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Long Time Behaviour of a Local Perturbation in the Isotropic XY Chain Under Periodic Forcing

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Abstract. We study the isotropic XY quantum spin chain with a time-periodic transverse magnetic field acting on a single site. The asymptotic dynamics is described by a highly resonant Floquet–Schrödinger equation, for which we show the existence of a periodic solution if the forcing frequency is away from a discrete set of resonances. This in turn implies the state of the quantum spin chain to be asymptotically a periodic function synchronised with the forcing, also at arbitrarily low non-resonant frequencies. The behaviour at the resonances remains a challenging open problem.

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1. Introduction

We investigate the isotropic XY quantum spin chain with a periodically time-dependent transverse external field acting only on one site, namely the κ-th, and free boundary conditions. The Hamiltonian reads

$$H_N(t) = -g \sum_{j=1}^{N-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right) - hV(\omega t)\sigma_\kappa^z, \quad 1 < \kappa < N. \quad (1.1)$$

For any $t \in \mathbb{R}$ and $N \in \mathbb{N}$, the operator $H_N(t)$ is self-adjoint on $\mathcal{H}_N := \mathbb{C}^{2 \otimes N}$, and the thermodynamic limit $N \to \infty$ is done as customary in the Fock space $\mathcal{F} := \bigoplus_N \mathcal{H}_N$. Here, $\sigma^x, \sigma^y, \sigma^z$ denote the Pauli matrices, $g, h, \omega > 0$ are parameters ruling, respectively, the spin–spin coupling, the magnitude of the external field and its frequency. We assume that $V(\omega t)$ is a real periodic analytic function with frequency ω

$$V(\omega t) = \sum_{k \in \mathbb{Z}} e^{ik\omega t} V_k, \qquad |V_k| \le C_0 e^{-\sigma|k|}. \tag{1.2}$$

The interaction of a small system (the impurity) with an environment (the rest of the chain) while it is irradiated by a monochromatic forcing is a question of primary interest in non-equilibrium statistical physics. Although more complicated systems have been considered [4,8,9], quantum spin chains are particularly appealing as they present a rich phenomenology along with a limited amount of technical difficulties. Indeed the lack of ergodicity of such systems has already been object of study since the '70s [17–23].

Also, there has been a resurgence of interest in periodically driven systems from the mathematical and theoretical physics community. When the external frequency is high enough compared to the natural frequencies of the system at rest, one can approximate the Floquet Hamiltonian by an effective time-independent Hamiltonian, which governs the asymptotic dynamics [10,12]. This idea has been exploited mathematically in [3], where the effective Hamiltonian is computed by a KAM-type reduction inspired by [7]. Consequences of this analysis for the behaviour of many-body interacting systems have been examined in [1,2].

The choice of considering a simple system such as the isotropic XY chain itself simplifies greatly the computations and allows us to perform a very detailed analysis of the dynamics. The motion of an impurity was first analysed in [6] with different forms of time-dependent external fields. In particular in the case $V(\omega t) = \cos \omega t$, the authors computed the transversal magnetisation of the perturbed spin at the first order in h, observing a divergence at $\omega = 2g$. The analysis of [11] shows that indeed all the values $\{2g/k\}_{k \in \mathbb{N}}$ are resonant (i.e. singular) in a sense that will be made clear below. Therefore in this context, it appears natural to distinguish between resonant and non-resonant frequencies more than between high and low ones (even though, since the set of resonant frequencies is bounded, the first characterisation induces the second one). Indeed, combining the results of the present and our previous paper [11], one can conclude that no matter how small the frequency is, away from resonances, the impurity asymptotically undergoes a periodic dynamics synchronised with the forcing. This simple picture breaks down for resonant frequencies, where some new phenomenon can occur.

It is well-known that the isotropic XY spin chain is equivalent to a system of quasi-free fermions on $\mathbb{Z}$ and therefore the N-particle state is fully described by a one-particle wave function (for more details about this derivation we refer to [5,6] or more recently [11,15]). At fixed t, the forcing $V(\omega t)$ is just a number which we can incorporate into h, and the spectrum is given by the standard analysis of the rank-one perturbation of the Laplacian on $\mathbb{Z}$ (see [5]). Precisely, as $N \to \infty$, we have a band $[-g, g]$ and an isolated eigenvalue given by

$$g\, \mathrm{sign}(h) \sqrt{1 + \frac{h^2}{g^2}}. \tag{1.3}$$

The study of the dynamics however is not as simple, because when t varies the eigenvalue moves and can touch the band, so creating resonances. More precisely, the dynamics in the time interval $[t_0, t]$ is governed by the following Floquet–Schrödinger equation on $\mathbb{Z}$ (again, the details on its derivation and its relation with the many-body system can be found in [5,6,11,15])

$$i\partial_t \psi(x,t) = gh\Delta\psi(x,t) + hH_F(t,t_0)\psi(x,t)\,, \quad \psi(x,t_0) = \delta(x)\,, \quad x \in \mathbb{Z}\,. \tag{1.4}$$

Here, $\delta(x)$ is the Kronecker delta centred in the origin, Δ is the Laplacian on $\mathbb{Z}$ with spectrum given by $\{-\cos q, q \in [-\pi,\pi]\}$, and

$$H_F(t,t_0)\psi(x,t) := V(\omega t)\psi(x,t) + ig\int_{t_0}^t dt' J_1(g(t-t'))e^{-i\Delta(t-t')}V(\omega t')\psi(x,t')\,, \tag{1.5}$$

where

$$J_k(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{ixk+it\cos x}\,, \quad k \in \mathbb{Z}\,. $$

The Floquet operator H_F acts as a memory term, accounting for the retarded effect of the rest of the chain on the site κ. This equation finds a more compact form in the Duhamel representation in the momentum space. We denote by $\xi \in [-1,1]$ the points of the spectrum of $-\Delta$. Moreover with a slight abuse of notation throughout the paper, we will systematically omit the customary $\hat{\ }$ to indicate either Fourier transforms (when transforming in space) and Fourier coefficients (when transforming in time).

Let $\psi(\xi,t)$, $\xi \in [-1,1]$, denote the Fourier transform of $\psi(x,t)$, $x \in \mathbb{Z}$. The corresponding equation (1.4) for $\psi(\xi,t)$ in its Duhamel form reads

$$(\mathbb{1} + ihW_{t_0})\psi(\xi,t) = 1\,, \tag{1.6}$$

where $\{W_{t_0}\}_{t_0 \in \mathbb{R}}$ is a family of Volterra operators for any $t > t_0$ and $\xi \in [-1,1]$, defined via

$$W_{t_0}f(\xi,t) := \int_{t_0}^t dt' J_0(g(t-t'))e^{ig\xi(t-t')}V(\omega t')f(\xi,t')\,. \tag{1.7}$$

Let $L_\xi^2 C_t^\omega([-1,1]\times[t_0,t])$ denote the space of square integrable functions in $[-1,1]$ and real analytic in the time interval $[t_0,t]$ (mind here the superscript ω denoting analyticity as customary, not to be confused with the frequency). Each W_{t_0} is a linear map from $L_\xi^2 C_t^\omega([-1,1]\times[t_0,t])$ into itself. For any t_0, finite W_{t_0} is a compact integral operator, which ensures the existence of a unique solution for $t - t_0 < \infty$ (see for instance [13]). We denote this one-parameter family of functions with $\psi_{t_0}(\xi,t)$. As $t_0 \to -\infty$, the limit of the W_{t_0} is an unbounded operator, denoted by W_∞, defined through

$$W_\infty f(\xi,t) := \int_{-\infty}^t dt' J_0(g(t-t'))e^{ig\xi(t-t')}V(\omega t')f(\xi,t')\,. \tag{1.8}$$

One can therefore use W_∞ to define an asymptotic version of equation (1.6) as $t_0 \to -\infty$

$$(\mathbb{1} + ihW_\infty)\psi(\xi,t) = 1\,, \tag{1.9}$$

whose solutions are denoted by $\psi_\infty(\xi, \omega t)$: indeed it is easy to check that W_∞ maps periodic functions of frequency ω into periodic functions of frequency ω, thus it is somehow expected to find solutions of (1.9) in this class of functions. Under the following assumption on the frequency, our analysis confirms this idea.

Hypothesis 1.1. *We assume that $\omega > 0$ is such that $\frac{2g}{\omega} \notin \mathbb{N}$.*

Remark 1.2. Note that if ω satisfies Hypothesis 1.1 above, there is $\bar{k} \in \mathbb{N}$ such that
$$\bar{\epsilon} := \inf_{k \in \mathbb{N}} \left| \frac{2g}{\omega} - k \right| = \left| \frac{2g}{\omega} - \bar{k} \right| > 0 \,. \tag{1.10}$$

Our main result reads as follows:

Theorem 1.3. *Let $\omega > 0$ satisfying Hypothesis 1.1 above, and let $\bar{\epsilon}$ as in (1.10). There is $\gamma_0 = \gamma_0(\frac{g}{\omega}, \bar{\epsilon}, V)$ small enough such that if $h < \gamma_0 \omega$, then there exists a periodic solution of (1.9) with frequency ω, $\psi_\infty(x, \omega t) \in \ell_x^2 C_t^\omega(\mathbb{Z} \times \mathbb{R})$. In particular, γ_0 is explicitly computable; see (3.24). Moreover, for all $x \in \mathbb{Z}$, one has*
$$\psi_{t_0}(x, t) = \psi_\infty(x, \omega t) + O\left(\frac{1}{\sqrt{t - t_0}} \right) \,.$$

Remark 1.4. To be more precise, we prove that the spatial Fourier transform of $\psi_\infty(x, \omega t)$ is bounded. This implies that ψ_∞ is square integrable for all t.

Remark 1.5. If $V_0 \neq 0$, we have in general $\gamma_0 \simeq \sqrt{\bar{\epsilon}}$. To fix the ideas, let us assume the frequency small enough, that is, $\omega \ll C_0^{\frac{3}{2}} g$, then $\gamma_0 \lesssim \sqrt{\frac{\bar{\epsilon}\omega}{g}}$. If $V_0 = 0$, then our worst estimate of the radius of convergence is $\gamma_0 \lesssim C_0^2 \bar{\epsilon}^2 \frac{g}{\omega}$, but this bound improves a bit under a specific assumption on the forcing V. A precise formula for the radius of convergence γ_0 is given in (3.24) below.

The main relevance of this result lies in its validity for low frequencies. To the best of our knowledge, a similar control of the convergence to the synchronised periodic state for a periodically forced small quantum system coupled with free fermion reservoirs has been achieved only in [4], for a different class of models. In general, it is known that the low-frequency assumption makes the dynamics harder to study.

On the other hand, the main limitation of the result is that we need to avoid the resonances: the closer $\frac{2g}{\omega}$ is to an integer, the smaller value of perturbation parameter h is allowed. This in particular prevents us to say anything on the behaviour of the system for resonant frequencies $\frac{2g}{k}$, leaving the problem open.

In [11], we proved the existence of periodic solutions of (1.9) with frequency ω if $V_0 = O\left(\frac{1}{h}\right)$ and h small or if $V_0 = 0$, $\omega > 2g$ (high frequencies) and h/ω small. The meaning of these conditions is clear: if V_0 is large and h is small, then the eigenvalue does not touch the band; if $\omega > 2g$, then the forcing cannot move energy levels within the band. In particular, the high-frequency

assumption appears in other related works in mathematical and theoretical physics [1–3, 10, 12, 14].

Note that differently from [11], here we allow also $V_0 \neq 0$ provided that Hypothesis 1.1 holds and h is small enough.

An important step of our analysis is [11, Proposition 3.1], in which we proved that if a periodic solutions of (1.9) ψ_∞ with non resonant frequency ω exists, then $\psi_{t_0}(\xi, t)$ must approach $\psi_\infty(\xi, \omega t)$ as $t_0 \to -\infty$ for all $t \in \mathbb{R}$, $\xi \in [-1, 1]$.

Proposition 1.6. *Let $\psi_\infty(\xi, \omega t)$ a periodic solution of (1.9) with frequency ω satisfying Hypothesis 1.1. For any $t \in \mathbb{R}$, $\xi \in [-1, 1]$, one has*

$$\psi_{t_0}(\xi, t) = \psi_\infty(\xi, \omega t) + O\left(\frac{1}{\sqrt{t - t_0}}\right). \tag{1.11}$$

Therefore for non resonant frequencies, the control on the long time behaviour of the solution of (1.6) amounts to establish the existence of a periodic solution of (1.9) for $\omega < 2g$, a condition defining the low frequency regime. This is a genuine PDE question, which is indeed the main focus of this paper.

More specifically, we are facing an unbounded time-dependent perturbation of the continuous spectrum of the Laplacian on $\mathbb{Z}$. Problems involving periodic forcing are typically dealt via a KAM-approach, namely one tries to reduce the perturbation to a constant operator by means of a sequence of bounded maps. This is for instance the approach adopted in [3, 12] in the context of interacting many-body system, in which a generalisation of the classical Magnus expansion is exploited via normal form methods. Indeed some salient features of periodically driven systems, as for instance pre-thermalisation or slow heating, from the mathematical point of view are essentially consequences of the KAM reduction. A similar approach has been used in [14] for the Klein–Gordon equation with a quasi-periodic forcing. All the aforementioned results are valid if the frequency is large enough, as usual in Magnus expansion approaches.

We cope here with two main sources of difficulty. First, we deal with a perturbation of operators with continuous spectrum. Secondly, the operator in (1.9) is a perturbation of the identity, which makes trivial the homological equation at each KAM step.

Thus, we have to use a different approach. As in [11], we explicitly construct a solution of (1.9) by resumming the Neumann series. More precisely, we formally write the solution to (1.9) as a power series in the small parameter h/ω; see (2.20). Such expression is plagued by the presence of the terms $j_{\mu_p}(\xi)$, which may be singular for finitely many values of ξ; see (2.6). The main problem then is that any divergent $j_{\mu_p}(\xi)$ appears raised to arbitrarily large powers, and this makes the series expansion very singular. (It does not belong to any L^p space.) For $\omega > 2g$, the function $j_0(\xi)$ is the only one having a singularity, while when $\omega \leq 2g$, also the function $j_{\mu_p}(\xi)$ can diverge for $|\mu_p| \leq 2g/\omega$.

We cure these divergences by a suitable renormalisation of the Neumann series, and one major advance of this work is that this is done regardless of the size of the frequency ω, once small intervals about countably many resonant

frequencies are excluded. To do so, we refine the method introduced in [11] in the case of large ω, which was based on the normalisation of the second-order term of the expansion, taking into account the subtle cancellations induced by the operator at the r.h.s. of (2.2). We stress that thanks to the careful analysis of subsequent Sects. 4 and 5, also in the low-frequency regime resumming the resonances of second order is sufficient to construct a meaningful solution and, from what we can see, no crucial advantages can come from higher-order resummations.

The rest of the paper is organised as follows. In Sect. 2, the main objects needed for the proof are introduced, while in Sect. 3, we prove the existence of a periodic solution of (1.9) with frequency ω. In Sects. 4 and 5, we prove few accessory results used in Sect. 3. Finally, we attach an "Appendix" in which we sketch the proof of Proposition 1.6.

2. Set-up

It is convenient to define

$$\varphi := \omega t \,, \qquad \alpha := \frac{g}{\omega} \,, \qquad \gamma := \frac{h}{\omega} \,, \tag{2.1}$$

so that we can rewrite (1.8) as

$$W'_\infty \psi(\xi, \varphi) := \int_{-\infty}^{\varphi} \mathrm{d}\varphi' \, J_0\left(\alpha(\varphi - \varphi')\right) e^{\mathrm{i}\alpha\xi(\varphi - \varphi')} V(\varphi') \psi(\xi, \varphi') \,,$$

and hence a periodic solution of (1.9) with frequency ω should satisfy

$$(\mathbb{1} + \mathrm{i}\gamma W'_\infty)\psi(\xi, \varphi) = 1 \,. \tag{2.2}$$

Such a solution will be explicitly constructed.

Note that by Remark 1.2, the inf in (1.10) is indeed a min, and it is attained either at $\bar{k} = \lfloor 2\alpha \rfloor$, i.e. the integer part of 2α, or at $\bar{k} = \lceil 2\alpha \rceil := \lfloor 2\alpha \rfloor + 1$. Moreover, $\bar{\epsilon} < 1$.

Recall the formula

$$j(\tau) := \int_0^{\infty} \mathrm{d}t J_0(t) e^{\mathrm{i}\tau t} = \frac{\chi(|\tau| \leq 1) + \mathrm{i}\,\mathrm{sign}(\tau)\chi(|\tau| > 1)}{\sqrt{|1 - \tau^2|}} \,. \tag{2.3}$$

The proof of (2.3) can be found for instance in [11, Lemma A.3]. Unfortunately in [11, (A.11)], the $\mathrm{sign}(\tau)$ in the imaginary part is mistakenly omitted, whereas it is clear from the proof that it should appear; see also [11, (A.12)].

Set

$$j_k(\xi) := \frac{1}{\alpha} j\left(\xi + \frac{k}{\alpha}\right) \tag{2.4}$$

and let us define $\xi_0 := 1$, $\xi_0^* := -1$, and for $k \in \mathbb{N}$,

$$\xi_k := \mathrm{sign}\,k - \frac{k}{\alpha} \,, \quad \xi_k^* := \mathrm{sign}\,k + \frac{k}{\alpha} \,. \tag{2.5}$$

Lemma 2.1. *For all $k \neq 0$, one has*

$$j_k(\xi) = \frac{\chi\left(\operatorname{sign}(k)(\xi - \xi_k) \leq 0\right) + \mathrm{i}\operatorname{sign}(k)\chi(\operatorname{sign}(k)(\xi - \xi_k) > 0)}{\alpha\sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}}. \qquad (2.6)$$

Proof. Using (2.3) and (2.4), we have

$$j_k(\xi) = \frac{\chi\left(|\alpha\xi + k| \leq \alpha\right) + \mathrm{i}\operatorname{sign}(\alpha\xi + k)\chi(|\alpha\xi + k| > \alpha)}{\alpha\sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}}.$$

Let us write

$$\chi(|\alpha\xi + k| \leq \alpha) = \chi\left(\xi \leq 1 - \frac{k}{\alpha}\right)\chi\left(\xi \geq -1 - \frac{k}{\alpha}\right)$$

and

$$\operatorname{sign}(\alpha\xi + k)\chi(|\alpha\xi + k| > \alpha) = \chi(\alpha\xi + k > \alpha) - \chi(\alpha\xi + k < -\alpha)$$

$$= \chi\left(\xi > 1 - \frac{k}{\alpha}\right) - \chi\left(\xi < -1 - \frac{k}{\alpha}\right).$$

Now, we note that since $\xi \in [-1, 1]$, if $k \geq 1$, then $\chi(\xi < -1 - \frac{k}{\alpha}) = 0$ and if $k \leq -1$ then $\chi(\xi > 1 - \frac{k}{\alpha}) = 0$. This implies

$$\chi\left(\xi \leq 1 - \frac{k}{\alpha}\right)\chi\left(\xi \geq -1 - \frac{k}{\alpha}\right) = \chi(\operatorname{sign}(k)(\xi - \xi_k) \leq 0),$$

and

$$\chi\left(\xi > 1 - \frac{k}{\alpha}\right) - \chi\left(\xi < -1 - \frac{k}{\alpha}\right) = \operatorname{sign}(k)\chi\left(\operatorname{sign}(k)(\xi - \xi_k) > 0\right)$$

so that the assertion is proven. $\qquad \square$

Note that by Lemma 2.1, $j_k(\xi)$ is either real or purely imaginary. On the other hand $j_0(\xi)$ is always real, while $j_k(\xi)$ is purely imaginary for $|k| > 2\alpha$.

We conveniently localise the functions j_k about their singularities. Let $r > 0$ and set

$$\begin{cases} Lj_0(\xi) & := j_0(\xi)(\chi(\xi < -1 + r) + \chi(\xi > 1 - r)), \\ Lj_k(\xi) & := j_k(\xi)\chi(|\xi - \xi_k| < r), \\ Rj_k(\xi) & := j_k(\xi) - Lj_k(\xi), \qquad k \in \mathbb{Z} \setminus \{0\}. \end{cases} \qquad (2.7)$$

The following properties are proved by straightforward computations.

Lemma 2.2. *(i) $\xi_k = -\xi_{-k}$ and $\xi_k^* = -\xi_{-k}^*$;*
(ii) One has

$$\min_{\substack{|k|,|k'| \leq \lfloor 2\alpha \rfloor \\ k \neq k'}} |\xi_k - \xi_{k'}| = \frac{\bar{\epsilon}}{\alpha};$$

(iii) $\xi_k > 0$ if and only if $k < -\alpha$ or $0 < k < \alpha$.

(iv) *One has*

$$\xi_k > \xi_{k'} \iff \begin{cases} k' > k > 0 \\ k < k' < 0 \\ k' > 0, k < 0, k' - k > 2\alpha \\ k' < 0, k > 0, k' - k > -2\alpha \end{cases}$$

(v) *If* $|k| > 2\alpha$, *then* $|\xi_k| > 1$;

(vi) *For* $k \geq 1$ *and* $r \in (0, (4\alpha)^{-1})$, *one has*

$$Lj_k(\xi) = \frac{\chi(\xi_k - r < \xi \leq \xi_k) + \mathrm{i}\chi(\xi_k < \xi < \xi_k + r)}{\alpha\sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}} \tag{2.8}$$

$$Rj_k(\xi) = \frac{\chi(\xi \leq \xi_k - r) + \mathrm{i}\chi(\xi \geq \xi_k + r)}{\alpha\sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}} \tag{2.9}$$

and for $k \leq -1$,

$$Lj_k(\xi) = \frac{\chi(\xi_k < \xi < \xi_k + r) - \mathrm{i}\chi(\xi_k - r < \xi \leq \xi_k)}{\alpha\sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}} \tag{2.10}$$

$$Rj_k(\xi) = \frac{\chi(\xi \geq \xi_k + r) - \mathrm{i}\chi(\xi \leq \xi_k - r)}{\alpha\sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}} . \tag{2.11}$$

(vii) *There exist* $c_1, c_2 > 0$ *such that for all* $k \in \mathbb{Z}$

$$\inf_{\xi \in [-1,1]} |Lj_k(\xi)| \geq \frac{c_1}{\alpha\sqrt{r}}, \quad \sup_{\xi \in [-1,1]} |Rj_k(\xi)| \leq \frac{c_2}{\alpha\sqrt{r}}. \tag{2.12}$$

(viii) *For* $|k| > 2\alpha$ *and* $\epsilon < \bar{\epsilon}$, *one has*

$$|j_k(\xi)| \leq \frac{c_0}{\sqrt{\alpha\epsilon}}, \tag{2.13}$$

Fix $\epsilon < \bar{\epsilon}$ (say $\epsilon = \bar{\epsilon}/2$) and take $r \in (0, r^*)$, with $r^* < \frac{\epsilon}{4\alpha}$ so that in particular, property (vi) in Lemma 2.2 is satisfied, and moreover, one has

$$Lj_k(\xi)Lj_{k'}(\xi) = 0 \qquad \text{for} \quad k \neq k' \tag{2.14}$$

by property (ii) of Lemma 2.2.

Combining a (formal) expansion as a power series in γ and Fourier series in φ (i.e. the so-called Lindstedt series), we can now obtain a formal series representation for the solution of (2.2) which is the starting point of our analysis. Precisely, we start by writing

$$\psi(\xi, \varphi) = \sum_{n \geq 0} \gamma^n \psi_n(\xi, \varphi), \tag{2.15}$$

so that inserting (2.15) into (2.2), we see that the coefficients ψ_n must satisfy

$$\psi_0 = 1, \quad \psi_n = -\mathrm{i}W'_\infty[\psi_{n-1}]. \tag{2.16}$$

We now expand

$$\psi_n(\xi, \varphi) = \sum_{k \in \mathbb{Z}} \psi_{n,k}(\xi) e^{\mathrm{i}k\varphi}.$$

Using that

$$(W_\infty \psi_n)_k(\xi) = j_k(\xi) \sum_{\mu \in \mathbb{Z}} V_{k-\mu} \psi_{n,\mu}(\xi), \tag{2.17}$$

by a direct computation, we obtain

$$\begin{cases} \psi_1(\xi,\varphi) = \displaystyle\sum_{k_1 \in \mathbb{Z}} j_{k_1}(\xi) V_{k_1} e^{\mathrm{i}k_1\varphi}, \\[2mm] \psi_2(\xi,\varphi) = \displaystyle\sum_{k_1,k_2 \in \mathbb{Z}} j_{k_1+k_2}(\xi) V_{k_2} j_{k_1}(\xi) V_{k_1} e^{\mathrm{i}(k_1+k_2)\varphi} \\[2mm] \qquad\vdots \\[2mm] \psi_n(\xi,\varphi) = \displaystyle\sum_{k_1,\dots,k_n \in \mathbb{Z}} \left(\prod_{i=1}^{n} j_{\mu_i}(\xi) V_{k_i} \right) e^{\mathrm{i}\mu_n\varphi}, \end{cases} \tag{2.18}$$

where we denoted

$$\mu_p = \mu(k_1,\dots,k_p) := \sum_{j=1}^{p} k_j. \tag{2.19}$$

Therefore, we arrive to write the formal series

$$\widetilde{\psi}(\xi,\varphi;\gamma) := \sum_{\mu \in \mathbb{Z}} e^{\mathrm{i}\mu\varphi} \psi_\mu(\xi;\gamma) = \sum_{\mu \in \mathbb{Z}} e^{\mathrm{i}\mu\varphi} \sum_{N \geq 0} (-\mathrm{i}\gamma)^N \psi_{N,\mu}(\xi)$$

$$= \sum_{\mu \in \mathbb{Z}} e^{\mathrm{i}\mu\varphi} \sum_{N \geq 0} \sum_{\substack{k_1,\dots,k_N \in \mathbb{Z} \\ \mu_N = \mu}} (-\mathrm{i}\gamma)^N \left(\prod_{p=1}^{N} j_{\mu_p}(\xi) V_{k_p} \right), \tag{2.20}$$

which solves (1.9) to all orders in γ. Note that for each $N \in \mathbb{N}$ the coefficient of γ^N is a sum of singular terms. This makes it difficult (if not impossible) to show the convergence of (2.20), and we will instead prove the convergence of a resummed series which solves the equation.

3. Proof of the Theorem

To explain our construction of the series giving a solution of (1.9), it is useful to introduce a slightly modified version of the graphical formalism of [11], inspired by the one developed in the context of KAM theory (for a review see for instance [16]).

Since our problem is linear, we shall deal with linear trees, or *reeds*. Precisely, an oriented tree is a finite graph with no cycle, such that all the lines are oriented towards a single point (the *root*) which has only one incident line (called *root line*). All the points in a tree except the root are called *nodes*. Note that in a tree, the orientation induces a natural total ordering ($\preceq$) on the set of the nodes $N(\rho)$ and lines. If a vertex v is attached to a line ℓ, we say that ℓ exits v if $v \preceq \ell$; otherwise, we say that ℓ enters v. Moreover, since a line ℓ may be identified by the node v which it exits, we have a natural total ordering also on the set of lines $L(\rho)$. We call *end-node* a node with no line

entering it, and *internal node* any other node. We say that a node has *degree*
d if it has exactly d incident lines. Of course an end-node has degree one. We
call *reed* a labelled rooted tree in which each internal node has degree two.

Given a reed ρ, we associate labels with each node and line as follows.
We associate with each node v a *mode label* $k_v \in \mathbb{Z}$ and with each line ℓ a
momentum $\mu_\ell \in \mathbb{Z}$ with the constraint

$$\mu_\ell = \sum_{v \prec \ell} k_v . \tag{3.1}$$

Note that (3.1) above is a reformulation of (2.19). We call *order* of a reed ρ
the number $\#N(\rho)$ and *total momentum* of a reed the momentum associated
with the root line.

$\Theta_{N,\mu}$ denotes the set of reeds of order N and total momentum μ. We say
that a line ℓ is *regular* if $|\mu_\ell| \geq \lceil 2\alpha \rceil$; otherwise, we say it is *singular*. With
every singular line ℓ, we attach a further *operator label* $\mathcal{O}_\ell \in \{L, R\}$; if ℓ is
singular, we say that it is *localised* if $\mathcal{O}_\ell = L$; otherwise, we say that it is
regularised.

We then associate with each node v a *node factor*

$$\mathcal{F}_v = V_{k_v} \tag{3.2}$$

and with each line ℓ a *propagator*

$$\mathcal{G}_\ell(\xi) = \begin{cases} j_{\mu_\ell}(\xi), & \ell \text{ is regular} \\ \mathcal{O}_\ell j_{\mu_\ell}(\xi), & \ell \text{ is singular,} \end{cases} \tag{3.3}$$

so that we can associate with each reed ρ a value as

$$\text{Val}(\rho) = \left(\prod_{v \in N(\rho)} \mathcal{F}_v \right) \left(\prod_{\ell \in L(\rho)} \mathcal{G}_\ell(\xi) \right) . \tag{3.4}$$

In particular, one has formally

$$\psi_{N,\mu} = \sum_{\rho \in \Theta_{N,\mu}} \text{Val}(\rho) . \tag{3.5}$$

Remark 3.1. If in a reed ρ with $\text{Val}(\rho) \neq 0$, there is a localised line ℓ, i.e.
if $\mathcal{O}_\ell = L$, then all the lines with momentum $\mu \neq \mu_\ell$ are either regular or
regularised. Indeed if ℓ is localised, then by (2.7), we have that ξ is r-close to
ξ_{μ_ℓ}, and hence it cannot be r-close to ξ_μ for $\mu \neq \mu_\ell$; see also (2.14).

Given a reed ρ, we say that a connected subset $\mathbf{s}$ of nodes and lines in
ρ is a *closed-subgraph* if $\ell \in L(\mathbf{s})$ implies that $v, w \in N(\mathbf{s})$ where v, w are the
nodes ℓ exits and enters, respectively. We say that a closed-subgraph $\mathbf{s}$ has
degree $d := |N(\mathbf{s})|$. We say that a line ℓ *exits* a closed-subgraph $\mathbf{s}$ if it exits a
node in $N(\mathbf{s})$ and enters either the root (so that ℓ is the root line) or a node
in $N(\rho) \setminus N(\mathbf{s})$. Similarly, we say that a line *enters* $\mathbf{s}$ if it enters a node in
$N(\mathbf{s})$ and exits a node in $N(\rho) \setminus N(\mathbf{s})$. We say that a closed-subgraph $\mathbf{s}$ is a
resonance if it has an exiting line $\ell_{\mathbf{s}}$ and an entering line $\ell'_{\mathbf{s}}$, both $\ell_{\mathbf{s}}$ and $\ell'_{\mathbf{s}}$ are
localised (so that in particular by Remark 3.1, the exiting and entering lines

of a resonance must carry the same momentum), while all lines $\ell \in L(\mathbf{s})$ have momentum $\mu_\ell \neq \mu_{\ell_\mathbf{s}}$.

Note that by (3.1), one has

$$\sum_{v \in N(\mathbf{s})} k_v = 0 \,, \tag{3.6}$$

We denote by $\mathcal{T}_{d,\mu}$ the set of resonances with degree d and entering and exiting lines with momentum μ. Note that if $d = 1$, then $\mathcal{T}_{1,\mu}$ is constituted by a single node v with mode $k_v = 0$.

Let us set

$$\mathcal{M}_{d,\mu}(\xi) := \sum_{\mathbf{s} \in \mathcal{T}_{d,\mu}} \mathrm{Val}(\mathbf{s}) \,, \tag{3.7}$$

where we define the value of a resonance $\mathbf{s}$ as in (3.4) but with the products restricted to nodes and lines in $\mathbf{s}$, namely

$$\mathrm{Val}(\mathbf{s}) := \left(\prod_{v \in N(\mathbf{s})} \mathcal{F}_v \right) \left(\prod_{\ell \in L(\mathbf{s})} R j_{\mu_\ell}(\xi) \right) \,.$$

Next, we proceed with the proof, which we divide into several steps.

Proof. (Step 1: resummation.) The idea behind resummation can be roughly described as follows. The divergence of the sum in (3.5) is due to the presence of localised lines (and their possible accumulation). If a reed $\rho_0 \in \Theta_{N,\mu}$ has a localised line ℓ, say exiting a node v, then we can consider another reed $\rho_1 \in \Theta_{N+1,\mu}$ obtained from ρ_0 by inserting an extra node v_1 with $k_{v_1} = 0$ and an extra localised line ℓ' between ℓ and v, i.e. ρ_1 has an extra resonace of degree one. Of course, while ρ_0 is a contribution to $\psi_N(\varphi)$, ρ_1 is a contribution to $\psi_{N+1}(\varphi)$, so when (formally) considering the whole sum, the value of ρ_1 will have an extra factor $(-\mathrm{i}\gamma)$. In other words, in the formal sum (2.20), there will be a term of the form

$$\mathrm{Val}(\rho_0) + (-\mathrm{i}\gamma)\mathrm{Val}(\rho_1) = (\text{common factor})\big(L j_{\mu_\ell}(\xi) + L j_{\mu_\ell}(\xi)(-\mathrm{i}\gamma)V_0 L j_{\mu_\ell}(\xi)\big)$$
$$= (\text{common factor})L j_{\mu_\ell}(\xi)\big(1 + (-\mathrm{i}\gamma)V_0 L j_{\mu_\ell}(\xi)\big)$$

Of course, we can indeed insert any chain of resonances of degree one, say of length p, so as to obtain a reed $\rho_p \in \Theta_{N+p,\mu}$, and when summing their values together we formally have

$$\sum_{p \geq 0} (-\mathrm{i}\gamma)^p \mathrm{Val}(\rho_p) = (\text{common factor})L j_{\mu_\ell}(\xi)\big(1 + (-\mathrm{i}\gamma)V_0 L j_{\mu_\ell}(\xi)$$
$$+ (-\mathrm{i}\gamma)V_0 L j_{\mu_\ell}(\xi)(-\mathrm{i}\gamma)V_0 L j_{\mu_\ell}(\xi) + \cdots\big)$$
$$= (\text{common factor})L j_{\mu_\ell}(\xi) \sum_{p \geq 0} ((-\mathrm{i}\gamma)V_0 L j_{\mu_\ell}(\xi))^p$$
$$= (\text{common factor})\frac{L j_\mu(x)}{1 + \mathrm{i}\gamma V_0 L j_\mu(\xi)}$$

In other words, we formally replace the sum over N of the sum of reeds in $\Theta_{N,\mu}$ with the sum of reeds where no resonance of degree one appear, but with the localised propagators replaced with

$$\frac{Lj_\mu(x)}{1 + i\gamma V_0 Lj_\mu(\xi)}.$$

Clearly in principle, we can perform this formal substitution considering resonances of any degree. Here, it is enough to consider resummations only of resonances of degree one and two. The advantage of such a formal procedure is that the localised propagators do not appear anymore. However, since the procedure is only formal, one has to prove not only that the new formally defined object is indeed well-defined, but also that it solves (2.2).

Having this in mind, let $\Theta_{N,\mu}^{\mathcal{R}}$ be the set of reeds in which no resonance of deree 1 nor 2 appear, and define

$$\mathcal{M}_\mu(\xi) = \mathcal{M}_\mu(\xi, \gamma) := (-i\gamma)\mathcal{M}_{0,\mu}(\xi) + (-i\gamma)^2 \mathcal{M}_{1,\mu}(\xi)$$
$$= -i\gamma V_0 - \gamma^2 \sum_{k \in \mathbb{Z}} V_k R j_{k+\mu}(\xi) V_{-k}. \tag{3.8}$$

$\square$

In Sect. 4, we prove the following result.

Proposition 3.2. *For all $\mu \in \mathbb{Z} \cap [-2\alpha, 2\alpha]$ and for*

$$\gamma \in \begin{cases} (0, +\infty) & V_0 \geq 0 \\ \left(0, c\sqrt{\dfrac{\epsilon}{\alpha}} \dfrac{|V_0|}{\|V\|_{L^2}^2}\right) & V_0 < 0 \end{cases} \tag{3.9}$$

where c is a suitable absolute constant, one has

$$\inf_{\xi \in [-1,1]} |1 - \mathcal{M}_\mu(\xi) Lj_\mu(\xi)| \geq \frac{1}{2}. \tag{3.10}$$

Proposition 3.2 allows us to set

$$Lj_\mu^{\mathcal{R}}(\xi) := \frac{Lj_\mu(\xi)}{1 - \mathcal{M}_\mu(\xi, \gamma) Lj_\mu(\xi)}. \tag{3.11}$$

For any $\rho \in \Theta_{N,\mu}^{\mathcal{R}}$, let us define the renormalised value of ρ as

$$\text{Val}^{\mathcal{R}}(\rho) := \left(\prod_{v \in N(\rho)} \mathcal{F}_v\right) \left(\prod_{\ell \in L(\rho)} \mathcal{G}_\ell^{\mathcal{R}}\right), \tag{3.12}$$

where

$$\mathcal{G}_{\ell_i}^{\mathcal{R}} = \begin{cases} Lj_\mu^{\mathcal{R}}(\xi), & |\mu_{\ell_i}| \leq \lfloor 2\alpha \rfloor, & \mathcal{O}_{\ell_i} = L, \\ Rj_\mu(\xi), & |\mu_{\ell_i}| \leq \lfloor 2\alpha \rfloor, & \mathcal{O}_{\ell_i} = R, \\ j_{\mu_{\ell_i}}(\xi), & |\mu_{\ell_i}| \geq \lceil 2\alpha \rceil. \end{cases} \tag{3.13}$$

In particular, if $\lfloor 2\alpha \rfloor = 0$, we have to renormalise only j_0, which is the case in our previous paper [11]. Then, we define

$$\psi_\mu^{\mathcal{R}}(\xi; \gamma) := \sum_{N \geq 1} (-\mathrm{i}\gamma)^N \sum_{\rho \in \Theta_{N,\mu}^{\mathcal{R}}} \mathrm{Val}^{\mathcal{R}}(\rho) , \tag{3.14}$$

so that

$$\psi^{\mathcal{R}}(\varphi; \xi, \gamma) := \sum_{\mu \in \mathbb{Z}} e^{\mathrm{i}\mu\varphi} \psi_\mu^{\mathcal{R}}(\xi; \gamma) , \tag{3.15}$$

is the renormalised series we want to prove to be a regular solution of (2.2).

Proof. (Step 2: radius of convergence.) First of all, we prove that the function (3.15) is well-defined.

We start by noting that the node factors are easily bounded by (1.2). The propagators defined in (3.13) are bounded as follows. If $|\mu_\ell| \geq \lceil 2\alpha \rceil$ formula (2.13) yields

$$|j_\mu(\xi)| \leq \frac{c_0}{\sqrt{2\lceil 2\alpha \rceil \epsilon}} ,$$

while for $|\mu| \leq \lfloor 2\alpha \rfloor$, by (2.12), we have

$$|Rj_\mu(\xi)| \leq \frac{c_2}{\alpha\sqrt{r}} .$$

Regarding the resummed propagators, the bound is more delicate. We start by denoting

$$\overline{V} := \begin{cases} 0 & \text{if } V_k = 0, \ \forall k \geq 1 \\ \max_{k \in \mathbb{Z} \setminus \{0\}} |V_k|^2 & \text{otherwise} , \end{cases} \tag{3.16}$$

and

$$\underline{V}_{\leq 2\alpha} := \begin{cases} 0 & \text{if } V_k = 0, \ \forall k = 1, \dots, \lfloor 2\alpha \rfloor \\ \min_{\substack{|k| \leq \lfloor 2\alpha \rfloor \\ V_k \neq 0}} |V_k| & \text{otherwise} , \end{cases} \qquad \overline{V}_{>2\alpha} := \begin{cases} 0 & \text{if } V_k = 0, \ \forall k \geq \lceil 2\alpha \rceil \\ \max_{\substack{|k| > \lceil 2\alpha \rceil \\ V_k \neq 0}} |V_k| & \text{otherwise} , \end{cases}$$

$$\tag{3.17}$$

$$\square$$

In Sect. 5, we prove the following result.

Proposition 3.3. *There is a constant $c > 0$ such that*

$$|Lj_\mu^{\mathcal{R}}(\xi)| \leq T(V, \epsilon, \alpha; \gamma) = T(\gamma) := \begin{cases} \frac{c}{\gamma |V_0|} & \text{if } V_0 \neq 0 \text{ and } \gamma \leq c\sqrt{\frac{\epsilon}{\alpha}} \frac{|V_0|}{\|V\|_{L^2}^2} ; \\ c\sqrt{\frac{\alpha}{\epsilon}} \gamma^{-2} \underline{V}_{\leq 2\alpha}^{-2} & \text{if } V_0 = 0, \ \underline{V}_{\leq 2\alpha} \neq 0 ; \\ c\sqrt{\alpha} \gamma^{-2} \overline{V}_{>2\alpha}^{-2} & \text{if } V_0 = 0, \ \underline{V}_{\leq 2\alpha} = 0 . \end{cases} \tag{3.18}$$

Let us set now

$$B = B(r, \alpha, \epsilon) := \max \left(\frac{1}{\alpha\sqrt{r}}, \frac{1}{\sqrt{\alpha\epsilon}} \right) = \frac{1}{\alpha\sqrt{r}}, \qquad C_1 := \max(c_0, c_2/2).$$

(3.19)

Note that if in a reed $\rho \in \Theta_{N,\mu}^{\mathcal{R}}$ there are l localised lines, we have

$$|\text{Val}^{\mathcal{R}}(\rho)| = \left(\prod_{v \in N(\rho)} |\mathcal{F}_v| \right) \left(\prod_{\ell \in L(\rho)} |\mathcal{G}_\ell| \right)$$

$$\leq \left(C_0 e^{-\sigma \sum_{v \in N(\rho)} |n_v|} \right) \left(\prod_{\ell \in L(\rho)} |\mathcal{G}_\ell| \right)$$

(3.20)

$$\leq C_0 C_1^N B^N T(\gamma)^l e^{-\sigma|\mu|},$$

for some constant $C_0 > 0$. By construction, in a renormalised reed, there must be at least two lines between two localised lines, since we resummed the resonances of degree one and two. This implies that a reed with N nodes can have at most $l = \lceil N/3 \rceil$ localised lines.

Then by (3.14), we obtain

$$|\psi_\mu^{\mathcal{R}}(\xi; \gamma)| \leq C \sum_{N \geq 1} \gamma^N B^N T(\gamma)^{\frac{N}{3}} e^{-\sigma|\mu|/2},$$

(3.21)

so that the series above converge for

$$\gamma^3 T(\gamma) B^3 < 1.$$

(3.22)

This entails

$$\gamma < \begin{cases} \min \left(\sqrt[3]{|V_0|}B^{-1}, c\sqrt{\frac{\epsilon}{\alpha}} \frac{|V_0|}{\|V\|_{L^2}^2} \right) & V_0 \neq 0; \\ c^{-1}B^{-3}\sqrt{\frac{\epsilon}{\alpha}} \, \underline{V}_{\leq 2\alpha}^2 & V_0 = 0, \underline{V}_{\leq 2\alpha} > 0; \\ c^{-1}B^{-3}\frac{1}{\sqrt{\alpha}}\overline{V}_{>2\alpha}^2 & V_0 = 0, \underline{V}_{\leq 2\alpha} = 0. \end{cases}$$

(3.23)

Therefore, under such smallness condition on γ, the function $\psi^{\mathcal{R}}(\varphi; \xi, \gamma)$ (recall (3.15)) is analytic w.r.t. $\varphi \in \mathbb{T}$, uniformly in $\xi \in [-1, 1]$ and for γ small enough

Choosing $\epsilon = \overline{\epsilon}/2$ and $r := \frac{\epsilon}{8\alpha}$, we have by (3.19)

$$B(r, \alpha, \epsilon)^{-3} = \sqrt{\frac{\alpha^3 \epsilon^3}{64}},$$

so that condition (3.23) implies that the series converges for $\gamma \leq \gamma_0 := c_1\gamma_1$ where

$$\gamma_1 = \gamma_1(\alpha, \overline{\epsilon}, V) := \begin{cases} \min \left(\sqrt[3]{|V_0|} \left(\frac{\sqrt{\alpha\overline{\epsilon}}}{4} \right), \sqrt{\frac{\overline{\epsilon}}{2\alpha}} \frac{|V_0|}{\|V\|_{L^2}^2} \right) & V_0 \neq 0; \\ \left(\frac{\sqrt{\alpha\overline{\epsilon}}}{4} \right)^3 \sqrt{\frac{\overline{\epsilon}}{2\alpha}} \, \underline{V}_{\leq 2\alpha}^2 & V_0 = 0, \underline{V}_{\leq 2\alpha} > 0; \\ \left(\frac{\sqrt{\alpha\overline{\epsilon}}}{4} \right)^3 \frac{1}{\sqrt{\alpha}}\overline{V}_{>2\alpha}^2 & V_0 = 0, \underline{V}_{\leq 2\alpha} = 0. \end{cases}$$

(3.24)

and $c_1 := \min\{c, c^{-1}\}$.

Proof. (Step 3: $\psi^{\mathcal{R}}(\varphi; \xi, \gamma)$ solves (2.2).) Now, we want to prove that

$$(\mathbb{1} + i\gamma W'_{\infty})\psi^{\mathcal{R}}(\varphi; \xi, \gamma) = 1 \,.$$

This is essentially a standard computation. Using (3.14) and (3.15), the last equation can be rewritten as

$$i\gamma W'_{\infty}\psi^{\mathcal{R}}(\varphi; \xi, \gamma) = 1 - \psi^{\mathcal{R}}(\varphi; \xi, \gamma) = -\sum_{\mu \in \mathbb{Z}} e^{i\mu\varphi} \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \Theta^{\mathcal{R}}_{N, \mu}} \mathrm{Val}^{\mathcal{R}}(\rho) \,. \tag{3.25}$$

Moreover, thanks to (2.17), we can compute

$$\begin{aligned} i\gamma W'_{\infty}\psi^{\mathcal{R}}(\varphi; \xi, \gamma) &= i\gamma \sum_{\mu \in \mathbb{Z}} \psi^{\mathcal{R}}(\xi; \gamma)(W'_{\infty}e^{i\mu\varphi}) \\ &= i\gamma \sum_{\mu \in \mathbb{Z}} e^{i\mu\varphi} j_{\mu}(\xi) \sum_{k \in \mathbb{Z}} V_{\mu-k}\psi^{\mathcal{R}}_k(\xi; \gamma) \\ &= i\gamma \sum_{\mu \in \mathbb{Z}} e^{i\mu\varphi} \sum_{N \geq 0} (-i\gamma)^N j_{\mu}(\xi) \sum_{\mu_1+\mu_2=\mu} V_{\mu_1} \sum_{\rho \in \Theta^{\mathcal{R}}_{N, \mu_2}} \mathrm{Val}^{\mathcal{R}}(\rho) \,. \end{aligned}$$

Thus, we can write (3.25) in terms of Fourier coefficients as

$$\sum_{N \geq 1} (-i\gamma)^N j_{\mu}(\xi) \sum_{\mu_1+\mu_2=\mu} V_{\mu_1} \sum_{\rho \in \Theta^{\mathcal{R}}_{N-1, \mu_2}} \mathrm{Val}^{\mathcal{R}}(\rho) = \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \Theta^{\mathcal{R}}_{N, \mu}} \mathrm{Val}^{\mathcal{R}}(\rho) \,. \tag{3.26}$$

Note that the root line ℓ of a reed has to be renormalised only if it carries momentum label $|\mu_\ell| \leq \lfloor 2\alpha \rfloor$ and operator $\mathcal{O}_\ell = L$; thus for $|\mu_\ell| \geq \lceil 2\alpha \rceil$, or $|\mu_\ell| \leq \lfloor 2\alpha \rfloor$ and $\mathcal{O}_\ell = R$, we see immediately that (3.26) holds.

Concerning the case $\mu_\ell = \mu$ with $|\mu| \leq \lfloor 2\alpha \rfloor$ and $\mathcal{O}_\ell = L$, we first note that

$$j_{\mu}(\xi) \sum_{\mu_1+\mu_2=\mu} V_{\mu_1} \sum_{\rho \in \Theta^{\mathcal{R}}_{N-1, \mu_2}} \mathrm{Val}^{\mathcal{R}}(\rho) = \sum_{\rho \in \overline{\Theta}^{\mathcal{R}}_{N, \mu}} \mathrm{Val}^{\mathcal{R}}(\rho)$$

where $\overline{\Theta}^{\mathcal{R}}_{N, \mu}$ is the set of reeds such that the root line may exits a resonance of degree ≤ 2, so that equation (3.26) reads

$$\psi^{\mathcal{R}}_{\mu}(\xi; \gamma) = \sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \overline{\Theta}^{\mathcal{R}}_{N, \mu}} \mathrm{Val}^{\mathcal{R}}(\rho) \,, \tag{3.27}$$

Let us split

$$\overline{\Theta}^{\mathcal{R}}_{N, \mu} = \widetilde{\Theta}^{\mathcal{R}}_{N, \mu} \cup \hat{\Theta}^{\mathcal{R}}_{N, \mu} \,, \tag{3.28}$$

where $\hat{\Theta}^{\mathcal{R}}_{N, \mu}$ are the reeds such that the root line indeed exits a resonance of degree ≤ 2, while $\widetilde{\Theta}^{\mathcal{R}}_{N, \mu}$ is the set of all other renormalised reeds. Therefore, we have

$$\sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \widetilde{\Theta}^{\mathcal{R}}_{N, \mu}} \mathrm{Val}^{\mathcal{R}}(\rho) = L j^{\mathcal{R}}_{\mu}(\xi) \sum_{\mu_1+\mu_2=\mu} (i\gamma V_{\mu_1})\psi^{\mathcal{R}}_{\mu_2}(\xi; \gamma) \tag{3.29}$$

and

$$\sum_{N \geq 1} (-i\gamma)^N \sum_{\rho \in \hat{\Theta}_{N,\mu}^{\mathcal{R}}} \mathrm{Val}^{\mathcal{R}}(\rho) = L j_\mu^{\mathcal{R}}(\xi) \mathcal{M}_\mu(\xi, \gamma) L j_\mu^{\mathcal{R}}(\xi) \sum_{\mu_1 + \mu_2 = \mu} (i\gamma V_{\mu_1}) \psi_{\mu_2}^{\mathcal{R}}(\xi; \gamma),$$

(3.30)

so that summing together (3.29) and (3.30), we obtain $\psi_\mu^{\mathcal{R}}(\xi; \gamma)$.   □

This concludes the proof of the Theorem.

4. Proof of Proposition 3.2

In this section, we prove Proposition 3.2. We will consider explicitly the case $\mu \in \mathbb{N}$, since negative μ are dealt with in a similar way.

Set for brevity

$$D_k(\xi) := \alpha \sqrt{|(\xi - \xi_k)(\xi + \xi_k^*)|}, \tag{4.1}$$

$$A_{\mu,k}(\xi) := -D_{\mu+k}^{-1}(\xi) + D_{\mu-k}^{-1}(\xi), \tag{4.2}$$

$$G_{\mu,k}(\xi) := (R j_{\mu+k}(\xi) + R j_{\mu-k}(\xi)) L j_\mu(\xi), \tag{4.3}$$

and note that we can write

$$\mathcal{M}_\mu(\xi) L j_\mu(\xi) = -i\gamma V_0 L j_\mu(\xi) - \gamma^2 \sum_{k \geq 1} |V_k|^2 G_{\mu,k}(\xi). \tag{4.4}$$

The next two lemmas establish useful properties of the functions $G_{\mu,k}(\xi)$ and $A_{\mu,k}(\xi)$.

Lemma 4.1. *Let $k \in \mathbb{N}$ and r sufficiently small. One has*

$$\inf_{\xi \in [\xi_\mu - r, \xi_\mu + r]} A_{\mu,k}(\xi) > 0 \tag{4.5}$$

Proof. By explicit calculation

$$A_{\mu,k}(\xi_\mu) = \begin{cases} \frac{2\sqrt{k}}{\sqrt{4\alpha^2 - k^2}(\sqrt{k(2\alpha - k)} + \sqrt{k(k + 2\alpha)})} > 0 & k < 2\alpha, \\ \frac{4\alpha}{\sqrt{k^2 - 4\alpha^2}(\sqrt{k(k - 2\alpha)} + \sqrt{k(k + 2\alpha)})} > 0 & k > 2\alpha \end{cases} \tag{4.6}$$

so we can conclude by continuity.   □

Lemma 4.2. *If $\xi \in (\xi_\mu, \xi_\mu + r)$, one has*

$$\mathrm{Re}(G_{\mu,k}(\xi)) = \begin{cases} -(D_\mu(\xi) D_{\mu+k}(\xi))^{-1} & 1 \leq k \leq \mu \\ -(D_\mu(\xi) D_{\mu+k}(\xi))^{-1} & \mu + 1 \leq k \leq \lfloor 2\alpha \rfloor \\ \frac{A_{\mu,k}(\xi)}{D_\mu(\xi)} & k \geq \lceil 2\alpha \rceil. \end{cases} \tag{4.7}$$

$$\mathrm{Im}(G_{\mu,k}(\xi)) = \begin{cases} (D_\mu(\xi) D_{\mu-k}(\xi))^{-1} & 1 \leq k \leq \mu \\ (D_\mu(\xi) D_{\mu-k}(\xi))^{-1} & \mu + 1 \leq k \leq \lfloor 2\alpha \rfloor \\ 0 & k \geq \lceil 2\alpha \rceil. \end{cases} \tag{4.8}$$

If $\xi \in (\xi_\mu - r, \xi_\mu]$, one has

$$\mathrm{Re}(G_{\mu,k}(\xi)) = \begin{cases} (D_\mu(\xi)D_{\mu-k}(\xi))^{-1} & 1 \le k \le \mu \\ (D_\mu(\xi)D_{\mu-k}(\xi))^{-1} & \mu \le k \le \lfloor 2\alpha \rfloor \\ 0 & k \ge \lceil 2\alpha \rceil. \end{cases} \qquad (4.9)$$

$$\mathrm{Im}(G_{\mu,k}(\xi)) = \begin{cases} (D_\mu(\xi)D_{\mu+k}(\xi))^{-1} & 1 \le k \le \mu \\ (D_\mu(\xi)D_{\mu+k}(\xi))^{-1} & \mu+1 \le k \le \lfloor 2\alpha \rfloor \\ -\dfrac{A_{\mu,k}(\xi)}{D_\mu(\xi)} & k \ge \lceil 2\alpha \rceil. \end{cases} \qquad (4.10)$$

Proof. Since we are considering the case $\mu \ge 1$, $Lj_\mu(\xi)$ is given by (2.10). Our analysis of $G_{\mu,k}(\xi)$ splits in several cases.

    i) $1 \le k \le \mu$

In this case, $\mu - k \ge 0$ and $\xi_{\mu+k} < \xi_\mu < \xi_{\mu-k}$. By (2.11), we write

$$Rj_{\mu+k}(\xi) + Rj_{\mu-k}(\xi) = D_{\mu+k}^{-1}(\xi)\chi(\xi < \xi_{\mu+k} - r) + D_{\mu-k}^{-1}(\xi)\chi(\xi < \xi_{\mu-k} - r)$$
$$+ iD_{\mu+k}^{-1}(\xi)\chi(\xi > \xi_{\mu+k} + r) + iD_{\mu-k}^{-1}(\xi)\chi(\xi > \xi_{\mu-k} + r). \qquad (4.11)$$

A direct computation gives

$$D_\mu(\xi)G_{k,\mu}(\xi) = D_{\mu-k}^{-1}(\xi)\chi(\xi_\mu - r < \xi \le \xi_\mu) - D_{\mu+k}^{-1}(\xi)\chi(\xi_\mu < \xi < \xi_\mu + r)$$
$$+ i(D_{\mu+k}^{-1}(\xi)\chi(\xi_\mu - r < \xi \le \xi_\mu) + D_{\mu-k}^{-1}(\xi)\chi(\xi_\mu < \xi < \xi_\mu + r)). \qquad (4.12)$$

    ii) $\mu + 1 \le k \le \lfloor 2\alpha \rfloor$

Now, $\mu - k < 0$ and $\max(\xi_{\mu+k}, \xi_{\mu-k}) < \xi_\mu$. Therefore by (2.11), (2.11)

$$Rj_{\mu+k}(\xi) + Rj_{\mu-k}(\xi) = D_{\mu+k}^{-1}(\xi)\chi(\xi < \xi_{\mu+k} - r) + D_{\mu-k}^{-1}(\xi)\chi(\xi > \xi_{\mu-k} + r)$$
$$+ iD_{\mu+k}^{-1}(\xi)\chi(\xi > \xi_{\mu+k} + r) - iD_{\mu-k}^{-1}(\xi)\chi(\xi < \xi_{\mu-k} - r). \qquad (4.13)$$

Moreover,

$$D_\mu(\xi)G_{k,\mu}(\xi) = D_{\mu-k}^{-1}(\xi)\chi(\xi_\mu - r < \xi \le \xi_\mu) - D_{\mu+k}^{-1}(\xi)\chi(\xi_\mu < \xi < \xi_\mu + r)$$
$$+ i(D_{\mu+k}^{-1}(\xi)\chi(\xi_\mu - r < \xi \le \xi_\mu) + D_{\mu-k}^{-1}(\xi)\chi(\xi_\mu < \xi < \xi_\mu + r)). \qquad (4.14)$$

    iii) $k \ge \lceil 2\alpha \rceil$ We have $\mu - k < 0$, $\xi_{\mu+k} < \xi_\mu < \xi_{\mu-k}$ and again

$$Rj_{\mu+k}(\xi) + Rj_{\mu-k}(\xi) = D_{\mu+k}^{-1}(\xi)\chi(\xi < \xi_{\mu+k} - r) + D_{\mu-k}^{-1}(\xi)\chi(\xi > \xi_{\mu-k} + r)$$
$$+ iD_{\mu+k}^{-1}(\xi)\chi(\xi > \xi_{\mu+k} + r) - iD_{\mu-k}^{-1}(\xi)\chi(\xi < \xi_{\mu-k} - r). \qquad (4.15)$$

Therefore,

$$G_{k,\mu}(\xi) = -iA_{\mu,k}(\xi)Lj_\mu(\xi) = \frac{A_{\mu,k}(\xi)}{D_\mu(\xi)}(\chi(\xi_\mu < \xi < \xi_\mu + r) - i\chi(\xi_\mu - r < \xi \le \xi_\mu)). \qquad (4.16)$$

Note that for $\xi \notin (\xi_\mu - r, \xi_\mu + r)$, all $G_{\mu,k}$ are identically zero, and by direct inspection, we deduce (4.7), (4.8), (4.9) and (4.10). $\qquad\square$

Let us introduce the notation

$$K_1(\xi) := \sum_{k=1}^{\lfloor 2\alpha \rfloor} |V_k|^2 (D_{\mu-k}(\xi))^{-1}$$

$$K_2(\xi) := \sum_{k>2\alpha} |V_k|^2 A_{\mu,k}(\xi), \qquad \widetilde{K}_2(\xi) := \sum_{k\geq 1} |V_k|^2 A_{\mu,k}(\xi) \qquad (4.17)$$

Note that $K_1(\xi)$ can vanish if and only if $V_k = 0$ for all $|k| \leq \lfloor 2\alpha \rfloor$ (that is $\underline{V}_{\leq 2\alpha} = 0$). Similarly, $K_2(\xi)$ can vanish if and only if $V_k = 0$ for all $|k| > \lfloor 2\alpha \rfloor$ (i.e. $\overline{V}_{>2\alpha} = 0$), while $\widetilde{K}_2(\xi)$ can vanish if and only if the forcing is constant in time, i.e. $V(\omega t) \equiv V_0$.

Recall the notations introduced in (3.16)–(3.17). We need the following bounds on $K_1(\xi)$, $K_2(\xi)$ and $\widetilde{K}_2(\xi)$.

Lemma 4.3. *There is $c > 0$ such that for all $\mu \in \mathbb{Z}$ and all $\xi \in (\xi_\mu - r, \xi_\mu + r)$ if $K_1(\xi) \neq 0$, then one has*

$$c\sqrt{\frac{\epsilon}{\alpha}}\, \underline{V}_{\leq 2\alpha}^2 \leq K_1(\xi) \leq c\sqrt{\frac{\alpha}{\epsilon}}\|V\|_{L^2}^2 \,. \qquad (4.18)$$

Proof. By the Lipschitz continuity of $D_{\mu-k}^{-1}(\xi)$ in $(\xi_\mu - r, \xi_\mu + r)$, there is $c_1 \geq 0$ such that for all $\xi \in (\xi_\mu - r, \xi_\mu + r)$, we have

$$|D_{\mu-k}^{-1}(\xi) - D_{\mu-k}^{-1}(\xi_\mu)| \leq c_1 \frac{r}{\sqrt{\alpha}} \,.$$

Furthermore, since $r \in (0, \frac{\epsilon}{2\alpha})$ and

$$D_{\mu-k}(\xi_\mu) = \sqrt{|2\alpha - k|k} \,,$$

we have

$$\sum_{k=1}^{\lfloor 2\alpha \rfloor} \frac{|V_k|^2}{D_{\mu-k}(\xi)} \leq C_1 \left(\sup_{k\in\mathbb{N}} |V_k|^2 \sum_{k=1}^{\lfloor 2\alpha \rfloor} \frac{1}{\sqrt{k|k-2\alpha|}} + \frac{r\|V\|_{L^2}^2}{\sqrt{\alpha}} \right)$$

$$\leq C_2 \|V\|_{L^2}^2 \left(\sqrt{\frac{\alpha}{\epsilon}} + \frac{\epsilon}{\alpha\sqrt{\alpha}} \right)$$

$$\leq C_3 \sqrt{\frac{\alpha}{\epsilon}} \|V\|_{L^2}^2 \,.$$

for some constants $C_1, C_2, C_3 > 0$. Similarly,

$$\sum_{k=1}^{\lfloor 2\alpha \rfloor} \frac{|V_k|^2}{D_{\mu-k}(\xi)} \geq C_1 \left(\underline{V}_{\leq 2\alpha}^2 \sum_{k=1}^{\lfloor 2\alpha \rfloor} \frac{1}{\sqrt{k|k-2\alpha|}} - \frac{r\|V\|_{L^2}^2}{\sqrt{\alpha}} \right)$$

$$\geq C_2 \sqrt{\frac{\epsilon}{\alpha}}\, \underline{V}_{\leq 2\alpha}^2 \,,$$

so the assertion follows. $\qquad\square$

Lemma 4.4. *There is $c > 0$ such that for all $\mu \in \mathbb{Z}$ and all $\xi \in (\xi_\mu - r, \xi_\mu + r)$ if $K_2(\xi) \neq 0$, then one has*

$$\frac{c}{\sqrt{\alpha}} \overline{V}_{>2\alpha}^2 \leq K_2(\xi) \leq c\sqrt{\frac{\alpha}{\epsilon}} \|V\|_{L^2}^2. \tag{4.19}$$

Proof. We use again the Lipschitz continuity of $D_{\mu \pm k}^{-1}(\xi)$ in $(\xi_\mu - r, \xi_\mu + r)$ to obtain that there is $c \geq 0$ such that for all $\xi \in (\xi_\mu - r, \xi_\mu + r)$, one has

$$|A_{\mu,k}(\xi) - A_{\mu,k}(\xi_\mu)| \leq c\frac{r}{\sqrt{\alpha}},$$

and hence for all $\xi \in (\xi_\mu - r, \xi_\mu + r)$

$$\Big| \sum_{k>2\alpha} |V_k|^2 A_{\mu,k}(\xi) - \sum_{k>2\alpha} |V_k|^2 A_{\mu,k}(\xi_\mu)\Big| \leq c\|V\|_{L^2} \frac{r}{\sqrt{\alpha}}. \tag{4.20}$$

Now, we have by (4.6)

$$\sum_{k>2\alpha} |V_k|^2 A_{\mu,k}(\xi_\mu) \leq \|V\|_{L^2}^2 \sum_{k>2\alpha} \frac{4\alpha}{\sqrt{k^2 - 4\alpha^2}(\sqrt{k(k-2\alpha)} + \sqrt{k(k+2\alpha)})}$$

$$\leq c_1 \sqrt{\frac{\alpha}{\epsilon}} \|V\|_{L^2}^2,$$

and similarly, if $\hat{k}$ denotes the Fourier mode at which the max in (3.17) is attained, we have

$$\sum_{k>2\alpha} |V_k|^2 A_{\mu,k}(\xi_\mu) \geq \overline{V}_{>2\alpha}^2 \frac{4\alpha}{\sqrt{\hat{k}^2 - 4\alpha^2}(\sqrt{\hat{k}(\hat{k}-2\alpha)} + \sqrt{\hat{k}(\hat{k}+2\alpha)})}$$

$$\geq \frac{c_2}{\sqrt{\alpha}} \overline{V}_{>2\alpha}^2,$$

so the assertion follows combining the latter two with (4.20). □

Remark 4.5. Note that if $V_{\lceil 2\alpha \rceil} \neq 0$, there is a further $1/\sqrt{\epsilon}$ in the lower bound in (4.21).

Lemma 4.6. *There is $c > 0$ such that for all $\mu \in \mathbb{Z}$ and all $\xi \in (\xi_\mu - r, \xi_\mu + r)$ if $\widetilde{K}_2(\xi) \neq 0$, then one has*

$$\frac{c}{\sqrt{\alpha}} \overline{V}^2 \leq \widetilde{K}_2(\xi) \leq c\sqrt{\frac{\alpha}{\epsilon}} \|V\|_{L^2}^2. \tag{4.21}$$

Proof. The lower bound follows exactly as the lower bound in Lemma 4.4. As for the upper bound, we simply add to the upper bound for $K_2(\xi)$ the quantity

$$\sum_{k=1}^{\lfloor 2\alpha \rfloor} |V_k|_2 A_{\mu,k}(\xi) \leq c_1 \|V\|_{L^2}^2 \left(\frac{r}{\sqrt{\alpha}} + \sum_{k=1}^{\lfloor 2\alpha \rfloor} \frac{4\alpha}{\sqrt{k^2 - 4\alpha^2}(\sqrt{k(k-2\alpha)} + \sqrt{k(k+2\alpha)})} \right)$$

$$\leq c_2 \|V\|_{L^2}^2 \frac{8\alpha^2}{\sqrt{\epsilon \lfloor 4\alpha \rfloor}(\sqrt{\epsilon \lfloor 2\alpha \rfloor} + 2\lfloor \alpha \rfloor)} \leq c_3 \sqrt{\frac{\alpha}{\epsilon}} \|V\|_{L^2}^2$$

$$\tag{4.22}$$

so the assertion follows. □

Now, we are in position to prove Proposition 3.2.

Proof of Proposition 3.2. The case $\mu = 0$ is easier and can be studied separately. Indeed by a direct computation, we see that

$$D_0(\xi)G_{0,k}(\xi) = D_{-k}^{-1}(\xi)\chi(\xi > 1 - r) + D_k^{-1}(\xi)\chi(\xi < -1 + r)$$
$$+ \mathrm{i}\left(-D_{-k}^{-1}(\xi)\chi(\xi < -1 + r) + D_k^{-1}(\xi)\chi(\xi > 1 - r)\right).$$

In particular, $\mathrm{Re}(G_{0,k}(\xi)) > 0$ for all $k \in \mathbb{Z}$, so that by (4.4), we have

$$|1 - \mathcal{M}_0(\xi)Lj_0(\xi)| \geq 1 - \mathrm{Re}\left(\mathcal{M}_0(\xi)Lj_0(\xi)\right) = 1 + \gamma^2 \sum_{k \geq 1} |V_k|^2 \mathrm{Re}(G_{0,k}(\xi)) > 1.$$

Now, we study the case $\mu \geq 1$. If $\xi \in (\xi_\mu - r, \xi_\mu]$ by Lemma 4.2, we see that

$$|1 - \mathrm{Re}(\mathcal{M}_\mu(\xi)Lj_\mu(\xi))| = 1 + \gamma^2 \sum_{k \geq 1} |V_k|^2 \mathrm{Re}(G_{\mu,k}) > 1, \tag{4.23}$$

which entails

$$\inf_{\xi \in (\xi_\mu, \xi_\mu + r)} |1 - \mathcal{M}_\mu(\xi)Lj_\mu(\xi)| > 1. \tag{4.24}$$

For all $\xi \in (\xi_\mu, \xi_\mu + r)$, we claim that

$$|1 - \mathcal{M}_\mu(\xi)Lj_\mu(\xi)| \geq \frac{1}{2}, \tag{4.25}$$

for γ small enough. To prove it, we use again Lemma 4.2.

We have

$$|1 - \mathcal{M}_\mu(\xi)Lj_\mu(\xi)|^2 = \left|1 + \gamma^2 \sum_{k \geq 1} |V_k|^2 \mathrm{Re}(G_{\mu,k}(\xi)) + \mathrm{i}\left(-\gamma \frac{V_0}{D_\mu(\xi)}\right.\right.$$

$$\left.\left. + \gamma^2 \sum_{k \geq 1} |V_k|^2 \mathrm{Im}(G_{\mu,k}(\xi))\right)\right|^2$$

$$= \left|1 + \gamma^2 \frac{\widetilde{K}_2(\xi)}{D_\mu(\xi)} - \gamma^2 \frac{K_1(\xi)}{D_\mu(\xi)} + \mathrm{i}\left(\gamma \frac{V_0}{D_\mu(\xi)} + \gamma^2 \frac{K_1(\xi)}{D_\mu(\xi)}\right)\right|^2$$

$$= \left(1 - \gamma^2 \frac{K_1(\xi)}{D_\mu(\xi)} + \gamma^2 \frac{\widetilde{K}_2(\xi)}{D_\mu(\xi)}\right)^2 + \left(\gamma \frac{V_0}{D_\mu(\xi)} + \gamma^2 \frac{K_1(\xi)}{D_\mu(\xi)}\right)^2. \tag{4.26}$$

Now if

$$\left|1 - \frac{\gamma^2}{D_\mu(\xi)}(K_1(\xi) - \widetilde{K}_2(\xi))\right| \geq \frac{1}{2},$$

we have

$$\text{r.h.s of (4.26)} \geq \frac{1}{4} + \left(\gamma \frac{V_0}{D_\mu(\xi)} + \gamma^2 \frac{K_1(\xi)}{D_\mu(\xi)}\right)^2 \geq \frac{1}{4}, \tag{4.27}$$

while if

$$\left| 1 - \frac{\gamma^2}{D_\mu(\xi)}(K_1(\xi) - \widetilde{K}_2(\xi)) \right| \leq \frac{1}{2},$$

then

$$\frac{\gamma^2}{D_\mu(\xi)}K_1(\xi) \geq \frac{1}{2} + \frac{\gamma^2}{D_\mu(\xi)}\widetilde{K}_2(\xi),$$

and moreover using Lemmata 4.3 and 4.6, we have also

$$\sqrt{\frac{\epsilon}{\alpha}}\frac{1}{4\|V\|_{L^2}^2} \leq \frac{\gamma^2}{D_\mu(\xi)} \leq \frac{3\sqrt{\alpha}}{2(-\underline{V}_{\leq 2\alpha}^2\sqrt{\epsilon} + \overline{V}^2)}. \tag{4.28}$$

But then

$$\text{r.h.s of (4.26)} \geq \left(\frac{\gamma}{D_\mu(\xi)}V_0 + \frac{\gamma^2}{D_\mu(\xi)}K_1(\xi) \right)^2 \geq \left(\frac{1}{2} + \frac{\gamma}{D_\mu(\xi)}V_0 + \frac{\gamma^2}{D_\mu(\xi)}\widetilde{K}_2(\xi) \right)^2$$

$$\geq \frac{1}{4}, \tag{4.29}$$

which is obvious if $V_0 \geq 0$, while if $V_0 < 0$, we need to impose

$$\gamma < c\sqrt{\frac{\epsilon}{\alpha}}\frac{|V_0|}{\|V\|_{L^2}^2} \tag{4.30}$$

where c is the constant appearing in Lemma 4.6, in order to obtain

$$\frac{1}{2} + \frac{\gamma}{D_\mu(\xi)}V_0 + \frac{\gamma^2}{D_\mu(\xi)}\widetilde{K}_2(\xi) < -\frac{1}{2}$$

Thus, the assertion follows. □

5. Proof of Proposition 3.3

Here, we prove Proposition 3.3.

Proof of Proposition 3.3. First of all we note that by (2.7) and (3.11), we have

$$\sup_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |j_\mu^{\mathcal{R}}(\xi)| = \sup_{\xi \in (\xi_\mu - r, \xi_\mu + r)} \left| \frac{j_\mu(\xi)}{1 - \mathcal{M}_\mu(\xi)j_\mu(\xi)} \right|$$

$$= \left(\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |D_\mu(\xi) - \mathcal{M}_\mu(\xi)| \right)^{-1}. \tag{5.1}$$

Note that

$$\mathcal{M}_\mu(\xi) = -\mathrm{i}\gamma V_0 - \gamma^2 D_\mu(\xi) \sum_{k \geq 1} |V_k|^2 G_{k,\mu}(\xi).$$

So thanks to Lemma 4.2 and using the notation in (4.17), we can write

$$\mathcal{M}_\mu(\xi) = -\mathrm{i}\gamma V_0 + \chi(\xi_\mu < \xi < \xi_\mu + r)\left(\gamma^2(K_1(\xi) - \widetilde{K}_2(\xi)) - \mathrm{i}\gamma^2 K_1(\xi) \right)$$

$$-\chi(\xi_\mu - r < \xi < \xi_\mu)\left(\mathrm{i}\gamma^2(K_1(\xi) - \widetilde{K}_2(\xi)) + \gamma^2 K_1(\xi) \right). \tag{5.2}$$

Therefore,

$$
\begin{aligned}
|D_\mu(\xi) - \mathcal{M}_\mu(\xi)| = {}& \chi(\xi_\mu < \xi < \xi_\mu + r)|D_\mu(\xi) - \gamma^2(K_1(\xi) - \widetilde{K}_2(\xi)) \\
& - \mathrm{i}(\gamma V_0 + \gamma^2 K_1(\xi))| \\
& + \chi(\xi_\mu - r < \xi < \xi_\mu)|D_\mu(\xi) + \gamma^2 K_1(\xi) \\
& - \mathrm{i}(\gamma V_0 + \gamma^2(K_1(\xi) - \widetilde{K}_2(\xi)))|,
\end{aligned}
$$

whence

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |D_\mu(\xi) - \mathcal{M}_\mu(\xi)| \geq \min\left(\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} \mathrm{d}(\xi), \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} s(\xi) \right),
\tag{5.3}
$$

with

$$
\mathrm{d}(\xi) := |D_\mu(\xi) - \gamma^2(K_1(\xi) - \widetilde{K}_2(\xi)) - \mathrm{i}(\gamma V_0 + \gamma^2 K_1(\xi))| \tag{5.4}
$$
$$
s(\xi) := |D_\mu(\xi) + \gamma^2 K_1(\xi) - \mathrm{i}(\gamma V_0 + \gamma^2(K_1(\xi) - \widetilde{K}_2(\xi)))|. \tag{5.5}
$$

To estimate $d(\xi)$ and $s(\xi)$, we treat separately the cases $V_0 = 0$ and $V_0 \neq 0$. Moreover for the first case, we consider two sub-cases, namely either $\underline{V}_{\leq 2\alpha} \neq 0$ or $\underline{V}_{\leq 2\alpha} = 0$.

Case I.1: $\underline{V_0 = 0}, \underline{V_{\leq 2\alpha} \neq 0}$. By Lemma 4.3, there is a constant $c > 0$ such that

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} d(\xi) \geq \gamma^2 \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |K_1(\xi)| \geq c\gamma^2 \sqrt{\frac{\epsilon}{\alpha}} \, \underline{V}_{\leq 2\alpha}^2 \tag{5.6}
$$

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} s(\xi) \geq \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |D_\mu(\xi) + \gamma^2 K_1(\xi)|
$$

$$
\geq \gamma^2 \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |K_1(\xi)| \geq c\gamma^2 \sqrt{\frac{\epsilon}{\alpha}} \, \underline{V}_{\leq 2\alpha}^2. \tag{5.7}
$$

Case I.2: $\underline{V_0 = 0}, \underline{V_{\leq 2\alpha} = 0}$. In this case, $K_1(\xi) = 0$. On the other hand by Lemma 4.4, there is a constant $c > 0$ such that

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} \mathrm{d}(\xi) \geq \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |D_\mu(\xi) + \gamma^2 K_2(\xi)|
$$

$$
\geq \gamma^2 \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |K_2(\xi)| \geq \gamma^2 \frac{c}{\sqrt{\alpha}} \overline{V}_{>2\alpha}^2 \tag{5.8}
$$

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} s(\xi) \geq \gamma^2 \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |K_2(\xi)| \geq \gamma^2 \frac{c}{\sqrt{\alpha}} \overline{V}_{>2\alpha}^2. \tag{5.9}
$$

Combining (5.1), (5.3), (5.6), (5.7), (5.8), (5.9) gives the second line of (3.18).

Case II: $\underline{V_0 \neq 0}$. We have

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} d(\xi) \geq \gamma \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |V_0 + \gamma K_1(\xi)| \geq \gamma \frac{|V_0|}{2} \tag{5.10}
$$

$$
\inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} s(\xi) \geq \gamma \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |V_0 + \gamma K_1(\xi) - \gamma K_2(\xi)|
$$

$$
\geq \gamma \inf_{\xi \in (\xi_\mu - r, \xi_\mu + r)} |V_0 - \gamma K_2(\xi)| \geq \frac{\gamma |V_0|}{2}. \tag{5.11}
$$

The last inequality is always satisfied if $\overline{V}_{>2\alpha} = 0$, while otherwise, we need to require

$$\gamma \leq \frac{1}{c}\sqrt{\frac{\epsilon}{\alpha}}\frac{|V_0|}{\|V\|_{L^2}^2} \tag{5.12}$$

by Lemma 4.4.

Combining (5.1), (5.3), (5.10), (5.11), and (5.12), the result follows. $\qquad\square$

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Appendix A. Proof of Proposition 1.6

In this "Appendix", we give a brief account on how to prove Proposition 1.6. We just outline the strategy and for most of the details we refer to our previous paper [11]. We consider only the case $V_0 = 0$, which is the most difficult case.

For any function $F = F(t, x_1, x_2, \ldots)$, we write

$$F = O\left(\frac{1}{\sqrt{t}}\right) \iff \frac{C_1}{\sqrt{t}} \leq \sup_{x_1, x_2, \ldots} F \leq \frac{C_2}{\sqrt{t}}$$

for some $C_1, C_2 > 0$.

Let us set for brevity

$$\mathcal{D} = \mathcal{D}(\xi, t, t_0, \omega) := \psi_\infty(\xi, \omega t) - \psi_{t_0}(\xi, t). \tag{A.1}$$

By (1.6) and (1.9), we readily obtain the following equation:

$$(\mathbb{1} + ihW_{t_0})\mathcal{D} = -q^{[0]}, \tag{A.2}$$

where

$$q^{[0]} = q^{[0]}(\xi, t, t_0, \omega) := \mathrm{i}h(W_\infty - W_{t_0})\psi_\infty$$

$$= \mathrm{i}h \int_{-\infty}^{t_0} \mathrm{d}\tau\, J_0(g(t-\tau))e^{\mathrm{i}g\xi(t-\tau)}V(\tau)\psi_\infty(\xi, \tau)\,.$$
(A.3)

Therefore, we have to prove that for any ω satisfying Hypothesis 1.1, there are $C_1, C_2 > 0$ such that

$$\mathcal{D} = O\left(\frac{1}{\sqrt{t-t_0}}\right)\,.$$

We write (here $\psi_{\infty,\mu}$ denotes the μ-th Fourier coefficient of ψ_∞)

$$q^{[0]} = \mathrm{i}h \sum_{\mu,k\in\mathbb{Z}} \psi_{\infty,\mu} V_k e^{\mathrm{i}\omega(\mu+k)t} \int_{t-t_0}^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega(\mu+k))\tau}$$

$$= \sum_{k\in\mathbb{Z}} \mathrm{i}h(\psi_\infty * V)_k e^{\mathrm{i}\omega k t} \int_{t-t_0}^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega k)\tau}$$

$$=: \sum_{k\in\mathbb{Z}} e^{\mathrm{i}\omega k t} q_k^{[0]}(t, t_0, \xi, \omega)\,,$$
(A.4)

where the last line is understood as the definition of the coefficients $q_k^{[0]}(t, t_0, \xi, \omega) = q_k^{[0]}$.

The first property to establish is the decay of $q^{[0]}$. This is done first singularly on each coefficient $q_k^{[0]}$, and then promoted to $q^{[0]}$ by analyticity. Using [11, Lemma A.6], we compute

$$\int_{t-t_0}^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega k)\tau} = \frac{c(g)}{\sqrt{t-t_0}} \sum_{\sigma=\pm 1} \sigma \frac{e^{\mathrm{i}(g(\xi+\sigma)-\omega k)(t-t_0)}}{\sqrt{|g(\xi+\sigma)-\omega k|}}$$

$$+ O\left(\frac{1}{(t-t_0)\sqrt{1-\tau^2}}\right)\,,$$
(A.5)

where $c(g) > 0$ is a constant. The divergences appearing in the above formula get however cancelled. Indeed we observe by (1.9)

$$\delta_{k,0} = \psi_{\infty,k} + \mathrm{i}h(\psi_\infty * V)_k \int_0^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega k)\tau}\,,$$

so the divergences of

$$\int_0^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(\xi-\omega k)\tau}$$

must coincide with the zeros of $(\psi_\infty * V)_k$, and we can write

$$\mathrm{i}h(\psi_\infty * V)_k = -(\psi_{\infty,k} - \delta_{k,0})\left(\int_0^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega k)\tau}\right)^{-1}$$

whence

$$-q_k^{[0]} = (\psi_{\infty,k} - \delta_{k,0})\frac{\int_{t-t_0}^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega k)\tau}}{\int_0^{\infty} \mathrm{d}\tau\, J_0(g\tau)e^{\mathrm{i}(g\xi-\omega k)\tau}}$$
(A.6)

is bounded with the desired decay. (Notice that the denominator can be computed as in (2.3).) We obtain

$$q_k^{[0]}(t, t_0, \xi, \omega)$$
$$= (\psi_{\infty,k} - \delta_{k,0}\psi_0) \left[\sum_{\sigma=\pm 1} \frac{e^{i(g\xi - \omega k + g\sigma)(t-t_0)} \widetilde{r}(t, t_0, \xi, \omega)}{\sqrt{t-t_0}} + O\left(\frac{1}{t-t_0}\right) \right].$$

(A.7)

Next, we invert the compact operator $\mathbb{1} + ihW_{t_0}$ in a standard way. It suffices to prove that successive applications of W_{t_0} preserve the decay of $q^{[0]}$. To this end, we define

$$q^{[1]} := W_{t_0} q^{[0]}$$

and represent (see [11, Lemma 3.4])

$$q^{[1]} = \sum_{k \in \mathbb{Z}} e^{ik\omega t} q_k^{[0]}(\xi, t, t_0, \omega), \tag{A.8}$$

$$q_k^{[1]}(\xi, t, t_0, \omega) := \int_0^{t-t_0} dt' J_0(gt') e^{i(g\xi - \omega k)t'} (V * q^{[0]})_k.$$

Combining (A.7) with the last definition, we get

$$q_k^{[1]} = \sum_{\mu \in \mathbb{Z}, \sigma = \pm 1} V_{k-\mu}(\psi_{\infty,k} - \delta_{n,0}) e^{i(g\xi - \omega\mu + \sigma)(t-t_0)}$$
$$\int_0^{t-t_0} dt' \frac{J_0(gt')}{\sqrt{t-t_0-t'}} e^{i(\omega(k-\mu)-\sigma)t'}. \tag{A.9}$$

To evaluate the inner integral, we use [11, Lemma A.6] according to which

$$\int_0^{t-t_0} dt' \frac{J_0(gt')}{\sqrt{t-t_0-t'}} e^{i(\omega(k-\mu)-\sigma)t'} = \begin{cases} O(1) & \omega(k-\mu) \in \{-2g, 0, 2g\} \\ O\left(\frac{1}{\sqrt{t-t_0}}\right) & \text{otherwise} \end{cases}.$$

(A.10)

The first case never occurs, since $|\omega(k-\mu)| = 2g$ is excluded by (1.10) and $\omega(k-\mu) = 0$ since we are considering the case $V_0 = 0$. Therefore,

$$q_k^{[1]}(t, t_0, \xi, \omega) = O\left(\frac{1}{\sqrt{t-t_0}}\right), \tag{A.11}$$

and again one can promote the decay to the entire expansion (A.8) by analyticity.

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