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Optimal Hardy Inequality for Fractional Laplacians on the Integers

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Abstract. We study the fractional Hardy inequality on the integers. We prove the optimality of the Hardy weight and hence affirmatively answer the question of sharpness of the constant.

1. Introduction

In [14], Hardy in 1919 proved an inequality which was shortly thereafter used to derive the *classical Hardy inequality*

$$\langle \Delta\varphi, \varphi\rangle \geq \langle w\varphi, \varphi\rangle$$

with

$$w(x) = \frac{1}{4x^2}$$

for $\varphi \in C_c(\mathbb{N}_0)$ with $\varphi(0) = 0$, where Δ is the standard Laplacian on $\mathbb{N}_0$. We refer to [17] for more on the "prehistory" of Hardy's inequality. Indeed, the constant $1/4$ is known to be optimal in the sense that it cannot be replaced with a larger constant. Despite the optimality of the constant, [19] recently showed that the weight can be improved to an optimal one, i.e., to one that is largest possible in a certain sense. By definition, this includes the optimality of the constant.

Here, we study a Hardy inequality

$$\langle \Delta^\sigma\varphi, \varphi\rangle \geq \langle w_\sigma\varphi, \varphi\rangle$$

of the fractional Laplacian Δ^σ, $0 < \sigma < 1/2$, on $\mathbb{Z}$ for $\varphi \in C_c(\mathbb{Z})$ with

$$w_\sigma(x) = C_\sigma \frac{1}{|x|^{2\sigma}} + O\left(\frac{1}{|x|^{1+2\sigma}}\right).$$

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The exact form of w_σ which is given in terms of the Gamma function was obtained by Ciaurri/Roncal in [4]. However, the question concerning the optimality of the constant remained open. Here, we show the optimality of their weight w_σ and as a consequence prove the optimality of the constant.

Hardy inequalities have a long mathematical tradition and are of great significance in various branches of mathematics such as functional analysis, partial differential equations, harmonic analysis, approximation theory and probability. To that end, a lot of work has been done to find optimal constants and critical weights in both the continuum and in the discrete setting (see, e.g., [3,6,7,12,21] and references therein). Typically, a Hardy inequality gives a lower bound of a quadratic form by some weight which is then called a *Hardy weight*.

The study of fractional Laplace operators can be traced back as far as to Riesz, see [29]. These operators appear as relativistic Schrödinger operators [27] in quantum mechanics and in the description of perturbed phenomena like turbulence [1], elasticity [8], laser design [25], and anomalous transport and diffusion [26]. There are various ways to define the fractional Laplacian; for example, there is a spectral definition as a Fourier multiplier, a definition via the heat semigroup or a definition through harmonic extension. In fact, [24] discusses ten equivalent definitions for the fractional Laplacian on $\mathbb{R}^d$. Here, we rely on the definition involving a certain time-integral over the heat semigroup.

In the continuum, Hardy inequalities for the fractional Laplacian are rather well-understood. The explicit constants are known to be optimal, see [10] or [15,31] for earlier references. Similar results have been shown for the half-space in [2]. In contrast, a fractional Hardy inequality in the discrete setting was obtained in [4], but optimality of the constant remained an open question. Our contribution here is to show that their weight is optimal and thus affirmatively answer their question about the optimality of the constant.

The structure of the paper is as follows. In the next section, we introduce the basic notions and formulate the main result. In Sect. 3, we present a family of Hardy weights arising from superharmonic functions via the ground state transform. For a specific choice within this family we show in Sect. 4 optimality via a so-called null-sequence argument.

2. Setup and Main Result

In this section, we introduce the main objects and concepts of this paper. We start by introducing the fractional Laplacian on the integers $\mathbb{Z}$. Moreover, we recall the notion of optimality of Hardy weights in the sense of [7,23]. Finally, we state the main result.

The Laplacian Δ on $\ell^2(\mathbb{Z})$ is the bounded operator acting as

$$\Delta f(x) = (f(x) - f(x+1)) + (f(x) - f(x-1))$$

for $x \in \mathbb{Z}$. The fractional Laplacian Δ^σ, $\sigma \in (0,1)$, on $\ell^2(\mathbb{Z})$ is then given by the spectral calculus. By the boundedness of Δ, the fractional Laplacian is a bounded operator as well. Moreover, the spectral calculus yields by direct computation that Δ^σ can be represented as

$$\Delta^\sigma f(x) = \frac{1}{|\Gamma(-\sigma)|}\int_0^\infty (1-e^{-t\Delta})f(x)\frac{\mathrm{d}t}{t^{1+\sigma}}$$

where $x \in \mathbb{Z}$, the operator $e^{-t\Delta}$, $t \geq 0$, is the semigroup of Δ and Γ is the Gamma function. This operator is thoroughly studied in [5]. We recall the most important facts here and refer to [5] or [28] for details.

Due to discreteness and homogeneity of $\mathbb{Z}$ the fractional Laplacian Δ^σ, $\sigma \in (0,1)$, can be written as a kernel operator. To see this we let, for $\alpha \in (-1/2, 1)$, $\alpha \neq 0$,

$$\kappa_\alpha(x) = \frac{1}{|\Gamma(-\alpha)|}\int_0^\infty e^{-t\Delta}1_0(x)\frac{\mathrm{d}t}{t^{1+\alpha}}$$

whenever $x \in \mathbb{Z}$, $|x| > \alpha$, as well as $\kappa_\alpha(0) = 0$ for $\alpha > 0$ and $\kappa_0 = 1_0$. Here, 1_y denotes the characteristic function of $y \in \mathbb{Z}$. Since the semigroup $e^{-t\Delta}$ is positivity improving (cf. [16]), we infer that κ_α is strictly positive apart from 0. It was shown in [5, Lemma 9.2] that

$$\kappa_\alpha(x) = \frac{4^\alpha\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}|\Gamma(-\alpha)|}\cdot\frac{\Gamma(|x|-\alpha)}{\Gamma(|x|+1+\alpha)} = \frac{4^\alpha\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}|\Gamma(-\alpha)|}\cdot\frac{1}{|x|^{1+2\alpha}} + O\left(\frac{1}{|x|^{2+2\alpha}}\right)$$

for $x \to \infty$. It turns out that the fractional Laplacian can be written as a graph Laplacian or a kernel operator with weights κ_α with $\alpha \in (0,1)$ (in which case we denote the parameter α by σ). Indeed, we use $e^{-t\Delta}1 = 1$, Fubini's theorem and $e^{-t\Delta}1_y(x) = e^{-t\Delta}1_0(x-y)$ to conclude

$$\begin{aligned}
\Delta^\sigma f(x) &= \frac{1}{|\Gamma(-\sigma)|}\int_0^\infty \left((e^{-t\Delta}1)(x)f(x) - (e^{-t\Delta}f)(x)\right)\frac{\mathrm{d}t}{t^{1+\sigma}} \\
&= \frac{1}{|\Gamma(-\sigma)|}\int_0^\infty \sum_{y\neq x} e^{-t\Delta}1_y(x)(f(x)-f(y))\frac{\mathrm{d}t}{t^{1+\sigma}} \\
&= \sum_{y\neq x}\left(\frac{1}{|\Gamma(-\sigma)|}\int_0^\infty e^{-t\Delta}1_y(x)\frac{\mathrm{d}t}{t^{1+\sigma}}\right)(f(x)-f(y)) \\
&= \sum_{y\in\mathbb{Z}} \kappa_\sigma(x-y)(f(x)-f(y))
\end{aligned}$$

for compactly supported f. From the asymptotics $\kappa_\sigma \in O(|\cdot|^{-1-2\sigma})$, it follows that the weights κ_σ are summable about any $x \in \mathbb{Z}$. Hence, Δ^σ is a graph Laplacian over $\mathbb{Z}$ with respect to a non-locally finite weighted graph in the sense of [16,30]. Moreover, the boundedness of Δ^σ allows to extend the formula above from the compactly supported functions to $\ell^2(\mathbb{Z})$. The asymptotics of κ_σ allow even an extension of the pointwise equality for $\Delta^\sigma f(x)$ for functions in the Banach space

$$B_\sigma = \ell^1(\mathbb{Z}, (1+|\cdot|)^{-1-2\sigma})$$

for $\sigma \in (0,1)$, see [4,28]. In particular, for $\alpha > -1/2$ and $\sigma \in (0,1)$, we have $\kappa_\alpha \in B_\sigma$. Note that we defined κ_α above also for $\alpha \in (-1/2, 0]$ as it will be needed below.

Let us now turn to Hardy weights. A non-trivial function $w : \mathbb{Z} \longrightarrow [0,\infty)$ is called a *Hardy weight* if for all compactly supported functions $\varphi \in C_c(\mathbb{Z})$,

$$\langle \Delta^\sigma \varphi, \varphi\rangle \geq \langle w\varphi, \varphi\rangle.$$

By boundedness of Δ^σ, this inequality then extends directly to all functions in $\ell^2(\mathbb{Z})$.

Next, we discuss optimality in the sense of [7,20]. A Hardy weight w is called *critical* if any Hardy weight $w' \geq w$ satisfies $w' = w$. If w is critical, then there exists a unique (up to multiplicative constants) positive harmonic function ψ (not necessarily in ℓ^2), i.e., $(\Delta^\sigma - w)\psi = 0$ which is called the *Agmon ground state*, cf. [22, Theorem 5.3]. A critical Hardy weight w is called *positive-critical* if $\psi \in \ell^2(\mathbb{Z}, w)$ and *null-critical* otherwise. A Hardy weight w is called *optimal* if the following two conditions are satisfied:

- w is critical,
- w is null-critical.

It turns out that if w is optimal, then it is also *optimal near infinity*, i.e., if

$$\langle \Delta^\sigma \varphi, \varphi\rangle \geq (1+\lambda)\langle w\varphi, \varphi\rangle$$

for all $\varphi \in C_c(\mathbb{Z}\backslash K)$ for some compact $K \subseteq \mathbb{Z}$, then $\lambda \leq 0$, cf. [9,18,20] for related models. This implies, in particular, that the constant appearing within the Hardy weight is optimal.

The following Hardy weight for Δ^σ was obtained by [4]

$$w_\sigma(x) = c_\sigma \cdot \frac{\Gamma(|x| + \frac{1-2\sigma}{4})\Gamma(|x| + \frac{3-2\sigma}{4})}{\Gamma(|x| + \frac{3+2\sigma}{4})\Gamma(|x| + \frac{1+2\sigma}{4})} = \frac{c_\sigma}{|x|^{2\sigma}} + O\left(\frac{1}{|x|^{1+2\sigma}}\right)$$
$$\text{with } c_\sigma = 4^\sigma \frac{\Gamma(\frac{1+2\sigma}{4})^2}{\Gamma(\frac{1-2\sigma}{4})^2},$$

where $\sigma \in (0, 1/2)$ which is strictly positive. We show w_σ is an optimal Hardy weight in the above sense. In particular, this affirmatively answers the open question in [4] on the optimality of the constant.

Theorem 1. *The function w_σ is an optimal Hardy weight of Δ^σ, $0 < \sigma < \frac{1}{2}$. In particular, w_σ is optimal near infinity.*

Let us discuss the strategy of the argument of the proof. The proof in [4] to show that w_σ is a Hardy weight uses the ground state transform for the Schrödinger operator $(\Delta^\sigma - w)$ with $w = (\Delta^\sigma u)/u$ where $u > 0$ is a superharmonic function of Δ^σ. To the best of our knowledge, this technique was first applied in the discrete setting in [11]. Indeed, [4] show that $\kappa_{-\alpha}$ are superharmonic functions of Δ^σ for $0 < \sigma \leq \alpha < 1/2$ and thus

$$w_{\sigma,\alpha} = \frac{\Delta^\sigma \kappa_{-\alpha}}{\kappa_{-\alpha}} = \frac{\kappa_{\sigma-\alpha}}{\kappa_{-\alpha}}$$

are Hardy weights. We also provide a short and concise proof of the second equality in Proposition 2.

By direct calculation one sees that $w_\sigma = w_{\sigma,\alpha}$ for

$$\alpha = \frac{1+2\sigma}{4}$$

and Theorem 1 states the Hardy weight is optimal in this case.

One may wonder how this particular choice of α comes about. From the asymptotics of κ_α one can read that all $w_{\sigma,\alpha}$ share the same asymptotics in x but come with different constants, i.e., outside of $x \neq 0$

$$w_{\sigma,\alpha}(x) = \Psi_\sigma(\alpha)\frac{1}{|x|^{2\sigma}} + O\left(\frac{1}{|x|^{1+2\sigma}}\right),$$

with $0 < \Psi_\sigma(\alpha) \leq c_\sigma$, cf. [4,10]. It was observed by [4] that the choice $\alpha = (1+2\sigma)/4$ yields the same constant c_σ as in the continuous setting. In Theorem 5, we show that $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$ is critical if and only if $\alpha \leq (1+2\sigma)/4$ and $\kappa_{-\alpha} \notin \ell^2(\mathbb{Z}, w_{\sigma,\alpha})$ if and only if $\alpha \geq (1+2\sigma)/4$. Thus, $w_{\sigma,\alpha}$ is optimal exactly for $\alpha = (1+2\sigma)/4$ in which case it is equal to w_σ from above.

There is also another structural reason for the particular choice of α. In [20] based on [7], it was shown that one obtains optimal Hardy weights via the super-solution construction

$$w = \frac{HG^{1/2}}{G^{1/2}}$$

for the Green's function G of a Schrödinger operator H whenever G is proper and satisfies an anti-oscillation condition. For $H = \Delta^\sigma$, the Green's function is $G = \kappa_{-\sigma}$. Furthermore, it turns out that $\kappa_{-\sigma}^{1/2}$ and $\kappa_{-\alpha}$ with $\alpha = (1+2\sigma)/4$ share the same asymptotics. Thus, it seems plausible to consider $\kappa_{-(1+2\sigma)/4}$ instead of $\kappa_{-\sigma}^{1/2}$ in the super-solution construction to obtain an optimal Hardy weight.

However, let us stress that the method developed in [20] to prove optimality for weighted graphs cannot be directly applied here. The existence of a positive superharmonic function which is proper and satisfies an anti-oscillation condition directly implies local finiteness of the underlying graph. Yet, the weighted graph of the fractional Laplacian is non-locally finite as discussed above. Indeed, even the refined method developed by Hake [13] does not seem to apply here. Instead, we use the explicit asymptotics of the edge weights in our proof below.

3. A Family of Hardy Weights

In this section, we present a family of Hardy weights for the fractional Laplacian. It was observed in [4] that $\kappa_{-\alpha}$ are superharmonic functions of Δ^σ for $0 < \sigma \leq \alpha < 1/2$. For the sake of being self-contained, we give a short alternative argument of this fact.

Proposition 2 (Superharmonic functions). *For all* $0 < \sigma \leq \alpha < 1/2$,

$$\Delta^{\sigma}\kappa_{-\alpha} = \kappa_{\sigma-\alpha}.$$

Proof. Define $\kappa_{\beta,\varepsilon} : \mathbb{Z} \longrightarrow [0,\infty]$ for $\beta \in \mathbb{R}$ and $\varepsilon > 0$ as

$$\kappa_{\beta,\varepsilon} = \frac{1}{|\Gamma(-\beta)|}\int_0^{\infty} e^{-t\varepsilon}e^{-t\Delta}1_0 \frac{\mathrm{d}t}{t^{1+\beta}}$$

for $\beta \neq 0$ and $\kappa_{0,\varepsilon} = 1_0$ for $\beta = 0$.

By the spectral calculus, we have for $\beta < 0$

$$(\Delta + \varepsilon)^{\beta}1_0 = \kappa_{\beta,\varepsilon}$$

and therefore, $\kappa_{\beta,\varepsilon} \in \ell^2(\mathbb{Z})$.

Moreover, for $\beta > 0$ and $f \in \ell^2(\mathbb{Z})$, we obtain by the spectral calculus

$$(\Delta + \varepsilon)^{\beta}f(x) = \frac{1}{|\Gamma(-\beta)|}\int_0^{\infty}(1 - e^{-t\varepsilon}e^{-t\Delta})f(x)\frac{\mathrm{d}t}{t^{1+\beta}}.$$

We use $\int_0^{\infty}(1 - e^{-t\varepsilon})t^{-1-\beta}\,\mathrm{d}t = \varepsilon^{\beta}|\Gamma(-\beta)|$ and $e^{-t\Delta}1 = 1$ to arrive at

$$\begin{aligned}(\Delta + \varepsilon)^{\beta}f(x) &= \frac{1}{|\Gamma(-\beta)|}\int_0^{\infty} e^{-t\varepsilon}(1 - e^{-t\Delta})f(x)\frac{\mathrm{d}t}{t^{1+\beta}} + \varepsilon^{\beta}f(x)\\ &= \sum_{y\in\mathbb{Z}}\kappa_{\beta,\varepsilon}(x-y)(f(x) - f(y)) + \varepsilon^{\beta}f(x).\end{aligned}$$

Now, let $0 < \sigma \leq \alpha < 1/2$. So, by the spectral calculus we see that

$$(\Delta + \varepsilon)^{\sigma}\kappa_{-\alpha,\varepsilon} = (\Delta + \varepsilon)^{\sigma}(\Delta + \varepsilon)^{-\alpha}1_0 = (\Delta + \varepsilon)^{-(\alpha-\sigma)}1_0 = \kappa_{\sigma-\alpha,\varepsilon}.$$

Since $\kappa_{\beta,\varepsilon}(x)$ converges monotonously to $\kappa_{\beta}(x)$, $x \in \mathbb{Z}$, as $\varepsilon \to 0$, the statement follows by the formulas for $\beta < 0$ and $\beta > 0$ above. □

Remark. From the proof, we can actually derive that $\kappa_{-\sigma}$ is the Green's function of Δ^{σ}, $0 < \sigma < 1/2$, i.e.,

$$\lim_{\varepsilon\to 0}(\Delta + \varepsilon)^{-\sigma}1_0 = \kappa_{-\sigma}.$$

Remark. One can define $\Delta^{-\alpha}$ for $0 < \alpha < 1/2$ on the Banach space $B_{-\alpha} = \ell^1(\mathbb{Z}, (1 + |\cdot|)^{-1+2\alpha})$ via

$$\Delta^{-\alpha}u(x) = \sum_{y\in\mathbb{Z}}\kappa_{-\alpha}(x-y)u(y) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-t\Delta}u(x)\frac{\mathrm{d}t}{t^{1-\alpha}}.$$

Then, the proof above can be extended to functions in $B_{-\alpha}$ for $0 < \sigma \leq \alpha < 1/2$, i.e., for $u \in B_{-\alpha}$,

$$\Delta^{\sigma}\Delta^{-\alpha}u = \Delta^{\sigma-\alpha}u.$$

Here, the details can be found in [28].

Next, we recall the ground state transform. First, observe that the quadratic form Q^{σ} associated to Δ^{σ} acts as

$$Q^{\sigma}(f) = \frac{1}{2}\sum_{x,y\in\mathbb{Z}}\kappa_{\sigma}(x-y)(f(x) - f(y))^2 = \langle\Delta^{\sigma}f, f\rangle$$

on $\ell^2(\mathbb{Z})$ which can be seen by the virtue of Green's formula (cf. [16]). Furthermore, for $0 < \sigma \leq \alpha < 1/2$ and $\varphi \in C_c(\mathbb{Z})$, denote

$$Q^{\sigma}_{-\alpha}(\varphi) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}} \kappa_{\sigma}(x-y)\kappa_{-\alpha}(x)\kappa_{-\alpha}(y)(\varphi(x) - \varphi(y))^2.$$

Proposition 3 (Ground state transform). *Let $0 < \sigma \leq \alpha < 1/2$ and $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$. Then, for $\varphi \in C_c(\mathbb{Z})$,*

$$(Q^{\sigma} - w_{\sigma,\alpha})(\kappa_{-\alpha}\varphi) = Q^{\sigma}_{-\alpha}(\varphi).$$

In particular, $w_{\sigma,\alpha}$ is a Hardy weight.

Proof. By Proposition 2, we have $\Delta^{\sigma}\kappa_{-\alpha} = \kappa_{\sigma-\alpha}$. Thus, $(\Delta^{\sigma} - w_{\sigma,\alpha})\kappa_{-\alpha} = 0$ and the statement follows by the ground state transform, cf. [22, Proposition 4.8]. □

4. Proof of Optimality

For the proof of optimality, we use the following criterion.

Proposition 4 (Generalized null sequences). *Let $0 < \sigma \leq \alpha < 1/2$. Then, $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$ is critical if and only if there is a sequence $0 \leq e_n \leq 1$ in $C_c(\mathbb{Z})$ such that $e_n \to 1$ pointwise and*

$$\sup_{n \in \mathbb{N}} Q^{\sigma}_{-\alpha}(e_n) < \infty.$$

Proof. The statement is in its essence a reformulation of the well-known criticality criterion via null sequences. The proof consists of three steps which are taken from the literature.

Step 1: Criticality of $(Q^{\sigma} - w_{\sigma,\alpha})$ is equivalent to existence of a null sequence for $(Q^{\sigma} - w_{\sigma,\alpha})$ and $\kappa_{-\alpha}$. A null sequence is a sequence $0 \leq \eta_n \leq \kappa_{-\alpha}$ in $C_c(\mathbb{Z})$ which converges pointwise to $\kappa_{-\alpha}$ and $(Q^{\sigma} - w_{\sigma,\alpha})(\eta_n) \to 0$, $n \to \infty$. This equivalence can be found, e.g., in [22, Theorem 5.3].

Step 2: Existence of a null sequence for $(Q^{\sigma} - w_{\sigma,\alpha})$ and $\kappa_{-\alpha}$ is equivalent to existence of a null sequence for the ground state transform $Q^{\sigma}_{-\alpha}$ and 1. This follows directly by Proposition 3.

Step 3: Existence of a null sequence for $Q^{\sigma}_{-\alpha}$ and 1 is equivalent to existence of sequence (e_n) in $C_c(\mathbb{Z})$ such that it converges pointwise to 1 and $Q^{\sigma}_{-\alpha}(e_n)$ stays bounded. This can be found in [16, Theorem 6.1, (i.d) $\Leftrightarrow$ (i.e.)] and is proven there via an elementary Banach–Saks-type argument. □

Theorem 1 is a direct consequence of the following theorem. This theorem elaborates on the criticality and null-criticality of the Hardy weights $w_{\sigma,\alpha}$ from above.

Theorem 5. *Let $0 < \sigma \leq \alpha < 1/2$ and $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$.*

(a) *The Hardy weights $w_{\sigma,\alpha}$ are critical if and only if $\alpha \leq (1 + 2\sigma)/4$.*
(b) *The super-solutions $\kappa_{-\alpha} \notin \ell^2(\mathbb{Z}, w_{\sigma,\alpha})$ if and only if $\alpha \geq (1 + 2\sigma)/4$.*

Specifically, $w_{\sigma,\alpha}$ is an optimal Hardy weight if and only if $\alpha = (1 + 2\sigma)/4$.

We split the proof into four lemmas. We start with an argument that allows to conclude optimality near infinity from criticality together with null-criticality. Although various arguments appear in the literature [9,18,20], these settings do not exactly cover our situation. Thus, we give a proof for the sake of being self-contained.

Lemma 6 (Optimality near infinity). *Let $0 < \sigma \leq \alpha < 1/2$ and assume that $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$ is critical. If for finite $K \subseteq \mathbb{Z}$ and $w' : \mathbb{Z} \longrightarrow [0,\infty)$*

$$\langle \Delta^\sigma \varphi, \varphi\rangle \geq \langle (w_{\sigma,\alpha} + w')\varphi, \varphi\rangle$$

for all $\varphi \in C_c(\mathbb{Z} \setminus K)$, then $\kappa_{-\alpha}$ satisfies

$$\kappa_{-\alpha} \in \ell^2(\mathbb{Z}, w').$$

In particular, $w_{\sigma,\alpha}$ is optimal near infinity, whenever it is also null-critical.

Proof. By the assumption, Green's formula and the ground state transform, Proposition 3, we have that for all $\eta \in C_c(\mathbb{Z})$ with $0 \leq \eta \leq 1$

$$\sum_{x\in\mathbb{Z}\setminus K} w'(x)(\kappa_{-\alpha}\eta)^2(x) \leq (Q^\sigma - w_{\sigma,\alpha})(\kappa_{-\alpha}\eta 1_{\mathbb{Z}\setminus K}) = Q^\sigma_{-\alpha}(\eta(1-1_K))$$

$$\leq 2Q^\sigma_{-\alpha}(\eta) + 2Q^\sigma_{-\alpha}(1_K),$$

where we used a discrete Leibniz rule and $(a+b)^2 \leq 2(a^2+b^2)$ in the last step. We choose η as a sequence (e_n) from Proposition 4 which exists as $(Q^\sigma - w_{\sigma,\alpha})$ is assumed to be critical. By Fatou's lemma and boundedness of the right-hand side, we infer

$$\sum_{x\in\mathbb{Z}\setminus K} w'(x)\kappa^2_{-\alpha}(x) < \infty.$$

Hence, $\kappa_{-\alpha} \in \ell^2(\mathbb{Z}, w')$ as K is finite.

Whenever $w_{\sigma,\alpha}$ is critical, then there exists a unique positive harmonic function (up to multiplicative constants) which is referred to as a (Agmon) ground state, [22, Theorem 5.3]. As $\kappa_{-\alpha}$ is a positive harmonic function by Proposition 2, $\kappa_{-\alpha}$ is the unique (Agmon) ground state. Now, the statement about optimality near infinity can be concluded with the choice $w' = \lambda w_{\sigma,\alpha}$ for $\lambda > 0$ as null-criticality implies $\kappa_{-\alpha} \notin \ell^2(\mathbb{Z}, w_{\sigma,\alpha})$. □

The next lemma shows criticality of $w_{\sigma,\alpha}$ for $\alpha \leq (1+2\sigma)/4$. Here we construct a sequence (e_n) as in Proposition 4.

Lemma 7 (Sufficient criterion for criticality). *Let $0 < \sigma \leq \alpha < 1/2$ and $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$. The Hardy weights $w_{\sigma,\alpha}$ are critical if $\alpha \leq (1+2\sigma)/4$ and $\kappa_{-\alpha}$ is the unique (up to multiplicative constants) ground state of $(\Delta^\sigma - w_{\sigma,\alpha})$ in this case.*

Proof. By Proposition 3 the function $w_{\sigma,\alpha}$ is a Hardy weight. To show criticality of $w_{\sigma,\alpha}$ for $\alpha \leq (1+2\sigma)/4$, we present a sequence (e_n) with the properties as in Proposition 4. For $n \in \mathbb{N}$, define $e_n : \mathbb{Z} \longrightarrow \mathbb{R}$, $e_n(0) = 1$,

$$e_n(x) = \left(1 - \sqrt{\frac{\log|x|}{\log n}}\right)_+, \quad x \neq 0.$$

Clearly, the support of e_n is $\{x \in \mathbb{Z} \mid |x| < n\}$, so (e_n) is a sequence in $C_c(\mathbb{Z})$, and $e_n \to 1$ pointwise. So, by the proposition above, criticality of $w_{\sigma,\alpha}$ follows if we can show the uniform boundedness of $Q^\sigma_{-\alpha}(e_n)$. It is straightforward to check using the asymptotics of $\kappa_\beta \in O(|\cdot|^{-1-2\beta})$ that

$$Q^\sigma_{-\alpha}(e_n) \le C \sum_{1\le y<x} \frac{(e_n(x)-e_n(y))^2}{(x-y)^{1+2\sigma}(xy)^{1-2\alpha}}.$$

We divide the sum above into two sums T_1 and T_2 over the sets $\{1 \le y < x \le n\}$ and $\{1 \le y < n < x\}$ since $(e_n(x)-e_n(y))^2$ vanishes outside.

For the first sum, we use the subadditivity of the square root to estimate $(e_n(x)-e_n(y))^2 \le \log(x/y)/\log n$ and obtain

$$\begin{aligned} T_1 &= \sum_{1\le y<x\le n} \frac{(e_n(x)-e_n(y))^2}{(x-y)^{1+2\sigma}(xy)^{1-2\alpha}} \\ &\le \frac{1}{\log n} \sum_{1\le y<x\le n} \frac{\log(x/y)}{(x-y)^{1+2\sigma}(xy)^{1-2\alpha}} \\ &\le \frac{1}{\log n} \sum_{1\le y<n} y^{4\alpha-2\sigma-3} \int_y^\infty \frac{\log(x/y)\,\mathrm{d}x}{(x/y-1)^{1+2\sigma}(x/y)^{1-2\alpha}} \\ &= \frac{1}{\log n} \sum_{1\le y<n} y^{4\alpha-2\sigma-2} \int_1^\infty \frac{\log t\,\mathrm{d}t}{(t-1)^{1+2\sigma}t^{1-2\alpha}}. \end{aligned}$$

We always have $\log t \le t-1$ and, moreover, for every $\beta > 0$, there exists $t_\beta > 0$ such that $\log t \le (t-1)^{2\beta}$ for $t > t_\beta + 1$. Hence, with $\beta = 1/2 - \alpha$,

$$\begin{aligned} T_1 &\le \frac{1}{\log n} \sum_{1\le y<n} y^{4\alpha-2\sigma-2} \left(\int_0^{t_\beta} t^{(1-2\sigma)-1}\,\mathrm{d}t + \int_{t_\beta}^\infty t^{(2(\beta+\alpha-\sigma)-1)-1}\,\mathrm{d}t \right) \\ &\le \frac{C_{\sigma,\alpha}}{\log n} \sum_{1\le y<n} y^{4\alpha-2\sigma-2} \end{aligned}$$

and the right-hand side stays bounded by the log-asymptotics of the harmonic series if $\alpha = (1+2\sigma)/4$ and even goes to zero if $\alpha < (1+2\sigma)/4$.

For the second sum T_2 over the set $\{1 \le y < n < x\}$, we estimate $(e_n(x)-e_n(y))^2 \le 1$ and $x^{2\alpha-1} \le (x-y)^{2\alpha-1}$ as $\alpha < 1/2$ and obtain

$$\begin{aligned} T_2 &= \sum_{1\le y<n<x} \frac{(e_n(x)-e_n(y))^2}{(x-y)^{1+2\sigma}(xy)^{1-2\alpha}} \\ &\le \int_0^n y^{2\alpha-1} \int_n^\infty (x-y)^{-2\sigma-1}(x-y)^{2\alpha-1}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{1+2\sigma-2\alpha} \int_0^n y^{2\alpha-1}(n-y)^{2\alpha-2\sigma-1}\,\mathrm{d}y \\ &= \frac{n^{4\alpha-2\sigma-1}}{1+2\sigma-2\alpha} \int_0^1 t^{2\alpha-1}(1-t)^{2\alpha-2\sigma-1}\,\mathrm{d}t \\ &= C_{\sigma,\alpha} n^{4\alpha-2\sigma-1} \end{aligned}$$

where we substituted $t = y/n$ and observe $C_{\sigma,\alpha} < \infty$ since $2\alpha, 2\alpha - 2\sigma > 0$. The right-hand side stays bounded for $\alpha = (1+2\sigma)/4$ and even goes to zero if $\alpha < (1+2\sigma)/4$ as $n \to \infty$.

Putting the two estimates together, we obtain $\sup_{n\in\mathbb{N}} Q^{\sigma}_{-\alpha}(e_n) < \infty$ which shows criticality of $(Q^{\sigma} - w_{\sigma,\alpha})$ for $\alpha \le (1+2\sigma)/4$ by Proposition 4.

As $\kappa_{-\alpha}$ is a positive harmonic function by Proposition 2, it is the (Agmon) ground state which is unique up to multiplicative constants for critical forms, cf. [22, Theorem 5.3]. □

To show the necessary criterion for criticality, we first need to prove (b) of Theorem 5.

Lemma 8. (Null-criticality) *Let $0 < \sigma \le \alpha < 1/2$. The super-solutions $\kappa_{-\alpha}$ are not in $\ell^2(\mathbb{Z}, w_{\sigma,\alpha})$ if and only if $\alpha \ge (1+2\sigma)/4$.*

Proof. From the asymptotics $\kappa_{-\alpha} \in O(|\cdot|^{-1+2\alpha})$ and $w_{\sigma,\alpha} \in O(|\cdot|^{-2\sigma})$, we infer that

$$\sum_{x\in\mathbb{Z}} \kappa_{-\alpha}^2(x) w_{\sigma,\alpha}(x) \asymp \sum_{x\ge 1} x^{(4\alpha-1-2\sigma)-1} \begin{cases} = \infty, & \alpha \ge \frac{1+2\sigma}{4} \\ < \infty, & \alpha < \frac{1+2\sigma}{4}, \end{cases}$$

where $\asymp$ means two-sided estimates with positive constants. □

With these preparations, we are in the position to show that $w_{\sigma,\alpha}$ cannot be critical for $\alpha > (1+2\sigma)/4$.

Lemma 9 (Necessary criterion for criticality). *Let $0 < \sigma \le \alpha < 1/2$ and $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$. The Hardy weights $w_{\sigma,\alpha}$ are critical only if $\alpha \le (1+2\sigma)/4$.*

Proof. The leading asymptotics of the Hardy weights $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$ in x share the same decay which is $|x|^{-2\sigma}$, cf. [5, Lemma 9.2] but differ in their constants which is given by the function

$$\Psi_\sigma : [\sigma, \tfrac{1}{2}) \longrightarrow (0,\infty), \quad \alpha \longmapsto 4^{\sigma} \frac{\Gamma(\frac{1}{2}+\sigma-\alpha)\Gamma(\alpha)}{\Gamma(\alpha-\sigma)\Gamma(\frac{1}{2}-\alpha)}.$$

We relate Ψ_σ the function $\Phi_{\sigma,1}$ given in Eq. (3.5) of [10] via

$$\Psi_\sigma(\alpha) = \Phi_{\sigma,1}(1-2\alpha) + c_\sigma,$$

where c_σ is constant of the Hardy weight w_σ defined on page 4. In [10, Lemma 3.2] it is shown that the function $\Phi_{\sigma,1}$ is strictly increasing in the interval $(0, (1-2\sigma)/2)$ with $\Phi_{\sigma,1}((1-2\sigma)/2) = 0$. Hence, Ψ_σ is strictly decreasing in $((1+2\sigma)/4, 1/2)$ with $\Psi_\sigma((1+2\sigma)/4) = c_\sigma$.

Assume the weights $w_{\sigma,\alpha}$ are critical for some $\alpha > (1+2\sigma)/4$. By the considerations above, we can choose λ such that

$$\frac{c_\sigma - \Psi_\sigma(\alpha)}{\Psi_\sigma(\alpha)} > \lambda > 0.$$

Thus, there is a finite set $K \subseteq \mathbb{Z}$ such that

$$w_{\sigma,\frac{1+2\sigma}{4}} \ge (1+\lambda) w_{\sigma,\alpha}$$

outside of K. Now, $\kappa_{-\alpha}$ is a positive harmonic function for $(\Delta^\sigma - w_{\sigma,\alpha})$ and, therefore, the unique ground state of the critical form $(Q^\sigma - w_{\sigma,\alpha})$. By Lemma 6 applied with $w' = \lambda w_{\sigma,\alpha}$, we obtain

$$\kappa_{-\alpha} \in \ell^2(\mathbb{Z}, w_{\sigma,\alpha}).$$

However, this contradicts Lemma 8. □

Putting these lemmas together yields the proof of Theorem 5.

Proof of Theorem 5. Statement (a) follows from Lemmas 7 and 9. Statement (b) is proven in Lemma 8. □

Finally, we deduce our main theorem from Theorem 5.

Proof of Theorem 1. By direct calculation one has $w_\sigma = w_{\sigma,\alpha}$ for $\alpha = (1 + 2\sigma)/4$ and by Theorem 5 one has that $w_{\sigma,\alpha} = \kappa_{\sigma-\alpha}/\kappa_{-\alpha}$ is critical and null-critical, and, thus, optimal for $\alpha = (1 + 2\sigma)/4$. Furthermore, optimality near infinity follows by Lemma 6. □

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