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Bulk Behaviour of Ground States for Relativistic Schrödinger Operators with Compactly Supported Potentials

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Abstract. We propose a probabilistic representation of the ground states of massive and massless Schrödinger operators with a potential well in which the behaviour inside the well is described in terms of the moment-generating function of the first exit time from the well and the outside behaviour in terms of the Laplace transform of the first entrance time into the well. This allows an analysis of their behaviour at short to mid-range from the origin. In a first part, we derive precise estimates on these two functionals for stable and relativistic stable processes. Next, by combining scaling properties and heat kernel estimates, we derive explicit local rates of the ground states of the given family of non-local Schrödinger operators both inside and outside the well. We also show how this approach extends to fully supported decaying potentials. By an analysis close-by to the edge of the potential well, we furthermore show that the ground state changes regularity, which depends qualitatively on the fractional power of the non-local operator.

Mathematics Subject Classification. Primary 47D08, 60G51; Secondary 47D03, 47G20.

Contents

Birkhäuser

1. Introduction

The purpose of this paper is to introduce and explore a relationship between the moment-generating functions and Laplace transforms of first hitting times of rotationally symmetric stable and relativistic stable processes, and the ground states of related non-local Schrödinger operators. Making use of this relationship, via precise estimates of these random time functionals we will be able to derive and prove the spatial localization properties of ground states in the bulk, i.e. for short to middle range from the origin. These non-local Schrödinger operators and their ground states closely relate with the generators and stationary measures, respectively, of random motion in a (possibly rugged) potential landscape, as it will be explained below.

The (semi-)relativistic Schrödinger operator $H = (-\Delta + m^2)^{1/2} - m + V$ on $L^2(\mathbb{R}^3)$, describing the Hamiltonian of an electrically charged particle with rest mass $m > 0$ moving under a Coulomb potential V is one of the fundamental models of mathematical quantum theory, and it has been studied extensively in the literature. Classic papers include [13,43,61] on the square-root Klein–Gordon equation, [20,21,32,63] on the properties of the spectrum, stability of the matter [26,27,46,47], and eigenfunction decay [15]. More recent developments further addressed low-energy scattering theory [54], embedded eigenvalues and Neumann–Wigner type potentials [50], decay rates when magnetic potentials and spin are included [33], a relativistic Kato-inequality [34], Carleman estimates and unique continuation [25,55], or nonlinear relativistic Schrödinger equations [1,19,59]. Given its relationship with random processes with jumps, the $V = 0$ case has received much attention also in potential theory [14,31,56].

There are only a very few examples around for which the spectrum and eigenfunctions of relativistic Schrödinger operators are explicitly determined [21,49], when the potential is confining rather than decaying, and interesting approximations of spectra and eigenfunctions for some other cases have been obtained in [40]. Thus, estimates on the eigenfunctions have a special relevance. While eigenfunction decay at infinity for a large class of non-local Schrödinger operators, including the relativistic operator, is now understood to a great detail in function of the asymptotic behaviour of the potential [15,33,37,38], very little is known on their local behaviour, i.e. for small to medium distances from the origin. Some information on local properties of eigenfunctions of non-local Schrödinger operators with Bernstein functions of the Laplacian and general potential wells has been obtained in [9, Sect. 4]. Specifically, these include estimates on the distance of the location of global extrema of eigenfunctions from the edge of the potential well or specific level sets. For domain operators results in a similar spirit have been obtained in [6,7].

Our goal in this paper is to make up for this hiatus and derive the local behaviour of the ground state of the relativistic operator when V is chosen to be a bounded potential of compact support and show how our technique extends to fully supported potentials. Instead of the above operator, we will consider more generally

$$H_{m,\alpha} = L_{m,\alpha} + V \quad \text{with} \quad L_{m,\alpha} = (-\Delta + m^{2/\alpha})^{\alpha/2} - m$$

on $L^2(\mathbb{R}^d)$, where $0 < \alpha < 2$, $m \geq 0$, and $d \in \mathbb{N}$, and for simplicity we call it in the $m > 0$ case the massive, and for $m = 0$ the massless relativistic Schrödinger operator. In case $V = -v\mathbf{1}_{\mathcal{K}}$ with a bounded set $\mathcal{K} \subset \mathbb{R}^d$ with non-empty interior, we say that V is a potential well with coupling constant (or depth) $v > 0$.

The main idea underlying our approach is simple, and it can be highlighted on the case of a spherical potential well $\mathcal{K} = \mathcal{B}_a$, where $\mathcal{B}_a$ is a ball of radius a centred in the origin. When the operator $H_{m,\alpha}$ has a ground state φ_0 at eigenvalue $\lambda_0 = \inf \operatorname{Spec} H_{m,\alpha}$, due to the eigenvalue equation we have $e^{-tH_{m,\alpha}}\varphi_0 = e^{-\lambda_0 t}\varphi_0$ and a path integral representation gives

$$\varphi_0(x) = e^{\lambda_0 t} e^{-tH_{m,\alpha}} \varphi_0(x) = e^{\lambda_0 t} \mathbb{E}^x [e^{-\int_0^t V(X_s)\mathrm{d}s} \varphi_0(X_t)], \quad t \geq 0,$$

for every point $x \in \mathbb{R}^d$ (see [48]), where $\mathbb{E}^x$ denotes expectation with respect to the path measure of the Markov process $(X_t)_{t \geq 0}$ starting at x, whose infinitesimal generator is $-L_{m,\alpha}$. This process is a rotationally symmetric α-stable process for $m = 0$ and a rotationally symmetric relativistic α-stable process for $m > 0$. Also, since V is a potential well supported on $\mathcal{B}_a$, now $\int_0^t V(X_s)\mathrm{d}s = -v \int_0^t \mathbf{1}_{\mathcal{B}_a}(X_s)\mathrm{d}s = -vU_t(a)$ is, apart from the constant prefactor, the occupation measure in the ball of the process $(X_t)_{t \geq 0}$. Clearly, the potential contributes as long as $X_t \in \mathcal{B}_a$ only, thus we may consider the first exit time $\tau_a = \inf\{t > 0 : X_t \in \mathcal{B}_a^c\}$ when starting from the inside, and the first entrance time $T_a = \inf\{t > 0 : X_t \in \mathcal{B}_a\}$ when starting from outside of the well. Since, crucially, $\left(e^{\lambda_0 t} e^{vU_t(a)} \varphi_0(X_t)\right)_{t \geq 0}$ can be shown to be a martingale, by optional stopping we get

$$\varphi_0(x) = \begin{cases} \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}\varphi_0(X_{\tau_a})] \text{ if } & x \in \mathcal{B}_a \\ \mathbb{E}^x[e^{-|\lambda_0|T_a}\varphi_0(X_{T_a})] \quad \text{if} & x \in \mathcal{B}_a^c. \end{cases} \tag{1.1}$$

When we work with a classical Schrödinger operator having $-\frac{1}{2}\Delta$ instead of the relativistic operator, so that $(X_t)_{t\geq 0} = (B_t)_{t\geq 0}$ is Brownian motion, due to path continuity the random variables B_{T_a} and B_{τ_a} are supported on the boundary of $\mathcal{B}_a$, and φ_0 can be determined exactly from (1.1). (This is shown in full detail in Sect. 5.1.) However, when we work with the non-local operators $H_{m,\alpha}$, the random process $(X_t)_{t\geq 0}$ is a jump process and now the supports of X_{T_a} and X_{τ_a} spread over the full sets $\mathcal{B}_a$ and $\mathcal{B}_a^c$, respectively, and obtaining φ_0 in an explicit form becomes very difficult, if not hopeless. Nevertheless, since $|X_{T_a}| \leq a$ and $|X_{\tau_a}| \geq a$ almost surely, using that φ_0 is (in a spherical potential well, radially) monotone decreasing, the expressions (1.1) yield good approximations of the form

$$\varphi_0(x) \asymp \begin{cases} \varphi_0(\mathbf{a})\mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}\right] \text{ if } & x \in \mathcal{B}_a \\ \varphi_0(\mathbf{a})\mathbb{E}^x[e^{-|\lambda_0|T_a}] \quad \text{if} & x \in \mathcal{B}_a^c, \end{cases} \tag{1.2}$$

where $\mathbf{a} = (a, 0, \dots, 0)$. Thus, the problem translates into deriving sharp estimates on the moment-generating function of τ_a and the Laplace transform of T_a. Our main goal in this paper is then to derive precise estimates of these functionals and show how they give tight two-sided bounds on the ground states.

We note that for the classical Schrödinger operator $H = -\frac{1}{2}\Delta - v\mathbf{1}_{\mathcal{B}_a}$ these random functionals (dependent on Brownian motion) can be determined exactly either by optional stopping methods, or via the solution of the eigenvalue equation, which is a convenient PDE. However, for $H_{m,\alpha}$ there is no available solution of the similar eigenvalue equation, due to the non-locality of the operator. Thus, the probabilistic alternative which we develop in this paper will prove to be particularly useful in studying the behaviour of ground states.

To derive bulk estimates of the ground state, we go through these steps systematically, leading to the following main results.

(1) *Symmetry Properties of the Ground State* It is intuitively clear that the ground state should inherit the symmetry properties of the potential well, which is also a technically relevant ingredient in deriving local estimates. In Theorem 4.1, we show rotational symmetry of the ground state when the potential well is a ball, and in Theorem 4.2 reflection symmetry when the potential well has the same symmetry with respect to a hyperplane.

(2) *Local Estimates of the Ground State* In Theorem 5.1, we prove that the expressions (1.1) provide the two-sided bounds (1.2) above, where the dependence of the comparability constants on the parameters of the non-local operator, potential well and spatial dimension can be tracked throughout. By deriving precise two-sided estimates on the moment-generating function of τ_a and the Laplace transform of T_a in Sect. 3,

we can make the expressions more explicit and obtain

$$\frac{\varphi_0(x)}{\varphi_0(\mathbf{a})} \asymp \begin{cases} 1 + \frac{v-|\lambda_0|}{\lambda_a - v + |\lambda_0|}\left(\frac{a-|x|}{a}\right)^{\alpha/2} & \text{if} \quad x \in \mathcal{B}_a \\ j_{m,\alpha}(|x|) & \text{if} \quad x \in \mathcal{B}_a^c, \end{cases} \tag{1.3}$$

see Corollary 5.1, where $j_{m,\alpha}$ denotes the jump kernel of the operator $L_{m,\alpha}$ (see details in Sect. 2.1), and $\lambda_a = \lambda_a(m,\alpha)$ is its principal Dirichlet eigenvalue for $\mathcal{B}_a$. While the comparability constants depend on m, inside the potential well the x-dependence is the same for both the massless and massive cases, reflecting the fact that the two processes are locally comparable. Since by using the L^2-normalization condition on the ground state the value $\varphi_0(\mathbf{a})$ can further be estimated from both sides (Proposition 5.3), the right hand side above actually provides pointwise bounds on φ_0 itself, with a new proportionality constant (Corollary 5.2). As an application of the information on the local behaviour, in Sect. 5.4 we estimate the ground state expectations $\Lambda_p(\varphi_0) = \left(\int_{\mathbb{R}^d} |x|^p \varphi_0^2(x)\mathrm{d}x\right)^{1/p}$, i.e. the moments of the position in the weighted space $L^2(\mathbb{R}^d, \varphi_0^2\mathrm{d}x)$ describing the "halo" or size of the ground state on different scales, while in Sect. 5.5 we discuss concentration properties of the ground state distribution $\varphi_0^2\mathrm{d}x$. Finally, in Theorems 5.3–5.4 we obtain counterparts of (1.2)–(1.3) to bounded decaying potentials supported everywhere in $\mathbb{R}^d$ with a general profile, giving estimates of φ_0 on appropriate level sets of the potential.

Using all this information, we also get some insight into the mechanisms driving these two regimes of behaviour:

(i) *Inside the Potential Well* Since we show that $(a-|x|)^{\alpha/2} \asymp \mathbb{E}^x[\tau_a]$, from (1.3) we see that the behaviour of $\varphi_0(x)/\varphi_0(\mathbf{a})$ is essentially determined by the ratio $\mathbb{E}^x[\tau_a]/\mathbb{E}^0[\tau_a]$ of mean exit times. Note that this is different from the case of the classical Schrödinger operator with the same potential well (see Sect. 5.1). For Brownian motion in $\mathbb{R}^d$, it is well known that $\mathbb{E}^x[\tau_a] = \frac{1}{d}(a^2 - |x|^2)$ and the moment-generating function of τ_a for $d = 1$ is given by $\mathbb{E}^x[e^{u\tau_a}] = \cos(\sqrt{2u}x)/\cos(\sqrt{2u}a)$ (and Bessel functions for higher dimensions, see Remark 5.1), thus the relation $\varphi_0(x)/\varphi_0(\mathbf{a}) \approx \mathbb{E}^x[\tau_a]/\mathbb{E}^0[\tau_a]$ no longer holds and the higher-order moments of τ_a contribute significantly. The reason for this can be appreciated to be that the α-stable and relativistically α-stable processes related to $L_{m,\alpha}$ and $L_{0,\alpha}$, respectively, have a different nature from Brownian motion. Indeed, we have shown previously that these two processes satisfy the jump-paring property, i.e. that all multiple large jumps are stochastically dominated by single large jumps, while Brownian motion evolves through typically small increments and builds up "backlog events" inflating sojourn times (for the definitions and discussion see [37, Sect. 2.1], [38, Def. 2.1, Rem. 4.4]). Furthermore, it is also seen from (1.3) that the ratio between the maximum $\varphi_0(0)$ of the ground state and $\varphi_0(\mathbf{a})$ is determined by $\frac{\lambda_a}{\lambda_a - (v - |\lambda_0|)}$, i.e. in fact the ratio of the gap between the ground state energy from the bottom value of the potential and the energy necessary to climb the well.

(ii) *Outside the Potential Well* The behaviour outside is governed by the Lévy measure which was shown in [38] for large enough $|x|$ and we see here by a different approach that this already sets in right from the boundary of the potential well. This is heuristically to be expected due to free motion everywhere outside the well, while to see a "second order" contribution of non-locality (distinguishing between polynomially vs exponentially decaying jump measures) around the boundary of the well would need more refined tools.

(iii) *At the Boundary of the Potential Well* From the profile functions given by (1.3), it can be conjectured that although the ground state is continuous (see Sect. 2.2), its change of behaviour around the edge of the potential well is rather abrupt. Indeed, in Theorem 5.2 and Remark 5.4 we show that at the boundary $\varphi_0 \notin C^{\alpha+\delta}_{\mathrm{loc}}(\mathcal{B}_{a+\varepsilon} \backslash \overline{\mathcal{B}}_{a-\varepsilon})$ for every $\delta \in (0, 1-\alpha)$ whenever $\alpha \in (0,1)$, and $\varphi_0 \notin C^{1,\alpha+\delta-1}_{\mathrm{loc}}(\mathcal{B}_{a+\varepsilon} \backslash \overline{\mathcal{B}}_{a-\varepsilon})$ for every $\delta \in (0, 2-\alpha)$ whenever $\alpha \in [1,2)$, for any small $\varepsilon > 0$. This implies that for the range of small α the ground state cannot be C^1 at the boundary, and for values of α starting from 1 it cannot be C^2 at the boundary.

(3) *Entrance/Exit Time Estimates* All these results depend on precise two-sided estimates on the moment-generating function for exit times from balls, and the Laplace transform of hitting times for balls, which we provide here (Sect. 3). Clearly, these are of independent interest in probabilistic potential theory; for further applications, see [24] on crossing times of subordinate Bessel processes.

For the remaining part of the paper, we proceed in Sect. 2 to a precise description of the operators and processes and in Sect. 3 to presenting the details of hitting/exit time estimates. Then, in Sect. 4 we show the martingale property mentioned above and symmetry of the ground state and in Sect. 5 derive the local estimates, regularity results and study the moments of the position in the ground states.

2. Preliminaries

2.1. The Massive and Massless Relativistic Operators

Let $\alpha \in (0,2)$, $m \geq 0$, $\Phi_{m,\alpha}(z) = (z + m^{2/\alpha})^{\alpha/2} - m$ for every $z \geq 0$, and denote

$$\begin{aligned} L_{m,\alpha} &= \Phi_{m,\alpha}(-\Delta) = (-\Delta + m^{2/\alpha})^{\alpha/2} - m \quad &\text{if} \quad m > 0 \\ L_{0,\alpha} &= \Phi_{0,\alpha}(-\Delta) = (-\Delta)^{\alpha/2} \quad &\text{if} \quad m = 0. \end{aligned}$$

We will combine the notation into just $L_{m,\alpha}$, $m \geq 0$, when a statement refers to both cases. These operators can be defined in several possible ways. We define them via the Fourier multipliers

$$(\widehat{L_{m,\alpha} f})(y) = \Phi_{m,\alpha}(|y|^2)\widehat{f}(y), \quad y \in \mathbb{R}^d,\ f \in \mathrm{Dom}(L_{m,\alpha}),$$

with domain

$$\mathrm{Dom}(L_{m,\alpha}) = \Big\{ f \in L^2(\mathbb{R}^d) : \Phi_{m,\alpha}(|\cdot|^2)\widehat{f} \in L^2(\mathbb{R}^d) \Big\}, \quad m \geq 0.$$

Then, for $f \in C_c^\infty(\mathbb{R}^d)$ the expressions

$$L_{m,\alpha} f(x) = -\lim_{\varepsilon \downarrow 0} \int_{|y-x|>\varepsilon} (f(y) - f(x))\, \nu_{m,\alpha}(\mathrm{d}y) \tag{2.1}$$

hold, with the Lévy measures

$$\nu_{m,\alpha}(\mathrm{d}x) = j_{m,\alpha}(|x|)\mathrm{d}x = \frac{2^{\frac{\alpha-d}{2}} m^{\frac{d+\alpha}{2\alpha}} \alpha}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})} \frac{K_{(d+\alpha)/2}(m^{1/\alpha}|x|)}{|x|^{(d+\alpha)/2}}\, \mathrm{d}x, \quad x \in \mathbb{R}^d \backslash \{0\},$$

for $m > 0$ (*relativistic fractional Laplacian*), and

$$\nu_{0,\alpha}(\mathrm{d}x) = j_{0,\alpha}(|x|)\mathrm{d}x = \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}|\Gamma(-\frac{\alpha}{2})|} \frac{\mathrm{d}x}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d \backslash \{0\}$$

for $m = 0$ (*fractional Laplacian*). Here

$$K_\rho(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\rho \int_0^\infty t^{-\rho-1} e^{-t-\frac{z^2}{4t}}\, \mathrm{d}t, \quad z > 0,\ \rho > -\frac{1}{2}.$$

is the standard modified Bessel function of the third kind. The operator $L_{m,\alpha}$ is positive and self-adjoint with core $C_c^\infty(\mathbb{R}^d)$, for every $0 < \alpha < 2$ and $m \geq 0$.

The difference of the massive and massless operators is bounded, and the relationship can be made explicit, which will be useful below. For $m, r > 0$ denote

$$\begin{aligned} \sigma_{m,\alpha}(r) &= \frac{\alpha 2^{1-\frac{d-\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}} \left(\frac{2^{\frac{d+\alpha}{2}-1}\Gamma\left(\frac{d+\alpha}{2}\right)}{r^{d+\alpha}} - \frac{m^{\frac{d+\alpha}{2\alpha}} K_{\frac{d+\alpha}{2}}\left(m^{1/\alpha} r\right)}{r^{\frac{d+\alpha}{2}}} \right) \\ &= \frac{\alpha 2^{1-\frac{d-\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right)\pi^{\frac{d}{2}}} \frac{1}{r^{d+\alpha}} \int_0^{m^{1/\alpha} r} w^{\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}-1}(w)\mathrm{d}w, \end{aligned}$$

and define the measure

$$\Sigma_{m,\alpha}(A) = \int_A \sigma_{m,\alpha}(|x|)\mathrm{d}x,$$

for all Borel sets $A \subset \mathbb{R}^d$. It can be shown that $\Sigma_{m,\alpha}$ is finite, positive and has full mass $\Sigma_{m,\alpha}(\mathbb{R}^d) = m$. For every function $f \in L^\infty(\mathbb{R}^d)$ consider the operator

$$G_{m,\alpha} f(x) = \frac{1}{2} \int_{\mathbb{R}^d} (f(x+h) - 2f(x) + f(x-h))\sigma_{m,\alpha}(|h|)\mathrm{d}h.$$

which is well defined and $\|G_{m,\alpha} f\|_\infty \leq 2\, m\, \|f\|_\infty$ holds. Then, the decomposition

$$j_{0,\alpha}(r) = j_{m,\alpha}(r) + \sigma_{m,\alpha}(r) \tag{2.2}$$

holds, which implies the formula

$$L_{m,\alpha} f = L_{0,\alpha} f - G_{m,\alpha} f,$$

for every function f belonging to the domain of $L_{m,\alpha}$. For the details and proofs we refer to [3, Sect. 2.3.2], see also [56, Lem. 2].

Next consider the multiplication operator $V : \mathbb{R}^d \to \mathbb{R}$ on $L^2(\mathbb{R}^d)$, which plays the role of the potential. In case $V = -v\mathbf{1}_{\mathcal{K}}$ with a bounded set $\mathcal{K} \subset \mathbb{R}^d$ having a non-empty interior, we say that V is a potential well with coupling constant $v > 0$. Since such a potential is relatively bounded with respect to $L_{m,\alpha}$, the operator

$$H_{m,\alpha} = L_{m,\alpha} - v\mathbf{1}_{\mathcal{K}} \tag{2.3}$$

can be defined by standard perturbation theory as a self-adjoint operator with core $C_c^\infty(\mathbb{R}^d)$. For simplicity, we call $H_{m,\alpha}$ the (massive or massless) *relativistic Schrödinger operator* with potential well supported in $\mathcal{K}$, no matter the value of $\alpha \in (0, 2)$.

Below we will use the following notations. For two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ we write $f(x) \asymp g(x)$ if there exists a constant $C \geq 1$ such that $(1/C)g(x) \leq f(x) \leq Cg(x)$. We denote $f(x) \sim g(x)$ as $|x| \to \infty$ (resp. if $|x| \downarrow 0$) if $\lim_{|x|\to\infty} \frac{f(x)}{g(x)} = 1$ (resp. if $\lim_{|x|\downarrow 0} \frac{f(x)}{g(x)} = 1$). Finally, we denote $f(x) \approx g(x)$ as $|x| \to \infty$ (analogously for $|x| \downarrow 0$) if there exists a constant $C \geq 1$ such that $(1/C) \leq \liminf_{|x|\to\infty} f(x)/g(x) \leq \limsup_{|x|\to\infty} f(x)/g(x) \leq C$. Also, we will use the notation $\mathcal{B}_r(x)$ for a ball of radius r centred in $x \in \mathbb{R}^d$, write just $\mathcal{B}_r$ when $x = 0$, and $\omega_d = |\mathcal{B}_1|$ for the volume of a d-dimensional unit ball. Moreover, for a domain $\mathcal{D} \subset \mathbb{R}^d$ we write $\mathcal{D}^c$ to denote $\mathbb{R}^d \setminus \overline{\mathcal{D}}$. In proofs we number the constants in order to be able to track them, but the counters will be reset in a subsequent statement and proof. Also, in the statements to follow, we will use the default assumptions $0 < \alpha < 2$ and $m \geq 0$ implicitly, unless specified otherwise.

2.2. Feynman–Kac Representation and the Related Random Processes

The operators $-L_{m,\alpha}$ are Markov generators and give rise to the following Lévy processes, which can be realized on the space of càdlàg paths (i.e. the space of functions that are continuous from the right with left limits), indexed by the positive semi-axis. To ease the notation, we denote these processes by $(X_t)_{t\geq 0}$ without subscripts, and it will be clear from the context which process it refers to. Also, we denote by $\mathbb{P}^x$ the probability measure on the space of càdlàg paths, induced by the process $(X_t)_{t\geq 0}$ starting from $x \in \mathbb{R}^d$, by $\mathbb{E}^x$ expectation with respect to $\mathbb{P}^x$, and simplify the notations to $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. We will also use the notation $\mathbb{E}^x[f(X_t); \text{conditions}]$ to mean $\mathbb{E}^x[f(X_t)\mathbf{1}_{\{\text{conditions}\}}]$.

If $m > 0$, the operator $-L_{m,\alpha}$ generates a rotationally symmetric relativistic α-stable process $(X_t)_{t\geq 0}$, and if $m = 0$, the operator $-L_{0,\alpha}$ generates a rotationally symmetric α-stable process $(X_t)_{t\geq 0}$. Thus, in either case

$$P_t f(x) := \left(e^{-tL_{m,\alpha}} f\right)(x) = \mathbb{E}^x[f(X_t)], \quad x \in \mathbb{R}^d,\ t \geq 0,\ f \in L^2(\mathbb{R}^d),$$

holds, giving rise to the Markov semigroup $\{P_t : t \geq 0\}$. Each P_t, $t > 0$, is an integral operator with translation invariant integral kernel $p(t, x, y) :=$

$p_t(x-y)$, i.e. $P_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y) \mathrm{d}y$ for all $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$. Also,

$$\mathbb{E}[e^{iu \cdot X_t}] = e^{t\Phi_{m,\alpha}(|u|^2)}, \quad u \in \mathbb{R}^d,\, m \ge 0,$$

so that $\Phi_{m,\alpha}(|u|^2) = (|u|^2 + m^{2/\alpha})^{\alpha/2} - m$, $m > 0$, gives the characteristic exponent of the rotationally symmetric relativistic α-stable process, which has the Lévy jump measure $\nu_{m,\alpha}(\mathrm{d}x)$, and $\Phi_{0,\alpha}(|u|^2) = |u|^\alpha$ gives the characteristic exponent of the rotationally symmetric α-stable process, which has the Lévy jump measure $\nu_{0,\alpha}(\mathrm{d}x)$. From a straightforward analysis, it can be seen that for small $|x|$ the Lévy intensity $j_{m,\alpha}(x)$ behaves like $j_{0,\alpha}(x)$, but due to $K_\rho(x) \sim C|x|^{-1/2} e^{-|x|}$ as $|x| \to \infty$ for a suitable constant $C > 0$, it decays exponentially at large distances, while $j_{0,\alpha}(x)$ is polynomial. This difference in the behaviours has a strong impact on the properties of the two processes.

The main object of interest in this paper are the ground states φ_0 of the operators $H_{m,\alpha}$ as given by (2.3), i.e. nonzero solutions of the eigenvalue equation

$$H_{m,\alpha}\varphi_0 = \lambda_0 \varphi_0$$

corresponding to the lowest eigenvalue, so that $\varphi_0 \in \mathrm{Dom}(H_{m,\alpha}) \setminus \{0\} \subset L^2(\mathbb{R}^d)$ and $\lambda_0 = \inf \operatorname{Spec} H_{m,\alpha}$, whenever they exist. As usual, we choose the normalization $\|\varphi_0\|_2 = 1$. Since the potentials $V = -v\mathbf{1}_{\mathcal{K}}$ are relatively compact perturbations of $H_{m,\alpha}$, the essential spectrum is preserved, and thus $\operatorname{Spec} H_{m,\alpha} = \operatorname{Spec}_{\mathrm{ess}} H_{m,\alpha} \cup \operatorname{Spec}_{\mathrm{d}} H_{m,\alpha}$, with $\operatorname{Spec}_{\mathrm{ess}} H_{m,\alpha} = \operatorname{Spec}_{\mathrm{ess}} L_{m,\alpha} = [0,\infty)$. The existence of a discrete component depends on further details of the potential. Generally, $\operatorname{Spec}_{\mathrm{d}} H_{m,\alpha} \subset (-v, 0)$, and $\operatorname{Spec}_{\mathrm{d}} H_{m,\alpha}$ consists of a finite set of isolated eigenvalues of finite multiplicity each.

The following result summarizes basic information on the ground states in our present set-up.

Proposition 2.1. *For the ground state φ_0 of $H_{m,\alpha}$, the following hold:*

(1) Existence
  - *(i)* Massive Case: *Let $m > 0$. Then, $H_{m,\alpha}$ has a ground state φ_0 for every $\alpha \in (0,2)$ and every $v > 0$, whenever $d = 1$ or $d = 2$. If $d \ge 3$, then there exists $0 < v^*_{m,\alpha,d} < \infty$ such that $H_{m,\alpha}$ has a ground state φ_0 for every $\alpha \in (0,2)$ and every $v > v^*_{m,\alpha,d}$.*
  - *(ii)* Massless Case: *Let $m = 0$. Then, $H_{0,\alpha}$ has a ground state φ_0 for every $\alpha \in [1,2)$ and every $v > 0$, whenever $d = 1$. If (a) $d = 1$ and $\alpha \in (0,1)$, or (b) $d \ge 2$ and $\alpha \in (0,2)$, then there exists $0 < v^*_{0,\alpha,d} < \infty$ such that $H_{0,\alpha}$ has a ground state φ_0 for every $v > v^*_{0,\alpha,d}$.*

(2) Uniqueness: *If a ground state φ_0 of $H_{m,\alpha}$ exists, then it is unique. Moreover, it has a version (with respect to Lebesgue measure on $\mathbb{R}^d$) such that $\varphi_0 > 0$.*

(3) Regularity: *For every $m \ge 0$ and $\alpha \in (0,2)$ the ground state φ_0 of $H_{m,\alpha}$ is a bounded continuous function, and has a pointwise decay to zero at*

infinity given by

$$\varphi_0(x) \approx j_{m,\alpha}(|x|) \begin{cases} = \mathcal{A}_{d,\alpha,0}|x|^{-d-\alpha} & \text{for } m = 0 \\ \sim \mathcal{A}_{d,\alpha,m}|x|^{-(d+\alpha+1)/2}e^{-m^{1/\alpha}|x|} & \text{for } m > 0, \end{cases}$$

for $|x| \to \infty$*, where*

$$\mathcal{A}_{d,\alpha,m} = \begin{cases} \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}|\Gamma(-\frac{\alpha}{2})|} & m = 0, \\ \frac{2^{(\alpha-d-1)/2}m^{(d+\alpha-1)/2}\alpha}{\pi^{(d-1)/2}\Gamma(1-\frac{\alpha}{2})} & m > 0. \end{cases}$$

(4) Feynman–Kac Representation: *Whenever a ground state* φ_0 *of the operator* $H_{m,\alpha}$ *exists, the expression*

$$e^{-tH_{m,\alpha}}\varphi_0(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)\mathrm{d}s}\varphi_0(X_t)] = \mathbb{E}^x[e^{vU_t^{\mathcal{K}}(X)}\varphi_0(X_t)] \tag{2.4}$$

holds for every $x \in \mathbb{R}^d$ *and* $t \geq 0$*, where* $U_t^{\mathcal{K}}(X) = \int_0^t \mathbf{1}_{\mathcal{K}}(X_s)\mathrm{d}s$ *is the occupation measure of the set* $\mathcal{K}$ *by* $(X_t)_{t\geq 0}$*.*

(5) Properties of the Feynman–Kac Semigroup: *For all* $m \geq 0$ *and* $\alpha \in (0, 2)$ *the semigroup* $\{T_t : t \geq 0\}$*,* $T_t = e^{-tH_{m,\alpha}}$*, is well-defined and strongly continuous. For all* $t > 0$*, every* T_t *is a bounded operator on every* $L^p(\mathbb{R}^d)$ *space,* $1 \leq p \leq \infty$*. The operators* $T_t : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ *for* $1 \leq p \leq \infty$*,* $T_t : L^p(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ *for* $1 < p \leq \infty$*, and* $T_t : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ *are bounded, for all* $t > 0$*. Also,* T_t *has a bounded measurable integral kernel* $q(t, x, y)$ *for all* $t > 0$*, i.e.* $T_t f(x) = \int_{\mathbb{R}^d} q(t, x, y)f(y)\mathrm{d}y$*, for all* $f \in L^p(\mathbb{R}^d)$*,* $1 \leq p \leq \infty$*.*

Proof. Since these properties are consequences of well-established general results, we only sketch a proof and refer the reader to the literature.

(1) For non-positive compactly supported potentials it is known that $\mathrm{Spec_d}\, H_{m,\alpha} \neq \emptyset$ if $(X_t)_{t\geq 0}$ is a recurrent process [15], [48, Th. 4.308]. Recall the Chung-Fuchs criterion of recurrence, which says that for a process with characteristic exponent Ψ the condition $\int_{|u|<r} \frac{\mathrm{d}u}{\Psi(u)} < \infty$ for some $r > 0$, is equivalent with the transience of the process [57, Cor. 37.17], [48, Th. 3.84]. Existence of φ_0 follows then by the Chung-Fuchs criterion applied to $\Psi(u) = \Phi_{m,\alpha}(|u|^2)$, which gives that the relativistic α-stable process is recurrent whenever $d = 1$ or $d = 2$, and transient for $d \geq 3$, while the α-stable process is recurrent in case $d = 1$ and $\alpha \geq 1$, and transient otherwise. In the transient cases, [4, Prop. 2.7] (or alternatively, the min-max principle) guarantees that for sufficiently large v a ground state exists. A sufficient condition is $v > \lambda_{\mathcal{K}}$, where $\lambda_{\mathcal{K}}$ is the principal Dirichlet eigenvalue of $L_{m,\alpha}$ over the well $\mathcal{K}$; we also note the bound $v + \lambda_0 < \lambda_{\mathcal{K}}$, see [4, Lem. 4.5].

(2) Uniqueness and strict positivity follow by the positivity improving property of the Feynman–Kac semigroup, for details see [48, Sects. 4.3.2, 4.9.1] and the references therein.

(3) By [48, Th. 4.107, Prop. 4.291], for all $t > 0$ and $f \in L^\infty(\mathbb{R}^d)$, we have that $T_t f$ is a bounded continuous function. Boundedness and continuity

of φ_0 are then immediate by property (5) due to the eigenvalue equation giving $\varphi_0 = e^{\lambda_0 t} T_t \varphi_0$, $t \geq 0$. The behaviour at infinity follows by the results for general decaying potentials obtained in [38].

(4) See [48, Sect. 4.6] and the references quoted there.
(5) See [48, Prop. 4.291].

Whenever a ground state φ_0 exists, throughout the remainder of this paper we will choose its strictly positive version guaranteed by the above results.

We note that φ_0 has an independent interest in probability. Since $e^{-tH_{m,\alpha}} \mathbf{1}_{\mathbb{R}^d} \neq \mathbf{1}_{\mathbb{R}^d}$ for any $t > 0$, the semigroup $\{T_t : t \geq 0\}$ is not Markovian and the Lévy process $(X_t)_{t \geq 0}$ perturbed by the potential V is no longer a random process. However, by a suitable Doob transform one can change the measure under which it does become a Markov process. Recall that $\varphi_0 > 0$ and $\|\varphi_0\|_2 = 1$, and define the unitary map $\mathrm{U} : L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x) \to L^2(\mathbb{R}^d, \mathrm{d}x)$, $f \mapsto \varphi_0 f$. This gives rise to the semigroup (known as the intrinsic Feynman–Kac semigroup)

$$\widetilde{T}_t f(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t(\varphi_0 f)(x)$$

associated with $\{T_t : t \geq 0\}$. Using the integral kernel $q(t,x,y)$ of T_t, we then have that $\widetilde{T}_t f(x) = \int_{\mathbb{R}^d} \tilde{q}(t,x,y) f(y) \varphi_0^2(y) \mathrm{d}y$, with

$$\widetilde{q}(t,x,y) = \frac{e^{\lambda_0 t} q(t,x,y)}{\varphi_0(x)\varphi_0(y)},$$

and infinitesimal generator $-\widetilde{H}_{m,\alpha}$, where $\widetilde{H}_{m,\alpha} = \mathrm{U}^{-1}(H_{m,\alpha} - \lambda_0)\mathrm{U}$ with domain $\mathrm{Dom}(\widetilde{H}_{m,\alpha}) = \left\{ f \in L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x) : \mathrm{U}f \in \mathrm{Dom}(H_{m,\alpha}) \right\}$. The operators $\widetilde{T}_t = e^{-t\widetilde{H}_{m,\alpha}}$ are contractions and we have $\widetilde{T}_t \mathbf{1}_{\mathbb{R}^d} = \mathbf{1}_{\mathbb{R}^d}$ for all $t \geq 0$, thus $\{\widetilde{T}_t : t \geq 0\}$ is a Markov semigroup on $L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)$. Then, it can be shown that for all $x \in \mathbb{R}^d$ there exists a probability measure $\widetilde{\mathbb{P}}^x$ on two-sided càdlàg path space $D(\mathbb{R}, \mathbb{R}^d)$ and a random process $(\widetilde{X}_t)_{t \in \mathbb{R}}$ satisfying the following properties:

(1) Let $-\infty < t_0 \leq t_1 \leq \cdots \leq t_n < \infty$ be an any division of the real line, for any $n \in \mathbb{N}$. The initial distribution of the process is $\widetilde{\mathbb{P}}^x(\widetilde{X}_0 = x) = 1$, and the finite dimensional distributions of $\widetilde{\mathbb{P}}^x$ with respect to the stationary distribution $\varphi_0^2 \mathrm{d}x$ are given by

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x} \left[\prod_{j=0}^n f_j(\widetilde{X}_{t_j}) \right] \varphi_0^2(x) \mathrm{d}x = \left(f_0, \widetilde{T}_{t_1 - t_0} f_1 ... \widetilde{T}_{t_n - t_{n-1}} f_n \right)_{L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)} \tag{2.5}$$

for all $f_0, f_n \in L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)$, $f_j \in L^\infty(\mathbb{R}^d)$, $j = 1, ..., n-1$.

(2) The finite dimensional distributions are time-shift invariant, i.e.

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x} \left[\prod_{j=0}^n f_j(\widetilde{X}_{t_j}) \right] \varphi_0^2(x) \mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x} \left[\prod_{j=0}^n f_j(\widetilde{X}_{t_j+s}) \right] \varphi_0^2(x) \mathrm{d}x, \quad s \in \mathbb{R}, \, n \in \mathbb{N}.$$

(3) $(\widetilde{X}_t)_{t\geq 0}$ and $(\widetilde{X}_t)_{t\leq 0}$ are independent, and $\widetilde{X}_{-t} \stackrel{\mathrm{d}}{=} \widetilde{X}_t$, for all $t \in \mathbb{R}$.

(4) Consider the filtrations $\left(\mathcal{F}_t^+\right)_{t\geq 0} = \sigma\left(\widetilde{X}_s : 0 \leq s \leq t\right)$ and $\left(\mathcal{F}_t^-\right)_{t\leq 0} = \sigma\left(\widetilde{X}_s : t \leq s \leq 0\right)$. Then, $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with respect to $\left(\mathcal{F}_t^+\right)_{t\geq 0}$, and $(\widetilde{X}_t)_{t\leq 0}$ is a Markov process with respect to $\left(\mathcal{F}_t^-\right)_{t\leq 0}$.

(5) The map $t \mapsto \widetilde{X}_t$ is $\widetilde{\mathbb{P}}^x$-almost surely càdlàg.

Furthermore, we have for all $f, g \in L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)$ the change-of-measure formula

$$\begin{aligned} (f, \widetilde{T}_t g)_{L^2(\mathbb{R}^d, \varphi_0^2 \mathrm{d}x)} = & \quad (f\varphi_0, e^{-t(H_{m,\alpha}-\lambda_0)} g\varphi_0)_{L^2(\mathbb{R}^d, \mathrm{d}x)} \\ = & \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}[f(\widetilde{X}_0) g(\widetilde{X}_t)] \varphi_0^2(x) \mathrm{d}x, \quad t \geq 0. \end{aligned} \tag{2.6}$$

It can also be shown that the probability measure $\widetilde{\mathbb{P}}^x$ is a Gibbs measure on the space of two-sided càdlàg paths with respect to the potential V. Furthermore, a calculation shows that under suitable conditions, the process $(\widetilde{X}_t)_{t\in\mathbb{R}}$ is a weak solution of the SDE with jumps

$$\begin{aligned} M_t = M_0 + B_t &+ \int_0^t \nabla \ln \varphi_0(M_s) \, \mathrm{d}s \\ &+ \int_0^t \int_{|z|\leq 1} \frac{\varphi_0(M_s + z) - \varphi_0(M_s)}{\varphi_0(M_s)} z \nu_{m,\alpha}(z) \mathrm{d}z \mathrm{d}s \\ &+ \int_0^t \int_{|z|\leq 1} \int_0^\infty z \mathbf{1}_{\left\{v \leq \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v) \\ &+ \int_0^t \int_{|z|>1} \int_0^\infty z \mathbf{1}_{\left\{v \leq \frac{\varphi_0(M_{s-}+z)}{\varphi_0(M_{s-})}\right\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}v), \end{aligned} \tag{2.7}$$

where $(B_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion and N is a Poisson random measure on $[0, \infty) \times \mathbb{R}_*^d \times [0, \infty)$, with intensity $\mathrm{d}t \nu_{m,\alpha}(z) \mathrm{d}z \mathrm{d}v$, where $\mathbb{R}_*^d = \mathbb{R}^d \backslash \{0\}$ and ν is the Lévy intensity of $(X_t)_{t\geq 0}$ generated by $-L_{m,\alpha}$. This shows that adding a potential V to $L_{m,\alpha}$ gives rise to a Feller process with drift terms and biases in the jump rates, with stationary density φ_0^2, which is thus no longer a Lévy process, and it gives a proper meaning to "motion in a potential landscape". For further details we refer to [36,39,47], in which also local and global sample path properties of $(\widetilde{X}_t)_{t\in\mathbb{R}}$ are discussed.

2.3. Heat Kernel of the Killed Feynman–Kac Semigroup

Let $\mathcal{D} \subset \mathbb{R}^d$ be an open set and consider the first exit time

$$\tau_{\mathcal{D}} = \inf \{t > 0 : X_t \notin \mathcal{D}\} \tag{2.8}$$

from $\mathcal{D}$. When $\mathcal{D} = \mathcal{B}_R$ we simplify the notation to τ_R, while if $\mathcal{D} = \mathcal{B}_R^c$ we use T_R. (From the context the reader will realize the meanings and not confuse this simple notation with the semigroup operators T_t.) The transition probability densities $p_{\mathcal{D}}(t,x,y)$ of the process killed on exiting $\mathcal{D}$ (or heat kernel of the killed semigroup) are given by the Dynkin–Hunt formula

$$p_{\mathcal{D}}(t,x,y) = p_t(x-y) - \mathbb{E}^x\left[p_{t-\tau_{\mathcal{D}}}(y - X_{\tau_{\mathcal{D}}}); \tau_{\mathcal{D}} < t\right], \quad x,y \in \mathcal{D}. \tag{2.9}$$

The heat kernel $p_{\mathcal{D}}(t,x,y)$ gives rise to the killed Feynman–Kac semigroup $\{P_t^{\mathcal{D}} : t \geq 0\}$ by $P_t^{\mathcal{D}} f(x) = \int_{\mathcal{D}} p_{\mathcal{D}}(t,x,y) f(y)\mathrm{d}y$, for all $x \in \mathcal{D}$, $t > 0$ and $f \in L^2(\mathbb{R}^d)$. It is known that $\{P_t^{\mathcal{D}} : t \geq 0\}$ is a strongly continuous semigroup of contraction operators on $L^2(\mathcal{D})$ and every operator $P_t^{\mathcal{D}}$, $t > 0$, is self-adjoint.

Below we will make frequent use of the Ikeda–Watanabe formula [35, Th. 1], which says that for every $\eta > 0$ and every bounded or non-negative Borel function f on $\mathbb{R}^d$, the equality

$$\mathbb{E}^x\left[e^{-\eta\tau_{\mathcal{D}}} f(X_{\tau_{\mathcal{D}}})\right] = \int_{\mathcal{D}} \int_0^\infty e^{-\eta t} p_{\mathcal{D}}(t,x,y)\mathrm{d}t \int_{\mathcal{D}^c} f(z) j_{m,\alpha}(z-y)\mathrm{d}z\mathrm{d}y, \quad x \in \mathcal{D},$$

holds. The same arguments leading to the above expression also allow the more general formulation (see, for instance, [10, eq. (1.58)] and [35, Th. 2])

$$\mathbb{E}^x\left[f(\tau_{\mathcal{D}}, X_{\tau_{\mathcal{D}}-}, X_{\tau_{\mathcal{D}}})\right] = \int_{\mathcal{D}} \int_{\mathcal{D}^c} \int_0^\infty p_{\mathcal{D}}(t,x,y) f(t,y,z) j_{m,\alpha}(z-y)\mathrm{d}t\mathrm{d}z\mathrm{d}y, \quad x \in \mathcal{D}, \tag{2.10}$$

which holds for every bounded Borel function $f : [0,\infty] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We will keep referring to this as the Ikeda–Watanabe formula.

In what follows, we will also rely on some estimates of the heat kernel of the killed semigroup. By (2.9), clearly $p_{\mathcal{D}}(t,x,y) \leq p_t(x-y)$ for all $t > 0$ and $x,y \in \mathcal{D}$. Recall that the semigroup $\{P_t^{\mathcal{D}} : t \geq 0\}$ is said to be intrinsically ultracontractive (IUC) whenever there exists $C_t^{\mathcal{D}} > 0$ such that $p_{\mathcal{D}}(t,x,y) \leq C_t^{\mathcal{D}} f_{\mathcal{D}}(x) f_{\mathcal{D}}(y)$, for all $t > 0$ and $x,y \in \mathcal{D}$, where $f_{\mathcal{D}}$ is the principal Dirichlet eigenfunction of the operator $L_{m,\alpha}$ in the domain $\mathcal{D}$. We also recall that $L_{m,\alpha}$ on a bounded open set $\mathcal{D}$ is meant to be the Friedrichs extension of $L_{m,\alpha}|_{C_c^\infty(\mathcal{D})}$, with form-domain containing those functions from the form-domain of $L_{m,\alpha}$ which almost surely vanish outside $\mathcal{D}$, see e.g. [28]. The so obtained operator $L_{m,\alpha}^{\mathcal{D}}$ is then the negative of the infinitesimal generator of the killed semigroup, i.e. $P_t^{\mathcal{D}} = e^{-tL_{m,\alpha}^{\mathcal{D}}}$ holds for all $t \geq 0$. It can be shown that if $\{P_t^{\mathcal{D}} : t \geq 0\}$ is IUC, then a similar lower bound holds with another constant. The following result provides a bound on $p_t(x)$, and will be useful for the IUC property of $\{P_t^{\mathcal{D}} : t \geq 0\}$ for a class of domains $\mathcal{D}$ that we will use below.

Lemma 2.1. *For every $\delta > 0$ there exists a constant $C_{d,m,\alpha}(\delta)$ such that*

$$\sup_{|x|\geq\delta,\ t>0} p_t(x) \leq C_{d,m,\alpha}(\delta).$$

Proof. Fix $\delta > 0$. By [56, eq. (9)] we know that

$$p_t(x) \leq C_\alpha^{(1)} e^{mt} t \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2}|x|^{d+\alpha}}.$$

Thus for $t \leq 1$ and $|x| \geq \delta$ we obtain

$$p_t(x) \leq C_\alpha^{(1)} e^m \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2}\delta^{d+\alpha}} =: C_{d,m,\alpha}^{(2)}(\delta).$$

For $t \geq 1$ we distinguish two cases. If $m = 0$, we use the estimate (see, for instance, [10])

$$p_t(x) \leq C^{(3)}_{d,\alpha} t^{-\frac{d}{\alpha}} \leq C^{(3)}_{d,\alpha}, \quad t > 1.$$

If $m > 0$, we can use [56, Lem. 3] to conclude that

$$p_t(x) \leq C^{(4)}_{d,m,\alpha} \left(m^{\frac{d}{\alpha}-\frac{d}{2}} t^{-\frac{d}{2}} + t^{-\frac{d}{\alpha}} \right) \leq C^{(5)}_{d,m,\alpha}, \quad t > 1.$$

Hence we can define

$$C_{d,m,\alpha}(\delta) = \begin{cases} \max\left\{C^{(2)}_{d,0,\alpha}(\delta), C^{(3)}_{d,\alpha}\right\} & m = 0 \\ \max\left\{C^{(2)}_{d,m,\alpha}(\delta), C^{(4)}_{d,m,\alpha}\right\} & m > 0, \end{cases}$$

giving $\sup_{|x|\geq\delta,\ t>0} p_t(x) \leq C_{d,m,\alpha}(\delta)$ for every $m \geq 0$.

Using that $\nu_{m,\alpha}(\mathcal{B}_r(x)) > 0$ for every $x \in \mathbb{R}^d$, $r > 0$ and $m \geq 0$, we immediately get the following result from the previous lemma and [30, Th. 3.1].

Corollary 2.1. *Let $\mathcal{D}$ be a bounded Lipschitz domain. The killed semigroup $\{P^{\mathcal{D}}_t : t \geq 0\}$ is IUC.*

We will denote the principal Dirichlet eigenfunction of $L_{m,\alpha}$ by f_R at eigenvalue λ_R whenever $\mathcal{D} = \mathcal{B}_R$. Using IUC and its implication of a similar lower bound, and the continuity of the killed heat kernel, it can be shown [22, Th. 4.2.5] that there exists a large enough $T > 0$ such that

$$\frac{1}{2}\, e^{-\lambda_R t} f_R(x) f_R(y) \leq p_{\mathcal{B}_R}(t,x,y) \leq \frac{3}{2}\, e^{-\lambda_R t} f_R(x) f_R(y), \tag{2.11}$$

for all $t > T$ and $x, y \in \mathcal{B}_R$.

3. Exit and Hitting Times Estimates

3.1. Estimates on the Survival Probability

As we will see below, the local behaviour of ground states depends on a function which can be estimated by using tools of potential theory for the stable and relativistic stable processes. We will denote this by $\mathcal{V}_{\alpha,m}$ and call it rate function. In this section we derive some key information on this function first. The results contained in this subsection have been obtained in a more general context in [11]. Since here we are considering two specific cases, which are widely used in applications, we reconsider some of the proofs in order to identify the values of the involved constants, which are not explicit in the cited work due to the greater generality of the arguments involved.

Lemma 3.1. *Let $\mathcal{D}$ be a $C^{1,1}$ bounded open set in $\mathbb{R}^d$, $(X^{(0)}_t)_{t\geq 0}$ be an isotropic α-stable process and $(X^{(m)}_t)_{t\geq 0}$ be an isotropic relativistic α-stable process with mass $m > 0$. Consider the first exit time $\tau^{(m)}_{\mathcal{D}} = \inf\{t > 0 : X^{(m)}_t \notin \mathcal{D}\}$, $m \geq 0$. Then $\mathbb{E}^x[\tau^{(m)}_{\mathcal{D}}] \asymp \mathbb{E}^x[\tau^{(0)}_{\mathcal{D}}]$, for every $m > 0$ and the comparability constant is independent of $\mathcal{D}$.*

Proof. The statement easily follows from [16, Cor. 1.2] and [17, Th. 1.3] due to the comparability of the respective Green functions.

As a consequence, we get the following upper bound. Recall the notation λ_R for the principal Dirichlet eigenvalue of $L_{m,\alpha}$ over the ball $\mathcal{B}_R$.

Corollary 3.1. *We have* $\lambda_R R^\alpha \leq C_{d,m,\alpha}$.

Proof. Denote $s(x) = \mathbb{E}^x[\tau_R]$ and $S = \|s\|_{L^2(\mathcal{B}_R)}$. First consider $m = 0$. Then the explicit formula due to M. Riesz (e.g. [10, eq. (1.56)])

$$s(x) = \frac{\pi^{1+d}\Gamma\left(\frac{d}{2}\right)\sin\left(\pi\frac{\alpha}{2}\right)\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|}{2^\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}(R^2 - |x|^2)^{\alpha/2}, \quad |x| \leq R,$$

holds. Hence we have

$$s(x) \geq C_{d,m,\alpha}R^\alpha, \quad |x| \leq \frac{R}{2}. \tag{3.1}$$

Lemma 3.1 guarantees that (3.1) holds even for $m > 0$. Thus, in general we have $S^2 \geq C_{d,m,\alpha}R^{2\alpha}|\mathcal{B}_{R/2}|$. By [5, Prop. 2.1] and Cauchy-Schwarz inequality we then obtain

$$\lambda_R \leq \int_{\mathcal{B}_R} \frac{s(x)}{S^2}\mathrm{d}x \leq \sqrt{\frac{|\mathcal{B}_R|}{S^2}} \leq \frac{C_{d,m,\alpha}}{R^\alpha}.$$

We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is (m,α)-*harmonic* on an open set $\mathcal{D} \subset \mathbb{R}^d$ if for every open set $\mathcal{U} \subset\subset \mathcal{D}$ (i.e. $\overline{\mathcal{U}} \subset \mathcal{D}$ is compact) the equality $f(x) = \mathbb{E}^x[f(X_{\tau_{\mathcal{U}}})]$ holds for every $x \in \mathcal{U}$. In the following we come back to the notation by $(X_t)_{t\geq 0}$ meaning either of the processes for the massless and massive cases, as used previously.

Lemma 3.2. *Let $d = 1$ and fix $r_0 > 0$. There exist an increasing concave (and thus subadditive) (m,α)-harmonic function $\mathcal{V}_{m,\alpha}(r) : (0,\infty) \to \mathbb{R}^+$ and constants $0 < C^{(1)}_{m,\alpha,r_0} < C^{(2)}_{m,\alpha,r_0}$ such that*

$$C^{(1)}_{m,\alpha,r_0} r^{\alpha/2} \leq \mathcal{V}_{m,\alpha}(r) \leq C^{(2)}_{m,\alpha,r_0} r^{\alpha/2}, \quad 0 \leq r \leq r_0.$$

Proof. Consider the running supremum $M_t = \sup_{0\leq s\leq t} X_t$, and let $Y_t = M_t - X_t$ be the process obtained by reflecting X_t on hitting the supremum. Let A_t be the local time at zero of Y_t, and $Z_t = \inf\{\tau > 0 : \ A_\tau > t\}$ its right-continuous inverse. Also, consider $H_t = M_{Z_t}$. By [60, eq. (1.8)] there exists a function $\psi_{m,\alpha}$ such that $\int_0^\infty \psi_{m,\alpha}(s)f(s)\mathrm{d}s = \int_0^\infty \mathbb{E}[f(H_s)]\mathrm{d}s$, for every non-negative Borel function f. Choosing in particular $f = \mathbf{1}_{[0,r]}$, we define

$$\mathcal{V}_{m,\alpha}(r) = \int_0^r \psi_{m,\alpha}(\rho)\mathrm{d}\rho = \int_0^\infty \mathbb{P}(H_\rho \leq r)\mathrm{d}\rho.$$

Note that $(H_t)_{t\geq 0}$ is a subordinator (see [8, Lem. VI.2]), different from a Poisson process since $(0,\infty)$ is a regular domain for $(X_t)_{t\geq 0}$. We can define its inverse subordinator $H_t^{-1} := \inf\{s > 0 : H_s > t\}$ and observe that $\mathcal{V}_{m,\alpha}(t) = \mathbb{E}[H_t^{-1}]$, implying subadditivity of $\mathcal{V}_{m,\alpha}$ (see [8, Ch. III]). The fact that $\mathcal{V}_{m,\alpha}$ is (m,α)-harmonic in $(0,\infty)$ follows from [60, Th. 2]. The comparability result follows by [41, Prop. 2.2, Ex. 2.3]. Concavity results by [41, Prop. 2.1] and [58, Th. 10.3] as $\psi_{m,\alpha} = \mathcal{V}'_{m,\alpha}$ is non-increasing.

Remark 3.1. In fact, $\mathcal{V}_{0,\alpha}(r) = r^{\alpha/2}$. Moreover, for $m > 0$ again by [41, Prop. 2.2 and Ex. 2.3] we get $\mathcal{V}_{m,\alpha}(r) \sim r$ as $r \to \infty$. As a direct consequence of the monotone density theorem, we furthermore have $\psi_{m,\alpha}(r) \sim r^{\frac{\alpha}{2}-1}$ as $r \downarrow 0$, for all $m \geq 0$.

As a consequence, we obtain the following Harnack-type inequality.

Lemma 3.3. *For every* $0 < x \leq y \leq z \leq 5x$ *we have*

$$\mathcal{V}_{m,\alpha}(z) - \mathcal{V}_{m,\alpha}(y) \leq 5\mathcal{V}'_{m,\alpha}(x)(z-y).$$

Proof. By Lemma 3.2 we know that $\mathcal{V}_{m,\alpha}$ is concave and thus, in particular, log-concave. Hence, the result follows by [11, Lem. 7.1].

Moreover, we can use the function $\mathcal{V}_{m,\alpha}$ to derive the following estimate.

Corollary 3.2. *Let* $d = 1$ *and define* $\tau_{(0,\infty)} = \inf\{t > 0 : \ X_t \leq 0\}$. *There exist constants* $C^{(1)}_{m,\alpha}$ *and* $C^{(2)}_{m,\alpha}$ *such that*

$$C^{(1)}_{m,\alpha}\left(\frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \leq \mathbb{P}^r(\tau_{(0,\infty)} > t) \leq C^{(2)}_{m,\alpha}\left(\frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1\right)$$

Proof. Immediate by [42, Cor. 3.2] and Lemma 3.2.

Remark 3.2. In the case $m = 0$, it is not difficult to determine explicitly the constant given in Corollary 3.1, while it is clear that the upper and lower bounds in Lemma 3.2 are actually identities. Furthermore, the constants obtained in Corollary 3.2 can be computed exactly to be $C^{(1)}_{m,\alpha} = \frac{1}{2e}\left(\frac{e-1}{8e^2}\right)^2$ and $C^{(2)}_{m,\alpha} = \frac{e}{e-1}$, which are independent of m and α. In fact, as observed in [11], these constants are universal for more general unimodal symmetric Lévy processes. The constants given in the following statements can be, at least in the case $m = 0$, tracked from the cited results or numerically evaluated via the principal Dirichlet eigenfunction.

As a direct consequence of Lemmas 3.2–3.3, we obtain the following lower bound.

Proposition 3.1. *For every* $R > 0$ *there exist constants* $C^{(1)}_{d,m,\alpha,R}, C^{(2)}_{d,m,\alpha}$ *such that*

$$\mathbb{P}^x(\tau_R > t) \geq C^{(1)}_{d,m,\alpha,R}\left(\frac{(R-|x|)^{\alpha/2}}{\sqrt{t}} \wedge 1\right), \quad t \leq C^{(2)}_{d,m,\alpha}\mathcal{V}^2_{m,\alpha}(R).$$

Proof. By Lemma 3.3 and [11, Prop. 6.1], we know that there exist constants $C^{(2)}_d, C^{(3)}_d > 0$ such that

$$\mathbb{P}^x(\tau_R > t) \geq C^{(3)}_{d,R}\left(\frac{\mathcal{V}_{m,\alpha}(R-|x|)}{\sqrt{t}} \wedge 1\right), \quad t \leq C^{(2)}_d\mathcal{V}^2_{m,\alpha}(R).$$

Lemma 3.2 then completes the proof.

Furthermore, we can derive an upper bound on the survival probability τ_R.

Lemma 3.4. *For every $x \in \mathcal{B}_R$ and $t > 0$ we have*

$$\mathbb{P}^x(\tau_R > t) \le 2\Big(\frac{(R-|x|)^{\alpha/2}}{\sqrt{t}} \wedge 1\Big).$$

Proof. Since $(X_t)_{t\ge 0}$ is rotationally symmetric, we may choose $x = r\mathbf{e}_1$ without loss of generality, where $\mathbf{e}_1 = (1,0,\dots,0)$ and $r \in (0,R)$. Define the set $\mathcal{H}_R^{\leftarrow} := \{x \in \mathbb{R}^d : \ x_1 < R\}$ and let $\widetilde{\tau}_R := \inf\{t > 0 : \ X_r \in (\mathcal{H}_R^{\leftarrow})^c\}$ be the first exit time from this set. Since $\mathcal{B}_R \subseteq \mathcal{H}_R$, we have $\tau_R \le \widetilde{\tau}_R$ almost surely. With the same notation $\tau_{(0,\infty)}$ as in Corollary 3.2, it follows that

$$\begin{aligned}\mathbb{P}^x(\tau_R > t) \le \ \mathbb{P}^x(\widetilde{\tau}_R > t) &= \mathbb{P}^{(r-R)\mathbf{e}_1}(\widetilde{\tau}_0 > t) = \mathbb{P}^{r-R}(\tau_{(0,\infty)} > t)\\ &\le \ 2\left(\tfrac{(R-r)^{\alpha/2}}{\sqrt{t}} \wedge 1\right).\end{aligned}$$

Using intrinsic ultracontractivity of the killed semigroup, we can improve these estimates.

Proposition 3.2. *For every $x \in \mathcal{B}_R$, we have*

$$\mathbb{P}^x(\tau_R > t) \asymp e^{-\lambda_R t}\left(\frac{(R-|x|)^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}} \wedge 1\right),$$

where the comparability constants depend on d, m, α, R, and λ_R is the principal Dirichlet eigenvalue of $L_{m,\alpha}$ in the ball $\mathcal{B}_R$.

Proof. Since we have already recalled Lemma 3.4 and Proposition 3.1, we only need to prove the exponential domination for large values of $t > 0$. Let f_R be the principal Dirichlet eigenfunction of $L_{m,\alpha}$ for the ball $\mathcal{B}_R$ and observe that, by [48, Prop. 4.289], f_R is continuous and bounded. Since the killed semigroup is IUC, see Lemma 2.1, we can choose $T > 0$ such that (2.11) holds for every $t \ge 0$ and $x, u \in \mathcal{B}_R$. For this fixed T, by [16, Th. 1.1] and [17, Th. 1.1], it follows that there exists a constant $C^{(1)}_{d,m,\alpha,R} > 0$ such that for every $t \ge T$ and $x, u \in \mathcal{B}_R$

$$\begin{aligned}\frac{1}{C^{(1)}_{d,m,\alpha,R}} e^{-\lambda_R t}(R-|x|)^{\alpha/2}(R-|u|)^{\alpha/2} &\le p_{\mathcal{B}_R}(t,x,u)\\ &\le \frac{3}{2} e^{-\lambda_R t}(R-|x|)^{\alpha/2}(R-|u|)^{\alpha/2} \qquad (3.2)\end{aligned}$$

holds. Combining (2.11) and (3.2) we have, for all $x, u \in \mathcal{B}_R$,

$$f_R(x) f_R(u) \ge \frac{2}{3C^{(1)}_{d,m,\alpha,R}} (R-|x|)^{\alpha/2}(R-|u|)^{\alpha/2}.$$

Taking $x = u = 0$, the previous inequality gives

$$f_R(0)^2 \ge \frac{2}{3C^{(1)}_{d,m,\alpha,R}} R^{\alpha} > 0. \qquad (3.3)$$

Furthermore, choosing $u = 0$ in (3.3) we get

$$f_R(x) \ge \frac{2}{3C^{(1)}_{d,m,\alpha,R} f_R(0)} R^{\alpha/2}(R-|x|)^{\alpha/2} =: C^{(2)}_{d,m,\alpha,R}(R-|x|)^{\alpha/2}. \qquad (3.4)$$

Finally, by (2.11) and (3.4) we obtain the lower bound

$$\begin{aligned}1 \geq \mathbb{P}^x(\tau_R > t) &= \int_t^\infty \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) p_{\mathcal{B}_R}(s,x,u) \mathrm{d}u \mathrm{d}z \mathrm{d}s \\ &\geq \frac{e^{-\lambda_R t}}{2\lambda_R} \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) f_R(x) f_R(u) \mathrm{d}u \mathrm{d}z \\ &\geq \frac{C^{(2)}_{d,m,\alpha,R}(R-|x|)^{\alpha/2} e^{-\lambda_R t}}{2\lambda_R} \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) f_R(u) \mathrm{d}u \mathrm{d}z.\end{aligned}$$

This guarantees that

$$\int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) f_R(u) \mathrm{d}u \mathrm{d}z < \infty$$

and, at the same time,

$$\begin{aligned}\mathbb{P}^x(\tau_R > t) &\geq \frac{C^{(2)}_{d,m,\alpha,R}(R-|x|)^{\alpha/2} e^{-\lambda_R t}}{2\lambda_R} \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) f_R(u) \mathrm{d}u \mathrm{d}z \\ &=: C^{(3)}_{d,m,\alpha,R}(R-|x|)^{\alpha/2} e^{-\lambda_R t},\end{aligned}$$

for every $x \in \mathcal{B}_R$ and $t \geq T$. Similarly, we have the estimate from above,

$$\begin{aligned}\mathbb{P}^x(\tau_R > t) &\leq \frac{3}{2} \int_t^\infty e^{-\lambda_R s} \mathrm{d}s \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) f_R(x) f_R(u) \mathrm{d}u \mathrm{d}z \\ &\leq \frac{3\,\|f_R\|_\infty}{2\lambda_R} e^{-\lambda_R t} \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z-u|) f_R(u) \mathrm{d}u \mathrm{d}z \\ &=: C^{(4)}_{d,m,\alpha,R} e^{-\lambda R t}.\end{aligned}$$

Next we derive an upper bound for the function $\mathbb{P}^x(T_R > t)$. First we need a technical lemma.

Lemma 3.5. *There exists a constant $C_{d,m,\alpha} > 0$ such that*

$$\nu_{m,\alpha}(\mathcal{B}_r^c) \sim C_{d,m,\alpha} r^{-\alpha}, \quad r \downarrow 0.$$

Proof. There is nothing to prove if $m = 0$, thus take $m > 0$ and for all $\varepsilon > 0$ let $t_0(\varepsilon)$ such that $(1-\varepsilon) C^{(1)}_{d,m,\alpha} \rho^{-d-\alpha} \leq j_{m,\alpha}(\rho) \leq (1+\varepsilon) C^{(1)}_{d,m,\alpha} \rho^{-d-\alpha}$ for every $0 < \rho < t_0(\varepsilon)$ (note that this holds by the 0+ asymptotics of the Bessel function). Consider $r < t_0(\varepsilon)$ and observe that

$$\nu_{m,\alpha}(\mathcal{B}_r^c) = \int_r^{t_0(\varepsilon)} \rho^{d-1} j_{m,\alpha}(\rho) \mathrm{d}\rho + \int_{t_0(\varepsilon)}^\infty \rho^{d-1} j_{m,\alpha}(\rho) \mathrm{d}\rho =: I_1(\varepsilon, r) + I_2(\varepsilon).$$

Clearly, $I_2(\varepsilon) < \infty$. Since

$$\begin{aligned}(1-\varepsilon)\frac{C^{(1)}_{d,m,\alpha}}{\alpha} r^{-\alpha} - (1-\varepsilon)\frac{C^{(3)}_{d,m,\alpha}}{\alpha} t_0(\varepsilon)^{-\alpha} &\leq I_1(\varepsilon, r) \\ \leq (1+\varepsilon)\frac{C^{(1)}_{d,m,\alpha}}{\alpha} r^{-\alpha} - (1+\varepsilon)\frac{C^{(3)}_{d,m,\alpha}}{\alpha} t_0(\varepsilon)^{-\alpha},\end{aligned}$$

the result follows directly.

Proposition 3.3. *For every* $0 < R < R_0$ *there exists a constant* $C_{d,m,\alpha,R,R_0} > 0$ *such that*

$$\mathbb{P}^x(T_R > t) \le C_{d,m,\alpha,R,R_0} \frac{(|x| - R)^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}}, \quad |x| \in [R, R_0). \tag{3.5}$$

Proof. Consider the function

$$\mathcal{J}_{m,\alpha}(R) = \inf_{0 \le r \le R} \nu_{m,\alpha}(\mathcal{B}_r^c)\mathcal{V}_{m,\alpha}^2(r).$$

Observe that $\nu_{m,\alpha}(\mathcal{B}_r^c)\mathcal{V}_{m,\alpha}^2(r) > 0$ for every $r > 0$. Moreover, by Lemmas 3.2 and 3.5 we know that $\nu_{m,\alpha}(\mathcal{B}_r^c)\mathcal{V}_{m,\alpha}^2(r) \ge C_{m,\alpha,r_0} > 0$ for $r_0 > 0$ and $r \in (0, r_0)$. This implies $\mathcal{J}_{m,\alpha}(R) > 0$. Lemma 3.3 guarantees that [11, Lem. 6.2] applies and we obtain

$$\mathbb{P}^x(T_R > t) \le \frac{5C_d}{(\mathcal{J}(R))^2} \frac{\mathcal{V}_{m,\alpha}(|x| - R)}{\sqrt{t} \wedge \mathcal{V}_{m,\alpha}(R)}.$$

Finally, for $|x| \in (R, R_0)$ we can use Lemma 3.2 to complete the proof.

Remark 3.3. Note that in case $m = 0$, there exists a constant $C_{d,\alpha} > 0$ such that $\mathcal{J}_{0,\alpha}(R) \ge C_{d,\alpha}$ for every R. This follows from the asymptotic behaviour of $\nu_{0,\alpha}(\mathcal{B}_r^c)$ as $r \to \infty$ given in [3, Cor. 2.1]. Thus, for the massless case (3.5) holds for all $|x| \ge R$, with no dependence on R_0. On the other hand, for $m > 0$ we have $\lim_{R\to\infty} \mathcal{J}_{m,\alpha}(R) = 0$. This is due to $\mathcal{V}_{m,\alpha}(R) \sim R$ as $R \to \infty$, as seen in Remark 3.1, while $\overline{\nu}_{m,\alpha}(\mathcal{B}_R^c)$ decays exponentially (see [3, Cor. 2.2]).

3.2. Estimates on the Moment-Generating Function for the Exit Time from a Ball

In view of deriving and using expressions of the type (1.1) in our main analysis below, in this section first we derive estimates of exponentials of exit times of the Lévy processes $(X_t)_{t\ge 0}$ for balls and their complements. Recall (2.8) and denote by

$$g_{\tau_R}(t) = \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m,\alpha}(|z - u|) p_{\mathcal{B}_R}(t, x, u) \mathrm{d}u \mathrm{d}z, \quad t > 0, \tag{3.6}$$

the probability density of τ_R. Now we prove the following estimate for the moment-generating function of τ_R.

Theorem 3.1. *Fix* $R > 0$. *Then, for every* $0 \le \lambda < \lambda_R$ *and* $x \in \mathcal{B}_R$ *we have*

$$\mathbb{E}^x[e^{\lambda\tau_R} - 1] \asymp \frac{\lambda}{\lambda_R - \lambda} \left(\frac{R - |x|}{R}\right)^{\alpha/2},$$

where the comparability constant depends on d, m, α, R. *Moreover we have,* $\mathbb{E}^x[e^{\lambda\tau_R}] = \infty$ *whenever* $\lambda \ge \lambda_R$.

Proof. First fix $0 \le \lambda < \lambda_R$. Using (3.6) and integrating by parts we obtain

$$\begin{aligned}\mathbb{E}^x[e^{\lambda\tau_R} - 1] &= \int_0^\infty (e^{\lambda t} - 1) g_{\tau_R}(t) \mathrm{d}t \\ &= -\lim_{s\to\infty} (e^{\lambda s} - 1)\mathbb{P}^x(\tau_R > s) + \lambda \int_0^\infty e^{\lambda t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t. \end{aligned} \tag{3.7}$$

Note that the limit is zero since by Proposition 3.2

$$e^{\lambda s}\mathbb{P}^x(\tau_R > s) \le C^{(1)}_{d,m,\alpha,R}\, e^{(\lambda-\lambda_R)s}\frac{(R-|x|)^{\alpha/2}}{\sqrt{s}\wedge R^{\alpha/2}},$$

and $\lambda < \lambda_R$.

First we show the lower bound of the remaining integral at the right hand side of (3.7). Using Proposition 3.2 again, we get

$$\begin{aligned}\int_0^\infty e^{\lambda t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t &\ge C^{(2)}_{d,m,\alpha,R}\left(\frac{R-|x|}{R}\right)^{\alpha/2}\int_{R^\alpha}^\infty e^{-(\lambda_R-\lambda)t}\mathrm{d}t \\ &\ge C^{(2)}_{d,m,\alpha,R}\left(\frac{R-|x|}{R}\right)^{\alpha/2}\frac{1}{\lambda_R-\lambda}e^{-\lambda_R R^\alpha}.\end{aligned} \tag{3.8}$$

Next note that by Corollary 3.1 we have $\lambda_R R^\alpha \le C^{(3)}_{d,m,\alpha}$ with a constant $C^{(3)}_{d,m,\alpha}$, thus $e^{-\lambda_R R^\alpha} \ge C^{(4)}_{d,m,\alpha}$. Using this lower bound in (3.8), we get

$$\int_0^\infty e^{\lambda t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t \ge C^{(5)}_{d,m,\alpha,R}\frac{1}{\lambda_R-\lambda}\left(\frac{R-|x|}{R}\right)^{\alpha/2}.$$

To get the upper bound, we estimate

$$\begin{aligned}&\int_0^\infty e^{\lambda t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t \\ &\le C^{(1)}_{d,m,\alpha,R}\int_0^\infty e^{-(\lambda_R-\lambda)t}\Big(1\wedge\frac{(R-|x|)^{\alpha/2}}{\sqrt{t}\wedge R^{\alpha/2}}\Big)\mathrm{d}t \\ &= C^{(1)}_{d,m,\alpha,R}\left(\int_0^{R^{\alpha/2}}\Big(1\wedge\frac{(R-|x|)^{\alpha/2}}{\sqrt{t}}\Big)e^{-(\lambda_R-\lambda)t}\mathrm{d}t + \Big(\frac{R-|x|}{R}\Big)^{\alpha/2}\frac{e^{-(\lambda_R-\lambda)R^\alpha}}{\lambda_R-\lambda}\right) \\ &\le C^{(1)}_{d,m,\alpha,R}\left((R-|x|)^{\alpha/2}\int_0^{R^\alpha}\frac{e^{-(\lambda_R-\lambda)t}}{\sqrt{t}}\mathrm{d}t + \Big(\frac{R-|x|}{R}\Big)^{\alpha/2}\frac{1}{\lambda_R-\lambda}\right) \\ &= C^{(1)}_{d,m,\alpha,R}\left((R-|x|)^{\alpha/2}\left(2R^{\alpha/2}e^{-(\lambda_R-\lambda)R^\alpha} + 2(\lambda_R-\lambda)\int_0^{R^\alpha}\sqrt{t}e^{-(\lambda_R-\lambda)t}\mathrm{d}t\right)\right. \\ &\qquad \left. + \left(\frac{R-|x|}{R}\right)^{\alpha/2}\frac{1}{\lambda_R-\lambda}\right) \\ &\le C^{(1)}_{d,m,\alpha,R}\left(4(R-|x|)^{\alpha/2}R^{\alpha/2} + \left(\frac{R-|x|}{R}\right)^{\alpha/2}\frac{1}{\lambda_R-\lambda}\right) \\ &\le C^{(1)}_{d,m,\alpha,R}\left(4R^\alpha\left(\frac{R-|x|}{R}\right)^{\alpha/2}\frac{\lambda_R}{\lambda_R-\lambda} + \left(\frac{R-|x|}{R}\right)^{\alpha/2}\frac{1}{\lambda_R-\lambda}\right) \\ &\le \frac{C^{(5)}_{d,m,\alpha,R}}{\lambda_R-\lambda}\left(\frac{R-|x|}{R}\right)^{\alpha/2},\end{aligned}$$

where we used the bound $\lambda_R R^\alpha \le C^{(3)}_{d,m,\alpha}$ again in the last line. This proves the first part of the claim.

To obtain the second statement we only need to prove that $\mathbb{E}[e^{\lambda_R \tau_R}] = \infty$. Notice that by Proposition 3.2

$$e^{\lambda_R s}\mathbb{P}^x(\tau_R > s) \le C^{(1)}_{d,m,\alpha,R} \frac{(R-|x|)^{\alpha/2}}{\sqrt{s}\wedge R^{\alpha/2}}.$$

For $s > R^\alpha$ we get

$$\begin{aligned}\mathbb{E}^x[e^{\lambda_R \tau_R}] &\ge \mathbb{E}^x[e^{\lambda_R \tau_R} - 1; \tau_R \le s] = \int_0^s (e^{\lambda t} - 1) g_{\tau_R}(t)\mathrm{d}t \\ &= -(e^{\lambda_R s} - 1)\mathbb{P}^x(\tau_R > s) + \lambda_R \int_0^s e^{\lambda_R t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t \\ &\ge -C^{(1)}_{d,m,\alpha,R}\frac{(R-|x|)^{\alpha/2}}{R^{\alpha/2}} + \lambda_R \int_{R^\alpha}^s e^{\lambda_R t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t.\end{aligned}$$

Taking the supremum over s on the right-hand side and using the lower bound in Proposition 3.2, we obtain

$$\begin{aligned}\mathbb{E}^x[e^{\lambda_R \tau_R}] &\ge -C^{(1)}_{d,m,\alpha,R}\frac{(R-|x|)^{\alpha/2}}{R^{\alpha/2}} + \lambda_R \int_{R^\alpha}^\infty e^{\lambda_R t}\mathbb{P}^x(\tau_R > t)\mathrm{d}t \\ &\ge -C^{(1)}_{d,m,\alpha,R}\frac{(R-|x|)^{\alpha/2}}{R^{\alpha/2}} + C^{(2)}_{d,m,\alpha,R}\lambda_R \int_{R^\alpha}^\infty \frac{(R-|x|)^{\alpha/2}}{R^{\alpha/2}}\mathrm{d}t = \infty.\end{aligned}$$

3.3. Estimates on the Laplace Transform of the Hitting Time for a Ball

Next we consider $T_R = \inf\{t > 0 : \ X_t \in \mathcal{B}_R\}$ and derive estimates on the Laplace transform $\mathbb{E}^x[e^{-\lambda T_R}]$, in which case there is no handy tool such as intrinsic ultracontractivity of the killed semigroup. We start with a lower bound for points in domains of the type $R \le |x| \le R'$, for the remaining choices of domains see Remark 3.4 (2).

Theorem 3.2. *Let λ, $R > 0$ and $R_2 > R_1 > R$. There exists a constant $C_{d,m,\alpha,R_1,R_2,R,\lambda} > 0$ such that*

$$\mathbb{E}^x[e^{-\lambda T_R}] \ge C_{d,m,\alpha,R_1,R_2,R,\lambda}\, j_{m,\alpha}(|x|), \quad R_1 \le |x| \le R_2.$$

Proof. Define

$$C^{(1)}_{d,m,\alpha,R_1,R_2} = \min_{R_1 \le |x| \le R_2} \frac{j_{m,\alpha}\left(|x| + \frac{5}{2}R\right)}{j_{m,\alpha}(|x|)}.$$

As before, fix $x = r\mathbf{e}_1$ for $r > 0$, and define

$$A(x) = \left\{u \in \mathbb{R}^d : \ |x| + R < |u| < |x| + 2R,\ \langle u, \mathbf{e}_1\rangle < 0\right\}.$$

Since $R_1 \le |x| \le R_2$, taking $\mathcal{D} = \mathcal{B}_{3R_2} \setminus \bar{\mathcal{B}}_R$ we see that $x \in \mathcal{D} \subset \bar{\mathcal{B}}^c_R$. In particular, $p_{\mathcal{B}^c_R}(t,x,u) \ge p_{\mathcal{D}}(t,x,u)$. Since $\mathcal{D}$ is a bounded and open Lipschitz set, the semigroup with kernel $p_{\mathcal{D}}(t,x,u)$ is IUC and we can apply to it the lower bound (2.11) with some $T > 0$, and the principal Dirichlet eigenvalue and eigenfunction $\lambda_{\mathcal{D}}$ and $f_{\mathcal{D}}$ of $L_{m,\alpha}$ on $\mathcal{D}$. Then, by using the Ikeda–Watanabe formula we get

$$\begin{aligned}\mathbb{E}^x[e^{-\lambda T_R}] &\geq \mathbb{E}^x\left[e^{-\lambda T_R}; X_{T_R-}\in A(x),\ |X_{T_R}|<\frac{R}{2},\ T_R>T\right]\\ &= \int_T^\infty \int_{\mathcal{B}_{R/2}}\int_{A(x)} e^{-\lambda t} j_{m,\alpha}(|u-z|)p_{\mathcal{B}_R^c}(t,x,u)\mathrm{d}t\mathrm{d}z\mathrm{d}u\\ &\geq \frac{R^d\omega_d}{2^{d+1}} j_{m,\alpha}\Big(|x|+\frac{5}{2}R\Big)\int_T^\infty\int_{A(x)} e^{-(\lambda+\lambda_{\mathcal{D}})t} f_{\mathcal{D}}(x)f_{\mathcal{D}}(u)\mathrm{d}t\mathrm{d}u\\ &\geq \frac{C^{(1)}_{d,m,\alpha,R_1,R_2}R^d\omega_d e^{-(\lambda+\lambda_{\mathcal{D}})T}}{(\lambda+\lambda_{\mathcal{D}})2^{d+1}} j_{m,\alpha}\left(|x|\right) f_{\mathcal{D}}(x)\int_{A(x)} f_{\mathcal{D}}(u)\mathrm{d}u.\end{aligned} \tag{3.9}$$

Note that since $\mathcal{D}$ is a bounded $C^{1,1}$ domain, by [16, Th. 1.1] and [17, Th. 1.1] there exists a constant $C^{(2)}_{d,m,\alpha,R_1,R_2}>1$ such that for every $t\geq 1$

$$p_{\mathcal{D}}(t,x,u)\leq C^{(2)}_{d,m,\alpha,R_1,R_2}e^{-\lambda_{\mathcal{D}}}\delta_{\mathcal{D}}^{\alpha/2}(x)\delta_{\mathcal{D}}^{\alpha/2}(u),$$

holds, where $\delta_{\mathcal{D}}(x)=\mathrm{dist}(x,\partial\mathcal{D})$. By definition of $f_{\mathcal{D}}(x)$ we get

$$\begin{aligned}f_{\mathcal{D}}(x) &= e^{\lambda_{\mathcal{D}}}\int_{\mathcal{D}} p_{\mathcal{D}}(1,x,u)f_{\mathcal{D}}(u)\mathrm{d}u\\ &\leq C^{(2)}_{d,m,\alpha,R_1,R_2}\left\|f_{\mathcal{D}}\right\|_{L^\infty(\mathcal{D})}\delta_{\mathcal{D}}^{\alpha/2}(x)\int_{\mathcal{D}}\delta_{\mathcal{D}}^{\alpha/2}(u)\mathrm{d}u\\ &\leq C^{(2)}_{d,m,\alpha,R_1,R_2}\left\|f_{\mathcal{D}}\right\|_{L^\infty(\mathcal{D})}(6R_2)^{\alpha/2}((3R_2)^d-R_1^d)\omega_d\delta_{\mathcal{D}}^{\alpha/2}(x).\end{aligned} \tag{3.10}$$

To obtain a lower bound on $f_{\mathcal{D}}(x)$, consider $\tau_{\mathcal{D}}=\inf\{t>0:\ X_t\in\bar{\mathcal{D}}^c\}$ and use again (2.10), (3.10) and the fact that $\delta_{\mathcal{D}}(u)\leq|u-z|$ for all $z\in\mathcal{D}^c$, giving

$$\begin{aligned}\mathbb{P}^x(\tau_{\mathcal{D}}>T) &= \int_T^\infty\int_{\mathcal{D}^c}\int_{\mathcal{D}} j_{m,\alpha}(|z-u|)p(t,x,u)\mathrm{d}u\mathrm{d}z\mathrm{d}t\\ &\leq \frac{3}{2}f_{\mathcal{D}}(x)\int_T^\infty\int_{\mathcal{D}^c}\int_{\mathcal{D}} j_{m,\alpha}(|z-u|)e^{-\lambda_{\mathcal{D}}t}f_{\mathcal{D}}(u)\mathrm{d}u\mathrm{d}z\mathrm{d}t\\ &\leq \frac{3}{2}f_{\mathcal{D}}(x)C^{(2)}_{d,m,\alpha,R_1,R_2}\left\|f_{\mathcal{D}}\right\|_{L^\infty(\mathcal{D})}(6R_2)^{\alpha/2}((3R_2)^d-R_1^d)\omega_d\\ &\quad\times\int_T^\infty\int_{\mathcal{D}^c}\int_{\mathcal{D}} j_{m,\alpha}(|z-u|)e^{-\lambda_{\mathcal{D}}t}\delta_{\mathcal{D}}(u)\mathrm{d}u\mathrm{d}z\mathrm{d}t\\ &\leq C^{(4)}_{d,\alpha}\frac{3e^{-\lambda_{\mathcal{D}}T}}{2\lambda_{\mathcal{D}}}f_{\mathcal{D}}(x)C^{(2)}_{d,m,\alpha,R_1,R_2}\left\|f_{\mathcal{D}}\right\|_{L^\infty(\mathcal{D})}(6R_2)^{\alpha/2}((3R_2)^d-R_1^d)\omega_d\\ &\quad\times\int_{\mathcal{D}^c}\int_{\mathcal{D}}\frac{\mathrm{d}u\mathrm{d}z}{|z-u|^{d+\frac{\alpha}{2}}}\\ &\leq \frac{3\,\mathrm{Per}_\alpha(\mathcal{D})}{2}C^{(3)}_{d,\alpha}C^{(2)}_{d,m,\alpha,R_1,R_2}\left\|f_{\mathcal{D}}\right\|_{L^\infty(\mathcal{D})}(6R_2)^{\alpha/2}\\ &\quad\times((3R_2)^d-R_1^d)\omega_d\frac{e^{-\lambda_{\mathcal{D}}T}}{\lambda_D}f_{\mathcal{D}}(x),\end{aligned}$$

where $\mathrm{Per}_{\alpha}(\mathcal{D}) = \int_{\mathcal{D}} \int_{\mathcal{D}^c} \frac{\mathrm{d}z\mathrm{d}u}{|z-u|^{d+\frac{\alpha}{2}}}$ is the fractional perimeter of $\mathcal{D}$ (see e.g. [29]), and we used that $j_{m,\alpha}(|z-u|) \le j_{0,\alpha}(|z-u|) = C^{(3)}_{d,\alpha}|z-u|^{-d-\alpha}$ by (2.2), see [56, Lem. 2]. Hence $f_{\mathcal{D}}(x) \ge C^{(4)}_{d,m,\alpha,R_1,R_2}\mathbb{P}^x(\tau_{\mathcal{D}} > T)$, where

$$C^{(4)}_{d,m,\alpha,R_1,R_2} = \frac{2\lambda_{\mathcal{D}}e^{\lambda_{\mathcal{D}}T}}{3C^{(3)}_{d,\alpha}\,\mathrm{Per}_{\alpha}(\mathcal{D})C^{(2)}_{d,m,\alpha,R_1,R_2}\,\|f_{\mathcal{D}}\|_{L^\infty(\mathcal{D})}\,(6R_2)^{\alpha/2}((3R_2)^d - R_1^d)\omega_d}.$$

Note that $\mathcal{D}$ is a $C^{1,1}$ bounded set with scaling radius $R_3 = (3R_2 + R)/2$. Fix $x \in \mathcal{D}$. Then, there exists a point $\bar{x} \in \mathcal{D}$ and a ball $\mathcal{B}_{R_3}(\bar{x})$ such that $x \in \mathcal{B}_{R_3}(\bar{x})$ and $\delta_{\mathcal{D}}(x) = R_3 - |x-\bar{x}|$. By Proposition 3.2 and the fact that $\mathcal{B}_{R_3}(\bar{x}) \subset \mathcal{D}$, we know that there exists a constant $C^{(5)}_{d,m,\alpha,R_1,R_2}$ such that

$$\begin{aligned}\mathbb{P}^x(\tau_{\mathcal{D}} > T) \ge\ & \mathbb{P}^x(\tau_{\mathcal{B}_{R_3}(\bar{x})} > T) = \mathbb{P}^{x-\bar{x}}(\tau_{R_3} > T)\\ \ge\ & C^{(5)}_{d,m,\alpha,R_1,R_2}e^{-\lambda_{\mathcal{D}}T}\left(\tfrac{\delta_{\mathcal{D}}(x)^{\alpha/2}}{\sqrt{T}\wedge R_3^{\alpha/2}} \wedge 1\right),\end{aligned}$$

and then

$$f_{\mathcal{D}}(x) \ge C^{(6)}_{d,m,\alpha,R_1,R_2}\left(\frac{\delta_{\mathcal{D}}(x)^{\alpha/2}}{\sqrt{T}\wedge R_3^{\alpha/2}} \wedge 1\right),$$

where $C^{(6)}_{d,m,\alpha,R_1,R_2} = C^{(5)}_{d,m,\alpha,R_1,R_2}C^{(5)}_{d,m,\alpha,R_1,R_2}e^{-\lambda_{\mathcal{D}}T}$. Applying this to (3.9) we have

$$\begin{aligned}\mathbb{E}^x[e^{-\lambda T_R}] \ge C^{(7)}_{d,m,\alpha,R_1,R_2,R,\lambda}&\left(\frac{\delta_{\mathcal{D}}(x)^{\alpha/2}}{\sqrt{T}\wedge R_3^{\alpha/2}} \wedge 1\right) j_{m,\alpha}(|x|)\\ &\times \int_{A(x)}\left(\frac{\delta_{\mathcal{D}}(u)^{\alpha/2}}{\sqrt{T}\wedge R_3^{\alpha/2}} \wedge 1\right)\mathrm{d}u,\end{aligned}$$

where

$$C^{(7)}_{d,m,\alpha,R_1,R_2,R,\lambda} = \frac{C^{(1)}_{d,m,\alpha}R^d\omega_d}{(\lambda+\lambda_{\mathcal{D}})2^{d+1}}e^{-(\lambda+\lambda_{\mathcal{D}})T}(C^{(6)}_{d,m,\alpha,R_1,R_2,R})^2.$$

Recall that $\min_{R_1\le|x|\le R_2}\delta_{\mathcal{D}}(x) = (C^{(8)}_{R_1,R_2,R})^{\frac{2}{\alpha}} > 0$ by definition of $\mathcal{D}$. Moreover, $u \in A(x)$ implies $R < R_1 + R \le |u| \le R_2 + 2R < 3R_2$, and hence $\min_{u\in A(x)}\delta_{\mathcal{D}}(u) \ge \min_{R_1+R\le|u|\le 2R+R_2}\delta_{\mathcal{D}}(u) = (C^{(9)}_{R_1,R_2,R})^{\alpha/2} > 0$. Finally, recall also that $|A(x)| \ge \frac{\omega_d}{2}d(R_1+R)^{d-1}R$ to conclude that

$$\mathbb{E}^x[e^{-\lambda T_R}] \ge C_{d,m,\alpha,R_1,R_2,R,\lambda}\,j_{m,\alpha}(|x|),$$

where

$$\begin{aligned}C_{d,m,\alpha,R_1,R_2,R,\lambda} = C^{(7)}_{d,m,\alpha,R_1,R_2,R,\lambda}&\left(\frac{C^{(8)}_{R_1,R_2,R}}{\sqrt{T}\wedge R_3^{\alpha/2}} \wedge 1\right)\left(\frac{C^{(9)}_{R_1,R_2,R}}{\sqrt{T}\wedge R_3^{\alpha/2}} \wedge 1\right)\\ &\times\frac{dR\omega_d}{2}(R_1+R)^{d-1}.\end{aligned}$$

To extend the lower bound up to the boundary of $\mathcal{B}_R$, we need the following result.

Proposition 3.4. *The following properties hold:*

(1) *There exist* $R^{(0)}_{d,m,\alpha,R,\lambda} > R$ *and* $C_{d,m,\alpha,R,\lambda} > 0$ *such that,* $\mathbb{E}^x[1-e^{-\lambda T_R}] \le C_{d,m,\alpha,R,\lambda}(|x|-R)^{\alpha/2}$ *such that for every* $R \le |x| \le R^{(0)}_{d,m,\alpha,R,\lambda}$.

(2) *There exists* $\widetilde{R}_{d,m,\alpha,R,\lambda} > R$ *such that* $\mathbb{E}^x[e^{-\lambda T_R}] \ge \frac{1}{2}$ *for every* $R \le |x| \le \widetilde{R}_{d,m,\alpha,R,\lambda}$.

Proof. By Proposition 3.3

$$\mathbb{P}^x(T_R = \infty) \le C^{(1)}_{d,m,\alpha,R}\left(1 \wedge \frac{(|x|-R)^{\alpha/2}}{R^{\alpha/2}}\right), \tag{3.11}$$

hence there exists $R^{(0)}_{d,m,\alpha,R,\lambda} > R$ such that, for $R < |x| < R^{(0)}_{d,m,\alpha,R,\lambda}$,

$$\mathbb{P}^x(T_R = \infty) \le C^{(1)}_{d,m,\alpha,R}\left(\frac{R^{(0)}_{d,m,\alpha,R,\lambda}}{R} - 1\right)^{\alpha/2} < \frac{1}{3},$$

so that

$$\mathbb{P}^x(T_R < \infty) \ge 1 - C^{(1)}_{d,m,\alpha,R}\left(\frac{R^{(0)}_{d,m,\alpha,R,\lambda}}{R} - 1\right)^{\alpha/2} := C^{(2)}_{d,m,\alpha,R} > \frac{2}{3}.$$

Notice that

$$\mathbb{E}^x[e^{-\lambda T_R}] = \mathbb{E}^x[e^{-\lambda T_R}; T_R < \infty] = \mathbb{E}^x[e^{-\lambda T_R} | T_R < \infty]\mathbb{P}^x(T_R < \infty).$$

Denote $\widetilde{\mathbb{P}}^x(\cdot) = \mathbb{P}^x(\cdot \,|\, T_R < \infty)$. We have

$$1 - \widetilde{\mathbb{E}}^x[e^{-\lambda T_R}] = \int_0^\infty \widetilde{\mathbb{P}}^x(1-e^{-\lambda T_R} > s)\mathrm{d}s = \int_0^1 \widetilde{\mathbb{P}}^x(1-e^{-\lambda T_R} > s)\mathrm{d}s.$$

Writing $s = 1 - e^{-\lambda t}$ we obtain

$$\begin{aligned}
1 - \widetilde{\mathbb{E}}^x[e^{-\lambda T_R}] &= \int_0^\infty \lambda e^{-\lambda t}\widetilde{\mathbb{P}}^x(1-e^{-\lambda T_R} > 1 - e^{-\lambda t})\mathrm{d}t \\
&= \int_0^\infty \lambda e^{-\lambda t}\widetilde{\mathbb{P}}^x(T_R > t)\mathrm{d}t \\
&= \frac{\lambda}{\mathbb{P}^x(T_R < \infty)}\int_0^\infty e^{-\lambda t}\mathbb{P}^x(T_R > t, T_R < \infty)\mathrm{d}t \\
&\le \frac{\lambda}{C^{(2)}_{d,m,\alpha,R}}\int_0^\infty e^{-\lambda t}\mathbb{P}^x(T_R > t)\mathrm{d}t.
\end{aligned}$$

Using Proposition 3.3 gives

$$\mathbb{P}^x(T_R > t) \le C^{(1)}_{d,m,\alpha,R}\left(1 \wedge \frac{(|x|-R)^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}}\right), \quad t > 0.$$

so that, setting $C^{(3)}_{d,m,\alpha,R} = C^{(1)}_{d,m,\alpha,R}/C^{(2)}_{d,m,\alpha,R}$, we get

$$
\begin{aligned}
1-\widetilde{\mathbb{E}}^x[e^{-\lambda T_R}] \le\ & \lambda C^{(3)}_{d,m,\alpha,R}\int_0^\infty e^{-\lambda t}\left(1\wedge\frac{(|x|-R)^{\alpha/2}}{\sqrt{t}\wedge R^{\alpha/2}}\right)\mathrm{d}t\\
=\ & \lambda C^{(3)}_{d,m,\alpha,R}\Bigg(\int_0^{(|x|-R)^\alpha} e^{-\lambda t}\mathrm{d}t+(|x|-R)^{\alpha/2}\\
& \times\left(\int_{(|x|-R)^\alpha}^{R^\alpha}\frac{e^{-\lambda t}}{\sqrt{t}}\mathrm{d}t+\int_{R^\alpha}^\infty\frac{e^{-\lambda t}}{R^{\alpha/2}}\mathrm{d}t\right)\Bigg)\\
=\ & \lambda C^{(3)}_{d,m,\alpha,R}\left(\frac{1-e^{-\lambda(|x|-R)^\alpha}}{\lambda}+\frac{(|x|-R)^{\alpha/2}}{\lambda R^{\alpha/2}}e^{-\lambda R^\alpha}\right.\\
& \left.+\int_{(|x|-R)^\alpha}^{R^\alpha} e^{-\lambda t}\frac{(|x|-R)^{\alpha/2}}{\sqrt{t}}\mathrm{d}t\right).
\end{aligned}
$$

The last term above can be further estimated as

$$
\begin{aligned}
&(|x|-R)^{\alpha/2}\int_{(|x|-R)^\alpha}^{R^\alpha}\frac{e^{-\lambda t}\mathrm{d}t}{\sqrt{t}}\\
&\quad=2(|x|-R)^{\alpha/2}(e^{-\lambda R^\alpha}R^{\alpha/2}-e^{-\lambda(|x|-R)^\alpha}(|x|-R)^{\alpha/2})\\
&\qquad+2(|x|-R)^{\alpha/2}\int_{(|x|-R)^\alpha}^{R^\alpha}\lambda e^{-\lambda t}\sqrt{t}\mathrm{d}t\\
&\quad\le 2(|x|-R)^{\alpha/2}(e^{-\lambda R^\alpha}R^{\alpha/2}-e^{-\lambda(|x|-R)^\alpha}(|x|-R)^{\alpha/2})\\
&\qquad+2(|x|-R)^{\alpha/2}R^{\alpha/2}(e^{-\lambda(|x|-R)^\alpha}-e^{-\lambda R^\alpha})\\
&\quad=2(|x|-R)^{\alpha/2}e^{-\lambda(|x|-R)^\alpha}(R^{\alpha/2}-(|x|-R)^{\alpha/2}).
\end{aligned}
$$

In sum, we obtain

$$
\begin{aligned}
1-\widetilde{\mathbb{E}}^x[e^{-\lambda T_R}]\le\lambda C^{(3)}_{d,m,\alpha,R}&\Bigg((|x|-R)^\alpha+\frac{(|x|-R)^{\alpha/2}}{\lambda R^{\alpha/2}}e^{-\lambda R^\alpha}\\
&+2(|x|-R)^{\alpha/2}e^{-\lambda(|x|-R)^\alpha}R^{\alpha/2}\Bigg)\\
:=\ & C^{(4)}_{d,m,\alpha,R,\lambda}(|x|-R)^\alpha.
\end{aligned}
$$

We can complete the proof of part (1) by observing that

$$
\begin{aligned}
\mathbb{E}^x[1-e^{-\lambda T_R}]&=\widetilde{\mathbb{E}}^x[1-e^{-\lambda T_R}]\mathbb{P}^x(T_R<\infty)+\mathbb{P}^x(T_R=\infty)\\
&\le C_{d,m,\alpha,R,\lambda}(|x|-R)^\alpha,\qquad R\le|x|\le R^{(0)}_{d,m,\alpha,R,\lambda},
\end{aligned}
$$

where we made use of (3.11). Part (2) follows from (1) by choosing $R<\widetilde{R}_{d,m,\alpha,R,\lambda}<R^{(0)}_{d,m,\alpha,R,\lambda}$ so that $\mathbb{E}^x[1-e^{-\lambda T_R}]\le 1/2$ holds for all $R\le|x|\le\widetilde{R}_{d,m,\alpha,R,\lambda}$.

Finally, we can combine Theorem 3.2 with Proposition 3.4 to obtain the following.

Corollary 3.3. *Let $R_2>R$. Then there exists a constant $C_{d,m,\alpha,R_2,R,\lambda}$ such that*

$$
\mathbb{E}^x[e^{-\lambda T_R}]\ge C_{d,m,\alpha,R_2,R,\lambda}j_{m,\alpha}(|x|),\quad R\le|x|<R_2.
$$

Proof. Let $\widetilde{R}_{d,m,\alpha,R,\lambda}$ be defined as in Proposition 3.4. Then, we have

$$\mathbb{E}^x[e^{-\lambda T_R}] \geq \frac{1}{2} = \frac{j_{m,\alpha}(|x|)}{2j_{m,\alpha}(|x|)} \geq \frac{j_{m,\alpha}(|x|)}{2j_{m,\alpha}(R)}, \quad R \leq |x| < \widetilde{R}_{d,m,\alpha,R,\lambda}.$$

Combining this estimate with Theorem 3.2 for $R_1 = \widetilde{R}_{d,m,\alpha,R,\lambda}$ the result follows.

To obtain an upper bound for the same quantities we can make use of [38, Th. 3.3], particularized to the massless and massive relativistic stable processes.

Theorem 3.3. *Let $\lambda, R > 0$. There exists a constant $C_{d,m,\alpha,R,\lambda} > 0$ such that*

$$\mathbb{E}^x[e^{-\lambda T_R}] \leq C_{d,m,\alpha,R,\lambda}\, j_{m,\alpha}(|x|), \quad |x| \geq R.$$

Proof. By [38, Th. 3.3] it follows that there exist constants $R^{(1)}_{d,\alpha,m,\lambda,R} > R$ and $C^{(1)}_{d,\alpha,m,\lambda,R} > 0$ such that

$$\mathbb{E}^x[e^{-\lambda T_R}] \leq C^{(1)}_{d,\alpha,m,\lambda,R}\, j_{m,\alpha}(|x|), \quad |x| \geq R^{(1)}_{d,\alpha,m,\lambda,R}.$$

Let $R^{(2)}_{d,\alpha,m,\lambda,R} = R^{(1)}_{d,\alpha,m,\lambda,R} + 1$ and notice that $j_{m,\alpha}(|x|) \geq j_{m,\alpha}(R^{(2)}_{d,\alpha,m,\lambda,R})$ whenever $R \leq |x| \leq R^{(2)}_{d,\alpha,m,\lambda,R}$. Hence for every $R \leq |x| \leq R^{(2)}_{d,\alpha,m,\lambda,R}$ we get

$$\mathbb{E}^x[e^{-\lambda T_R}] \leq 1 \leq \frac{j_{m,\alpha}(|x|)}{j_{m,\alpha}(R^{(2)}_{d,\alpha,m,\lambda,R})}.$$

Setting $C_{d,m,\alpha,R,\lambda} = \max\left\{C^{(1)}_{d,\alpha,m,\lambda,R}, \frac{1}{j_{m,\alpha}(R^{(2)}_{d,\alpha,m,\lambda,R})}\right\}$ completes the proof.

Remark 3.4. 1. A similar estimate follows by using the Ikeda–Watanabe formula. In this approach we can derive a bound which is uniform with respect to $\alpha \in [\alpha_0, 2]$ for a suitable $\alpha_0 > 0$.
2. Above we obtained a global upper and a local lower bound for $\mathbb{E}^x[e^{-\lambda T_R}]$. A global lower bound for $\mathbb{E}^x[e^{-\lambda T_R}]$ outside the well will be obtained as a consequence of the estimates of the ground states.

4. Basic Qualitative Properties of Ground States

4.1. Martingale Representation of Ground States

For our purposes below, it will be useful to consider a variant of the Feynman–Kac representation (2.4) with general stopping times. In order to obtain this, the following martingale property will be important. Define the random process $(M^x_t)_{t\geq 0}$,

$$M^x_t = e^{\lambda_0 t} e^{-\int_0^t V(X_r + x)\mathrm{d}r} \varphi_0(X_t + x), \quad x \in \mathbb{R}^d. \tag{4.1}$$

Let $(\mathcal{F}^X_t)_{t\geq 0}$ be the natural filtration of the Lévy process $(X_t)_{t\geq 0}$. A version of the following result dates back at least to Carmona's work (see [48, Sect. 4.6.3] for a detailed discussion and references, as well as [15]), but since it is of fundamental interest in this paper, we provide a proof for a self-contained presentation.

Lemma 4.1. *$(M_t^x)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t^X)_{t\geq 0}$.*

Proof. We have

$$\mathbb{E}[|M_t^x|] = \mathbb{E}[M_t^x] \leq e^{\lambda_0 t}\|\varphi_0\|_\infty \mathbb{E}\left[e^{-\int_0^t V(X_r+x)\mathrm{d}r}\right]$$
$$\leq \qquad e^{(v-|\lambda_0|)t}\|\varphi_0\|_\infty < \infty, \quad t \geq 0.$$

Let $0 \leq s \leq t$. By the Markov property of $(X_t)_{t\geq 0}$, we have that

$$\begin{aligned}\mathbb{E}[M_t^x|\mathcal{F}_s^X] = &\quad e^{\lambda_0 t}e^{-\int_0^s V(X_r+x)\mathrm{d}r}\mathbb{E}[e^{-\int_s^t V(X_r+x)\mathrm{d}r}\varphi_0(X_t+x)|\mathcal{F}_s^X]\\ &= e^{\lambda_0 s}e^{-\int_0^s V(X_r+x)\mathrm{d}r}\mathbb{E}^{X_s}[e^{\lambda_0(t-s)}e^{-\int_0^{t-s} V(X_r+x)\mathrm{d}r}\varphi_0(X_{t-s}+x)]\\ &= \quad e^{\lambda_0 s}e^{-\int_0^s V(X_r+x)\mathrm{d}r}\varphi_0(X_s+x) = M_s^x.\end{aligned}$$

Hence, the lemma follows.

Note that by the martingale property $\mathbb{E}[M_t^x] = \mathbb{E}[M_0^x] = \varphi_0(x)$, for all $t \geq 0$ and $x \in \mathbb{R}^d$.

The above martingale property easily leads to the following Feynman–Kac type formula for the stopped process.

Proposition 4.1. *Let τ be a $\mathbb{P}$-almost surely finite stopping time with respect to the filtration $(\mathcal{F}_t^X)_{t\geq 0}$. Then,*

$$\varphi_0(x) = \mathbb{E}^x\left[e^{-\int_0^\tau (V(X_s)-\lambda_0)\mathrm{d}s}\varphi_0(X_\tau)\right].$$

Proof. Since φ_0 is strictly positive, clearly M_t^x is almost surely non-negative. Thus, by the Feynman–Kac formula

$$\mathbb{E}[(M_t^x)^+] = \mathbb{E}[M_t^x] = \varphi_0(x) \leq \|\varphi_0\|_\infty.$$

The martingale convergence theorem (see e.g. [53, Th. 2.10]) implies that $(M_t^x)_{t\geq 0}$ has a final element M_∞^x with $\mathbb{E}[|M_\infty^x|] < \infty$, and the optional stopping theorem (see e.g. [53, Th. 3.2]) then gives

$$\varphi_0(x) = \mathbb{E}[M_0^x] = \mathbb{E}[M_\tau^x] = \mathbb{E}^x\left[e^{-\int_0^\tau (V(X_s)-\lambda_0)\mathrm{d}s}\varphi_0(X_\tau)\right].$$

4.2. Symmetry Properties

Next we discuss some shape properties of ground states, specifically, symmetry and monotonicity, which will be essential ingredients in the study of their local behaviour. First we show radial symmetry of the ground states for rotationally symmetric potential wells. This result can also be obtained by purely analytic methods, see [4, Prop. 4.3].

Theorem 4.1. *Let $\mathcal{K} = \mathcal{B}_a$ with a given $a > 0$ and suppose that $H_{m,\alpha}$ has a ground state φ_0. Then, φ_0 is rotationally symmetric.*

Proof. First observe that if another function $\widetilde{\varphi}_0$ existed satisfying (2.4), $\|\widetilde{\varphi}_0\|_2 = 1$ and $\widetilde{\varphi}_0 > 0$, then by the uniqueness of the ground state we would have $\widetilde{\varphi}_0 \equiv \varphi_0$ almost surely.

Fix a rotation $\mathsf{R} \in \mathrm{SO}(d)$ and consider $\widetilde{\varphi}_0(x) = \varphi_0(\mathsf{R}x)$. Clearly, since R is an isometry, it is immediate that $\|\widetilde{\varphi}_0\|_2 = 1$, $\widetilde{\varphi}_0 > 0$, and $\widetilde{\varphi}_0(x) =$

$\mathbb{E}[e^{-\int_0^t (V(X_s+\mathsf{R}x)-\lambda_0)\mathrm{d}s}\varphi_0(X_t + \mathsf{R}x)]$ by (2.4). By rotational invariance of $(X_t)_{t\geq 0}$ we may furthermore write

$$\begin{aligned}\widetilde{\varphi}_0(x) &= \mathbb{E}\big[e^{-\int_0^t (V(\mathsf{R}X_s+\mathsf{R}x)-\lambda_0)\mathrm{d}s}\varphi_0(\mathsf{R}X_t + \mathsf{R}x)\big] \\ &= \mathbb{E}\big[e^{-\int_0^t (V(X_s+x)-\lambda_0)\mathrm{d}s}\widetilde{\varphi}_0(X_t + x)\big],\end{aligned}$$

where we used the fact that also V is rotationally invariant and $\mathcal{K} = \mathcal{B}_a$. Then, by the observation above, $\widetilde{\varphi}_0 \equiv \varphi_0$ almost surely. Since $\mathsf{R} \in \mathrm{SO}(d)$ is arbitrary, the claim follows.

We can also prove a reduced symmetry of φ_0 for cases when $\mathcal{K}$ is not spherically symmetric.

Theorem 4.2. *Let $\mathcal{K}$ be reflection symmetric with respect to a hyperplane $\mathcal{H}$ such that $0 \in \mathcal{H}$, and let $\mathsf{S} : \mathbb{R}^d \to \mathbb{R}^d$, $\mathsf{S}x$, be such that $\mathsf{S}x$ is the reflection of x with respect to $\mathcal{H}$. Suppose that v is chosen such that $H_{m,\alpha}$ has a ground state φ_0. Then, $\varphi_0(\mathsf{S}x) = \varphi_0(x)$, for all $x \in \mathbb{R}^d$.*

Proof. We can argue similarly to Theorem 4.1. Consider $\widetilde{\varphi}_0(x) = \varphi_0(\mathsf{S}x)$. By the isometry property of S we have again $\|\widetilde{\varphi}_0\|_2 = 1$, $\widetilde{\varphi}_0 > 0$, and $\widetilde{\varphi}_0(x) = \mathbb{E}[e^{-\int_0^t (V(X_s+\mathsf{S}x)-\lambda_0)\mathrm{d}s}\varphi_0(X_t+\mathsf{S}x)]$ by (2.4). Since $(X_t)_{t\geq 0}$ is isotropic, we get

$$\begin{aligned}\widetilde{\varphi}_0(x) &= \mathbb{E}\big[e^{-\int_0^t (V(\mathsf{S}X_s+\mathsf{S}x)-\lambda_0)\mathrm{d}s}\varphi_0(\mathsf{S}X_t + \mathsf{S}x)\big] \\ &= \mathbb{E}\big[e^{-\int_0^t (V(X_s+x)-\lambda_0)\mathrm{d}s}\widetilde{\varphi}_0(X_t + x)\big],\end{aligned}$$

where we used the fact that if $x \in \mathcal{K}$, then also $\mathsf{S}x \in \mathcal{K}$. Arguing as before, we obtain $\varphi_0(\mathsf{S}x) = \widetilde{\varphi}_0(x) = \varphi_0(x)$ for all $x \in \mathbb{R}^d$.

Remark 4.1. We note that Theorems 4.1 and 4.2 hold, respectively, for any rotationally or reflection symmetric potential V once a ground state exists and is unique. Moreover, they can be seen as converses to [3, Th. 7.1−7.2], by using the expression

$$V = -\frac{1}{\varphi_0} L_{m,\alpha}\varphi_0 + \lambda_0,$$

provided $L_{m,\alpha}\varphi_0$ can be defined pointwise.

We fix $\mathcal{K} = \mathcal{B}_a$ for some $a > 0$ and assume that $H_{m,\alpha}$ has a ground state. Furthermore, we will make extensive use of the following, for a proof see [4].

Proposition 4.2. *There exists a non-increasing function $\rho_0 : [0,\infty) \to \mathbb{R}$ such that $\varphi_0(x) = \rho_0(|x|)$ for every $x \in \mathbb{R}^d$.*

5. Local Estimates

5.1. A Prime Example: Classical Laplacian and Brownian Motion

First we present the case of the classical Schrödinger operator with a potential well, for which not only estimates can be obtained but a full reconstruction of the ground state is possible by using the martingale $(M_t)_{t\geq 0}$ in (4.1). Alternatively, this can be done by an explicit solution of the Schrödinger eigenvalue equation, which in this case is a textbook example; however, our point here is that while the eigenvalue problem cannot in general be solved for non-local cases, the probabilistic approach is a useful alternative and this example shows best how this can be done by using occupation times.

Proposition 5.1. *Let*

$$H = -\frac{1}{2}\frac{d^2}{dx^2} - v\mathbf{1}_{\{|x|\le a\}}$$

be given on $L^2(\mathbb{R})$*. Then, the normalized ground state of* H *is*

$$\varphi_0(x) = A_0 e^{-\sqrt{2|\lambda_0|}|x|}\mathbf{1}_{\{|x|>a\}} + B_0\cos\left(\sqrt{2(v-|\lambda_0|)}\,x\right)\mathbf{1}_{\{|x|\le a\}},$$

with

$$A_0 = \sqrt{\frac{\sqrt{2|\lambda_0|}}{1+a\sqrt{2|\lambda_0|}}e^{a\sqrt{2|\lambda_0|}}\cos\left(a\sqrt{2(v-|\lambda_0|)}\right)}, \quad B_0 = \sqrt{\frac{\sqrt{2|\lambda_0|}}{1+a\sqrt{2|\lambda_0|}}}.$$

Proof. Consider for any $b, c \in \mathbb{R}$ with $b < 0 < c$, the first hitting times

$$T_b = \inf\{t>0 : B_t = b\}, \quad T_c = \inf\{t>0 : B_t = c\}, \quad \text{and} \quad T_{b,c} = T_b \wedge T_c,$$

for Brownian motion $(B_t)_{t\ge 0}$ starting at zero, and recall the general formula by Lévy [45]

$$\mathbb{E}^x[e^{iuT_{b,c}}] = \frac{e^{(1+i)x\sqrt{u}}}{e^{(1+i)c\sqrt{u}}+e^{(1+i)b\sqrt{u}}} + \frac{e^{-(1+i)x\sqrt{u}}}{e^{-(1+i)b\sqrt{u}}+e^{-(1+i)c\sqrt{u}}},$$

with $b < x < c$, and

$$\mathbb{E}[e^{-uT_b}] = e^{-\sqrt{2u}|b|} \quad \text{and} \quad \mathbb{E}[e^{-uT_{b,c}}] = \frac{\cosh\left(\sqrt{2u}\,\frac{c+b}{2}\right)}{\cosh\left(\sqrt{2u}\,\frac{c-b}{2}\right)}, \quad u \ge 0. \tag{5.1}$$

It is well known that all these hitting times are almost surely finite stopping times with respect to the natural filtration. From (2.4) we have

$$\varphi_0(x) = \mathbb{E}[e^{-|\lambda_0|t+vU^x_{0,t}(a)}\varphi_0(B_t+x)],$$

where we denote

$$U^x_{0,t}(a) = \int_0^t \mathbf{1}_{\{|B_s+x|\le a\}}\mathrm{d}s = \int_0^t \mathbf{1}_{\{-a-x\le B_s\le a-x\}}\mathrm{d}s.$$

Then, $U^x_{T_{-a-x,a-x}}(a) = T_{-a-x,a-x}$ whenever $|x| < a$, and is zero otherwise. Using Proposition 4.1, we obtain

$$\varphi_0(x) = \left[e^{-|\lambda_0|T_{-a-x,a-x}+vU^x_{T_{-a-x,a-x}}(a)}\varphi_0(B_{T_{-a-x,a-x}}+x)\right].$$

Now suppose $x > a$. By path continuity $T_{-a-x,a-x} = T_{a-x}$ and thus

$$\begin{aligned}\varphi_0(x) &= \mathbb{E}[e^{-|\lambda_0|T_{a-x}}\varphi_0(B_{T_{a-x}}+x)] = \varphi_0(a)\mathbb{E}[e^{-|\lambda_0|T_{a-x}}] \\ &= \varphi_0(a)e^{-\sqrt{2|\lambda_0|}(x-a)}.\end{aligned}$$

We obtain similarly for $x < -a$ that $T_{-a-x,a-x} = T_{-a-x}$ and

$$\begin{aligned}\varphi_0(x) &= \mathbb{E}[e^{-|\lambda_0|T_{a-x}}\varphi_0(B_{T_{a-x}}+x)] = \varphi_0(a)\mathbb{E}[e^{-|\lambda_0|T_{a-x}}] \\ &= \varphi_0(a)e^{-\sqrt{2|\lambda_0|}(x-a)}.\end{aligned}$$

using $\varphi_0(-a) = \varphi_0(a)$. When $-a < x < a$, the two-barrier formula in (5.1) gives

$$\begin{aligned}\varphi_0(x) =& \ \mathbb{E}[e^{(v-|\lambda_0|)T_{-a-x,a-x}}\varphi_0(B_{T_{-a-x,a-x}}+x)]\\ =& \ \mathbb{E}[e^{(v-|\lambda_0|)T_{-a-x,a-x}}\varphi_0(B_{T_{-a-x,a-x}}+x)\mathbf{1}_{\{T_{-a-x}<T_{a-x}\}}]\\ &+\mathbb{E}[e^{(v-|\lambda_0|)T_{-a-x,a-x}}\varphi_0(B_{T_{-a-x,a-x}}+x)\mathbf{1}_{\{T_{-a-x}>T_{a-x}\}}]\\ =& \ \varphi_0(a)\frac{\cos(\sqrt{2(v-|\lambda_0|)}x)}{\cos(\sqrt{2(v-|\lambda_0|)}a)}.\end{aligned}$$

The constant $\varphi_0(a)$ can be determined by the normalization condition $\|\varphi_0\|_2 = 1$, which then yields the claimed expression of the ground state.

Remark 5.1. The argument can also be extended to higher dimensions. For instance, for $d \geq 3$, denote by $\mathcal{B}_r(z)$ a ball of radius r centred in z, write $\mathcal{B}_r = \mathcal{B}_r(0)$, and define the stopping times

$$T_r = \inf\{t > 0 : X_t \in \bar{\mathcal{B}}_r\} \quad \text{and} \quad \tau_r = \inf\{t > 0 : X_t \notin \mathcal{B}_r\}.$$

Using the Ciesielski–Taylor formulae (see e.g. [18, eq. (2.15)] and [12, formula 2.0.1])

$$\mathbb{E}^x[e^{-u\tau_r}] = \left(\frac{r}{|x|}\right)^{\frac{d-2}{2}} \frac{I_{\frac{d-2}{2}}(|x|\sqrt{2u})}{I_{\frac{d-2}{2}}(r\sqrt{2u})} \quad \text{and}$$

$$\mathbb{E}^x[e^{-uT_r}] = \left(\frac{r}{|x|}\right)^{\frac{d-2}{2}} \frac{K_{\frac{d-2}{2}}(|x|\sqrt{2u})}{K_{\frac{d-2}{2}}(r\sqrt{2u})},$$

and the properties of the Bessel function $J_{(d-2)/2}$ and modified Bessel functions $I_{(d-2)/2}$ and $K_{(d-2)/2}$ in standard notation (for properties of the Bessel functions, we refer to [62]), by a similar argument as above for the potential well $-v\mathbf{1}_{\mathcal{B}_a}$ we obtain

$$\begin{aligned}\varphi_0(x) =& \ A_0\left(\frac{a}{|x|}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}\big(\sqrt{2|\lambda_0|}\,|x|\big)\mathbf{1}_{\{|x|>a\}}\\ &+B_0\left(\frac{a}{|x|}\right)^{\frac{d-2}{2}} J_{\frac{d-2}{2}}\big(\sqrt{2(v-|\lambda_0|)}\,|x|\big)\mathbf{1}_{\{|x|\leq a\}},\end{aligned}$$

where the constants A_0, B_0 can be determined from L^2-normalization as before. The details are left to the reader.

5.2. Local Behaviour of the Ground State

To come to our main point in this section, we need some scaling estimates on the Lévy measure $\nu_{m,\alpha}$ of the exterior of a ball.

Lemma 5.1. *For every $R > 0$ there exists a constant $C_{d,m,\alpha,R} > 1$ such that*

$$\int_{\mathcal{B}^c_{C_{d,m,\alpha,R}R}} j_{m,\alpha}(|x-y|)\mathrm{d}y \leq \frac{1}{2}\int_{\mathcal{B}^c_R} j_{m,\alpha}(|x-y|)\mathrm{d}y.$$

Moreover, if $m = 0$, then $C_{d,0,\alpha,R}$ does not depend on R.

Proof. Since $j_{m,\alpha}$ is non-increasing, for every $\theta > 0$ the set $\{j_{m,\alpha}(|x|) \geq \theta\}$ is a ball and then $\nu_{m,\alpha}(\mathrm{d}x)$ is unimodal. As a consequence of Anderson's inequality [2, Th. 1] we get $\int_{\mathcal{B}_R^c} j_{m,\alpha}(|x-y|)\mathrm{d}y \geq \int_{\mathcal{B}_R^c} j_{m,\alpha}(|y|)\mathrm{d}y$, for every $R > 0$ and $x \in \mathcal{B}_R$. Taking $R > 0$, $x \in \mathcal{B}_R$ and $k > 2$, we obtain

$$\begin{aligned}\int_{\mathcal{B}_{kR}^c} j_{m,\alpha}(|x-y|)\mathrm{d}y &\leq \int_{\mathcal{B}_{(k-1)R}^c(x)} j_{m,\alpha}(|x-y|)\mathrm{d}y \\ &= \int_{\mathcal{B}_{(k-1)R}^c} j_{m,\alpha}((k-1)|y|)\mathrm{d}y \\ &= (k-1)^d \int_{\mathcal{B}_R^c} j_{m,\alpha}\left((k-1)|y|\right)\mathrm{d}y.\end{aligned}$$

First consider $m = 0$. We have

$$\int_{\mathcal{B}_R^c} j_{0,\alpha}((k-1)|y|)\mathrm{d}y = \frac{1}{(k-1)^{d+\alpha}} \int_{\mathcal{B}_R^c} j_{0,\alpha}(|y|)\mathrm{d}y,$$

and thus

$$\begin{aligned}\int_{\mathcal{B}_{kR}^c} j_{0,\alpha}(|x-y|)\mathrm{d}y &\leq \frac{1}{(k-1)^{\alpha}} \int_{\mathcal{B}_R^c} j_{0,\alpha}\left(|y|\right)\mathrm{d}y \\ &\leq \frac{1}{(k-1)^{\alpha}} \int_{\mathcal{B}_R^c} j_{0,\alpha}\left(|x-y|\right)\mathrm{d}y.\end{aligned}$$

We can then set $C_{d,0,\alpha} = 1 + 2^{1/\alpha}$ to complete the proof.

Next consider $m > 0$. Using that $j_{m,\alpha}(r) \sim C^{(2)}_{d,m,\alpha} r^{-\frac{d+\alpha+1}{2}} e^{-m^{1/\alpha} r}$ as $r \to \infty$, we have

$$\begin{aligned}j_{m,\alpha}((k-1)|y|) &\leq C^{(3)}_{d,\alpha,R} C^{(2)}_{d,m,\alpha} (k-1)^{-\frac{d+\alpha+1}{2}} |y|^{-\frac{d+\alpha+1}{2}} \frac{e^{-m^{1/\alpha}(k-1)|y|}}{e^{-m^{1/\alpha}|y|}} e^{-m^{1/\alpha}|y|} \\ &\leq (C^{(3)}_{d,\alpha,R})^2 (k-1)^{-\frac{d+\alpha+1}{2}} e^{-m^{1/\alpha} kR} j_{m,\alpha}(|y|),\end{aligned}$$

with some $C^{(3)}_{d,\alpha,R} > 1$, and hence

$$\begin{aligned}\int_{\mathcal{B}_{kR}^c} j_{m,\alpha}(|y|)\mathrm{d}y &\leq (C^{(3)}_{d,\alpha,R})^2 (k-1)^{-\frac{d-\alpha-1}{2}} e^{-m^{1/\alpha} kR} \int_{\mathcal{B}_R^c} j_{m,\alpha}(|y|)\mathrm{d}y \\ &\leq (C^{(3)}_{d,\alpha,R})^2 (k-1)^{-\frac{d-\alpha-1}{2}} e^{-m^{1/\alpha} kR} \int_{\mathcal{B}_R^c} j_{m,\alpha}(|x-y|)\mathrm{d}y.\end{aligned}$$

Choosing $C_{d,m,\alpha,R} > 2$ such that $(C^{(3)}_{d,\alpha,R})^2 (C_{d,m,\alpha,R} - 1)^{-\frac{d-\alpha-1}{2}} e^{-m^{1/\alpha} C_{d,m,\alpha,R} R} \leq \frac{1}{2}$ and using it instead of k, the claim follows.

Combining the last estimate with the Ikeda–Watanabe formula, we obtain the following result.

Lemma 5.2. *For every $R > 0$ there exists a constant $C_{d,m,\alpha,R} > 0$ such that*

$$\mathbb{E}^x\left[g(\tau_R); R \leq |X_{\tau_R}| \leq C_{d,m,\alpha,R} R\right] \geq \frac{1}{2}\mathbb{E}^x[g(\tau_R)]$$

for every non-negative function g and all $x \in \mathcal{B}_R$.

Proof. First consider $g \in L^\infty(\mathbb{R}^d)$ and let $C_{d,m,\alpha,R} > 0$ be defined as in Lemma 5.1. By the Ikeda–Watanabe formula

$$\begin{aligned}&\mathbb{E}^x[g(\tau_R); |X_{\tau_R}| > C_{d,m,\alpha,R}R] \\ &\quad = \int_0^\infty \int_{\mathcal{B}_R} g(t)p_{\mathcal{B}_R}(t,x,y) \int_{\mathcal{B}^c_{C_{d,m,\alpha,R}R}} j_{m,\alpha}(|y-z|)\mathrm{d}z\mathrm{d}y\mathrm{d}t.\end{aligned}$$

Using Lemma 5.1, we thus have

$$\begin{aligned}&\mathbb{E}^x[g(\tau_R); |X_{\tau_R}| > C_{d,m,\alpha,R}R] \\ &\quad \le \frac{1}{2}\int_0^\infty \int_{\mathcal{B}_R} g(t)p_{\mathcal{B}_R}(t,x,y) \int_{\mathcal{B}^c_R} j_{m,\alpha}(|y-z|)\mathrm{d}z\mathrm{d}y\mathrm{d}t = \mathbb{E}^x[g(\tau_R)].\end{aligned}$$

Next suppose that g is unbounded and let $g_N(t) = g(t) \wedge N$ for $N \in \mathbb{N}$. Then, $g_N \uparrow g$ pointwise, moreover

$$\mathbb{E}^x[g_N(\tau_R); R \le |X_{\tau_R}| \le C_{d,m,\alpha,R}R] \ge \frac{1}{2}\mathbb{E}^x[g_N(\tau_R)], \quad N \in \mathbb{N}.$$

As $N \to \infty$, by monotone convergence we then have

$$\mathbb{E}^x[g(\tau_R); R \le |X_{\tau_R}| \le C_{d,m,\alpha,R}R] \ge \frac{1}{2}\mathbb{E}^x[g(\tau_R)].$$

Now we can turn to local estimates of the ground state. Consider the spherical potential well supported in $\mathcal{K} = \mathcal{B}_a$ with some $a > 0$.

Theorem 5.1. *Let φ_0 be the ground state of $H_{m,\alpha}$ with $V = -v\mathbf{1}_{\mathcal{B}_a}$ and denote $\mathbf{a} = (a, 0, \ldots, 0)$. Then, the estimates*

$$\varphi_0(x) \asymp \varphi_0(\mathbf{a}) \times \begin{cases} \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}] & \text{if} \quad |x| \le a \\ \mathbb{E}^x[e^{-|\lambda_0|T_a}] & \text{if} \quad |x| \ge a \end{cases}$$

hold, where the comparability constant depends on $d, m, \alpha, a, v, \lambda_0$.

Proof. Note that φ_0 is rotationally symmetric by Theorem 4.1 and non-increasing by Proposition 4.2. We first prove the bound inside and next outside the well.

Step 1: First consider $|x| \le a$. Using Proposition 4.1 with the almost surely finite stopping time τ_a, and that $X_{\tau_a} \in \mathcal{B}^c_a$ and $\varphi_0(X_{\tau_a}) \le \varphi_0(\mathbf{a})$, we have

$$\varphi_0(x) = \mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}\varphi_0(X_{\tau_a})\right] \le \varphi_0(\mathbf{a})\mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}\right]. \tag{5.2}$$

On the other hand, using that $|X_{\tau_a}| \le C^{(1)}_{d,m,\alpha,a}a$, where $C^{(1)}_{d,m,\alpha,a}$ is defined in Lemma 5.2, we furthermore obtain

$$\begin{aligned}\varphi_0(x) \ge&\ \mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}\varphi_0(X_{\tau_a}); a \le |X_{\tau_a}| \le C^{(1)}_{d,m,\alpha,a}a\right] \\ \ge&\ \varphi_0(C^{(1)}_{d,m,\alpha,a}\mathbf{a})\mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}; a \le |X_{\tau_a}| \le C^{(1)}_{d,m,\alpha,a}a\right].\end{aligned}$$

Recall that $C^{(1)}_{d,m,\alpha,a} > 1$. Consider T_a and $T_M = T_a \wedge M$ for any positive integer $M \in \mathbb{N}$. By Proposition 4.1 applied to the almost surely finite stopping time T_M, note that

$$\varphi_0(C^{(1)}_{d,m,\alpha,a}\mathbf{a}) = \mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_M}\varphi_0(X_{T_M})] \le \varphi_0(0)\mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_M}].$$

By dominated convergence, in the limit $M \to \infty$ we then get

$$0 < \varphi_0(C^{(1)}_{d,m,\alpha,a}\mathbf{a}) \le \varphi_0(0)\mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_a}],$$

implying $C^{(2)}_{d,m,\alpha,a} := \mathbb{P}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}(T_a = \infty) < 1$. In particular, there exists a constant $C^{(3)}_{d,m,\alpha,a} > 0$ such that $\mathbb{P}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}(T_a > C^{(3)}_{d,m,\alpha,a}) < C^{(2)}_{d,m,\alpha,a}$. Furthermore, by using Proposition 4.1 again, we get

$$\begin{aligned}\varphi_0(C^{(1)}_{d,m,\alpha,a}\mathbf{a}) =\ & \mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_M}\varphi_0(X_{T_M})] \\ \ge\ & \mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_a}\varphi_0(X_{T_M})] \\ \ge\ & \mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_a}\varphi_0(X_{T_M}); T_a \le C^{(3)}_{d,m,\alpha,a}].\end{aligned}$$

Since on the set $\{T_a \le C^{(3)}_{d,m,\alpha,a}\}$ the random time T_M is almost surely constant as $M \to \infty$, in the limit

$$\begin{aligned}\varphi_0(C^{(1)}_{d,m,\alpha,a}\mathbf{a}) \ge\ & \mathbb{E}^{C^{(1)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0|T_a}\varphi_0(X_{T_a}); T_a \le C^{(3)}_{d,m,\alpha,a}] \\ \ge\ & (1 - C^{(2)}_{d,m,\alpha,a})e^{-|\lambda_0|C^{(3)}_{d,m,\alpha,a}}\varphi_0(\mathbf{a}) \end{aligned} \tag{5.3}$$

follows, where we also used Proposition 4.2. On the other hand, by Lemma 5.2, we have

$$\mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}; a \le |X_{\tau_a}| \le C^{(1)}_{d,m,\alpha,a}a\right] \ge \frac{1}{2}\mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}]. \tag{5.4}$$

Combining (5.3)–(5.4) and choosing $C^{(4)}_{d,m,\alpha,a,|\lambda_0|} = (1 - C^{(2)}_{d,m,\alpha,a})e^{-|\lambda_0|C^{(3)}_{d,m,\alpha,a}}$ we obtain

$$\varphi_0(x) \ge \frac{C^{(4)}_{d,m,\alpha,a,|\lambda_0|}}{2}\varphi_0(\mathbf{a})\mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}\right],$$

thus

$$\varphi_0(x) \asymp \varphi_0(\mathbf{a})\mathbb{E}^x\left[e^{(v-|\lambda_0|)\tau_a}\right], \quad |x| \le a,$$

where the comparability constant depends on $d, m, \alpha, a, |\lambda_0|$.
Step 2: Next consider $|x| > a$, and let T_a and T_M be defined as before. By Proposition 4.1, we have

$$\begin{aligned}\varphi_0(x) &= \mathbb{E}^x[e^{-|\lambda_0|T_M}\varphi_0(X_{T_M})] \ge \mathbb{E}^x[e^{-|\lambda_0|T_a}\varphi_0(X_{T_M})] \\ &\ge \mathbb{E}^x[e^{-|\lambda_0|T_a}\varphi_0(X_{T_M}); T_a < \infty],\end{aligned}$$

due to $T_M \le T_a$. Taking the limit $M \to \infty$ and observing that T_M is a definite constant if $T_a < \infty$, we get

$$\begin{aligned}\varphi_0(x) &\ge \mathbb{E}^x[e^{-|\lambda_0|T_a}\varphi_0(X_{T_a}); T_a < \infty] \ge \varphi_0(\mathbf{a})\mathbb{E}^x[e^{-|\lambda_0|T_a}; T_a < \infty] \\ &= \varphi_0(\mathbf{a})\mathbb{E}^x[e^{-|\lambda_0|T_a}].\end{aligned} \tag{5.5}$$

On the other hand,

$$\varphi_0(x) \le \varphi_0(0)\mathbb{E}^x[e^{-|\lambda_0|T_M}] \to \varphi_0(0)\mathbb{E}^x[e^{-|\lambda_0|T_a}],$$

as $M \to \infty$, by using dominated convergence. By Step 1, Theorem 3.1 and (5.7) we find a constant $C^{(5)}_{d,m,\alpha,a,|\lambda_0|}$ such that

$$\varphi_0(0) \le C^{(5)}_{d,m,\alpha,a,|\lambda_0|}\varphi_0(\mathbf{a})\left(1+\frac{v-|\lambda_0|}{\lambda_a-v+|\lambda_0|}\right) =: C^{(6)}_{d,m,\alpha,a,v,|\lambda_0|}\varphi_0(\mathbf{a}).$$

and thus

$$\varphi_0(x) \le C^{(6)}_{d,m,\alpha,a,v,|\lambda_0|}\varphi_0(\mathbf{a})\mathbb{E}^x[e^{-|\lambda_0|T_a}]. \tag{5.6}$$

This leads to

$$\varphi_0(x) \asymp \varphi_0(\mathbf{a})\mathbb{E}^x[e^{-|\lambda_0|T_a}], \quad |x| \ge a,$$

where the comparability constants depend on $d, m, \alpha, a, v, |\lambda_0|$.

Remark 5.2. 1. In fact, along the way we also proved that

$$\begin{aligned} &C^{(1)}_{d,m,\alpha,a}\varphi_0(\mathbf{a})e^{-C^{(2)}_{d,m,\alpha,a}|\lambda_0|}\mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}] \\ &\quad \le \varphi_0(x) \le C^{(3)}_{d,m,\alpha,a}\varphi_0(\mathbf{a})\mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}], \end{aligned}$$

for every $|x| \le a$, with constants dependent only on d, m, α, a (and independent of v and λ_0).

2. We point out that we have shown in particular that

$$\mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}] \le \frac{2}{C^{(3)}_{d,m,\alpha,a,|\lambda_0|}}\frac{\varphi_0(x)}{\varphi_0(\mathbf{a})} < \infty.$$

However, from (3.8) we know that $\mathbb{E}^x[e^{\lambda\tau_a}]$ is finite if and only if $\lambda < \lambda_a$. Thus we have also shown that

$$v - |\lambda_0| < \lambda_a. \tag{5.7}$$

We note that to prove this only monotonicity of φ_0 outside the potential well is a required input, which has been proven in [4] without using (5.7) (which is, on the other hand, indispensable to obtain monotonicity inside the well). Thus this argument provides an alternative, purely probabilistic, proof of [4, Lem. 4.5].

Using the following estimate in conjunction with the estimates in Sect. 3, we can derive explicit local estimates for the ground states of the massless and massive relativistic operators.

Corollary 5.1. *With the same notations as in Theorem 5.1, we have*

$$\varphi_0(x) \asymp \varphi_0(\mathbf{a}) \times \begin{cases} 1+\frac{v-|\lambda_0|}{\lambda_a-v+|\lambda_0|}\left(\frac{a-|x|}{a}\right)^{\alpha/2} & \text{if} \quad |x| \le a \\ j_{m,\alpha}(|x|) & \text{if} \quad |x| \ge a, \end{cases}$$

where the comparability constant depends on $d, m, \alpha, a, v, |\lambda_0|$.

Proof. For $|x| \le a$ the result is immediate by a combination of Theorems 5.1 and 3.1, using (5.7). For $|x| \ge a$ we distinguish two cases. First, if $m = 0$, by [38, Cor. 4.1] there exists $R_{d,0,\alpha,a}$ such that

$$\varphi_0(x) \ge C^{(1)}_{d,0,\alpha}|x|^{-d-\alpha} \ge C^{(2)}_{d,0,\alpha}j_{0,\alpha}(|x|), \quad |x| \ge R_{d,0,\alpha,a},$$

where $C^{(1)}_{d,0,\alpha}$ is defined in the quoted result and $C^{(2)}_{d,0,\alpha} = C^{(1)}_{d,0,\alpha}\frac{\pi^{d/2}\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|}{2^\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}$. Secondly, when $m > 0$ we use [38, Cor. 4.3(1)] to find that there exists $R_{d,m,\alpha,a}$ such that

$$\varphi_0(x) \geq C^{(1)}_{d,m,\alpha,a}|x|^{-\frac{d+\alpha+1}{2}}e^{-m^{1/\alpha}|x|}, \quad |x| \geq R_{d,m,\alpha,a}.$$

Moreover, we know that $j_{m,\alpha}(x) \sim |x|^{-\frac{d+\alpha+1}{2}}e^{-m^{1/\alpha}|x|}$ as $|x| \to \infty$, hence there exists a constant $C^{(2)}_{d,m,\alpha}$ such that $\varphi_0(x) \geq C^{(2)}_{d,m,\alpha,a}j_{m,\alpha}(|x|)$ for $|x| \geq R_{d,m,\alpha,a}$. Thus, by (5.6)

$$\mathbb{E}^x[e^{-|\lambda_0|T_a}] \geq C^{(3)}_{d,m,\alpha,a}j_{m,\alpha}(|x|), \quad |x| \geq R_{d,m,\alpha,a}.$$

Combining this with Corollary 3.3 and Theorem 3.3, we obtain

$$\mathbb{E}^x[e^{-|\lambda_0|T_a}] \asymp j_{m,\alpha}(|x|), \quad |x| \geq a,$$

where the comparability constants depend on $d, \alpha, m, a, v, |\lambda_0|$.

Remark 5.3. By Remark 5.2 we have similarly

$$C^{(1)}_{d,m,\alpha,a}\varphi_0(\mathbf{a})e^{-C^{(2)}_{d,m,\alpha,a}|\lambda_0|}\left(1 + \frac{v-|\lambda_0|}{\lambda_a - v + |\lambda_0|}\left(\frac{a-|x|}{a}\right)^{\alpha/2}\right)$$
$$\leq \varphi_0(x) \leq C^{(3)}_{d,m,\alpha,a}\varphi_0(\mathbf{a})\left(1 + \frac{v-|\lambda_0|}{\lambda_a - v + |\lambda_0|}\left(\frac{a-|x|}{a}\right)^{\alpha/2}\right),$$

for $|x| \leq a$ it holds and with constants which depend only on d, m, α, a (and not on v and λ_0).

The local estimates on φ_0 can further be improved to see the behaviour as $|x| \to a$.

Proposition 5.2. *There exist $\varepsilon = \varepsilon_{d,m,\alpha,a,v}, C_{d,m,\alpha,a,v} > 0$ such that for every $x \in \mathcal{B}_{R+\varepsilon}\backslash\mathcal{B}_{R-\varepsilon}$*

$$\left|\frac{\varphi(x)}{\varphi(\mathbf{a})} - 1\right| \leq C_{d,m,\alpha,a,v}\big||x| - a\big|^{\alpha/2}$$

holds.

Proof. The estimate is clear once $x \in \partial\mathcal{B}_a$. Consider first the case $x \in \mathcal{B}_a$. By (5.2), we have

$$\frac{\varphi(x)}{\varphi(\mathbf{a})} - 1 \leq \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a} - 1] \leq C_{d,m,\alpha,a,v}(a-|x|)^{\alpha/2},$$

where we used Theorem 3.1. Taking $x \in \mathcal{B}^c_a$, we have by (5.5),

$$1 - \frac{\varphi(x)}{\varphi(\mathbf{a})} \leq \mathbb{E}^x[1 - e^{-|\lambda_0|T_a}].$$

Choosing $R^{(0)}_{d,m,\alpha,a,v}$ as in Proposition 3.4 and defining $\varepsilon = (R^{(0)}_{d,m,\alpha,a,v} - a) \wedge a$ the result follows.

By using the normalization condition $\|\varphi_0\|_2 = 1$, we are able to provide a two-sided bound on $\varphi_0(\mathbf{a})$.

Proposition 5.3. *Denote* $\mathcal{I} = \int_1^\infty r^{d-1} j_{m,\alpha}^2(ar)\,dr$ *and by* $B(x,y)$ *the usual Beta-function. Then, with the same comparability constant as in Corollary* 5.1*,*

$$\varphi_0(\boldsymbol{a}) \asymp \left(a^d d\omega_d \left(\frac{1}{d} + 2\frac{v-|\lambda_0|}{\lambda_a - v + |\lambda_0|} B\left(d, 1+\frac{\alpha}{2}\right) + \left(\frac{v-|\lambda_0|}{\lambda_a - v+|\lambda_0|}\right)^2 B(d, 1+\alpha) + \mathcal{I}\right)\right)^{-\frac{1}{2}},$$

Proof. We write $\kappa = \frac{v-|\lambda_0|}{\lambda_a - v+|\lambda_0|}$ for a shorthand. Consider $|x| \le a$. By Corollary 5.1 we have

$$\frac{1}{C_{d,m,\alpha,a,v,|\lambda_0|}} \varphi_0(\mathbf{a}) \left(1 + \kappa \left(\frac{a-|x|}{a}\right)^{\alpha/2}\right) \le \varphi_0(x) \le C_{d,m,\alpha,a,v,|\lambda_0|} \varphi_0(\mathbf{a})\,(1+\kappa)\left(\frac{a-|x|}{a}\right)^{\alpha/2},$$

which gives

$$\frac{1}{C_{d,m,\alpha,a,v,|\lambda_0|}} \varphi_0(x) \le \varphi_0(\mathbf{a}) \left(1 + \kappa \left(\frac{a-|x|}{a}\right)^{\alpha/2}\right) \le C_{d,m,\alpha,a,v,|\lambda_0|} \varphi_0(x).$$

Taking the square on both sides and integrating over $\mathcal{B}_a$ we get

$$\frac{1}{(C_{d,m,\alpha,a,v,|\lambda_0|})^2} \int_{\mathcal{B}_a} \varphi_0^2(x)\mathrm{d}x \le \varphi_0^2(\mathbf{a}) \int_{\mathcal{B}_a} \left(1 + \kappa\left(\frac{a-|x|}{a}\right)^{\alpha/2}\right)^2 \mathrm{d}x \le (C_{d,m,\alpha,a,v,|\lambda_0|})^2 \int_{\mathcal{B}_a} \varphi_0^2(x)\mathrm{d}x \qquad (5.8)$$

Consider next $|x| > a$. Proceeding similarly, we have

$$\frac{1}{(C_{d,m,\alpha,a,v,|\lambda_0|})^2} \int_{\mathcal{B}_a^c} \varphi_0^2(x)\mathrm{d}x \le \varphi_0^2(\mathbf{a}) \int_{\mathcal{B}_a^c} j_{m,\alpha}^2(|x|)\mathrm{d}x \le (C_{d,m,\alpha,a,v,|\lambda_0|})^2 \int_{\mathcal{B}_a^c} \varphi_0^2(x)\mathrm{d}x. \qquad (5.9)$$

Adding up (5.8)–(5.9) and using that $\|\varphi_0\|_2 = 1$, we get

$$\frac{1}{(C_{d,m,\alpha,a,v,|\lambda_0|})^2} \le \varphi_0^2(\mathbf{a}) \left(\int_{\mathcal{B}_a} \left(1 + \kappa\left(\frac{a-|x|}{a}\right)^{\alpha/2}\right)^2 \mathrm{d}x + \int_{\mathcal{B}_a^c} j_{m,\alpha}^2(|x|)\mathrm{d}x\right) \le (C_{d,m,\alpha,a,v,|\lambda_0|})^2.$$

By evaluating the integrals and taking the square root, we obtain the required result.

As a direct consequence, we can rewrite Corollary 5.1 as follows.

Corollary 5.2. *With the same notations as in Theorem* 5.1*, we have*

$$\varphi_0(x) \asymp \begin{cases} 1 + \frac{v-|\lambda_0|}{\lambda_a - v+|\lambda_0|}\left(\frac{a-|x|}{a}\right)^{\alpha/2} & \text{if } |x| \le a \\ j_{m,\alpha}(|x|) & \text{if } |x| \ge a, \end{cases}$$

where the comparability constant depends on $d, m, \alpha, a, v, |\lambda_0|$ and is independent of φ_0.

5.3. Lack of Regularity at the Boundary of the Potential Well

From a quick asymptotic analysis of the profile functions appearing in the estimates in Corollary 5.1, the difference of the leading terms suggests that while the regime change around the boundary of the potential well is continuous, it cannot be smooth beyond a degree. To describe this quantitatively, we show next a lack of regularity of the ground state arbitrarily close to the boundary. For a result on Hölder regularity of solutions of related non-local Schrödinger equations, see [44].

Lemma 5.3. *Consider the operator $L_{m,\alpha}$ and the following two cases:*

(1) $\alpha \in (0,1)$ *and* $f \in C^{\alpha+\delta}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *for some* $\delta \in (0, 1-\alpha)$

(2) $\alpha \in [1,2)$ *and* $f \in C^{1,\alpha+\delta-1}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *for some* $\delta \in (0, 2-\alpha)$.

In either case above, the function $\mathbb{R}^d \ni x \mapsto L_{m,\alpha} f(x)$ *is continuous.*

Proof. Note that under the assumptions above, $L_{m,\alpha} f$ is well-defined pointwise via the integral representation (2.1). We show the statement for $m = 0$ only, for $m > 0$ the proof is similar by using the asymptotic behaviour of $j_{m,\alpha}(r)$ around zero and at infinity.

To prove (1), we use the integral representation (2.1) and claim that in this case

$$L_{0,\alpha} f(x) = -C^{(1)}_{d,\alpha} \lim_{\varepsilon \downarrow 0} \left(\int_{\varepsilon < |x-y| < 1} + \int_{|x-y|>1} \right) \tfrac{f(y)-f(x)}{|x-y|^{d+\alpha}} \mathrm{d}y,$$

with the constant $C^{(1)}_{d,\alpha}$ entering the definition of the massless operator. Indeed, note that the second integral in the split is independent of ε, while for the first integral we can use the Hölder inequality giving

$$\begin{aligned} \int_{\varepsilon<|x-y|<1} \frac{|f(y)-f(x)|}{|x-y|^{d+\alpha}} \mathrm{d}y &\leq C^{(2)} \int_{\varepsilon<|x-y|<1} \frac{1}{|x-y|^{d-\delta}} \\ &\leq dC^{(2)} \omega_d \int_0^1 \frac{1}{\rho^{1-\delta}} \mathrm{d}\rho = \frac{dC^{(2)}\omega_d}{\delta}. \end{aligned}$$

The claimed right hand side follows then by dominated convergence. Next choosing $h \in \mathbb{R}^d$, $|h| < 1$, we show that $\lim_{h\to 0} L_{0,\alpha} f(x+h) = L_{0,\alpha} f(x)$. We write

$$\begin{aligned} L_{0,\alpha} f(x+h) &= -C^{(1)}_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(y)-f(x+h)}{|x+h-y|^{d+\alpha}} \mathrm{d}y \\ &= -C^{(1)}_{d,\alpha} \left(\int_{\mathcal{B}_3(x+h)} + \int_{\mathcal{B}_3^c(x+h)} \right) \frac{f(y)-f(x+h)}{|x+h-y|^{d+\alpha}} \mathrm{d}y. \end{aligned}$$

To estimate the first integral, note that $\mathcal{B}_3(x+h) \subseteq \mathcal{B}_4(x)$ for every $h \in \mathcal{B}_1$. Let $C^{(3)}$ be the Hölder constant associated with $\overline{\mathcal{B}}_4(x)$ and observe that

$$\int_{\mathcal{B}_3(x+h)} \frac{|f(y)-f(x+h)|}{|x+h-y|^{d+\alpha}}\mathrm{d}y = \int_{\mathcal{B}_3} \frac{|f(x+h+y)-f(x+h)|}{|y|^{d+\alpha}}\mathrm{d}y$$
$$\leq C^{(3)}\int_{\mathcal{B}_3}\frac{\mathrm{d}y}{|y|^{d-\delta}} = \frac{3^\delta C^{(3)} d\omega_d}{\delta}.$$

For the second integral, observe that if $y \in \mathcal{B}_2(x)$, then $|x+h-y| \leq |x-y| + |h| < 3$ so that $y \in \mathcal{B}_3(x+h)$ for any $h \in \mathcal{B}_1$. This means that $\mathcal{B}_3^c(x+h) \subseteq \mathcal{B}_2^c(x)$ for all h and then

$$\int_{\mathcal{B}_3^c(x+h)} \frac{|f(y)-f(x+h)|}{|x+h-y|^{d+\alpha}}\mathrm{d}y \leq \int_{\mathcal{B}_2^c(x)} \frac{|f(y)-f(x+h)|}{|x+h-y|^{d+\alpha}}\mathrm{d}y$$
$$\leq 2\|f\|_\infty \int_{\mathcal{B}_2^c(x)} \frac{\mathrm{d}y}{(|x-y|-|h|)^{d+\alpha}}$$
$$\leq 2\|f\|_\infty d\omega_d \int_2^\infty \frac{\rho^{d-1}}{(\rho-1)^{d+\alpha}}\mathrm{d}\rho < \infty.$$

Thus, again we can use dominated convergence to prove the claim.

Next consider (2). Fix $x \in \mathbb{R}^d$ and define the function

$$\mathcal{B}_1 \ni h \mapsto D_h f(x) := f(x+h) - 2f(x) + f(x-h).$$

By Lagrange's theorem there exist $\xi_\pm(h) \in [x, x \pm h]$, where $[x,y]$ denotes the segment with endpoints x, y, such that

$$f(x+h) - 2f(x) + f(x-h) = \langle \nabla f(\xi_+(h)) - \nabla f(\xi_-(h)), h\rangle$$

and thus $|D_h f(x)| \leq |\nabla f(\xi_+(h)) - \nabla f(\xi_-(h))||h|$. Since $\xi_\pm(h) \in [x, x\pm h]$, in particular $\xi_\pm(h) \in \mathcal{B}_1(x)$, and we can use the Hölder property of the gradient to conclude that

$$|\nabla f(\xi_+(h)) - \nabla f(\xi_-(h))| \leq C^{(1)}(x)|\xi_+(h) - \xi_-(h)||h|^{\alpha+\delta-1}.$$

Moreover, $|\xi_+(h) - \xi_-(h)| \leq 2$, and thus $|D_h f(x)| \leq 2C^{(1)}(x)|h|^{\alpha+\delta}$. Using that $\int_0^1 \frac{1}{\rho^{1-\delta}}\mathrm{d}\rho = \frac{1}{\delta}$, by an application of [3, Prop. 2.6, Rem. 2.4] we then obtain

$$L_{0,\alpha} f(x) = -\frac{C_{d,\alpha}^{(2)}}{2}\int_{\mathbb{R}^d} \frac{D_h f(x)}{|h|^{d+\alpha}}\mathrm{d}h, \quad x \in \mathbb{R}^d.$$

Taking $k \in \mathcal{B}_1$, we show that $\lim_{k\to 0} L_{0,\alpha} f(x+k) = L_{0,\alpha} f(x)$. Write

$$L_{0,\alpha} f(x+k) = -\frac{C_{d,\alpha}^{(2)}}{2}\int_{\mathcal{B}_3} \frac{D_h f(x+k)}{|h|^{d+\alpha}}\mathrm{d}h - \frac{C_{d,\alpha}^{(2)}}{2}\int_{\mathcal{B}_3^c} \frac{D_h f(x+k)}{|h|^{d+\alpha}}\mathrm{d}h.$$

In the first integral, we have $x + k \pm h \in \mathcal{B}_4(x)$ for every $k \in \mathcal{B}_1$ and $h \in \mathcal{B}_3$, hence $|D_h f(x+k)| \leq 8C^{(3)}(x)|h|^{\alpha+\delta}$, similarly to in the previous case, where $C^{(3)}(x)$ is the Hölder constant of ∇f in $\overline{\mathcal{B}}_4(x)$. Thus, we obtain

$$\int_{\mathcal{B}_3} \frac{|D_h f(x+k)|}{|h|^{d+\alpha}}\mathrm{d}h \leq \frac{8C^{(3)} d\omega 3^\delta}{\delta}\int_{\mathcal{B}_3}\frac{\mathrm{d}h}{|h|^{d-\delta}}.$$

For the second integral, using that $f \in L^\infty(\mathbb{R}^d)$ we get

$$\int_{\mathcal{B}_3^c} \frac{|D_h f(x+k)|}{|h|^{d+\alpha}} \mathrm{d}h \le 4\,\|f\|_\infty \int_{\mathcal{B}_3^c} \frac{\mathrm{d}h}{|h|^{d+\alpha}} < \infty.$$

The proof is then completed by dominated convergence.

Theorem 5.2. *Let φ_0 be the ground state of $H_{m,\alpha}$. The following hold:*

(1) If $\alpha \in (0,1)$, then $\varphi_0 \notin C^{\alpha+\delta}_{\mathrm{loc}}(\mathbb{R}^d)$ for every $\delta \in (0, 1-\alpha)$.

(2) If $\alpha \in [1,2)$, then $\varphi_0 \notin C^{1,\alpha+\delta-1}_{\mathrm{loc}}(\mathbb{R}^d)$ for every $\delta \in (0, 2-\alpha)$.

Proof. We rewrite the eigenvalue equation like

$$L_{m,\alpha}\varphi_0 = (v\mathbf{1}_{\mathcal{B}_a} + \lambda_0)\varphi_0. \tag{5.10}$$

Suppose that $\alpha \in (0,1)$ and $\varphi_0 \in C^{\alpha+\delta}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\delta \in (0, 1-\alpha)$. Then, by (1) of Lemma 5.3 we have that the left-hand side of (5.10) is continuous. On the other hand, take $\mathbf{e}_1 = (1, 0, \ldots, 0)$ and notice that

$$\lim_{\varepsilon \downarrow 0}(v\mathbf{1}_{\mathcal{B}_a}((a+\varepsilon)\mathbf{e}_1) + \lambda_0)\varphi_0((a+\varepsilon)\mathbf{e}_1) = \lambda_0\varphi_0(a\mathbf{e}_1)$$
$$\lim_{\varepsilon \downarrow 0}(v\mathbf{1}_{\mathcal{B}_a}((a-\varepsilon)\mathbf{e}_1) + \lambda_0)\varphi_0((a-\varepsilon)\mathbf{e}_1) = (v+\lambda_0)\varphi_0(a\mathbf{e}_1),$$

thus the right-hand side is continuous in $a\mathbf{e}_1$ if and only if $\varphi_0(a\mathbf{e}_1) = 0$, which is in contradiction with the fact that φ_0 is positive. In particular, the same argument holds for any point $x \in \partial\mathcal{B}_a$; thus, the right-hand side of (5.10) has a jump discontinuity on $\partial\mathcal{B}_a$, which is impossible since the left-hand side is continuous. The same arguments hold for $\alpha \in [1,2)$ by using part (2) of Lemma 5.3.

Remark 5.4. (1) Instead of using $C^{\alpha+\delta}_{\mathrm{loc}}(\mathbb{R}^d)$ we also can prove part (1) of Lemma 5.3 with $f \in C^{\alpha+\delta}(\overline{\mathcal{B}}_r(x))$ for some $x \in \mathbb{R}^d$, implying that $L_{m,\alpha}f$ is continuous in x. With this localization argument, we obtain for $\alpha \in (0,1)$ that $\varphi_0 \notin C^{\alpha+\delta}_{\mathrm{loc}}(\mathcal{B}_{a+\varepsilon}\setminus\overline{\mathcal{B}}_{a-\varepsilon})$, for all $\varepsilon \in (0,a)$ and $\delta \in (0, 1-\alpha)$. In particular, this implies that φ_0 cannot be C^1 on $\partial\mathcal{B}_a$. The same arguments apply to part (2) of Lemma 5.3 and the case $\alpha \ge 1$, implying that φ_0 cannot be C^2 on $\partial\mathcal{B}_a$. We note that for the classical case the ground state is C^1 but fails to be C^2 at the boundary of the potential well.

(2) It is reasonable to expect that φ_0 has at least a $C^{\alpha-\varepsilon}$-regularity, for all $\varepsilon > 0$ small enough, both inside and outside the potential well (away from the boundary). However, this needs different tools and we do not pursue this point here.

5.4. Moment Estimates of the Position in the Ground State

As an application of the local estimates of ground states, we consider now the behaviour of the following functional. Note that when the ground state is chosen to satisfy $\|\varphi_0\|_2 = 1$, the expression $\varphi_0^2(x)\mathrm{d}x$ defines a probability measure on $\mathbb{R}^d$. Let $p > 0$ and define

$$\Lambda_p(\varphi_0) = \left(\int_{\mathbb{R}^d} |x|^p \varphi_0^2(x)\mathrm{d}x\right)^{1/p},$$

which can then be interpreted for $p \geq 1$ as the pth moment of an $\mathbb{R}^d$-valued random variable under this probability distribution. In the physics literature, the ground state expectation for $p = 2$ is called the size of the ground state.

Let $m \geq 0$, $\alpha \in (0, 2)$, and define

$$p_*(m,\alpha) := \begin{cases} d + 2\alpha & \text{if} \quad m = 0 \\ \infty & \text{if} \quad m > 0. \end{cases}$$

Also, we write for a shorthand

$$\mathcal{J}_a = \frac{\lambda_a}{\lambda_a - v + |\lambda_0|} = \frac{1}{1 - \frac{v - |\lambda_0|}{\lambda_a}}, \tag{5.11}$$

which is a constant related to the ratio between the energy gap separating the ground state eigenvalue from the bottom value of the potential and the energy needed to climb the potential well.

Lemma 5.4. *The following cases occur:*

1. *If* $0 < p < p_*(m, \alpha)$, *then* $\Lambda_p(\varphi_0) < \infty$.
2. *If* $p \geq p_*(m, \alpha)$, *then* $\Lambda_p(\varphi_0) = \infty$.

Proof. It is a direct consequence of Corollary 5.1, using that $j_{0,\alpha}(r) = C_{d,\alpha} r^{-d-\alpha}$, and $j_{m,\alpha}(r) \approx r^{-(d+\alpha+1)/2} e^{-m^{1/\alpha} r}$ as $r \to \infty$ if $m > 0$. Indeed, while for $m > 0$ it is immediate, for $m = 0$ we have $\rho^{d-1+p} j^2_{0,\alpha}(\rho) = C_{d,\alpha} \rho^{-(d+1+2\alpha-p)}$, so that it is integrable at infinity if and only if $d + 2\alpha > p$.

Proposition 5.4. *Let* $0 < p < p_*(m, \alpha)$. *There exist constants* $C^{(1)}_{d,m,\alpha,a,p}$, $C^{(2)}_{d,m,\alpha,a} > 0$ *such that*

$$\Lambda_p(\varphi_0) \geq C^{(1)}_{d,m,\alpha,a,p} \mathcal{J}_a^{2/p} e^{-\frac{2}{p} C^{(2)}_{d,m,\alpha,a} |\lambda_0|} \varphi_0^{2/p}(\mathbf{a}). \tag{5.12}$$

Proof. By Remark 5.3, we get

$$\begin{aligned} \varphi_0^2(x) &\geq \varphi_0^2(\mathbf{a}) (C^{(3)}_{d,m,\alpha,a})^2 \left(1 + 2 \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \left(\frac{a - |x|}{a} \right)^{\alpha/2} \right. \\ &\quad \left. + \left(\frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \right)^2 \left(\frac{a - |x|}{a} \right)^{\alpha} \right) e^{-2C^{(2)}_{d,m,\alpha,a} |\lambda_0|} \\ &\geq \varphi_0^2(\mathbf{a}) (C^{(3)}_{d,m,\alpha,a})^2 \mathcal{J}_a^2 \left(\frac{a - |x|}{a} \right)^{\alpha} e^{-2C^{(2)}_{d,m,\alpha,a} |\lambda_0|}, \quad |x| \leq a, \end{aligned}$$

where the last step follows by the fact that $\frac{a-|x|}{a} \leq 1$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^p \varphi_0^2(x) \mathrm{d}x &\geq \int_{\mathcal{B}_a} |x|^p \varphi_0^2(x) \mathrm{d}x \\ &\geq \varphi_0^2(\mathbf{a}) (C^{(3)}_{d,m,\alpha,a})^2 \mathcal{J}_a^2 e^{-2C^{(2)}_{d,m,\alpha,a} |\lambda_0|} \int_{\mathcal{B}_a} |x|^p \left(\frac{a - |x|}{a} \right)^{\alpha} \mathrm{d}x. \end{aligned}$$

Setting $(C^{(1)}_{d,m,\alpha,a,p})^p = (C^{(3)}_{d,m,\alpha,a})^2 \int_{\mathcal{B}_a} |x|^p \left(\frac{a-|x|}{a} \right)^{\alpha} \mathrm{d}x$, the result follows.

Proposition 5.5. *Let $0 < p < p_*(m, \alpha)$ and $v > \lambda_a + \delta$ for some $\delta > 0$. Then, there exists a constant $C_{d,m,\alpha,\delta,a,p} > 0$ such that*

$$\Lambda_p(\varphi_0) \le C_{d,m,\alpha,\delta,a,p}\, \mathcal{J}_a{}^{2/p} \varphi_0^{2/p}(\mathbf{a}).$$

Proof. As in Theorem 5.1, observe that for $|x| \ge a$ we have by Proposition 4.2

$$\varphi_0(x) \le \varphi_0(0)\mathbb{E}^x[e^{-|\lambda_0|T_a}]. \tag{5.13}$$

Moreover, by Remark 5.3,

$$\varphi_0(0) \le C^{(1)}_{d,m,\alpha,a} \mathcal{J}_a \varphi_0(\mathbf{a}). \tag{5.14}$$

On the other hand, from $v - |\lambda_0| < \lambda_a$ we get $|\lambda_0| > v - \lambda_a > \delta$ and then

$$\mathbb{E}^x[e^{-|\lambda_0|T_a}] \le \mathbb{E}^x[e^{-\delta T_a}] \le C^{(2)}_{d,m,\alpha,\delta,a} j_{m,\alpha}(|x|), \quad |x| \ge a, \tag{5.15}$$

where we used also Theorem 3.3. Combining (5.14)–(5.15) with (5.13), we obtain

$$\varphi_0(x) \le C^{(3)}_{d,m,\alpha,\delta,a} \mathcal{J}_a \varphi_0(\mathbf{a}) j_{m,\alpha}(|x|), \quad |x| \ge a, \tag{5.16}$$

where $C^{(3)}_{d,m,\alpha,\delta,a} = C^{(1)}_{d,m,\alpha,a} C^{(2)}_{d,m,\alpha,\delta,a}$. For $|x| \le a$, we have directly by Remark 5.3

$$\begin{aligned} \varphi_0(x) &\le C^{(1)}_{d,m,\alpha,a} \varphi_0(\mathbf{a}) \left(1 + \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \left(\frac{a - |x|}{a}\right)^{\alpha/2}\right) \\ &\le C^{(1)}_{d,m,\alpha,a} \mathcal{J}_a \varphi_0(\mathbf{a}), \end{aligned} \tag{5.17}$$

where again we used that $\frac{a-|x|}{a} \le 1$. Hence, by (5.16)–(5.17) we get

$$\int_{\mathbb{R}^d} |x|^p \varphi_0^2(x)\mathrm{d}x = \int_{\mathcal{B}_a} |x|^p \varphi_0^2(x)\mathrm{d}x + \int_{\mathcal{B}_a^c} |x|^p \varphi_0^2(x)\mathrm{d}x \le C^p_{d,m,\alpha,\delta,a,p} \mathcal{J}_a^2 \varphi_0^2(\mathbf{a}),$$

where

$$C^p_{d,m,\alpha,\delta,a,p} = \max\left\{ (C^{(1)}_{d,m,\alpha,a})^2 \int_{\mathcal{B}_a} |x|^p \mathrm{d}x,\ (C^{(3)}_{d,m,\alpha,\delta,a})^2 \int_{\mathcal{B}_a^c} |x|^p j^2_{m,\alpha}(x)\mathrm{d}x \right\}.$$

Remark 5.5. As discussed in Sect. 2.2, a ground state exists for all $v > 0$ when the process $(X_t)_{t\ge 0}$ is recurrent, and it only exists for $v > v^*$ with a given $v^* = v^*(\alpha, m, a, d) > 0$ when the process is transient. An interesting question is to analyse the blow-up rate of $\Lambda_p(\varphi_0)$ for some p as $v \downarrow v^*$. This would require a good control of the v-dependence of λ_0 and the comparability constants, however, both appear to be rather involved. An expression of $\lambda_0 = \lambda_0(v)$ may in principle be expected to follow from the continuity condition $\varphi_0(\mathbf{a}-) = \varphi_0(\mathbf{a}+)$; however, this seems to be difficult to obtain in any neat explicit form. In fact, even in the classical Schrödinger eigenvalue problem this is a transcendental equation which can only numerically be solved, and the similar blow-up problem also becomes untractable in terms of closed form expressions.

A further interesting question is what are the convexity properties of the ground state φ_0. To study this would need tools which go beyond the scope of this paper; however, we conjecture that inside the potential well φ_0 is a concave function (at least for $\alpha \geq 1$), and outside the potential well it becomes convex for all α. Instead of this property, now we estimate the contribution to the moments of the probability mass separately inside and outside the potential well. For every $p \geq 1$ define

$$\Lambda_p^{\mathrm{in}}(\varphi_0) = \Lambda_p(\varphi_0 \mathbf{1}_{\mathcal{B}_a}) = \left(\int_{\mathcal{B}_a} |x|^p \varphi_0^2(x) \mathrm{d}x\right)^{1/p} \quad \text{and}$$

$$\Lambda_p^{\mathrm{out}}(\varphi_0) = \Lambda_p(\varphi_0 \mathbf{1}_{\mathcal{B}_a^c}) = \left(\int_{\mathcal{B}_a^c} |x|^p \varphi_0^2(x) \mathrm{d}x\right)^{1/p}.$$

Since φ_0 is continuous, it is clear that $\Lambda_p^{\mathrm{in}}(\varphi_0) < \infty$ for all $p \geq 1$, $m \geq 0$ and $\alpha \in (0,2)$. We can provide a simple two-sided bound of $\Lambda_p^{\mathrm{in}}(\varphi_0)$.

Proposition 5.6. *Let $p > 0$. Then, there exist three constants $C^{(1)}_{d,m,\alpha,a,p}$, $C^{(2)}_{d,m,\alpha,a}, C^{(3)}_{d,m,\alpha,a}$ such that*

$$C^{(1)}_{d,m,\alpha,a,p} e^{-\frac{2}{p} C^{(2)}_{d,m,\alpha,a} |\lambda_0|} \mathcal{J}_a^{2/p} \varphi_0^{2/p}(\mathbf{a}) \leq \Lambda_p^{\mathrm{in}}(\varphi_0) \leq C^{(3)}_{d,m,\alpha,a,p} \mathcal{J}_a^{2/p} \varphi_0^{2/p}(\mathbf{a}).$$

Proof. The lower bound follows by a similar argument as in Proposition 5.4. For the upper bound, observe that (5.17) continues to hold and then

$$\int_{\mathcal{B}_a} |x|^p \varphi_0^2(x) \mathrm{d}x \leq \left(C^{(4)}_{d,m,\alpha,a}\right)^2 \frac{d}{d+p} \omega_d a^{d+p} \mathcal{J}_a^2 \varphi_0^2(\mathbf{a}).$$

By setting $\left(C^{(3)}_{d,m,\alpha,a,p}\right)^p = \left(C^{(4)}_{d,m,\alpha,a}\right)^2 \frac{d}{d+p} \omega_d a^{d+p}$, we get the statement.

Remark 5.6. Arguing as in Theorem 5.1, we can prove that there exist constants $C^{(1)}_{d,m,\alpha,a} > 0$ and $C^{(2)}_{d,m,\alpha,a} > 1$ such that

$$\varphi_0(x) \geq C^{(1)}_{d,m,\alpha,a} \mathcal{J}_a \varphi_0(\mathbf{a}) \left(\frac{a - |x|}{a}\right)^{\alpha/2} \mathbb{E}^{C^{(2)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0| T_a}]$$

and thus

$$\Lambda_p^{\mathrm{in}}(\varphi_0) \geq C_{d,m,\alpha,a,p} \mathcal{J}_a^{2/p} \varphi_0^{2/p}(\mathbf{a}) \left(\mathbb{E}^{C^{(2)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0| T_a}]\right)^{2/p}.$$

Furthermore, by Lemma 5.4, we know that $\Lambda_p^{\mathrm{out}}(\varphi_0) < \infty$ if and only if $p < p_*(m,\alpha)$. Since we are in lack of a lower bound on $\varphi_0(x)$ outside the well in which the dependence on $|\lambda_0|$ is explicit, we only focus on an upper bound for $\Lambda_p^{\mathrm{out}}(\varphi_0)$.

Proposition 5.7. *Let $0 < p < p_*(m,\alpha)$ and $v > \lambda_a + \delta$ for some $\delta > 0$. There exists a constant $C_{d,m,\alpha,\delta,a,p}$ such that*

$$\Lambda_p^{\mathrm{out}}(\varphi_0) \leq C_{d,m,\alpha,\delta,a,p} \mathcal{J}_a^{2/p} (\varphi_0(\mathbf{a}))^{2/p}.$$

Moreover, $C_{d,m,\alpha,\delta,a,p} \to 0$ as $\delta \to \infty$.

Proof. By (5.13) and (5.14), we have for $|x| \geq a$

$$\varphi_0(x) \leq C^{(1)}_{d,m,\alpha,a} \mathcal{J}_a \varphi_0(\mathbf{a}) \mathbb{E}^x[e^{-|\lambda_0| T_a}] \leq C^{(1)}_{d,m,\alpha,a} \mathcal{J}_a \varphi_0(\mathbf{a}) \mathbb{E}^x[e^{-\delta T_a}]$$

and thus

$$\int_{\mathcal{B}_a^c} |x|^p \varphi_0^2(x) \mathrm{d}x \leq C^{(1)}_{d,m,\alpha,a} \mathcal{J}_a^2 \varphi_0^2(\mathbf{a}) \int_{\mathcal{B}_a^c} |x|^p \left(\mathbb{E}^x[e^{-\delta T_a}]\right)^2 \mathrm{d}x,$$

where we note that by Theorem 3.3 and the fact that $p < p_*(m, \alpha)$ the integral at the right-hand side above is bounded above by $\int_{\mathcal{B}_a^c} |x|^p j^2_{m,\alpha}(|x|) \mathrm{d}x < \infty$. The upper bound then follows by setting the constant $C^p_{d,m,\alpha,\delta,a,p} = \left(C^{(1)}_{d,m,\alpha,a}\right)^2 \int_{\mathcal{B}_a^c} |x|^p \left(\mathbb{E}^x[e^{-\delta T_a}]\right)^2 \mathrm{d}x$. The second part of the statement follows by a direct application of the dominated convergence theorem.

Remark 5.7. 1. We can choose $\delta = |\lambda_0|$ so that

$$\Lambda_p^{\mathrm{out}}(\varphi_0) \leq C_{d,m,\alpha,a,p} \mathcal{J}_a^{2/p} (\varphi_0(\mathbf{a}))^{2/p} \left(\int_{\mathcal{B}_a^c} |x|^p \left(\mathbb{E}^x[e^{-|\lambda_0| T_a}]\right)^2 \mathrm{d}x\right)^{1/p}$$

follows with a constant $C_{d,m,\alpha,a,p} > 0$.

2. Finally, we note that by Propositions 5.6–5.7 and Remarks 5.6–5.7 we also have

$$\begin{aligned} \frac{\Lambda_p^{\mathrm{out}}(\varphi_0)}{\Lambda_p^{\mathrm{in}}(\varphi_0)} &\leq K^{(1)} \frac{\left(\int_{\mathcal{B}_a^c} |x|^p \left(\mathbb{E}^x[e^{-|\lambda_0| T_a}]\right)^2 \mathrm{d}x\right)^{1/p}}{\left(\mathbb{E}^{C^{(2)}_{d,m,\alpha,a}\mathbf{a}}[e^{-|\lambda_0| T_a}]\right)^{2/p}} \\ &\leq K^{(2)} \frac{\left(\int_{\mathcal{B}_a^c} |x|^p j^2_{m,\alpha}(|x|) \mathrm{d}x\right)^{1/p}}{j^{2/p}_{m,\alpha}(C^{(2)}_{d,m,\alpha,a}\mathbf{a})}, \end{aligned}$$

where $K^{(1)} = K^{(1)}_{d,m,\alpha,a,p} > 0$, $K^{(2)} = K^{(2)}_{d,m,\alpha,a,p,|\lambda_0|} > 0$ are suitable constants, and $C^{(2)}_{d,m,\alpha,a}$ is defined in Remark 5.6.

5.5. Concentration Properties of the Ground State

In this section, we discuss the R-dependence of the probability distribution function

$$F(R) = \int_{\mathcal{B}_R} \varphi_0^2(x) \mathrm{d}x, \quad R \geq 0. \tag{5.18}$$

We derive a two-sided bound for F by using the local estimates of the ground state obtained above. Below we write $B(u; x, y) = \int_0^u r^{x-1}(1-r)^{y-1} \mathrm{d}r$, $0 \leq u \leq 1$, for the incomplete Beta-function, $B(x, y)$ for the standard Beta-function as above (which coincides with the incomplete Beta-function for $u = 1$), and keep using the shorthands (5.11) and $\bar{\mathcal{J}}_a = \mathcal{J}_a - 1 = \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|}$.

Proposition 5.8. *1. Let $R \le a$. There exist constants $C^{(1)}_{d,m,\alpha,a}, C^{(2)}_{d,m,\alpha,a}, C^{(3)}_{d,m,\alpha,a} > 0$ such that*

$$K^{(1)}_{d,m,\alpha,a}\left(\frac{1}{d}\left(\frac{R}{a}\right)^d + 2\bar{\mathcal{J}}_a B\left(\frac{R}{a}; d, 1+\frac{\alpha}{2}\right) + \bar{\mathcal{J}}_a{}^2 B\left(\frac{R}{a}; d, 1+\alpha\right)\right) \le F(R) \le K^{(2)}_{d,m,\alpha,a}\left(\frac{1}{d}\left(\frac{R}{a}\right)^d + 2\bar{\mathcal{J}}_a B\left(\frac{R}{a}; d, 1+\frac{\alpha}{2}\right) + \bar{\mathcal{J}}_a{}^2 B\left(\frac{R}{a}; d, 1+\alpha\right)\right),$$

where $K^{(1)}_{d,m,\alpha,a} = (C^{(1)}_{d,m,\alpha,a}\varphi_0(\boldsymbol{a}))^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} d\omega_d a^d$ and $K^{(2)}_{d,m,\alpha,a} = (C^{(3)}_{d,m,\alpha,a}\varphi_0(\boldsymbol{a}))^2 d\omega_d a^d$.

2. Let $R \ge a$. There exists a constant $C^{(4)}_{d,m,\alpha,a,|\lambda_0|} > 0$ such that

$$F(R) \ge C^{(4)}_{d,m,\alpha,a,|\lambda_0|}\varphi_0^2(\boldsymbol{a}) d\omega_d a^d \left(\tfrac{1}{d} + 2\bar{\mathcal{J}}_a B\left(d, 1+\tfrac{\alpha}{2}\right) + \bar{\mathcal{J}}_a{}^2 B(d, 1+\alpha) + \mathcal{I}(R)\right),$$

where $\mathcal{I}(R) = \int_1^{R/a} r^{d-1} j^2_{m,\alpha}(ar)\,\mathrm{d}r$.

3. Let $R \ge a$. Suppose that $v > \lambda_a$ and write $\delta = v - \lambda_a$. Then, there exists a constant $C^{(5)}_{d,m,\alpha,\delta,a} > 0$ such that

$$F(R) \le C^{(5)}_{d,m,\alpha,\delta,a}\varphi_0^2(\mathbf{a})\left(\frac{1}{d} + 2\bar{\mathcal{J}}_a B\left(d, 1+\frac{\alpha}{2}\right) + \bar{\mathcal{J}}_a{}^2 B(d, 1+\alpha) + \mathcal{J}_a{}^2 \mathcal{I}(R)\right).$$

Proof. (1) By Remark 5.3, we have

$$\begin{aligned} F(R) &\ge (C^{(1)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} d\omega_d \times \int_0^R r^{d-1}\left(1 + 2\bar{\mathcal{J}}_a\left(\frac{a-r}{a}\right)^{\alpha/2} + \bar{\mathcal{J}}_a\left(\frac{a-r}{a}\right)^{\alpha}\right)\mathrm{d}r \\ &= (C^{(1)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} d\omega_d a^d \times \left(\frac{1}{d}\left(\frac{R}{a}\right)^d + 2\bar{\mathcal{J}}_a B\left(\frac{R}{a}; d, 1+\frac{\alpha}{2}\right) + \bar{\mathcal{J}}_a{}^2 B\left(\frac{R}{a}; d, 1+\alpha\right)\right), \end{aligned}$$

and

$$F(R) \le (C^{(3)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 d\omega_d \int_0^R r^{d-1}\left(1 + 2\bar{\mathcal{J}}_a\left(\frac{a-r}{a}\right)^{\alpha/2} + \bar{\mathcal{J}}_a{}^2\left(\frac{a-r}{a}\right)^{\alpha}\right)\mathrm{d}r$$

$$= (C^{(3)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 d\omega_d a^d \left(\frac{1}{d}\left(\frac{R}{a}\right)^d + 2\bar{\mathcal{J}}_a B\left(\frac{R}{a}; d, 1+\frac{\alpha}{2}\right)\right.$$
$$\left. + \bar{\mathcal{J}_a}^2 B\left(\frac{R}{a}; d, 1+\alpha\right)\right),$$

where $C^{(1)}_{d,m,\alpha,a}, C^{(2)}_{d,m,\alpha,a}, C^{(3)}_{d,m,\alpha,a}$ are the constants defined in Remark 5.3.
(2) Recall that by Corollary 5.1 we have $\varphi_0(x) \geq C^{(6)}_{d,m,\alpha,a,|\lambda_0|}\varphi_0(\mathbf{a}) j_{m,\alpha}(|x|)$ for $|x| > a$, with a constant $C^{(6)}_{d,m,\alpha,a,|\lambda_0|} > 0$. This gives

$$\begin{aligned} F(R) &= \int_{\mathcal{B}_a} \varphi_0^2(x)\mathrm{d}x + \int_{\mathcal{B}_R\setminus\mathcal{B}_a} \varphi_0^2(x)\mathrm{d}x \\ &\geq (C^{(1)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} d\omega_d \int_0^a r^{d-1}\left(1 + 2\bar{\mathcal{J}}_a\left(\frac{a-r}{a}\right)^{\alpha/2}\right. \\ &\quad \left. + \bar{\mathcal{J}}_a^2\left(\frac{a-r}{a}\right)^{\alpha}\right)\mathrm{d}r \\ &\quad + (C^{(6)}_{d,m,\alpha,a,|\lambda_0|}\varphi_0(\mathbf{a}))^2 d\omega_d \int_a^R r^{d-1} j_{m,\alpha}(r)\mathrm{d}r \\ &\geq C^{(4)}_{d,m,\alpha,a,|\lambda_0|}\varphi_0^2(\mathbf{a})\left(\frac{1}{d} + 2\bar{\mathcal{J}}_a B\Big(d, 1+\frac{\alpha}{2}\Big) + \bar{\mathcal{J}}_a^2 B\Big(d, 1+\alpha\Big) + \mathcal{I}(R)\right). \end{aligned}$$

where

$$C^{(4)}_{d,m,\alpha,a,|\lambda_0|} = d\omega_d a^d \min\left\{(C^{(1)}_{d,m,\alpha,a})^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|}, (C^{(6)}_{d,m,\alpha,a,|\lambda_0|})^2\right\}.$$

(3) By (5.16) we have $\varphi_0(x) \leq C^{(7)}_{d,m,\alpha,\delta,a}\mathcal{J}_a\varphi_0(\mathbf{a}) j_{m,\alpha}(|x|)$ for $|x| \geq a$, with a constant $C^{(7)}_{d,m,\alpha,\delta,a} > 0$. Hence, we get

$$\begin{aligned} F(R) &= \int_{\mathcal{B}_a} \varphi_0^2(x)\mathrm{d}x + \int_{\mathcal{B}_R\setminus\mathcal{B}_a} \varphi_0^2(x)\mathrm{d}x \\ &\leq (C^{(3)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 d\omega_d \int_0^a r^{d-1}\left(1 + 2\bar{\mathcal{J}}_a\left(\frac{a-r}{a}\right)^{\alpha/2} + \bar{\mathcal{J}_a}^2\left(\frac{a-r}{a}\right)^{\alpha}\right)\mathrm{d}r \\ &\quad + (C^{(7)}_{d,m,\alpha,\delta,a}\varphi_0(\mathbf{a}))^2 d\omega_d \bar{\mathcal{J}_a}^2 \int_a^R r^{d-1} j_{m,\alpha}(r)\mathrm{d}r \\ &\leq C^{(5)}_{d,m,\alpha,\delta,a}\varphi_0^2(\mathbf{a})\left(\frac{1}{d} + 2\bar{\mathcal{J}}_a B\Big(d, 1+\frac{\alpha}{2}\Big) + \bar{\mathcal{J}_a}^2 B\Big(d, 1+\alpha\Big) + \bar{\mathcal{J}_a}^2\mathcal{I}(R)\right), \end{aligned}$$

where $C^{(5)}_{d,m,\alpha,\delta,a} = d\omega_d a^d \max\left\{(C^{(3)}_{d,m,\alpha,a})^2, (C^{(7)}_{d,m,\alpha,\delta,a})^2\right\}$.

Remark 5.8.
1. To get a closer idea of the behaviour in leading order of the estimates in Proposition 5.8, one can make use of the estimate $B(u; x, y) \leq \frac{1}{x}u^x(1-u)^y$, which results from [52, 8.17.22].
2. Part (1) of Proposition 5.8 indicates that inside the potential well, for fixed R the paths select from three different independent sampling strategies in order to build up the stationary distribution φ_0^2. One corresponds to the uniform distribution $\left(\frac{R}{a}\right)^d$, and the other two are Beta-distributions

$\mathrm{B}(d,\beta_1)$, $\mathrm{B}(d,\beta_2)$, contributing with different weights. Their respective probability distribution functions are

$$I_{R/a}(d,\beta_1) = \frac{B(\frac{R}{a};d,\beta_1)}{B(d,\beta_1)},\ \beta_1 = 1+\frac{\alpha}{2} \quad \text{and}$$
$$I_{R/a}(d,\beta_2) = \frac{B(\frac{R}{a};d,\beta_2)}{B(d,\beta_2)},\ \beta_2 = 1+\alpha.$$

The contribution of $\mathrm{B}(d,\beta_1)$ with weight proportional to $w_1 = 2\bar{\mathcal{J}}_a B(d,\beta_1)$ corresponds to relatively few large jumps, while the contribution of $\mathrm{B}(d,\beta_2)$ with weight proportional to $w_2 = \bar{\mathcal{J}_a}^2 B(d,\beta_2)$ corresponds to relatively many small jumps. A similar observation can be made about the mix of random jump strategies outside the potential well.

3. It is clear that for $R \geq a$ we have by part (1) of Proposition 5.8

$$\begin{aligned} F(R) &\geq F(a) \\ &\geq (C^{(1)}_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} d\omega_d a^d \\ &\quad \times \left(\frac{1}{d} + 2\bar{\mathcal{J}}_a B\Big(d, 1+\frac{\alpha}{2}\Big) + \bar{\mathcal{J}_a}^2 B\Big(d,1+\alpha\Big)\right), \end{aligned}$$

where the constants $C^{(1)}_{d,m,\alpha,a}$ and $C^{(2)}_{d,m,\alpha,a}$ are defined in Remark 5.3.

4. We can also derive an estimate of the probability mass outside the well. Indeed, by a similar argument as in part (2) of Proposition 5.8 we get for $R \geq a$ that

$$F(R) - F(a) \geq (C_{d,m,\alpha,a,|\lambda_0|}\varphi_0(\mathbf{a}))^2 d\omega_d a^d \mathcal{I}(R),$$

where the constant $C_{d,m,\alpha,a,|\lambda_0|}$ is the comparability constant in Corollary 5.1. Taking the limit as $R \to \infty$, we also get

$$\int_{B_a^c} \varphi_0^2(x)\mathrm{d}x \geq (C_{d,m,\alpha,a,|\lambda_0|}\varphi_0(\mathbf{a}))^2 d\omega_d a^d \mathcal{I}_a,$$

where $\mathcal{I}_a = \int_1^\infty r^{d-1} j^2_{m,\alpha}(ar)\,\mathrm{d}r$. On the other hand, if $v > \lambda_a$, we can set $\delta = v - \lambda_a$ and argue as in part (3) of Proposition 5.8 to show for $R \geq a$ that

$$F(R) - F(a) \leq (C_{d,m,\alpha,\delta,a}\varphi_0(\mathbf{a}))^2 d\omega_d a^d \mathcal{J}_a^2 \mathcal{I}(R),$$

and on taking the limit as $R \to \infty$ obtain

$$\int_{B_a^c} \varphi_0^2(x)\mathrm{d}x \leq (C_{d,m,\alpha,\delta,a}\varphi_0(\mathbf{a}))^2 d\omega_d a^d \mathcal{J}_a^2 \mathcal{I},$$

where the constant $C_{d,m,\alpha,\delta,a}$ is defined in (5.16). We can also choose $\delta = |\lambda_0|$.

5. We also have

$$\int_{B_a^c} \varphi_0^2(x)\mathrm{d}x \leq (C_{d,m,\alpha,a}\varphi_0(\mathbf{a}))^2 d\omega_d a^d \mathcal{J}_a^2 \int_{\mathcal{B}_a^c} \mathbb{E}^x\left[e^{-|\delta|T_a}\right]\mathrm{d}x,$$

where the integral at the right-hand side converges to 0 as $\delta \to \infty$.

6. Arguing as in Proposition 5.4, we can use the bound

$$\varphi_0^2(x) \geq \varphi_0^2(\mathbf{a})(C^{(3)}_{d,m,\alpha,a})^2 \mathcal{J}_a^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} \left(\frac{a-|x|}{a}\right)^{\alpha},$$

to prove like in Proposition 5.8 that there exists a constant $C^{(1)}_{d,m,\alpha,a} = (C^{(3)}_{d,m,\alpha})^2 \mathrm{d}\omega_d a^d$ such that

$$F(a) \geq C^{(1)}_{d,m,\alpha,a} e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} \varphi_0^2(\mathbf{a}) \mathcal{J}_a^2 B\left(d, 1+\alpha\right).$$

A less refined version

$$F(a) \geq C^{(1)}_{d,m,\alpha,a} \left(\mathbb{E}^{C^{(2)}_{d,m,\alpha,a}\mathbf{a}}\left[e^{-|\lambda_0|T_a}\right]\right)^2 \varphi_0^2(\mathbf{a}) \mathcal{J}_a^2 B\left(d, 1+\alpha\right),$$

of this estimate follows by Remark 5.6.

7. For the quantities

$$P_{\mathrm{in}}(\varphi_0) = F(a) \quad \text{and} \quad P_{\mathrm{out}}(\varphi_0) = \lim_{R\to\infty} (F(R) - F(a))$$

we obtain

$$\frac{P_{\mathrm{out}}(\varphi_0)}{P_{\mathrm{in}}(\varphi_0)} \leq C_{d,m,\alpha,a} \frac{\int_{\mathcal{B}_a^c} \mathbb{E}^x\left[e^{-|\lambda_0|T_a}\right] \mathrm{d}x}{\left(\mathbb{E}^{C^{(2)}_{d,m,\alpha,a}\mathbf{a}}\left[e^{-|\lambda_0|T_a}\right]\right)^2} \leq C_{d,m,\alpha,a,|\lambda_0|} \frac{\mathcal{I}}{j_{m,\alpha}(C^{(2)}_{d,m,\alpha,a}\mathbf{a})}.$$

We give one last application of Proposition 5.8. Suppose that $p_*(m,\alpha) > 2$, so that the variance $\Lambda_2(\varphi_0)$ of $\varphi_0^2(x)\mathrm{d}x$ exists. Then we can provide a lower bound on $\Lambda_2(\varphi_0)$ by means of Proposition 5.4. Denote by $\underline{\varphi}_0(\mathbf{a})$ the lower bound of $\varphi_0(\mathbf{a})$ provided in Proposition 5.3, and by $\underline{\sigma}$ the left-hand side of (5.12) for $p=2$ with $\underline{\varphi}_0(\mathbf{a})$ in place of $\varphi_0(a)$. Note that both $\underline{\varphi}_0(\mathbf{a})$ and $\underline{\sigma}$ can be numerically estimated. For $R \leq a$ define the function

$$\begin{aligned}\underline{F}(R) &:= (C^{(1)}_{d,m,\alpha,a}\underline{\varphi}_0(\mathbf{a}))^2 e^{-2C^{(2)}_{d,m,\alpha,a}|\lambda_0|} d\omega_d a^d \\ &\quad \times \left(\frac{1}{d}\left(\frac{R}{a}\right)^d + 2\bar{\mathcal{J}}_a B\Big(\frac{R}{a}; d, 1+\frac{\alpha}{2}\Big) + \bar{\mathcal{J}}_a{}^2 B\Big(\frac{R}{a}; d, 1+\alpha\Big)\right),\end{aligned}$$

and for $R > a$

$$\begin{aligned}\underline{F}(R) := C^{(4)}_{d,m,\alpha,a,|\lambda_0|} \underline{\varphi}_0^2(\mathbf{a}) d\omega_d a^d \Big(\tfrac{1}{d} + 2\bar{\mathcal{J}}_a B\Big(d, 1+\tfrac{\alpha}{2}\Big) \\ + \bar{\mathcal{J}}_a{}^2 B\Big(d, 1+\alpha\Big) + \mathcal{I}(R)\Big),\end{aligned}$$

where the constants $C^{(1)}_{d,m,\alpha,a}, C^{(2)}_{d,m,\alpha,a}, C^{(4)}_{d,m,\alpha,a,|\lambda_0|}$ are defined in Proposition 5.8 and can be numerically estimated. Then, for every $n \in \mathbb{N}$ we have the lower bound

$$\underline{F}(n\underline{\sigma}) \leq F(n\Lambda_2(\varphi_0))$$

which can be used to derive an $n\sigma$-rule, i.e. an estimate of the probability mass within a radius of $n\sigma$ from the mean. The details are left to the reader.

5.6. Extension to Fully Supported Decaying Potentials

Our technique to derive local estimates on the ground state of a non-local Schrödinger operator with a compactly supported potential can be extended to potentials supported everywhere on $\mathbb{R}^d$. This is of interest since apart from decay rates as $|x| \to \infty$ (see [38]), there is no information on the behaviour of the ground state from small to mid range.

Consider a potential $V(x) = -v(|x|)$ with a continuous non-increasing function $v : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{r\to\infty} v(r) = 0$. We assume that $H_{m,\alpha}$ has a ground state φ_0 with eigenvalue $\lambda_0 < 0$. We already know from Remark 4.1 that φ_0 is radially symmetric; thus, we can write $\varphi_0(x) = \varrho_0(|x|)$ with a suitable $\varrho_0 : \mathbb{R}^+ \to \mathbb{R}^+$. Also in this case we will suppose the following condition to hold.

Assumption 5.1. The function $\varrho_0 : [0,\infty) \to \mathbb{R}$ is non-increasing.

A first main result of this section is as follows.

Theorem 5.3. *Let φ_0 be the ground state of $H_{m,\alpha}$ with potential $V(x) = -v(|x|)$, $v : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous. Let Assumption 5.1 hold and consider any $\gamma > 0$ such that the level set $\mathcal{K}_\gamma = \{x \in \mathbb{R}^d : \ V(x) < -\gamma\} \neq \emptyset$. Then there exists a constant $C^{(1)}_{d,m,\alpha,a} = (C^{(3)}_{d,m,\alpha})^2 d\omega_d a^d$ such that*

$$C_{d,m,\alpha,\gamma,|\lambda_0|}\varphi_0(x_\gamma)\mathbb{E}^x[e^{(\gamma-|\lambda_0|)\tau_{r_\gamma}}] \\ \leq \varphi_0(x) \leq \varphi_0(x_\gamma)\mathbb{E}^x[e^{(v(0)-|\lambda_0|)\tau_{r_\gamma}}], \quad x \in \mathcal{K}_\gamma,$$

where $\tau_{r_\gamma} = \inf\{t > 0 : \ X_t \in \mathcal{K}^c_\gamma\}$, $x_\gamma \in \partial\mathcal{K}_\gamma$ is arbitrary and $r_\gamma = |x_\gamma|$.

Proof. Take $x \in \mathcal{K}_\gamma$ and notice that since v is non-increasing and continuous, $\mathcal{K}_\gamma$ is an open ball centred at the origin, i.e. there exists $r_\gamma > 0$ such that $\mathcal{K}_\gamma = \mathcal{B}_{r_\gamma}$. Consider the stopping time

$$\tau_{r_\gamma} = \inf\{t > 0 : \ X_t \in \mathcal{B}^c_{r_\gamma}\}.$$

Since φ_0 is radially symmetric, it is constant on $\partial\mathcal{B}_{r_\gamma}$. Take any $x_\gamma \in \partial\mathcal{B}_{r_\gamma}$. By Proposition 4.1 and Assumption 5.1 we have

$$\varphi_0(x) = \mathbb{E}^x[e^{\int_0^{\tau_{r_\gamma}} v(|X_s|)ds - |\lambda_0|\tau_{r_\gamma}}\varphi_0(X_{\tau_{r_\gamma}})] \leq \varphi_0(x_\gamma)\mathbb{E}^x[e^{(v(0)-|\lambda_0|)\tau_{r_\gamma}}].$$

Consider $C^{(1)}_{d,m,\alpha,r_\gamma} > 1$ defined in Lemma 5.2 and observe that

$$\varphi_0(x) \geq \mathbb{E}^x[e^{\int_0^{\tau_{r_\gamma}} v(|X_s|)ds - |\lambda_0|\tau_{r_\gamma}}\varphi_0(X_{\tau_{r_\gamma}}); r_\gamma \leq X_{\tau_{r_\gamma}} \leq C^{(1)}_{d,m,\alpha,r_\gamma} r_\gamma] \\ \geq \varphi_0(C^{(1)}_{d,m,\alpha,r_\gamma} x_\gamma)\mathbb{E}^x[e^{(\gamma-|\lambda_0|)\tau_{r_\gamma}}; r_\gamma \leq X_{\tau_{r_\gamma}} \leq C^{(1)}_{d,m,\alpha,r_\gamma} r_\gamma]. \tag{5.19}$$

Also, notice that by the definition of $C^{(1)}_{d,m,\alpha,r_\gamma}$,

$$\mathbb{E}^x[e^{(\gamma-|\lambda_0|)\tau_{r_\gamma}}; r_\gamma \leq X_{\tau_{r_\gamma}} \leq C^{(1)}_{m,\alpha,r_\gamma} r_\gamma] \geq \frac{1}{2}\mathbb{E}^x[e^{(\gamma-|\lambda_0|)\tau_{r_\gamma}}]. \tag{5.20}$$

On the other hand, arguing as in Theorem 5.1, we have

$$\varphi_0(C^{(1)}_{d,m,\alpha,r_\gamma} x_\gamma) \geq \mathbb{E}^{C^{(1)}_{d,m,\alpha,a} x_\gamma}[e^{-|\lambda_0|T_{r_\gamma}}\varphi_0(X_{T_{r_\gamma}})],$$

where $T_{r_\gamma} = \inf\{t > 0 : \ X_t \in \mathcal{K}_\gamma\}$ and we used the fact that $v(|x|) \geq 0$ for all $x \in \mathbb{R}^d$. By Assumption 5.1 we have

$$\varphi_0(C^{(1)}_{m,\alpha,r_\gamma} x_\gamma) \geq \varphi_0(x_\gamma) \mathbb{E}^{C^{(2)}_{m,\alpha,a} x_\gamma}[e^{-|\lambda_0| T_{r_\gamma}}] \geq C^{(2)}_{d,m,\alpha,r_\gamma,|\lambda_0|} \varphi_0(x_\gamma), \quad (5.21)$$

where

$$C^{(2)}_{d,m,\alpha,r_\gamma,|\lambda_0|} := C^{(3)}_{d,m,\alpha,r_\gamma,|\lambda_0|} j_{m,\alpha}(C^{(1)}_{d,m,\alpha,r_\gamma} r_\gamma),$$

$C^{(3)}_{d,m,\alpha,r_\gamma,|\lambda_0|}$ is defined in Corollary 3.3 by choosing $R_2 > C^{(1)}_{d,m,\alpha,r_\gamma} r_\gamma$. Combining (5.20)–(5.21) with (5.19) the claim follows.

Remark 5.9. We note that when $v(0) - |\lambda_0| \geq \lambda_{r_\gamma}$, the upper bound is trivial as $\mathbb{E}^x[e^{(v(0)-|\lambda_0|)\tau_{r_\gamma}}] = \infty$. Also, if $|\lambda_0| \geq \gamma$, then the lower bound is trivial since $\mathbb{E}^x[e^{(\gamma-|\lambda_0|)\tau_{r_\gamma}}] \leq 1$ and $\varphi_0(x) \geq \varphi_0(x_\gamma)$ by Assumption 5.1. Furthermore, by a similar argument as in Step 1 of Theorem 5.1, the implication is that $\gamma - |\lambda_0| < \lambda_{r_\gamma}$ whenever $\mathcal{K}_\gamma \neq \emptyset$. In particular, due to $\lim_{\gamma \to v(0)} \lambda_{r_\gamma} = \infty$, there is a constant $\gamma_0 > 0$ such that $v(0) - |\lambda_0| < \lambda_{r_\gamma}$ for every $\gamma \in (\gamma_0, v(0))$.

Exploiting the asymptotic behaviour of the moment-generating function involved as above for the spherical potential well, we have the following result.

Corollary 5.3. *Let φ_0 be the ground state of $H_{m,\alpha}$ with $V(x) = -v(|x|)$, $v : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous. Let Assumption 5.1 hold, and consider any $\gamma > 0$ such that the set $\mathcal{K}_\gamma = \{x \in \mathbb{R}^d : \ V(x) < -\gamma\} \neq \emptyset$, $|\lambda_0| < \gamma$ and $v(0) - |\lambda_0| < \lambda_{\mathcal{K}_\gamma}$. Then there exists a constant $C^{(1)}_{d,m,\alpha,\gamma,|\lambda_0|}$ such that*

$$C^{(1)}_{d,m,\alpha,\gamma,|\lambda_0|} \varphi_0(x_\gamma) \left(1 + \frac{\gamma - |\lambda_0|}{\lambda_{\mathcal{K}_\gamma} - \gamma + |\lambda_0|} \left(\frac{r_\gamma - |x|}{r_\gamma}\right)^{\alpha/2}\right)$$
$$\leq \varphi_0(x) \leq \varphi_0(x_\gamma) \left(1 + \frac{v(0) - |\lambda_0|}{\lambda_{\mathcal{K}_\gamma} - v(0) + |\lambda_0|} \left(\frac{r_\gamma - |x|}{r_\gamma}\right)^{\alpha/2}\right),$$

for every $x \in \mathcal{K}_\gamma$, where $\tau_{r_\gamma} = \inf\{t > 0 : \ X_t \in \mathcal{K}^c_\gamma\}$, $x_\gamma \in \partial\mathcal{K}_\gamma$ is arbitrary, and $r_\gamma = |x_\gamma|$.

Proof. Starting from (5.3) and recalling that $\mathcal{K}_\gamma = \mathcal{B}_{r_\gamma}$, the upper bound follows from the assumption that $v(0) - |\lambda_0| < \lambda_{r_\gamma}$ and Theorem 3.1. The lower bound follows from Remark 5.9 guaranteeing $\gamma - |\lambda_0| < \lambda_{r_\gamma}$, and furthermore by an application of Theorem 3.1.

Theorem 5.4. *Let φ_0 be the ground state of $H_{m,\alpha}$ with $V(x) = -v(|x|)$, $v : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous, and let Assumption 5.1 hold. Let $\gamma_1 \leq |\lambda_0|$ and $\gamma_2 \in (\gamma_0, v(0))$, where γ_0 is defined as in Remark 5.9, such that $\gamma_1 \leq \gamma_2$. Define $\mathcal{K}_{\gamma_i} = \{x \in \mathbb{R}^d, V(x) < -\gamma_i\}$, $i = 1, 2$. Then*

$$\varphi_0(x_{\gamma_1}) \mathbb{E}^x[e^{-|\lambda_0| T_{r_{\gamma_1}}}] \leq \varphi_0(x)$$
$$\leq C_{d,m,\alpha,\gamma_2,|\lambda_0|} \varphi_0(x_{\gamma_2}) \mathbb{E}^x[e^{(\gamma_1 - |\lambda_0|) T_{r_{\gamma_1}}}], \quad x \in \mathcal{K}_{\gamma_1},$$

where $x_{\gamma_i} \in \partial\mathcal{K}_{\gamma_i}$ and $r_{\gamma_i} = |x_{\gamma_i}|$, $i = 1, 2$.

Proof. By a similar argument as in Theorem 5.3, there exist r_{γ_i} such that $\mathcal{K}_{\gamma_i} = \mathcal{B}_{r_{\gamma_i}}$, $i = 1, 2$. Moreover, $\mathcal{K}^c_{\gamma_1} \subseteq \mathcal{K}^c_{\gamma_2}$ since v is non-increasing. Let $x \in \mathcal{K}^c_{\gamma_1}$ and observe that, as in Theorem 5.1,

$$\varphi_0(x_{\gamma_1})\mathbb{E}^x[e^{-|\lambda_0|T_{r_{\gamma_1}}}] \leq \varphi_0(x) \leq \varphi_0(0)\mathbb{E}^x[e^{(\gamma_1-|\lambda_0|)T_{r_{\gamma_1}}}],$$

where $x_{\gamma_1} \in \partial\mathcal{B}_{r_{\gamma_1}}$. Using that $0 \in \mathcal{K}_{\gamma_2}$, by Corollary 5.3 we get

$$\begin{aligned}\varphi_0(0) \leq \varphi_0(x_{\gamma_2}) &\left(1 + \frac{v(0) - |\lambda_0|}{\lambda_{r_{\gamma_2}} - v(0) + |\lambda_0|}\right) \\ &=: C_{d,m,\alpha,\gamma_2,|\lambda_0|}\varphi_0(x_{\gamma_2}), \quad x_{\gamma_2} \in \partial\mathcal{B}_{r_{\gamma_2}}.\end{aligned}$$

Again, by using the asymptotics of the Laplace transform of the hitting times, we get the following.

Corollary 5.4. *Let φ_0 be the ground state of $H_{m,\alpha}$ with $V(x) = -v(|x|)$, $v : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous, and let Assumption 5.1 hold. Choose $\gamma_1 \leq |\lambda_0|$ and $\gamma_2 \in (\gamma_0, v(0))$, where γ_0 is defined in Remark 5.9, such that $\gamma_1 \leq \gamma_2$. Define $\mathcal{K}_{\gamma_i} = \{x \in \mathbb{R}^d, V(x) < -\gamma_i\}$, $i = 1, 2$. Then*

$$C^{(1)}_{d,m,\alpha,\gamma_1,|\lambda_0|}\varphi_0(x_{\gamma_2})j_{m,\alpha}(|x|) \leq \varphi_0(x) \leq C^{(2)}_{d,m,\alpha,\gamma_2,\gamma_1,|\lambda_0|}\varphi_0(x_{\gamma_2})j_{m,\alpha}(|x|),$$

where $x_{\gamma_i} \in \partial\mathcal{K}_{\gamma_i}$ and $r_{\gamma_i} = |x_{\gamma_i}|$, $i = 1, 2$.

Proof. The upper bound follows directly by Theorems 5.4 and 3.3. For the lower bound first consider the potential well $\widetilde{V} = -\widetilde{v}\mathbf{1}_{\mathcal{K}_{\gamma_1}}$, where $\widetilde{v}$ is chosen to be large enough to guarantee the existence of a ground state $\widetilde{\varphi}_0$. Recall that $\mathcal{K}_{\gamma_1}$ is an open ball. By Corollary 5.1 we know that

$$\frac{\widetilde{\varphi}_0(x)}{\widetilde{\varphi}_0(x_{\gamma_1})} \geq C^{(3)}_{d,m,\alpha,\gamma_1,|\lambda_0|}j_{m,\alpha}(|x|), \quad x \in \mathcal{K}^c_{\gamma_1}.$$

On the other hand, by Theorem 5.1 we get

$$\mathbb{E}^x[e^{-|\lambda_0|T_{r_{\gamma_1}}}] \geq C^{(4)}_{d,m,\alpha,\gamma_1,|\lambda_0|}\frac{\widetilde{\varphi}_0(x)}{\widetilde{\varphi}_0(x_{\gamma_1})}, \quad x \in \mathcal{K}^c_{\gamma_1}.$$

Combining the previous estimates with the lower bound in Theorem 5.4, the statement follows.

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