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Infraparticle States in the Massless Nelson Model: Revisited

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Dedicated to the memory of Krzysztof Gawędzki who was an example to many.

Abstract. We provide a new construction of infraparticle states in the massless Nelson model. The approximating sequence of our infraparticle state does not involve any infrared cut-offs. Its derivative w.r.t. the time parameter t is given by a simple explicit formula. The convergence of this sequence as $t \to \infty$ to a nonzero limit is then obtained by the Cook method combined with stationary phase estimates. To apply the latter technique, we exploit recent results on regularity of ground states in the massless Nelson model, which hold in the low coupling regime.

Keywords. Nelson model, Scattering theory, Infrared problems.

1. Introduction

The massless Nelson model is a time-honoured theoretical laboratory for the infrared aspects of QED. One of its variants, which we consider in this work, describes one non-relativistic massive particle ('the electron'), interacting with massless scalar bosons ('the photons'). The coupling between the electrons and photons is chosen in such a way that the model exhibits the *infraparticle problem*, i.e. it does not contain physical states describing the electron in empty space. In other words, the electron is always encircled by an ever larger halo of ever softer photons, and it is a challenge to mathematically describe the resulting composite object, usually called an infraparticle. Two milestones in rigorous understanding of this problem are works of Fröhlich [19, 21] and Pizzo [32, 33]. The latter two papers actually give a complete discussion of the infraparticle in the Nelson model and of its collisions with (hard) photons. Also collisions of an infraparticle with a Wigner-type particle ('an atom') in a Nelson model with two massive particles are under control [17]. However, scattering of several infraparticles appears steeply difficult in the conventional approach from [33], as discussed in detail in [17, Introduction]. One reason is that the approximating sequence of the infraparticle state from [33] and the proof of its convergence are technically quite intricate, which may be due to limited spectral information on the model available back then. Given intervening advances in spectral theory [1,15,16], we revisit the subject and propose a simpler approximating sequence of the infraparticle in the Nelson model. Its convergence to a non-trivial limit is relatively straightforward, given the currently available spectral ingredients. Needless to say, our discussion above only touched upon the broad topic of spectral and scattering theory in non-relativistic QED, see e.g. [3,6,8,9,13,18,36], and no systematic review is attempted here.

To explain our construction, let us recall the definition of the Nelson model. The Hilbert space of the model is $\mathcal{H} = L^2(\mathbb{R}^3_x;\mathcal{F})$, where $\mathcal{F}$ is the symmetric Fock space over $L^2(\mathbb{R}^3_k)$. For introductory material on the theory of Fock spaces, we refer the reader to [34, Sect. X.7]. Thus, we will treat $\psi \in \mathcal{H}$ as $\mathcal{F}$-valued square-integrable functions $\{\psi(x)\}_{x\in\mathbb{R}^3}$, whose scalar product has the form

$$\langle \psi_1, \psi_2\rangle_{\mathcal{H}} = \int d^3x\, \langle \psi_1(x), \psi_2(x)\rangle_{\mathcal{F}}. \tag{1.1}$$

The creation and annihilation operators on $\mathcal{F}$ are denoted by $a^{(*)}(f)$, $f \in L^2(\mathbb{R}^3_k)$, and their sharp variants by $k \mapsto a^{(*)}(k)$. We will also occasionally write

$$\Phi(f) := a^*(-if) + a(-if). \tag{1.2}$$

The Hamiltonian of the Nelson model has the form

$$H = \frac{(-i\nabla_x)^2}{2} + H_{\mathrm{f}} + a(v_x) + a^*(v_x). \tag{1.3}$$

Here, x and $-i\nabla_x$ are the position and momentum operators on $L^2(\mathbb{R}^3_x)$, $(H_{\mathrm{f}}, P_{\mathrm{f}}) := (\mathrm{d}\Gamma(|k|), \mathrm{d}\Gamma(k))$ denote the energy-momentum operators of non-interacting photons and $v_x(k) = v(k)e^{-ik\cdot x}$, where $v(k) := \lambda\frac{\chi_\kappa(k)}{\sqrt{2|k|}}$ and $|\lambda| \in (0, \lambda_0]$ is the coupling constant, whose maximal value λ_0 will be sufficiently small but nonzero. Here, $\chi_\kappa \in C_0^\infty(\mathbb{R}^3)$ is a smooth approximate characteristic function of the ball of radius $\kappa = 1$.[1] We choose this function rotation invariant, supported in the ball of radius κ and equal to one on a ball of a slightly smaller radius $(1-\varepsilon_0)\kappa$ for some $0 < \varepsilon_0 < 1$. By the Kato–Rellich theorem, H is a self-adjoint operator on $D(\frac{1}{2}(-i\nabla_x)^2 + H_{\mathrm{f}})$. This elementary observation dates back to [30], for a textbook discussion in a similar model we refer to [36, Sect. 13.3]. The origin of the infrared problem lies in the fact that $v(k)/|k| \notin L^2(\mathbb{R}^3_k)$, as will be recalled later in Sect. 2.

Recalling that the model is translation invariant, we denote by $\{H_p\}_{p\in\mathbb{R}^3}$ the usual fibre Hamiltonians acting on the fibre Fock space $\mathcal{F}_{\mathrm{fi}}$, satisfying

$$H = \Pi^* \left(\int^{\oplus} d^3p\, H_p \right) \Pi, \quad \Pi = F e^{iP_{\mathrm{f}}\cdot x}, \tag{1.4}$$

[1] Although $\kappa = 1$, it is convenient to keep it in the notation.

where F is the Fourier transform in the x variable. In our construction of infraparticle scattering states we will identify the fibre Fock space $\mathcal{F}_{\mathrm{fi}}$ with the physical Fock space $\mathcal{F}$ which is the reason for the appearance of the unitary Π explicitly in formula (1.8). After this identification, the fibre Hamiltonians are the following self-adjoint operators on $D(P_{\mathrm{f}}^2 + H_{\mathrm{f}}) \subset \mathcal{F}$

$$H_p := \frac{1}{2}(p - P_{\mathrm{f}})^2 + H_{\mathrm{f}} + a^*(v) + a(v), \quad p \in \mathbb{R}^3. \tag{1.5}$$

We denote the infimum of the spectrum of H_p by E_p. One manifestation of the infraparticle problem is that E_p is not an eigenvalue. This has been established in considerable generality in [11,21,32]. For $p \in S$, where

$$S := \{\, p' \in \mathbb{R}^3 \,|\, |p'|<1/3 \,\} \tag{1.6}$$

and λ_0 sufficiently small we know in addition from [1] that $p \mapsto E_p$ is real analytic and $|\nabla E_p| < 1/2$ for $p \in S$ (cf. Lemma 3.1). It is also well-known that the *modified Hamiltonian* H_p^{w}, obtained from H_p by the Bogolubov transformation

$$a^{(*)}(k) \mapsto a^{(*)}(k) - f_p(k), \quad f_p(k) := \lambda \frac{\chi_\kappa(k)}{\sqrt{2|k|}} \frac{1}{|k|(1 - e_k \cdot \nabla E_p)}, e_k := k/|k|, \tag{1.7}$$

is self-adjoint on $D(P_{\mathrm{f}}^2 + H_{\mathrm{f}})$ and E_p is its eigenvalue at the bottom of the spectrum corresponding to an eigenvector ϕ_p [32]. (Its phase is chosen in the following in accordance with [15, Definition 5.2].) Such a change in the character of E_p is possible because $f_p \notin L^2(\mathbb{R}^3)$, and hence, the Bogolubov transformation (1.7) is not unitarily implementable.

After these preparations we are ready to define the approximating sequences of the infraparticle states. Motivation for this formula is given in Sect. 2, and in Conclusions, we relate it to the Faddeev–Kulish approach. For any $h \in C_0^\infty(\mathbb{R}^3)$ supported in S and any time parameter $t \in \mathbb{R}$, we set

$$\begin{aligned} \psi_t(x) &:= e^{iHt} e^{-iP_{\mathrm{f}} \cdot x} \\ &\quad \times \frac{1}{(2\pi)^{3/2}} \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p(e^{-i|k|t + ik\cdot x} - 1)\big)\phi_p, \end{aligned} \tag{1.8}$$

$$\gamma(p,x,t) := \int d^3k f_p(k)^2 \sin(|k|t - k\cdot x), \tag{1.9}$$

where $W(g) := e^{a^*(g) - a(g)}, g \in L^2(\mathbb{R}_k^3)$, is a Weyl operator on $\mathcal{F}$. It is well-defined for $g(k) := f_p(k)(e^{-i|k|t + ik\cdot x} - 1)$ for any $(t,x) \in \mathbb{R}^4$. This is due to the fact that $|e^{-i|k|t+ik\cdot x} - 1| \le |k|(|x| + |t|)$ and hence $k \mapsto f_p(k)(e^{-i|k|t+ik\cdot x} - 1)$ is square integrable, unlike f_p, cf. Lemma E.1. The integral in (1.8) is well-defined as a Bochner integral in $\mathcal{F}$, since $S \ni p \mapsto \phi_p$ is Hölder continuous in norm by [32] (which can also be seen by [15, formulas (1.8), (A.4) and Corollary 5.6] combined with Lemma C.3 below). This integral belongs to $L^2(\mathbb{R}_x^3; \mathcal{F})$ by Lemma 4.5. Our main result is the following:

Theorem 1.1. *There is such* $\lambda_0 > 0$ *that the following holds: For any* $t \in \mathbb{R}$, *the vector* ψ_t *given by* (1.8) *belongs to* $L^2(\mathbb{R}^3_x; \mathcal{F})$. *The derivative* $\partial_t \psi_t$ *exists in norm in* $L^2(\mathbb{R}^3_x; \mathcal{F})$, *and we have*

$$\partial_t \psi_t(x) = e^{iHt} e^{-iP_{\mathrm{f}} \cdot x} \times \frac{1}{(2\pi)^{3/2}} \int d^3p \, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} i\gamma_{\mathrm{int}}(p,x,t) h(p) W(f_p(e^{-i|k|t + ik\cdot x} - 1))\phi_p, \tag{1.10}$$

where $\gamma_{\mathrm{int}}(p,x,t) := 2\int d^3k \, f_p(k)^2(|k| - k\cdot\nabla E_p)\cos(|k|t - k\cdot x)$ *is rapidly decreasing in the region* $|x|/t < 1$ *(cf. Lemma* 4.7*). Furthermore,*

$$\int_0^\infty dt \, \|\partial_t \psi_t\|_{\mathcal{H}} < \infty, \tag{1.11}$$

hence $\psi^+ := \lim_{t\to\infty} \psi_t$ *exists in the norm of* $L^2(\mathbb{R}^3_x; \mathcal{F})$. *For* $h \neq 0$ *and* $|\lambda| \in (0, \lambda_0]$ *sufficiently small,* $\psi^+ \neq 0$. *Analogous statements hold for incoming scattering states.*

The most remarkable part of the theorem is the explicit formula for $\partial_t \psi_t$ given in (1.10). It can be anticipated by formal computations on $\mathcal{F}$ noting the key relation

$$\begin{aligned} T(p,x,t)^* \, (-i\nabla_x - P_{\mathrm{f}}) \, T(p,x,t) &= -i\nabla_x - P_{\mathrm{f}}^{\mathrm{w}}, \\ T(p,x,t) &:= W(f_p(e^{-i|k|t + ik\cdot x} - 1)) e^{i\gamma(p,x,t)}, \end{aligned} \tag{1.12}$$

where $P_{\mathrm{f}}^{\mathrm{w}}$ is obtained from P_{f} via the Bogolubov transformation (1.7). Relation (1.12) allows to reconstruct H_p^{w} in front of ϕ_p and make use of $H_p^{\mathrm{w}}\phi_p = E_p\phi_p$. It dictates our choice of the phase γ, and it is noteworthy that the resulting γ_{int} enjoys a rapid decay in t in the physical region of velocities of the electron. This coincidence suggests that our approximating vector (1.8) captures optimally the asymptotic dynamics of the Nelson model in the infrared regime. The decay of γ_{int} is the driving force of our convergence argument based on the Cook method [10,35]. It also allows for a simple proof of non-triviality of the limit for small $|\lambda|$.

Given formula (1.10) and the above remarks, it may seem very easy to prove the theorem. But it should be kept in mind that estimate (1.11) must hold in the norm of $L^2(\mathbb{R}^3_x; \mathcal{F})$, which involves the integral over the whole space, cf. formula (1.1). To control this integral, we use the stationary phase method, which generates derivatives w.r.t. p up to the second order (cf. Lemma 4.1). Since differentiability of $p \mapsto \phi_p$ is not settled, we have to approximate ϕ_p with $\phi_{p,\sigma}$, which come from the Nelson model with an infrared cut-off $\sigma > 0$ in the interaction. The function $p \mapsto \phi_{p,\sigma}$ is differentiable, and its derivatives up to the second order have only a mild infrared divergence of the form

$$\|\partial_p^\alpha \phi_{p,\sigma}\|_{\mathcal{F}} \leq c\sigma^{-\delta_{\lambda_0}}, \quad |\alpha| = 0, 1, 2, \tag{1.13}$$

where $\delta_{\lambda_0} > 0$ tends to zero with $\lambda_0 \to 0$. This estimate, and similar bounds for the wave functions of $\phi_{p,\sigma}$, rely on technical advances from [15,16]. Thus, at our present level of understanding, we can eliminate the infrared cut-off from the formulation of Theorem 1.1, but not from its proof. As mentioned above, to eliminate the cut-off also from the proof it seems necessary to establish differentiability of $p \mapsto \phi_p$. For some ideas in this direction, we refer to [15, formula (1.9)] and a cancellation of infrared singularities conjectured in this formula.

This paper is organized as follows: In Sect. 2, we provide motivation for our infraparticle ansatz (1.8). In Sect. 3, we give some technical information, in particular about the model with infrared cut-off. Section 4 is devoted to the proof of Theorem 1.1. In Conclusions, we provide a brief comparison of our infraparticle states with the Faddeev–Kulish approach. More technical parts of the discussion are postponed to Appendices.

2. Motivation for the Infraparticle Ansatz (1.8)

The fibre decomposition and the associated transformation, both seen in (1.4), are due to Lee, Low and Pines [27] and have been used ever since. We find it convenient to phrase that transformation as a superposition instead. To convey the quite trivial idea, let us first consider the simpler case of functions $\Psi \in L^2(\mathbb{R}_x)$, which can be written as $\Psi = (2\pi)^{-1/2} \int dp\, \Psi_p$ where $\Psi_p = \Psi_p(x)$ is $\Psi_p(x) = \hat{\Psi}_p e^{ipx}$. While $\Psi \mapsto \hat{\Psi}$ is the Fourier transform, the integral itself is a superposition of improper elements of $L^2(\mathbb{R}_x)$. The former transformation is more precise, the latter is closer to physical intuition because it displays Ψ as the superposition of plane waves.

In the case of $\mathcal{H} = L^2(\mathbb{R}_x; \mathcal{F})$, the decomposition of $\Psi \in \mathcal{H}$ is

$$\Psi = (2\pi)^{-3/2} \int_{\mathbb{R}^3} d^3p\, \Psi_p, \tag{2.1}$$

where Ψ_p is an improper element of $\mathcal{H}$. It is singled out by its character $y \mapsto e^{-ip\cdot y}$ (in the sense of representation theory) of the (abelian) translation group $e^{-iP\cdot y}$, $P = -i\nabla_x + P_{\mathrm{f}}$,

$$e^{-iP\cdot y}\Psi_p = e^{-ip\cdot y}\Psi_p. \tag{2.2}$$

An informal expression for Ψ_p is provided by the Wigner projection onto the isotypical component associated with the character:

$$\Psi_p = (2\pi)^{-3/2} \int d^3y\, \overline{e^{-ip\cdot y}} e^{-iP\cdot y}\Psi = (2\pi)^{-3/2} \int d^3y\, e^{-i(P-p)\cdot y}\Psi. \tag{2.3}$$

Indeed, by $\int d^3p\, e^{ip\cdot y} = (2\pi)^3\delta(y)$, we have

$$(2\pi)^{-3/2} \int d^3p\, \Psi_p = \int d^3y\, \delta(y) e^{-iP\cdot y}\Psi = \Psi. \tag{2.4}$$

Let us also note that Ψ_p "$\in$" $L^2(\mathbb{R}^3_x; \mathcal{F})$ takes values $\Psi_p(x)$ in $\mathcal{F}$ for $x \in \mathbb{R}^3$. They are related to one another by

$$\Psi_p(x-y) = (e^{-i(-i\nabla_x)\cdot y}\Psi_p)(x) = (e^{-i(p-P_{\mathrm{f}})\cdot y}\Psi_p)(x) = e^{-i(p-P_{\mathrm{f}})\cdot y}\Psi_p(x) \tag{2.5}$$

because P_{f} acts on $\mathcal{F}$ alone. In particular, setting x to zero and then renaming y to $-x$, we get

$$\Psi_p(x) = e^{i(p-P_{\mathrm{f}})\cdot x}\Psi_p(0). \tag{2.6}$$

For a rigorous treatment, it is better to restate (2.1) as a unitary map

$$\Pi : \mathcal{H} \to L^2(\mathbb{R}_p;\mathcal{F}), \quad \Psi \mapsto (\Pi_p\Psi)_{p\in\mathbb{R}^3}. \tag{2.7}$$

This is achieved by evaluation at, say, $x = 0$

$$\Pi_p\Psi = \Psi_p(0). \tag{2.8}$$

By (2.1), (2.6) the inverse map of Π, i.e. $\Pi^{-1} : (\Psi_p(0))_{p\in\mathbb{R}^3} \mapsto \Psi$ is

$$\Psi = (2\pi)^{-3/2}\int d^3p\, e^{i(p-P_{\mathrm{f}})\cdot x}\Psi_p(0). \tag{2.9}$$

We note that (2.3) can also be written as

$$\Psi_p(x) = (2\pi)^{-3/2}\int d^3y\, e^{ip\cdot y}e^{-iP_{\mathrm{f}}\cdot y}\Psi(x-y) \tag{2.10}$$

because of $(e^{-iP\cdot y}\Psi)(x) = e^{-iP_{\mathrm{f}}\cdot y}\Psi(x-y)$. Thus

$$\Pi_p\Psi = \Psi_p(0) = (2\pi)^{-3/2}\int d^3x\, e^{-ip\cdot x}e^{iP_{\mathrm{f}}\cdot x}\Psi(x) \tag{2.11}$$

by the substitution $y =: -x$, which is possible because x has been disposed of by setting $x = 0$ first. We conclude that $\Pi_p\Psi = F(e^{iP_{\mathrm{f}}\cdot x}\Psi)$.

The understanding of the ansatz (1.8) benefits from a comparison with the van Hove model. Compared to the Nelson model, the (dynamical) electron is replaced by an external source. More formally, the Hilbert space is $\mathcal{F}$ and the Hamiltonian is

$$H = \int d^3k\,\omega(k)(a^*(k) - \bar{f}(k))(a(k) - f(k)). \tag{2.12}$$

We observe that it resembles (1.3) once x and ∇_x are omitted and $\omega(k) = |k|$ (though up to zero point energy irrelevantly differing by $\int d^3k\,\omega(k)|f(k)|^2$). Regarding f, let us first assume that $f \in L^2(\mathbb{R}^3_k)$ and comment on the physical choice $f(k) = -v(k)/|k| \notin L^2(\mathbb{R}^3_k)$ later on.

The free Hamiltonian H_0, corresponding to $f = 0$, and the Weyl operators $W(g)$ $(g \in L^2(\mathbb{R}^3_k))$ satisfy:

$$W(g_1)W(g_2) = e^{-i\mathrm{Im}\langle g_1,g_2\rangle}W(g_1+g_2), \tag{2.13}$$
$$e^{-itH_0}W(g) = W(e^{-it\omega}g)e^{-itH_0}; \tag{2.14}$$

moreover, H_0 is unitarily equivalent to the Hamiltonian

$$H = W(f)H_0W(-f), \tag{2.15}$$

since $f \in L^2(\mathbb{R}^3_k)$ and thus $W(f)$ well-defined. In particular, $W(f)\Omega$ is the ground state of H:

$$HW(f)\Omega = E_0W(f)\Omega \tag{2.16}$$

with $E_0 = 0$ for the above choice of zero point energy. Equations (2.13, 2.14) imply the identities:

$$e^{-itH} = e^{-i\langle f,\sin(\omega t) f\rangle} W((1-e^{-it\omega})f)e^{-itH_0}, \tag{2.17}$$

$$e^{-itH}W(g) = e^{2i\mathrm{Im}\langle f,(1-e^{-it\omega})g\rangle} W(e^{-it\omega}g)e^{-itH}. \tag{2.18}$$

Using (2.17), we get

$$e^{-itH}\Omega = e^{-itE_0}e^{-i\langle f,\sin(\omega\, t) f\rangle} W((1-e^{-it\omega})f)\Omega, \tag{2.19}$$

which describes the evolution of the bare vacuum Ω. Somewhat more suggestively, it is restated as

$$e^{-itH}\Omega = e^{-itE_0}e^{-2i\langle f,\sin(\omega\, t) f\rangle} W(-e^{-it\omega}f)W(f)\Omega. \tag{2.20}$$

In this guise, it describes the approach of the unperturbed vacuum Ω to the perturbed ground state $W(f)\Omega$ as $t \to +\infty$ (up to a numerical factor $e^{-\frac{1}{2}\|f\|_2^2}$). In fact, we note that $e^{-it\omega}f \to 0$ weakly in $L^2(\mathbb{R}^3_k)$ as $t \to \infty$, whence $W(e^{-it\omega}f) \to e^{-\frac{1}{2}\|f\|_2^2}$ in the weak topology[2] of operators on $\mathcal{F}$. The decreased norm can be attributed to photons lost at infinity in $\mathbb{R}^3_x$.

Equation (2.18), when applied to $\Psi \in \mathcal{F}$, states that out of the trajectory $e^{-itH}\Psi$ another one can be obtained by adding a coherent bunch of freely moving bosons by means of $W(e^{-it\omega}g)$; in fact up to phase, the trajectory $e^{-itH}W(g)\Psi$ results. For example, we can choose $\Psi = W(f)\Omega$, i.e. the perturbed ground state, in which case

$$e^{-itH}W(g)W(f)\Omega = e^{2i\mathrm{Im}\langle f,(1-e^{-it\omega})g\rangle} W(e^{-it\omega}g)e^{-itE_0}W(f)\Omega. \tag{2.21}$$

For $g = -f$, we simply recover (2.20). The unperturbed ground state Ω is referred to as bare and the perturbed one $W(f)\Omega$ then arises in the same picture by the dressing transformation $W(f)$. In its own "infrared" picture the perturbed ground state is still given by Ω, because the Weyl operator intertwines exactly between H and H_0.

The origin of the infrared problem, which arises when $f \notin L^2(\mathbb{R}^3_k)$, is now manifest: So to speak, the ground state $W(f)\Omega$ in (2.16) leaves the Fock space $\mathcal{F}$. While $\Psi = W(f)\Omega$ is not well-defined, a trajectory in $\mathcal{F}$ can be defined via (2.21), provided that $g + f \in L^2(\mathbb{R}^3_k)$. To this end, the pair of Weyl operators seen there or in (2.20) should be merged to one, as done in (2.19). Then, by restating (2.21) for $g + f = 0$ as follows

$$\Omega = e^{itH}e^{-itE_0}e^{i\mathrm{Im}\langle f,e^{-it\omega}f\rangle} W((1-e^{-it\omega})f)\Omega \tag{2.22}$$

we note a similar structure as in our infraparticle vector (1.8).

These conclusions can be transposed to the Nelson model, which fibrewise resembles the van Hove model of coupling $f(k) = -v(k)/|k| \notin L^2(\mathbb{R}^3_k)$.

[2] We use that $W(e^{-it\omega}f) = e^{-\frac{1}{2}\|f\|_2^2}e^{a^*(e^{-it\omega}f)}e^{-a(e^{-it\omega}f)}$ in terms of quadratic forms on the dense domain of finite particle vectors from $\mathcal{F}$. Now, the claim follows from the Riemann–Lebesgue lemma and the uniform boundedness of $t \mapsto W(e^{-it\omega}f)$.

In the Nelson model, the ground state of H_p^{w} of total momentum p is ϕ_p in its own (infrared) picture,

$$H_p^W \phi_p = E_p \phi_p. \tag{2.23}$$

That state is perturbatively close to, but no longer identical to Ω, because no Weyl operator removes the interaction terms in (1.3) exactly.

In the bare picture that state is $W(-f_p)\phi_p$, where

$$f_p(k) = -\frac{f(k)}{1 - e_k \cdot \nabla E_p}. \tag{2.24}$$

Since $f_p \notin L^2(\mathbb{R}^3_k)$, this vector is not well-defined (in contrast to ϕ_p).

Collecting fibres, cf. (2.9), we obtain (still not well-defined) states on the mass shell $p \mapsto E_p$,

$$\phi(x) = (2\pi)^{-3/2} \int d^3p\, h(p) e^{i(p-P_{\mathrm{f}})\cdot x} W(-f_p)\phi_p, \tag{2.25}$$

where the support of h is contained in the set S, cf. (1.6). Its trajectory $\phi_t = e^{-itH}\phi$ is

$$\phi_t(x) = (2\pi)^{-3/2} \int d^3p\, h(p) e^{i(p\cdot x - E_p t)} e^{-iP_{\mathrm{f}}\cdot x} W(-f_p)\phi_p. \tag{2.26}$$

The goal is to add a bunch of photons to ϕ_t in a way that is simple and explicit, though not strictly compatibly with e^{-iHt} as in (2.18), and yet in such a way that:

- unlike (2.26), the resulting state $\Psi_t = \Psi_t(x)$ lies in $\mathcal{H}$,
- the addition is asymptotically compatible with dynamics, in the sense that the limit

$$\lim_{t\to\infty} e^{itH}\Psi_t = \psi^+ \tag{2.27}$$

exists. That in fact means that $e^{-iHt}\psi^+$ has Ψ_t as its explicit asymptote.

In line with (2.18), its interpretation and its use for $g = -f$, we modify (2.26) to

$$\Psi_t(x) = (2\pi)^{-3/2} \int d^3p\, h(p) e^{2i\tilde{\gamma}} W(e^{-i\omega t} f_p) e^{i(p\cdot x - E_p t)} e^{-iP_{\mathrm{f}}\cdot x} W(-f_p)\phi_p, \tag{2.28}$$

where the phase $\tilde{\gamma} = \tilde{\gamma}(x,p,t)$ is going to be chosen in a moment. By

$$e^{-iP_{\mathrm{f}}\cdot x} W(g) = W(e^{-ik\cdot x} g) e^{-iP_{\mathrm{f}}\cdot x}, \tag{2.29}$$

cf. (2.13), (2.14), we get

$$\Psi_t(x) = (2\pi)^{-3/2} \int d^3p\, h(p) e^{2i\tilde{\gamma}} e^{i(p\cdot x - E_p t)} e^{-iP_{\mathrm{f}}\cdot x} W(e^{i(k\cdot x - \omega t)} f_p) W(-f_p)\phi_p. \tag{2.30}$$

Now, we choose the phase by comparing (2.30) with (2.21). We recall (2.21)

$$e^{-itH} W(g) W(f)\Omega = e^{-itE_0} e^{2i\mathrm{Im}\langle f,(1-e^{-it\omega})g\rangle} W(e^{-it\omega} g) W(f)\Omega, \tag{2.31}$$

and make substitutions $g \to e^{ik\cdot x} f_p$, $f \to -f_p$. This suggests

$$\tilde{\gamma} = \mathrm{Im}\langle f_p, -(1 - e^{-i(\omega t - k\cdot x)}) f_p\rangle = \mathrm{Im}\langle f_p, e^{-i(\omega t - k\cdot x)} f_p\rangle \\ = -\langle f_p, \sin(\omega t - k\cdot x) f_p\rangle. \tag{2.32}$$

Finally, we merge the Weyl operators in (2.30),

$$W(e^{-i(\omega t - k\cdot x)} f_p) W(-f_p) = e^{-i\tilde{\gamma}} W(-(1 - e^{-i(\omega t - k\cdot x)}) f_p). \tag{2.33}$$

We note that the argument of the last Weyl operator lies in $L^2(\mathbb{R}^3_k)$ for each x, and we obtain from (2.30)

$$\Psi_t(x) = (2\pi)^{-3/2} \int d^3p\, h(p) e^{i\tilde{\gamma}} e^{i(p\cdot x - E_p t)} e^{-iP_{\mathrm{f}}\cdot x} W(-(1 - e^{-i(\omega t - k\cdot x)}) f_p) \phi_p. \tag{2.34}$$

The result is similar but not identical to (1.8): The phases (1.9) and (2.32) have opposite signs. The first phase is so chosen to make identity (1.12) possible. For comparison, had the same choice been made for the (exactly solvable) van Hove model, then the approximant to $e^{-itH}\Omega$ would not be $e^{-itH}\Omega$ itself, in the form of the r.h.s. of (2.19), but $e^{-2i\tilde{\gamma}} e^{-itH}\Omega$, leading to $\psi_t = e^{-2i\tilde{\gamma}}\Omega$ as a counterpart to (1.8). By $\tilde{\gamma} \to 0$ $(t \to \infty)$, we still have $\psi_t \to \Omega$, but $\partial_t \psi_t = -2i(\partial_t \tilde{\gamma})\psi_t = i\tilde{\gamma}_{\mathrm{int}} \Psi_t$ with $\tilde{\gamma}_{\mathrm{int}} = 2\langle f_p, \omega \cos(\omega t) f_p\rangle$, in line with (1.10).

3. Preliminaries

Recall that $\{H_p\}_{p\in\mathbb{R}^3}$ are the fibre Hamiltonians (1.5) and let $\{H_{p,\sigma}\}_{p\in\mathbb{R}^3}$ be their counterparts at an infrared cut-off $0 < \sigma \le \kappa$. This means that the form factor v, defined below (1.3), is replaced with v^σ given by

$$v^\sigma(k) := \lambda \frac{\chi_{[\sigma,\kappa)}(k)}{\sqrt{2|k|}}. \tag{3.1}$$

Here, $\chi_{[\sigma,\kappa)}(k) = \mathbf{1}_{\mathcal{B}'_\sigma}(k)\chi_\kappa(k)$, $\mathcal{B}'_\sigma$ is the complement of the ball of radius σ and $\mathbf{1}_\Delta$ is the characteristic function of a set Δ. We remark that $\{H_{p,\sigma}\}_{p\in\mathbb{R}^3}$ act on a dense domain in $\mathcal{F}$, that is, no infrared cut-off is introduced on the Fock space. We will work in the range of parameters for which the technical results of [15–17] hold. That is,

$$|\lambda| \le \lambda_0, \quad \sigma \in (0, \kappa_{\lambda_0}], \quad p \in S := \{\, p' \in \mathbb{R}^3 \,|\, |p'| < 1/3\}, \tag{3.2}$$

where λ_0 is sufficiently small and $0 < \kappa_{\lambda_0} \le \kappa$. As the fibre Hamiltonians $H_p, H_{p,\sigma}$ are bounded from below, we can define

$$E_p := \inf \boldsymbol{\sigma}(H_p), \quad E_{p,\sigma} := \inf \boldsymbol{\sigma}(H_{p,\sigma}), \tag{3.3}$$

where $\boldsymbol{\sigma}$ denotes the spectrum. (Occasionally, we will write $E_p^{(\lambda)}$, $E_{p,\sigma}^{(\lambda)}$ etc. if the dependence on λ will play a role.) E_p enters our definition of the infraparticle state (1.10), and our analysis relies on the following result:

Lemma 3.1 [1]. *The function $S \times \mathcal{B}_{\lambda_0} \ni (p, \lambda) \mapsto E_p^{(\lambda)}$ is real-analytic and non-constant. It satisfies $|\nabla_p E_p^{(\lambda)}| \le 1/2$ and its Hessian matrix in the p-variable is bounded from below for $p \in S$ by a positive constant, uniformly in λ.*

We recall that the modified Hamiltonians H_p^{w} are obtained from H_p by the Bogolubov transformation (1.7) and their ground states are denoted ϕ_p. Similarly, the modified Hamiltonians $H_{p,\sigma}^{\mathrm{w}}$ are obtained from $H_{p,\sigma}$ by the transformation

$$a^{(*)}(k) \mapsto a^{(*)}(k) - f_{p,\sigma}(k), \quad f_{p,\sigma}(k) := \lambda \frac{\chi_{[\sigma,\kappa)}(k)}{\sqrt{2|k|}} \frac{1}{|k|(1 - e_k \cdot \nabla E_{p,\sigma})} \tag{3.4}$$

and their ground states are denoted $\phi_{p,\sigma}$. Both ϕ_p and $\phi_{p,\sigma}$ are in the domain of any power of H_{f} (cf. Lemma C.3) and in addition $\phi_{p,\sigma}$ are in the domain of any power of the number operator $N := \mathrm{d}\Gamma(1)$, cf. Lemma C.2. For a choice of the phases of $\phi_p, \phi_{p,\sigma}$ as in [15, Definition 5.2], the following estimate holds

$$\|(H_{\mathrm{f}})^{\ell}(\phi_p - \phi_{p,\sigma})\|_{\mathcal{F}} \leq c\sigma^{1/5}, \quad p \in S, \quad \ell \in \mathbb{N}_0, \tag{3.5}$$

provided that $\lambda_0 > 0$ is readjusted for each ℓ. It is well-known for $\ell = 0$ [32], [17, Corollary 5.6 (a)] and for $\ell \in \mathbb{N}$, it is shown in Appendix C. We will also need the following lemma:

Lemma 3.2. *Fix $\ell_1, \ell_2 \in \mathbb{N}_0$. Then, there exists $\tilde{\lambda}_0 > 0$ and a positive function $[-\tilde{\lambda}_0, \tilde{\lambda}_0] \ni \lambda_0 \mapsto \delta_{\lambda_0}$ s.t. $\lim_{\lambda_0 \to 0} \delta_{\lambda_0} = 0$ with the following property: For any fixed $\lambda_0 \in [-\tilde{\lambda}_0, \tilde{\lambda}_0]$ and all $\sigma \in (0, \kappa_{\lambda_0}]$,*

$$\|H_{\mathrm{f}}^{\ell_1} N^{\ell_2} \partial_p^{\alpha} \phi_{p,\sigma}\|_{\mathcal{F}} \leq \frac{c}{\sigma^{\delta_{\lambda_0}}} \quad \textit{for} \quad |\alpha| = 0, 1, 2. \tag{3.6}$$

The constant c is independent of p, σ, λ within the restrictions (3.2) but may depend on ℓ_1, ℓ_2.

In Appendix B, we show how to extract the proof of Lemma 3.2 from [15,16]. We remark that Lemma 3.1, bound (3.5), and Lemma 3.2 are the technical basis for our discussion in the next section.

We remark that a possible dependence of the constants c in (3.6) and (3.5) on ℓ, ℓ_1, ℓ_2 does not cause complications, because it suffices to consider $\ell, \ell_1, \ell_2 \leq L$ for some finite L fixed throughout the proof. This can be seen from the discussion below (4.55) and from the proof of Lemma 4.4.

Notation. As we will discuss only outgoing scattering states, we set $t \geq 1$. We denote by c numerical constants which may change from line to line. These constants are independent of σ, p, λ, t, x within the assumed restrictions, but may depend on h, λ_0, ε_0, where ε_0 was defined below (1.3). The functions denoted $\lambda_0 \mapsto \delta_{\lambda_0}$ are positive and satisfy $\lim_{\lambda_0 \to 0} \delta_{\lambda_0} = 0$. They are independent of σ, p within the assumed restrictions but may depend on ε_0. These functions may change from line to line.

4. Infraparticle States

The goal of this section is to provide a proof of Theorem 1.1. Our main tool will be the stationary phase method. The estimates suitable for our purposes are stated in the following lemma, which is proven in Appendix D.

Lemma 4.1. *Let $p \mapsto g(p) \in \mathcal{F}$ be weakly infinitely differentiable on some dense domain and compactly supported in S. Let c_0 be s.t. $|\nabla E_p| < c_0 < 1$ for $p \in \operatorname{supp} g$. Then, for any $0 \le \varepsilon \le 1/2$,*

$$\left(\int_{|x|/t \le c_0} d^3x \left\| \int d^3p\, e^{i(p\cdot x - E_p t)} g(p) \right\|_{\mathcal{F}}^2 \right)^{1/2} \le c \sum_{|\alpha|\le 2} \sup_{p,|x|\le c_0 t} \|\partial_p^\alpha g(p)\|_{\mathcal{F}}, \tag{4.1}$$

$$\left(\int_{|x|/t \ge c_0} d^3x \left\| \int d^3p\, e^{i(p\cdot x - E_p t)} g(p) \right\|_{\mathcal{F}}^2 \right)^{1/2} \le ct^{-1/2+\varepsilon} \sum_{|\alpha|\le 2} \sup_{p,|x|\ge c_0 t} \left(\frac{1}{(1+t+|x|)^{\varepsilon}} \|\partial_p^\alpha g(p)\|_{\mathcal{F}} \right). \tag{4.2}$$

The function g above may depend on (x,t).

Lemma 4.1 immediately gives the following estimate

$$\left(\int d^3x \left\| \int d^3p \;\; e^{i(p\cdot x - E_p t)} g(p) \right\|_{\mathcal{F}}^2 \right)^{1/2} \le ct^{1/2} \sum_{|\alpha|\le 2} \sup_{p,x} \left(\frac{1}{(1+|x|)^{1/2}} \|\partial_p^\alpha g(p)\|_{\mathcal{F}} \right), \tag{4.3}$$

which will be useful for analysing vectors (1.8) at finite t. Like in Lemma 4.1, the function g may depend on (x,t). We note that we cannot apply (4.3) or Lemma 4.1 directly to the infraparticle vector (1.8), since differentiability of $p \mapsto \phi_p$ is out of control. In the course of our discussion, we will approximate ϕ_p with $\phi_{p,\sigma}$ in a suitable manner, which will introduce an x-dependence of g.

As a first step of our analysis, we compute and estimate derivatives of $e^{i\gamma(p,x,t)}$ w.r.t. p, x, t. The following is a result of a straightforward computation:

$$\partial_t e^{i\gamma(p,x,t)} = e^{i\gamma(p,x,t)} i \int d^3k\, f_p(k)^2 |k| \cos(|k|t - k\cdot x), \tag{4.4}$$

$$\partial_{x_i} e^{i\gamma(p,x,t)} = -e^{i\gamma(p,x,t)} i \int d^3k\, f_p(k)^2 k_i \cos(|k|t - k\cdot x), \tag{4.5}$$

$$\partial_{x_j}\partial_{x_i} e^{i\gamma(p,x,t)} = -e^{i\gamma(p,x,t)} \int d^3k\, f_p(k)^2 k_j \cos(|k|t - k\cdot x) \times \int d^3k f_p(k)^2 k_i \cos(|k|t - k\cdot x) \tag{4.6}$$

$$- e^{i\gamma(p,x,t)} i \int d^3k f_p(k)^2 k_i k_j \sin(|k|t - k\cdot x). \tag{4.7}$$

Now, we estimate the above expressions together with their derivatives w.r.t. p.

Lemma 4.2. *The following bounds hold*

$$|\partial_p^\alpha \partial_t^\ell e^{i\gamma(p,x,t)}| \le c(1+\log(1+t+|x|))^2, \tag{4.8}$$

$$|\partial_p^\alpha \partial_x^\beta e^{i\gamma(p,x,t)}| \le c(1+\log(1+t+|x|))^2, \tag{4.9}$$

for $|\alpha|, |\beta| \le 2$, $\ell \le 1$.

Proof. We see from (4.4)–(4.7) that the derivatives w.r.t. x, t produce expressions which are uniformly bounded in x, t due to the additional factors $k_i, |k|$, which regularize the singularity of f_p^2 at $|k| = 0$. Hence, it suffices to study the expression

$$\begin{aligned}\partial_{p_j}\partial_{p_i} e^{i\gamma(p,x,t)} &= \partial_{p_j}\big(e^{i\gamma(p,x,t)} i\partial_{p_i}\gamma(p,x,t)\big)\\ &= e^{i\gamma(p,x,t)}\big(i\partial_{p_j}\gamma(p,x,t)\big)\big(i\partial_{p_i}\gamma(p,x,t)\big)\\ &\quad + e^{i\gamma(p,x,t)} i\partial_{p_j}\partial_{p_i}\gamma(p,x,t).\end{aligned} \tag{4.10}$$

Making use of (E.4), we obtain

$$|\partial_{p_j}\partial_{p_i} e^{i\gamma(p,x,t)}| \le c(1+\log(1+t+|x|))^2, \tag{4.11}$$

where the dependence of c on parameters is as discussed in Sect. 3. This concludes the proof. □

As a next step of our discussion, we compute derivatives of the following auxiliary vector

$$\hat{g}_{(t,x)}(p) := W\big(f_p m(t,x)\big)\phi_p, \quad m(t,x) := u(t,x) - 1, \quad u(t,x) := e^{-i|k|t + ik\cdot x} \tag{4.12}$$

w.r.t. (t, x) up to the second order. We will abbreviate $m := m(t, x), u := u(t, x)$.

Lemma 4.3. *The function* $(t, x) \mapsto \hat{g}_{(t,x)}(p)$ *is infinitely often partially differentiable in the norm of* $\mathcal{F}$ *and the following formulas hold*

$$\partial_t \hat{g}_{(t,x)}(p) = W\big(f_p m\big) i\big(\Phi(f_p\partial_t m) + \mathrm{Im}\langle f_p m, f_p \partial_t m\rangle\big)\phi_p, \tag{4.13}$$

$$\begin{aligned}\partial_t^2 \hat{g}_{(t,x)}(p) &= -W(f_p m)\big(\Phi(f_p\partial_t m) + \mathrm{Im}\langle f_p m, f_p\partial_t m\rangle\big)^2\phi_p\\ &\quad + W(f_p m) i\big(\Phi(f_p\partial_t^2 m) + \mathrm{Im}\langle f_p m, f_p\partial_t^2 m\rangle\big)\phi_p\end{aligned} \tag{4.14}$$

$$\partial_{x_i}\hat{g}_{(t,x)}(p) = W\big(f_p m\big) i\big(\Phi(f_p\partial_{x_i} m) + \mathrm{Im}\langle f_p m, f_p\partial_{x_i} m\rangle\big)\phi_p, \tag{4.15}$$

$$\begin{aligned}\partial_{x_j}\partial_{x_i}\hat{g}_{(t,x)}(p) &= W\big(f_p m\big) i\big(\Phi(f_p\partial_{x_j} m) + \mathrm{Im}\langle f_p m, f_p\partial_{x_j} m\rangle\big)\\ &\quad \times i\big(\Phi(f_p\partial_{x_i} m) + \mathrm{Im}\langle f_p m, f_p\partial_{x_i} m\rangle\big)\phi_p\\ &\quad + W\big(f_p m\big) i\big(\Phi(f_p\partial_{x_j}\partial_{x_i} m) + \mathrm{Im}\langle f_p\partial_{x_j} m, f_p\partial_{x_i} m\rangle\\ &\quad + \mathrm{Im}\langle f_p m, f_p\partial_{x_j}\partial_{x_i} m\rangle\big)\phi_p,\end{aligned} \tag{4.16}$$

where $\Phi(F) := a^*(-iF) + a(-iF)$, $F \in L^2(\mathbb{R}^3_k)$, *as defined in* (1.2).

Proof. We note that, by Lemma C.3, ϕ_p belongs to $D(H_\mathrm{f}^\ell)$ for any $\ell \in \mathbb{N}$. We observe that for any fixed (t, x) the function $f_p m(t, x) \in L^2_\omega(\mathbb{R}^3_k)$, and it is infinitely often differentiable in (t,x) in the norm of $L^2_\omega(\mathbb{R}^3_k)$ (see Appendix A). For the first derivative w.r.t. x_i, this follows from

$$m(t,x+(\Delta x)_i e_i) = m(t,x) + (\Delta x)_i (\partial_{x_i} m)(t,x) + (\Delta x)_i^2 \int_0^1 ds\,(1-s)\,(\partial_{x_i}^2 m)(t,x+s(\Delta x)_i e_i), \tag{4.17}$$

and the fact that $|k|^{-1}\partial^\ell_{x_i} m(t,x)$ is bounded in k for any $\ell \in \mathbb{N}_0$. For higher derivatives, we simply replace m with $\partial^\ell_{x_i} m$ in (4.17). The arguments regarding the derivatives w.r.t. t are analogous. Thus, we can compute the derivatives using Lemma A.2, which gives the formulas from the statement of the lemma. □

Now, we analyse the regularized variants of the vectors from (4.12)

$$\hat{g}^\sigma_{(t,x)}(p) := W\big(f_p m(t,x)\big)\phi_{p,\sigma}. \tag{4.18}$$

We note the following fact:

Lemma 4.4. *There hold the bounds*

$$\|\partial^\alpha_p \partial^\ell_t \hat{g}^\sigma_{(t,x)}(p)\|_{\mathcal{F}} \le c\frac{(1+\log(1+|x|+t))^3}{\sigma^{\delta_{\lambda_0}}}, \tag{4.19}$$

$$\|\partial^\alpha_p \partial^\beta_x \hat{g}^\sigma_{(t,x)}(p)\|_{\mathcal{F}} \le c\frac{(1+\log(1+|x|+t))^3}{\sigma^{\delta_{\lambda_0}}}, \tag{4.20}$$

for $\ell, |\alpha|, |\beta| \le 2$ and $\sigma \in (0,\kappa_{\lambda_0}]$. The x and t derivatives exist in the norm of $\mathcal{F}$. The derivatives w.r.t. p exist in the weak sense on the domain of finite particle vectors with compactly supported wave functions (cf. [34, p. 208]*). The bound* (4.20) *still holds if ∂^β_x is replaced with $H_{\mathrm{f}}, P_{\mathrm{f},i}, P^2_{\mathrm{f},i}$ or $\partial_{x_i} P_{\mathrm{f},i}$.*

Proof. We consider only (4.20) for $|\alpha| = 2, |\beta| = 2$ as the remaining cases are analogous and simpler. To handle the resulting expressions, it is convenient to define, for $s \mapsto F_s$ as in Lemma A.2,

$$\widetilde{\Phi}_s(F) := \Phi(\partial_s F_s) + \mathrm{Im}\langle F_s, \partial_s F_s\rangle. \tag{4.21}$$

Using this notation and recalling (4.16), we can write

$$\partial_{x_j}\partial_{x_i}\hat{g}^\sigma_{(t,x)}(p) = W\big(f_p m\big)\Big\{ i\widetilde{\Phi}_{x_j}(f_p m) i\widetilde{\Phi}_{x_i}(f_p m) + i\partial_{x_j}\widetilde{\Phi}_{x_i}(f_p m)\Big\}\phi_{p,\sigma} = W\big(f_p m\big)\mathrm{Pol}_{x_i,x_j}(f_p m)\phi_{p,\sigma}, \tag{4.22}$$

where in the last step we denoted the expression in curly brackets by the symbol $\mathrm{Pol}_{x_i,x_j}(f_p m)$ to further abbreviate the notation. Now, we compute the first derivative w.r.t. momentum. We recall that these derivatives must only exist weakly on the domain of finite particle vectors, i.e. after taking a scalar product with such vectors. This will control the unbounded operators acting on $\phi_{p,\sigma}$ below and, in particular, allow us to differentiate $p \mapsto \phi_{p,\sigma}$ in (4.25) below. In this sense, we compute:

$$\partial_{p_{\hat{i}}}\partial_{x_j}\partial_{x_i}\hat{g}^\sigma_{(t,x)}(p) = W\big(f_p m\big) i\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\mathrm{Pol}_{x_i,x_j}(f_p m)\phi_{p,\sigma} \tag{4.23}$$

$$+ W\big(f_p m\big)\partial_{p_{\hat{i}}}\big(\mathrm{Pol}_{x_i,x_j}(f_p m)\big)\phi_{p,\sigma} \tag{4.24}$$

$$+W\big(f_p m\big)\mathrm{Pol}_{x_i,x_j}(f_p m)\partial_{p_{\hat{i}}}\phi_{p,\sigma}. \tag{4.25}$$

Now, we compute the respective contributions to $\partial_{p_{\hat{j}}}\partial_{p_{\hat{i}}}\partial_{x_j}\partial_{x_i}\hat{g}^{\sigma}_{(t,x)}(p)$: (4.23) gives

$$\begin{aligned}
&\partial_{p_{\hat{j}}}\big(W\big(f_p m\big)i\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\mathrm{Pol}_{x_i,x_j}(f_p m)\phi_{p,\sigma}\big)\\
&\quad = W\big(f_p m\big)i\widetilde{\Phi}_{p_{\hat{j}}}(f_p m)i\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\mathrm{Pol}_{x_i,x_j}(f_p m)\phi_{p,\sigma} &(4.26)\\
&\qquad + W\big(f_p m\big)i\partial_{p_{\hat{j}}}(\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\mathrm{Pol}_{x_i,x_j}(f_p m))\phi_{p,\sigma} &(4.27)\\
&\qquad + W\big(f_p m\big)i\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\mathrm{Pol}_{x_i,x_j}(f_p m)\partial_{p_{\hat{j}}}\phi_{p,\sigma}. &(4.28)
\end{aligned}$$

From (4.24), we obtain

$$\begin{aligned}
\partial_{p_{\hat{j}}}\big(W\big(f_p m\big)\partial_{p_{\hat{i}}}\mathrm{Pol}_{x_i,x_j}(f_p m)\phi_{p,\sigma}\big) &= W\big(f_p m\big)i\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\partial_{p_{\hat{i}}}\mathrm{Pol}_{x_i,x_j}(f_p m)\phi_{p,\sigma}\\
&\quad + W\big(f_p m\big)\partial_{p_{\hat{j}}}\partial_{p_{\hat{i}}}\big(\mathrm{Pol}_{x_i,x_j}(f_p m)\big)\phi_{p,\sigma}\\
&\quad + W\big(f_p m\big)\partial_{p_{\hat{i}}}\mathrm{Pol}_{x_i,x_j}(f_p m)\partial_{p_{\hat{j}}}\phi_{p,\sigma}.
\end{aligned} \tag{4.29}$$

From (4.25), we get

$$\begin{aligned}
\partial_{p_{\hat{j}}}\big(W\big(f_p m\big)\mathrm{Pol}_{x_i,x_j}(f_p m)\partial_{p_{\hat{i}}}\phi_{p,\sigma}\big) &= W\big(f_p m\big)i\widetilde{\Phi}_{p_{\hat{i}}}(f_p m)\ \mathrm{Pol}_{x_i,x_j}(f_p m)\partial_{p_{\hat{i}}}\phi_{p,\sigma}\\
&\quad + W\big(f_p m\big)\partial_{p_{\hat{j}}}\big(\mathrm{Pol}_{x_i,x_j}(f_p m)\big)\partial_{p_{\hat{i}}}\phi_{p,\sigma}\\
&\quad + W\big(f_p m\big)\mathrm{Pol}_{x_i,x_j}(f_p m)\partial_{p_{\hat{i}}}\partial_{p_{\hat{j}}}\phi_{p,\sigma}.
\end{aligned} \tag{4.30}$$

To estimate these expressions, we recall from Lemma 3.2 that $\partial_p^{\alpha}\phi_{p,\sigma}$ are in the domain of any power of N and $\|N^{\ell}\partial_p^{\alpha}\phi_{p,\sigma}\|_{\mathcal{F}} \leq c_{\ell}\sigma^{-\delta_{\lambda_0}}$. Thus making use of the number bounds (A.2), we have

$$\|\partial_{p_{\hat{j}}}\partial_{p_{\hat{i}}}\partial_{x_j}\partial_{x_i}\hat{g}^{\sigma}_{(t,x)}(p)\|_{\mathcal{F}} \leq \mathrm{Pol}(\|f_p m\|_2, \|f_p\partial_{x_i}m\|_2, \|f_p\partial_{x_j}\partial_{x_i}m\|_2)\sigma^{-\delta_{\lambda_0}}. \tag{4.31}$$

Here Pol is a certain polynomial in the specified norms, which also includes $\|\partial_p^{\alpha}f_p m\|_2$. We recall, however, that $f_p(k) := \lambda\frac{\chi_{\kappa}(k)}{\sqrt{2|k|}}\frac{1}{|k|(1-e_k\cdot\nabla E_p)}$, thus derivatives of f_p w.r.t. p only change the behaviour of this function in the angular variable e_k but not in the $|k|$-variable. As our estimates are insensitive to the angular behaviour, we omitted these derivatives in the notation in (4.31). We have

$$\|f_p m\|_2 \leq c|\lambda|(1+\log(1+|x|+t))^{1/2},\ \|f_p\partial_{x_i}m\|_2 \leq c|\lambda|,\ \ \|f_p\partial_{x_i}\partial_{x_j}m\|_2 \leq c|\lambda|, \tag{4.32}$$

where the first inequality follows from Lemma E.3 and the last two follow directly from the definition of f_p in (1.7), since $m(t,x) := e^{-i|k|t+ik\cdot x} - 1$. By inspection, we see that Pol is at most of the sixth order in $\|f_p m\|_2$ (cf. (4.26)), which concludes the proof of estimates (4.19), (4.20).

As for the last statement of the lemma, the case of $H_{\mathrm{f}}, P_{\mathrm{f},i}, P_{\mathrm{f},i}^2$ is covered by the fact that the derivatives w.r.t. p should exist only weakly on vectors which belong to domains of these operators. After computing these derivatives, one pulls $H_{\mathrm{f}}, P_{\mathrm{f},i}, P_{\mathrm{f},i}^2$ to the right through the Weyl operator according to

$$H_{\mathrm{f}} W(f_p m) = W(f_p m)(H_{\mathrm{f}} + a^*(|k| f_p m) + a(|k| f_p m) + \||k|^{1/2} f_p m\|_2^2), \tag{4.33}$$

for which we refer to [12, eq. (3.20) and Prop. 3.11]. Next one applies Lemmas 3.2 and A.1. The case of $P_{\mathrm{f},i}\partial_{x_i}$ requires more consideration as the derivative w.r.t. x_i should exist in the norm of $\mathcal{F}$. To check that $P_{\mathrm{f},i} W(f_p m)\phi_{p,\sigma}$ is partially differentiable w.r.t. x in the norm of $\mathcal{F}$, we write, analogously to (4.33),

$$P_{\mathrm{f},i} W(f_p m)\phi_{p,\sigma} = W(f_p m)(P_{\mathrm{f},i} + a^*(k_i f_p m) + a(k_i f_p m) + \langle f_p, k_i f_p\rangle)\phi_{p,\sigma} \tag{4.34}$$

and refer to Lemma A.2. By a computation, we obtain

$$\partial_{x_i} P_{\mathrm{f},i} W(f_p m)\phi_{p,\sigma} = P_{\mathrm{f},i}\partial_{x_i}\big(W(f_p m)\big)\phi_{p,\sigma}, \tag{4.35}$$

where $\partial_{x_i}\big(W(f_p m)\big)$ is the explicit formula from Lemma A.2, and then proceed as in the discussion of H_{f}, $P_{\mathrm{f},i}$, $P_{\mathrm{f},i}^2$ above. □

Now, we are ready to analyse the infraparticle vector (1.8).

Lemma 4.5. *There is such $\lambda_0 > 0$ that for any $|\lambda| \in (0, \lambda_0]$ and $t \in \mathbb{R}$, the integral*[3]

$$\Psi_t(x) := \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m(t,x)\big)\phi_p \tag{4.36}$$

has the following properties:

(a) $\Psi_t \in L^2(\mathbb{R}^3_x; \mathcal{F})$.

(b) Ψ_t *is differentiable in t in the norm of $L^2(\mathbb{R}^3_x; \mathcal{F})$ and*

$$\begin{aligned}\partial_t \Psi_t(x) = &\int d^3p\, e^{i(p\cdot x - E_p t)}\big(-iE_p + i\partial_t\gamma(p,x,t) + i\mathrm{Im}\langle f_p m, f_p\partial_t m\rangle\big)\\ &\times\, e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big)\phi_p\\ &+ \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big)(a^*(f_p\partial_t m) - a(f_p\partial_t m))\phi_p.\end{aligned} \tag{4.37}$$

Proof. As for (a), to prove that $x \mapsto \Psi_t(x)$ is square integrable, we intend to apply Lemma 4.1. However, we lack information about the differentiability of $p \mapsto \phi_p$. To circumvent this problem, we introduce an x-dependent cut-off $\sigma_x := \kappa_{\lambda_0}/(1+|x|)^M$, where M is sufficiently large but fixed. We insert into (4.36)

$$\phi_p = (\phi_p - \phi_{p,\sigma_x}) + \phi_{p,\sigma_x} \tag{4.38}$$

and obtain

$$\Psi_t(x) = \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big)(\phi_p - \phi_{p,\sigma_x}) + \Psi_t^{\sigma_x}(x). \tag{4.39}$$

[3] This $\Psi_t(x)$ differs from the one defined in (2.30) by the factor $(2\pi)^{-3/2}e^{-iP_{\mathrm{f}}\cdot x}$.

Here, $\Psi_t^{\sigma_x}(x)$ is given by (4.36) with ϕ_p replaced with ϕ_{p,σ_x}. Concerning the first term on the r.h.s. of (4.39), we have by (3.5)

$$\left\| \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big)(\phi_p - \phi_{p,\sigma_x}) \right\|_{\mathcal{F}} \leq \frac{c(\kappa_{\lambda_0})^{1/5}}{(1+|x|)^{M/5}}. \tag{4.40}$$

Thus, this term is manifestly in $L^2(\mathbb{R}^3_x;\mathcal{F})$ for $2M/5 > 3$. As for the last term on the r.h.s. of (4.39), estimate (4.3) gives

$$\|\Psi_t^{\sigma_x}\|_{\mathcal{H}} \leq ct^{1/2} \sum_{|\alpha|\leq 2} \sup_{p,x} \left(\frac{1}{(1+|x|)^{1/2}} \|\partial_p^\alpha (e^{i\gamma(p,x,t)} \hat{g}^{\sigma_x}_{(t,x)}(p))\|_{\mathcal{F}} \right). \tag{4.41}$$

The expression on the r.h.s. above is finite for any fixed t by Lemmas 4.2, 4.4, provided δ_{λ_0} of Lemma 4.4 satisfies $M\delta_{\lambda_0} < 1/2$. This concludes the proof of part (a).

Part (b) is a straightforward computation, provided we can show differentiability in the norm of $L^2(\mathbb{R}^3_x;\mathcal{F})$. To this end, we use the Taylor theorem (cf. formula (4.17))

$$\begin{aligned}
&\int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) \\
&\qquad \times \left(\frac{W\big(f_p m(t+\Delta t, x)\big) - W\big(f_p m(t,x)\big)}{\Delta t} - \partial_t W\big(f_p m(t,x)\big) \right) \phi_p \\
&\quad = \Delta t \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) \\
&\qquad \times \int_0^1 ds\, (1-s) \big\{ \partial_\tau^2 W\big(f_p m(\tau,x)\big)|_{\tau = t + s\Delta t} \big\} \phi_p.
\end{aligned} \tag{4.42}$$

Now, we obtain from Lemma A.3 that (4.42) tends to zero with $\Delta t \to 0$ in the norm of $L^2(\mathbb{R}^3_x;\mathcal{F})$. Differentiability in the norm of $L^2(\mathbb{R}^3_x;\mathcal{F})$ of other ingredients of (4.36) can be shown by analogous and simpler arguments. Now, formula (4.37) follows by an application of Lemma A.2. □

Lemma 4.6. *The vectors $\Psi_t \in L^2(\mathbb{R}^3_x;\mathcal{F})$, $t \in \mathbb{R}$, defined in (4.36) have the following properties:*

(a) Ψ_t is in the domain of $P_{\mathrm{f},i}, P_{\mathrm{f},i}^2$, H_f and the following formula holds

$$\begin{aligned}
(H_\mathrm{f}\Psi_t)(x) = &\int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big) \\
&\times \big(H_\mathrm{f}^\mathrm{w} + a^*(|k| f_p u) + a(|k| f_p u) \\
&+ \langle f_p, |k| f_p\rangle - 2\mathrm{Re}\langle f_p, |k| f_p u\rangle \big) \phi_p.
\end{aligned} \tag{4.43}$$

(b) Ψ_t is in the domain of $-i\partial_{x_i}$, $(-i\partial_{x_i})^2$, $-i\partial_{x_i} P_{\mathrm{f},i}$ and the following formula holds

$$(-i\partial_{x_i} - P_{\mathrm{f},i})^2 \Psi_t(x) = \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big) \big(p_i - P^\mathrm{w}_{\mathrm{f},i}\big)^2 \phi_p. \tag{4.44}$$

(c) Ψ_t *is in the domain of* $(a^*(v)+a(v))$ *and the following formula holds*

$$(a^*(v)+a(v))\Psi_t(x) = \int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)W\big(f_pm\big) \times((a^*(v)+a(v))^{\mathrm{w}} + 2\mathrm{Re}\langle f_pu,v\rangle)\phi_p, \quad (4.45)$$

where $(a^*(v)+a(v))^{\mathrm{w}} = a^*(v)+a(v)-2\langle f_p,v\rangle$ *in accordance with* (1.7)*.*

Proof. We start with some computations on $\mathcal{F}$ which are justified by Lemma A.2. Since $W\big(f_pm\big)\phi_p$ is in the domain of H_{f}, we can write for any fixed t

$$\begin{aligned} H_{\mathrm{f}}\Psi_t(x) &= \int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)H_{\mathrm{f}}W\big(f_pm\big)\phi_p \\ &= \int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)W\big(f_pm\big) \\ &\quad\times\big(H_{\mathrm{f}}+a^*(|k|f_pm)+a(|k|f_pm)+\||k|^{1/2}f_pm\|_2^2\big)\phi_p \\ &= \int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)W\big(f_pm\big) \\ &\quad\times\big(H_{\mathrm{f}}^{\mathrm{w}}+a^*(|k|f_pu)+a(|k|f_pu) \\ &\qquad+\langle f_p,|k|f_p\rangle - 2\mathrm{Re}\langle f_p,|k|f_pu\rangle\big)\phi_p, \end{aligned} \quad (4.46)$$

where we made use of $H_{\mathrm{f}}^{\mathrm{w}} = H_{\mathrm{f}} - a^*(|k|f_p) - a(|k|f_p) + \|\,|k|^{1/2}f_p\|_2^2$ (cf. formula (4.33)) and

$$-\||k|^{1/2}f_p\|_2^2 + \||k|^{1/2}f_pm\|_2^2 = \langle f_p,|k|f_p\rangle - 2\mathrm{Re}\langle f_p,|k|f_pu\rangle. \quad (4.47)$$

Analogously, we obtain for $\ell\in\{1,2\}$,

$$(P_{\mathrm{f},i})^\ell\Psi_t(x) = \int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)W\big(f_pm\big) \times\big(P_{\mathrm{f},i}^{\mathrm{w}}+a^*(k_if_pu)+a(k_if_pu)+\langle f_p,k_if_p\rangle - 2\mathrm{Re}\langle f_p,k_if_pu\rangle\big)^\ell\phi_p. \quad (4.48)$$

Furthermore, we can exchange $-i\partial_{x_i}$ with the p-integral defining Ψ_t. In fact, similarly as in (4.42), we write

$$\int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)\bigg(\frac{W\big(f_pm(t,x+(\Delta x_i)e_i)\big)-W\big(f_pm(t,x)\big)}{\Delta x_i} -\partial_{x_i}W\big(f_pm(t,x)\big)\bigg)\phi_p$$

$$= (\Delta x_i) \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p)$$
$$\times \int_0^1 ds\, (1-s) \big\{ \partial^2_{x'_i} W\big(f_p m(t,x')\big)|_{x'=x+s(\Delta x_i)e_i} \big\} \phi_p \tag{4.49}$$

and make use of Lemma A.3 to take the limit $\Delta x_i \to 0$. Thus, we can write

$$\begin{aligned} -i\partial_{x_i}\Psi_t(x) = &\int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) \\ &\times \big(p_i + \partial_{x_i}\gamma(p,x,t) + \mathrm{Im}\langle f_p m, f_p \partial_{x_i} m\rangle\big) W(f_p m)\phi_p \\ &+ \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) \\ &\times (a^*(k_i f_p u) + a(k_i f_p u))\phi_p. \end{aligned} \tag{4.50}$$

Combining the above computations, we also obtain

$$\begin{aligned} &-i\partial_{x_i} P_{\mathrm{f},i}\Psi_t(x) \\ &= \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) \big(p_i + \partial_{x_i}\gamma(p,x,t) + \mathrm{Im}\langle f_p m, f_p\partial_{x_i} m\rangle\big) \\ &\quad \times W\big(f_p m\big)\big(P_{\mathrm{f},i} + a^*(k_i f_p m) + a(k_i f_p m) + \langle f_p m, k_i f_p m\rangle\big)\phi_p \\ &\quad + \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big) \\ &\quad \times \big(P_{\mathrm{f},i} + a^*(k_i f_p m) + a(k_i f_p m) + \langle f_p m, k_i f_p m\rangle\big) \\ &\quad \times (a^*(k_i f_p u) + a(k_i f_p u))\phi_p = P_{\mathrm{f},i}(-i\partial_{x_i})\Psi_t(x). \end{aligned} \tag{4.51}$$

Thus, we get from (4.48) and (4.50)

$$\begin{aligned} (-i\partial_{x_i} - P_{\mathrm{f},i})\Psi_t(x) = &\int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big) \\ &\times \big(- P^{\mathrm{w}}_{\mathrm{f},i} + p_i + \partial_{x_i}\gamma(p,x,t) + \mathrm{Im}\langle f_p m, f_p \partial_{x_i} m\rangle - \langle f_p, k_i f_p\rangle \\ &\quad + 2\mathrm{Re}\langle f_p, k_i f_p u\rangle\big)\phi_p \\ = &\int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big)\big(p_i - P^{\mathrm{w}}_{\mathrm{f},i}\big)\phi_p, \end{aligned} \tag{4.52}$$

where we used that

$$\begin{aligned} &\mathrm{Im}\langle f_p m, f_p\partial_{x_i} m\rangle - \langle f_p, k_i f_p\rangle + 2\mathrm{Re}\langle f_p, k_i f_p u\rangle \\ &= \mathrm{Re}\langle f_p, k_i f_p u\rangle = -\partial_{x_i}\gamma(p,x,t). \end{aligned} \tag{4.53}$$

By iteration of (4.52), we get

$$(-i\partial_{x_i} - P_{\mathrm{f},i})^\ell \Psi_t(x) = \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W\big(f_p m\big)\big(p_i - P^{\mathrm{w}}_{\mathrm{f},i}\big)^\ell \phi_p. \tag{4.54}$$

We remark that at the level of formal computations, relations (4.52), (4.54) can also be obtained from (1.12). Finally, we obtain

$$(a^*(v)+a(v))\Psi_t(x) = \int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)W\big(f_pm\big)((a^*(v)+a(v))^{\mathrm{w}} + 2\mathrm{Re}\langle f_pu,v\rangle)\phi_p. \tag{4.55}$$

One can see, by analogous arguments as in the proof of Lemma 4.5 (a), that all vectors above are in $L^2(\mathbb{R}^3_x;\mathcal{F})$: First, we apply the shift (4.39) and estimate the term involving $\phi_p - \phi_{p,\sigma_x}$ with the help of the bound (3.5). The presence of H_{f}^{ℓ} in (3.5) allows us to control both $P_{\mathrm{f},i}$ and the creation and annihilation operators acting on $\phi_p - \phi_{p,\sigma_x}$ as for example in the case of (4.51). To the latter operators, we apply the energy bounds (A.3) and note that all the resulting $\|\cdot\|_\omega$-norms are finite. Next, we study the term proportional to ϕ_{p,σ_x} using Lemma 4.1. Staying with the case of (4.51), we can rewrite the relevant vector as $\{P_{\mathrm{f},i}(-i\partial_{x'_i}\Psi_t^{\sigma_x}(x'))|_{x'=x}\}_{x\in\mathbb{R}^3}$ and estimate the r.h.s. of (4.3) using Lemmas 4.2, 4.4. In particular, the last part of Lemma 4.4 plays a role here, since estimate (4.3) gives

$$\begin{aligned}&\|\{P_{\mathrm{f},i}(-i\partial_{x'_i}\Psi_t^{\sigma_x}(x'))|_{x'=x}\}_{x\in\mathbb{R}^3}\|_{\mathcal{H}}\\ &\quad\le ct^{1/2}\sum_{|\alpha|\le 2}\sup_{p,x}\left(\frac{1}{(1+|x|)^{1/2}}\|\partial_p^\alpha(\{\partial_{x'_i}e^{i\gamma(p,x',t)}P_{\mathrm{f},i}\hat{g}^{\sigma_x}_{(t,x')}(p)\}|_{x'=x})\|_{\mathcal{F}}\right).\end{aligned} \tag{4.56}$$

From (4.54), (4.51) we also obtain that $\{(-i\partial_{x_i})^2\Psi_t(x)\}_{x\in\mathbb{R}^3}$ is in $L^2(\mathbb{R}^3_x;\mathcal{F})$. This concludes the proof. □

Proof of Theorem 1.1. We recall that $\psi_t(x) := \frac{1}{(2\pi)^{3/2}}e^{iHt}e^{-iP_{\mathrm{f}}\cdot x}\Psi_t(x)$ as seen in (1.8). By Lemma 4.5, $t\mapsto\Psi_t$ is differentiable in the norm in $L^2(\mathbb{R}^3_x;\mathcal{F})$. Next, by applying the Stone theorem to e^{iHt}, we obtain the differentiability of $t\mapsto\psi_t$ in the norm of $L^2(\mathbb{R}^3_x;\mathcal{F})$, provided that the vector $\{e^{-iP_{\mathrm{f}}\cdot x}\Psi_t(x)\}_{x\in\mathbb{R}^3}\in L^2(\mathbb{R}^3_x;\mathcal{F})$ is in the domain of H. This is easily checked using Lemma 4.6. In particular, to verify that this vector is in the domain of $(-i\nabla_x)^2$, we apply the Stone theorem to $x\mapsto e^{-iP_{\mathrm{f}}\cdot x}$ and use that Ψ_t is in the domain of P_{f}^2. Now, we compute

$$\begin{aligned}\partial_t\psi_t(x) &= \frac{1}{(2\pi)^{3/2}}e^{iHt}iHe^{-iP_{\mathrm{f}}\cdot x}\Psi_t(x) + \frac{1}{(2\pi)^{3/2}}e^{iHt}e^{-iP_{\mathrm{f}}\cdot x}\partial_t\Psi_t(x)\\ &= \frac{1}{(2\pi)^{3/2}}e^{iHt}e^{-iP_{\mathrm{f}}\cdot x}i\Big(\frac{1}{2}(-i\nabla_x - P_{\mathrm{f}})^2\Psi_t(x) + H_{\mathrm{f}}\Psi_t(x)\\ &\qquad +(a^*(v)+a(v))\Psi_t(x) - i\partial_t\Psi_t(x)\Big)\\ &= \frac{1}{(2\pi)^{3/2}}e^{iHt}e^{-iP_{\mathrm{f}}\cdot x}\int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}i\gamma_{\mathrm{int}}(p,x,t)h(p)W(f_pm)\phi_p,\end{aligned} \tag{4.57}$$

where in the last step we made use of the formulas in Lemmas 4.5, 4.6, the fact that $H_p^{\mathrm{w}}\phi_p = E_p\phi_p$, and of the relations

$$\begin{aligned}&\langle f_p,|k|f_p\rangle - 2\mathrm{Re}\langle f_p,|k|f_pu\rangle + \partial_t\gamma(p,x,t) + \mathrm{Im}\langle f_pm, f_p\partial_tm\rangle = 0,\\ &2\mathrm{Re}\langle f_pu(t,x),v\rangle = \gamma_{\mathrm{int}}(p,x,t),\end{aligned} \tag{4.58}$$

where v was defined below (1.3). To show (1.11), we proceed similarly as in the proof of Lemma 4.5: We choose a (t,x)-dependent cut-off as follows: $\sigma_{(t,x)} = \kappa_{\lambda_0}/(1+t+|x|)^M$ where $M \in \mathbb{N}$ is fixed. We make a shift $\phi_p = (\phi_p - \phi_{p,\sigma_{(t,x)}}) + \phi_{p,\sigma_{(t,x)}}$ and insert it into the formula for the norm of $\partial_t \psi_t$:

$$\begin{aligned}\|\partial_t \psi_t\|_{\mathcal{H}} \leq \frac{1}{(2\pi)^{3/2}} \bigg\| \bigg\{ \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} i\gamma_{\text{int}}(p,x,t) h(p) \\ \times W(f_p(e^{-i|k|t + ik\cdot x} - 1))(\phi_p - \phi_{p,\sigma_{(t,x)}}) \bigg\}_{x\in\mathbb{R}^3} \bigg\|_{\mathcal{H}} \\ + \frac{1}{(2\pi)^{3/2}} \bigg\| \bigg\{ \int d^3p\, e^{i(p\cdot x - E_p t)} e^{i\gamma(p,x,t)} i\gamma_{\text{int}}(p,x,t) h(p) \\ \times W(f_p(e^{-i|k|t + ik\cdot x} - 1))\phi_{p,\sigma_{(t,x)}} \bigg\}_{x\in\mathbb{R}^3} \bigg\|_{\mathcal{H}}. \end{aligned} \tag{4.59}$$

We note that by (3.5) the term involving $(\phi_p - \phi_{p,\sigma_{(t,x)}})$ is integrable in t in the norm of $L^2(\mathbb{R}^3_x; \mathcal{F})$ for M sufficiently large. Our strategy to estimate the second term on the r.h.s. of (4.59) is to combine Lemmas 4.1, 4.7 and 3.2. In our case, g of Lemma 4.1 has the form

$$g_{(t,x)}(p) := e^{i\gamma(p,x,t)} i\gamma_{\text{int}}(p,x,t) h(p) W(f_p(e^{-i|k|t+ik\cdot x} - 1))\phi_{p,\sigma_{(t,x)}}. \tag{4.60}$$

We rewrite this expression as follows:

$$\begin{aligned} g_{(t,x)}(p) &= e^{i\gamma(p,x,t)} i\gamma_{\text{int}}(p,x,t) h(p) \hat{g}^{\sigma_{(t,x)}}_{(t,x)}(p), \\ \hat{g}^{\sigma}_{(t,x)}(p) &:= W(f_p(e^{-i|k|t+ik\cdot x} - 1))\phi_{p,\sigma}. \end{aligned} \tag{4.61}$$

First, we note that by Lemma 4.7, for c_0 as in Lemma 4.1,

$$|\partial_p^\alpha \gamma_{\text{int}}(p,x,t)| \leq |\lambda|^2 \frac{c_{\tilde{M}}}{t^{\tilde{M}}} \quad \text{for} \quad |x|/t \leq c_0 < 1, \tag{4.62}$$

$$|\partial_p^\alpha \gamma_{\text{int}}(p,x,t)| \leq |\lambda|^2 \frac{c}{t} |\log(t)| \quad \text{for} \quad |x|/t \geq c_0, \tag{4.63}$$

and $|\alpha| = 0, 1, 2$. Furthermore, we have by Lemma 4.2

$$|\partial_p^\alpha e^{i\gamma(p,x,t)}| \leq c(1 + \log(1+t+|x|))^2. \tag{4.64}$$

Given (4.62)–(4.64), Lemmas 4.1 and 4.4, for any $0 < \varepsilon < 1/2$, we can choose λ_0 so small, that

$$\|\partial_t \psi_t\|_{\mathcal{H}} \leq |\lambda|^2 \frac{c}{t^{3/2-\varepsilon}} \tag{4.65}$$

which concludes the proof of (1.11). Hence, by the Cook method [10], we obtain the existence of the limit ψ^+.

To see that $\psi^+ \neq 0$ under the specified conditions, we write

$$\|\psi^{+,(\lambda)}\|_{\mathcal{H}} \geq \|\psi^{(\lambda)}_{t=0}\|_{\mathcal{H}} - \int_0^\infty \mathrm{d}t\, \|\partial_t \psi^{(\lambda)}_t\|_{\mathcal{H}}, \tag{4.66}$$

where we included the dependence on λ explicitly in the notation. We recall that all constants in our discussion are uniformly bounded in $|\lambda| \in (0, \lambda_0]$. Thus, by estimate (4.65), the second term on the r.h.s. of (4.66) tends to zero as $\lambda \to 0$. So it suffices to show that $\|\psi^{(\lambda)}_{t=0}\|_{\mathcal{H}}$ is bounded from below uniformly in λ from some neighbourhood of zero. We collect the relevant ingredients: First, we recall that by [15, formula (5.2)]

$$\|\phi_p^{(\lambda)} - \Omega\|_{\mathcal{F}} \leq c|\lambda|^{1/4}. \tag{4.67}$$

Furthermore, we obtain from (4.32), (E.4)

$$\begin{aligned} \|f_p^{(\lambda)}(e^{-i|k|t+ik\cdot x} - 1)\|_2 \leq& c|\lambda|(1+\log(1+t+|x|))^{1/2}, \\ |\gamma(p,x,t)| \leq& c|\lambda|^2(1+\log(1+t+|x|)). \end{aligned} \tag{4.68}$$

Considering the above, we have

$$\begin{aligned} &\psi_{t=0}^{(\lambda)}(x) \\ &\quad = \frac{1}{(2\pi)^{3/2}}\int d^3p\, e^{ip\cdot x} e^{i\gamma^{(\lambda)}(p,x,0)} h(p) W(f_p^{(\lambda)}(e^{ik\cdot x}-1))\phi_p^{(\lambda)} && (4.69) \\ &\quad = \frac{1}{(2\pi)^{3/2}}\int d^3p\, e^{ip\cdot x} e^{i\gamma^{(\lambda)}(p,x,0)} h(p) W(f_p^{(\lambda)}(e^{ik\cdot x}-1))(\phi_p^{(\lambda)} - \Omega) && (4.70) \\ &\qquad + \frac{1}{(2\pi)^{3/2}}\int d^3p\, e^{ip\cdot x} e^{i\gamma^{(\lambda)}(p,x,0)} h(p) \big(W(f_p^{(\lambda)}(e^{ik\cdot x}-1)) - 1\big)\Omega && (4.71) \\ &\qquad + \frac{1}{(2\pi)^{3/2}}\int d^3p\, e^{ip\cdot x} \big(e^{i\gamma^{(\lambda)}(p,x,0)} - 1\big) h(p)\Omega && (4.72) \\ &\qquad + \frac{1}{(2\pi)^{3/2}}\int d^3p\, e^{ip\cdot x} h(p)\Omega. && (4.73) \end{aligned}$$

Thus, it is manifest from estimates (4.68), (4.67), combined with an argument as in (A.9) that

$$\psi_{t=0}^{(\lambda)}(x) = (F^{-1}h)(x)\Omega + O(|\lambda|^{1/4}(1+\log(1+|x|))), \tag{4.74}$$

where F is the Fourier transform and we have $\|O(|\lambda|^{1/4}(1+\log(1+|x|)))\|_{\mathcal{F}} \leq c|\lambda|^{1/4}(1+\log(1+|x|))$. Clearly, we can write for any compact subset $\Delta \subset \mathbb{R}^3$

$$\begin{aligned} \|\psi_{t=0}^{(\lambda)}\|_{\mathcal{H}} \geq& \left(\int_\Delta d^3x\, \|\psi_{t=0}^{(\lambda)}(x)\|_{\mathcal{F}}^2\right)^{1/2} \\ \geq& \left(\int_\Delta d^3x\, |(F^{-1}h)(x)|^2\right)^{1/2} - c|\lambda|^{1/4}\left(\int_\Delta d^3x\, (1+\log(1+|x|))^2\right)^{1/2}. \end{aligned} \tag{4.75}$$

For any Δ intersecting with the support of $F^{-1}h$, the first term in the second line of (4.75) is positive and independent of λ. As the second term tends to zero as $\lambda \to 0$, this concludes the proof. □

It remains to prove the following estimates.

Lemma 4.7. *Consider the expression*

$$\gamma_{\mathrm{int}}(p,x,t) := 2\int d^3k\, f_p(k)^2(|k| - k\cdot\nabla E_p)\cos(|k|t - k\cdot x). \tag{4.76}$$

The following bounds hold:

(a) *Fix some* $0 < c_0 < 1$. *For any* $M \in \mathbb{N}$, *there exists a constant* c_M, *uniform in* $p \in S$, *s.t.*

$$\sup_{(|x|/t)\leq c_0} |\gamma_{\rm int}(p,x,t)| \leq |\lambda|^2 \frac{c_M}{t^M}. \tag{4.77}$$

(b) *For all* $p \in S$ *and* $(t,x) \in \mathbb{R}^4$

$$|\gamma_{\rm int}(p,x,t)| \leq |\lambda|^2 \frac{c}{t} |\log t|. \tag{4.78}$$

Analogous estimates hold if we replace $p \mapsto f_p(k)^2(|k| - k\cdot\nabla E_p)$ *in* (4.76) *by its arbitrary derivatives w.r.t. p.*

Proof. Proceeding to spherical coordinates $d^3k = d\Omega(e_k)|k|^2 d|k|$, we have

$$\begin{aligned}\gamma_{\rm int}(p,x,t) &= \int d\Omega(e_k)\int_0^\infty d|k|\, f(|k|,e_k,p)\cos(|k|t(1-e_k\cdot \mathrm{v})),\\ & \qquad f(|k|,e_k,p) := |\lambda|^2 \frac{\chi_\kappa(k)^2}{2}\frac{1}{(1-e_k\cdot\nabla E_p)},\end{aligned} \tag{4.79}$$

where we set $\mathrm{v} := x/t$. We suppose that $|\mathrm{v}| \leq c_0 < 1$ and consider part (a) of the lemma. By integrating by parts w.r.t. $|k|$ and exploiting that sine vanishes at zero, we obtain

$$\gamma_{\rm int}(p,x,t) = -\int d\Omega(e_k)\int_0^\infty d|k|\,\partial_{|k|}f(|k|,e_k,p)\frac{1}{t(1-e_k\cdot\mathrm{v})}\sin(|k|t(1-e_k\cdot\mathrm{v})). \tag{4.80}$$

Now, we can continue integrating by parts, exploiting that $\partial_{|k|}f$ vanishes in a fixed neighbourhood of zero due to our assumptions on χ_κ (1.3). The fact that $(1-e_k\cdot\mathrm{v})$ is never zero in this case gives the claim.

Proceeding to (b), we suppose that $|\mathrm{v}| \geq c_0 > 0$ as the case $|\mathrm{v}| \leq c_0$ is settled by (a). We choose the third axis in the direction of v and write

$$\begin{aligned}&\gamma_{\rm int}(p,x,t)\\ &\quad= \int_{|k|\geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta)\, f(|k|, e(\cos(\theta),\varphi),p)\\ &\qquad \times \cos(|k|t(1-|\mathrm{v}|\cos(\theta))) + O(t^{-1})\\ &\quad= -\int_{|k|\geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta)\, f(|k|, e(\cos(\theta),\varphi),p)\\ &\qquad \times \frac{1}{t|k||v|}\frac{d}{d\cos(\theta)}\sin(|k|t(1-|\mathrm{v}|\cos(\theta))) + O(t^{-1})\\ &\quad= -\int_{|k|\geq 1/t} d|k| \int_0^{2\pi} d\varphi\, f(|k|, e(\cos(\theta),\varphi),p)\\ &\qquad \times \frac{1}{t|k||v|}\sin(|k|t(1-|\mathrm{v}|\cos(\theta)))\Big|_{\cos\theta=-1}^{\cos\theta=1} + O(t^{-1})\\ &\qquad + \int_{|k|\geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta)\left(\frac{d}{d\cos(\theta)} f(|k|, e(\cos(\theta),\varphi),p)\right)\end{aligned}$$

$$\times \frac{1}{t|k||v|} \sin(|k|t(1 - |\mathrm{v}| \cos(\theta))). \tag{4.81}$$

By estimating $|\sin(|k|t(1 - |\mathrm{v}| \cos(\theta)))| \leq 1$ everywhere above and using that the integration in $|k|$ is over a compact set, the claim follows from

$$\frac{1}{t} \int_{\kappa \geq |k| \geq 1/t} d|k| \frac{1}{|k|} \leq \frac{c'}{t} |\log(t)|. \tag{4.82}$$

This concludes the proof. □

5. Conclusions

In this paper, we proposed a new construction of infraparticle states in the massless Nelson model. The approximating sequence does not involve infrared cut-offs and the proof of convergence is relatively simple: Taking the spectral results from [1,15,16] for granted, it amounts to the Cook method combined with the stationary phase method, like for basic Schrödinger operators. It is legitimate to ask how the new infraparticle state compares with the established knowledge on the infrared problem in the Nelson model. To partially answer this question, we provide some heuristic remarks on the relation of our states to the Faddeev–Kulish approach. First, we note that the asymptotically dominant part of the wave packet (1.8) should propagate along the ballistic trajectory $x = \nabla E_p t$, thus ψ_t should have the same limit as

$$\psi_t^{\mathrm{D}}(x) := e^{iHt} \int d^3p\, h(p)\, e^{-i(E_p + H_{\mathrm{f}})t} e^{i\gamma(p, \nabla E_p t, t)} \times W\big(f_p(1 - e^{i|k|t - ik\cdot\nabla E_p t})\big) e^{iH_{\mathrm{f}}t} \frac{1}{(2\pi)^{3/2}} e^{i(p - P_{\mathrm{f}})\cdot x} \phi_p. \tag{5.1}$$

To proceed, let us second quantize also the electrons, denote their creation and annihilation operators by $b^{(*)}$ and the common vacuum of the electrons and photons by Ω. Expressing $\phi_p \in \mathcal{F}$ by its n-particle wave functions ϕ_p^n, we define its renormalized creation operator in a standard manner [2]:

$$\hat{b}_{\mathrm{w}}^*(p) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int d^{3n}k\, \phi_p^n(k_1, \ldots, k_n) a^*(k_1) \ldots a^*(k_n)\, b^*(p - (k_1 + \cdots + k_n)), \tag{5.2}$$

so that $\frac{1}{(2\pi)^{3/2}} e^{i(p - P_{\mathrm{f}})\cdot x} \phi_p$ can be identified with $\hat{b}_{\mathrm{w}}^*(p)\Omega$. Now recalling that $f_p(k) = v(k) \frac{1}{|k| - k\cdot\nabla E_p}$, we can write

$$\begin{aligned} &W\big(f_p(1 - e^{i|k|t - ik\cdot\nabla E_p t})\big) \\ &= \exp\Big(- i \int_0^t d\tau\, e^{iH_{\mathrm{f}}\tau} \{a^*(ve^{-ik\cdot\nabla E_p \tau}) + a(ve^{-ik\cdot\nabla E_p\tau})\} e^{-iH_{\mathrm{f}}\tau}\Big) \\ &= e^{iC_p t} e^{-i\gamma(p, \nabla E_p t, t)} \mathrm{Texp}\Big(-i \int_0^t d\tau\, e^{iH_{\mathrm{f}}\tau} \{a^*(ve^{-ik\cdot\nabla E_p\tau}) + a(ve^{-ik\cdot\nabla E_p\tau})\} e^{-iH_{\mathrm{f}}\tau}\Big), \end{aligned} \tag{5.3}$$

where $C_p := \int d^3k \, \frac{v(k)^2}{|k| - k \cdot \nabla E_p}$ is finite and the time-ordered exponential $U_{\mathrm{D}}(t) := \mathrm{Texp}(\dots)$ is the Dollard modifier of the Nelson model, cf [14, formula (3.6)]. Thus, (5.1) can be rewritten as

$$\psi_t^{\mathrm{D}} = e^{iHt} \int d^3p \, h(p) \, e^{-i(E_p + H_{\mathrm{f}} - C_p)t} U_{\mathrm{D}}(t) e^{iH_{\mathrm{f}}t} \hat{b}_{\mathrm{w}}^*(p) \Omega. \tag{5.4}$$

We recall from [14], that a direct application of the Faddeev–Kulish prescription to the Nelson model leads to a formula which differs from (5.4) only by a substitution $\hat{b}_{\mathrm{w}}^*(p) \to b^*(p)$. We believe that this discrepancy can be attributed to the quantum mechanical origin of the Dollard formalism which makes it difficult to reconcile with the electron mass renormalization present in the model. We think that formula (5.4) is a correct implementation of the Faddeev–Kulish formalism in the Nelson model and hope that the findings of the present paper will lead to a rigorous proof of convergence of ψ_t^{D} as $t \to \infty$.

There are several other future research directions, which we would like to point out. They include a proof of expected properties of infraparticle states familiar from [33] such as the convergence of asymptotic electron velocity and asymptotic photon fields on our states and the clustering relation, i.e. $\langle \psi_1^+, \psi_2^+ \rangle = \langle h_1, h_2 \rangle$, where the former scalar product is in $\mathcal{H}$, the latter in $L^2(\mathbb{R}_k^3)$ and h_j are related to $\psi_j^+ := \lim_{t\to\infty} \psi_{j,t}$, $j = 1, 2$, via (1.8). These problems appear to be within reach of available methods and are not treated here mainly to keep this paper within reasonable limits. A more intriguing, but still quite tractable problem, is to provide a non-perturbative proof of the Weinberg's soft photon theorem [38] in the massless Nelson model. Such a proof would provide a useful benchmark to test various perturbative versions, e.g. [23,31], currently considered in the context of the Strominger's 'infrared triangle' [37]. Another meaningful direction is to apply our construction of infraparticle scattering states to more sophisticated models, such as the Nelson model without the UV cut-off or the Pauli–Fierz model. This direction faces, however, a technical challenge of generalizing the spectral results from [1,15,16] to these theories. There is a solid basis for such endeavours, e.g. [4,5,8,9,20,24,25,28,29], but also a lot of work remains to be done.

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A Energy Bounds and Derivatives of the Weyl Operators

We introduce the following subspace of $L^2(\mathbb{R}^3_k)$:

$$L^2_\omega(\mathbb{R}^3_k) := \{f \in L^2(\mathbb{R}^3_k) \,|\, \|f\|_\omega := \|(1+|k|^{-1/2})f\|_2 < \infty\}. \tag{A.1}$$

We recall that $N := \mathrm{d}\Gamma(1), H_\mathrm{f} := \mathrm{d}\Gamma(|k|)$ and state the standard energy and number bounds [7]:

Lemma A.1. *Let $f_1, \dots, f_n \in L^2(\mathbb{R}^3_k)$. Then*

$$\|a^{(*)}(f_1)\dots a^{(*)}(f_n)(1+N)^{-n/2}\|_\mathcal{F} \le c_n \|f_1\|_2 \dots \|f_n\|_2. \tag{A.2}$$

Let $f_1, \dots, f_n \in L^2_\omega(\mathbb{R}^3_k)$. Then,

$$\|a^{(*)}(f_1)\dots a^{(*)}(f_n)(1+H_\mathrm{f})^{-n/2}\|_\mathcal{F} \le c_n \|f_1\|_\omega \dots \|f_n\|_\omega. \tag{A.3}$$

Formula (A.4) is also well-known, but we provide a proof for the reader's convenience.

Lemma A.2. *Let $\mathbb{R} \ni s \mapsto F_s \in L^2_\omega(\mathbb{R}^3_k)$ be differentiable in the norm $\|\cdot\|_\omega$. Then, $s \mapsto W(F_s)\psi$, $\psi \in D(H_\mathrm{f}^{1/2})$, is differentiable in the norm of $\mathcal{F}$ and*

$$\partial_s W(F_s)\psi = W(F_s)(a^*(\partial_s F_s) - a(\partial_s F_s) + i\mathrm{Im}\langle F_s, \partial_s F_s\rangle)\psi. \tag{A.4}$$

Also, $s \mapsto a^{(*)}(F_s)\psi$ is differentiable w.r.t. s in the norm of $\mathcal{F}$ and $\partial_s a^{(*)}(F_s)\psi = a^{(*)}(\partial_s F_s)\psi$. If $\psi \in D(N^{1/2})$ then analogous statements hold for $s \mapsto F_s$ differentiable in the norm $\|\cdot\|_2$.

Proof. Using the Weyl relations $W(F)W(G) = e^{-i\mathrm{Im}\langle F,G\rangle}W(F+G)$, $F, G \in L^2(\mathbb{R}^3_k)$,

$$\begin{aligned}
&\frac{1}{\Delta s}\big(W(F_{s+\Delta s}) - W(F_s)\big)\\
&\quad = W(F_s)\frac{1}{\Delta s}(W(-F_s)W(F_{s+\Delta s}) - 1)\\
&\quad = W(F_s)\frac{1}{\Delta s}(e^{i\mathrm{Im}\langle F_s, F_{s+\Delta s}-F_s\rangle}W(F_{s+\Delta s}-F_s) - 1)\\
&\quad = W(F_s)\frac{1}{\Delta s}\big(e^{i\mathrm{Im}\langle F_s, F_{s+\Delta s}-F_s\rangle} - 1\big)W(F_{s+\Delta s}-F_s) &\text{(A.5)}\\
&\qquad + W(F_s)\frac{1}{\Delta s}\big(W(F_{s+\Delta s}-F_s) - 1\big). &\text{(A.6)}
\end{aligned}$$

Considering (A.5), we obtain immediately

$$\lim_{\Delta s\to 0}\frac{1}{\Delta s}\big(e^{i\mathrm{Im}\langle F_s, F_{s+\Delta s}-F_s\rangle}-1\big)=i\mathrm{Im}\langle F_s,\partial_s F_s\rangle. \tag{A.7}$$

Furthermore, it is easy to see that in the norm of $\mathcal{F}$

$$\lim_{\Delta s\to 0} W(F_{s+\Delta s}-F_s)\psi=\psi. \tag{A.8}$$

In fact, denoting $\Phi(F):=a^*(-iF)+a(-iF)$, we can write

$$W(F_{s+\Delta s}-F_s)\psi=\psi+\left\{\frac{e^{i\Phi(F_{s+\Delta s}-F_s)}-1}{\Phi(F_{s+\Delta s}-F_s)}\right\}\Phi(F_{s+\Delta s}-F_s)\psi. \tag{A.9}$$

By the spectral theorem, the norm of the expression in curly bracket above is bounded uniformly in Δs. On the other hand, by the assumed form of differentiability

$$\Phi(F_{s+\Delta s}-F_s)\psi=\Delta s\,\Phi(\partial_s F_s)\psi+\Phi(o(\Delta s))\psi, \tag{A.10}$$

where $\partial_s F_s\in L^2_\omega(\mathbb{R}^3_k)$ and the rest term satisfies

$$\lim_{\Delta s\to 0}\frac{\|o(\Delta s)\|_\omega}{\Delta s}=0. \tag{A.11}$$

Thus, by the energy bounds of Lemma A.1, we obtain that (A.10) tends to zero in the norm of $\mathcal{F}$ as $\Delta s\to 0$ which gives (A.8).

Concerning (A.6), we write again $F_{s+\Delta s}-F_s=\Delta s\,\partial_s F_s+o(\Delta s)$, which gives

$$\begin{aligned}&\frac{1}{\Delta s}\big(W(F_{s+\Delta s}-F_s)-1\big)\psi\\&\quad=\frac{1}{\Delta s}\big(e^{-i\mathrm{Im}\langle\Delta s\,\partial_s F_s,o(\Delta s)\rangle}W(\Delta s\partial_s F_s)W(o(\Delta s))-1\big)\psi.\end{aligned} \tag{A.12}$$

To take the limit $\Delta s\to 0$ above, we note

$$\begin{aligned}\lim_{\Delta s\to 0}\frac{1}{\Delta s}\big(e^{-i\mathrm{Im}\langle\Delta s\partial_s F_s,o(\Delta s)\rangle}-1\big)&=0,\\ \lim_{\Delta s\to 0}\frac{1}{\Delta s}\big(W(o(\Delta s))-1\big)\psi&=0,\end{aligned} \tag{A.13}$$

where the latter limit is computed as in (A.10) using (A.11). Also, we exploit that by the Stone theorem

$$\lim_{\Delta s\to 0}\frac{1}{\Delta s}(W(\Delta s\partial_s F_s)-1)\psi=i\Phi(\partial_s F_s)\psi. \tag{A.14}$$

Finally, substituting (A.14), (A.7) to (A.5), (A.6), we obtain (A.4). The last statement of the lemma is proven by analogous arguments. □

Lemma A.3. *Under the assumptions of Lemma 4.5, for any fixed t,*

$$\begin{aligned}&\sup_{0\le\Delta x_i\le 1}\Bigg\|\Bigg\{\int d^3p\,e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)\\&\quad\times\int_0^1 ds(1-s)\big\{\partial^2_{x_i'}W\big(f_pm(t,x')\big)|_{x'=x+s(\Delta x_i)e_i}\big\}\phi_p\Bigg\}_{x\in\mathbb{R}^3}\Bigg\|_{\mathcal{H}}<\infty,\end{aligned} \tag{A.15}$$

$$\sup_{0\leq\Delta t\leq 1}\Big\|\Big\{\int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p) \times\int_0^1 ds(1-s)\big\{\partial_\tau^2 W\big(f_pm(\tau,x)\big)|_{\tau=t+s\Delta t}\big\}\phi_p\Big\}_{x\in\mathbb{R}^3}\Big\|_{\mathcal{H}}<\infty. \tag{A.16}$$

Proof. Let us prove (A.15). Since ϕ_p is in the domain of any power of H_{f} (cf. Lemma C.3), we can compute $\partial_{x'}^2 W\big(f_pm(\tau,x')\big)$ using Lemma A.2:

$$\begin{aligned}
&\partial_{x'}^2 W\big(f_pm(\tau,x')\big)\phi_p\\
&\quad= W\big(f_pm(\tau,x')\big)(a^*(\partial_{x'}f_pm(\tau,x'))-a(\partial_{x'}f_pm(\tau,x'))\\
&\qquad+i\mathrm{Im}\langle f_pm(\tau,x'),\partial_{x'}f_pm(\tau,x')\rangle)^2\phi_p \qquad\qquad (A.17)\\
&\qquad+W\big(f_pm(\tau,x')\big)(a^*(\partial_{x'}^2f_pm(\tau,x'))-a(\partial_{x'}^2f_pm(\tau,x'))\\
&\qquad+i\partial_{x'}\mathrm{Im}\langle f_pm(\tau,x'),\partial_{x'}f_pm(\tau,x')\rangle)\phi_p. \qquad\qquad (A.18)
\end{aligned}$$

Next, exploiting the energy bounds (A.3) to control the creation and annihilation operators acting on ϕ_p, we apply the shift (4.38) and estimate (3.5). More explicitly, we rewrite the expression under the norm in (A.16) as follows:

$$\begin{aligned}
&\int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)\\
&\qquad\times\int_0^1 ds\,(1-s)\big\{\partial_{x'}^2W\big(f_pm(\tau,x')\big)|_{x'=x+s(\Delta x_i)e_i}\big\}\phi_p\\
&\quad=\int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)\\
&\qquad\times\int_0^1 ds\,(1-s)\big\{\partial_{x'}^2W\big(f_pm(\tau,x')\big)|_{x'=x+s(\Delta x_i)e_i}\big\}\\
&\qquad\times(1+H_{\mathrm{f}})^{-1}(1+H_{\mathrm{f}})(\phi_p-\phi_{p,\sigma_x}) \qquad\qquad (A.19)\\
&\qquad+\int d^3p\, e^{i(p\cdot x-E_pt)}e^{i\gamma(p,x,t)}h(p)\\
&\qquad\times\int_0^1 ds\,(1-s)\big\{\partial_{x'}^2W\big(f_pm(\tau,x')\big)|_{x'=x+s(\Delta x_i)e_i}\big\}\phi_{p,\sigma_x}. \qquad\qquad (A.20)
\end{aligned}$$

By (3.5) and $\sigma_x:=\kappa_{\lambda_0}/(1+|x|)^M$, M sufficiently large, we obtain that (A.19) remains bounded for $0\leq\Delta x_i\leq 1$ in the norm of $L^2(\mathbb{R}^3_x;\mathcal{F})$. As for (A.20), estimate (4.3) gives

$$\|(A.20)\|_{\mathcal{H}}\leq ct^{1/2} \times\sum_{|\alpha|\leq 2}\sup_{p,x}\sup_{s\in[0,1]}\Big(\frac{1}{(1+|x|)^{1/2}}\|\partial_p^\alpha(e^{i\gamma(p,x,t)}\{\partial_{x'}^2\hat{g}^{\sigma_x}_{(\tau,x')}(p)\}|_{x'=x+s(\Delta x_i)e_i})\|_{\mathcal{F}}\Big), \tag{A.21}$$

where $\hat{g}^{\sigma_x}_{(\tau,x')}$ was defined in (4.18). Expression (A.21) also remains bounded for $0\leq\Delta x_i\leq 1$ by Lemma 4.4. (From the discussion above, it is manifest that $\partial_{x_i'}^2$ in (A.21) does not act on σ_x. In fact, $\partial_{x_i'}^2$ appears already in (A.15), whereas the shift $\phi_p\to\phi_{p,\sigma_x}$ was applied later in (A.20)). The proof of (A.16) is analogous.

B Proof of Lemma 3.2

We write $\phi_{p,\sigma} = \{f^n_{\mathrm{w},p,\sigma}\}_{n\in\mathbb{N}_0}$ in terms of the Fock space wave functions. Given Lemma B.1 and formula (B.3), we can write

$$\begin{aligned}\|H_{\mathrm{f}}^{\ell_1} N^{\ell_2} \partial_p^\alpha \phi_{p,\sigma}\|_{\mathcal{F}} &\leq \left(\sum_{n=0}^{\infty} n^{2(\ell_1+\ell_2)} \|\partial_p^\alpha f^n_{\mathrm{w},p,\sigma}\|_2^2\right)^{1/2}\\ &\leq \frac{c}{\sigma^{\delta'_{\lambda_0}}}\left(\sum_{n=0}^{\infty}\frac{1}{n!}(c_{\ell_1,\ell_2}\lambda)^n |\log\sigma|^n\right)^{1/2}\\ &\leq \frac{c}{\sigma^{\delta_{\lambda_0}}}, \qquad\qquad (\mathrm{B.1})\end{aligned}$$

for some constants c_{ℓ_1,ℓ_2} and $\delta_{\lambda_0} > 0$ which tends to zero as $\lambda_0 \to 0$. To handle the powers of H_{f} we used that the UV cut-off $\kappa = 1$ and consequently the wave functions $f^n_{\mathrm{w},p,\sigma}$ are supported in unit balls in each variable $k_1, \dots, k_n$ separately. This gives Lemma 3.2.

In preparation for the proof of Lemma B.1, we state a general relation for wave functions of a Fock space vector:

$$f^n_{\mathrm{w},p,\sigma}(k_1,\dots,k_n) = \frac{1}{\sqrt{n!}}\langle\Omega, a(k_1)\dots a(k_n)\phi_{p,\sigma}\rangle. \qquad (\mathrm{B.2})$$

This formula is meaningful by considerations in [15, Appendix D]. Let us now introduce the following auxiliary functions:

$$g^0_\sigma := c \quad \text{and} \quad g^n_\sigma(k_1,\dots,k_n) := \prod_{i=1}^{n}\frac{c\lambda\chi_{[\sigma,\kappa_*)}(k_i)}{|k_i|^{3/2}}, n \geq 1, \qquad (\mathrm{B.3})$$

where $\kappa_* := (1-\varepsilon_0)^{-1}\kappa$ is slightly larger than κ and $0 < \varepsilon_0 < 1$ was introduced below (1.3).

Lemma B.1. *The following estimates hold*

$$|\partial_p^\alpha f^n_{\mathrm{w},p,\sigma}(k_1,\dots,k_n)| \leq \frac{1}{\sqrt{n!}}\left(\frac{1}{\sigma^{\delta_{\lambda_0}}}\right)^{|\alpha|} g^n_\sigma(k_1,\dots,k_n) \quad \textit{for} \quad |\alpha| \leq 2. \qquad (\mathrm{B.4})$$

Proof. In [16, formula (4.42)], the following functions are introduced[4]

$$\hat{f}^n_{p,\sigma}(k_1,\dots,k_n) := W^*(f_{p,\sigma})a(k_1)\dots a(k_n)\phi_{p,\sigma}, \qquad (\mathrm{B.5})$$

where $W(f_{p,\sigma}) := e^{a^*(f_{p,\sigma})-a(f_{p,\sigma})}$ and the r.h.s. above is well-defined by considerations from [15, Appendix D]. In Proposition 4.7 of [16] and in the subsequent discussion in this reference, the following bounds are shown

$$\|\partial_p^\alpha \hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}} \leq \left(\frac{1}{\sigma^{\delta_{\lambda_0}}}\right)^{|\alpha|} g^n_\sigma(k_1,\dots,k_n) \quad \text{for} \quad |\alpha| \leq 2. \qquad (\mathrm{B.6})$$

In view of (B.3), the r.h.s. depends on numerical constants, whose dependence on various parameters is specified at the end of Sect. 3. We note that for $|\alpha| = 0$ (B.4) follows immediately from (B.6) and (B.2).

[4] For consistency with the notation from [15–17], we use similar symbols for several different functions: $f_{p,\sigma}$ are defined in (3.4) and $\hat{f}^n_{p,\sigma}$ in (B.5).

As for the case $|\alpha| = 1$, we can write

$$\partial_{p_j}(a(k_1)\dots a(k_n)\phi_{p,\sigma}) = \partial_{p_j}(W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)). \tag{B.7}$$

The term in which the derivative acts on $\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)$ is immediately estimated using (B.6) for $|\alpha| = 1$. As for the remaining term, we estimate

$$\begin{aligned}
&\|\partial_{p_j}(W(f_{p,\sigma}))\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}} \\
&\quad \le 2\|a(\partial_{p_j} f_{p,\sigma})W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}} \\
&\qquad + \|\partial_{p_j} f_{p,\sigma}\|_2\, \|\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}} \qquad (B.8)\\
&\quad \le 2\int d^3k_0\, |(\partial_{p_j} f_{p,\sigma})(k_0)|\, \|\hat{f}^{n+1}_{p,\sigma}(k_0,k_1,\dots,k_n)\|_{\mathcal{F}} \\
&\qquad + c|\log(\sigma)|^{1/2}\, \|\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}}. \qquad (B.9)
\end{aligned}$$

For differentiability of the Weyl operator, we refer to Lemma A.2 and the fact that $\hat{f}^n_{p,\sigma}$ is in the domain of $H_{\mathrm{f}}^{1/2}$ (cf. [15, formula (D.8)]). The bound on $\|\partial_{p_j} f_{p,\sigma}\|_2$ follows from Lemma E.4. This, together with (B.6), gives (B.4) for $|\alpha| = 1$.

Now, we consider the case $|\alpha| = 2$. Again, we can write

$$\partial_{p_j}\partial_{p_i}\big(a(k_1)\dots a(k_n)\phi_{p,\sigma}\big) = \partial_{p_j}\partial_{p_i}\big(W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\big). \tag{B.10}$$

The term in which both derivatives act on $\hat{f}^n_{p,\sigma}$ is immediately estimated using (B.6). Let us consider the term in which one derivative acts on $W(f_{p,\sigma})$ and another on $\hat{f}^n_{p,\sigma}$. Similarly as in (B.8), we have

$$\begin{aligned}
&\|\partial_{p_j}\big(W(f_{p,\sigma})\big)\partial_{p_i}\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}} \\
&\quad \le 2\|a(\partial_{p_j} f_{p,\sigma})\partial_{p_i}\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}} + \|\partial_{p_j} f_{p,\sigma}\|_2\, \|\partial_{p_i}\hat{f}^n_{p,\sigma}(k_1,\dots,k_n)\|_{\mathcal{F}}.
\end{aligned} \tag{B.11}$$

The last term on the r.h.s. of (B.11) clearly satisfies the required bound by (B.6) and Lemma E.4. As for the first term above, we note that

$$\begin{aligned}
&a(\partial_{p_j} f_{p,\sigma})\partial_{p_i}\hat{f}^n_{p,\sigma}(k_1,\dots,k_n) \\
&= -\bigg(\int d^3k_0\, (\partial_{p_j} f_{p,\sigma})(k_0) f_{p,\sigma}(k_0)\bigg)\partial_{p_i}(W(f_{p,\sigma})^* a(k_1)\dots a(k_n)\phi_{p,\sigma}) \\
&+ \int d^3k_0\, (\partial_{p_j} f_{p,\sigma})(k_0)\partial_{p_i}(W(f_{p,\sigma})^* a(k_0)a(k_1)\dots a(k_n)\phi_{p,\sigma}),
\end{aligned} \tag{B.12}$$

where we first computed the derivative of $\hat{f}^n_{p,\sigma}$ and then used $a(g)W(f_{p,\sigma})^* = W(f_{p,\sigma})^*(a(g) - \langle g, f_{p,\sigma}\rangle)$ for $g = \partial_{p_j} f_{p,\sigma}$. The last expression is immediately estimated using (B.6) for $|\alpha| = 1$. We still have to estimate a contribution to (B.10), where both derivatives act on $W(f_{p,\sigma})$:

$$\begin{aligned}&\|\partial_{p_i}\partial_{p_j}(W(f_{p,\sigma}))\hat{f}^n_{p,\sigma}(k_1,\ldots,k_n)\|_{\mathcal{F}}\\&\le \|(a^*(\partial_{p_j}f_{p,\sigma})-a(\partial_{p_j}f_{p,\sigma}))(a^*(\partial_{p_i}f_{p,\sigma})-a(\partial_{p_i}f_{p,\sigma}))W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\ldots,k_n)\|_{\mathcal{F}}\\&\quad+\|(a^*(\partial_{p_i}\partial_{p_j}f_{p,\sigma})-a(\partial_{p_i}\partial_{p_j}f_{p,\sigma}))W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\ldots,k_n)\|_{\mathcal{F}}.\end{aligned}\tag{B.13}$$

This expression can be estimated by a linear combination of terms of the form:

$$\|\partial_p^{\alpha_1}f_{p,\sigma}\|_2\,\|\partial_p^{\alpha_2}f_{p,\sigma}\|_2\,\|\hat{f}^n_{p,\sigma}(k_1,\ldots,k_n)\|_{\mathcal{F}},\tag{B.14}$$

$$\|\partial_p^{\alpha_1}f_{p,\sigma}\|_2\,\|a(\partial_p^{\alpha_2}f_{p,\sigma})W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\ldots,k_n)\|_{\mathcal{F}},\tag{B.15}$$

$$\|a(\partial_p^{\alpha_1}f_{p,\sigma})a(\partial_p^{\alpha_2}f_{p,\sigma})W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1,\ldots,k_n)\|_{\mathcal{F}},\tag{B.16}$$

where $|\alpha_1|,|\alpha_2|\le 2$. Expression (B.14) is estimated using (B.6) for $|\alpha|=0$ and Lemma E.4. Expression (B.15) is estimated as in (B.9). As for (B.16), it can be bounded by

$$(B.16)\le\int d^3k'd^3k''\,|\partial_p^{\alpha_1}f_{p,\sigma}(k')|\,|\partial_p^{\alpha_2}f_{p,\sigma}(k'')|\,\|\hat{f}^{n+2}_{p,\sigma}(k',k'',k_1,\ldots,k_n)\|_{\mathcal{F}},\tag{B.17}$$

which is estimated with the help of (B.6) for $|\alpha|=0$ and Lemma E.4. □

C Proof of Estimate (3.5)

Lemma C.1. *For any $\ell\in\mathbb{N}_0$, the maximal coupling constant $\lambda_0>0$ can be chosen sufficiently small, so that there exists a constant c such that*

$$\|H_{\mathrm{f}}^{\ell}(\phi_p-\phi_{p,\sigma})\|_{\mathcal{F}}\le c\sigma^{1/5}.\tag{C.1}$$

Proof. First, we will prove

$$\|(H_p^{\mathrm{w}})^{\ell}(\phi_p-\phi_{p,\sigma})\|_{\mathcal{F}}\le c\sigma^{1/5}.\tag{C.2}$$

To this end, we recall from [32], [15, Lemma 3.6] the form of the modified Hamiltonian on $D(P_{\mathrm{f}}^2+H_{\mathrm{f}})$

$$H^{\mathrm{w}}_{p,\sigma}=\frac{1}{2}\Gamma^2_{p,\sigma}+\int d^3k\,\alpha_{p,\sigma}(e_k)|k|a^*(k)a(k)+c_p^{\sigma},\tag{C.3}$$

where

$$\begin{aligned}\Gamma_{p,\sigma}&:=\nabla E_{p,\sigma}-(p-P^{\mathrm{w}}_{\mathrm{f},\sigma}),\quad P^{\mathrm{w}}_{\mathrm{f},\sigma}:=W(f_{p,\sigma})P_{\mathrm{f}}W(f_{p,\sigma})^*,\\\alpha_{p,\sigma}(e_k)&:=(1-e_k\cdot\nabla E_{p,\sigma}),\end{aligned}\tag{C.4}$$

$$c_p^{\sigma}:=\frac{1}{2}p^2-\frac{1}{2}(p-\nabla E_{p,\sigma})^2-\lambda^2\int d^3k\frac{\chi_{[\sigma,\kappa)}(k)}{2|k|^2\alpha_{p,\sigma}(e_k)}.\tag{C.5}$$

The corresponding quantities at $\sigma=0$ are denoted by dropping σ in the notation. We will use the standard bounds from [32]

$$|E_p-E_{p,\sigma}|\le c\sigma,\quad|\nabla E_p-\nabla E_{p,\sigma}|\le c\sigma^{1/4},\quad\|\phi_p-\phi_{p,\sigma}\|_{\mathcal{F}}\le c\sigma^{1/2},\tag{C.6}$$

see also [15, Theorem 2.1 (b), Corollary 5.6], [17, Proposition A.2].

Now we proceed by induction: for $\ell = 0$ the estimate holds by the third bound in (C.6). For the inductive step, we compute

$$\begin{aligned} \|(H_p^{\mathrm{w}})^{\ell}(\phi_p - \phi_{p,\sigma})\|_{\mathcal{F}} &= \|(H_p^{\mathrm{w}})^{\ell-1}(E_p\phi_p - H_p^{\mathrm{w}}\phi_{p,\sigma})\|_{\mathcal{F}} \\ &\leq \|(H_p^{\mathrm{w}})^{\ell-1}(E_p\phi_p - E_{p,\sigma}\phi_{p,\sigma})\|_{\mathcal{F}} + \|(H_p^{\mathrm{w}})^{\ell-1}(H_p^{\mathrm{w}} - H_{p,\sigma}^{\mathrm{w}})\phi_{p,\sigma}\|_{\mathcal{F}}. \end{aligned} \tag{C.7}$$

The first term on the r.h.s. of (C.7) is $O(\sigma^{1/5})$ by the induction hypothesis and the first estimate in (C.6). Concerning the last term on the r.h.s. of (C.7), we note that there are three contributions to $H_p^{\mathrm{w}} - H_{p,\sigma}^{\mathrm{w}}$ coming from the three terms in the Hamiltonian (C.3). They have the following properties: First, by (C.6), $|c_p - c_p^{\sigma}| \leq c\sigma^{1/4}$. Thus, by Lemma C.2,

$$\|(H_p^{\mathrm{w}})^{\ell-1}(c_p - c_p^{\sigma})\phi_{p,\sigma}\|_{\mathcal{F}} \leq c\sigma^{1/4-\delta_{\lambda_0}}. \tag{C.8}$$

Clearly, for $\lambda_0 > 0$ sufficiently small, the last expression is $O(\sigma^{1/5})$. The second contribution is $\mathrm{d}\Gamma((\alpha_{p,\sigma}(e_k) - \alpha_p(e_k))|k|)$, where $|\alpha_{p,\sigma}(e_k) - \alpha_p(e_k)| \leq c\sigma^{1/4}$ by (C.6). Thus, Lemma C.2 gives

$$\|(H_p^{\mathrm{w}})^{\ell-1}\mathrm{d}\Gamma((\alpha_{p,\sigma}(e_k) - \alpha_p(e_k))|k|)\phi_{p,\sigma}\|_{\mathcal{F}} \leq c\sigma^{1/4-\delta_{\lambda_0}}. \tag{C.9}$$

Concerning the third contribution, $(\Gamma_p^2 - \Gamma_{p,\sigma}^2)$, we note that on $D(P_{\mathrm{f}}^2 + H_{\mathrm{f}})$,

$$\begin{aligned} (P_{\mathrm{f},\sigma}^{\mathrm{w}})_i &= W(f_{p,\sigma})P_{\mathrm{f},i}W(f_{p,\sigma})^* = (P_{\mathrm{f}})_i - a^*(k_i f_{p,\sigma}) - a(k_i f_{p,\sigma}) + \langle f_{p,\sigma}, k_i f_{p,\sigma}\rangle \\ &= (P_{\mathrm{f}}^{\mathrm{w}})_i - a^*(k_i(f_{p,\sigma} - f_p)) - a(k_i(f_{p,\sigma} - f_p)) + (\langle f_{p,\sigma}, k_i f_{p,\sigma}\rangle - \langle f_p, k_i f_p\rangle). \end{aligned} \tag{C.10}$$

Consequently, $\Gamma_{p,\sigma} = \Gamma_p + \Delta\Gamma_{p,\sigma}$, where

$$\begin{aligned} (\Delta\Gamma_{p,\sigma})_i = &-a^*(k_i(f_{p,\sigma} - f_p)) - a(k_i(f_{p,\sigma} - f_p)) + \big(\langle f_{p,\sigma}, k_i f_{p,\sigma}\rangle - \langle f_p, k_i f_p\rangle\big) \\ &+(\nabla E_{p,\sigma} - \nabla E_p)_i. \end{aligned} \tag{C.11}$$

Considering that $(\Gamma_p^2 - \Gamma_{p,\sigma}^2) = -\Gamma_p \cdot \Delta\Gamma_{p,\sigma} - \Delta\Gamma_{p,\sigma} \cdot \Gamma_p - (\Delta\Gamma_{p,\sigma})^2$, Lemmas C.2 and E.5 give

$$\|(H_p^{\mathrm{w}})^{\ell-1}(\Gamma_p^2 - \Gamma_{p,\sigma}^2)\phi_{p,\sigma}\|_{\mathcal{F}} \leq c\sigma^{1/4-\delta_{\lambda_0}}. \tag{C.12}$$

This concludes the proof of (C.2). Now, (C.1) follows from Lemma C.3. □

Lemma C.2. *Let $h_1, \ldots, h_\ell$ be real-valued measurable functions (in momentum space) which are bounded on compact sets. Then,*

$$\|\mathrm{d}\Gamma(h_1)\ldots\mathrm{d}\Gamma(h_\ell)\phi_{p,\sigma}\|_{\mathcal{F}} \leq \frac{c_\ell}{\sigma^{\delta_{\lambda_0}}}\left(\sup_{|k_1|\leq\kappa_*}|h_1(k_1)|\ldots\sup_{|k_\ell|\leq\kappa_*}|h_\ell(k_\ell)|\right), \tag{C.13}$$

where c_ℓ, δ_{λ_0} may depend on ℓ. Furthermore, if $f_1, \ldots, f_{\tilde{\ell}} \in L^2(\mathbb{R}_k^3)$ are supported in a ball of radius κ_*, then we get

$$\begin{aligned} &\|\mathrm{d}\Gamma(h_1)\ldots\mathrm{d}\Gamma(h_\ell)a^{(*)}(f_1)\ldots a^{(*)}(f_{\tilde{\ell}})\phi_{p,\sigma}\|_{\mathcal{F}} \\ &\quad\leq \frac{c_{\ell,\tilde{\ell}}}{\sigma^{\delta_{\lambda_0}}}\left(\sup_{|k_1|\leq\kappa_*}|h_1(k_1)|\ldots\sup_{|k_\ell|\leq\kappa_*}|h(k_\ell)|\right)\left(\|f_1\|_2\ldots\|f_{\tilde{\ell}}\|_2\right), \end{aligned} \tag{C.14}$$

where $c_{\ell,\tilde{\ell}}$ and δ_{λ_0} may depend on ℓ, $\tilde{\ell}$. The estimate also holds for an arbitrary permutation of the $(\ell+\tilde{\ell})$-element set of operators on the l.h.s. of (C.14).

Proof. We consider (C.13) for $\ell = 1$. Let $f^n_{\mathrm{w},p,\sigma}$ be the n-photon wave functions of $\phi_{p,\sigma}$. Then, by [17, Proposition A.4], we have

$$|f^n_{\mathrm{w},p,\sigma}(k_1,\dots,k_n)| \le \frac{1}{\sqrt{n!}} g^n_\sigma(k_1,\dots,k_n), \tag{C.15}$$

where g^n_σ are defined as in (B.3). Thus,

$$\begin{aligned}\|\mathrm{d}\Gamma(h_1)\phi_{p,\sigma}\|^2_{\mathcal{F}} &\le \sum_{n=1}^{\infty} \frac{1}{n!}\int d^{3n}k\,(h_1(k_1)+\cdots+h_1(k_n))^2 |g^n_\sigma(k_1,\dots,k_n)|^2\\ &\le \left(\sup_{|k|\le\kappa_*}|h(k)|\right)^2 \sum_{n=1}^{\infty}\frac{n^2}{n!}\int d^{3n}k\,|g^n_\sigma(k_1,\dots,k_n)|\\ &\le \frac{c}{\sigma^{\delta_{\lambda_0}}}\left(\sup_{|k|\le\kappa_*}|h(k)|\right)^2, \end{aligned}\tag{C.16}$$

where we estimated as in (B.1). Generalization to arbitrary ℓ is straightforward.

As for (C.14), we first commute all the operators $a^{(*)}(f_j)$ to the left and thus get a linear combination of terms of the form

$$\|a^{(*)}(f'_1)\dots a^{(*)}(f'_{\tilde{\ell}})(1+N)^{-\tilde{\ell}}(1+N)^{\tilde{\ell}}(\mathrm{d}\Gamma(h_{i_1})\dots\mathrm{d}\Gamma(h_{i_{\ell'}}))\phi_{p,\sigma}\|_{\mathcal{F}}, \tag{C.17}$$

where $f'_i(k) = h_{j_1}(k)\dots h_{j_l}(k)f_i(k)$ and the functions h_j included in f'_i do not appear in the product of $\mathrm{d}\Gamma(h_{i_1})$ in (C.17). Now using the number bounds (A.2) on creation and annihilation operators, assumption on the supports of f_i and (C.13), we obtain the claim.

Lemma C.3. *For any $\ell \in \mathbb{N}$, the operators $H^\ell_{\mathrm{f}}(i+H^{\mathrm{w}}_p)^{-\ell}$ are bounded.*

Proof. Let $\psi \in \mathcal{F}$, $\|\psi\|_{\mathcal{F}} = 1$, be in the domain of H^ℓ_{f}. Then, we can write

$$\begin{aligned}\|(1+H^{\mathrm{w}}_p)^{-\ell}H^\ell_{\mathrm{f}}\psi\| \le& \|\big((1+H^{\mathrm{w}}_p)^{-\ell}-(1+H^{\mathrm{w}}_{p,\sigma})^{-\ell}\big)H^\ell_{\mathrm{f}}\psi\|_{\mathcal{F}}\\ &+\|(1+H_{p,\sigma})^{-\ell}(W(f_{p,\sigma})^* H_{\mathrm{f}} W(f_{p,\sigma}))^\ell\|_{\mathcal{F}}.\end{aligned}\tag{C.18}$$

Exploiting the concrete expression for $W(f_{p,\sigma})^* H_{\mathrm{f}} W(f_{p,\sigma})$ and standard energy bounds for the Hamiltonian $H_{p,\sigma}$, i.e. the boundedness of $(1+H_{p,\sigma})^{-\ell}H^\ell_{\mathrm{f}}$ (cf. [22, Appendix D]), we obtain that the last term is uniformly bounded in σ. Now since $\lim_{\sigma\to 0} H^{\mathrm{w}}_{p,\sigma} = H^{\mathrm{w}}_p$ in the norm-resolvent sense, we complete the proof by first taking $\sigma \to 0$ on the r.h.s. and then taking supremum over ψ. (The statement about the norm-resolvent convergence is verified using the resolvent identity and explicit formulas for $H^{\mathrm{w}}_{p,\sigma} - H^{\mathrm{w}}_p$, appearing in the proof of Lemma C.1).

D Proof of Lemma 4.1

D.1 Proof of (4.2)

Let $V := \overline{\bigcup_{|\lambda|\leq\lambda_0}\{\,\nabla E_p^{(\lambda)}\,|\, p\in \operatorname{supp} g\}}$ and V_δ be an open set containing the closed set V. Since $|\nabla E_p^{(\lambda)}| < c_0 < 1$ for $p\in S$, we can ensure that V_δ is in the interior of the ball of radius c_0 centered at zero. As we show below by the non-stationary phase method, for $x/t \notin V_\delta$ and any ψ of norm one from the dense domain in the statement of Lemma 4.1

$$\left|\chi_{\{|x|/t\notin V_\delta\}}(x)\int d^3p\, e^{i(p\cdot x - E_p^{(\lambda)}t)}\langle\psi, g(p)\rangle_{\mathcal{F}}\right|$$
$$\leq c(1+|x|+t)^{-2}\chi_{\{|x|/t\notin V_\delta\}}(x)\sum_{|\alpha|\leq 2}\sup_p|\langle\psi,\partial_p^\alpha g(p)\rangle_{\mathcal{F}}|, \qquad \text{(D.1)}$$

where c is independent of g, x, t. Hence, considering that $|x|/t \geq c_0$ implies $(x/t)\notin V_\delta$, we obtain for any $0<\varepsilon<1$,

$$\int_{|x|\geq c_0 t} d^3x \sup_{\|\psi\|_{\mathcal{F}}\leq 1}\left|\left\langle\psi, \int d^3p\, e^{i(p\cdot x - E_p^{(\lambda)}t)} g(p)\right\rangle_{\mathcal{F}}\right|^2$$
$$\leq \int_{|x|\geq c_0 t} d^3x\, c(1+|x|)^{-4+2\varepsilon}\left(\sum_{|\alpha|\leq 2}\sup_{p,|x'|\geq c_0 t}\frac{1}{(1+|x'|+t)^{\varepsilon}}\|\partial_p^\alpha g(p)\|_{\mathcal{F}}\right)^2, \qquad \text{(D.2)}$$

which gives (4.2).

Estimate (D.1) is obtained by a slight generalization of Corollary of Theorem XI.14 from [35], which makes the dependence of the r.h.s. on g explicit. Namely, we set $\mathrm{v} := x/t$ and write

$$p\cdot x - E_p^{(\lambda)}t = (|x|+t)F_{\mathrm{v}}^{(\lambda)}(p), \quad F_{\mathrm{v}}^{(\lambda)}(p) := \frac{p\cdot \mathrm{v} - E_p^{(\lambda)}}{|\mathrm{v}|+1}, \qquad \text{(D.3)}$$

and note that

$$|\nabla_p F_{\mathrm{v}}^{(\lambda)}(p)| = \left|\frac{\mathrm{v} - \nabla E_p^{(\lambda)}}{|\mathrm{v}|+1}\right| > \epsilon > 0. \qquad \text{(D.4)}$$

Here, ϵ can be chosen uniformly in $p\in S$, $|\lambda|\leq\lambda_0$ and $\mathrm{v}\notin V_\delta$ since $V := \overline{\bigcup_{|\lambda|\leq\lambda_0}\{\,\nabla E_p^{(\lambda)}\,|\,p\in\operatorname{supp} g\}}$ is a compact set contained in the open set V_δ. We can thus write the following identity

$$e^{i(p\cdot x - E_p^{(\lambda)}t)} = e^{i(|x|+t)F_{\mathrm{v}}^{(\lambda)}(p)} = \frac{1}{i(|x|+t)}\frac{\nabla_p F_{\mathrm{v}}^{(\lambda)}(p)\cdot\nabla_p e^{i(|x|+t)F_{\mathrm{v}}^{(\lambda)}(p)}}{|\nabla_p F_{\mathrm{v}}^{(\lambda)}(p)|^2}, \qquad \text{(D.5)}$$

substitute it to the l.h.s. of (D.1) and integrate by parts w.r.t. p. By repeating this procedure, we obtain (D.1). The uniformity of the constant in (D.1) in $\mathrm{v}\notin V_\delta$, $|\lambda|\leq\lambda_0$ and g follows from the uniformity of ϵ.

D.2 Proof of (4.1)

As we show below by the the stationary phase method, we have for all $x \in \mathbb{R}^3$

$$\left|\chi_{\{|x|/t\le c_0\}}(x)\int d^3p\, e^{i(p\cdot x - E_p^{(\lambda)}t)}\langle\psi, g(p)\rangle_{\mathcal{F}}\right| \le ct^{-3/2}\chi_{\{|x|/t\le c_0\}}(x)\sum_{|\alpha|\le 2}\sup_p |\langle\psi, \partial_p^\alpha g(p)\rangle_{\mathcal{F}}|. \tag{D.6}$$

Thus, we get

$$\int_{|x|\le c_0 t} d^3x \sup_{\|\psi\|_{\mathcal{F}}\le 1}\left|\left\langle\psi, \int d^3p\, e^{i(p\cdot x - E_p t)} g(p)\right\rangle_{\mathcal{F}}\right|^2 \le \int_{|x|\le c_0 t} d^3x\, c(t^{-3})\left(\sum_{|\alpha|\le 2}\sup_{p,|x'|\le c_0 t}\|\partial_p^\alpha g(p)\|_{\mathcal{F}}\right)^2, \tag{D.7}$$

which gives (4.1).

Estimate (D.6) is obtained by generalizing the Corollary of Theorem XI.15 from [35] so as to make the dependence of the r.h.s. of (D.6) on g explicit. Again, the non-trivial part is to make sure that the constant c in (D.6) is independent of g, x, t and uniform in $|\lambda| \le \lambda_0$. To carefully keep track of this aspect, let us recall Theorem XI.15 of [35]:

Theorem D.1 [35]. *Let f be a C^∞ real-valued function defined in a neighbourhood of 0 in $\mathbb{R}^n$. Suppose that $(\nabla f)(0) = 0$ and that its Hessian matrix at zero is invertible. Then, there is a neighbourhood $\mathcal{O}$ of 0 s.t. for any $s > n/2$ there is a c so that for all $u \in C_0^\infty(\mathcal{O})$ and $\tau \ge 1$*

$$\left|\int d^np\, e^{if(p)\tau}u(p)\right| \le c\tau^{-n/2}\sum_{|\alpha|\le s}\sup_p |\partial_p^\alpha u(p)|. \tag{D.8}$$

Moreover, given such an f_0, there exist neighbourhoods $\mathcal{O}_1$ and $\mathcal{O}_2$ of zero with $\overline{\mathcal{O}_1} \subset \mathcal{O}_2 \subset \mathcal{O}$ and a neighbourhood $\mathcal{N}$ of f_0 in the $C^\ell(\mathcal{O}_2)$ topology (for some ℓ) so that (D.8) holds for all $u \in C_0^\infty(\mathcal{O}_1)$ and $f \in \mathcal{N}$.

We denote $u(p) := \langle\psi, g(p)\rangle_{\mathcal{F}}$, $\mathrm{v} := x/t \in \overline{V}_\delta$ and consider the following integrals

$$I_{\mathrm{v},t}^{(\lambda)}(u) := \int d^3p\, e^{iF^{(\lambda),\mathrm{v}}t}u(p), \quad F^{(\lambda),\mathrm{v}}(p) := p\cdot\mathrm{v} - E_p^{(\lambda)}. \tag{D.9}$$

We note for future reference that for any fixed $|\lambda| \le \lambda_0, \mathrm{v} \in \overline{V}_\delta$ the function $F^{(\lambda),\mathrm{v}}$ has exactly one critical point in S by invertibility of the relation $p \mapsto \nabla E_p^{(\lambda)}$ given by Lemma 3.1. Now for any given $F^{(\lambda),\mathrm{v}}$ we consider neighbourhoods $\mathcal{O}_1, \mathcal{O}_2$ of its critical point and a neighbourhood $\mathcal{N}$ of $F^{(\lambda),\mathrm{v}}$ in the $C^\ell(\mathcal{O}_2)$ topology as in the second part of Theorem D.1. Let $\mathcal{B}_{\mathrm{v},r}$ be an open ball of radius r centered at v and $J_{\lambda,r} := (\lambda - r, \lambda + r)$. It is easy to see using Lemma 3.1 that for r sufficiently small,

$$\{\, F^{(\lambda'),\mathrm{v}'} \,|\, (\lambda', \mathrm{v}') \in J_{\lambda,r} \times \mathcal{B}_{\mathrm{v},r}\} \subset \mathcal{N}. \tag{D.10}$$

In fact, it is enough to check that

$$\sup_{p\in S} |\partial_p^\alpha \big(p\cdot(\mathrm{v}-\mathrm{v}') - (E_p^{(\lambda)} - E_p^{(\lambda')})\big)| \le cr \tag{D.11}$$

for $|\alpha| \le \ell$ which follows from analyticity of $(p,\lambda) \mapsto E_p^{(\lambda)}$. By reducing r again, we can also ensure that the critical points do not approach the boundary of $\mathcal{O}_1$, that is,

$$\overline{\{p \in \mathcal{O}_1 \mid \nabla_p F^{(\lambda'),\mathrm{v}'} = 0,\ (\lambda',\mathrm{v}') \in J_{\lambda,r} \times \mathcal{B}_{\mathrm{v},r}\}} \subset \mathcal{O}_1. \tag{D.12}$$

In fact, if this relation was violated, there would be a sequence of critical points $\mathcal{O}_1 \ni p_n \to p \in \overline{\mathcal{O}_1}\backslash\mathcal{O}_1$ corresponding to a sequence $(\lambda'_n,\mathrm{v}'_n) \in J_{\lambda,r} \times \mathcal{B}_{\mathrm{v},r}$. Since r is such that (D.10) holds (and all $F^{(\lambda'),\mathrm{v}'}$ in (D.10) have critical points in $\mathcal{O}_1$) this latter sequence must have accumulation points at the boundary of $J_{\lambda,r} \times \mathcal{B}_{\mathrm{v},r}$. By passing to $J_{\lambda,r/2} \times \mathcal{B}_{\mathrm{v},r/2}$, we exclude the sequence.

Since the sets $J_{\lambda,r} \times \mathcal{B}_{\mathrm{v},r}$ form a covering of the compact set $[-\lambda_0,\lambda_0] \times \overline{V}_\delta$, we can choose a finite sub-covering, labelled by the resulting centers $(\lambda_1,\mathrm{v}_1),\ldots,(\lambda_m,\mathrm{v}_m)$. We denote the neighbourhoods $\mathcal{O}_1$, corresponding to $F^{(\lambda_1),\mathrm{v}_1},\ldots,F^{(\lambda_m),\mathrm{v}_m}$, as $\mathcal{O}_{1,1},\ldots,\mathcal{O}_{1,m}$. We denote by $\chi_{\mathcal{O}_{1,j}} \in C_0^\infty(\mathcal{O}_{1,j})$ the approximate characteristic function of $\mathcal{O}_{1,j}$. Using (D.12), we can choose it in such a way, that the critical points of $F^{(\lambda'),\mathrm{v}'}$, $(\lambda',\mathrm{v}') \in J_{\mathrm{v}_j,r_j} \times \mathcal{B}_{\mathrm{v}_j,r_j}$, are outside of the support of $1-\chi_{O_{1,j}}$. Then, for each $\lambda \in [-\lambda_0,\lambda_0]$, $\mathrm{v} \in \overline{V}_\delta$, we write

$$I_{\mathrm{v},t}^{(\lambda)}(u) = I_{\mathrm{v},t}^{(\lambda)}(\chi_{\mathcal{O}_{1,j}}u) + I_{\mathrm{v},t}^{(\lambda)}((1-\chi_{\mathcal{O}_{1,j}})u), \tag{D.13}$$

where j is chosen so that $(\lambda,\mathrm{v}) \in J_{\lambda_j,r} \times \mathcal{B}_{\mathrm{v}_j,r}$. Now, since the Hessian matrix of $p \mapsto E_p^{(\lambda)}$ has the positivity property from Lemma 3.1, the first term on the r.h.s. of (D.13) satisfies the estimate from Theorem D.1,

$$|I_{\mathrm{v},t}^{(\lambda)}(\chi_{\mathcal{O}_{1,j}}u)| \le ct^{-3/2} \sum_{|\alpha|\le 2} \sup_p |\partial_p^\alpha u(p)| \tag{D.14}$$

with a constant chosen from a finite set, corresponding to $F^{(\lambda_1),\mathrm{v}_1},\ldots,F^{(\lambda_m),\mathrm{v}_m}$. To treat the second term on the r.h.s. of (D.13), we write similarly as in (D.5)

$$e^{i(p\cdot x - E_p^{(\lambda)}t)} = e^{iF^{(\lambda),\mathrm{v}}(p)t} = \frac{1}{it}\frac{\nabla_p F^{(\lambda),\mathrm{v}}(p)\cdot\nabla_p e^{iF^{(\lambda),\mathrm{v}}(p)t}}{|\nabla_p F^{(\lambda),\mathrm{v}}(p)|^2} \tag{D.15}$$

and integrate twice by parts. For p in the support of $1-\chi_{\mathcal{O}_{1,j}}$ and $(\lambda,\mathrm{v}) \in J_{\lambda_j,r} \times \mathcal{B}_{\mathrm{v}_j,r}$, the denominator satisfies

$$|\nabla_p F^{(\lambda),\mathrm{v}}(p)| = |\mathrm{v} - \nabla E_p^{(\lambda)}| > \epsilon_j > 0, \tag{D.16}$$

by our choice of $\chi_{\mathcal{O}_{1,j}}$ and relation (D.12). Thus, we obtain

$$|I_{\mathrm{v},t}^{(\lambda)}((1-\chi_{\mathcal{O}_{1,j}})u)| \le ct^{-2} \sum_{|\alpha|\le 2} \sup_p |\partial_p^\alpha u(p)|. \tag{D.17}$$

Summing up, for $\mathrm{v} \in \overline{V}_\delta$, $\lambda \in [-\lambda_0, \lambda_0]$,

$$\left| \int d^3p\, e^{iF^{(\lambda),\mathrm{v}} t} u(p) \right| \leq ct^{-3/2} \sum_{|\alpha| \leq 2} \sup_p |\partial_p^\alpha u(p)|, \tag{D.18}$$

with a constant independent of u, x, t and uniform in $|\lambda| \leq \lambda_0$. Using this and (D.1) we verify (D.6).

E Properties of Functions f_p, $f_{p,\sigma}$

We recall from (1.7), (3.4) the definitions:

$$f_p(k) := \lambda \frac{\chi_\kappa(k)}{\sqrt{2|k|}} \frac{1}{|k|(1 - e_k \cdot \nabla E_p)},\; f_{p,\sigma}(k) := \lambda \frac{\chi_{[\sigma,\kappa)}(k)}{\sqrt{2|k|}} \frac{1}{|k|(1 - e_k \cdot \nabla E_{p,\sigma})}. \tag{E.1}$$

We start with the following preparatory lemma.

Lemma E.1. *For $n = 1, 2, \ldots$, there holds the bound*

$$\int \frac{\chi_\kappa(k)^2}{|k|^3} |e^{-i|k|t + ik\cdot x} - 1|^n d^3k \leq c_n(1 + \log(1 + t + |x|)). \tag{E.2}$$

Proof. We estimate

$$\begin{aligned} &\int \frac{\chi_\kappa(k)^2}{|k|^3} |e^{-i|k|t + ik\cdot x} - 1|^n d^3k \\ &\leq \sum_{\varepsilon = \pm} \int_{\varepsilon(t - e_k\cdot x) \geq 0} d\Omega(e_k) \int_0^\kappa \frac{d|k|}{|k|} |e^{-i\varepsilon|k|\varepsilon(t - e_k\cdot x)} - 1|^n \\ &\leq \sum_{\varepsilon = \pm} \int_{\varepsilon(t - e_k\cdot x) \geq 0} d\Omega(e_k) \int_0^{\kappa(1 + t + |x|)} \frac{d|k|}{|k|} |e^{-i\varepsilon|k|} - 1|^n \\ &\leq c \int_0^\kappa \frac{d|k|}{|k|} |e^{-i|k|} - 1|^n + c2^n \int_\kappa^{\kappa(1 + t + |x|)} \frac{d|k|}{|k|} \\ &\leq c_n(1 + \log(1 + t + |x|)). \end{aligned} \tag{E.3}$$

This completes the proof.

Lemma E.2. *There holds the following bound for $|\alpha| = 0, 1, 2$*

$$|\partial_p^\alpha \gamma(p, x, t)| \leq c|\lambda|^2 (1 + \log(1 + t + |x|)). \tag{E.4}$$

Proof. We have

$$\partial_{p_i} f_p(k) = \lambda \frac{\chi_\kappa(k)}{\sqrt{2}|k|^{3/2}} \frac{1}{(1 - e_k \cdot \nabla E_p)^2} \partial_{p_i}(e_k \cdot \nabla E_p), \tag{E.5}$$

$$\begin{aligned} \partial_{p_j}\partial_{p_i} f_p(k) = \lambda \frac{\chi_\kappa(k)}{\sqrt{2}|k|^{3/2}} \Big\{ & 2\frac{1}{(1 - e_k \cdot \nabla E_p)^3} \partial_{p_j}(e_k \cdot \nabla E_p) \partial_{p_i}(e_k \cdot \nabla E_p) \\ & + \frac{1}{(1 - e_k \cdot \nabla E_p)^2} \partial_{p_j}\partial_{p_i}(e_k \cdot \nabla E_p) \Big\}. \end{aligned} \tag{E.6}$$

Thus by Lemma 3.1, we have

$$|f_p(k)|,\ |\partial_{p_i} f_p(k)|,\ |\partial_{p_j}\partial_{p_i} f_p(k)| \le c\lambda \frac{\chi_\kappa(k)}{\sqrt{2}|k|^{3/2}}. \tag{E.7}$$

Now, we write

$$\partial_p^\alpha \gamma(p,x,t) = \int d^3k\, \partial_p^\alpha (f_p(k)^2) \sin(|k|t - k\cdot x). \tag{E.8}$$

Clearly, estimates (E.7) and Lemma E.1 give (E.4). (Here, we made use of $|\sin y| = |\mathrm{Im}\, e^{iy}| = |\mathrm{Im}(e^{iy}-1)| \le |e^{iy}-1|$.)

Lemma E.3. *Let* $m(t,x) := (e^{-i|k|t+ik\cdot x} - 1)$*. Then, for* $|\alpha| \le 2, |\beta| \le 2$,

$$|\langle \partial_p^\alpha f_p m(t,x), \partial_p^\beta f_p m(t,x)\rangle| \le c|\lambda|^2 (1 + \log(1+t+|x|)). \tag{E.9}$$

Proof. Follows immediately from (E.7) and Lemma E.1. Indeed, we have

$$\begin{aligned} &\int d^3k\, |(\partial_p^\alpha f_p)(k)(\partial_p^\beta f_p)(k)|\, |e^{-i|k|t+ik\cdot x} - 1|^2 \\ &\le c|\lambda|^2 \int d^3k\, \frac{\chi_\kappa(k)^2}{2|k|^3} |e^{-i|k|t+ik\cdot x} - 1|^2 \\ &\le c|\lambda|^2 (1+\log(1+t+|x|)), \end{aligned} \tag{E.10}$$

which concludes the proof.

Lemma E.4. *The following bounds hold*

$$\begin{aligned} |\partial_p^\alpha f_{p,\sigma}(k)| &\le c\lambda \frac{\chi_{[\sigma,\kappa)}(k)}{\sqrt{2}|k|^{3/2}} \quad \text{for} \quad |\alpha| = 0,1, \\ |\partial_p^\alpha f_{p,\sigma}(k)| &\le \frac{c}{\sigma^{\delta_{\lambda_0}}} \lambda \frac{\chi_{[\sigma,\kappa)}(k)}{\sqrt{2}|k|^{3/2}} \quad \text{for} \quad |\alpha| = 2. \end{aligned} \tag{E.11}$$

Proof. The estimates follow from definition (3.4) via computations analogous to (E.5)–(E.6). For the relevant estimates on derivatives of $S \ni p \mapsto E_{p,\sigma}$ up to the third order, see [15, Theorem 2.1].

Lemma E.5. *The following bounds hold*

$$\|k_i(f_{p,\sigma} - f_p)\|_2 \le c\sigma^{1/4}, \quad |\langle f_{p,\sigma}, k_i f_{p,\sigma}\rangle - \langle f_p, k_i f_p\rangle| \le c\sigma^{1/4}. \tag{E.12}$$

Proof. Definitions (E.1) give

$$\begin{aligned} f_p(k) - f_{p,\sigma}(k) &= \lambda \frac{\chi_{[0,\sigma]}(k)}{\sqrt{2}|k|^{3/2}} \frac{1}{(1 - e_k\cdot\nabla E_p)} \\ &\quad + \lambda \frac{\chi_{[\sigma,\kappa]}(k)}{\sqrt{2}|k|^{3/2}} \left(\frac{1}{(1-e_k\cdot\nabla E_p)} - \frac{1}{(1-e_k\cdot\nabla E_{p,\sigma})} \right), \end{aligned} \tag{E.13}$$

where $\chi_{[0,\sigma]}$ is the characteristic function of a ball of radius σ centered at zero. Now considering that $|\nabla E_p - \nabla E_{p,\sigma}| \le c\sigma^{1/4}$ (cf. (C.6) above), we can write for any $\beta > 0$

$$\|\, |k|^\beta (f_{p,\sigma} - f_p)\|_2 \le c(\sigma^\beta + \sigma^{1/4}). \tag{E.14}$$

Setting $\beta = 1$, we obtain the first bound in (E.12). As for the second estimate, we note that

$$|\langle f_{p,\sigma}, k_i f_{p,\sigma}\rangle - \langle f_p, k_i f_p\rangle| \leq \int d^3k\, |k| \big|(f_{p,\sigma}(k) - f_p(k))(f_{p,\sigma}(k) + f_p(k))\big|$$
$$\leq \| \, |k|^{1/2}(f_{p,\sigma} - f_p)\|_2 \big(\| \, |k|^{1/2} f_{p,\sigma}\|_2 + \| \, |k|^{1/2} f_p\|_2\big). \tag{E.15}$$

Applying (E.14) with $\beta = 1/2$ and considering that $\| \, |k|^{1/2} f_{p,\sigma}\|_2, \| \, |k|^{1/2} f_p\|_2 \leq c$, we conclude the proof. □

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