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Smooth 1-Dimensional Algebraic Quantum Field Theories

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Abstract. This paper proposes a refinement of the usual concept of algebraic quantum field theories (AQFTs) to theories that are smooth in the sense that they assign to every smooth family of spacetimes a smooth family of observable algebras. Using stacks of categories, this proposal is realized concretely for the simplest case of 1-dimensional spacetimes, leading to a stack of smooth 1-dimensional AQFTs. Concrete examples of smooth AQFTs, of smooth families of smooth AQFTs and of equivariant smooth AQFTs are constructed. The main open problems that arise in upgrading this approach to higher dimensions and gauge theories are identified and discussed.

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1. Introduction and Summary

An m-dimensional algebraic quantum field theory (AQFT) is a functor $\mathfrak{A} : \mathbf{Loc}_m \to {}^*\mathbf{Alg}_{\mathbb{C}}$ from a suitable category of m-dimensional Lorentzian spacetimes to the category of associative and unital $*$-algebras over $\mathbb{C}$. The algebra $\mathfrak{A}(M)$ that is assigned by this functor to a spacetime M is interpreted as the algebra of quantum observables of the theory $\mathfrak{A}$ that can be measured in M. Such functors are also required to satisfy a list of physically motivated axioms, see, e.g., [10,15], which includes most notably the Einstein causality axiom.

Even though this axiomatic definition of AQFTs is by now widely used in the relevant research community and has led to interesting model-independent results, we would like to point out the following issue that is usually not discussed: Suppose that we consider a family of spacetimes $\{M_s \in \mathbf{Loc}_m\}_{s\in\mathbb{R}}$ that depends smoothly (in some appropriate sense as explained in this paper) on a parameter $s \in \mathbb{R}$. This s-dependence could, for example, be due to changing smoothly the coefficients of the metric tensor. Evaluating an AQFT $\mathfrak{A} : \mathbf{Loc}_m \to {}^*\mathbf{Alg}_{\mathbb{C}}$ on this smooth family results in a family of algebras $\{\mathfrak{A}(M_s) \in {}^*\mathbf{Alg}_{\mathbb{C}}\}_{s\in\mathbb{R}}$ which, however, will in general not be smooth in any appropriate sense because smoothness is not covered by the usual AQFT axioms. Similarly, given a smooth family $\{f_s : M_s \to N_s\}_{s\in\mathbb{R}}$ of spacetime morphisms, the associated family $\{\mathfrak{A}(f_s) : \mathfrak{A}(M_s) \to \mathfrak{A}(N_s)\}_{s\in\mathbb{R}}$ of ${}^*\mathbf{Alg}_{\mathbb{C}}$-morphisms will in general not be smooth. In our opinion, encoding a suitable concept of smoothness for these families as part of the axioms of AQFT is desirable for several reasons: (1) Physically speaking, smoothness of these families means that a small variation at the level spacetimes does not have a too drastic effect on the observable content of the theory; hence, it excludes models with unpleasant discontinuous behavior. (2) Certain standard constructions in AQFT, such as the computation of the stress-energy tensor as a derivative of the relative Cauchy evolution [10], only exist for models that react sufficiently smoothly to metric perturbations. (3) In the context of AQFT on spacetimes with background fields, see, e.g., [32], such a smooth dependence may be used to describe an adiabatic switching of the interaction with the background fields. Similarly, it enables us to introduce AQFTs that are smoothly equivariant with respect to an action of a Lie group.

The main goal of this paper is to make some first steps toward developing a refinement of the axiomatic foundations of AQFT that encodes the preservation of smooth families as part of its structure. The key idea behind our approach is to refine the ordinary categories $\mathbf{Loc}_m$ and ${}^*\mathbf{Alg}_{\mathbb{C}}$ that enter the definition of AQFTs to *stacks of categories*, which will encode precise concepts of smooth families of spacetimes and algebras, and to introduce a concept of *smooth AQFTs* in terms of morphisms between these stacks. A similar program of smooth refinements of QFTs has been developed successfully within other approaches, in particular for functorial QFTs in the sense of Atiyah and Segal [9,11,25,30], however as of now this idea seems to be unexplored in the context of AQFT. In order to outline our proposal in the simplest possible setting and to circumvent in this first paper certain technical challenges (of both analytical and algebraic nature, see Sect. 6), we consider only the simplest case of dimension $m = 1$, which physically represents AQFTs on time intervals (i.e., quantum mechanics). We believe that, due to its simplicity, the case of 1-dimensional AQFTs is perfectly suited to explain the main ideas and features of our proposed framework for smooth AQFTs and to illustrate this formalism through the simplest possible examples.

Our framework for smooth AQFTs introduces naturally a second layer of smoothness. Because we realize smooth 1-dimensional AQFTs as the points of a stack $\mathbf{AQFT}_1^\infty$, we can also make precise sense of questions like what are "smooth families of smooth AQFTs" and in particular what are "smooth curves of smooth AQFTs." We shall illustrate through simple examples that smooth variations (e.g., an adiabatic switching) of the external parameters of a theory, such as the mass parameter, define such smooth families of smooth AQFTs. Furthermore, we show that each smooth AQFT has an associated smooth automorphism group, refining the discrete automorphism groups of ordinary AQFTs [14], and explain how these are related to smooth AQFTs that are equivariant with respect to a smooth action of a Lie group. We construct a concrete example of the latter that captures the global $U(1)$-symmetry of the 1-dimensional massless Dirac field.

The outline of the remainder of this paper is as follows: Sect. 2 contains a brief review of some relevant aspects of the theory of stacks of categories that we shall need in this work. In Sect. 3 we introduce the stacks of categories $\mathbf{Loc}_1^\infty$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ that provide smooth refinements of the category $\mathbf{Loc}_1$ of 1-dimensional spacetimes and of the category ${}^*\mathbf{Alg}_{\mathbb{C}}$ of associative and unital $*$-algebras. The stack of smooth 1-dimensional AQFTs is then defined as the mapping stack $\mathbf{AQFT}_1^\infty := \mathsf{Map}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$ and we shall explore some interesting consequences of this definition, including a natural notion of smooth automorphism group of a smooth AQFT and its relation to G-equivariant smooth AQFTs, for G a Lie group. Section 4 develops two stack morphisms $\mathfrak{CCR}$ and $\mathfrak{CAR}$ that are smooth refinements of the usual canonical (anti-)commutation relation quantization functors for Bosonic (resp., Fermionic) theories. These are later used for constructing explicit examples in Sect. 5, which illustrate our proposed approach to smooth AQFT. In Sect. 5.1, we introduce smooth refinements of retarded/advanced Green operators and prove their existence in simple cases through explicit formulas. We then construct in Sect. 5.2 a concrete example of a smooth family of smooth 1-dimensional AQFTs, which can be interpreted physically as (a smooth analog of) the 1-dimensional massive scalar field (quantum harmonic oscillator) in the presence of a smooth variation of the mass (frequency) parameter. In Sect. 5.3, we construct a concrete example of a $U(1)$-equivariant smooth 1-dimensional AQFT, which can be interpreted physically as (a smooth analog of) the 1-dimensional massless Dirac field, together with its global $U(1)$-symmetry. Section 6 provides a concise list of open problems that have to be solved to upgrade our approach to encompass higher dimensions $m \geq 2$ and gauge theories. Most pressingly, Open Problem 6.1 poses the question of existence of smoothly parameterized retarded/advanced Green operators for vertical normally hyperbolic operators on smooth families of Lorentzian spacetimes, which goes beyond the standard results developed, e.g., in [1] and might be of interest to researchers in hyperbolic PDE theory.

2. Preliminaries on Stacks of Categories

We shall briefly review some basic concepts from the theory of *stacks of categories* that we need to describe a smooth refinement of algebraic quantum field theories (AQFTs). Our perspective on smoothness is through the functor of points approach, see, e.g., [4] and [5, Section 3.2] for introductions in the context of AQFT and also [28] for a more detailed overview. We also refer to [22, 24] for the relevant 2-categorical background and to [31] for a detailed introduction to the theory of stacks.

Let **Man** denote the category of (finite-dimensional) smooth manifolds and smooth maps. The usual open cover Grothendieck topology endows **Man** with the structure of a site. We choose the site **Man** because our aim is to formalize *smooth* families. The framework of stacks is however very flexible and can be adapted to model other types of families, such as continuous or algebraic, by an appropriate choice of site.

Definition 2.1. A *prestack* (of categories) is a pseudo-functor $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ to the 2-category **Cat** of categories, functors and natural transformations. Explicitly, this consists of the following data:

(1) For each object $U \in \mathbf{Man}$, a category $X(U)$.
(2) For each morphism $h : U \to U'$ in **Man**, a functor $X(h) : X(U') \to X(U)$.
(3) For each pair of composable morphisms $h : U \to U'$ and $h' : U' \to U''$ in **Man**, a natural isomorphism $X_{h',h} : X(h)\,X(h') \Rightarrow X(h'\,h)$ of functors from $X(U'')$ to $X(U)$.
(4) For each object $U \in \mathbf{Man}$, a natural isomorphism $X_U : \mathrm{id}_{X(U)} \Rightarrow X(\mathrm{id}_U)$ of functors from $X(U)$ to $X(U)$.

These data have to satisfy the following axioms:

(i) For all triples of composable morphisms $h : U \to U'$, $h' : U' \to U''$ and $h'' : U'' \to U'''$ in **Man**, the diagram

$$\begin{array}{ccc} X(h)\,X(h')\,X(h'') & \xRightarrow{X_{h',h} * \mathrm{Id}} & X(h'\,h)\,X(h'') \\ {\scriptstyle \mathrm{Id} * X_{h'',h'}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle X_{h'',h'h}} \\ X(h)\,X(h''\,h') & \xRightarrow[X_{h''h',h}]{} & X(h''\,h'\,h) \end{array} \tag{2.1}$$

of natural transformations commutes. (The capital Id denotes identity natural transformations, and $*$ denotes horizontal composition of natural transformations.)

(ii) For all morphisms $h : U \to U'$ in **Man**, the two diagrams

$$\begin{array}{ccc} \mathrm{id}_{X(U)}\, X(h) & & \\ {\scriptstyle X_U * \mathrm{Id}}\Big\Downarrow & \diagdown\!\!\diagdown & \\ X(\mathrm{id}_U)\, X(h) & \overset{}{\underset{X_{h,\mathrm{id}_U}}{\Longrightarrow}} & X(h) \end{array} \qquad \begin{array}{ccc} X(h)\, \mathrm{id}_{X(U')} & & \\ {\scriptstyle \mathrm{Id} * X_{U'}}\Big\Downarrow & \diagdown\!\!\diagdown & \\ X(h)\, X(\mathrm{id}_{U'}) & \underset{X_{\mathrm{id}_{U'},h}}{\Longrightarrow} & X(h) \end{array} \tag{2.2}$$

of natural transformations commute.

Remark 2.2. From now on we shall often follow the usual convention of suppressing the symbols $X_{h',h}$ and X_U denoting the coherence isomorphisms of a prestack X. Definition 2.1 should help readers unfamiliar with this convention to extrapolate from the context which coherence isomorphism is relevant at any given point.

Given any prestack $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, one can define, for every manifold $U \in \mathbf{Man}$ and every open cover $\{U_\alpha \subseteq U\}$, the associated *descent category* $X(\{U_\alpha \subseteq U\}) \in \mathbf{Cat}$, see, e.g., [31, Definition 4.2]. An object in this category is a tuple

$$\Big(\big\{x_\alpha\big\}, \big\{\varphi_{\alpha\beta} : x_\beta|_{U_{\alpha\beta}} \to x_\alpha|_{U_{\alpha\beta}}\big\}\Big) \in X(\{U_\alpha \subseteq U\}) \tag{2.3a}$$

of families of objects $x_\alpha \in X(U_\alpha)$ and isomorphisms $\varphi_{\alpha\beta}$ in $X(U_{\alpha\beta})$ satisfying

$$\begin{array}{ccccc} & & x_\beta|_{U_{\beta\gamma}}|_{U_{\alpha\beta\gamma}} & & \\ & {\scriptstyle \varphi_{\beta\gamma}|_{U_{\alpha\beta\gamma}}}\nearrow & & \searrow{\scriptstyle \cong} & \\ x_\gamma|_{U_{\beta\gamma}}|_{U_{\alpha\beta\gamma}} & & & & x_\beta|_{U_{\alpha\beta}}|_{U_{\alpha\beta\gamma}} \\ {\scriptstyle \cong}\big\downarrow & & & & \big\downarrow{\scriptstyle \varphi_{\alpha\beta}|_{U_{\alpha\beta\gamma}}} \\ x_\gamma|_{U_{\alpha\gamma}}|_{U_{\alpha\beta\gamma}} & & & & x_\alpha|_{U_{\alpha\beta}}|_{U_{\alpha\beta\gamma}} \\ & {\scriptstyle \varphi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}}\searrow & & \swarrow{\scriptstyle \cong} & \\ & & x_\alpha|_{U_{\alpha\gamma}}|_{U_{\alpha\beta\gamma}} & & \end{array} \qquad \begin{array}{ccc} x_\alpha|_{U_{\alpha\alpha}} & \overset{\varphi_{\alpha\alpha}}{\longrightarrow} & x_\alpha|_{U_{\alpha\alpha}} \\ {\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ x_\alpha & \underset{\mathrm{id}_{x_\alpha}}{\longrightarrow} & x_\alpha \end{array} \tag{2.3b}$$

for all α, β, γ. Here we denote by $U_{\alpha_1\alpha_2\cdots\alpha_n} := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n}$ the intersection of open subsets and by $|_{\tilde{U}} := X(\iota_{\tilde{U}}^{U}) : X(U) \to X(\tilde{U})$ the functor associated with a subset inclusion morphism $\iota_{\tilde{U}}^{U} : \tilde{U} \to U$ in **Man**. The unlabeled isomorphisms in (2.3b) are given by the coherence isomorphisms associated with the pseudo-functor $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$. A morphism

$$\{\psi_\alpha\} : \big(\{x_\alpha\}, \{\varphi_{\alpha\beta}\}\big) \longrightarrow \big(\{x'_\alpha\}, \{\varphi'_{\alpha\beta}\}\big) \tag{2.4a}$$

in the descent category $X(\{U_\alpha \subseteq U\})$ is a family of morphisms $\psi_\alpha : x_\alpha \to x'_\alpha$ in $X(U_\alpha)$ satisfying

$$\begin{array}{ccc} x_\beta|_{U_{\alpha\beta}} & \xrightarrow{\psi_\beta|_{U_{\alpha\beta}}} & x'_\beta|_{U_{\alpha\beta}} \\ {\scriptstyle \varphi_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \varphi'_{\alpha\beta}} \\ x_\alpha|_{U_{\alpha\beta}} & \xrightarrow[\psi_\alpha|_{U_{\alpha\beta}}]{} & x'_\alpha|_{U_{\alpha\beta}} \end{array} \tag{2.4b}$$

for all α, β. There exists a canonical functor

$$\begin{aligned} X(U) &\longrightarrow X(\{U_\alpha \subseteq U\}), \\ x &\longmapsto \left(\{x|_{U_\alpha}\}, \{x|_{U_\beta}|_{U_{\alpha\beta}} \xrightarrow{\cong} x|_{U_\alpha}|_{U_{\alpha\beta}}\}\right), \\ \psi &\longmapsto \{\psi|_{U_\alpha}\}. \end{aligned} \tag{2.5}$$

Definition 2.3. A *stack* (of categories) is a prestack $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ that satisfies the following descent condition: For every $U \in \mathbf{Man}$ and every open cover $\{U_\alpha \subseteq U\}$, the functor in (2.5) is an equivalence of categories.

Example 2.4. A simple example is the stack of vector bundles $\mathbf{VecBun}_{\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, where $\mathbb{K}$ denotes either the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. It assigns to each manifold $U \in \mathbf{Man}$ the category $\mathbf{VecBun}_{\mathbb{K}}(U)$ of (locally trivializable and finite rank) $\mathbb{K}$-vector bundles over U, with morphisms given by vector bundle maps that preserve the base space. To a morphism $h : U \to U'$ in $\mathbf{Man}$ it assigns the functor $h^* : \mathbf{VecBun}_{\mathbb{K}}(U') \to \mathbf{VecBun}_{\mathbb{K}}(U)$ that forms pullback bundles. The coherence isomorphisms are canonically given by the universal property of pullback bundles. The descent condition expresses the following local-to-global (or gluing) properties of vector bundles and their morphisms: Vector bundles on U can be described equivalently in terms of families of vector bundles on an open cover $\{U_\alpha \subseteq U\}$ and transition functions on the overlaps $U_{\alpha\beta}$, see, e.g., [23, Problem 10-6]. (Observe that the diagrams in (2.3b) are the cocycle conditions for the transition functions.) From this perspective, vector bundle maps on U can be described equivalently in terms of families of vector bundle maps on an open cover $\{U_\alpha \subseteq U\}$ that are compatible (in the sense of (2.4b)) with the transition functions.

Definition 2.5. A *morphism* $F : X \to Y$ between two stacks is a pseudo-natural transformation between the underlying pseudo-functors $X, Y : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$. Explicitly, this consists of the following data:

(1) For each $U \in \mathbf{Man}$, a functor $F_U : X(U) \to Y(U)$.
(2) For each morphism $h : U \to U'$ in $\mathbf{Man}$, a natural isomorphism

$$\begin{array}{ccc} X(U') & \xrightarrow{F_{U'}} & Y(U') \\ {\scriptstyle X(h)}\downarrow & \swarrow_{F_h} & \downarrow{\scriptstyle Y(h)} \\ X(U) & \xrightarrow[F_U]{} & Y(U) \end{array} \tag{2.6}$$

These data have to satisfy the following axioms:

(i) For all pairs of composable morphisms $h : U \to U'$ and $h' : U' \to U''$ in **Man**, the diagram

$$\begin{array}{ccccc}
Y(h)\,Y(h')\,F_{U''} & \xRightarrow{\mathrm{Id}*F_{h'}} & Y(h)\,F_{U'}\,X(h') & \xRightarrow{F_h*\mathrm{Id}} & F_U\,X(h)\,X(h') \\
{\scriptstyle Y_{h',h}*\mathrm{Id}}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \mathrm{Id}*X_{h',h}} \\
Y(h'\,h)F_{U''} & & \xRightarrow[F_{h'h}]{\qquad\qquad\qquad} & & F_U\,X(h'\,h)
\end{array} \tag{2.7}$$

of natural transformations commutes.

(ii) For all $U \in$ **Man**, the diagram

$$\begin{array}{ccc}
\mathrm{id}_{Y(U)}\,F_U & = & F_U\,\mathrm{id}_{X(U)} \\
{\scriptstyle Y_U*\mathrm{Id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{Id}*X_U} \\
Y(\mathrm{id}_U)\,F_U & \xRightarrow[F_{\mathrm{id}_U}]{} & F_U\,X(\mathrm{id}_U)
\end{array} \tag{2.8}$$

of natural transformations commutes.

Definition 2.6. A 2-*morphism* $\zeta : F \Rightarrow G$ between two stack morphisms $F, G : X \to Y$ is a modification between the underlying pseudo-natural transformations. Explicitly, this consists of the following data:

(1) For each $U \in$ **Man**, a natural transformation $\zeta_U : F_U \Rightarrow G_U$ of functors from $X(U)$ to $Y(U)$.

These data have to satisfy the following axioms:

(i) For all morphisms $h : U \to U'$ in **Man**, the diagram

$$\begin{array}{ccc}
Y(h)\,F_{U'} & \xRightarrow{\mathrm{Id}*\zeta_{U'}} & Y(h)\,G_{U'} \\
{\scriptstyle F_h}\Big\Downarrow & & \Big\Downarrow{\scriptstyle G_h} \\
F_U\,X(h) & \xRightarrow[\zeta_U*\mathrm{Id}]{} & G_U\,X(h)
\end{array} \tag{2.9}$$

of natural transformations commutes.

It is well-known that pseudo-functors, pseudo-natural transformations and modifications form a 2-category, see, e.g., [16, Chapter 3] and [27, Appendix A.1]. Selecting only those pseudo-functors that satisfy descent leads to the 2-category defined below.

Definition 2.7. We denote by **St** the 2-*category of stacks* (of categories). Its objects are stacks (see Definition 2.3), 1-morphisms are stack morphisms (see Definition 2.5) and 2-morphisms are given in Definition 2.6.

We conclude this section by recalling briefly some important constructions involving stacks that will be needed in the bulk of our paper.

2-Yoneda Lemma: There exists a 2-functor (called 2-*Yoneda embedding*)

$$\underline{(-)} \;:\; \mathbf{Man} \;\longrightarrow\; \mathbf{St} \tag{2.10}$$

from the category of manifolds to the 2-category of stacks. It assigns to a manifold $N \in \mathbf{Man}$ the stack $\underline{N} := \mathbf{Man}(-,N) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}\,,\ U \mapsto \mathbf{Man}(U,N)$, where the set $\mathbf{Man}(U,N) = C^\infty(U,N)$ of smooth maps is regarded as a category with only identity morphisms. This 2-functor is fully faithful, i.e., manifolds can be equivalently regarded as stacks. Even more, for every $U \in \mathbf{Man}$ and $X \in \mathbf{St}$, there exists a natural equivalence

$$\mathbf{St}(\underline{U},X) \;\simeq\; X(U) \tag{2.11}$$

between the category of morphisms from $\underline{U}$ to X and the category obtained by evaluating X on U. As a consequence, the category $X(U)$ admits a useful interpretation as the category of "smooth maps" $\underline{U} \to X$ from the manifold U to the stack X. In particular, for $U = \{*\}$ the point, we can interpret $X(\{*\})$ as the category of "global points" $\underline{\{*\}} \to X$ and similarly, for $U = \mathbb{R}$ the line, we can interpret $X(\mathbb{R})$ as the category of "smooth curves" $\underline{\mathbb{R}} \to X$ in the stack X.

Products of stacks: Given any two stacks $X, Y \in \mathbf{St}$, one defines the *product stack* $X \times Y \in \mathbf{St}$ in terms of the pseudo-functor

$$\begin{aligned} X \times Y \;:\; \mathbf{Man}^{\mathrm{op}} &\longrightarrow \mathbf{Cat}, \\ U &\longmapsto X(U) \times Y(U), \\ \big(h : U \to U'\big) &\longmapsto \big(X(h) \times Y(h) : X(U') \times Y(U') \to X(U) \times Y(U)\big), \end{aligned} \tag{2.12}$$

together with the obvious coherence isomorphisms induced from X and Y. For the particular case of two manifolds $M, N \in \mathbf{Man}$, one finds that the product stack $\underline{M} \times \underline{N} \simeq \underline{M \times N}$ is equivalent to the stack associated with the product manifold.

Mapping stacks: Given any two stacks $X, Y \in \mathbf{St}$, one defines the *mapping stack* $\mathsf{Map}(X,Y) \in \mathbf{St}$ in terms of the (strict) 2-functor

$$\begin{aligned} \mathsf{Map}(X,Y) \;:\; \mathbf{Man}^{\mathrm{op}} &\longrightarrow \mathbf{Cat}, \\ U &\longmapsto \mathbf{St}(X \times \underline{U}, Y), \\ \big(h : U \to U'\big) &\longmapsto \big((\mathrm{id} \times \underline{h})^* : \mathbf{St}(X \times \underline{U'}, Y) \to \mathbf{St}(X \times \underline{U}, Y)\big), \end{aligned} \tag{2.13}$$

where $(\mathrm{id} \times \underline{h})^* := (-) \circ (\mathrm{id} \times \underline{h})$ denotes pre-composition. For the particular case of two manifolds $M, N \in \mathbf{Man}$, one finds that the mapping stack $\mathsf{Map}(\underline{M}, \underline{N})$ is equivalent to the usual functor of points for the mapping space of manifolds. See, e.g., [4] and [5, Section 3.2] for more details on the latter.

3. Smooth 1-Dimensional AQFTs

An m-dimensional algebraic quantum field theory (AQFT) [10,15] is a functor $\mathfrak{A} : \mathbf{Loc}_m \to {}^*\mathbf{Alg}_{\mathbb{C}}$ from the category $\mathbf{Loc}_m$ of m-dimensional (globally hyperbolic) Lorentzian spacetimes to the category ${}^*\mathbf{Alg}_{\mathbb{C}}$ of associative and unital $*$-algebras over $\mathbb{C}$. This functor is required to satisfy certain physically motivated axioms, most notably the Einstein causality axiom expressing that every two causally disjoint observables must commute with each other. Such structures can be described most effectively in terms of operad theory and one observes that the category $\mathbf{AQFT}_m$ of m-dimensional AQFTs is the category of algebras over a suitable colored operad, see [6,7] for the details. The case of $m = 1$ dimensions, which physically represents AQFTs on time intervals (i.e., quantum mechanics), is structurally much simpler because causal disjointness, and hence the associated Einstein causality axiom, is a phenomenon arising only in dimension $m \geq 2$. As a consequence, the category $\mathbf{AQFT}_1 = \mathbf{Fun}(\mathbf{Loc}_1, {}^*\mathbf{Alg}_{\mathbb{C}})$ of 1-dimensional AQFTs is simply a functor category.

The aim of this section is to introduce a smooth refinement of 1-dimensional AQFTs. This means that we will upgrade the categories $\mathbf{Loc}_1$ and ${}^*\mathbf{Alg}_{\mathbb{C}}$ to stacks of categories, which encode suitable concepts of smoothly U-parameterized families of spacetimes and algebras, for all manifolds $U \in \mathbf{Man}$. A smooth 1-dimensional AQFT will then be defined as a stack morphism between these two stacks, which in particular means that smooth AQFTs map smooth U-families of spacetimes to smooth U-families of algebras. Loosely speaking, one may say that "smooth AQFTs respond smoothly to smooth variations of spacetimes."

Our approach to smooth AQFTs introduces also a further layer of smoothness, namely we can define a stack $\mathbf{AQFT}_1^\infty \in \mathbf{St}$ of smooth 1-dimensional AQFTs. Through this stack we obtain a natural concept of smoothly U-parameterized families of smooth AQFTs, which for the special case $U = \mathbb{R}$ leads to a notion of smooth curves of smooth AQFTs. We shall illustrate later in Sect. 5 that smooth variations of the external parameters of a theory, such as the mass parameter, gives rise to such smooth families.

Throughout the whole paper we restrict our attention to the simplest case given by 1-dimensional AQFTs. We expect that a generalization to higher-dimensional AQFTs is possible by using similar techniques; however, there are certain additional technical difficulties and challenges that we explain in more detail in Sect. 6.

3.1. The Stack $\mathbf{Loc}_1^\infty$

In this subsection we introduce a stack $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ that provides a smooth refinement of the category $\mathbf{Loc}_1$ of 1-dimensional spacetimes. Let us first recall how the latter category is defined. A 1-dimensional spacetime (i.e., time interval) may be described in terms of a pair (I, e), where $I \subseteq \mathbb{R}$ is an open interval and $e \in \Omega^1(I)$ is a nondegenerate 1-form that encodes the geometry and orientation of I. (In physics terminology, one may call e a

1-*bein*, i.e., a 1-dimensional vielbein.) A morphism $f : (I, e) \to (I', e')$ in $\mathbf{Loc}_1$ is an open embedding $f : I \to I'$ of intervals that preserves the 1-forms, i.e., $f^*(e') = e$.

Given any manifold $U \in \mathbf{Man}$, the category $\mathbf{Loc}_1^\infty(U)$ is supposed to describe smooth U-families of 1-dimensional spacetimes. A suitable way to formalize those is through fiber bundles and their vertical geometry.

Definition 3.1. A *smooth U-family of 1-dimensional spacetimes* is a pair $(\pi : M \to U, E)$ consisting of a (locally trivializable) fiber bundle $\pi : M \to U$ with typical fiber an open interval $I \subseteq \mathbb{R}$ and a nondegenerate vertical 1-form $E \in \Omega^1_{\mathrm{v}}(M)$ on the total space.

Remark 3.2. The interpretation of this definition is as follows: Given any pair $(\pi : M \to U, E)$ as in Definition 3.1, one obtains, for every point $x \in U$, a 1-dimensional spacetime $(M|_x, E|_x) := (\pi^{-1}(\{x\}), E|_{\pi^{-1}(\{x\})}) \in \mathbf{Loc}_1$ by restricting to the fiber over $x \in U$. Due to the smooth fiber bundle structure, it makes sense to interpret this pair as depending smoothly on $x \in U$.

A natural concept of morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ between smooth U-families of 1-dimensional spacetimes is given by fiber bundle maps

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & U & \end{array} \tag{3.1}$$

that preserve the 1-forms, i.e., $f^*(E') = E$, and that are in a suitable sense "open embeddings of fiber bundles." Indeed, from the AQFT point of view, it is quite natural to consider open embeddings as they allow to push forward compactly supported sections of vector bundles, which is crucial to construct examples. We would like to emphasize that there exist a priori different concepts of what open embeddings of fiber bundles could be. For example, we could demand the *point-wise* condition that the restriction $f|_x : M|_x \to M'|_x$ to the fiber over every point $x \in U$ is an open embedding of manifolds. Unfortunately, this simple point-wise condition is incompatible with pushing forward vertically compactly supported functions on the total spaces. Hence, the correct concept of "open embeddings of fiber bundles" should be in some sense more uniform on U. There exist a priori different options to formalize this, but fortunately the three main candidates are equivalent.

Lemma 3.3. *Let $f : (\pi : M \to U) \to (\pi' : M' \to U)$ be a fiber bundle map. Then the following three statements are equivalent:*

1. *For each $x \in U$, there exists an open neighborhood $U_x \subseteq U$, such that the restriction $f|_{U_x} : M|_{U_x} \to M'|_{U_x}$ is an open embedding of manifolds.*
2. *The map of total spaces $f : M \to M'$ is an open embedding of manifolds.*
3. *For each open subset $\tilde{U} \subseteq U$, the restriction $f|_{\tilde{U}} : M|_{\tilde{U}} \to M'|_{\tilde{U}}$ is an open embedding of manifolds.*

Proof. 1. $\Rightarrow$ 2.: From the hypothesis it is clear that $f : M \to f(M)$ is a bijection of sets. Furthermore, for each $x \in U$, there exists an open neighborhood $U_x \subseteq U$ such that $f|_{U_x} : M|_{U_x} \to M'|_{U_x}$ is an open embedding, which implies that $f : M \to f(M)$ is a diffeomorphism. To show that the image $f(M) \subseteq M'$ is open, observe that $f|_{U_x}(M|_{U_x}) \subseteq M'|_{U_x}$ is by hypothesis open and that $M'|_{U_x} \subseteq M'$ is open too. Hence, $f(M) = \bigcup_{x\in U} f|_{U_x}(M|_{U_x}) \subseteq M'$ is open.

2. $\Rightarrow$ 3.: The open embedding $f : M \to M'$ factors as a diffeomorphism $f : M \to f(M)$ followed by an open inclusion $f(M) \subseteq M'$. Take any open subset $\tilde{U} \subseteq U$ and consider the restriction $f|_{\tilde{U}} : M|_{\tilde{U}} \to M'|_{\tilde{U}}$, which factors as a map $f|_{\tilde{U}} : M|_{\tilde{U}} \to f(M|_{\tilde{U}})$ followed by an inclusion $f(M|_{\tilde{U}}) \subseteq M'|_{\tilde{U}}$. Because f is a fiber bundle map, we have that $f(M|_{\tilde{U}}) = f(M) \cap M'|_{\tilde{U}}$, which implies that $f|_{\tilde{U}} : M|_{\tilde{U}} \to f(M|_{\tilde{U}})$ is a diffeomorphism and that $f(M|_{\tilde{U}}) \subseteq M'|_{\tilde{U}}$ is an open inclusion. Hence, $f|_{\tilde{U}} : M|_{\tilde{U}} \to M'|_{\tilde{U}}$ is an open embedding.

3. $\Rightarrow$ 1.: Trivial. □

Definition 3.4. For any manifold $U \in \mathbf{Man}$, we denote by $\mathbf{Loc}_1^\infty(U)$ the category whose objects are smooth U-families of 1-dimensional spacetimes $(\pi : M \to U, E)$, see Definition 3.1, and whose morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ are fiber bundle maps (3.1) that preserve the 1-forms, i.e., $f^*(E') = E$, and that satisfy any of the three equivalent conditions in Lemma 3.3.

In order to define a (pre)stack $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ (see Definition 2.1), we have to assign to every morphism $h : U \to U'$ in $\mathbf{Man}$ a functor

$$h^* := \mathbf{Loc}_1^\infty(h) : \mathbf{Loc}_1^\infty(U') \longrightarrow \mathbf{Loc}_1^\infty(U). \tag{3.2}$$

Let us recall that, given any fiber bundle $\pi : M \to U'$, one may form the *pullback bundle*

$$\begin{array}{ccc} h^*M & \overset{\overline{h}^M}{\dashrightarrow} & M \\ {\scriptstyle \pi_h}\,\big\downarrow & & \big\downarrow\,{\scriptstyle \pi} \\ U & \underset{h}{\longrightarrow} & U' \end{array} \tag{3.3}$$

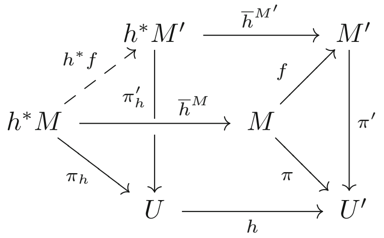
which is a locally trivializable fiber bundle with the same typical fiber as $\pi : M \to U'$. We then define the functor in (3.2) on objects as

$$h^*\big(\pi : M \to U', E\big) := \big(\pi_h : h^*M \to U, \overline{h}^{M*}(E)\big) \tag{3.4a}$$

and on morphisms $f : (\pi : M \to U', E) \to (\pi' : M' \to U', E')$ as

$$h^*f : \big(\pi_h : h^*M \to U, \overline{h}^{M*}(E)\big) \longrightarrow \big(\pi'_h : h^*M' \to U, \overline{h}^{M'*}(E')\big), \tag{3.4b}$$

where the fiber bundle map h^*f is defined uniquely through the universal property of pullback bundles by the commutative diagram



(3.4c)

The fact that h^*f preserves the 1-forms, i.e., $(h^*f)^*\overline{h}^{M'*}(E') = \overline{h}^{M*}(E)$, is a direct consequence of this diagram and $f^*(E') = E$. Furthermore, the condition 1. of Lemma 3.3 for h^*f can be easily proven using that f satisfies this condition. Hence, (3.4b) defines a morphism in $\mathbf{Loc}_1^\infty(U)$.

Proposition 3.5. *The prestack* $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ *defined by Definition* 3.4*,* (3.2)*,* (3.4) *and the canonical coherence isomorphisms given by the universal property of pullback bundles is a stack, i.e., it satisfies the descent condition from Definition* 2.3*.*

Proof. This is a direct consequence of descent for fiber bundles and differential forms and of the fact that the first condition on the fiber bundle morphisms stated in Lemma 3.3 is a local condition on $U \in \mathbf{Man}$. In more detail, spelling out descent for objects $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$, one observes that it involves descent for the underlying fiber bundles $\pi : M \to U$ and also for the underlying 1-forms E, which are straightforward consequences of descent for fiber bundles and differential forms. Similarly, descent for $\mathbf{Loc}_1^\infty(U)$-morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ involves descent for the underlying fiber bundle maps $f : (\pi : M \to U) \to (\pi' : M' \to U)$ and the verification that the 1-forms are preserved and that any one of the equivalent conditions from Lemma 3.3 holds, which are again both consequences of descent for fiber bundles and differential forms and the fact that the descent data fulfill these properties. □

Remark 3.6. Observe that the category $\mathbf{Loc}_1^\infty(\{*\})$ of global points $\underline{\{*\}} \to \mathbf{Loc}_1^\infty$ of the stack $\mathbf{Loc}_1^\infty$ is the ordinary category $\mathbf{Loc}_1$ of 1-dimensional spacetimes.

3.2. The Stack ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty$

The aim of this subsection is to develop a stack ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ that provides a smooth refinement of the usual category ${}^*\mathbf{Alg}_{\mathbb{C}}$ of associative and unital $*$-algebras over $\mathbb{C}$. Let us recall that the latter category may be defined as the category ${}^*\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Vec}_{\mathbb{C}})$ of order-reversing $*$-monoids in the *involutive* symmetric monoidal category $\mathbf{Vec}_{\mathbb{C}}$ of complex vector spaces, see, e.g., [7,19] for the relevant background on involutive category theory. Our strategy is to introduce first a stack (of involutive symmetric monoidal categories) that refines the category $\mathbf{Vec}_{\mathbb{C}}$ of vector spaces over $\mathbb{C}$ and then discuss how to form order-reversing $*$-monoids at the level of stacks.

Let us consider for now the case where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. As a first attempt to introduce a smooth refinement of the category $\mathbf{Vec}_{\mathbb{K}}$, we could consider the stack $\mathbf{VecBun}_{\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of $\mathbb{K}$-vector bundles introduced in Example 2.4. To a manifold $U \in \mathbf{Man}$, this stack assigns the category $\mathbf{VecBun}_{\mathbb{K}}(U)$ of (locally trivializable and finite rank) $\mathbb{K}$-vector bundles over U. Considering as in Remark 3.2 the fibers over points $x \in U$, every vector bundle can be interpreted as a smooth U-family of $\mathbb{K}$-vector spaces. The problem with this first attempt is that the fibers of vector bundles are (by definition) *finite-dimensional* vector spaces, while examples of AQFTs, even in dimension 1, require infinite-dimensional vector spaces, such as the vector spaces underlying the canonical commutation relation algebras. A natural way to enlarge the category $\mathbf{VecBun}_{\mathbb{K}}(U)$ in order to capture such infinite-dimensional aspects is to pass (via the sheaf of sections functor) to the category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ of sheaves of $C^\infty_{\mathbb{K},U}$-modules over $U \in \mathbf{Man}$. Here $C^\infty_{\mathbb{K},U} : \mathbf{Open}(U)^{\mathrm{op}} \to \mathbf{Alg}_{\mathbb{K}}\,,\ (\tilde{U} \subseteq U) \mapsto C^\infty_{\mathbb{K}}(\tilde{U})$ denotes the sheaf of $\mathbb{K}$-valued smooth functions on U. Indeed, $\mathbf{VecBun}_{\mathbb{K}}(U)$ embeds fully faithfully in $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ and the essential image consists of locally free $C^\infty_{\mathbb{K},U}$-modules of finite rank, see, e.g., [26, Chapter 2].

Remark 3.7. We would like to note that there are also alternative candidates to enlarge the category $\mathbf{VecBun}_{\mathbb{K}}(U)$ to include such infinite-dimensional aspects. For example, one could imagine to work with bundles over U whose fibers are, e.g., locally convex, bornological or diffeological vector spaces. However, to make this a valid choice, one would have to confirm that such categories assemble into a stack, as it is the case for the sheaf categories $U \mapsto \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$, see Proposition 3.8. As another alternative, one could search directly for a stack providing a smooth refinement of the category of C^*-algebras. To the best of our knowledge, such a stack has not yet been studied, but we believe that this may be related to the concept of continuous bundles/fields of C^*-algebras, see, e.g., [13, Section 10.3] and [21].

Following [20], let us now describe the (pre)stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of sheaves of $C^\infty_{\mathbb{K}}$-modules in more detail. To each manifold $U \in \mathbf{Man}$, it assigns the category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ of sheaves of $C^\infty_{\mathbb{K},U}$-modules over $U \in \mathbf{Man}$, with morphisms given by $C^\infty_{\mathbb{K},U}$-linear sheaf morphisms. To a morphism $h : U \to U'$ in $\mathbf{Man}$, it assigns the functor

$$h^* \,:=\, \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(h) \,:\, \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U') \;\longrightarrow\; \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U) \tag{3.5a}$$

that acts on $V \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U')$ as

$$h^*V \,:=\, h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U}, \tag{3.5b}$$

where h^{-1} is the inverse image sheaf functor and $\otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})}$ denotes the relative tensor product of sheaves of modules. Together with the canonical coherence isomorphisms associated with relative tensor products and inverse image functors, this defines a prestack $\mathbf{Sh}_{C^\infty_\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ in the sense of Definition 2.1. The following result is well-known, see, e.g., [20, Proposition 19.4.7].

Proposition 3.8. *For $\mathbb{K}$ being either $\mathbb{R}$ or $\mathbb{C}$, the prestack $\mathbf{Sh}_{C^\infty_\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ defined above is a stack, i.e., it satisfies the descent condition from Definition 2.3.*

Remark 3.9. Observe that the category $\mathbf{Sh}_{C^\infty_\mathbb{K}}(\{*\})$ of global points $\underline{\{*\}} \to \mathbf{Sh}_{C^\infty_\mathbb{K}}$ of the stack $\mathbf{Sh}_{C^\infty_\mathbb{K}}$ is the ordinary category $\mathbf{Vec}_\mathbb{K}$ of vector spaces over $\mathbb{K}$.

As explained at the beginning of this subsection, we interpret the stack $\mathbf{Sh}_{C^\infty_\mathbb{K}}$ as a smooth refinement of the category $\mathbf{Vec}_\mathbb{K}$ of vector spaces over $\mathbb{K}$. In order to introduce a smooth refinement of the category ${}^*\mathbf{Alg}_\mathbb{C} = {}^*\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Vec}_\mathbb{C})$ of associative and unital $*$-algebras over $\mathbb{C}$, we have to define an involutive symmetric monoidal structure on $\mathbf{Sh}_{C^\infty_\mathbb{C}}$. To achieve this goal, let us first observe that, for both $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the category $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$ of sheaves of $C^\infty_{\mathbb{K},U}$-modules over each $U \in \mathbf{Man}$ is symmetric monoidal with respect to the relative tensor product

$$V \otimes_{C^\infty_{\mathbb{K},U}} V' \in \mathbf{Sh}_{C^\infty_\mathbb{K}}(U), \tag{3.6}$$

for all $V, V' \in \mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$. (The monoidal unit is $C^\infty_{\mathbb{K},U} \in \mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$, regarded as a sheaf of $C^\infty_{\mathbb{K},U}$-modules.) Furthermore, for each morphism $h : U \to U'$ in $\mathbf{Man}$, the functor in (3.5) is strong symmetric monoidal via the coherence isomorphisms

$$\begin{aligned} h^*\big(V \otimes_{C^\infty_{\mathbb{K},U'}} V'\big) &= h^{-1}\big(V \otimes_{C^\infty_{\mathbb{K},U'}} V'\big) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \\ &\cong h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} h^{-1}(V') \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \\ &\cong \big(h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U}\big) \otimes_{C^\infty_{\mathbb{K},U}} \big(h^{-1}(V') \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U}\big) \\ &= (h^*V) \otimes_{C^\infty_{\mathbb{K},U}} (h^*V') \end{aligned} \tag{3.7a}$$

and

$$h^* C^\infty_{\mathbb{K},U'} = h^{-1}(C^\infty_{\mathbb{K},U'}) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \cong C^\infty_{\mathbb{K},U}. \tag{3.7b}$$

One can check that the canonical coherence isomorphisms of the stack $\mathbf{Sh}_{C^\infty_\mathbb{K}}$ are monoidal natural transformations.

Corollary 3.10. *The stack $\mathbf{Sh}_{C^\infty_\mathbb{K}}$ in Proposition 3.8 is canonically a stack $\mathbf{Sh}_{C^\infty_\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$ with values in the 2-category $\mathbf{SMCat}$ of symmetric monoidal categories, strong symmetric monoidal functors and monoidal natural transformations.*

In the case of $\mathbb{K} = \mathbb{C}$, we can define further, for each $U \in \mathbf{Man}$, an involution endofunctor $\overline{(-)} : \mathbf{Sh}_{C^\infty_\mathbb{C}}(U) \to \mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$. It assigns to an object $V \in \mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ the complex conjugate sheaf of $C^\infty_{\mathbb{C},U}$-modules $\overline{V} \in \mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ which, as a sheaf, coincides with V, but the $C^\infty_{\mathbb{C},U}$-module structure is defined via complex conjugation of $\mathbb{C}$-valued functions as $\overline{v} \cdot a := \overline{v \cdot a^*}$, for all $\overline{v} \in \overline{V}$

and $a \in C^\infty_{\mathbb{C},U}$. Clearly, the endofunctor $\overline{(-)}$ squares to the identity and hence defines an involutive structure on the category $\mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$, see [7,19]. Observe that $\overline{(-)}$ is canonically a strong symmetric monoidal functor with respect to the symmetric monoidal structure on $\mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ introduced above. Hence, we obtain that $\mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ is an involutive symmetric monoidal category, for every $U \in \mathbf{Man}$. Furthermore, for each morphism $h : U \to U'$ in $\mathbf{Man}$, the symmetric monoidal functor in (3.5) is involutive via the coherence isomorphisms

$$\begin{aligned} h^*\overline{V} &= h^{-1}(\overline{V}) \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} C^\infty_{\mathbb{C},U} \cong \overline{h^{-1}(V)} \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} \overline{C^\infty_{\mathbb{C},U}} \\ &\cong \overline{h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} C^\infty_{\mathbb{C},U}} = \overline{h^*V}, \end{aligned} \tag{3.8}$$

for all $V \in \mathbf{Sh}_{C^\infty_\mathbb{C}}(U')$, where in the second step we used complex conjugation $* : C^\infty_{\mathbb{C},U} \to \overline{C^\infty_{\mathbb{C},U}}$. Summing up, we obtain

Corollary 3.11. *For $\mathbb{K} = \mathbb{C}$, the stack $\mathbf{Sh}_{C^\infty_\mathbb{C}}$ in Proposition 3.8 is canonically a stack $\mathbf{Sh}_{C^\infty_\mathbb{C}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{ISMCat}$ with values in the 2-category $\mathbf{ISMCat}$ of involutive symmetric monoidal categories, involutive strong symmetric monoidal functors and involutive monoidal natural transformations.*

With these preparations, it is now straightforward to introduce a (pre)stack ${}^*\mathbf{Alg}^\infty_\mathbb{C}$ that provides a smooth refinement of the ordinary category ${}^*\mathbf{Alg}_\mathbb{C} = {}^*\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Vec}_\mathbb{C})$ of associative and unital $*$-algebras over $\mathbb{C}$. Using that forming order-reversing $*$-monoids is a 2-functor ${}^*\mathbf{Mon}_{\mathrm{rev}} : \mathbf{ISMCat} \to \mathbf{Cat}$, see [7,19], we define a prestack (in the sense of Definition 2.1) by the composition

$${}^*\mathbf{Alg}^\infty_\mathbb{C} \, := \, {}^*\mathbf{Mon}_{\mathrm{rev}} \circ \mathbf{Sh}_{C^\infty_\mathbb{C}} \, : \, \mathbf{Man}^{\mathrm{op}} \, \longrightarrow \, \mathbf{Cat}. \tag{3.9}$$

More explicitly, this prestack assigns, to each manifold $U \in \mathbf{Man}$, the category ${}^*\mathbf{Alg}^\infty_\mathbb{C}(U) = {}^*\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Sh}_{C^\infty_\mathbb{C}}(U))$ of order-reversing $*$-monoids in the involutive symmetric monoidal category $\mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$. An object in this category is a quadruple $(A, \mu, \eta, *)$, where $A \in \mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ is a sheaf of $C^\infty_{\mathbb{C},U}$-modules on $U \in \mathbf{Man}$ and

$$\mu \, : \, A \otimes_{C^\infty_{\mathbb{C},U}} A \, \longrightarrow \, A, \qquad \eta \, : \, C^\infty_{\mathbb{C},U} \, \longrightarrow \, A, \qquad * \, : \, A \, \longrightarrow \, \overline{A} \tag{3.10}$$

are morphisms in $\mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ that satisfy the axioms of an associative and unital $*$-algebra. A morphism $\kappa : (A, \mu, \eta, *) \to (A', \mu', \eta', *')$ in ${}^*\mathbf{Alg}^\infty_\mathbb{C}(U)$ is a morphism $\kappa : A \to A'$ in $\mathbf{Sh}_{C^\infty_\mathbb{C}}(U)$ that preserves the multiplications, units and involutions. To each morphism $h : U \to U'$ in $\mathbf{Man}$, the prestack ${}^*\mathbf{Alg}^\infty_\mathbb{C}$ assigns the functor

$$h^* \, := \, {}^*\mathbf{Alg}^\infty_\mathbb{C}(h) \, : \, {}^*\mathbf{Alg}^\infty_\mathbb{C}(U') \, \longrightarrow \, {}^*\mathbf{Alg}^\infty_\mathbb{C}(U) \tag{3.11}$$

that maps $(A, \mu, \eta, *) \in {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U')$ to the object $h^*A \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ given in (3.5b), endowed with the structure maps

$$(h^*A) \otimes_{C^\infty_{\mathbb{C},U}} (h^*A) \,\cong\, h^*\big(A \otimes_{C^\infty_{\mathbb{C},U'}} A\big) \xrightarrow{\;h^*\mu\;} h^*A\;, \tag{3.12a}$$

$$C^\infty_{\mathbb{C},U} \,\cong\, h^* C^\infty_{\mathbb{C},U'} \xrightarrow{\;h^*\eta\;} h^*A\;, \tag{3.12b}$$

$$h^*A \xrightarrow{\;h^**\;} h^*\overline{A} \,\cong\, \overline{h^*A}, \tag{3.12c}$$

obtained by using the coherence isomorphisms of the involutive symmetric monoidal stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ from Corollary 3.11.

Proposition 3.12. *The prestack* ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ *defined in* (3.9) *is a stack, i.e., it satisfies the descent condition from Definition* 2.3.

Proof. Let $\{U_\alpha \subseteq U\}$ be any open cover of any $U \in \mathbf{Man}$. The key step is to realize that the descent category ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(\{U_\alpha \subseteq U\})$ coincides with the category ${}^*\mathbf{Mon}_{\mathrm{rev}}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(\{U_\alpha \subseteq U\})\big)$ of order-reversing $*$-monoids in the descent category $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(\{U_\alpha \subseteq U\})$, which we endow with the involutive symmetric monoidal structure given by

$$\big(\{V_\alpha\}, \{\varphi_{\alpha\beta}\}\big) \otimes \big(\{V'_\alpha\}, \{\varphi'_{\alpha\beta}\}\big) \,:=\, \big(\{V_\alpha \otimes_{C^\infty_{\mathbb{C},U_\alpha}} V'_\alpha\}, \{\varphi_{\alpha\beta} \otimes_{C^\infty_{\mathbb{C},U_{\alpha\beta}}} \varphi'_{\alpha\beta}\}\big) \tag{3.13a}$$

$$\overline{\big(\{V_\alpha\}, \{\varphi_{\alpha\beta}\}\big)} \,:=\, \big(\{\overline{V_\alpha}\}, \{\overline{\varphi_{\alpha\beta}}\}\big), \tag{3.13b}$$

where we have suppressed the coherence isomorphisms in (3.7) and (3.8). Fully explicitly, the conjugated cocycle $\overline{\varphi_{\alpha\beta}}$ is given by

$$\overline{V_\beta}|_{U_{\alpha\beta}} \cong \overline{V_\beta|_{U_{\alpha\beta}}} \xrightarrow{\;\overline{\varphi_{\alpha\beta}}\;} \overline{V_\alpha|_{U_{\alpha\beta}}} \cong \overline{V_\alpha}|_{U_{\alpha\beta}}\;, \tag{3.14}$$

and similarly for the tensor product cocycle $\varphi_{\alpha\beta} \otimes_{C^\infty_{\mathbb{C},U_{\alpha\beta}}} \varphi'_{\alpha\beta}$. The functor to the descent category $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U) \to \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(\{U_\alpha \subseteq U\})$ given in (2.5) carries a canonical involutive symmetric monoidal structure and it is an equivalence in $\mathbf{ISMCat}$ because $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ is a stack. Applying the 2-functor ${}^*\mathbf{Mon}_{\mathrm{rev}} : \mathbf{ISMCat} \to \mathbf{Cat}$ that takes order-reversing $*$-monoids then yields the equivalence of categories ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(\{U_\alpha \subseteq U\})$ that proves descent for ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}$. □

Remark 3.13. Observe that the category ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(\{*\})$ of global points $\underline{\{*\}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ of the stack ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ is the ordinary category ${}^*\mathbf{Alg}_{\mathbb{C}}$ of associative and unital $*$-algebras over $\mathbb{C}$.

3.3. The Stack $\mathbf{AQFT}^\infty_1$

With these preparations, we are now ready to introduce a natural concept of smooth 1-dimensional AQFTs. Recalling that the category of ordinary 1-dimensional AQFTs is described as the functor category $\mathbf{AQFT}_1 := \mathbf{Fun}(\mathbf{Loc}_1, {}^*\mathbf{Alg}_{\mathbb{C}})$, we propose the following

Definition 3.14. The *stack of smooth* 1*-dimensional AQFTs* is defined as the mapping stack (see (2.13))

$$\mathbf{AQFT}_1^\infty := \mathsf{Map}\big(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_\mathbb{C}^\infty\big) \in \mathbf{St} \tag{3.15}$$

from the stack $\mathbf{Loc}_1^\infty$ of 1-dimensional spacetimes developed in Sect. 3.1 to the stack ${}^*\mathbf{Alg}_\mathbb{C}^\infty$ of associative and unital $*$-algebras developed in Sect. 3.2.

This very simple definition is incredibly rich and powerful, as we shall explain throughout the rest of this subsection. Before discussing some of its more sophisticated consequences, we believe that it is worth spelling out explicitly what a smooth 1-dimensional AQFT is. By definition, it is a global point $\underline{\{*\}} \to \mathbf{AQFT}_1^\infty$ of the stack introduced in Definition 3.14 which, by the 2-Yoneda Lemma, is equivalently an object $\mathfrak{A} \in \mathbf{AQFT}_1^\infty(\{*\})$. According to (2.13), which defines mapping stacks, we find that a smooth 1-dimensional AQFT is then simply a stack morphism $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_\mathbb{C}^\infty$. Even more explicitly, this consists of a family of functors

$$\mathfrak{A}_U \,:\, \mathbf{Loc}_1^\infty(U) \longrightarrow {}^*\mathbf{Alg}_\mathbb{C}^\infty(U), \tag{3.16a}$$

for all manifolds $U \in \mathbf{Man}$, and natural isomorphisms

$$\begin{array}{ccc} \mathbf{Loc}_1^\infty(U') & \xrightarrow{\mathfrak{A}_{U'}} & {}^*\mathbf{Alg}_\mathbb{C}^\infty(U') \\ {\scriptstyle h^*}\big\downarrow & \swArrow{\scriptstyle \mathfrak{A}_h} & \big\downarrow{\scriptstyle h^*} \\ \mathbf{Loc}_1^\infty(U) & \xrightarrow[\mathfrak{A}_U]{} & {}^*\mathbf{Alg}_\mathbb{C}^\infty(U) \end{array} \tag{3.16b}$$

for all morphisms $h : U \to U'$ in $\mathbf{Man}$, that satisfy the coherence axioms listed in Definition 2.5. When we interpret, as explained in the previous subsections, $\mathbf{Loc}_1^\infty(U)$ as the category of smooth U-families of 1-dimensional spacetimes and ${}^*\mathbf{Alg}_\mathbb{C}^\infty(U)$ as the category of smooth U-families of algebras, the role of the functor $\mathfrak{A}_U : \mathbf{Loc}_1^\infty(U) \to {}^*\mathbf{Alg}_\mathbb{C}^\infty(U)$ is to capture the response of the observable algebras to "smooth variations of spacetimes." Hence, smooth AQFTs have built in a suitable concept of smooth dependence on smooth variations of spacetimes, which we will illustrate in more detail via simple examples in Sect. 5. Let us also note that the functor $\mathfrak{A}_{\{*\}} : \mathbf{Loc}_1^\infty(\{*\}) = \mathbf{Loc}_1 \to {}^*\mathbf{Alg}_\mathbb{C}^\infty(\{*\}) = {}^*\mathbf{Alg}_\mathbb{C}$ associated with the point $U = \{*\}$ defines an ordinary 1-dimensional AQFT. Hence, every smooth AQFT has an underlying ordinary AQFT and it therefore provides a refinement of the ordinary concept.

Another interesting consequence of Definition 3.14 is that it introduces a natural concept of "smooth curves of smooth AQFTs," or more generally of smooth $\tilde{U}$-families of smooth AQFTs, for every manifold $\tilde{U} \in \mathbf{Man}$. By definition, a smooth $\tilde{U}$-family of smooth AQFTs is a $\tilde{U}$-point $\underline{\tilde{U}} \to \mathbf{AQFT}_1^\infty$ of the stack from Definition 3.14 which, by the 2-Yoneda Lemma, is equivalently an object $\mathfrak{B} \in \mathbf{AQFT}_1^\infty(\tilde{U})$. From the definition of mapping stacks (2.13), we obtain that this is simply a stack morphism $\mathbf{Loc}_1^\infty \times \underline{\tilde{U}} \to {}^*\mathbf{Alg}_\mathbb{C}^\infty$, or equivalently a stack morphism

$$\mathfrak{B} \,:\, \mathbf{Loc}_1^\infty \longrightarrow \mathsf{Map}\big(\underline{\tilde{U}}, {}^*\mathbf{Alg}_\mathbb{C}^\infty\big) \tag{3.17}$$

to the mapping stack from $\underline{\tilde{U}}$ to ${}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}$. Even more explicitly, using again the 2-Yoneda Lemma, this is a family of functors

$$\mathfrak{B}_U \,:\, \mathbf{Loc}_1^{\infty}(U) \;\longrightarrow\; {}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}(U \times \tilde{U}), \tag{3.18a}$$

for all manifolds $U \in \mathbf{Man}$, and natural isomorphisms

$$\begin{array}{ccc} \mathbf{Loc}_1^{\infty}(U') & \xrightarrow{\;\mathfrak{B}_{U'}\;} & {}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}(U' \times \tilde{U}) \\ {\scriptstyle h^*}\big\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \mathfrak{B}_h} & \big\downarrow{\scriptstyle (h\times \mathrm{id})^*} \\ \mathbf{Loc}_1^{\infty}(U) & \xrightarrow[\;\mathfrak{B}_U\;]{} & {}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}(U \times \tilde{U}) \end{array} \tag{3.18b}$$

for all morphisms $h : U \to U'$ in $\mathbf{Man}$, that satisfy the coherence axioms listed in Definition 2.5. The role of the functor $\mathfrak{B}_U : \mathbf{Loc}_1^{\infty}(U) \to {}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}(U \times \tilde{U})$ is now twofold: Firstly, it captures the response of the observable algebras to "smooth U-variations of spacetimes." Secondly, it captures the response of the observable algebras to "smooth $\tilde{U}$-variations of the smooth AQFT itself." Again, this concept is best illustrated via simple examples, see Sect. 5.

As another interesting consequence of Definition 3.14, let us note that every smooth AQFT $\mathfrak{A} : \underline{\{*\}} \to \mathbf{AQFT}_1^{\infty}$ has a smooth automorphism group. (We refer to [14] for automorphism groups in ordinary AQFT, which in general are not smooth groups.) This can be defined in terms of the *loop stack*

$$\begin{array}{ccc} \mathrm{Aut}(\mathfrak{A}) & \dashrightarrow & \underline{\{*\}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{A}} \\ \underline{\{*\}} & \xrightarrow[\;\mathfrak{A}\;]{} & \mathbf{AQFT}_1^{\infty} \end{array} \tag{3.19}$$

which is a bicategorical pullback in the 2-category $\mathbf{St}$ of stacks of categories.[1] By a direct computation of this bicategorical pullback, one finds that $\mathrm{Aut}(\mathfrak{A}) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set} \subset \mathbf{Cat}$ is equivalent to a sheaf of sets (i.e., discrete categories), which due to the universal property of bicategorical pullbacks comes endowed with a group structure. This implies that $\mathrm{Aut}(\mathfrak{A}) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grp}$ is a sheaf of groups on $\mathbf{Man}$, i.e., a smooth group from the functor of points perspective.

Let us briefly explain how this concept of smooth automorphism groups is related to the more practical concept of smooth AQFTs with a smooth action of a Lie group. Given any Lie group G, we use the 2-Yoneda embedding to define a group object $\underline{G} \in \mathbf{St}$ in the 2-category of stacks and construct the quotient stack

$$[\{*\}/G] \,:=\, \mathrm{bicolim}_{\mathbf{St}}\Big(\underline{\{*\}} \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \underline{G} \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \underline{G}^2 \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow}\; \cdots \Big) \in \mathbf{St} \tag{3.20}$$

associated with the trivial action of G on the point $\{*\}$ via a bicategorical colimit. A *G-equivariant smooth 1-dimensional AQFT* is then defined to be a stack morphism

[1] The bicategorical pullback in (3.19) exists because the 2-category $\mathbf{Cat}$ admits all bicategorical limits, see, e.g., [16, Theorem 5.1], and hence so does $\mathbf{St}$.

$$\mathfrak{A}^{\mathrm{eq}} \,:\, [\{*\}/G] \;\longrightarrow\; \mathbf{AQFT}_1^{\infty}. \tag{3.21}$$

By the universal property of the bicategorical colimit in (3.20), this datum is equivalent to a smooth AQFT $\mathfrak{A} : \underline{\{*\}} \to \mathbf{AQFT}_1^{\infty}$ together with a 2-automorphism $\mathfrak{A}_2$ of the stack morphism $\underline{G} \to \underline{\{*\}} \stackrel{\mathfrak{A}}{\to} \mathbf{AQFT}_1^{\infty}$ that satisfies certain compatibility conditions arising from the face and degeneracy maps in (3.20).[2] From this we obtain a bicategorical cone

$$\begin{array}{ccc} \underline{G} & \longrightarrow & \underline{\{*\}} \\ \big\downarrow & \overset{\mathfrak{A}_2}{\swarrow\!\!\swarrow} & \big\downarrow{\scriptstyle \mathfrak{A}} \\ \underline{\{*\}} & \underset{\mathfrak{A}}{\longrightarrow} & \mathbf{AQFT}_1^{\infty} \end{array} \tag{3.22}$$

and hence, by the universal property of the loop stack in (3.19), a stack morphism $\underline{G} \to \mathrm{Aut}(\mathfrak{A})$ to the smooth automorphism group. Due to the compatibility conditions of $\mathfrak{A}_2$ this is a morphism of group objects.

We conclude this subsection by providing an equivalent, but more explicit, description of G-equivariant smooth AQFTs. The quotient stack in (3.20) can also be described as the stackyfication of the prestack $[\{*\}/G]_{\mathrm{pre}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ that assigns to each $U \in \mathbf{Man}$ the groupoid

$$[\{*\}/G]_{\mathrm{pre}}(U) \;=\; \begin{cases} \mathrm{Obj}: & * \\ \mathrm{Mor}: & C^{\infty}(U,G) \end{cases} \tag{3.23}$$

with a single object $*$ and morphisms the smooth functions to the Lie group. (Composition of morphisms is given by the point-wise group structure of $C^{\infty}(U,G)$.) On $\mathbf{Man}$-morphisms $h : U \to U'$ this prestack acts via pullback of functions $[\{*\}/G]_{\mathrm{pre}}(h) := h^*$. Because ${}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}$ is a stack by Proposition 3.12, the universal property of stackyfication implies that the datum of a stack morphism $\mathfrak{A}^{\mathrm{eq}} : [\{*\}/G] \to \mathbf{AQFT}_1^{\infty}$ is equivalent to a pseudo-natural transformation $[\{*\}/G]_{\mathrm{pre}} \to \mathbf{AQFT}_1^{\infty}$ between prestacks, or equivalently a pseudo-natural transformation

$$\tilde{\mathfrak{A}} \,:\, \mathbf{Loc}_1^{\infty} \times [\{*\}/G]_{\mathrm{pre}} \;\longrightarrow\; {}^*\mathbf{Alg}_{\mathbb{C}}^{\infty}. \tag{3.24}$$

We will show in Sect. 5.3 that the latter perspective on G-equivariant smooth AQFTs is not very complicated to describe in concrete examples.

[2] Recalling Definition 2.6, let us also state these conditions explicitly at the level of the component natural automorphisms $\mathfrak{A}_{2U}$ of the functors $\underline{G}(U) \to \{*\} \stackrel{\mathfrak{A}_U}{\to} \mathbf{AQFT}_1^{\infty}(U)$, for all $U \in \mathbf{Man}$. Because $\underline{G}(U)$ is a discrete category, i.e., it only has identity morphisms, $\mathfrak{A}_{2U}$ is simply a family of $\mathbf{AQFT}_1^{\infty}(U)$-isomorphisms $\mathfrak{A}_{2U\,g} : \mathfrak{A}_U(*) \to \mathfrak{A}_U(*)$ labeled by elements $g \in \underline{G}(U) = C^{\infty}(U,G)$. The compatibility conditions then state that this labeling is compatible with the point-wise group structure on $\underline{G}(U) = C^{\infty}(U,G)$, i.e., $\mathfrak{A}_{2U\,g\cdot g'} = \mathfrak{A}_{2U\,g} \circ \mathfrak{A}_{2U\,g'}$, for all $g, g' \in \underline{G}(U)$, and $\mathfrak{A}_{2U\,e} = \mathrm{id}$, for the identity element $e \in \underline{G}(U)$.

4. Smooth Canonical Quantization

The construction of free field theories in ordinary AQFT crucially relies on the existence of canonical (anti-)commutation relation quantization functors, see, e.g., [1,2]. The goal of this section is to show that these quantization functors admit a smooth refinement, which will allow us to construct both Bosonic and Fermionic examples of smooth 1-dimensional AQFTs in Sect. 5.

4.1. Canonical Commutation Relations

Ordinary canonical commutation relation (CCR) quantization is described by a functor $\mathfrak{CCR} : \mathbf{PoVec}_{\mathbb{R}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ from the category of Poisson vector spaces to the category of associative and unital $*$-algebras. Recall that an object in the category $\mathbf{PoVec}_{\mathbb{R}}$ is a tuple (W, τ), where $W \in \mathbf{Vec}_{\mathbb{R}}$ is a real vector space and $\tau : W \otimes_{\mathbb{R}} W \to \mathbb{R}$ is an antisymmetric morphism in $\mathbf{Vec}_{\mathbb{R}}$, and that a morphism $\psi : (W, \tau) \to (W', \tau')$ in $\mathbf{PoVec}_{\mathbb{R}}$ is a $\mathbf{Vec}_{\mathbb{R}}$-morphism $\psi : W \to W'$ satisfying $\tau' \circ (\psi \otimes_{\mathbb{R}} \psi) = \tau$. (The objects in $\mathbf{PoVec}_{\mathbb{R}}$ are interpreted physically as vector spaces of linear observables, endowed with a Poisson structure.) The CCR functor assigns to a Poisson vector space $(W, \tau) \in \mathbf{PoVec}_{\mathbb{R}}$ the associative and unital $*$-algebra

$$\mathfrak{CCR}(W, \tau) := \bigoplus_{n \geq 0} \big(W \otimes_{\mathbb{R}} \mathbb{C}\big)^{\otimes_{\mathbb{C}} n} \Big/ \mathcal{I}^{\mathrm{CCR}}_{(W,\tau)} \in {}^*\mathbf{Alg}_{\mathbb{C}}, \tag{4.1}$$

where $\mathcal{I}^{\mathrm{CCR}}_{(W,\tau)}$ is the 2-sided $*$-ideal generated by the canonical commutation relations $w \otimes w' - w' \otimes w = \mathrm{i}\ \tau(w, w')$, for all $w, w' \in W$, where $\mathrm{i} \in \mathbb{C}$ denotes the imaginary unit. The $*$-involution on $\mathfrak{CCR}(W, \tau)$ is specified by $w^* = w$, for all $w \in W$. Let us reformulate (4.1) in a slightly more abstract language. For this it is useful to observe that the construction of CCR algebras (4.1) consists of three steps:

1. Complexify the real vector space $W \in \mathbf{Vec}_{\mathbb{R}}$ to the complex vector space $W \otimes_{\mathbb{R}} \mathbb{C} \in \mathbf{Vec}_{\mathbb{C}}$, which may be endowed with a $*$-involution $\mathrm{id} \otimes_{\mathbb{R}} * : W \otimes_{\mathbb{R}} \mathbb{C} \to \overline{W \otimes_{\mathbb{R}} \mathbb{C}} = W \otimes_{\mathbb{R}} \overline{\mathbb{C}}$ determined by complex conjugation on $\mathbb{C}$. Hence, $(W \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{id} \otimes_{\mathbb{R}} *) \in {}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}})$ defines a $*$-object in the involutive symmetric monoidal category of complex vector spaces.
2. Take the free order-reversing $*$-monoid of $(W \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{id} \otimes_{\mathbb{R}} *) \in {}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}})$, which defines the associative and unital $*$-algebra $\bigoplus_{n \geq 0} \big(W \otimes_{\mathbb{R}} \mathbb{C}\big)^{\otimes_{\mathbb{C}} n} \in {}^*\mathbf{Alg}_{\mathbb{C}}$.
3. Implement the canonical commutation relations associated with the Poisson structure τ by a coequalizer in the category ${}^*\mathbf{Alg}_{\mathbb{C}}$.

Before we can generalize this construction to the context of stacks, we have to find a smooth refinement of the category $\mathbf{PoVec}_{\mathbb{R}}$. As explained in Sect. 3.2, we consider the stack $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$ of sheaves of $C^\infty_{\mathbb{R}}$-modules as a smooth refinement of the category $\mathbf{Vec}_{\mathbb{R}}$; hence, a smooth refinement of the category $\mathbf{PoVec}_{\mathbb{R}}$ should be built from this stack. Concretely, we define the (pre)stack $\mathbf{PoVec}^\infty_{\mathbb{R}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ by the following data. To each manifold $U \in \mathbf{Man}$, it assigns the category $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$ whose objects are tuples (W, τ) with $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ and $\tau : W \otimes_{C^\infty_{\mathbb{R},U}} W \to C^\infty_{\mathbb{R},U}$ an antisymmetric morphism in $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$, called Poisson structure. The morphisms $\psi : (W,\tau) \to (W',\tau')$ in this category are $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms $\psi : W \to W'$ satisfying $\tau' \circ (\psi \otimes_{C^\infty_{\mathbb{R},U}} \psi) = \tau$. To a morphism $h : U \to U'$ in **Man**, the prestack $\mathbf{PoVec}^\infty_{\mathbb{R}}$ assigns the functor

$$h^* := \mathbf{PoVec}^\infty_{\mathbb{R}}(h) \,:\, \mathbf{PoVec}^\infty_{\mathbb{R}}(U') \longrightarrow \mathbf{PoVec}^\infty_{\mathbb{R}}(U) \tag{4.2}$$

that assigns to $(W,\tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U')$ the object in $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$ determined by the object $h^*W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ (see (3.5b)) and the Poisson structure

$$(h^*W) \otimes_{C^\infty_{\mathbb{R},U}} (h^*W) \cong h^*\big(W \otimes_{C^\infty_{\mathbb{R},U'}} W\big) \xrightarrow{h^*\tau} h^*C^\infty_{\mathbb{R},U'} \cong C^\infty_{\mathbb{R},U} \ , \tag{4.3}$$

where $\cong$ are the coherence isomorphisms in (3.7).

Proposition 4.1. *The prestack* $\mathbf{PoVec}^\infty_{\mathbb{R}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ *defined above is a stack, i.e., it satisfies the descent condition from Definition 2.3.*

Proof. This follows from the fact that $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$ is a stack, see Proposition 3.8. Indeed, spelling out descent for objects $(W,\tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U)$, one observes that it involves descent for the underlying objects $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ and also for the underlying $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms $\tau : W \otimes_{C^\infty_{\mathbb{R},U}} W \to C^\infty_{\mathbb{R},U}$, which are both simple consequences of descent for the stack $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$. Similarly, descent for $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$-morphisms $\psi : (W,\tau) \to (W',\tau')$ involves descent for the underlying $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms $\psi : W \to W'$ and the verification that $\tau' \circ (\psi \otimes_{C^\infty_{\mathbb{R},U}} \psi) = \tau$ coincide as $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms, which are again both consequences of descent for the stack $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$ and of the fact that the descent data have this property. □

Adopting an analogous three step construction as in the case of the ordinary CCR functor, we shall now define a stack morphism

$$\mathfrak{CCR} \,:\, \mathbf{PoVec}^\infty_{\mathbb{R}} \longrightarrow {}^*\mathbf{Alg}^\infty_{\mathbb{C}} \tag{4.4}$$

that provides a smooth refinement of CCR quantization. By Definition 2.5, this consists of functors

$$\mathfrak{CCR}_U \,:\, \mathbf{PoVec}^\infty_{\mathbb{R}}(U) \longrightarrow {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U), \tag{4.5}$$

for each manifold $U \in \mathbf{Man}$, together with coherence isomorphisms. Regarding the first step, we observe that, for each $U \in \mathbf{Man}$, there exists an adjunction

$$L_U \,:\, \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U) \rightleftarrows {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big) \,:\, R_U \ . \tag{4.6}$$

The left adjoint functor L_U assigns to $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ its complexification $W \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U} \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$, with the $*$-object structure $\mathrm{id} \otimes_{C^\infty_{\mathbb{R},U}} *$ determined by complex conjugation $* : C^\infty_{\mathbb{C},U} \to \overline{C^\infty_{\mathbb{C},U}}$. The right adjoint functor R_U assigns to a $*$-object $(V,*)$ in $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ the sheaf of $*$-invariants $R_U(V,*) = \ker\big(V_{\mathbb{R}} \xrightarrow{*-\mathrm{id}} V_{\mathbb{R}}\big) \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$, where by $V_{\mathbb{R}} \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ we denote the restriction of $V \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ to a sheaf of $C^\infty_{\mathbb{R},U}$-modules via the morphism $C^\infty_{\mathbb{R},U} \to C^\infty_{\mathbb{C},U}$ from real to complex-valued functions.

Regarding the second step, we observe that, for each $U \in \mathbf{Man}$, there exists an adjunction

$$F_U \,:\, {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big) \rightleftarrows {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \,:\, G_U. \tag{4.7}$$

The right adjoint functor G_U assigns to an associative and unital $*$-algebra $(A,\mu,\eta,*)$ in $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ its underlying $*$-object $(A,*)$, i.e., it forgets the multiplication μ and unit η. The left adjoint functor F_U is the free order-reversing $*$-monoid functor. Explicitly, it assigns to a $*$-object $(V,*)$ the free order-reversing $*$-monoid $F_U(V,*) := \bigoplus_{n\geq 0} V^{\otimes n}$, where tensor products and coproducts are formed in the symmetric monoidal category $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$. The order-reversing $*$-structure of $F_U(V,*)$ is defined by the canonical extension of the $*$-structure on the generators $(V,*)$.

With these preparations, we can now define the values of (4.5) on objects by carrying out the third step. Explicitly, given any object $(W,\tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U)$, i.e., $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ and $\tau : W \otimes_{C^\infty_{\mathbb{R},U}} W \to C^\infty_{\mathbb{R},U}$ an antisymmetric morphism in $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$, we define

$$\mathfrak{CCR}_U(W,\tau) := \mathrm{colim}\Big(F_U L_U\big(W \otimes_{C^\infty_{\mathbb{R},U}} W\big) \underset{r_2}{\overset{r_1}{\rightrightarrows}} F_U L_U\big(W\big) \Big) \in {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \tag{4.8}$$

by a coequalizer in ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$. The relations r_1, r_2 are defined in terms of their adjuncts under the adjunctions in (4.6) and (4.7) by

$$\begin{array}{ccccc}
W \otimes_{C^\infty_{\mathbb{R},U}} W & \xrightarrow{\qquad\qquad\qquad \tilde{r}_1 \qquad\qquad\qquad} & & & RGFL(W) \\
\Big\downarrow{\scriptstyle \text{unit } L\dashv R} & & & & \Big\uparrow{\scriptstyle R(\mu-\mu^{\mathrm{op}})} \\
RL(W \otimes_{C^\infty_{\mathbb{R},U}} W) & \xrightarrow[\cong]{} & R\big(L(W) \otimes_{C^\infty_{\mathbb{C},U}} L(W)\big) & \xrightarrow[\text{unit } F\dashv G]{} & R\big(GFL(W) \otimes_{C^\infty_{\mathbb{C},U}} GFL(W)\big)
\end{array} \tag{4.9a}$$

and

$$\begin{array}{ccccc}
W \otimes_{C^\infty_{\mathbb{R},U}} W & \xrightarrow{\qquad\qquad \tilde{r}_2 \qquad\qquad} & & & RGFL(W) \\
\Big\downarrow{\scriptstyle \tau} & & & & \Big\uparrow{\scriptstyle R(\,\mathrm{i}\;\eta)} \\
C^\infty_{\mathbb{R},U} & \xrightarrow[\text{unit } L\dashv R]{} & RL\big(C^\infty_{\mathbb{R},U}\big) & \xrightarrow[\cong]{} & R\big(C^\infty_{\mathbb{C},U},*\big)
\end{array} \tag{4.9b}$$

where we suppressed for notational convenience the subscripts $_U$ on the functors. Here $\mu^{(\mathrm{op})}$ denotes the (opposite) multiplication and η the unit element in $FL(W)$. Because the coequalizer in (4.8) is clearly functorial with respect to morphisms $\psi : (W,\tau) \to (W',\tau')$ in $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$, we have successfully defined the desired functor in (4.5).

Remark 4.2. For $U = \{*\}$ a point, (4.8) gives precisely the usual CCR algebra in (4.1).

To complete our construction of the desired stack morphism $\mathfrak{CCR} : \mathbf{PoVec}^\infty_{\mathbb{R}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$, it remains to define coherence isomorphisms (see Definition 2.5)

$$\begin{array}{ccc} \mathbf{PoVec}^\infty_{\mathbb{R}}(U') & \xrightarrow{\;\mathfrak{CCR}_{U'}\;} & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \\ {\scriptstyle h^*}\big\downarrow & \swarrow\!\!\!\!\swarrow {\scriptstyle \mathfrak{CCR}_h} & \big\downarrow{\scriptstyle h^*} \\ \mathbf{PoVec}^\infty_{\mathbb{R}}(U) & \xrightarrow[\;\mathfrak{CCR}_U\;]{} & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \end{array} \tag{4.10}$$

for all morphisms $h : U \to U'$ in **Man**. These can be built from the analogous coherence isomorphisms for the left adjoint functors in (4.6) and (4.7), i.e.,

$$\begin{array}{ccccc} \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U') & \xrightarrow{\;L_{U'}\;} & {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U')\big) & \xrightarrow{\;F_{U'}\;} & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \\ {\scriptstyle h^*}\big\downarrow & \swarrow\!\!\!\!\swarrow {\scriptstyle L_h} & \big\downarrow{\scriptstyle h^*} & \swarrow\!\!\!\!\swarrow {\scriptstyle F_h} & \big\downarrow{\scriptstyle h^*} \\ \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U) & \xrightarrow[\;L_U\;]{} & {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big) & \xrightarrow[\;F_U\;]{} & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \end{array} \tag{4.11}$$

Explicitly, for $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U')$, the isomorphism L_h is given by

$$\begin{aligned} h^* L_{U'}(W) &= h^*\big(W \otimes_{C^\infty_{\mathbb{R},U'}} C^\infty_{\mathbb{C},U'}, \mathrm{id} \otimes *\big) \\ &\cong \Big(h^{-1}(W) \otimes_{h^{-1}(C^\infty_{\mathbb{R},U'})} h^{-1}(C^\infty_{\mathbb{C},U'}) \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes * \otimes *\Big) \\ &\cong \Big(h^{-1}(W) \otimes_{h^{-1}(C^\infty_{\mathbb{R},U'})} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes *\Big) \\ &\cong \Big(h^{-1}(W) \otimes_{h^{-1}(C^\infty_{\mathbb{R},U'})} C^\infty_{\mathbb{R},U} \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes \mathrm{id} \otimes *\Big) \\ &= L_U h^*(W). \end{aligned} \tag{4.12a}$$

For $(V, *) \in {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U')\big)$, the isomorphism F_h is given by

$$\begin{aligned} h^* F_{U'}(V,*) &= h^*\Big(\bigoplus_{n\geq 0} V^{\otimes_{C^\infty_{\mathbb{C},U'}} n}\Big) \cong \bigoplus_{n \geq 0} h^*\Big(V^{\otimes_{C^\infty_{\mathbb{C},U'}} n}\Big) \cong \bigoplus_{n\geq 0} \big(h^*(V)\big)^{\otimes_{C^\infty_{\mathbb{C},U}} n} \\ &= F_U h^*(V,*), \end{aligned} \tag{4.12b}$$

where in the second step we have used that h^* preserves coproducts because it is a left adjoint functor and in the third step we have used the coherence isomorphisms of the involutive symmetric monoidal stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ from Corollary 3.11. Pasting the natural isomorphisms in (4.11) defines a natural isomorphism $(FL)_h : h^* F_{U'} L_{U'} \Rightarrow F_U L_U h^*$. For every object $(W, \tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U')$, the associated isomorphism $h^* F_{U'} L_{U'}(W) \cong F_U L_U h^*(W)$ descends to the CCR algebras in (4.8) and thereby defines the natural isomorphism $\mathfrak{CCR}_h$ in (4.10).

Proposition 4.3. *The construction above defines a stack morphism* $\mathfrak{CCR} : \mathbf{PoVec}^\infty_{\mathbb{R}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$.

4.2. Canonical Anti-commutation Relations

A smooth refinement of the canonical anti-commutation relation (CAR) quantization functor for Fermionic theories can be developed along the same lines as in Sect. 4.1. Before we spell out some of the details, let us briefly recall the ordinary CAR functor $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ following the presentation in [12]. The category $\mathbf{IPVec}_{\mathbb{C}}$ has objects $(V, *, \langle\cdot,\cdot\rangle)$ consisting of a $*$-object $(V,*) \in {}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}})$ in the involutive symmetric monoidal category $\mathbf{Vec}_{\mathbb{C}}$ and a symmetric $*$-morphism $\langle\cdot,\cdot\rangle : (V,*) \otimes (V,*) \to (\mathbb{C},*)$. More explicitly, the latter is a symmetric $\mathbb{C}$-linear map $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{C}$ satisfying

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\quad\langle\cdot,\cdot\rangle\quad} & \mathbb{C} \\ {\scriptstyle *\otimes *}\big\downarrow & & \big\downarrow{\scriptstyle *} \\ \overline{V} \otimes \overline{V} \cong \overline{V \otimes V} & \xrightarrow[\overline{\langle\cdot,\cdot\rangle}]{\quad\quad} & \overline{\mathbb{C}} \end{array} \tag{4.13}$$

or at the level of elements $\langle v, v'\rangle^* = \langle v^*, v'^*\rangle$, for all $v, v' \in V$. Morphisms $\psi : (V, *, \langle\cdot,\cdot\rangle) \to (V', *', \langle\cdot,\cdot\rangle')$ in $\mathbf{IPVec}_{\mathbb{C}}$ are $*$-morphisms $\psi : (V,*) \to (V',*')$ satisfying $\langle\cdot,\cdot\rangle' \circ (\psi \otimes \psi) = \langle\cdot,\cdot\rangle$. The CAR functor assigns to $(V, *, \langle\cdot,\cdot\rangle) \in \mathbf{IPVec}_{\mathbb{C}}$ the associative and unital $*$-algebra

$$\mathfrak{CAR}\big(V, *, \langle\cdot,\cdot\rangle\big) := \bigoplus_{n\geq 0} V^{\otimes n} \Big/ \mathcal{I}^{\mathrm{CAR}}_{(V,*,\langle\cdot,\cdot\rangle)} \in {}^*\mathbf{Alg}_{\mathbb{C}}, \tag{4.14}$$

where $\mathcal{I}^{\mathrm{CAR}}_{(V,*,\langle\cdot,\cdot\rangle)}$ is the 2-sided $*$-ideal generated by the canonical anti-commutation relations $v \otimes v' + v' \otimes v = \langle v, v'\rangle$, for all $v, v' \in V$. Observe that this construction consists of two steps:

1. Take the free order-reversing $*$-monoid of $(V,*) \in {}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}})$, which defines the associative and unital $*$-algebra $\bigoplus_{n\geq 0} V^{\otimes n} \in {}^*\mathbf{Alg}_{\mathbb{C}}$.
2. Implement the canonical anti-commutation relations associated with $\langle\cdot,\cdot\rangle$ by a coequalizer in the category ${}^*\mathbf{Alg}_{\mathbb{C}}$.

To obtain a smooth refinement of the category $\mathbf{IPVec}_{\mathbb{C}}$, we follow the same strategy as in Sect. 4.1. We define a (pre)stack $\mathbf{IPVec}^\infty_{\mathbb{C}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ by the following data. To each manifold $U \in \mathbf{Man}$, it assigns the category $\mathbf{IPVec}^\infty_{\mathbb{C}}(U)$ whose objects $(V, *, \langle\cdot,\cdot\rangle)$ consist of a $*$-object $(V,*) \in {}^*\mathbf{Obj}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$ and a symmetric $*$-morphism $\langle\cdot,\cdot\rangle : (V,*) \otimes_{C^\infty_{\mathbb{C},U}} (V,*) \to (C^\infty_{\mathbb{C},U}, *)$. The morphisms $\psi : (V, *, \langle\cdot,\cdot\rangle) \to (V', *', \langle\cdot,\cdot\rangle')$ in this category are ${}^*\mathbf{Obj}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$-morphisms $\psi : (V,*) \to (V',*')$ satisfying $\langle\cdot,\cdot\rangle' \circ (\psi \otimes_{C^\infty_{\mathbb{C},U}} \psi) = \langle\cdot,\cdot\rangle$. To a morphism $h : U \to U'$ in $\mathbf{Man}$, the prestack $\mathbf{IPVec}^\infty_{\mathbb{C}}$ assigns the functor

$$h^* := \mathbf{IPVec}^\infty_{\mathbb{C}}(h) \;:\; \mathbf{IPVec}^\infty_{\mathbb{C}}(U') \longrightarrow \mathbf{IPVec}^\infty_{\mathbb{C}}(U) \tag{4.15}$$

that assigns to $(V, *, \langle\cdot,\cdot\rangle) \in \mathbf{IPVec}^\infty_{\mathbb{C}}(U')$ the object in $\mathbf{IPVec}^\infty_{\mathbb{C}}(U)$ determined by the object $h^*(V,*) \in {}^*\mathbf{Obj}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$ and the morphism

$$(h^*(V,*)) \otimes_{C^\infty_{\mathbb{C},U}} (h^*(V,*)) \cong h^*\big((V,*) \otimes_{C^\infty_{\mathbb{C},U'}} (V,*)\big) \xrightarrow{\;h^*\langle\cdot,\cdot\rangle\;} h^*(C^\infty_{\mathbb{C},U'}, *) \cong (C^\infty_{\mathbb{C},U}, *) \;, \tag{4.16}$$

where we have used the coherence isomorphisms of the involutive symmetric monoidal stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ from Corollary 3.11. The proof of the following statement is completely analogous to the one of Proposition 4.1.

Proposition 4.4. *The prestack* $\mathbf{IPVec}^\infty_{\mathbb{C}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ *defined above is a stack, i.e., it satisfies the descent condition from Definition 2.3.*

We shall now define a stack morphism

$$\mathfrak{CAR} : \mathbf{IPVec}^\infty_{\mathbb{C}} \longrightarrow {}^*\mathbf{Alg}^\infty_{\mathbb{C}} \tag{4.17}$$

that provides a smooth refinement of CAR quantization. In analogy to (4.8), we define the component functors $\mathfrak{CAR}_U : \mathbf{IPVec}^\infty_{\mathbb{C}}(U) \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$, for all $U \in \mathbf{Man}$, by the coequalizer

$$\mathfrak{CAR}_U\big(V,*,\langle\cdot,\cdot\rangle\big) := \mathrm{colim}\Big(F_U\big((V,*)\otimes_{C^\infty_{\mathbb{C},U}} (V,*)\big) \overset{s_1}{\underset{s_2}{\rightrightarrows}} F_U(V,*) \Big) \in {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U), \tag{4.18}$$

where the relations s_1, s_2 are defined in terms of their adjuncts under (4.7) by

$$\begin{array}{ccc} (V,*)\otimes_{C^\infty_{\mathbb{C},U}} (V,*) & \xrightarrow{\tilde{s}_1} & GF(V,*) \\ {\scriptstyle \text{unit } F\dashv G}\big\downarrow & \nearrow{\scriptstyle \mu+\mu^{\mathrm{op}}} & \\ GF(V,*)\otimes_{C^\infty_{\mathbb{C},U}} GF(V,*) & & \end{array} \tag{4.19a}$$

and

$$\begin{array}{ccc} (V,*)\otimes_{C^\infty_{\mathbb{C},U}} (V,*) & \xrightarrow{\tilde{s}_2} & GF(V,*) \\ {\scriptstyle \langle\cdot,\cdot\rangle}\big\downarrow & \nearrow{\scriptstyle \eta} & \\ (C^\infty_{\mathbb{C},U},*) & & \end{array} \tag{4.19b}$$

where we suppressed for notational convenience the subscripts $_U$ on the functors. The coherence isomorphisms

$$\begin{array}{ccc} \mathbf{IPVec}^\infty_{\mathbb{C}}(U') & \xrightarrow{\mathfrak{CAR}_{U'}} & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \\ {\scriptstyle h^*}\big\downarrow & \swarrow\!\!\swarrow{\scriptstyle \mathfrak{CAR}_h} & \big\downarrow{\scriptstyle h^*} \\ \mathbf{IPVec}^\infty_{\mathbb{C}}(U) & \xrightarrow[\mathfrak{CAR}_U]{} & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \end{array} \tag{4.20}$$

associated with $\mathbf{Man}$-morphisms $h : U \to U'$ are built similarly to those in Sect. 4.1. Summing up, we have

Proposition 4.5. *The construction above defines a stack morphism* $\mathfrak{CAR} : \mathbf{IPVec}^\infty_{\mathbb{C}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$.

5. Illustration Through Free Theories

We shall illustrate our formalism by constructing concrete examples of smooth 1-dimensional AQFTs. The models we study are smooth refinements of the Bosonic and Fermionic free field theories discussed in, e.g., [1,2]. Similarly to the ordinary case, our Bosonic models will be described by stack morphisms

$$\begin{array}{ccc} \mathbf{Loc}_1^\infty & \xrightarrow{\quad \mathfrak{A}^{\mathsf{b}} \quad} & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \\ & {\scriptstyle \mathfrak{L}^{\mathsf{b}}} \searrow \quad \nearrow {\scriptstyle \mathfrak{CCR}} & \\ & \mathbf{PoVec}_{\mathbb{R}}^\infty & \end{array} \tag{5.1}$$

obtained as the composition of a stack morphism $\mathfrak{L}^{\mathsf{b}}$ assigning the linear observables with their Poisson structure and the $\mathfrak{CCR}$-quantization stack morphism developed in Sect. 4.1. The Fermionic models will be described similarly by stack morphisms

$$\begin{array}{ccc} \mathbf{Loc}_1^\infty & \xrightarrow{\quad \mathfrak{A}^{\mathsf{f}} \quad} & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \\ & {\scriptstyle \mathfrak{L}^{\mathsf{f}}} \searrow \quad \nearrow {\scriptstyle \mathfrak{CAR}} & \\ & \mathbf{IPVec}_{\mathbb{C}}^\infty & \end{array} \tag{5.2}$$

factorizing through the $\mathfrak{CAR}$-quantization stack morphism from Sect. 4.2.

Inspired by the standard constructions in ordinary AQFT [1,2], we shall obtain examples of the stack morphisms $\mathfrak{L}^{\mathsf{b}/\mathsf{f}}$ assigning linear observables by using a suitable smooth refinement of the concept of retarded/advanced Green operators $G^{\pm}$ to be developed in Sect. 5.1. Recall that the role of such Green operators is to determine the Poisson structure τ of a Bosonic theory and the bilinear map $\langle \cdot, \cdot \rangle$ of a Fermionic theory. In Sect. 5.2 we will spell out this construction for the simplest case of a 1-dimensional massive scalar field, which is equivalent to the harmonic oscillator.[3] We shall even construct a smooth $\tilde{U}$-family of smooth AQFTs (in the sense of (3.17)) that describes a family of 1-dimensional massive scalar fields with a smoothly varying mass parameter $m \in C^\infty(\tilde{U}, \mathbb{R}^{>0})$. In Sect. 5.3 we construct the 1-dimensional massless Dirac field as a smooth AQFT and show that its global $U(1)$-symmetry is realized in terms of smooth automorphisms in the sense of (3.19).

[3] In the context of quantum mechanics, the harmonic oscillator is usually described via the equal time canonical commutation relations $[\hat{x}, \hat{p}] = \mathrm{i}\, \hat{\mathbb{1}}$ and the Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{m^2}{2}\hat{x}^2$ with frequency/mass parameter $m > 0$. In the context of AQFT, one works instead with the covariant commutation relations $[\hat{\Phi}(\varphi), \hat{\Phi}(\varphi')] = \mathrm{i} \int_{\mathbb{R}} \varphi\, G(\varphi')\, \mathrm{d}t\, \hat{\mathbb{1}}$, where $\hat{\Phi}(\varphi^{(\prime)})$ are the field operators smeared by compactly supported functions $\varphi, \varphi' \in C_{\mathrm{c}}^\infty(\mathbb{R})$ on the time line $\mathbb{R}$ and $G = G^+ - G^-$ is the retarded-minus-advanced Green operator for the equation of motion operator $P = \partial_t^2 + m^2$. Due to the well-posed initial value problem, one can show that both approaches are equivalent, see, e.g., [3, Remark 3.3.4].

5.1. Green Operators, Solutions and Initial Data

Let us consider a manifold $U \in \mathbf{Man}$ and a smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$. We introduce the functor

$$C_\pi^\infty : \mathbf{Open}(U)^{\mathrm{op}} \longrightarrow \mathbf{Set} \tag{5.3}$$

that assigns to each open subset $U' \subseteq U$ the set $C_\pi^\infty(U') := C_\mathbb{R}^\infty(M|_{U'})$ of real valued smooth functions on the restricted total space $M|_{U'} = \pi^{-1}(U') \subseteq M$ and to each open subset inclusion $U' \subseteq U'' \subseteq U$ the restriction map $C_\mathbb{R}^\infty(M|_{U''}) \to C_\mathbb{R}^\infty(M|_{U'})$. Together with the $C_{\mathbb{R},U}^\infty$-module structure induced by pullback of functions along the projection map $\pi : M \to U$, this defines an object $C_\pi^\infty \in \mathbf{Sh}_{C_\mathbb{R}^\infty}(U)$ that we shall interpret as the field configuration space of a real scalar field on the smooth family of spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$. More generally, the configuration space of vector-valued fields is given by

$$C_\pi^\infty \otimes \mathbb{K}^n \in \mathbf{Sh}_{C_\mathbb{K}^\infty}(U), \tag{5.4}$$

where $n \in \mathbb{Z}_{\geq 1}$ is the number of field components, and we take $\mathbb{K} = \mathbb{R}$ for real fields and $\mathbb{K} = \mathbb{C}$ for complex fields. As equation of motion we will consider a $\mathbf{Sh}_{C_\mathbb{K}^\infty}(U)$-morphism $P : C_\pi^\infty \otimes \mathbb{K}^n \to C_\pi^\infty \otimes \mathbb{K}^n$ given by a vertical differential operator on $\pi : M \to U$, i.e., a differential operator on M that differentiates only along the fibers of $\pi : M \to U$. See our Examples 5.9 and 5.11.

In order to define a concept of Green operators for such P, we introduce certain subsheaves of the sheaf of functions C_π^∞ on $\pi : M \to U$ that describe functions with restrictions on their vertical support. In the following definition, we shall use that the fiber bundle $\pi : M \to U$ underlying any object $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ admits sections because the fibers are open intervals, see, e.g., [29, Sections 12.2 and 6.7]. Furthermore, given any subset $S \subseteq M$ of the total space, we denote by $J_\mathrm{v}^\pm(S) \subseteq M$ the *vertical future/past* of S, i.e., the subset of all points that can be reached from S by future/past directed vertical curves with respect to the orientation induced by $E \in \Omega_\mathrm{v}^1(M)$.

Definition 5.1. Let $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ and $U' \subseteq U$ an open subset. We say that a function $\varphi \in C_\pi^\infty(U') = C_\mathbb{R}^\infty(M|_{U'})$ is *vertically past/future compactly supported* if there exists a section $\sigma : U' \to M|_{U'}$ such that $\mathrm{supp}(\varphi) \subseteq J_\mathrm{v}^\pm(\sigma(U'))$. We say that $\varphi \in C_\pi^\infty(U')$ is *vertically compactly supported* if it is both vertically past and future compactly supported, i.e., there exist two sections $\sigma_1, \sigma_2 : U' \to M|_{U'}$ such that $\mathrm{supp}(\varphi) \subseteq J_\mathrm{v}^+(\sigma_1(U')) \cap J_\mathrm{v}^-(\sigma_2(U'))$.

Remark 5.2. Note that our definition of vertically compactly supported functions uses manifestly the fact that we consider smooth families of 1-*dimensional* spacetimes. In this 1-dimensional case, we have bundles $\pi : M \to U$ whose fibers are intervals; hence, it makes sense to define vertical compactness through vertical boundedness from above and below. A dimension-independent definition for $\varphi \in C_\pi^\infty(U')$ to be vertically compactly supported is given by the condition that $\mathrm{supp}(\varphi) \cap \pi^{-1}(K)$ is compact, for all $K \subseteq U'$ compact. Upon sheafification (see Definition 5.3), this coincides in the 1-dimensional case with our more practical Definition 5.1.

From this definition we obtain sub-*pre*sheaves $\widetilde{C}^{\infty}_{\pi\,\mathrm{vc}}$, $\widetilde{C}^{\infty}_{\pi\,\mathrm{vpc}}$ and $\widetilde{C}^{\infty}_{\pi\,\mathrm{vfc}}$ of C^{∞}_{π} that assign vertically compactly supported, vertically past compactly supported and vertically future compactly supported functions. Note that these presheaves are separated, but they do not satisfy the descent condition for sheaves and hence have to be sheafified.

Definition 5.3. We denote by $C^{\infty}_{\pi\,\mathrm{vc}}, C^{\infty}_{\pi\,\mathrm{vpc}}, C^{\infty}_{\pi\,\mathrm{vfc}} \in \mathbf{Sh}_{C^{\infty}_{\mathbb{R}}}(U)$ the sheafifications of the presheaves $\widetilde{C}^{\infty}_{\pi\,\mathrm{vc}}$ of vertically compactly supported functions, $\widetilde{C}^{\infty}_{\pi\,\mathrm{vpc}}$ of vertically past compactly supported functions and $\widetilde{C}^{\infty}_{\pi\,\mathrm{vfc}}$ of vertically future compactly supported functions.

Remark 5.4. These sheaves admit the following explicit description as subsheaves of C^{∞}_{π}. To each open subset $U' \subseteq U$, the sheaf $C^{\infty}_{\pi\,\mathrm{v(p/f)c}}$ assigns the subset $C^{\infty}_{\pi\,\mathrm{v(p/f)c}}(U') \subseteq C^{\infty}_{\pi}(U')$ consisting of all functions $\varphi \in C^{\infty}_{\pi}(U')$ that satisfy the following local support condition: For every point $x \in U'$, there exists an open neighborhood $U_x \subseteq U'$ of x such that the restriction $\varphi|_{U_x} \in C^{\infty}_{\pi}(U_x)$ is vertically (past/future) compactly supported in the sense of Definition 5.1.

With these preparations we can now introduce a concept of Green operators.

Definition 5.5. Let $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ be a $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism that is determined from a vertical differential operator on $\pi : M \to U$, i.e., a differential operator on M that differentiates only along the fibers of $\pi : M \to U$. A *retarded/advanced Green operator* for P is a $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism $G^{\pm} : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ that satisfies the following properties:

(i) $G^{\pm}$ is the inverse of the restriction $P : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ of P to the subsheaves of vertically past/future compactly supported functions.

(ii) For each open subset $U' \subseteq U$ and $\varphi \in C^{\infty}_{\pi\,\mathrm{vpc/vfc}}(U') \otimes \mathbb{K}^n$, we have $\mathrm{supp}(G^{\pm}\varphi) \subseteq J^{\pm}_{\mathrm{v}}(\mathrm{supp}(\varphi))$.

We refer to the $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism $G := G^{+} - G^{-} : C^{\infty}_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ as the *causal propagator*.

Remark 5.6. Observe that, as a consequence of item (i), retarded and advanced Green operators are unique, provided they exist. Their existence is instead a condition on P, namely the restrictions $P : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ must be invertible and their inverses must fulfill also item (ii). Examples 5.9, 5.10 and 5.11 present vertical differential operators that fulfill these conditions.

The usual exact sequence for P and G, see, e.g., [1], generalizes to our context.

Proposition 5.7. *Let $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ be a $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism that is determined from a vertical differential operator on $\pi : M \to U$ and*

$G^{\pm} : C^\infty_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^\infty_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ retarded/advanced Green operators for P. Then the associated sequence

$$0 \longrightarrow C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \xrightarrow{P} C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \xrightarrow{G} C^\infty_{\pi} \otimes \mathbb{K}^n \xrightarrow{P} C^\infty_{\pi} \otimes \mathbb{K}^n \longrightarrow 0 \quad (5.5)$$

in $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ is exact. Even stronger, the corresponding sequence of presheaves is exact, i.e., for each open subset $U' \subseteq U$, the sequence

$$0 \longrightarrow C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n \xrightarrow{P} C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n \xrightarrow{G} C^\infty_{\pi}(U') \otimes \mathbb{K}^n \xrightarrow{P} C^\infty_{\pi}(U') \otimes \mathbb{K}^n \longrightarrow 0 \quad (5.6)$$

of $C^\infty_{\mathbb{K}}(U')$-modules is exact.

Proof. We prove the second (stronger) statement, which implies the first. Let $U' \subseteq U$ be any open subset. To prove exactness at the first node, consider any $\varphi \in C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ such that $P\varphi = 0$ and note that $0 = G^{\pm} P\varphi = \varphi$ by Definition 5.5 (i). For the second node, let $\varphi \in C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ be such that $G\varphi = 0$. Then $G^+\varphi = G^-\varphi =: \rho \in C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ because of the support properties of Green operators and the definition of vertically compact support. Hence, $P\rho = PG^{\pm}\varphi = \varphi$ by Definition 5.5 (i).

For the third node, let $\Phi \in C^\infty_{\pi}(U') \otimes \mathbb{K}^n$ be such that $P\Phi = 0$. Choosing two nonintersecting sections $\sigma_{\pm} : U' \to M|_{U'}$ such that σ_+ lies in the vertical future of σ_-, we obtain an open cover $\{M|_{U'} \backslash J^-_{\mathrm{v}}(\sigma_-(U')),\, M|_{U'} \backslash J^+_{\mathrm{v}}(\sigma_+(U'))\}$ of $M|_{U'}$. Choosing a partition of unity subordinate to this cover, we can decompose $\Phi = \Phi_+ + \Phi_-$ with $\Phi_{\pm} \in C^\infty_{\pi\,\mathrm{vpc/vfc}}(U') \otimes \mathbb{K}^n$. Then $\rho := P\Phi_+ = -P\Phi_- \in C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ is vertically compactly supported and $G\rho = G^+\rho - G^-\rho = G^+P\Phi_+ + G^-P\Phi_- = \Phi_+ + \Phi_- = \Phi$ by Definition 5.5 (i).

For the last node, take any $\Phi \in C^\infty_{\pi}(U') \otimes \mathbb{K}^n$ and decompose as before $\Phi = \Phi_+ + \Phi_-$ with $\Phi_{\pm} \in C^\infty_{\pi\,\mathrm{vpc/vfc}}(U') \otimes \mathbb{K}^n$. Defining $\rho := G^+\Phi_+ + G^-\Phi_-$, we obtain $P\rho = PG^+\Phi_+ + PG^-\Phi_- = \Phi_+ + \Phi_- = \Phi$ by Definition 5.5 (i). □

Remark 5.8. As a direct consequence of this proposition, we obtain that the cokernel sheaf

$$\tfrac{C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n)} := \mathrm{coker}\big(P : C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \to C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n\big) \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U) \quad (5.7a)$$

may be computed as a presheaf quotient, i.e.,

$$\tfrac{C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n)}(U') = C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n \big/ P(C^\infty_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n), \quad (5.7b)$$

for every open subset $U' \subseteq U$. Furthermore, this sheaf is isomorphic via the causal propagator

$$G : \tfrac{C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n)} \xrightarrow{\cong} \mathsf{Sol}_\pi \quad (5.8)$$

to the solution sheaf $\mathsf{Sol}_\pi := \ker(P : C^\infty_{\pi} \otimes \mathbb{K}^n \to C^\infty_{\pi} \otimes \mathbb{K}^n) \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$.

Let us now illustrate these concepts by examples.

Example 5.9. Consider any $U \in \mathbf{Man}$ and any smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$. As equation of motion we take the vertical differential operator

$$P_{(\pi,E)} := *_\mathrm{v}\, \mathrm{d}_\mathrm{v} *_\mathrm{v} \mathrm{d}_\mathrm{v} + m^2 \,:\, C^\infty_\pi \longrightarrow C^\infty_\pi, \tag{5.9}$$

where $m \in (0,\infty)$ is a fixed parameter and $*_\mathrm{v}\, \mathrm{d}_\mathrm{v} *_\mathrm{v} \mathrm{d}_\mathrm{v}$ is the vertical Laplacian, which is obtained from the vertical de Rham differential d_v on $\pi : M \to U$ and the vertical Hodge operator $*_\mathrm{v}$ induced by $E \in \Omega^1_\mathrm{v}(M)$. This differential operator describes a smooth U-family of 1-dimensional scalar fields (or equivalently harmonic oscillators) with a fixed mass/frequency parameter m on time intervals whose geometry (i.e., length) depends on the point $x \in U$. To prove that (5.9) admits a retarded and an advanced Green operator, it is sufficient to prove existence of local retarded and advanced Green operators $G^\pm_\alpha : C^\infty_{\pi\,\mathrm{vpc/vfc}}|_{U_\alpha} \to C^\infty_{\pi\,\mathrm{vpc/vfc}}|_{U_\alpha}$ for an arbitrary choice of open cover $\{U_\alpha \subseteq U\}$. This is because uniqueness of retarded/advanced Green operators entails that the family $\{G^\pm_\alpha\}$ satisfies the relevant compatibility conditions on all overlaps $U_{\alpha\beta}$ and hence, recalling from Proposition 3.8 that $\mathbf{Sh}_{C^\infty_\mathbb{R}}$ is a stack, it defines a global retarded/advanced Green operator.

To prove local existence, consider any open cover $\{U_\alpha \subseteq U\}$ in which the restricted bundles $M|_{U_\alpha} \to U_\alpha$ admit a trivialization $M|_{U_\alpha} \cong \mathbb{R} \times U_\alpha$. In this trivialization, we have that $E|_{U_\alpha} \cong \rho\, \mathrm{d}t$ for a positive function $\rho \in C^\infty(\mathbb{R} \times U_\alpha, \mathbb{R}^{>0})$, where $t \in \mathbb{R}$ is a time coordinate on $\mathbb{R}$, and the equation of motion operator reads as $P_\alpha = \rho^{-1}\, \partial_t\, \rho^{-1} \partial_t + m^2$. We can simplify this differential operator even further by introducing a new ($x \in U_\alpha$ dependent) time coordinate $T(t,x)$ such that $\mathrm{d}_\mathrm{v} T = \rho\, \mathrm{d}t$. Note that in these coordinates the fiber over $x \in U_\alpha$ is the interval $(T(-\infty,x), T(\infty,x))$, i.e., the geometry/length of the interval may depend on x. The equation of motion operator then reads as $P_\alpha = \partial_T^2 + m^2$, which admits the retarded/advanced Green operator

$$(G^\pm_\alpha \varphi)(T,x) = \int_{T(\mp\infty,x)}^{T} m^{-1} \sin\big(m\,(T-S)\big)\, \varphi(S,x)\, \mathrm{d}S. \tag{5.10}$$

Because $\varphi \in C^\infty_{\pi\,\mathrm{vpc/vfc}}|_{U_\alpha}$ is vertically past/future compactly supported, this integral exists and it depends smoothly on both $T \in (T(-\infty,x), T(\infty,x))$ and $x \in U_\alpha$.

Example 5.10. In order to construct in Sect. 5.2 an example of a smooth $\tilde{U}$-family of smooth AQFTs, we generalize Example 5.9 to the case where the mass parameter is not a constant but rather a smooth positive function $m \in C^\infty(\tilde{U}, \mathbb{R}^{>0})$ on $\tilde{U} \in \mathbf{Man}$. Given any $U \in \mathbf{Man}$ and any smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$, we consider the object $(\pi \times \mathrm{id} : M \times \tilde{U} \to U \times \tilde{U}, \mathrm{pr}_M^*(E)) \in \mathbf{Loc}_1^\infty(U \times \tilde{U})$ and define on it the vertical differential operator

$$\tilde{P}_{(\pi,E)} := *_\mathrm{v}\, \mathrm{d}_\mathrm{v} *_\mathrm{v} \mathrm{d}_\mathrm{v} + \mathrm{pr}_{\tilde{U}}^*(m^2) \,:\, C^\infty_{\pi\times\mathrm{id}} \longrightarrow C^\infty_{\pi\times\mathrm{id}}, \tag{5.11}$$

where $\mathrm{pr}_M : M \times \tilde{U} \to M$ and $\mathrm{pr}_{\tilde{U}} : M \times \tilde{U} \to \tilde{U}$ denote the projection maps. Proceeding in complete analogy to Example 5.9, it is sufficient to choose an open cover $\{U_\alpha \subseteq U\}$ for which there exist bundle trivializations $(M \times \tilde{U})|_{U_\alpha \times \tilde{U}} \cong \mathbb{R} \times U_\alpha \times \tilde{U}$ and prove the existence of Green operators locally. Choosing again a convenient time coordinate T by solving $\mathrm{d_v} T = \rho\, \mathrm{d}t \cong \mathrm{pr}_M^*(E)|_{U_\alpha \times \tilde{U}}$, the local equation of motion operator reads as $\tilde{P}_\alpha = \partial_T^2 + m^2(\tilde{x})$, where we made the dependence on $\tilde{x} \in \tilde{U}$ explicit. This operator admits a retarded/advanced Green operator given by

$$(\tilde{G}_\alpha^\pm \varphi)(T, x, \tilde{x}) \;=\; \int_{T(\mp\infty,x)}^{T} m(\tilde{x})^{-1} \sin\big(m(\tilde{x})\,(T - S)\big)\, \varphi(S, x, \tilde{x})\, \mathrm{d}S, \quad (5.12)$$

for all $\varphi \in C^\infty_{\pi \times \mathrm{id}\,\mathrm{vpc/vfc}}|_{U_\alpha \times \tilde{U}}$.

Example 5.11. Consider any $U \in \mathbf{Man}$ and any smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$. The 1-dimensional massless Dirac field is described by the vertical differential operator

$$D_{(\pi,E)} \;:=\; \begin{pmatrix} \mathrm{i} *_{\mathrm{v}} \mathrm{d_v} & 0 \\ 0 & -\mathrm{i} *_{\mathrm{v}} \mathrm{d_v} \end{pmatrix} \,:\, C^\infty_\pi \otimes \mathbb{C}^2 \;\longrightarrow\; C^\infty_\pi \otimes \mathbb{C}^2, \quad (5.13)$$

where $\mathrm{i} \in \mathbb{C}$ is the imaginary unit and the elements $\binom{\Psi}{\overline{\Psi}} \in C^\infty_\pi \otimes \mathbb{C}^2$ should be interpreted as the Dirac field Ψ and its Dirac conjugate $\overline{\Psi}$. The existence of retarded/advanced Green operators can be proven as in the previous examples by a local argument. Indeed, restricting again to a trivializing cover $\{U_\alpha \subseteq U\}$ and introducing the local time coordinate T, the local Dirac operator reads as

$$D_\alpha \;=\; \begin{pmatrix} \mathrm{i}\, \partial_T & 0 \\ 0 & -\mathrm{i}\, \partial_T \end{pmatrix} \quad (5.14)$$

and its associated retarded/advanced Green operator is given by fiber integration

$$\Big(S_\alpha^\pm \Big(\frac{\psi}{\overline{\psi}}\Big)\Big)(T, x) \;=\; \int_{T(\mp\infty,x)}^{T} \begin{pmatrix} -\mathrm{i}\; \psi(S, x) \\ \mathrm{i}\; \overline{\psi}(S, x) \end{pmatrix} \, \mathrm{d}S, \quad (5.15)$$

where $T(\mp\infty, x)$ was defined in Example 5.9.

We conclude this subsection with a few remarks about smoothly parameterized initial value problems. Let us start with the case where the $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$-morphism $P : C^\infty_\pi \otimes \mathbb{K}^n \to C^\infty_\pi \otimes \mathbb{K}^n$ corresponds to a second-order vertical differential operator on $\pi : M \to U$, as it is the case in our Examples 5.9 and 5.10. Choosing any section $\sigma : U \to M$, we can define a $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$-morphism

$$\mathrm{data}_\sigma^{\mathrm{2nd}} \,:\, \mathsf{Sol}_\pi \;\longrightarrow\; \big(C^\infty_{\mathbb{R},U} \otimes \mathbb{K}^n\big)^{\oplus 2} \quad (5.16a)$$

that assigns, for each open subset $U' \subseteq U$, to a solution $\Phi \in \mathsf{Sol}_\pi(U') \subseteq C^\infty_\pi(U') \otimes \mathbb{K}^n = C^\infty_\mathbb{R}(M|_{U'}) \otimes \mathbb{K}^n$ (see Remark 5.8) its initial data

$$\mathrm{data}_\sigma^{\mathrm{2nd}}(\Phi) \;:=\; \big(\sigma^*(\Phi), \sigma^*(*_\mathrm{v} \mathrm{d_v} \Phi)\big) \;\in\; \big(C^\infty_\mathbb{R}(U') \otimes \mathbb{K}^n\big)^{\oplus 2} \quad (5.16b)$$

on $\sigma(U') \subseteq M|_{U'}$. We say that P has a well-posed initial value problem if (5.16) is an isomorphism in $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$. Note that if P has retarded/advanced

Green operators and a well-posed initial value problem, it follows by using also Remark 5.8 that

$$\frac{C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n)} \xrightarrow{\;G\;} \mathsf{Sol}_\pi \xrightarrow{\;\mathrm{data}^{2\mathrm{nd}}_\sigma\;} \big(C^\infty_{\mathbb{R},U}\otimes\mathbb{K}^n\big)^{\oplus 2} \tag{5.17}$$

is an isomorphism in $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ and hence that $C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n/P(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n)\in\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ is a free $C^\infty_{\mathbb{K},U}$-module of rank $2n$. This observation will be useful in Sect. 5.2.

Example 5.12. The equation of motion operators in Examples 5.9 and 5.10 have a well-posed initial value problem. Let us show this for the more general operator in (5.11) of the latter example, which reduces for $\tilde{U}=\{*\}$ a point to the operator in (5.9) of the former example. Using again that $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$ is a stack, it is sufficient to prove the isomorphism property of the sheaf morphism in (5.16) in each patch U_α of an arbitrary open cover $\{U_\alpha\subseteq U\}$. Using as in Example 5.10 a trivializing cover and suitable time coordinates T, we obtain the local equation of motion operator $\tilde{P}_\alpha=\partial_T^2+m^2(\tilde{x})$. The inverse of the restriction of the initial data map $\mathrm{data}^{2\mathrm{nd}}_\sigma$ to $U_\alpha\subseteq U$ is then given by

$$\begin{aligned}\mathrm{solve}_\sigma\big(\Phi_0,\Phi_1\big)(T,x,\tilde{x}) := \Phi_0(x,\tilde{x})\,\cos\big(m(\tilde{x})\,(T-T_\sigma(x))\big)\\ +\Phi_1(x,\tilde{x})\,m(\tilde{x})^{-1}\,\sin\big(m(\tilde{x})\,(T-T_\sigma(x))\big),\end{aligned} \tag{5.18}$$

for all $(\Phi_0,\Phi_1)\in(C^\infty_{\mathbb{R},U_\alpha\times\tilde{U}})^{\oplus 2}$, where the initial time $T_\sigma(x)\in(T(-\infty,x),T(\infty,x))$ is determined from the local coordinate expression $\sigma(x)=(T_\sigma(x),x)$, for all $x\in U_\alpha$, of the section σ.

Remark 5.13. The case of a first-order vertical differential operator $P: C^\infty_\pi\otimes\mathbb{K}^n\to C^\infty_\pi\otimes\mathbb{K}^n$ on $\pi: M\to U$ works similarly. The analog of (5.16) is given by the $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$-morphism

$$\mathrm{data}^{1\mathrm{st}}_\sigma\;:\;\mathsf{Sol}_\pi\;\longrightarrow\;C^\infty_{\mathbb{R},U}\otimes\mathbb{K}^n \tag{5.19a}$$

that assigns, for each open subset $U'\subseteq U$, to a solution $\Phi\in\mathsf{Sol}_\pi(U')$ its initial data

$$\mathrm{data}^{1\mathrm{st}}_\sigma(\Phi)\;:=\;\sigma^*(\Phi)\;\in\;C^\infty_{\mathbb{R}}(U')\otimes\mathbb{K}^n \tag{5.19b}$$

on $\sigma(U')\subseteq M|_{U'}$. We again say that P has a well-posed initial value problem if (5.19) is an isomorphism. It is easy to check that the Dirac operator from Example 5.11 has a well-posed initial value problem in this sense.

5.2. 1-Dimensional Scalar Field

The aim of this subsection is to construct an explicit smooth $\tilde{U}$-family of smooth AQFTs that can be interpreted as a smooth refinement of the 1-dimensional scalar field with a smoothly varying mass parameter $m\in C^\infty(\tilde{U},\mathbb{R}^{>0})$. Our construction will be based on Example 5.10 and we will carry out (smooth generalizations of) the usual steps in the construction of Bosonic free field theories, see, e.g., [1,2]. Since we are interested in smooth $\tilde{U}$-families (see (3.17)), we have to define instead of (5.1) a stack morphism

$$\begin{array}{ccc} \mathbf{Loc}_1^\infty & \xrightarrow{\mathfrak{B}} & \mathsf{Map}\big(\underline{\tilde{U}}, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty\big) \\ & {\scriptstyle \mathfrak{W}}\searrow \quad \nearrow{\scriptstyle \mathsf{Map}(\underline{\tilde{U}},\mathfrak{CCR})} & \\ & \mathsf{Map}\big(\underline{\tilde{U}}, \mathbf{PoVec}_{\mathbb{R}}^\infty\big) & \end{array} \tag{5.20}$$

which for the case of $\tilde{U} = \{*\}$ a point reduces to (5.1). For our example of interest, the stack morphism $\mathfrak{W} : \mathbf{Loc}_1^\infty \to \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}_{\mathbb{R}}^\infty)$ is given by the following data: For each manifold $U \in \mathbf{Man}$, we define the functor

$$\mathfrak{W}_U \;:\; \mathbf{Loc}_1^\infty(U) \;\longrightarrow\; \mathbf{PoVec}_{\mathbb{R}}^\infty(U \times \tilde{U}) \tag{5.21}$$

that assigns to $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ the object

$$\mathfrak{W}_U\big(\pi : M \to U, E\big) \;:=\; \Big(\tfrac{C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{(\pi,E)}C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}}}, \tau_{(\pi\times\mathrm{id},\mathrm{pr}_M^*(E))}\Big) \;\in\; \mathbf{PoVec}_{\mathbb{R}}^\infty(U \times \tilde{U}), \tag{5.22a}$$

where $\tilde{P}_{(\pi,E)}$ is the equation of motion operator in (5.11). The Poisson structure reads as

$$\tau_{(\pi\times\mathrm{id},\mathrm{pr}_M^*(E))} \;=\; \langle\cdot, \tilde{G}_{(\pi,E)}(\cdot)\rangle_{(\pi\times\mathrm{id},\mathrm{pr}_M^*(E))}, \tag{5.22b}$$

where $\tilde{G}_{(\pi,E)}$ is the causal propagator for $\tilde{P}_{(\pi,E)}$, whose existence was established in Example 5.10, and

$$\begin{aligned} &\langle\cdot,\cdot\rangle_{(\pi\times\mathrm{id},\mathrm{pr}_M^*(E))} \;: \\ &C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \otimes_{C^\infty_{\mathbb{R},U\times\tilde{U}}} C^\infty_{\pi\times\mathrm{id}} \xrightarrow{\;\mu\;} C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \xrightarrow{\;\int_{\pi\times\mathrm{id}}(-)\mathrm{pr}_M^*(E)\;} C^\infty_{\mathbb{R},U\times\tilde{U}}\;. \end{aligned} \tag{5.22c}$$

is the $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U\times\tilde{U})$-morphism obtained by composing the multiplication map μ of functions and fiber integration on $(\pi \times \mathrm{id} : M \times \tilde{U} \to U \times \tilde{U}, \mathrm{pr}_M^*(E)) \in \mathbf{Loc}^\infty(U \times \tilde{U})$. Less formally, we can write for (5.22b) also the more familiar looking expression

$$\tau_{(\pi\times\mathrm{id},\mathrm{pr}_M^*(E))}\big(\varphi, \varphi'\big) \;=\; \int_\pi \varphi\, \tilde{G}_{(\pi,E)}(\varphi')\, \mathrm{pr}_M^*(E). \tag{5.23}$$

Note that the Poisson structure is well-defined on the quotient in (5.22a) because $\tilde{P}_{(\pi,E)}$ is formally self-adjoint with respect to the pairing in (5.22c) and $\tilde{G}_{(\pi,E)} \circ \tilde{P}_{(\pi,E)} = 0 = \tilde{P}_{(\pi,E)} \circ \tilde{G}_{(\pi,E)}$ due to the definition of Green operators, see Definition 5.5.

Given any morphism $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ in $\mathbf{Loc}_1^\infty(U)$, we can define a morphism $f \times \mathrm{id} : (\pi \times \mathrm{id} : M \times \tilde{U} \to U \times \tilde{U}, \mathrm{pr}_M^*(E)) \to (\pi' \times \mathrm{id} : M' \times \tilde{U} \to U \times \tilde{U}, \mathrm{pr}_{M'}^*(E'))$ in $\mathbf{Loc}_1^\infty(U \times \tilde{U})$. This yields a pushforward (i.e., extension by zero) $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U})$-morphism $(f \times \mathrm{id})_* : C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \to C^\infty_{\pi'\times\mathrm{id}\,\mathrm{vc}}$ of vertically compactly supported functions (recall from Definition 3.4 that $f \times \mathrm{id}$ is an open embedding of fiber bundles) that intertwines the equation of motion operators, i.e., $\tilde{P}_{(\pi',E')}\,(f\times \mathrm{id})_* = (f\times \mathrm{id})_*\,\tilde{P}_{(\pi,E)}$. We can then define the values of (5.21) on morphisms by

$$\mathfrak{W}_U(f) \,:=\, (f\times \mathrm{id})_* \,:\, \mathfrak{W}_U\big(\pi : M\to U, E\big) \;\longrightarrow\; \mathfrak{W}_U\big(\pi' : M'\to U, E'\big). \tag{5.24}$$

Note that the preservation of Poisson structures follows from the uniqueness of retarded/advanced Green operators.

To complete the definition of the stack morphism $\mathfrak{W}$, it remains to provide, for each morphism $h : U\to U'$ in **Man**, a natural isomorphism (see Definition 2.5)

$$\begin{array}{ccc} \mathbf{Loc}_1^\infty(U') & \xrightarrow{\;\mathfrak{W}_{U'}\;} & \mathbf{PoVec}_{\mathbb{R}}^\infty(U'\times\tilde{U}) \\ {\scriptstyle h^*}\downarrow & \overset{\mathfrak{W}_h}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle (h\times\mathrm{id})^*} \\ \mathbf{Loc}_1^\infty(U) & \xrightarrow[\;\mathfrak{W}_U\;]{} & \mathbf{PoVec}_{\mathbb{R}}^\infty(U\times\tilde{U}) \end{array} \tag{5.25a}$$

To define the component

$$\mathfrak{W}_h \;:\; (h\times \mathrm{id})^*\mathfrak{W}_{U'}\big(\pi : M\to U', E\big) \;\longrightarrow\; \mathfrak{W}_U\big(h^*(\pi : M\to U', E)\big) \tag{5.25b}$$

at $(\pi : M\to U', E)\in \mathbf{Loc}_1^\infty(U')$, we recall the pullback bundle construction in (3.3) and (3.4) and consider the $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U\times\tilde{U})$-morphism

$$(\overline{h}^M\times \mathrm{id})^* \;:\; (h\times\mathrm{id})^* C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \;\longrightarrow\; C^\infty_{\pi_h\times\mathrm{id}\,\mathrm{vc}} \tag{5.26}$$

that is defined through its adjunct under $(h\times\mathrm{id})^* : \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U\times\tilde{U}) \rightleftarrows \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U'\times\tilde{U})$ by the components (denoted with abuse of notation by the same symbol)

$$(\overline{h}^M\times \mathrm{id})^* \;:\; C^\infty_{\mathbb{R}}\big((M\times\tilde{U})|_{U''}\big) \;\longrightarrow\; C^\infty_{\mathbb{R}}\big((h^*M\times\tilde{U})|_{(h\times\mathrm{id})^{-1}(U'')}\big), \tag{5.27}$$

for all open subsets $U''\subseteq U'\times\tilde{U}$, which describe the pullback of functions along the map of total spaces. Due to the universal property of pullback bundles, one easily checks that each section $\sigma : U'\times\tilde{U}\to M\times\tilde{U}$ induces a section $\sigma_h : U\times\tilde{U}\to h^*M\times\tilde{U}$ of the pullback bundle that satisfies $\sigma\,(h\times\mathrm{id}) = (\overline{h}^M\times\mathrm{id})\,\sigma_h$; hence, the maps in (5.27) preserve vertically compact support. Due to naturality of the vertical differential operators $\tilde{P}$ in (5.11), we obtain the commutative diagram

$$\begin{array}{ccc} (h\times\mathrm{id})^* C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} & \xrightarrow{\;(\overline{h}^M\times\mathrm{id})^*\;} & C^\infty_{\pi_h\times\mathrm{id}\,\mathrm{vc}} \\ {\scriptstyle (h\times\mathrm{id})^*\tilde{P}_{(\pi,E)}}\downarrow & & \downarrow{\scriptstyle \tilde{P}_{h^*(\pi,E)}} \\ (h\times\mathrm{id})^* C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} & \xrightarrow[\;(\overline{h}^M\times\mathrm{id})^*\;]{} & C^\infty_{\pi_h\times\mathrm{id}\,\mathrm{vc}} \end{array} \tag{5.28}$$

in $\mathbf{Sh}_{C^\infty_\mathbb{R}}(U \times \tilde{U})$, which allows us to induce (5.26) to the quotients

$$(\overline{h}^M \times \mathrm{id})^* \,:\, (h \times \mathrm{id})^* \Big(\frac{C^\infty_{\pi \times \mathrm{id}\ \mathrm{vc}}}{\tilde{P}_{(\pi,E)} C^\infty_{\pi \times \mathrm{id}\ \mathrm{vc}}} \Big) \longrightarrow \frac{C^\infty_{\pi_h \times \mathrm{id}\ \mathrm{vc}}}{\tilde{P}_{h^*(\pi,E)} C^\infty_{\pi_h \times \mathrm{id}\ \mathrm{vc}}}. \tag{5.29}$$

Here we also used that $(h \times \mathrm{id})^*$ is a left adjoint functor; hence, it commutes with the colimit defining these quotients. From the explicit expression in (5.27) for (the adjunct of) this morphism and observing that a diagram similar to (5.28) involving retarded/advanced Green operators commutes due to their uniqueness, one checks that (5.29) preserves the relevant Poisson structures and thereby defines the desired $\mathbf{PoVec}^\infty_\mathbb{R}(U' \times \tilde{U})$-morphism $\mathfrak{W}_h := (\overline{h}^M \times \mathrm{id})^*$ in (5.25b).

It remains to confirm that (5.29) is an isomorphism in $\mathbf{PoVec}^\infty_\mathbb{R}(U' \times \tilde{U})$. Using the causal propagators in (5.8) and the initial data morphisms in (5.16) corresponding to any choice of section $\sigma : U' \times \tilde{U} \to M \times \tilde{U}$ and its induced section $\sigma_h : U \times \tilde{U} \to h^* M \times \tilde{U}$ of the pullback bundle, we obtain the commutative diagram

$$\begin{array}{ccc} (h \times \mathrm{id})^* \Big(\frac{C^\infty_{\pi \times \mathrm{id}\ \mathrm{vc}}}{\tilde{P}_{(\pi,E)} C^\infty_{\pi \times \mathrm{id}\ \mathrm{vc}}} \Big) & \xrightarrow{(\overline{h}^M \times \mathrm{id})^*} & \frac{C^\infty_{\pi_h \times \mathrm{id}\ \mathrm{vc}}}{\tilde{P}_{h^*(\pi,E)} C^\infty_{\pi_h \times \mathrm{id}\ \mathrm{vc}}} \\ \Big\downarrow {\scriptstyle (h\times\mathrm{id})^* \tilde{G}_{(\pi,E)}} & & \Big\downarrow {\scriptstyle \tilde{G}_{h^*(\pi,E)}} \\ (h \times \mathrm{id})^* \mathsf{Sol}_{\pi \times \mathrm{id}} & \xrightarrow{(\overline{h}^M \times \mathrm{id})^*} & \mathsf{Sol}_{\pi_h \times \mathrm{id}} \\ \Big\downarrow {\scriptstyle (h\times\mathrm{id})^* \mathrm{data}^{2\mathrm{nd}}_\sigma} & & \Big\downarrow {\scriptstyle \mathrm{data}^{2\mathrm{nd}}_{\sigma_h}} \\ (h \times \mathrm{id})^* \big(C^\infty_{\mathbb{R}, U' \times \tilde{U}} \big)^{\oplus 2} & \xrightarrow[\cong]{} & \big(C^\infty_{\mathbb{R}, U \times \tilde{U}} \big)^{\oplus 2} \end{array} \tag{5.30}$$

in $\mathbf{Sh}_{C^\infty_\mathbb{R}}(U \times \tilde{U})$, where the bottom horizontal isomorphism uses that $(h \times \mathrm{id})^*$ preserves coproducts (as it is a left adjoint functor) and the symmetric monoidal coherence isomorphism for the monoidal unit in (3.7). By Remark 5.8 and Examples 5.10 and 5.12, all vertical arrows in this diagram are isomorphisms; hence, the top horizontal arrow is an isomorphism too. This implies that $\mathfrak{W}_h$ is an isomorphism in $\mathbf{PoVec}^\infty_\mathbb{R}(U \times \tilde{U})$.

Summing up, the main result of this section is

Proposition 5.14. *The construction described above defines a stack morphism* $\mathfrak{W} : \mathbf{Loc}^\infty_1 \to \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R})$.

As a consequence, we obtain an explicit example of a smooth $\tilde{U}$-family of smooth 1-dimensional AQFTs $\mathfrak{B} := \mathsf{Map}(\underline{\tilde{U}}, \mathfrak{CCR}) \circ \mathfrak{W} : \mathbf{Loc}^\infty_1 \to \mathsf{Map}(\underline{\tilde{U}}, {}^*\mathbf{Alg}^\infty_\mathbb{C})$ describing a smooth refinement of the 1-dimensional scalar field with a smoothly varying mass parameter $m \in C^\infty(\tilde{U}, \mathbb{R}^{>0})$. In the special case where $\tilde{U} = \{*\}$ is a point, our construction describes a smooth refinement of the 1-dimensional scalar field with a fixed mass $m > 0$.

5.3. 1-Dimensional Dirac Field

The construction of Sect. 5.2 can be easily adapted to the case of the 1-dimensional Dirac field introduced in Example 5.11. We will spell out the relevant steps to construct the corresponding stack morphism $\mathfrak{L}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \to \mathbf{IPVec}_{\mathbb{C}}^\infty$ such that $\mathfrak{A}^{\mathsf{f}} := \mathfrak{CAR} \circ \mathfrak{L}^{\mathsf{f}}$ in (5.2) describes a smooth refinement of the massless 1-dimensional Dirac field. For this we will carry out (smooth generalizations of) the usual steps in the construction of Fermionic free field theories, see, e.g., [12]. After that we will show that the smooth automorphism group in (3.19) of this model includes the global $U(1)$-symmetry of the Dirac field.

The stack morphism $\mathfrak{L}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \to \mathbf{IPVec}_{\mathbb{C}}^\infty$ is given by the following data: For each manifold $U \in \mathbf{Man}$, we define the functor

$$\mathfrak{L}^{\mathsf{f}}_U \,:\, \mathbf{Loc}_1^\infty(U) \longrightarrow \mathbf{IPVec}_{\mathbb{C}}^\infty(U) \tag{5.31}$$

that assigns to each $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ the object

$$\mathfrak{L}^{\mathsf{f}}_U\big(\pi : M \to U, E\big) := \Big(\tfrac{C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{C}^2}{D_{(\pi,E)}(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{C}^2)}, *_{(\pi,E)}, \langle\cdot,\cdot\rangle_{(\pi,E)}\Big) \in \mathbf{IPVec}_{\mathbb{C}}^\infty(U), \tag{5.32a}$$

where $D_{(\pi,E)}$ is the Dirac operator from Example 5.11. The $*$-involution

$$*_{(\pi,E)}\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix} := \begin{pmatrix}\overline{\psi}^*\\ \psi^*\end{pmatrix} \tag{5.32b}$$

is given by swapping the components followed by complex conjugation, which descends to the quotient since $*_{(\pi,E)} \circ D_{(\pi,E)} = D_{(\pi,E)} \circ *_{(\pi,E)}$. The symmetric pairing

$$\Big\langle \begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix}, \begin{pmatrix}\psi'\\ \overline{\psi}'\end{pmatrix} \Big\rangle_{(\pi,E)} := \int_\pi \big(\psi\ \overline{\psi}\big)\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix} S_{(\pi,E)}\begin{pmatrix}\psi'\\ \overline{\psi}'\end{pmatrix} E \tag{5.32c}$$

is given by fiber integration, the causal propagator $S_{(\pi,E)}$ for $D_{(\pi,E)}$ and the displayed matrix multiplications. It is easy to check that $\langle\cdot,\cdot\rangle_{(\pi,E)}$ descends to the quotient in (5.32a) and that it satisfies the compatibility condition in (4.13) for $*$-involutions. The definition of the functor in (5.31) on morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ is as in (5.24) via pushforward of vertically compactly supported functions. The coherence isomorphisms for $\mathbf{Man}$-morphisms $h : U \to U'$ are constructed in complete analogy to (5.25).

Summing up, we obtain

Proposition 5.15. *The construction described above defines a stack morphism* $\mathfrak{L}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \to \mathbf{IPVec}_{\mathbb{C}}^\infty$.

As a consequence, we obtain another example of a smooth 1-dimensional AQFT $\mathfrak{A}^{\mathsf{f}} := \mathfrak{CAR} \circ \mathfrak{L}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ describing a smooth refinement of the 1-dimensional massless Dirac field.

To conclude this section, we will show that our construction can be refined to define a $U(1)$-equivariant smooth AQFT, showing that the global $U(1)$-symmetry of the Dirac field is smooth in our sense. Recalling from (3.24) their explicit description, we will define a $U(1)$-equivariant smooth AQFT

$$\begin{array}{ccc} \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} & \xrightarrow{\;\tilde{\mathfrak{A}}^{\mathsf{f}}\;} & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \\ & {\scriptstyle \tilde{\mathfrak{L}}^{\mathsf{f}}} \searrow \quad \nearrow {\scriptstyle \mathfrak{CAR}} & \\ & \mathbf{IPVec}_{\mathbb{C}}^\infty & \end{array} \tag{5.33}$$

by specifying a pseudo-natural transformation $\tilde{\mathfrak{L}}^{\mathsf{f}}$. For each manifold $U \in$ **Man**, we define the functor

$$\tilde{\mathfrak{L}}^{\mathsf{f}}_U \,:\, \mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U) \;\longrightarrow\; \mathbf{IPVec}_{\mathbb{C}}^\infty(U) \tag{5.34}$$

that acts on objects $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U)$ precisely as in (5.32), i.e.,

$$\begin{aligned} \tilde{\mathfrak{L}}^{\mathsf{f}}_U\big(\pi : M \to U, E\big) \;&:=\; \mathfrak{L}^{\mathsf{f}}_U\big(\pi : M \to U, E\big) \\ &= \Big(\tfrac{C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{C}^2}{D_{(\pi,E)}(C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{C}^2)}, *_{(\pi,E)}, \langle\cdot,\cdot\rangle_{(\pi,E)}\Big). \end{aligned} \tag{5.35}$$

(Note that the objects of $\mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U)$ are canonically identified with the objects of $\mathbf{Loc}_1^\infty(U)$ because $[\{*\}/U(1)]_{\mathrm{pre}}(U)$ has only a single object, see (3.23).) Things get more interesting at the level of morphisms, because the morphisms in $\mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U)$ are pairs (f, g) consisting of a $\mathbf{Loc}_1^\infty(U)$-morphism $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ and a $U(1)$-valued smooth function $g \in C^\infty(U, U(1))$. These morphisms are defined to act by a combination of the pushforward of vertically compactly supported functions and a complex phase rotation

$$\tilde{\mathfrak{L}}^{\mathsf{f}}_U(f, g) \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \;:=\; \begin{pmatrix} f_*\big(\pi^*(g)\,\psi\big) \\ f_*\big(\pi^*(g)^{-1}\,\overline{\psi}\big) \end{pmatrix} \;=\; \begin{pmatrix} \pi^*(g)\, f_*(\psi) \\ \pi^*(g)^{-1}\, f_*(\overline{\psi}) \end{pmatrix}, \tag{5.36}$$

where $\pi^*(g)$ denotes the pullback of $g \in C^\infty(U, U(1))$ along the projection map $\pi : M \to U$. (The second equality in (5.36) follows from the fact that f_* only acts along the fibers where $\pi^*(g)$ is constant.) These maps clearly preserve the quotient in (5.35), the $*$-involution in (5.32b) and the pairing in (5.32c); hence, they define $\mathbf{IPVec}_{\mathbb{C}}^\infty(U)$-morphisms. The coherence isomorphisms for **Man**-morphisms $h : U \to U'$ are constructed in complete analogy to our previous examples.

Summing up, we obtain

Proposition 5.16. *The construction described above defines a pseudo-natural transformation* $\tilde{\mathfrak{L}}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \to \mathbf{IPVec}_{\mathbb{C}}^\infty$.

As a consequence, we obtain an example of a $U(1)$-equivariant smooth 1-dimensional AQFT $\tilde{\mathfrak{A}}^{\mathsf{f}} := \mathfrak{CAR} \circ \tilde{\mathfrak{L}}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ describing a smooth refinement of the 1-dimensional massless Dirac field together with its global $U(1)$-symmetry.

6. Outlook: Toward Higher Dimensions and Gauge Theories

The aim of this section is to outline the way we believe the results of this paper could be generalized to higher-dimensional AQFTs and also to gauge theories. In particular, we shall explain the additional technical challenges and open questions that arise from such a generalization.

Let us first discuss possible generalizations of the stack $\mathbf{Loc}_1^\infty$ of 1-dimensional spacetimes from Sect. 3.1 to the case of higher dimensions $m \geq 2$. For $U \in \mathbf{Man}$ a manifold, we can define a smooth U-family of m-dimensional Lorentzian manifolds to be a tuple $(\pi : M \to U, g)$ consisting of a (locally trivializable) fiber bundle $\pi : M \to U$ with typical fiber an m-manifold N and a metric g of signature $(+-\cdots-)$ on the vertical tangent bundle of $\pi : M \to U$. For illustrative purposes, we note that in a local trivialization $M|_{U'} \cong N \times U'$ and in local coordinates y^μ on the fiber N, the vertical metric takes the form $g|_{U'} \cong g_{\mu\nu}(y,x)\, \mathrm{d}y^\mu \otimes \mathrm{d}y^\nu$, i.e., it has only vertical components along N that however are allowed to depend smoothly on $x \in U' \subseteq U$. There are obvious notions of vertical orientation $\mathfrak{o}$ and vertical time-orientation $\mathfrak{t}$; hence, we can introduce a concept of smooth U-families of m-dimensional oriented and time-oriented Lorentzian manifolds $(\pi : M \to U, g, \mathfrak{o}, \mathfrak{t})$. What is less obvious is the correct generalization of the important concept of global hyperbolicity to this smoothly parameterized context. One could either impose the point-wise condition that each fiber $(M|_x, g|_x)$ is globally hyperbolic in the usual sense or seek for a condition that is more uniform on U. The role of this condition should be to ensure that vertical normally hyperbolic operators, such as the vertical Klein–Gordon operator

$$P \,:=\, *_{\mathrm{v}}\mathrm{d}_{\mathrm{v}}*_{\mathrm{v}}\mathrm{d}_{\mathrm{v}} + m^2 \,:\, C^\infty_\pi \;\longrightarrow\; C^\infty_\pi, \tag{6.1}$$

admit retarded and advanced Green operators and a well-posed initial value problem, both described in terms of morphisms of sheaves of $C^\infty_{\mathbb{R},U}$-modules. This can be interpreted saying that both the retarded/advanced Green operators and the initial value problem are smoothly parameterized. Again for illustrative purposes, we note that in a local trivialization $M|_{U'} \cong N \times U'$ and in local coordinates y^μ on the fiber N, the vertical differential operator in (6.1) takes the form

$$P|_{U'} \,\cong\, g^{\mu\nu}(y,x)\frac{\partial^2}{\partial y^\mu \partial y^\nu} + B^\mu(y,x)\frac{\partial}{\partial y^\mu} + A(y,x), \tag{6.2}$$

i.e., there are no derivatives along $x \in U'$ but the coefficients may be x-dependent. Summing up, we record

Open Problem 6.1. Find a suitable generalization of global hyperbolicity to smooth U-families of m-dimensional oriented and time-oriented Lorentzian manifolds $(\pi : M \to U, g, \mathfrak{o}, \mathfrak{t})$ such that vertical normally hyperbolic operators admit smoothly parameterized retarded and advanced Green operators and a well-posed smoothly parameterized initial value problem.

Successfully solving this problem will lead to a sensible definition of a stack $\mathbf{Loc}_m^\infty$ of m-dimensional globally hyperbolic spacetimes. One can then attempt to construct examples of smooth m-dimensional AQFTs in terms of stack morphisms $\mathfrak{A} : \mathbf{Loc}_m^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ by using the same strategy as in (5.1). We note that most of our constructions in Sect. 5 only rely on the existence (and uniqueness) of Green operators; hence, they would generalize directly to the higher-dimensional case, provided that Open Problem 6.1 is solved. There is however one exception: In the higher-dimensional case, the space of initial data is infinite-dimensional; hence, we cannot argue as in (5.30) to conclude that the assignment of linear observables $\mathfrak{L} : \mathbf{Loc}_m^\infty \to \mathbf{PoVec}_{\mathbb{R}}^\infty$ is a stack morphism. More specifically, this could lead to the problem that the coherence maps $\mathfrak{L}_h$ are only natural transformations, but not natural isomorphisms, which means that $\mathfrak{L}$ is only a *lax* stack morphism. At the moment we do not know whether it will be more convenient to enlarge the 2-category $\mathbf{St}$ of stacks to include also lax morphisms or to replace the stack $\mathbf{Sh}_{C_{\mathbb{R}}^\infty}$ of sheaves of $C_{\mathbb{R}}^\infty$-modules by a stack describing sheaves of topological (or bornological) modules in order to obtain a better control on these infinite-dimensional aspects. Summing up, we record

Open Problem 6.2. Find a suitable framework such that the assignment $\mathfrak{L} : \mathbf{Loc}_m^\infty \to \mathbf{PoVec}_{\mathbb{R}}^\infty$ of linear observables for a smooth m-dimensional free AQFT is a morphism between stacks. Possible options could be enlarging the 2-category $\mathbf{St}$ of stacks to allow for lax morphisms or replacing the stack $\mathbf{Sh}_{C_{\mathbb{R}}^\infty}$ of sheaves of $C_{\mathbb{R}}^\infty$-modules by a stack describing sheaves of topological (or bornological) modules.

As already emphasized at the beginning of Sect. 3, higher-dimensional AQFTs are sensitive to the phenomenon of Einstein causality, which means that they are not simply functors but rather algebras over a suitable colored operad [6,7]. Encoding this aspect in our smooth setting leads to the following

Open Problem 6.3. Develop a theory of stacks of colored operads in order to define the stack $\mathbf{AQFT}_m^\infty$ of smooth m-dimensional AQFTs in terms of a suitable mapping stack between stacks of colored operads.

To conclude, we would like to comment briefly on a potential generalization of our framework to gauge theories. The latter are most appropriately described by the BV-formalism, which is captured by a concept of AQFTs taking values in cochain complexes, see, e.g., [8,17,18]. This necessarily introduces to an ∞-categorical context because the natural notion of equivalence between cochain complexes is given by quasi-isomorphisms, as opposed to isomorphisms. This in particular means that, instead of the smooth refinement $\mathbf{Sh}_{C_{\mathbb{K}}^\infty}$ of the ordinary category $\mathbf{Vec}_{\mathbb{K}}$ from Sect. 3.2, one has to consider a smooth refinement of the ∞-*category* $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes. A natural candidate for this purpose is the ∞-*stack* $\mathbf{Ch}(\mathbf{Sh}_{C_{\mathbb{K}}^\infty})$ of cochain complexes of sheaves of modules.

Open Problem 6.4. Show that, by replacing the stack $\mathbf{Sh}_{C_{\mathbb{K}}^\infty}$ of sheaves of modules by the ∞-stack $\mathbf{Ch}(\mathbf{Sh}_{C_{\mathbb{K}}^\infty})$ of cochain complexes of sheaves of modules, the relevant definitions and constructions from Sect. 3 generalize to the context of ∞-stacks.

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