Spectral Asymptotics for Kinetic Brownian Motion on Surfaces of Constant Curvature

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Abstract. The kinetic Brownian motion on the sphere bundle of a Riemannian manifold $\mathbb{M}$ is a stochastic process that models a random perturbation of the geodesic flow. If $\mathbb{M}$ is an orientable compact constantly curved surface, we show that in the limit of infinitely large perturbation the L^2 -spectrum of the infinitesimal generator of a time-rescaled version of the process converges to the Laplace spectrum of the base manifold.

1. Introduction

Kinetic Brownian motion is a stochastic process that describes a stochastic perturbation of the geodesic flow and has the property that the perturbation affects only the direction of the velocity but preserves its absolute value. It has been studied in the past years by several authors in pure mathematics [1, 5, 10, 14, 26] but versions of this diffusion process have been developed independently as surrogate models for certain textile production processes (see, e.g., [18, 19, 23]).

Kinetic Brownian motion $(Y_t^\gamma)_{t \geq 0}$ in the setting of a compact Riemannian manifold $(\mathbb{M}, g)$ can be informally described in the following way: $(Y_t^\gamma)_{t \geq 0}$ is a stochastic process with continuous paths described by a stochastic perturbation of the geodesic flow on the sphere bundle $S\mathbb{M} = \{\xi \in T\mathbb{M}, \|\xi\|_g = 1\}$. More precisely, if we denote the geodesic flow vector field by X and the (nonnegative) Laplace operator on the fibers of $S\mathbb{M}$ by $\Delta_{\mathbb{S}}$, then the kinetic Brownian motion is generated by the differential operator

$$-X + \frac{1}{2}\gamma\Delta_{\mathbb{S}}: L^2(S\mathbb{M}) \rightarrow L^2(S\mathbb{M}).$$

The connection to the stochastic process $(Y_t^\gamma)_{t \geq 0}$ is given via

$$e^{-t(-X + \frac{1}{2}\gamma\Delta_{\mathbb{S}})}f(x) = \mathbb{E}_x[f(Y_t^\gamma)] \quad \text{with} \quad f \in L^2(S\mathbb{M}), x \in S\mathbb{M},$$

where $\mathbb{E}_x$ denotes the expected value if the stochastic process is started at $x \in S\mathbb{M}$. Observe that the parameter $\gamma > 0$ controls the strength of the stochastic perturbation and it is a natural question to study the behavior of $-X + \frac{1}{2}\gamma\Delta_S$ and Y_t^γ in the regimes $\gamma \rightarrow 0$ as well as $\gamma \rightarrow \infty$. Drouot [10] has studied the convergence of the discrete spectrum of $-X + \frac{1}{2}\gamma\Delta_S$ in the limit $\gamma \rightarrow 0$ for negatively curved manifolds and has shown that it converges to the Pollicott–Ruelle resonances of the geodesic flow. These resonances are a replacement of the spectrum of X since its L^2 -spectrum is equal to $i\mathbb{R}$ and they can be defined in various generalities of hyperbolic flows as pole of the meromorphically continued resolvent [6, 8, 9, 11, 15, 27]. A more general framework of semiclassical subelliptic operators that includes the kinetic Brownian motion for $\gamma \rightarrow 0$ has been established by Smith [30]. In the limit of large random noise Li [26] and Angst-Bailleul-Tardif [1] proved that $\pi(Y_{\gamma t}^\gamma)$ converges weakly to the Brownian motion on $\mathbb{M}$ with speed 2 as $\gamma \rightarrow \infty$ where $\pi: S\mathbb{M} \rightarrow \mathbb{M}$ is the projection. This rescaled kinetic Brownian motion is generated by $P_\gamma = -\gamma X + \frac{1}{2}\gamma^2\Delta_S$ whereas the Brownian motion on the base manifold is generated by the Laplace operator $\frac{1}{2}\Delta_{\mathbb{M}}$. Therefore, one may conjecture that the discrete spectrum of P_γ converges to the Laplace spectrum. We will give a proof of this fact in the case of constant curvature surfaces:

Theorem 1. *Let $(\mathbb{M}, g)$ be an orientable compact surface of constant curvature. For every $\eta \in \sigma(\Delta_{\mathbb{M}})$ with multiplicity n there is an analytic function $\lambda_\eta:]r_\eta, \infty[\rightarrow \mathbb{C}$ such that $\lambda_\eta(\gamma)$ is an eigenvalue of P_γ with multiplicity at least n and $\lambda_\eta(\gamma) \rightarrow \eta$ as $\gamma \rightarrow \infty$.*

Note that this theorem does not imply that in a compact set all eigenvalues of P_γ are close to eigenvalues of the Laplacian (see Remark 3.2 for a discussion of the problems that prevent us from proving this stronger statement).

Another question to ask is whether the kinetic Brownian motion converges to equilibrium, i.e.,

$$\mathbb{E}_x[f(Y_{\gamma t}^\gamma)] \xrightarrow{t \rightarrow \infty} \int_{S\mathbb{M}} f.$$

Baudoin–Tardif [5] showed exponential convergence, i.e.

$$\left\| e^{-tP_\gamma} f - \int_{S\mathbb{M}} f \right\| \leq C e^{-C_\gamma t} \left\| f - \int_{S\mathbb{M}} f \right\|, \quad f \in L^2(S\mathbb{M}).$$

We should point out that the given rate C_γ converges to 0 as $\gamma \rightarrow \infty$, but they conjecture that the optimal rate converges to the spectral gap of $\Delta_{\mathbb{M}}$ which is the smallest nonzero Laplace eigenvalue η_1 (see [5, Section 3.1]). A direct consequence of Theorem 1 shows that the optimal rate C_γ is less than $\Re \lambda_{\eta_1}(\gamma)$ for surfaces of constant curvature. Hence $\limsup_{\gamma \rightarrow \infty} C_\gamma \leq \eta_1$. For a more explicit study of the convergence towards equilibrium, we prove a spectral expansion and explicit error estimates in the case of constant negative curvature in [25].

Note that a problem related to the kinetic Brownian motion in $S\mathbb{M}$ is the study of the hypoelliptic Laplacian on $T\mathbb{M}$ introduced by Bismut [3]. Like

the kinetic Brownian motion the hypoelliptic Laplacian interpolates between the geodesic flow and the Brownian motion. In [4, Chapter 17], Bismut and Lebeau prove the convergence of the spectrum of the hypoelliptic Laplacian to the spectrum of the Laplacian on $\mathbb{M}$ using semiclassical analysis. It seems plausible that their techniques can also be transferred to the setting of kinetic Brownian motion and might give the spectral convergence without any curvature restriction. The purpose of this article is, however, not to attack this general setting but show that the assumption of constant curvature allows to drastically reduce the analytical difficulties. In fact, we are able to reduce the problem to standard perturbation theory. This is also the reason why we are able to obtain explicit error estimates in [25].

In the limit $\gamma \rightarrow \infty$, one would like to consider the geodesic vector field as a perturbation of the spherical Laplacian. The major difficulties of the proof are that $\frac{1}{\gamma}X$ is not a small perturbation in comparison with $\Delta_{\mathbb{S}}$ and that the unperturbed eigenspaces are infinite dimensional. After the spectral decomposition with respect to some operator Ω described below there is a precise way to consider X as small operator in any eigenspace of Ω . Moreover, the eigenspaces of $\Delta_{\mathbb{S}}$ become finite dimensional which simplifies the situation drastically. We also want to point out that the eigenvalues of $\Delta_{\mathbb{M}}$ appear in the second derivatives of the evolving eigenvalues in the perturbation expansion, not as their limits. This is one reason why we do not obtain that in a compact set all eigenvalues of P_{γ} are close to eigenvalues of the Laplacian $\Delta_{\mathbb{M}}$.

Let us sketch the spectral decomposition mentioned above: By the assumption of constant curvature we have a three-dimensional Lie algebra $\mathfrak{g} = \langle X, X_{\perp}, V \rangle_{\mathbb{C}}$ of vector fields on $S\mathbb{M}$. Denoting the Gaussian curvature by K , the operator $\Omega = -X^2 - X_{\perp}^2 - KV^2$ commutes with $\mathfrak{g}$ and P_{γ} and its spectrum is a discrete set, but the multiplicities might be infinite. Because of the commutation the generator P_{γ} preserves the decomposition of $L^2(S\mathbb{M})$ in eigenspaces of Ω , and we can study the restriction of P_{γ} on each occurring eigenspace separately. In each of these eigenspaces, the spectral asymptotics in the limit $\gamma \rightarrow \infty$ of P_{γ} can then be handled by standard perturbation theory of an operator family of type (A) in the sense of Kato. For the calculations, it will be important that each eigenspace of Ω can be further split into the eigenspaces of the vector field V which correspond to the Fourier modes in the fibers of $S\mathbb{M} \rightarrow \mathbb{M}$ which are just circles as $\dim \mathbb{M} = 2$.

The article is organized as follows: We will give a short overview over the kinetic Brownian motion and the connection between constant curvature surfaces and the global analysis of sphere bundles of constant curvature surfaces in Sects. 2.1 and 2.2. After that we will recall a few results of perturbation theory for unbounded linear operators (Sect. 2.3) which are mostly taken from [22]. Afterward, we will give the proof of the convergence of the spectra (Sect. 3).

2. Preliminaries

2.1. Kinetic Brownian Motion

Let $\mathbb{M}$ be a compact Riemannian manifold of dimension $d \geq 2$ with sphere bundle $S\mathbb{M} = \{(x, v) \in T\mathbb{M} \mid \|v\| = 1\}$. We introduce the spherical Laplacian Δ_S as follows: For every $x \in \mathbb{M}$ the tangent space $T_x\mathbb{M}$ is a Euclidean vector space via the Riemannian metric and $S_x\mathbb{M} = \{v \in T_x\mathbb{M} \mid \|v\| = 1\}$ is a submanifold of $T_x\mathbb{M}$. The inner product on $T_x\mathbb{M}$ induces a Riemannian structure on $S_x\mathbb{M}$. Hence, the (positive) Laplace-Beltrami operator $\Delta_S(x) := \Delta_{S_x\mathbb{M}}$ of $S_x\mathbb{M}$ defines an operator $C^\infty(S_x\mathbb{M}) \rightarrow C^\infty(S_x\mathbb{M})$. We now obtain the spherical Laplace operator Δ_S by

$$\Delta_S : C^\infty(S\mathbb{M}) \rightarrow C^\infty(S\mathbb{M}), \quad \Delta_S f(x, v) := (\Delta_S(x)f(x, \cdot))(v).$$

For $(x, v) \in S\mathbb{M}$ and $w \in T_{(x,v)}S\mathbb{M}$, we define $\theta_{(x,v)}(w) = g_x(v, (T_{(x,v)}\pi)w)$ where $\pi : S\mathbb{M} \rightarrow \mathbb{M}$ is the projection, $T_{(x,v)}\pi$ its tangent map at the point (x, v) , and g is the Riemannian metric on $\mathbb{M}$. Then θ is a 1-form on $S\mathbb{M}$ and $\nu = \theta \wedge (d\theta)^{d-1}$ defines the Liouville measure on $S\mathbb{M}$ which is invariant under the geodesic flow $\phi_t : S\mathbb{M} \rightarrow S\mathbb{M}$. The vector field $X = \frac{d}{dt}|_{t=0} \phi_t^*$ is called the geodesic vector field.

Let us consider the operator $P_\gamma = -\gamma X + \frac{1}{2}\gamma^2 \Delta_S$ with domain $\text{dom}(P_\gamma) = \{u \in L^2(S\mathbb{M}) \mid P_\gamma u \in L^2(S\mathbb{M})\}$ for $\gamma > 0$. Note that the action of P_γ has to be interpreted in the sense of distributions. We first want to collect some properties of P_γ .

Proposition 2.1. *P_γ is a hypoelliptic operator with*

$$\|f\|_{H^{2/3}} \leq C(\|f\|_{L^2} + \|P_\gamma f\|_{L^2}) \quad \text{for } f \in \text{dom}(P_\gamma).$$

P_γ is accretive (i.e., $\Re\langle P_\gamma f, f \rangle \geq 0$) and coincides with the closure of $P_\gamma|_{C^\infty}$. Therefore, P_γ has compact resolvent on $L^2(S\mathbb{M})$, discrete spectrum with eigenspaces of finite dimension, and the spectrum is contained in the right half plane. P_γ generates a positive strongly continuous contraction semigroup e^{-tP_γ} .

Proof. See the appendix. □

2.2. Surfaces of Constant Curvature

Let $\mathbb{M}$ be an orientable compact Riemannian manifold of dimension 2 and constant curvature and let K be the Gaussian curvature. Since $\mathbb{M}$ has finitely many connected components, let us assume without loss of generality that $\mathbb{M}$ is connected. We follow the notation of [28]. Let X be the geodesic vector field on $S\mathbb{M}$, and let V be a vertical vector field such that $\Delta_S = -V^2$. We define $X_\perp = [X, V]$. We then have the commutator relations $X = [V, X_\perp]$ and $[X, X_\perp] = -KV$. In particular, $\mathfrak{g} := \mathbb{C}X \oplus \mathbb{C}X_\perp \oplus \mathbb{C}V$ is a Lie algebra. The Casimir operator Ω is defined as $\Omega = -X^2 - X_\perp^2 - KV^2$, and it is routine to check that

$$[\Omega, X] = [\Omega, X_\perp] = [\Omega, V] = 0$$

using the above commutator relations. The Laplace operator $\Delta_{S\mathbb{M}}$ of $S\mathbb{M}$ for the metric which is declared by the requirement that the frame $\{X, X_\perp, V\}$ is an orthonormal basis (i.e. $\Delta_{S\mathbb{M}} = -X^2 - X_\perp^2 - V^2$) is an elliptic operator on the compact manifold $S\mathbb{M}$ and hence it admits a discrete spectrum. Since $\Delta_{S\mathbb{M}}$ is symmetric the eigenspaces are orthogonal and since $\Delta_{S\mathbb{M}}$ is nonnegative each eigenvalue is nonnegative. Both operators Ω and V leave these eigenspaces invariant. Thus we have the following decomposition:

$$L^2(S\mathbb{M}) = \bigoplus_{k \in \mathbb{Z}, \eta \in \sigma(\Omega)} V_{\eta, k}$$

where $V_{\eta, k} = \{u \in C^\infty(S\mathbb{M}) \mid Vu = iku, \Omega u = \eta u\}$ is finite-dimensional and the sum is countable. Notice that $V_{\eta, k}$ is a joint eigenspace of $\Delta_{S\mathbb{M}}$ and V , which we choose to label with the eigenvalue η of Ω , instead of using the eigenvalue $\eta + (1-K)k^2$ of $\Delta_{S\mathbb{M}}$. Since V is skew-symmetric, the decomposition is orthogonal. The subspace $V_\eta = \bigoplus_{k \in \mathbb{Z}} V_{\eta, k}$ is an eigenspace of Ω which is invariant under all three vector fields $X, X_\perp, V$ and in particular invariant under Δ_S .

Remark 2.2. Note that $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{so}(3)$, $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$, $\mathfrak{sl}_2(\mathbb{R})$ if $K > 1$, $K = 0$, $K < 0$, respectively. We are essentially decomposing the representation of $\mathfrak{g}$ on $L^2(S\mathbb{M})$ into irreducible ones. In fact in all three cases $S\mathbb{M}$ can be written as $\Gamma \backslash G$ for some torsion free, discrete, cocompact subgroup $\Gamma \subseteq G$ with $G \in \{SO(3), \mathbb{R}^2 \rtimes SO(2), PSL_2(\mathbb{R})\}$. The decomposition $L^2(S\mathbb{M}) = \bigoplus V_\eta$ can be seen as a Plancherel decomposition of this space and $V_\eta = \bigoplus V_{\eta, k}$ as the decomposition into K -types or weights, respectively. In all three cases the irreducible representations have explicit realizations on certain L^2 -spaces (see, e.g., [31, Ch. 8] for $\mathfrak{sl}_2$), and one could go on by analyzing those but they do not contain more information than the abstract decomposition we provided here for all three cases at once. We would like to note that this harmonic analysis point of view was our original approach motivated by previous works that used similar techniques for geodesic flows [7, 13, 16, 17, 24].

Let us furthermore define $X_\pm := \frac{1}{2}(X \pm iX_\perp)$. We then have the commutator relations

$$[V, X_\pm] = \pm iX_\pm \quad \text{and} \quad [X_+, X_-] = \frac{1}{2}iKV.$$

Hence, $X_\pm: V_{\eta, k} \rightarrow V_{\eta, k \pm 1}$. Moreover, $X_\pm^* = -X_\mp$ and $\Omega = -2X_+X_- - 2X_-X_+ - KV^2 = -4X_+X_- + iKV - KV^2$.

The next lemma is crucial for our main result as it connects the eigenvalues η of Ω to the spectrum of the Laplace operator of the base manifold $\mathbb{M}$.

Lemma 2.3. *Let $\Delta_{\mathbb{M}}$ be the Laplace–Beltrami operator of $\mathbb{M}$. Then*

$$\sigma(\Delta_{\mathbb{M}}) = \{\eta \in \sigma(\Omega) \mid V_{\eta, 0} \neq 0\} = \sigma(\Omega) \cap \mathbb{R}_{\geq 0}.$$

Moreover, $\mathfrak{g}$ acts trivially on $V_{0, 0}$ and for $\eta \in \sigma(\Delta_{\mathbb{M}})$, $\eta > 0$, there is a closed $\mathfrak{g}$ -invariant subspace V'_η of V_η such that $V'_{\eta, k} := V_{\eta, k} \cap V'_\eta$ satisfies $\dim V'_{\eta, k} \leq 1$

and $\dim V'_{\eta,0} = 1$ and V_η is isomorphic to m_η orthogonal copies of V'_η as a $\mathfrak{g}$ -space where m_η is the multiplicity of the eigenvalue η of $\Delta_{\mathbb{M}}$. Different choices for V'_η are isomorphic as $\mathfrak{g}$ -spaces.

Proof. Let $\eta \in \sigma(\Omega)$ with $V_{\eta,0} \neq 0$. On $V_{\eta,0} \neq 0$ the operator Ω equals the nonnegative operator $\Delta_{S\mathbb{M}}$ and acts through multiplication by η . Therefore, $\eta \geq 0$. Conversely, pick $\eta \geq 0$ in $\sigma(\Omega)$, i.e., $V_{\eta,k_0} \neq 0$ for some $k_0 \in \mathbb{Z}$. If $k_0 = 0$ we are done. For $\eta = 0$, the space $V_{0,0}$ consists of constant functions and therefore $V_{0,0}$ is nonzero. Equation (1) shows that $X_\pm$ vanish on $V_{0,0}$ and therefore $V_{0,0}$ is a trivial $\mathfrak{g}$ -space. For $\eta > 0$ consider the operators $X_\pm X_\mp$: On $V_{\eta,k}$ they act as a multiple of the identity with the multiple given by

$$X_+ X_-|_{V_{\eta,k}} = -\frac{1}{4}(\eta + Kk - Kk^2) \quad \text{and} \quad X_- X_+|_{V_{\eta,k}} = -\frac{1}{4}(\eta - Kk - Kk^2).$$

Since $X_\pm^* = -X_\mp$ we have

$$\|X_+ u\|^2 = \frac{1}{4}(\eta - Kk - Kk^2)\|u\|^2 \quad \text{and} \quad \|X_- u\|^2 = \frac{1}{4}(\eta + Kk - Kk^2)\|u\|^2 \quad (1)$$

for $u \in V_{\eta,k}$. In particular, if $V_{\eta,k} \neq 0$, then $\eta \mp Kk - Kk^2 \geq 0$ and if $\eta \mp Kk - Kk^2 > 0$ then $X_\pm: V_{\eta,k} \rightarrow V_{\eta,k \pm 1}$ is injective. More specifically, $X_\pm$ are both injective for all $k \in \mathbb{Z}$ in the case of $K \leq 0$. This implies that $V_{\eta,0} \neq 0$ in this case.

If K is positive, we may assume $k_0 > 0$. The case $k_0 < 0$ is handled similarly. As mentioned above it holds that $\eta - Kk_0 - Kk_0^2 \geq 0$. Analyzing the quadratic equations, we observe $\eta + Kk - Kk^2 > 0$ for $k = 1, \dots, k_0$. Hence

$$V_{\eta,k_0} \xrightarrow{X_-} V_{\eta,k_0-1} \xrightarrow{X_-} \dots \longrightarrow V_{\eta,1} \xrightarrow{X_-} V_{\eta,0}$$

are all injective. We infer $V_{\eta,0} \neq 0$.

Pick $f \in V_{\eta,0}$ and define $\mathcal{H}_f$ as closure of the $\mathfrak{g}$ -invariant subspace generated by f . Since $X_\pm X_\mp$ are scalar on $V_{\eta,k}$, we see that $\mathcal{H}_f = \overline{\text{span}}\{X_+^k f, X_-^k f \mid k \in \mathbb{N}_0\}$ and it follows that $\mathcal{H}_f \cap V_{\eta,k}$ is at most one-dimensional. Moreover, $\mathcal{H}_f$ for different $f \in V_{\eta,0}$ are isomorphic as $\mathfrak{g}$ -spaces and we claim $\mathcal{H}_g \perp \mathcal{H}_f$ for $g \perp f$. Since $V_{\eta,k}$ are orthogonal for different k , we only have to verify $(\mathcal{H}_f \cap V_{\eta,k}) \perp (\mathcal{H}_g \cap V_{\eta,k})$. But as this subspace is given by $\mathbb{C}X_{\text{sign } k}^{|k|} f$ and $\mathbb{C}X_{\text{sign } k}^{|k|} g$ respectively we need to show $\langle X_\pm^l f, X_\pm^l g \rangle = 0$ for $l \geq 0$. As $\langle X_\pm^l f, X_\pm^l g \rangle = (-1)^l \langle X_\mp^l X_\pm^l f, g \rangle$ and $X_\pm X_\mp$ is scalar on $V_{\eta,k}$ the claim follows by induction on l .

If we pick an orthonormal basis $f_1, \dots, f_n$ of $V_{\eta,0}$, then $\bigoplus \mathcal{H}_{f_i} \subseteq V_\eta$ and the above argument showing the injectivity $V_{\eta,k} \rightarrow V_{\eta,0}$ shows that equality holds. Hence, we can choose $V'_\eta = \mathcal{H}_{f_1}$. If we choose a different closed $\mathfrak{g}$ -invariant subspace V'_η of V_η such that $V'_{\eta,k} := V_{\eta,k} \cap V'_\eta$ satisfies $\dim V'_{\eta,k} \leq 1$ and $\dim V'_{\eta,0} = 1$, then V'_η equals $\mathcal{H}_f$ for $f \in V'_{\eta,0}$ again by injectivity of $V'_{\eta,k} \rightarrow V'_{\eta,0}$. Therefore, all choices are isomorphic as $\mathfrak{g}$ -spaces.

It remains to verify that $\Omega = \Delta_{\mathbb{M}}$ on $V_{\eta,0} \subseteq L^2(\mathbb{M})$. Since $V = 0$ on this space, we need to calculate $-4X_+X_-$. We use isothermal coordinates, i.e., coordinates (x, y) such that the metric on $\mathbb{M}$ is given by $ds^2 = e^{2\lambda}(dx^2 + dy^2)$ where λ is a smooth real-valued function of (x, y) . Then $\Delta_{\mathbb{M}} = -e^{-2\lambda} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$. Furthermore, we have (see [28]):

$$\begin{aligned} X_+(u) &= e^{(k-1)\lambda} \partial (h e^{-k\lambda}) e^{i(k+1)\theta} \quad \text{and} \\ X_-(u) &= e^{(-k-1)\lambda} \bar{\partial} (h e^{k\lambda}) e^{i(k-1)\theta} \end{aligned}$$

where $u(x, y, \theta) = h(x, y) e^{ik\theta} \in V_{\eta,k}$ and θ is the angle between a unit vector in $T\mathbb{M}$ and $\frac{\partial}{\partial x}$ so that (x, y, θ) are local coordinates on $S\mathbb{M}$. Furthermore, $\partial = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$ and $\bar{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$. With this notation we have for $h \in V_{\eta,0}$

$$X_+X_-h = X_+(e^{-\lambda}\bar{\partial}(h)e^{-i\theta}) = e^{-2\lambda}\partial\bar{\partial}(e^{-\lambda}he^{\lambda}) = e^{-2\lambda}\partial\bar{\partial}h = -\frac{1}{4}\Delta_{\mathbb{M}}h.$$

This completes the proof. $\square$

2.3. Perturbation Theory

We want to collect some basic results from perturbation theory for linear operators that can be found in [22]. First, we introduce families of operators we want to deal with.

Definition 2.4 (see [22, Ch. VII §2.1]). A family $T(z)$ of closed operators on a Banach space X where z is an element in a domain $D \subseteq \mathbb{C}$ is called *holomorphic of type (A)* if the domain of $T(z)$ is independent of z and $T(z)u$ is holomorphic for every $u \in \text{dom}(T(z))$.

Without loss of generality let us assume that 0 is contained in the domain D . We call $T = T(0)$ the unperturbed operator and $A(z) = T(z) - T$ the perturbation. Furthermore, let $R(\zeta, z) = (T(z) - \zeta)^{-1}$ be the resolvent of $T(z)$ and $R(\zeta) = R(\zeta, 0)$. If $\zeta \notin \sigma(T)$ and $1 + A(z)R(\zeta)$ is invertible, then $\zeta \notin \sigma(T(z))$ and the following identity holds:

$$R(\zeta, z) = R(\zeta)(1 + A(z)R(\zeta))^{-1}. \quad (2)$$

Let us assume that $\sigma(T)$ splits into two parts by a closed simple C^1 -curve Γ . Then there is $r > 0$ such that $R(\zeta, z)$ exists for $\zeta \in \Gamma$ and $|z| < r$ (see [22, Ch. VII Thm. 1.7]). If the perturbation is linear (i.e., $T(z) = T + zA$), then a possible choice for r is given by $\min_{\zeta \in \Gamma} \|AR(\zeta)\|^{-1}$. Note that $AR(\zeta)$ is automatically bounded by the closed graph theorem. In particular, we obtain that $\Gamma \subseteq \mathbb{C} \setminus \sigma(T(z))$ for $|z| < r$, i.e. the spectrum of $T(z)$ still splits into two parts by Γ . Let us define $\sigma_{\text{int}}(z)$ as the part of $\sigma(T(z))$ lying inside Γ and $\sigma_{\text{ext}}(z) = \sigma(T(z)) \setminus \sigma_{\text{int}}(z)$. The decomposition of the spectrum gives a $T(z)$ -invariant decomposition of the space $X = M_{\text{int}}(z) \oplus M_{\text{ext}}(z)$ where $M_{\text{int}}(z) = P(z)X$ and $M_{\text{ext}}(z) = \ker P(z)$ with the bounded-holomorphic projection

$$P(z) = -\frac{1}{2\pi i} \int_{\Gamma} R(\zeta, z) d\zeta.$$

Furthermore, $\sigma(T(z)|_{M_{\text{int}}(z)}) = \sigma_{\text{int}}(z)$ and $\sigma(T(z)|_{M_{\text{ext}}(z)}) = \sigma_{\text{ext}}(z)$. To get rid of the dependence of z in the space $M_{\text{int}}(z)$, we will use the following proposition.

Proposition 2.5 (see [22, Ch. II §4.2]). *Let $P(z)$ be a bounded-holomorphic family of projections on a Banach space X defined in a neighborhood of 0. Then there is a bounded-holomorphic family of operators $U(z): X \rightarrow X$ such that $U(z)$ is an isomorphism of Banach spaces for every z and $U(z)P(0) = P(z)U(z)$. In particular, $U(z)P(0)X = P(z)X$ and $U(z)\ker P(0) = \ker P(z)$.*

Denoting $U(z)^{-1}T(z)U(z)$ as $\tilde{T}(z)$ we observe

$$\sigma(\tilde{T}(z)|_{M_{\text{int}}(0)}) = \sigma(\tilde{T}(z)) \cap \text{int}(\Gamma) = \sigma(T(z)) \cap \text{int}(\Gamma)$$

since $U(z)$ is an isomorphism. Here we denote the interior of Γ by $\text{int}(\Gamma)$.

Let us from now on suppose that Γ encloses an eigenvalue μ of T with finite multiplicity and no other eigenvalues of T . Then $\sigma_{\text{int}}(0) = \{\mu\}$ and $M_{\text{int}}(0)$ is finite dimensional. Hence, $\tilde{T}(z)|_{M_{\text{int}}(0)}$ is a holomorphic family of operators on a finite-dimensional vector space. It follows that the eigenvalues of $T(z)$ are continuous as a function in z . In addition to the previous assumptions, let us suppose that the eigenvalue μ is simple. Then $M_{\text{int}}(0)$ is one-dimensional and $\tilde{T}(z)|_{M_{\text{int}}(0)}$ is a scalar operator. We obtain that there is a holomorphic function $\mu: B_r \rightarrow \mathbb{C}$ (with $r = \min_{\zeta \in \Gamma} \|AR(\zeta)\|^{-1}$ as above) such that $\mu(z)$ is an eigenvalue of $T(z)$, $\mu(z)$ is inside Γ and $\mu(z)$ is the only part of $\sigma(T(z))$ inside Γ since $\sigma_{\text{int}}(z) = \sigma(\tilde{T}(z)|_{M_{\text{int}}(0)})$.

We now want to calculate the Taylor coefficients of $\mu(z)$ in order to get an approximation of $\mu(z)$ in the case where $X = \mathcal{H}$ is a Hilbert space and $T(z)$ is a holomorphic family of type (A) with symmetric T but not necessarily symmetric $T(z)$ for $z \neq 0$. To this end let $\varphi^{(0)} \in M_{\text{int}}(0)$, $\varphi^{(0)} \neq 0$, i.e., $\varphi^{(0)}$ is an eigenvector of T , and define $\varphi(z) = U(z)\varphi^{(0)}$. Then $\varphi(z)$ is an eigenvector of $T(z)$ with eigenvalue $\mu(z)$ and it is holomorphic in z . Consider the Taylor series $\mu(z) = \sum z^n \mu^{(n)}$, $\varphi(z) = \sum z^n \varphi^{(n)}$. The holomorphy of $T(z)$ produces a sequence of linear operators $T^{(n)}$ such that $T(z)u = \sum z^n T^{(n)}u$ for every $u \in \text{dom}(T)$ which converges on a disc of positive radius independent of u . This is due to the fact that Taylor series of holomorphic functions converge on every disc that is contained in the domain.

We compare the Taylor coefficients in

$$(T(z) - \mu(z))\varphi(z) = 0 \quad \text{and} \quad \langle (T(z) - \mu(z))\varphi(z), \varphi(z) \rangle = 0$$

and obtain

$$(T - \mu^{(0)})\varphi^{(k)} = - \sum_{n=1}^k (T^{(n)} - \mu^{(n)})\varphi^{(k-n)} \quad \text{for } k \geq 1$$

and

$$\mu^{(k)} = \langle T^{(k)}\varphi^{(0)}, \varphi^{(0)} \rangle + \sum_{n=1}^{k-1} \langle (T^{(n)} - \mu^{(n)})\varphi^{(k-n)}, \varphi^{(0)} \rangle \quad \text{for } k \geq 1.$$

In particular,

$$\mu^{(1)} = \langle T^{(1)} \varphi^{(0)}, \varphi^{(0)} \rangle \quad (3)$$

$$\mu^{(2)} = \langle T^{(2)} \varphi^{(0)}, \varphi^{(0)} \rangle + \langle (T^{(1)} - \mu^{(1)}) \varphi^{(1)}, \varphi^{(0)} \rangle, \quad (4)$$

where $\varphi^{(1)}$ fulfills

$$(T - \mu) \varphi^{(1)} = -(T^{(1)} - \mu^{(1)}) \varphi^{(0)}. \quad (5)$$

Although $\varphi^{(1)}$ is not uniquely determined by Eq. (5), $\mu^{(2)}$ can be calculated in our setting. Indeed, $\varphi^{(1)} = v + c\varphi^{(0)}$ with $v \in (\varphi^{(0)})^\perp$ and $c \in \mathbb{C}$. By Eq. (3) the right hand side of Eq. (5) is an element of $(\varphi^{(0)})^\perp$ which is a $(T - \mu)$ -invariant subspace since T is symmetric and $\mathbb{C}\varphi^{(0)} = \ker(T - \mu)$ is $(T - \mu)$ -invariant. Since the μ is a simple eigenvalue and an isolated point of $\sigma(T)$, we infer that $T - \mu$ is invertible on $(\varphi^{(0)})^\perp$ and

$$v = -((T - \mu)|_{(\varphi^{(0)})^\perp})^{-1} (T^{(1)} - \mu^{(1)}) \varphi^{(0)}.$$

Using Eq. (4) we infer that

$$\begin{aligned} \mu^{(2)} &= \langle T^{(2)} \varphi^{(0)}, \varphi^{(0)} \rangle \\ &\quad - \langle (T^{(1)} - \mu^{(1)}) ((T - \mu)|_{(\varphi^{(0)})^\perp})^{-1} (T^{(1)} - \mu^{(1)}) \varphi^{(0)}, \varphi^{(0)} \rangle, \end{aligned} \quad (6)$$

since $(T^{(1)} - \mu^{(1)}) \varphi^{(0)}$ is orthogonal to $\varphi^{(0)}$ as before.

3. Perturbation Theory of the Kinetic Brownian Motion

We want to establish the limit $\gamma \rightarrow \infty$ of the spectrum of P_γ . To do so we write $P_\gamma = \frac{\gamma^2}{2}(\Delta_{\mathbb{S}} - 2\gamma^{-1}X) = \frac{\gamma^2}{2}T(-2\gamma^{-1})$ where $T(z) = \Delta_{\mathbb{S}} + zX$ and we want to use the methods established in Chapter 2.3.

In order to have finite-dimensional eigenspaces and holomorphic families of type (A), we will use the orthogonal eigenspace decomposition of $L^2(S\mathbb{M})$ derived in Sect. 2.2:

$$L^2(S\mathbb{M}) = \bigoplus_{\eta, k} V_{\eta, k}$$

where $V_{\eta, k} = \{u \in C^\infty(S\mathbb{M}) \mid \Omega u = \eta u, Vu = iku\}$.

Proposition 3.1. *The family of operators $T(z)$, $z \in \mathbb{C}$, restricted to V_η defines a holomorphic family of type (A) with domain $H^2(S\mathbb{M}) \cap V_\eta$. The same is true for V'_η .*

Proof. Since $\Delta_{S\mathbb{M}}$ is a second-order elliptic differential operator, we have $H^2(S\mathbb{M}) = \{u \in L^2(S\mathbb{M}) \mid (\Delta_{S\mathbb{M}} + zX)u \in L^2(S\mathbb{M})\}$ for each $z \in \mathbb{C}$. The space V_η is invariant under $\Delta_{S\mathbb{M}}$ so that $H^2(S\mathbb{M}) \cap V_\eta = \{u \in V_\eta \mid (\Delta_{S\mathbb{M}} + zX)u \in V_\eta\}$. We now use that $\Delta_{S\mathbb{M}} = \Omega + (1 - K)\Delta_{\mathbb{S}}$ and $\Omega = \eta$ on V_η . Therefore, $H^2(S\mathbb{M}) \cap V_\eta = \{u \in V_\eta \mid ((1 - K)\Delta_{\mathbb{S}} + zX)u \in V_\eta\} = \text{dom}(T(z)/(1 - K)|_{V_\eta})$ for $K \neq 1$. For $K = 1$ the same argument works if we replace $\Delta_{S\mathbb{M}}$ by $\Delta_{S\mathbb{M}} + \Delta_{\mathbb{S}}$. Since $T(z)|_{V_\eta}$ is closed as a restriction of a closed operator the proposition is proven. The proof for V'_η is identical. $\square$

We denote the restriction of $T(z)$ to V_η by $T_\eta(z)$.

The eigenspaces of the unperturbed operator $\Delta_{\mathbb{S}}|_{V_\eta}$ are $V_{\eta,0}$ and $V_{\eta,k} \oplus V_{\eta,-k}$ which are finite dimensional. As we have seen in Sect. 2.3 the eigenvalues of a holomorphic family of type (A) are continuous as a function of z in this case. We deduce that for the eigenvalues $\mu(z)$ of $T_\eta(z)$ that arise from nonzero eigenvalues $\mu = \mu(0)$ of $\Delta_{\mathbb{S}}|_{V_\eta}$ the limit $\gamma \rightarrow \infty$ of $\frac{\gamma^2}{2}\mu(-2\gamma^{-1})$, which is an eigenvalue of P_γ , is ∞ . Therefore, we are only interested in eigenvalues of $T_\eta(z)$ which arise from the unperturbed eigenvalue 0. In order to have that 0 is an eigenvalue of $T_\eta(0)$, we must have $V_{\eta,0} \neq 0$, i.e., $\eta \in \sigma(\Delta_{\mathbb{M}})$ by Lemma 2.3. Let us first deal with $\eta = 0$. Here $\mathfrak{g}$ acts trivially on $V_{0,0}$ by Lemma 2.3 and therefore the eigenvalue of $T_0(z)$ that arises from the eigenvalue 0 is 0. If $\eta > 0$ we restrict $T_\eta(z)$ to V'_η . We observe that the eigenvalues of $T_\eta(z)|_{V'_\eta}$ are independent from the choice of V'_η since all choices are isomorphic $\mathfrak{g}$ -spaces (see Lemma 2.3). The restriction of $T_\eta(z)$ to V'_η is a holomorphic family of type (A) and the eigenspace of the unperturbed eigenvalue is $V'_{\eta,0}$ which is one-dimensional. Hence, we are in the precise setting of Sect. 2.3. We obtain that there is a holomorphic function μ defined on a neighborhood of 0 (depending on η) such that $\mu(z)$ is an eigenvalue of $T(z)|_{V'_\eta}$ with $\mu(0) = 0$.

Let $\varphi(z)$ be a corresponding holomorphic eigenvector, in particular $\varphi(0) \in V'_{\eta,0}$. We can use Eq. (3) from Sect. 2.3:

$$\mu'(0) = \langle X\varphi(0), \varphi(0) \rangle = \frac{1}{2} \langle (X_+ + X_-)\varphi(0), \varphi(0) \rangle.$$

Due to the fact that $X_\pm$ are raising respectively lowering operators, i.e. $X_\pm V_{\eta,k} \subseteq V_{\eta,k\pm 1}$, we conclude that $\mu'(0) = 0$.

We now want to find the second derivative $\mu''(0)$ of μ . Notice that $X\varphi(0) \in V'_{\eta,-1} \oplus V'_{\eta,1} = \{u \mid \Delta_{\mathbb{S}}u = u\}$. Therefore, $(\Delta_{\mathbb{S}}|_{\varphi(0)^\perp})^{-1}X\varphi(0) = X\varphi(0)$. Consequently by Eq. (6),

$$\begin{aligned} \mu''(0) &= -2\langle X(X\varphi(0)), \varphi(0) \rangle = -\frac{1}{2} \langle (X_+ + X_-)^2 \varphi(0), \varphi(0) \rangle \\ &= -\frac{1}{2} \langle (X_+^2 + X_+X_- + X_-X_+ + X_-^2) \varphi(0), \varphi(0) \rangle. \end{aligned}$$

Again, $X_\pm$ are raising/lowering operators. Therefore,

$$\begin{aligned} \mu''(0) &= -\frac{1}{2} \langle (X_+X_- + X_-X_+) \varphi(0), \varphi(0) \rangle \\ &= \langle \Omega \varphi(0), \varphi(0) \rangle = \eta \end{aligned}$$

as the Casimir operator Ω equals $-2X_+X_- - 2X_-X_+ - KV^2$ and $V\varphi(0) = 0$.

If we now substitute $z = -2\gamma^{-1}$ we obtain that $\lambda_\eta(\gamma) = \frac{\gamma^2}{2}\mu(-2\gamma^{-1})$ is an eigenvalue of P_γ with uniform multiplicity $m_\eta = \dim \ker(\Delta_{\mathbb{M}} - \eta)$ which is analytic in γ^{-1} and converges to η as $\gamma \rightarrow \infty$. This proves Theorem 1.

Remark 3.2. In order to obtain uniform convergence of the eigenvalues in compact sets, i.e. for each compact set K there is γ_K such that $K \cap \sigma(P_\gamma) \subseteq \{\lambda_\eta(\gamma) \mid \eta \in K \cap \sigma(\Delta_{\mathbb{M}})\}$ for all $\gamma > \gamma_K$, we would like to deal with all $\eta \in \sigma(\Delta_{\mathbb{M}})$ simultaneously in a uniform way since the eigenvalues of $\Delta_{\mathbb{M}}$ appear in the

second derivatives of the evolving eigenvalues in the perturbation expansion, not as their limits. More precisely, we want to separate converging eigenvalues (which arise from 0 as an element of $\sigma(\Delta_{\mathbb{S}})$) from non-converging eigenvalues. For this to happen, we must have that $1 + zX(\Delta_{\mathbb{S}} - \zeta)^{-1}$ on V'_η is invertible for small $|z|$ and for ζ in some closed curve enclosing 0 but no other element of $\sigma(\Delta_{\mathbb{S}}) = \{k^2 \mid k \in \mathbb{Z}\}$. In particular, $1 + zX(\Delta_{\mathbb{S}} - \zeta)^{-1}$ has to be invertible for some $\zeta \in (0, 1)$ but we can only ensure this for $|z| < \|X(\Delta_{\mathbb{S}} - \zeta)^{-1}|_{V'_\eta}\|^{-1}$. As

$$\begin{aligned} \|X(\Delta_{\mathbb{S}} - \zeta)^{-1}|_{V'_\eta}\| &\geq \|X(\Delta_{\mathbb{S}} - \zeta)^{-1}|_{V'_{\eta,0}}\| = |\zeta|^{-1} \|X|_{V'_{\eta,0}}\| \\ &= |\zeta|^{-1} \sqrt{\|X_+|_{V'_{\eta,0}}\|^2 + \|X_-|_{V'_{\eta,0}}\|^2} \\ &= |\zeta|^{-1} \sqrt{\frac{1}{2}\eta} \geq \sqrt{\frac{1}{2}\eta} \end{aligned}$$

this is impossible for all $\eta \in \sigma(\Delta_{\mathbb{M}})$ at once.

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Appendix A. Proof of Proposition 2.1

The proof that P_γ is hypoelliptic with the subelliptic estimate can be found in [10, Section 2.2]. There exist vector fields X_j on $S\mathbb{M}$ such that $\Delta_{\mathbb{S}} = -\sum_{j=1}^d X_j^2$ and $\operatorname{div} X_j = 0$ (see [10, §2.2.6]). Hence, the X_j as well as X are skew-symmetric with respect to the inner product of $L^2(S\mathbb{M})$. It follows

that $\Re\langle P_\gamma f, f \rangle = \sum \frac{1}{2} \gamma^2 \langle X_j f, X_j f \rangle - \gamma \Re\langle X f, f \rangle \geq 0$ since $\langle X f, f \rangle \in i\mathbb{R}$, i.e. $P_\gamma|_{C^\infty}$ is accretive.

We show that $\text{Ran}(P_\gamma|_{C^\infty} + I)$ is dense in $L^2(\text{SM})$ following the proof of [21, Prop. 5.5]. Let $f \in \text{Ran}(P_\gamma|_{C^\infty} + I)^\perp$. Then we have $\langle f, (P_\gamma + I)u \rangle = 0$ for all $u \in C^\infty$, hence $(P_{-\gamma} + I)f = 0$ in $\mathcal{D}'$. Since $P_{-\gamma}$ is hypoelliptic, it follows that $f \in C^\infty$ and with $u = f$ in the above formula we have $\langle f, (P_\gamma + I)f \rangle = 0$. Therefore, $\|f\|^2 = -\Re\langle f, P_\gamma f \rangle \leq 0$. Thus $f = 0$.

We obtain that the closure $\overline{P_\gamma|_{C^\infty}}$ is maximal-accretive (see e.g. [21, Thm. 5.4]). In the same way, $\overline{P_{-\gamma}|_{C^\infty}}$ is maximal-accretive. An operator A on a Hilbert space is maximal-accretive iff it generates a contraction semigroup e^{-tA} (see [29, p. 241]). Hence, $\overline{P_{-\gamma}|_{C^\infty}}$ generates a contraction semigroup $e^{-t\overline{P_{-\gamma}|_{C^\infty}}}$. The adjoint semigroup $(e^{-t\overline{P_{-\gamma}|_{C^\infty}}})^*$ is generated by $(P_{-\gamma}|_{C^\infty})^*$ that is $\frac{1}{2}\gamma^2\Delta_S - \gamma X$ with domain $\{f \in L^2 \mid (\frac{1}{2}\gamma^2\Delta_S - \gamma X)f \in L^2\}$ (see [12, I.5.14 and II.2.5]), i.e. $(P_{-\gamma}|_{C^\infty})^* = P_\gamma$. In particular, the operator P_γ is maximal-accretive as it generates a contraction semigroup. Since both operators $\overline{P_\gamma|_{C^\infty}}$ and P_γ are maximal-accretive and we conclude that they coincide. Similar arguments can be found in [20].

For the positivity of the generated contraction semigroup we have to check if

$$\langle (\text{sign } f)P_\gamma f, u \rangle \geq \langle |f|, (P_\gamma)^* u \rangle$$

for all real $f \in C^\infty$ and a strictly positive subeigenvector u of $(P_\gamma)^*$ (see [2, C-II Cor. 3.9]). Note that 1 is a strictly positive eigenvector of $(P_\gamma)^*$ and $\frac{1}{2}\Delta_S(x)$ as well as $-X$ generate stochastic Feller processes on $S_x\mathbb{M}$ and SM respectively (namely the Brownian motion on $S_x\mathbb{M}$ and the geodesic flow). Hence, $e^{-t\Delta_S(x)}$ and e^{tX} define positive semigroups so that $\langle (\text{sign } f)\Delta_S(x)f, 1 \rangle_{S_x\mathbb{M}} \geq 0$ for $f \in C^\infty(S_x\mathbb{M})$ and $\langle (\text{sign } f)(-X)f, 1 \rangle \geq 0$ for $f \in C^\infty(\text{SM})$ (see [2, C-II Thm.2.4]). Combining both statements completes the proof.

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