Ann. Henri Poincaré 24 (2023), 1355–1376

1424-0637/23/041355-22
published online November 8, 2022
https://doi.org/10.1007/s00023-022-01245-2

Annales Henri Poincaré

H^1 Scattering for Mass-Subcritical NLS with Short-Range Nonlinearity and Initial Data in Σ

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Abstract. We consider short-range mass-subcritical nonlinear Schrödinger equations, and we show that the corresponding solutions with initial data in Σ scatter in H^1. Hence we up-grade the classical scattering result proved by Yajima and Tsutsumi from L^2 to H^1. We also provide some partial results concerning the scattering of the first order moments, as well as a short proof via lens transform of a classical result due to Tsutsumi and Cazenave–Weissler on the scattering in Σ.

1. Introduction

In this paper, we are interested in the long-time behavior of solutions to the following Cauchy problems associated with the defocusing nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u + \Delta u - u|u|^p = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0,.) = \varphi. \end{cases} \tag{1.1}$$

It is well known, by combining Strichartz estimates and a contraction argument, that the Cauchy problems above are locally well posed for every initial datum $\varphi \in H^1(\mathbb{R}^n)$ with time of existence which depends only on the size of the initial datum in $H^1(\mathbb{R}^n)$, provided that $0 < p < \frac{4}{n-2}$ if $n \geq 3$ and

N.T. was supported by ANR grant ODA (ANR-18-CE40-0020-01) and V.G. and N.V. by PRIN grant 2020XB3EFL; moreover, they acknowledge the Gruppo Nazionale per l' Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituzione Nazionale di Alta Matematica (INDAM). V.G. is also partially supported by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, and by Top Global University Project, Waseda University.

$0 < p < \infty$ if $n = 1, 2$. Then the conservation of mass and conservation of the energy:

$$E(u(t,x)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t,x)|^2 dx + \frac{1}{p+2}\int_{\mathbb{R}^n} |u(t,x)|^{p+2} dx,$$

(the energy is positive since we consider the defocusing NLS) imply that the $H^1(\mathbb{R}^n)$ norm of the solution is uniformly bounded and hence the local theory can be iterated in order to provide a global well-posedness result. There is a huge literature around this topic; for simplicity we quote the very complete book [7] and all the references therein. We also recall that the much more difficult critical nonlinearity $p = \frac{4}{n-2}$ for $n \geq 3$ has been extensively studied more recently starting from the pioneering paper [3] in the radial case and its extension in the non-radial setting in [11]. In the sequel, in order to emphasize the dependence of the nonlinear solution from the initial datum, we shall write $u_\varphi(t,x)$ to denote the unique global solution to (1.1) where $\varphi \in H^1(\mathbb{R}^n)$ and p is assumed to be given.

Once the existence of global solutions is established, it is natural to look at the long-time behavior. In the range of mass-supercritical and energy-subcritical nonlinearities, namely $\frac{4}{n} < p < \frac{4}{n-2}$ for $n \geq 3$ and $\frac{4}{n} < p < \infty$ for $n = 1, 2$, it has been proved that nonlinear solutions to NLS behave as free waves as $t \to \pm\infty$. More precisely, we have the following property:

$$\forall \varphi \in H^1(\mathbb{R}^n) \quad \exists \varphi_\pm \in H^1(\mathbb{R}^n) \text{ s.t. } \|u_\varphi(t,x) - e^{it\Delta}\varphi_\pm\|_{H^1(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0,$$
$$\text{provided that } \frac{4}{n} < p < \frac{4}{n-2} \text{ for } n \geq 3, \quad \frac{4}{n} < p < \infty \text{ for } n = 1, 2. \tag{1.2}$$

We point out that the scattering property (1.2) can be stated in the following equivalent form

$$\|e^{-it\Delta}u_\varphi(t,x) - \varphi_\pm\|_{H^1(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0 \tag{1.3}$$

by using the fact that the group $e^{it\Delta}$ is an isometry in $H^1(\mathbb{R}^n)$. The property (1.2) is known in the literature as the asymptotic completeness of the wave operator in $H^1(\mathbb{R}^n)$, or more quickly $H^1(\mathbb{R}^n)$ scattering. Roughly speaking, (1.2) implies that for large times (both positive and negative) the nonlinear evolution can be approximated in $H^1(\mathbb{R}^n)$ by a linear one with a suitably modified initial data which represents the nonlinear effect. The literature around $H^1(\mathbb{R}^n)$ scattering in the mass-supercritical and energy-subcritical case is huge. Beside the already quoted reference [7] and the bibliography therein, we mention at least [17] in the case $n \geq 3$ and [22] for $n = 1, 2$. More recently shorter proof of scattering in the energy space $H^1(\mathbb{R}^n)$ for mass-supercritical and energy-subcritical NLS has been achieved by using the interaction Morawetz estimates, first introduced in [11]. We mention in this direction [9,10,24,31] and all the references therein. We recall that the scattering of nonlinear solutions to free waves in the energy space has been extended to the energy critical case, namely $p = \frac{4}{n-2}$ when $n \geq 3$, in a series of papers starting from the pioneering articles [3] and [11] for $n = 3$. Its extension in higher dimension is provided in [26,30]. In the mass critical case $p = \frac{4}{n}$, the $H^1(\mathbb{R}^n)$ scattering property follows from [13–15].

Notice that the mass-subcritical nonlinearities, namely $0 < p < \frac{4}{n}$, do not enter in the analysis above. In fact, we can introduce the intermediate nonlinearity $p = \frac{2}{n}$ which is a discriminant between short-range ($\frac{2}{n} < p < \frac{4}{n}$) and long-range nonlinearity ($0 < p \leq \frac{2}{n}$). More specifically, one can prove that in the long-range mass-subcritical setting nonlinear solutions do not behave as free waves. In this direction, we mention [1] and [7], where it is proved that the scattering property fails in the $L^2(\mathbb{R}^n)$ topology for every nontrivial solution to NLS, even for initial datum which is very smooth. The precise statement can be given in the following form:

$$\limsup_{t\to\pm\infty} \|u_\varphi(t,x) - e^{it\Delta}\psi\|_{L^2(\mathbb{R}^n)} > 0,$$
$$\forall(\varphi,\psi) \in C_0^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \quad (\varphi,\psi) \neq (0,0), \text{ provided that } 0 < p \leq \frac{2}{n}.$$

On the contrary in the short-range mass-subcritical case, following [28], one can show the following version of scattering:

$$\begin{aligned} &\forall\varphi \in \Sigma_n \quad \exists\varphi_\pm \in L^2(\mathbb{R}^n) \text{ s.t.} \\ &\quad \|u_\varphi(t,x) - e^{it\Delta}\varphi_\pm\|_{L^2(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0, \text{ provided that } \frac{2}{n} < p < \frac{4}{n} \end{aligned} \tag{1.4}$$

where the space Σ_n is the following one:

$$\Sigma_n = \left\{\varphi \in H^1(\mathbb{R}^n) | \int_{\mathbb{R}^n} |x|^2|\varphi|^2 dx < \infty\right\},$$

endowed with the norm

$$\|\varphi\|^2_{\Sigma_n} = \int_{\mathbb{R}^n} (|\nabla\varphi|^2 + |\varphi|^2 + |x|^2|\varphi|^2)dx.$$

Some properties of the data-to-scattering-states map have been considered in [21] in the mass-subcritical case. As an outcome, one can deduce a mild ill-posedness of this operator in the L^2 topology.

Notice that the result in [28] is very general, in the sense that the full set of short-range mass-subcritical nonlinearities $\frac{2}{n} < p < \frac{4}{n}$ is covered, and is sharp in view of the aforementioned result in [1]. However, the weakness of (1.4) is that although the initial datum is assumed to belong to the space Σ_n, the convergence to free waves is proved only in the $L^2(\mathbb{R}^n)$ sense.

The main aim of this paper is to overcome, at least partially, this fact and to up-grade the convergence from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$ by assuming that the initial datum belongs to the space Σ_n. We can now state the main result of this paper

Theorem 1.1. *Assume $\frac{2}{n} < p < \frac{4}{n}$, then for every $\varphi \in \Sigma_n$ there exist $\varphi_\pm \in H^1(\mathbb{R}^n)$ such that*

$$\|u_\varphi(t,x) - e^{it\Delta}\varphi_\pm\|_{H^1(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0. \tag{1.5}$$

We remark that by the conservation of mass and energy we get

$$\sup_t \|u_\varphi(t,x)\|_{H^1(\mathbb{R}^n)} < \infty. \tag{1.6}$$

By using this fact, along with an interpolation argument and (1.4), it is easy to deduce that one can conclude

$$\|u_\varphi(t,x) - e^{it\Delta}\varphi_\pm\|_{H^s(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0, \quad s\in[0,1) \tag{1.7}$$

for $\frac{2}{n} < p < \frac{4}{n}$. However, the convergence in $H^1(\mathbb{R}^n)$ stated in Theorem 1.1 is more delicate and is the main contribution of the paper.

We point out that Theorem 1.1 covers the full set of short-range mass-subcritical nonlinearities $\frac{2}{n} < p < \frac{4}{n}$, despite previous results where only a subset of short-range nonlinearities was treated. We quote in this direction [8] and [27], where the following property (which is stronger than $H^1(\mathbb{R}^n)$ scattering) is proved:

$$\begin{aligned} &\forall\varphi\in\Sigma_n \quad \exists\varphi_\pm\in\Sigma_n \text{ s.t.} \\ &\qquad \|e^{-it\Delta}u_\varphi(t,x) - \varphi_\pm\|_{\Sigma_n} \overset{t\pm\infty}{\longrightarrow} 0, \text{ provided that } p_n \le p < \frac{4}{n} \end{aligned} \tag{1.8}$$

where

$$p_n = \frac{2-n+\sqrt{n^2+12n+4}}{2n}, \tag{1.9}$$

i.e., p_n is the larger root of the polynomial $nx^2+(n-2)x-4=0$ (see Appendix for a short proof of (1.8) via the lens transform). One can check that $p_n > \frac{2}{n}$ for every $n\ge 1$, and hence the results in [8] and [27] do not cover the full set of short-range mass-subcritical nonlinearities. Notice also that, despite the fact that (1.2) and (1.3) are equivalent, it is not clear whether or not (1.8) implies

$$\|u_\varphi(t,x) - e^{it\Delta}\varphi_\pm\|_{\Sigma_n} \overset{t\to\pm\infty}{\longrightarrow} 0. \tag{1.10}$$

In fact it is well known that, due to the dispersion, the Σ_n norm grows quadratically in time along free waves and hence the group $e^{it\Delta}$ is not uniformly bounded w.r.t. the Σ_n topology. Only in some very few special cases, it is proved that (1.8) implies (1.10) (see [2]). Summarizing the main point in Theorem 1.1 is that we cover the full range of nonlinearities $\frac{2}{n} < p < \frac{4}{n}$; however, our conclusion is weaker than (1.8) which on the other hand is available for a more restricted set of nonlinearities.

We point out that our approach to prove Theorem 1.1 is based only on Hilbert space considerations and we don't rely on Strichartz estimates. In fact, Strichartz estimates in collaboration with boundedness of a family of space-time Lebesgue norms that arise from the pseudoconformal energy are the key tools in [8] and [27]. However, in order to close the estimates, following this approach, some restrictions appear on the nonlinearity and hence the lower bound $p\ge p_n$ is needed.

We also underline that in order to prove Theorem 1.1 we take the result in [28] (see (1.4)) as a black-box and we prove how to go from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$ convergence. The proof of Theorem 1.1 is obtained as a combination of [28] and the following result.

Theorem 1.2. *Let $\varphi \in \Sigma_n$, $0 < p < \frac{4}{n}$ and assume that there exist $\varphi_\pm \in H^1(\mathbb{R}^n)$ such that*

$$\big\| |u_\varphi(t,x)| - |e^{it\Delta}\varphi_\pm| \big\|_{L^2(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0, \tag{1.11}$$

then

$$\|\nabla u_\varphi(t,x)\|_{L^2(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} \|\nabla \varphi_\pm\|_{L^2(\mathbb{R}^n)}. \tag{1.12}$$

Notice that in Theorem 1.2 on the one hand we allow the nonlinearity p to be mass-subcritical (both short-range and long-range); on the other hand, we assume (1.11) which is granted in the short-range setting by the stronger condition (1.4).

Next we do some considerations about the convergence of the second-order moments of solutions to (1.1) to free waves, if the initial datum belongs to Σ_n. We recall that the classical definition of scattering in Σ_n (see [7] and all the references therein) is provided by (1.8), which unfortunately we are not able to show in the full set of short-range mass-subcritical nonlinearities. Moreover, as already mentioned above it is unclear how, even if (1.8) is established, one can compare the nonlinear solutions to free waves as described in (1.10). On the other hand, notice that (1.10) is a very strong request since it requires to compare asymptotically quantities which diverge for large times. In fact, it is well known that for free waves the second-order moments grow quadratically and hence the request (1.10) seems to be very hard to prove (in fact it is known in very few cases, see [2]). On the other hand, for free waves with initial datum in Σ_n we have that the renormalized second-order moments $\int_{\mathbb{R}^n} \frac{|x|^2}{t^2} |e^{it\Delta}\varphi|^2 dx$ are bounded and we have a precise limit as $t \to \pm\infty$ (see for instance [29]). As a consequence, it seems quite natural to understand whether or not we can compare the renormalized second-order moment of the nonlinear solution with the renormalized second-order moment of the free wave. The aim of next result is to show that scattering of renormalized second-order moments is equivalent to the regularity of the scattering state $\varphi_\pm$.

Theorem 1.3. *Let $p, \varphi, \varphi_\pm$ as in Theorem 1.1, then we have the following equivalence:*

$$\Big\| \frac{|x|}{t}(u_\varphi(t,x) - e^{it\Delta}\varphi_\pm) \Big\|_{L^2(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0 \iff \varphi_\pm \in \Sigma_n.$$

Unfortunately, we can prove the property $\varphi_\pm \in \Sigma_n$ only for a subset of short-range mass-subcritical NLS, namely the ones treated in the references [8,27]. Indeed once (1.8) is established, we get for free $\varphi_\pm \in \Sigma_n$ provided that $\varphi \in \Sigma_n$. We believe that the property $\varphi_\pm \in \Sigma_n$, which appears in Theorem 1.3, is an interesting question of intermediate difficulty compared with the proof of scattering in Σ_n as described in (1.8). We think it deserves to be investigated in the full set of short-range mass-subcritical nonlinearities.

Next we make a further comment about the condition $\varphi_\pm \in \Sigma_n$; in particular, we show its connection with a question of regularity for a family of

Cauchy problems. First we introduce the pesudo-conformal transformation of $u_\varphi(t,x)$, which will play a crucial role in the sequel:

$$w_\varphi(t,x)=\frac{1}{t^{n/2}}\bar{u}_\varphi\left(\frac{1}{t},\frac{x}{t}\right)e^{i\frac{|x|^2}{4t}}. \tag{1.13}$$

We recall that the key point in [28] is the proof of the existence of the functions $w_\varphi^\pm\in L^2(\mathbb{R}^n)$ such that

$$\|w_\varphi(t,x)-w_\varphi^\pm\|_{L^2(\mathbb{R}^n)}\overset{t\to0^\pm}{\longrightarrow}0. \tag{1.14}$$

The following connection is well known (see (3.1) in [27], (14) in [28]) for $\frac{2}{n}<p<\frac{4}{n}$:

$$\hat{\varphi}_\pm(\xi)=(2i)^{\frac{n}{2}}\bar{w}_\varphi^\pm(2\xi). \tag{1.15}$$

As a consequence, we get, based on elementary Fourier analysis, the following equivalence:

$$\int_{\mathbb{R}^n}|x|^2|\varphi_\pm|^2dx<\infty \iff w_\varphi^\pm\in\dot{H}^1(\mathbb{R}^n). \tag{1.16}$$

By using (1.16) and writing the partial differential equation solved by $w_\varphi(t,x)$ (see Sect. 2), we get that the property $\varphi_\pm\in\Sigma_n$ (appearing in Theorem 1.3) is equivalent to study up to the time $t=0$ the $H^1(\mathbb{R}^n)$ regularity of solutions to the following Cauchy problem:

$$\begin{cases} i\partial_t w+\Delta w-t^{-\alpha(n,p)}w|w|^p=0, & (t,x)\in(0,\infty)\times\mathbb{R}^n, \quad \alpha(n,p)=2-\frac{np}{2}.\\ w(1,.)\in\Sigma_n. \end{cases}$$

We believe that the analysis of the Cauchy problem above, up to time $t=0$ and for $\frac{2}{n}<p<\frac{4}{n}$, has its own independent interest.

We conclude the introduction by quoting the papers [18] and [19] where the question of scattering theory is studied in negative Sobolev spaces for a family of long-range mass-subcritical nonlinearities. In particular, a series of conditional scattering results are achieved in the aforementioned papers. We finally mention [4] where the authors prove in dimension $n=1$ new probabilistic results about scattering and smoothing effect of the scattering states in weighted negative Sobolev spaces in the mass-subcritical short-range regime. The result has been extended in higher dimensions under the radiality condition in [20].

2. The Pseudo-Conformal Transformation

Let $u_\varphi(t,x)$ be the unique global solution to (1.1) with initial condition $\varphi\in\Sigma_n$, then following [28] we introduce the pseudo-conformal transformation $w_\varphi(t,x)$ defined by (1.13). Notice that $w_\varphi(t,x)$ is well defined for $(t,x)\in(0,\infty)\times\mathbb{R}^n$ and $(t,x)\in(-\infty,0)\times\mathbb{R}^n$. We shall focus mainly on the restriction of $w_\varphi(t,x)$ on the strip $(t,x)\in(0,\infty)\times\mathbb{R}^n$, which is of importance in order to prove Theorem 1.1 as $t\to\infty$, by a similar argument we can treat the case $t\to-\infty$ by using the restriction of $w_\varphi(t,x)$ on the strip $(t,x)\in(-\infty,0)\times\mathbb{R}^n$. One can check by direct computation that $w_\varphi(t,x)$ is solution to the following partial differential equation:

$$i\partial_t w_\varphi + \Delta w_\varphi - t^{-\alpha(n,p)} w_\varphi |w_\varphi|^p = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \quad \alpha(n,p) = 2 - \frac{np}{2}. \tag{2.1}$$

Notice that in the regime of short-range nonlinearity we have that $t^{-\alpha(n,p)} \in L^1(0,1)$ and in the regime of long-range nonlinearity we have that $t^{-\alpha(n,p)} \notin L^1(0,1)$. Hence the nonlinearity $p = \frac{2}{n}$ is borderline to guarantee local integrability in a neighborhood of the origin of the weight $t^{-\alpha(n,p)}$ which appears in front of the nonlinearity in (2.1). As already mentioned in the introduction, the key idea in [28] is to deduce the $L^2(\mathbb{R}^n)$ scattering property for the solution $u_\varphi(t,x)$ as $t \to \infty$ by showing that the following limit exists

$$\lim_{t \to 0^+} w_\varphi(t,x) \text{ in } L^2(\mathbb{R}^n).$$

Notice that even if $w_\varphi(t,x) \in \Sigma_n$ for $t \neq 0$, it is not well defined at $t = 0$ and hence to show the existence of the limit above as $t \to 0^+$ is not obvious. In order to achieve this property in [28], it is first proved that the limit above exists in $L^2(\mathbb{R}^n)$ in the weak sense, and then in a second step the convergence is up-graded to strong convergence in $L^2(\mathbb{R}^n)$.

We collect in the next proposition the key properties of $w_\varphi(t,x)$ that will be useful in the sequel.

Proposition 2.1. *Let $\varphi \in \Sigma_n$, $0 < p < \frac{4}{n-2}$ for $n \geq 3$ and $0 < p < \infty$ for $n = 1, 2$. Let $w_\varphi(t,x)$ be the pseudoconformal transformation associated with $u_\varphi(t,x)$ as in (1.13), then we have the following properties:*

$$w_\varphi(t,x) \in C((0,\infty); \Sigma_n) \tag{2.2}$$

and

$$\begin{aligned} t^{\alpha(n,p)} \|\nabla w_\varphi(t,x)\|^2_{L^2(\mathbb{R}^n)} + \frac{2}{p+2} \|w_\varphi(t,x)\|^{p+2}_{L^{p+2}(\mathbb{R}^n)} \\ = \|\nabla w_\varphi(t,x)\|^2_{L^2} \frac{d}{dt} t^{\alpha(n,p)} > 0. \end{aligned} \tag{2.3}$$

In particular for $0 < p < \frac{4}{n}$ we have

$$\sup_{t \in (0,1]} \left(t^{\alpha(n,p)} \|\nabla w_\varphi(t,x)\|^2_{L^2(\mathbb{R}^n)} + \|w_\varphi(t,x)\|^{p+2}_{L^{p+2}(\mathbb{R}^n)} \right) < \infty. \tag{2.4}$$

The estimate (2.4) plays a fundamental role in [28] and will be of crucial importance in the sequel along the proof of Theorem 1.2. The basic idea to establish (2.4) is to multiply first the equation (2.1) by $t^{\alpha(n,p)}$, and then in a second step the corresponding equation is tested with the function $\partial_t \bar{w}_\varphi(t,x)$. Then the proof follows by integration by parts and by considering the real part of the identity obtained. Concerning the property (2.2), it follows from the definition of $w_\varphi(t,x)$ and from the fact that $\varphi \in \Sigma_n$ implies $u_\varphi(t,x) \in C((0,\infty); \Sigma_n)$ (see [7] for a proof of this fact).

3. Proof of Theorem 1.1

We shall treat in detail the case $t \to \infty$ (by a similar argument one can treat $t \to -\infty$). Let $\varphi_+ \in L^2(\mathbb{R}^n)$ be given in (1.4), then we shall prove $\varphi_+ \in H^1(\mathbb{R}^n)$ and also

$$e^{-it\Delta} u_\varphi(t,x) \overset{t\to\infty}{\longrightarrow} \varphi_+ \text{ in } H^1(\mathbb{R}^n). \tag{3.1}$$

This will complete the proof of (1.5) since $e^{-it\Delta}$ are isometries in $H^1(\mathbb{R}^n)$. Notice also that by (1.6) in conjunction with the fact that $e^{it\Delta}$ are isometries in $L^2(\mathbb{R}^n)$ commuting with the operator ∇ we get:

$$\sup_t \|e^{-it\Delta} u_\varphi(t,x)\|_{H^1(\mathbb{R}^n)} < \infty. \tag{3.2}$$

On the other hand, (1.4) implies

$$\|e^{-it\Delta} u_\varphi(t,x) - \varphi_+\|_{L^2(\mathbb{R}^n)} = \|u_\varphi(t,x) - e^{it\Delta}\varphi_+\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} 0,$$

namely we have convergence of $e^{-it\Delta} u_\varphi(t,x)$ to φ_+ in $L^2(\mathbb{R}^n)$. By combining this fact with (3.2), we conclude on the one hand $\varphi_+ \in H^1(\mathbb{R}^n)$, on the other hand we get the weak convergence

$$e^{-it\Delta} u_\varphi(t,x) \overset{t\to\infty}{\rightharpoonup} \varphi_+ \text{ in } H^1(\mathbb{R}^n). \tag{3.3}$$

We claim that (3.1) follows provided that we show

$$\|\nabla(e^{-it\Delta} u_\varphi(t,x))\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} \|\nabla\varphi_+\|_{L^2(\mathbb{R}^n)}. \tag{3.4}$$

In fact by combining (3.4) with the following convergence

$$\|e^{-it\Delta} u_\varphi(t,x)\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} \|\varphi_+\|_{L^2(\mathbb{R}^n)}$$

(which in turn follows from (1.4)), we get

$$\|e^{-it\Delta} u_\varphi(t,x)\|_{H^1(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} \|\varphi_+\|_{H^1(\mathbb{R}^n)}. \tag{3.5}$$

Then we have weak convergence in $H^1(\mathbb{R}^n)$ by (3.3) and convergence of the norms by (3.5); hence, we get strong convergence in $H^1(\mathbb{R}^n)$. We conclude since (3.4) follows by (1.12) in Theorem 1.2 in conjunction with the fact that $e^{it\Delta}$ are isometries in $L^2(\mathbb{R}^n)$ commuting with the operator ∇. Notice that the assumptions of Theorem 1.2 are satisfied, in fact (1.11) is weaker than (1.4) and $\varphi_+ \in H^1(\mathbb{R}^n)$ has been established above.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 follows from two lemma in conjunction with an argument that we borrow from [29], where the precise long-time behavior of moments is considered for a family of mass-supercritical NLS.

Lemma 4.1. *For $p \in (0, \frac{4}{n})$, we have the following property:*

$$\left\|\nabla u_\varphi(t,x) - i\frac{x}{2t} u_\varphi(t,x)\right\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. By using (1.13), we get

$$\nabla w_\varphi(t,x) = \frac{1}{t^{n/2+1}}\nabla \bar{u}_\varphi\left(\frac{1}{t},\frac{x}{t}\right)e^{i\frac{|x|^2}{4t}} + \frac{i}{2t^{n/2+1}}x\bar{u}_\varphi\left(\frac{1}{t},\frac{x}{t}\right)e^{i\frac{|x|^2}{4t}}$$

and hence

$$\|\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)} = \left\|\frac{1}{t}\nabla\bar{u}_\varphi\left(\frac{1}{t},x\right) + i\frac{x}{2}\bar{u}_\varphi\left(\frac{1}{t},x\right)\right\|_{L^2(\mathbb{R}^n)}.$$

Then we get

$$\left\|\nabla u_\varphi\left(\frac{1}{t},x\right) - i\frac{tx}{2}u_\varphi\left(\frac{1}{t},x\right)\right\|_{L^2(\mathbb{R}^n)} = t\|\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}, \quad \forall t\in(0,1]$$

and by (2.4) we have

$$\left\|\nabla u_\varphi\left(\frac{1}{t},x\right) - i\frac{tx}{2}u_\varphi\left(\frac{1}{t},x\right)\right\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{\alpha(n,p)}{2}+1}).$$

We conclude by considering the limit as $t\to 0^+$ (and hence $\frac{1}{t}\to\infty$) and by noticing the $-\frac{\alpha(n,p)}{2}+1>0$. $\square$

Lemma 4.2. *For $p\in(0,\frac{4}{n})$, we have:*

$$\forall\varepsilon>0 \quad \exists t_\varepsilon, R_\varepsilon>0 \ s.t. \ \sup_{t>t_\varepsilon}\int_{|x|>R_\varepsilon t}\frac{|x|^2}{t^2}|u_\varphi(t,x)|^2dx<\varepsilon. \tag{4.1}$$

Proof. We have the identity

$$|w_\varphi(s,x)|^2 = \frac{1}{s^n}\left|u_\varphi\left(\frac{1}{s},\frac{x}{s}\right)\right|^2$$

and hence

$$\int_{|x|>R}|x|^2|w_\varphi(s,x)|^2 = \int_{|x|>R}|x|^2\left|u_\varphi\left(\frac{1}{s},\frac{x}{s}\right)\right|^2\frac{dx}{s^n}$$
$$= s^2\int_{s|x|>R}|x|^2\left|u_\varphi\left(\frac{1}{s},x\right)\right|^2dx.$$

If we denote $s=\frac{1}{t}$, we get the following identity:

$$\int_{|x|>Rt}\frac{|x|^2}{t^2}|u_\varphi(t,x)|^2dx = \int_{|x|>R}|x|^2\left|w_\varphi\left(\frac{1}{t},x\right)\right|^2dx, \quad \forall R>0.$$

Hence in order to get the conclusion (4.1) we are reduced to prove:

$$\forall\varepsilon>0 \quad \exists\tilde{t}_\varepsilon,\tilde{R}_\varepsilon \text{ s.t. } \sup_{t\in(0,\tilde{t}_\varepsilon]}\int_{|x|>\tilde{R}_\varepsilon}|x|^2|w_\varphi(t,x)|^2dx<\varepsilon. \tag{4.2}$$

More precisely showing smallness of the contribution to the renormalized second-order moment of $u_\varphi(t,x)$ for large times in the exterior of a cone is equivalent to showing smallness of the contribution to the second-order moment of $w_\varphi(t,x)$ for small times in the exterior of a cylinder. In order to prove (4.2), first we introduce a non-negative function $\psi\in C^\infty(\mathbb{R}^n)$ such that:

(1) $\psi(x) = \psi(|x|)$;
(2) $\psi(x) = |x|, \quad \forall |x| > 1$;
(3) $\psi(x) = 0, \quad \forall |x| < \frac{1}{2}$.

Along with ψ, we introduce the rescaled functions $\psi_R(x) = R\psi\big(\frac{x}{R}\big)$. First notice that there exists $C > 0$ such that:

$$\sup_{\substack{t\in(0,1)\\ R>0}} \left(\int_{\mathbb{R}^n} (\psi_R(x))^2 |w_\varphi(t,x)|^2 dx\right)^{\frac{1}{2}} \le C \sup_{t\in(0,1)} \|x w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}$$
$$\le C \sup_{t\in(0,1)} \|x w_\varphi(t,x) + 2it\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}$$
$$+ 2C \sup_{t\in(0,1)} \|t\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)} < \infty, \quad (4.3)$$

where at the last step we have used (2.4) to control the term $\|t\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}$, and the following identity (in turn coming from the definition of w_φ)

$$\sup_{t\in(0,1)} \|x w_\varphi(t,x) + 2it\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}$$
$$= \sup_{t\in(0,1)} \left\|\nabla u_\varphi\left(\frac{1}{t}, x\right)\right\|_{L^2(\mathbb{R}^n)}$$

along with (1.6) to control $\|x w_\varphi(t,x) + 2it\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}$. Next by elementary computations we get:

$$\left|\frac{d}{dt}\int_{\mathbb{R}^n} (\psi_R(x))^2 |w_\varphi(t,x)|^2 dx\right| \le C \int_{\mathbb{R}^n} \psi_R(x)|\nabla\psi_R(x)||w_\varphi(t,x)||\nabla w_\varphi(t,x)|dx$$
$$\le C\|\nabla w_\varphi(t,x)\|_{L^2(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} (\psi_R(x))^2|w_\varphi(t,x)|^2dx\right)^{\frac{1}{2}} \le Ct^{-\frac{\alpha}{2}}, \quad \forall t\in(0,1), \quad \forall R>0, \quad (4.4)$$

where we have used at the last step (2.4) along with the bound (4.3). Notice that in order to conclude (4.2) it is sufficient to show that for every $\varepsilon > 0$ there exist $\tilde{t}_\varepsilon, \tilde{R}_\varepsilon > 0$ such that

$$\sup_{t\in(0,\tilde{t}_\varepsilon]} \int_{\mathbb{R}^n} (\psi_{\tilde{R}_\varepsilon}(x))^2 |w_\varphi(t,x)|^2 dx < \varepsilon. \quad (4.5)$$

In order to select $\tilde{t}_\varepsilon, \tilde{R}_\varepsilon > 0$ with this property, notice that by (4.4) we get

$$\sup_{t\in(0,\bar{t}]} \int_{\mathbb{R}^n} (\psi_R(x))^2|w_\varphi(t,x)|^2dx \le \int_{\mathbb{R}^n} (\psi_R(x))^2|w_\varphi(\bar{t},x)|^2dx$$
$$+ C\int_0^{\bar{t}} \tau^{-\frac{\alpha}{2}} d\tau, \quad \forall \bar{t}\in(0,1), \quad \forall R>0$$

and hence it is sufficient to choose $\bar{t} = \tilde{t}_\varepsilon$ such that $C\int_0^{\tilde{t}_\varepsilon} \tau^{-\frac{\alpha}{2}} d\tau < \frac{\varepsilon}{2}$ and $R = \tilde{R}_\varepsilon$ in such a way that $\int_{\mathbb{R}^n} (\psi_{\tilde{R}_\varepsilon}(x))^2 |w_\varphi(\tilde{t}_\varepsilon, x)|^2 dx < \frac{\varepsilon}{2}$ (notice that this choice of $\tilde{R}_\varepsilon$ is possible by (2.2)). □

Proof of Theorem 1.2. By Lemma 4.1, we have that (1.12) is equivalent to

$$\left\|\frac{x}{2t}u_\varphi(t,x)\right\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} \|\nabla\varphi_+\|_{L^2(\mathbb{R}^n)}. \tag{4.6}$$

Next we show that (4.6) is almost satisfied if we compute the L^2 norm in the more restricted region inside the cone $|x|<Rt$, for $R>0$ that will be chosen larger and larger. More precisely, we shall prove the following fact:

$$\int_{|x|<Rt}\frac{|x|^2}{t^2}|u_\varphi(t,x)|^2dx \overset{t\to\infty}{\longrightarrow} 4\int_{|x|<\frac{R}{2}}|x|^2|\hat{\varphi}_+(x)|^2dx. \tag{4.7}$$

By combining this property with (4.1) and by noticing that

$$\int_{|x|<\frac{R}{2}}|x|^2|\hat{\varphi}_+(x)|^2dx \overset{R\to\infty}{\longrightarrow} \int_{\mathbb{R}^n}|x|^2|\hat{\varphi}_+(x)|^2dx = \|\nabla\varphi_+\|^2_{L^2},$$

we conclude (4.6).

In order to prove (4.7), we shall use the following asymptotic formula to describe free waves (see [16] and [25]):

$$\left\|e^{it\Delta}h - \frac{e^{i\frac{|x|^2}{4t}}}{(2it)^{\frac{n}{2}}}\hat{h}\left(\frac{x}{2t}\right)\right\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} 0, \quad \forall\, h\in L^2(\mathbb{R}^n), \tag{4.8}$$

where $\hat{h}(\xi)$ denotes the Fourier transform of h, which in turn implies

$$\left\||e^{it\Delta}h| - \frac{\left|\hat{h}(\frac{x}{2t})\right|}{(2t)^{\frac{n}{2}}}\right\|_{L^2(\mathbb{R}^n)} \overset{t\to\infty}{\longrightarrow} 0, \quad \forall\, h\in L^2(\mathbb{R}^n). \tag{4.9}$$

Next for every $R>0$ fixed we get by the Minkowski inequality

$$\begin{aligned}\left\|\frac{|x|}{t}\left(|u_\varphi(t,x)| - \frac{\left|\hat{\varphi}_+(\frac{x}{2t})\right|}{(2t)^{\frac{n}{2}}}\right)\right\|_{L^2(|x|<Rt)} &\leq \left\|\frac{|x|}{t}\left(|u_\varphi(t,x)| - |e^{it\Delta}\varphi_+|\right)\right\|_{L^2(|x|<Rt)}\\ &+\left\|\frac{|x|}{t}\left(|e^{it\Delta}\varphi_+| - \frac{\left|\hat{\varphi}_+(\frac{x}{2t})\right|}{(2t)^{\frac{n}{2}}}\right)\right\|_{L^2(|x|<Rt)}.\end{aligned} \tag{4.10}$$

Next notice that for any fixed $R>0$ inside the cone $|x|<Rt$ we have that the weight $\frac{|x|}{t}$ is uniformly bounded and hence by combining (4.9) with (1.11) we conclude that both terms on the r.h.s. in (4.10) converge to zero as $t\to\infty$, and hence:

$$\left\|\frac{|x|}{t}\left(|u_\varphi(t,x)| - \frac{\left|\hat{\varphi}_+(\frac{x}{2t})\right|}{(2t)^{\frac{n}{2}}}\right)\right\|_{L^2(|x|<Rt)} \overset{t\to\infty}{\longrightarrow} 0,$$

which in turn implies

$$\int_{|x|<Rt}\frac{|x|^2}{t^2}|u_\varphi(t,x)|^2dx - \int_{|x|<Rt}\frac{|x|^2}{t^2}\left|\hat{\varphi}_+\left(\frac{x}{2t}\right)\right|^2\frac{dx}{(2t)^n} \overset{t\to\infty}{\longrightarrow} 0.$$

By a change of variable, we get (4.7).

5. Proof of Theorem 1.3

We show separately the two implications.

Proof of $\Rightarrow$. Recall that we have the property $u_\varphi(t,x)\in\Sigma_n$ for every $t>0$, since $\varphi\in\Sigma_n$ (see [7]). Moreover, by assumption we have

$$\left\|\frac{|x|}{t}(u_\varphi(t,x)-e^{it\Delta}\varphi_+)\right\|_{L^2(\mathbb{R}^n)}\overset{t\to\infty}{\longrightarrow}0$$

and in particular there exists $\bar t$ such that

$$\||x|(u_\varphi(\bar t,x)-e^{i\bar t\Delta}\varphi_+)\|_{L^2(\mathbb{R}^n)}<\infty. \tag{5.1}$$

Since we know that $u_\varphi(\bar t,x)\in\Sigma_n$ necessarily we have by (5.1) and the Minkowski inequality that $e^{i\bar t\Delta}\varphi_+\in\Sigma_n$ and hence, by the invariance of the space Σ_n under the linear flow $e^{it\Delta}$, we deduce

$$\varphi_+=e^{-i\bar t\Delta}(e^{i\bar t\Delta}\varphi_+)\in\Sigma_n.$$

Proof of $\Leftarrow$. We use the following well-known identity:

$$\frac{x}{t}(u_\varphi-e^{it\Delta}\varphi_+)=\frac{1}{t}(x+2it\nabla)u_\varphi-2i\nabla(u_\varphi-e^{it\Delta}\varphi_+)-\frac{1}{t}e^{it\Delta}(x\varphi_+).$$

Notice that by Lemma 4.1 the L^2 norm of the first term on the r.h.s. converges to zero as $t\to\infty$. The same property holds for the second term on the r.h.s. due to Theorem 1.1. The conclusion follows since $e^{it\Delta}$ is an isometry in L^2 and hence also the third term on the r.h.s. converges to zero as $t\to\infty$.

6. Appendix: Scattering in Σ_n Via Lens Transform, $p_n\le p<\frac{4}{n}$

The aim of this appendix is to provide alternative proof of the results established in [8] and [27] by using the lens transform (instead of the pseudoconformal energy which is the key tool in [8] and [27]). We introduce, following [4–6], for every time $t\in(-\frac{\pi}{4},\frac{\pi}{4})$ the lens transform acting as follows on time independent function $G:\mathbb{R}^n\to\mathbb{R}$:

$$\mathcal{L}_tG(x)=(\cos(2t))^{-\frac{n}{2}}G\Big(\frac{x}{\cos(2t)}\Big)e^{-i\frac{|x|^2\tan(2t)}{2}},\quad x\in\mathbb{R}^n.$$

By direct computation, we have that if we denote

$$H=-\Delta+|x|^2, \tag{6.1}$$

then we get the following identity:

$$e^{i(t(s))H}=\mathcal{L}_{t(s)}\circ e^{is\Delta},\ \text{where } t(s)=\frac{\arctan(2s)}{2}. \tag{6.2}$$

Moreover, we have that if the function $u_\varphi(t,x)$ is solution to (1.1), then the function $v_\varphi(t,x)$ defined as follows:

$$v_\varphi(t(s),x):=\mathcal{L}_{t(s)}\Big(u_\varphi(s,\cdot)\Big)(x) \tag{6.3}$$

solves the following Cauchy problem

$$\begin{cases} i\partial_t v_\varphi - Hv_\varphi + \cos(2t)^{-\alpha(n,p)} v_\varphi |v_\varphi|^p = 0, & (t,x) \in (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^n, \\ \quad \alpha(n,p) = 2 - \frac{np}{2} \\ v_\varphi(0,x) = \varphi \in \Sigma_n, \end{cases} \tag{6.4}$$

where H is defined in (6.1). Notice that the main advantage of the lens transform compared with the pseudoconformal transform is that the full norm Σ_n is involved in the energy associated with (6.4), and not only the $H^1(\mathbb{R}^n)$ norm. Therefore, the lens transform seems to be a suitable tool to study the scattering in Σ_n.

We recall that the Cauchy problem (6.4) admits one unique solution

$$v_\varphi(t,x) \in C\left([0, \frac{\pi}{4}\right); \Sigma_n) \cap L^r_{loc}\left([0, \frac{\pi}{4}\right); \mathcal{W}^{1,s}(\mathbb{R}^n)) \tag{6.5}$$

where (r,s) is an admissible Strichartz couple (namely $\frac{2}{r} + \frac{n}{s} = \frac{n}{2}$ and $r \geq 2$ for $n \geq 3$, $r > 2$ for $n = 2$, $r \geq 4$ for $n = 1$) and $\mathcal{W}^{1,s}(\mathbb{R}^n)$ denotes the harmonic Sobolev spaces associated, namely

$$\mathcal{W}^{1,s}(\mathbb{R}^n) = \{w \in L^s(\mathbb{R}^n) \text{ s.t. } H^{\frac{1}{2}} w \in L^s(\mathbb{R}^n)\}$$

endowed with the norm $\|w\|_{\mathcal{W}^{1,s}(\mathbb{R}^n)} = \|w\|_{L^s} + \|H^{\frac{s}{2}} w\|_{L^s}$. Following [12], one can show that for $1 < s < \infty$ there exists $C > 0$ such that

$$\frac{1}{C}(\|\nabla u\|_{L^s(\mathbb{R}^n)} + \|\langle x \rangle u\|_{L^s(\mathbb{R}^n)}) \leq \|u\|_{\mathcal{W}^{1,s}(\mathbb{R}^n)} \leq C(\|\nabla u\|_{L^s(\mathbb{R}^n)} \\ + \|\langle x \rangle u\|_{L^s(\mathbb{R}^n)}). \tag{6.6}$$

Moreover, it is well known that Strichartz estimates are available (locally in time) for the group e^{-itH}, under the same numerology for which they are satisfied (globally in time) for $e^{-it\Delta}$ (they can be obtained simply by applying the lens transform). Hence we have all the tools necessary to construct local solutions to (6.4) by repeating *mutatis mutandis* the same computations necessary to construct local solutions for the usual NLS. Notice that the chain rule in the framework of the harmonic Sobolev spaces is essentially reduced to the classical chain rule in the usual Sobolev spaces by (6.6). In order to show that the solution can be extended on the full interval $[0, \frac{\pi}{4}]$ with regularity (6.5) we can rely on the following conservation law:

$$\frac{d}{dt}\Big(\cos(2t)^{\alpha(n,p)} \|v_\varphi(t,x)\|^2_{\Sigma_n} + \frac{2}{p+2} \|v_\varphi(t,x)\|^{p+2}_{L^{p+2}(\mathbb{R}^n)} \Big) \\ = \|v_\varphi(t,x)\|^2_{\Sigma_n} \frac{d}{dt} \cos(2t)^{\alpha(n,p)} < 0, \quad \forall t \in \left[0, \frac{\pi}{4}\right) \tag{6.7}$$

whose proof follows the same argument to get (2.4) in the context of the pseudoconformal transformation. Since the weight $\cos(2t)^{\alpha(n,p)}$ has no zero in the interval $[0, \frac{\pi}{4})$, we have a control of the Σ_n norm of the solution up to time $t = \frac{\pi}{4}$ and hence we can globalize in $[0, \frac{\pi}{4}]$. A similar discussion holds in the interval $[-\pi/4, 0]$.

We have the following result that reduces the question of scattering in Σ_n for $u_\varphi(t,x)$ solution to (1.1) (see (1.8)) to the extendibility (by continuity) of the function $v_\varphi(t,x)$ up to time $t = \frac{\pi}{4}$ in the space Σ_n.

Proposition 6.1. *Let $\varphi \in \Sigma_n$, $0 < p < \frac{4}{n-2}$ for $n \geq 3$ and $0 < p < \infty$ for $n = 1, 2$. Then we have the following equivalence:*

$$\exists \varphi_+ \in \Sigma_n \text{ s.t. } \|e^{-is\Delta}(u_\varphi(s,y)) - \varphi_+\|_{\Sigma_n} \overset{s\to\infty}{\longrightarrow} 0 \iff \exists v_+ \in \Sigma_n \text{ s.t. } \|v_\varphi(t,x) - v_+\|_{\Sigma_n} \overset{t\to\frac{\pi}{4}^-}{\longrightarrow} 0.$$

Proof. The identity (6.2) is equivalent to

$$e^{-is\Delta} = e^{-i(t(s))H} \circ \mathcal{L}_{t(s)}$$

and hence

$$e^{-is\Delta}(u_\varphi(s,y)) = e^{-i(t(s))H}\Big(\mathcal{L}_{t(s)}(u_\varphi(s,\cdot))\Big) = e^{-i(t(s))H}(v_\varphi(t(s),y))$$

where we have used at the last step (6.3). We conclude since $e^{-i(t(s))H}$ are isometries in Σ_n and $\lim_{s\to\infty} t(s) = \frac{\pi}{4}$. □

6.1. The Case $p_n < p < \frac{4}{n-2}$

In this subsection, we provide an alternative proof of the following result first established in [27] (see also [7]).

Theorem 6.1 ([27]). *Assume $p_n < p < \frac{4}{n-2}$ for $n \geq 3$ and $p_n < p < \infty$ for $n = 1, 2$ (here p_n is defined in (1.9)). Then for every $\varphi \in \Sigma_n$ there exists $\varphi_+ \in \Sigma_n$ such that*

$$\|e^{-it\Delta}(u_\varphi(t,x)) - \varphi_+\|_{\Sigma_n} \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. By Proposition 6.1, we have to prove $\|v_\varphi(t,x) - v_+\|_{\Sigma_n} \overset{t\to\frac{\pi}{4}^-}{\longrightarrow} 0$, where $v_+ \in \Sigma_n$. In the rest of the proof, we shall denote $v = v_\varphi$. Next we denote by $(r, p+2)$, the couple of exponents such that

$$\frac{2}{r} + \frac{n}{p+2} = \frac{n}{2} \tag{6.8}$$

and we shall first prove $v \in L^r((0,\frac{\pi}{4}); \mathcal{W}^{1,p+2}(\mathbb{R}^n))$. It is easy to check that the couple $(r, p+2)$ is Strichartz admissible in any dimension $n \geq 1$. In view of (6.5), it is sufficient to prove the existence of $t_0 \in (0, \frac{\pi}{4})$ such that $v \in L^r((t_0, \frac{\pi}{4}); \mathcal{W}^{1,p+2}(\mathbb{R}^n))$, and in turn it is sufficient to show that $\sup_{\tau\in(t_0,\frac{\pi}{4})} \|v\|_{L^r((t_0,\tau);\mathcal{W}^{1,p+2}(\mathbb{R}^n))} < \infty$. Notice that the main advantage of working with $\tau < \frac{\pi}{4}$ is that in the following computation we deal with finite quantities. By Strichartz estimates available for the propagator e^{itH}, we get:

$$\|v\|_{L^r((t_0,\tau);\mathcal{W}^{1,p+2}(\mathbb{R}^n))} \leq C\|v(t_0)\|_{\Sigma_n} + C\|\cos(2t)^{-\alpha(n,p)} v|v|^p\|_{L^{r'}((t_0,\tau);\mathcal{W}^{1,(p+2)'}(\mathbb{R}^n))} \tag{6.9}$$

where t_0 is an arbitrary point in the interval $[0, \frac{\pi}{4})$ that we shall fix later, r', p' denote conjugate exponents and τ is arbitrary in $(t_0, \frac{\pi}{4})$. Notice that by the chain rule and Hölder inequality w.r.t. space and time we can continue the estimate as follows:

$$\begin{aligned}\dots &\leq C\|v(t_0)\|_{\Sigma_n} \\ &\quad + C\|\cos(2t)^{-\alpha(n,p)}\|_{L^{\frac{r}{r-2}}(t_0,\frac{\pi}{4})}\|v\|_{L^r((t_0,\tau);\mathcal{W}^{1,(p+2)}(\mathbb{R}^n))}\|v\|^p_{L^\infty((t_0,\tau);L^{p+2}(\mathbb{R}^n))}.\end{aligned} \tag{6.10}$$

Due to (6.7) (which implies $\sup_{t\in(0,\frac{\pi}{4})}\|v(t,x)\|^{p+2}_{L^{p+2}} < \infty$), we can absorb the second term on the r.h.s. in (6.10) in the l.h.s. in (6.9) provided we have $\cos(2t)^{-\alpha(n,p)} \in L^{\frac{r}{r-2}}(0,\frac{\pi}{4})$ and t_0 is close enough to $\frac{\pi}{4}$. This integrability condition is equivalent to $\frac{\alpha(n,p)r}{r-2} < 1$ which in turn, thanks to (6.8), is equivalent to $np^2 + (n-2)p - 4 > 0$ (recall that $\alpha(n,p) = 2 - \frac{np}{2}$). We conclude since we recall p_n is the larger root of the algebraic equation $nx^2 + (n-2)x - 4 = 0$. To deduce the existence of the limit v_+ by the Duhamel formulation and dual of Strichartz estimates we have for any couple $0 < \tau < \sigma < \frac{\pi}{4}$:

$$\begin{aligned}\|v(\tau) - v(\sigma)\|_{\Sigma_n} &= \Big\|\int_\tau^\sigma e^{i(t-s)H}\cos(2s)^{-\alpha(n,p)}v(s)|v(s)|^p ds\Big\|_{\Sigma_n} \\ &= \Big\|\int_\tau^\sigma e^{-isH}\cos(2s)^{-\alpha(n,p)}v(s)|v(s)|^p ds\Big\|_{\Sigma_n} \\ &\leq C\Big\|\cos(2s)^{-\alpha(n,p)}v(s)|v(s)|^p\Big\|_{L^{r'}((\tau,\sigma));\mathcal{W}^{1,(p+2)'}(\mathbb{R}^n))}.\end{aligned}$$

Arguing as above and by using $v \in L^r((t_0,\frac{\pi}{4});\mathcal{W}^{1,p+2}(\mathbb{R}^n))$, we can continue as follows:

$$\dots \leq C\|\cos(2t)^{-\alpha(n,p)}\|_{L^{\frac{r}{r-2}}(\tau,\sigma)}\|v\|_{L^r((\tau,\sigma);\mathcal{W}^{1,(p+2)}(\mathbb{R}^n))} \overset{\tau,\sigma\to\frac{\pi}{4}^-}{\longrightarrow} 0.$$

□

6.2. The Case $p_n \leq p < \frac{4}{n-2}$, $n \geq 3$

Next result includes the one in [27] with the extra bonus that it covers the limit case $p = p_n$. We shall give the proof for $n \geq 3$; however, the result is true also for $n = 1, 2$. We recall that compared with the original proof in [8] we deal with the equation obtained after the lens transform, which is adapted to work in the Σ_n space, rather than the pseudoconformal transformation that seems to perform better in the $H^1(\mathbb{R}^n)$ setting. Another point is that we give a proof of the key alternative (6.11) or (6.12) below, based on a continuity argument. This is different of the proof given in [8] based on a fixed point. We restrict below to the case $n \geq 3$; however, following [8] the proof can be adapted to the case $n = 1$, and the case $n = 2$ has been treated in [23].

Theorem 6.2 ([8]). *Assume $n \geq 3$ and $p_n \leq p < \frac{4}{n-2}$ (p_n is defined in (1.9)), then for every $\varphi \in \Sigma_n$ there exists $\varphi_+ \in \Sigma_n$ such that*

$$\|e^{-it\Delta}(u_\varphi(t,x)) - \varphi_+\|_{\Sigma_n} \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. By Proposition 6.1, we are reduced to prove $\|v_\varphi(t,x) - v_+\|_{\Sigma_n} \overset{t\to\frac{\pi}{4}^-}{\longrightarrow} 0$, where $v_+ \in \Sigma_n$. In the rest of the proof, we shall denote $v = v_\varphi$.

We claim that we have the following alternative for every $\frac{4}{n+2} < p < \frac{4}{n-2}$ (notice $\frac{4}{n+2} < p_n$): - either there exists $v_+ \in \Sigma_n$ such that

$$\|v(t,x) - v_+\|_{\Sigma_n} \overset{t\to\frac{\pi}{4}^-}{\longrightarrow} 0; \tag{6.11}$$

- or we have the lower bound

$$\inf_{t\in[0,\frac{\pi}{4})} \|v(t,x)\|_{\Sigma_n}^p \left(\int_t^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} > 0. \tag{6.12}$$

We shall prove first how the alternative (6.11) or (6.12) implies the result. We need to exclude the scenario (6.12) under the extra condition $p_n \le p < \frac{4}{n-2}$. Indeed if by the absurd (6.12) is true, then we get by (6.7)

$$\begin{aligned}
&\frac{d}{dt}\left(\cos(2t)^{\alpha(n,p)}\|v(t,x)\|_{\Sigma_n}^2 + \frac{2}{p+2}\|v(t,x)\|_{L^{p+2}(\mathbb{R}^n)}^{p+2}\right)\\
&\le -2\varepsilon_0^{\frac{2}{p}}\alpha(n,p)\sin(2t)\cos(2t)^{\alpha(n,p)-1}\left(\int_t^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{-4+p(n-2)}{2p}},\\
&t \in \left(0,\frac{\pi}{4}\right),
\end{aligned}$$

where $\varepsilon_0 > 0$ is the infimum in (6.12). Notice that $\cos(2t)$ behaves as $(\frac{\pi}{4} - t)$ when $t \to \frac{\pi}{4}^-$ and hence we get by elementary computations

$$\begin{aligned}
&\frac{d}{dt}\Big(\cos(2t)^{\alpha(n,p)}\|v(t,x)\|_{\Sigma_n}^2 + \frac{2}{p+2}\|v(t,x)\|_{L^{p+2}(\mathbb{R}^n)}^{p+2}\Big)\\
&\le -c_0\left(t-\frac{\pi}{4}\right)^{-1-\frac{np^2+p(n-2)-4}{2p}}, \quad t\in\left(0,\frac{\pi}{4}\right)
\end{aligned}$$

for a suitable $c_0 > 0$. Notice that the function at the r.h.s. fails to be integrable on $(0,\frac{\pi}{4})$ as long as $p \ge p_n$, and hence by integration on the interval $(0,\frac{\pi}{4})$ we easily get a contradiction.

Next we give a proof of the alternative (6.11) or (6.12) which is based on the following remark.

Lemma 6.1. *Given two sequences $a_k, b_k > 0$ for $k \in \mathbb{N}$ and $p > 0$, define $f_k : \mathbb{R}^+ \to \mathbb{R}$ as follows $f_k(s) = s - a_k - b_k s^{1+p}$. Assume that $a_k b_k^{\frac{1}{p}} \overset{k\to\infty}{\longrightarrow} 0$, then there exists $\bar{k}$ such that for every $k > \bar{k}$ there exist $0 < c_k < d_k < \infty$ such that*

$$\{s \in \mathbb{R}^+ \ s.t.\ f_k(s) \le 0\} = [0, c_k] \cup [d_k, \infty).$$

Proof. One can check that the function f_k has one unique maximum at the point $\bar{s}_k > 0$ given by the condition $f_k'(\bar{s}_k) = 0$, namely $\bar{s}_k^p = \frac{1}{(p+1)b_k}$. Moreover, f_k is increasing for $s < \bar{s}_k$, decreasing for $s > \bar{s}_k$, $\lim_{s\to\infty} f_k(s) = -\infty$ and $f_k(0) < 0$. We conclude provided that we show that for k large enough we have

$f_k(\bar{s}_k) > 0$. By direct computation, we get $f_k(\bar{s}_k) = \frac{1}{(p+1)^{\frac{1}{p}} b_k^{\frac{1}{p}}} - a_k - \frac{1}{(p+1)^{1+\frac{1}{p}} b_k^{\frac{1}{p}}}$ and hence the condition $f_k(\bar{s}_k) > 0$ is equivalent to $\frac{1}{(p+1)^{\frac{1}{p}}} - \frac{1}{(p+1)^{1+\frac{1}{p}}} > a_k b_k^{\frac{1}{p}}$ which is satisfied for k large enough due to the assumption $a_k b_k^{\frac{1}{p}} \overset{k\to\infty}{\longrightarrow} 0$. □

We can now complete the proof of the alternative (6.11) or (6.12) in the general setting $\frac{4}{n+2} < p < \frac{4}{n-2}$. Since now on we shall use that under this condition on p we have $\int_0^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau < \infty$. We shall prove that if (6.12) is false then (6.11) is satisfied. If (6.12) is false, then there exists a sequence $t_k \in (0, \frac{\pi}{4})$ and $\varepsilon_k > 0$ such that

$$t_k \overset{k\to\infty}{\longrightarrow} \frac{\pi}{4} \quad \text{and} \quad \|v(t_k,x)\|_{\Sigma_n}^p \left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} = \varepsilon_k \overset{k\to\infty}{\longrightarrow} 0. \tag{6.13}$$

In the sequel, we denote $v(t_k,x) = v_k$. Next we choose the Strichartz admissible couple (r,q) such that

$$1 - \frac{2}{q} = \frac{p(n-2)}{2n}$$

and by Strichartz estimates and Hölder inequalities (in space and time)

$$\begin{aligned} \|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))} &\le C\|v_k\|_{\Sigma_n} + C\|\cos(2t)^{-\alpha(n,p)} v|v|^p\|_{L^{r'}((t_k,t);\mathcal{W}^{1,\frac{2nq}{2n+pq(n-2)}}(\mathbb{R}^n))} \\ &\le C\|v_k\|_{\Sigma_n} \\ &+C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} \|v\|^p_{L^\infty((t_k,t);L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))} \end{aligned} \tag{6.14}$$

and by the Sobolev embedding and elementary inequalities, we can continue the estimate as follows

$$\begin{aligned} \cdots &\le C\|v_k\|_{\Sigma_n} \\ &\quad +C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} \|v\|^p_{L^\infty((t_k,t);\Sigma_n)} \|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))} \\ &\le C\|v_k\|_{\Sigma_n} \\ &+C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} \|v-v_k\|^p_{L^\infty((t_k,t);\Sigma_n)} \|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))} \\ &+C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} \|v_k\|^p_{\Sigma_n} \|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))}. \end{aligned}$$

Due to (6.13), we have that if we choose k large enough, then the last term on the r.h.s. can be estimated by $\frac{1}{2}\|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))}$. Hence we can absorb it on the l.h.s. and we get

$$\|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))} \leq C\|v_k\|_{\Sigma_n} + C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4p}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} \|v-v_k\|^p_{L^\infty((t_k,t);\Sigma_n)}\|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))}. \tag{6.15}$$

Again by Strichartz estimates and triangular inequality, we get

$$\begin{aligned}\|v-v_k\|_{L^\infty((t_k,t);\Sigma_n)} &\leq \|v_k\|_{\Sigma_n} + \|v\|_{L^\infty((t_k,t);\Sigma_n)}\\ &\leq C\|v_k\|_{\Sigma_n} + C\|\cos(2t)^{-\alpha(n,p)}v|v|^p\|_{L^{r'}((t_k,t);\mathcal{W}^{1,\frac{2nq}{2n+pq(n-2)}}(\mathbb{R}^n))}\end{aligned}$$

and hence we can estimate the r.h.s. as above and we get for k large enough

$$\begin{aligned}\|v-v_k\|_{L^\infty((t_k,t);\Sigma_n)} &\leq C\|v_k\|_{\Sigma_n}\\ &+C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} \|v-v_k\|^p_{L^\infty((t_k,t);\Sigma_n)}\|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))}\\ &+\frac{1}{2}\|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))}.\end{aligned} \tag{6.16}$$

Next we introduce the functions $X_k : (t_k, \frac{\pi}{4}) \to \mathbb{R}^+$ defined as follows: $X_k(t) = \|v-v_k\|_{L^\infty((t_k,t);\Sigma_n)} + \|v\|_{L^r((t_k,t);\mathcal{W}^{1,q}(\mathbb{R}^n))}$. Notice that by (6.15) and (6.16) we get

$$X_k(t) \leq C\|v_k\|_{\Sigma_n} + C\left(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}} (X_k(t))^{p+1}$$

and hence $X_k(t)$ belongs to the sublevel $\{f_k(s) \leq 0\}$ where $f_k(s)$ is as in Lemma 6.1, with $a_k = C\|v_k\|_{\Sigma_n}$ and $b_k = C(\int_{t_k}^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau)^{\frac{4-p(n-2)}{4}}$. Notice that $a_k b_k^{\frac{1}{p}} \overset{t\to\infty}{\longrightarrow} 0$ by (6.13) and hence if we choose $k = \bar{k}+1$ (following the notations of the Lemma 6.1) we get, since $X_{\bar{k}+1}(t)$ are continuous functions and $X_{\bar{k}+1}(t_{\bar{k}+1}) = 0$, that $X_{\bar{k}+1}(t)$ leaves for every $t \in (t_{\bar{k}+1}, \frac{\pi}{4})$ in the corresponding bounded connected component $[0, c_{\bar{k}+1}]$ provided by Lemma 6.1. Summarizing, we get $v(t,x) \in L^r((0,\frac{\pi}{4});\mathcal{W}^{1,q}(\mathbb{R}^n)) \cap L^\infty((0,\frac{\pi}{4});\Sigma_n)$. Going back to the Duhamel formulation, using Strichartz estimates and Hölder inequality in space and time (in the same spirit as in (6.14)) we get for every $0 < \tau < \sigma < \frac{\pi}{4}$:

$$\begin{aligned}\|v(\tau)-v(\sigma)\|_{\Sigma_n} &= \left\|\int_\tau^\sigma e^{i(t-s)H}\cos(2s)^{-\alpha(n,p)}v(s)|v(s)|^p ds\right\|_{\Sigma_n}\\ &= \left\|\int_\tau^\sigma e^{-isH}\cos(2s)^{-\alpha(n,p)}v(s)|v(s)|^p ds\right\|_{\Sigma_n}\\ &\leq C\|\cos(2s)^{-\alpha(n,p)}v(s)|v(s)|^p\|_{L^{r'}((\tau,\sigma);\mathcal{W}^{1,\frac{2nq}{2n+pq(n-2)}}(\mathbb{R}^n))}\end{aligned}$$

$$\leq C\left(\int_0^{\frac{\pi}{4}} |\cos(2\tau)|^{-\frac{4\alpha(n,p)}{4-p(n-2)}} d\tau\right)^{\frac{4-p(n-2)}{4}}$$

$$\|v\|^p_{L^\infty((\tau,\sigma);L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \|v\|_{L^r((\tau,\sigma);\mathcal{W}^{1,q}(\mathbb{R}^n))} \overset{\tau,\sigma\to\frac{\pi}{4}^-}{\longrightarrow} 0.$$

where at the last step we have used the property $v(t,x) \in L^r((0,\frac{\pi}{4});\mathcal{W}^{1,q}(\mathbb{R}^n)) \cap L^\infty((0,\frac{\pi}{4});\Sigma_n)$ established above and the Sobolev embedding $\Sigma_n \subset L^{\frac{2n}{n-2}}$. □

Acknowledgements

The authors are grateful to Thierry Cazenave for interesting discussions during the preparation of this paper. They are also grateful to the anonymous referee for interesting comments and suggestions.

Funding Open access funding provided by Universitá di Pisa within the CRUI-CARE Agreement.

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Communicated by Nader Masmoudi.
Received: December 3, 2021.
Accepted: October 24, 2022.