 + L\ell) + O(1)$, which relies on $L\ell \geq 1$. □

Now we are in position to present and prove the main result in this section. The dependency of our error term on Ω is not just $O(1)$ as in [36] but explicit in terms of the number, lengths, and distances of the constituent intervals of Ω. Sobolev in [32, Chapter 8] has a similar error term, which, however, does not seem to suffice for our purposes.

Corollary C.3. *We assume the same conditions on the set Ω as in Lemma C.1, $\mu > 0$ and $m \in \mathbb{N}$. Then, with $\mathsf{l}(m)$ explained after (2.15), we have for any*

$L \geq 1$,

$$\operatorname{tr}\left(\mathbb{1}_{L\Omega}\mathbb{1}[(-\mathrm{i}\nabla^{\|})^2 \leq \mu]\mathbb{1}_{L\Omega}\right)^m = \frac{\sqrt{\mu}}{\pi}L|\Omega| + 4k\,\mathsf{I}(m)\ln(1+L) \tag{C.20}$$

$$+ O\left(k + |\ln(\ell_k)| + \sum_{j=1}^{k-1}|\ln(\ell_j)| + |\ln(d_j)|\right). \tag{C.21}$$

Proof. For the case of a single interval, we use Lemma C.2

$$\operatorname{tr}\left(\mathbb{1}_{LI_j}Q\mathbb{1}_{LI_j}\right)^m = \frac{\sqrt{\mu}}{\pi}L\ell_j + 4\,\mathsf{I}(m)\ln(1+L\ell_j) + O(1) \tag{C.22}$$

$$= \frac{\sqrt{\mu}}{\pi}L\ell_j + 4\,\mathsf{I}(m)\Big[\ln(1+L) + \ln\Big(\frac{L}{1+L}\Big) + \ln\Big(\frac{1}{L}+\ell_j\Big)\Big] + O(1). \tag{C.23}$$

As $L \geq 1$, we have

$$\left|\ln\left(\frac{L}{1+L}\right) + \ln\left(\frac{1}{L}+\ell_j\right)\right| < 3 + |\ln(\ell_j)|. \tag{C.24}$$

Next, we observe[9] that for any $a > 0$ and $b > 0$, we have

$$\ln(1+ab) < |\ln(a)| + |\ln(b)| + 1. \tag{C.25}$$

Now, we only need to rewrite the error term from Lemma C.1 in the form we claim in this corollary. Thus, we estimate

$$\ln\left(1 + \frac{L\ell_j}{1+Ld_j}\right) = \ln\left(1 + \frac{\ell_j}{\frac{1}{L}+d_j}\right) \leq \ln(1+\frac{\ell_j}{d_j}) < |\ln(\ell_j)| + |\ln(d_j)| + 1. \tag{C.26}$$

Once we sum the error term estimate in (C.26) for $i = 1, \ldots, k-1$ and the one in (C.24) for $i = 1, \ldots, k$, we arrive at the claimed error estimate in (C.21). We also see that the sum of the main terms in (C.23) for $i = 1, \ldots, k$ agrees with the main term in (C.21). This finishes the proof. □

Appendix D. A Technical Lemma on Decaying Functions

This section contains a technical lemma that was useful to construct the sequence $(a_i)_{i\in\mathbb{N}}$ and the region Λ in the proof of Theorem 6.2.

Lemma D.1. *Let* $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be bounded and satisfy* $\lim_{L\to\infty} f(L) = 0$. *Then, there is a convex, non-increasing function* $\operatorname{Env}(f)\colon \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying* $\operatorname{Env}(f)(0) = 1, \lim_{L\to\infty}\operatorname{Env}(f)(L) = \lim_{L\to\infty} f(L)/\operatorname{Env}(f)(L) = 0$ *and* $\operatorname{Env}(f)(L) \geq C/\sqrt{\ln(2+L)}$ *for some* $C > 0$.

[9] If $ab \leq 1$, then $\ln(1+ab) \leq ab \leq 1$ and (C.25) holds. If $ab \geq 1$, then we distinguish between the case that both $a \geq 1$ and $b \geq 1$ and the case where one of them is smaller than 1. In the first case (C.25) is equivalent to $\ln(1+ab) - \ln(ab) = \ln(1+1/(ab)) \leq 1$ which holds because $\ln(1+1/(ab)) \leq 1/(ab) \leq 1$. In the remaining case, we may assume $a \leq 1$ and $b \geq 1$ (but still $ab \geq 1$). Then, $ab \leq b/a$ and $\ln(1+ab) = \int_1^{ab} \mathrm{d}x/x \leq \int_1^{b/a} \mathrm{d}x/x = \ln(b/a)$.

Proof. The conditions on $\mathrm{Env}(f)$ only get worse if we increase f. Hence, we can replace f by

$$\hat{f}(s) := \sup_{t>s} f(t). \tag{D.1}$$

This is non-increasing. To achieve the $f(L)/\mathrm{Env}(f)(L)$ condition we consider $\sqrt{\hat{f}}$. However, we still need to make sure that $\mathrm{Env}(f)$ is convex. For this reason, we need to consider the lower convex envelope $\breve{\sqrt{\hat{f}}}$. It is given by the supremum over all convex functions below $\sqrt{\hat{f}}$. Another way to think of it is that the area above the graph of $\breve{\sqrt{\hat{f}}}$ is the convex hull of the area above $\sqrt{\hat{f}}$. Finally, we define

$$\mathrm{Env}(f)(t) := N\left(\breve{\sqrt{\hat{f}}}\left(\frac{t}{2}\right) + \frac{1}{\sqrt{\ln(2+t)}}\right), \quad t \geq 0, \tag{D.2}$$

where N is a normalization constant to be chosen below. As the lower convex envelope and $\frac{1}{\sqrt{\ln(2+t)}}$ are convex, so is $\mathrm{Env}(f)$. As the lower convex envelope lies below the function, we have $\mathrm{Env}(f)(t) \leq N\Big(\sqrt{\hat{f}(t/2)} + \frac{1}{\sqrt{\ln(2+t)}}\Big) \to 0$ as $t \to \infty$. As $\mathrm{Env}(f)$ is convex and $\lim_{L\to\infty} \mathrm{Env}(f)(L) = 0$, $\mathrm{Env}(f)$ is non-increasing. The condition $\mathrm{Env}(f)(L) \geq C/\sqrt{\ln(2+L)}$ is trivially satisfied and implies $\mathrm{Env}(f)(0) > 0$ and hence allows us to choose N such that $\mathrm{Env}(f)(0) = 1$. We are only left with the condition $\lim_{L\to\infty} f(L)/\mathrm{Env}(f)(L) = 0$. To show this, it is sufficient to prove

$$\mathrm{Env}(f)(L) \geq C\sqrt{f(L)}. \tag{D.3}$$

By the definition of the convex envelope, for any $t \geq 0$, there are $0 \leq t_1 \leq t < t_2$ such that

$$\mathrm{Env}(f)(t)/N = \frac{t_2 - t}{t_2 - t_1}\sqrt{\hat{f}}\left(\frac{t_1}{2}\right) + \frac{t - t_1}{t_2 - t_1}\sqrt{\hat{f}}\left(\frac{t_2}{2}\right) + \frac{1}{\sqrt{\ln(2+t)}}. \tag{D.4}$$

If $t_2 \leq 2t$, as $\hat{f}$ is non-increasing, we get $\sqrt{\hat{f}}\left(\frac{t_j}{2}\right) \geq \sqrt{\hat{f}}(t)$ for $j = 1, 2$ and thus are finished. If $t_2 > 2t$, then, as $t_1 \geq 0$, we have $\frac{t_2 - t}{t_2 - t_1} > \frac{1}{2}$. Thus, it suffices to bound the first summand from below, which we already did. $\square$

References

[1] Abdul-Rahman, H., Fischbacher, C., Stolz, G.: Entanglement bounds in the XXZ quantum spin chain. Ann. Henri Poincaré **21**, 2327–2366 (2020)

[2] Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)

[3] Amico, L., Fazio, R., Osterloh, A., Vedral, V.: Entanglement in many-body systems. Rev. Mod. Phys. **80**, 517–576 (2008)

[4] Beaud, V., Warzel, S.: Bounds on the entanglement entropy of droplet states in the XXZ spin chain. J. Math. Phys. **59**, 012109 (2018). https://doi.org/10.1063/1.5007035

[5] Charles, L., Estienne, B.: Entanglement entropy and Berezin–Toeplitz operators. Commun. Math. Phys. **376**, 521–554 (2020)

[6] Eisert, J., Cramer, M., Plenio, M.B.: Area laws for the entanglement entropy—a review. Rev. Mod. Phys. **82**, 277 (2010)

[7] Elgart, A., Klein, A., Stolz, G.: Many-body localization in the droplet spectrum of the random XXZ quantum spin chain. J. Funct. Anal. **275**, 211–258 (2018)

[8] Elgart, A., Pastur, L., Shcherbina, M.: Large block properties of the entanglement entropy of free disordered Fermions. J. Stat. Phys. **166**, 1092–1127 (2017)

[9] Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. Revised Edition (1st edn.), Chapman and Hall/CRC (2015)

[10] Fock, V.: Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld. Z. Phys. **47**, 446–448 (1928)

[11] Fischbacher, C., Schulte, R.: Lower bound on the entanglement entropy of the XXZ spin ring. arXiv:2007.00735

[12] Fischbacher, C., Ogunkoya, O.: Entanglement entropy bounds in the higher spin XXZ chain. J. Math. Phys. **62**, 101901 (2021)

[13] Gioev, D., Klich, I.: Entanglement entropy of fermions in any dimension and the Widom conjecture. Phys. Rev. Lett. **96**, 100503 (2006)

[14] Guillemin, V., Pollack, A.: Differential Topology. Prentice-Hall (1974)

[15] Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865–942 (2009)

[16] Jin, B.-Q., Korepin, V.E.: Quantum spin chain, Toeplitz determinants and the Fisher–Hartwig conjecture. J. Stat. Phys. **116**, 79 (2004)

[17] Klich, I.: Lower entropy bounds and particle number fluctuations in a Fermi sea. J. Phys. A Math. Gen. **39**, L85 (2006)

[18] Landau, L.: Diamagnetismus der Metalle. Z. Phys. **64**, 629–637 (1930)

[19] Landau, H.J., Widom, H.: Eigenvalue distribution of time and frequency limiting. J. Math. Anal. Appl. **77**, 469–481 (1980)

[20] Leschke, H., Sobolev, A.V., Spitzer, W.: Scaling of Rényi entanglement entropies of the free Fermi-gas ground state: a rigorous proof. Phys. Rev. Lett. **112**, 160403 (2014)

[21] Leschke, H., Sobolev, A.V., Spitzer, W.: Asymptotic growth of the local ground-state entropy of the ideal Fermi gas in a constant magnetic field. Commun. Math. Phys. **381**, 673–705 (2021)

[22] Leschke, H., Sobolev, A.V., Spitzer, W.: Trace formulas for Wiener–Hopf operators with applications to entropies of free fermionic equilibrium states. J. Funct. Anal. **273**, 1049–1094 (2017)

[23] Leschke, H., Sobolev, A.V., Spitzer, W.: Rényi entropies of the free Fermi gas in multi-dimensional space at high temperature. Oper. Theory Adv. Appl. **289**, 477–508 (2022)

[24] Müller, P., Pastur, L., Schulte, R.: How much delocalisation is needed for an enhanced area law of the entanglement entropy? Commun. Math. Phys. **376**, 649–679 (2020)

[25] Müller, P., Schulte, R.: Stability of the enhanced area law of the entanglement entropy. Ann. Henri Poincaré **21**, 3639–3658 (2020)

[26] Müller, P., Schulte, R.: Stability of a Szegő-type asymptotics. J. Math. Phys. **64**, 022101 (2023)

[27] NIST Digital Library of Mathematical Functions. In: Olver, F.W.J., Olde Daalhuis, A.B., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A. (eds.). http://dlmf.nist.gov/. Release 1.1.5 of 2022-03-15

[28] Pastur, L., Slavin, V.: Area law scaling for the entropy of disordered quasifree fermions. Phys. Rev. Lett. **113**, 150404 (2014)

[29] Pfeiffer, P.: On the stability of the area law for the entanglement entropy of the Landau Hamiltonian. arXiv:2102.07287

[30] Pfirsch, B., Sobolev, A.V.: Formulas of Szegő type for the periodic Schrödinger operator. Commun. Math. Phys. **358**, 675–704 (2018)

[31] Rodríguez, I.D., Sierra, G.: Entanglement entropy of integer Quantum Hall states. Phys. Rev. B **80**, 153303 (2009)

[32] Sobolev, A.V.: Pseudo-differential operators with discontinuous symbols: Widom's Conjecture. Mem. AMS **222**, 1043 (2013)

[33] Sobolev, A.V.: Wiener–Hopf operators in higher dimensions: the Widom conjecture for piece-wise smooth domains. Integr. Equ. Oper. Theory **81**, 435–449 (2015)

[34] Sobolev, A.V.: On the Schatten-von Neumann properties of some pseudo-differential operators. J. Funct. Anal. **266**, 5886–5911 (2014)

[35] Sobolev, A.V.: Quasi-classical asymptotics for functions of Wiener–Hopf operators: smooth vs non-smooth symbols. Geom. Funct. Anal. **27**, 676–725 (2017)

[36] Widom, H.: On a class of integral operators with discontinuous symbol. Toeplitz centennial (Tel Aviv, 1981), pp. 477–500, Operator Theory: Adv. Appl., **4**, Birkhäuser, Basel-Boston, Mass (1982)

[37] Widom, H.: On a class of integral operators on a half-space with discontinuous symbol. J. Funct. Anal. **88**(1), 166–193 (1990)

Paul Pfeiffer and Wolfgang Spitzer
Fakultät für Mathematik und Informatik
FernUniversität in Hagen
Universitätsstraße 1
58097 Hagen
Germany
e-mail: paul.pfeiffer@fernuni-hagen.de;
wolfgang.spitzer@fernuni-hagen.de

Communicated by Alain Joye.
Received: February 28, 2023.
Accepted: October 6, 2023.