On Pleijel's Nodal Domain Theorem for Quantum Graphs

Matthias Hofmann, James B. Kennedy, Delio Mugnolo and Marvin Plümer

Abstract. We establish metric graph counterparts of Pleijel's theorem on the asymptotics of the number of nodal domains ν_n of the n th eigenfunction(s) of a broad class of operators on compact metric graphs, including Schrödinger operators with L^1 -potentials and a variety of vertex conditions as well as the p -Laplacian with natural vertex conditions, and without any assumptions on the lengths of the edges, the topology of the graph, or the behaviour of the eigenfunctions at the vertices. Among other things, these results characterise the accumulation points of the sequence $(\frac{\nu_n}{n})_{n \in \mathbb{N}}$, which are shown always to form a finite subset of $(0, 1]$. This extends the previously known result that $\nu_n \sim n$ *generically*, for certain realisations of the Laplacian, in several directions. In particular, in the special cases of the Laplacian with natural conditions, we show that for graphs any graph with pairwise commensurable edge lengths and at least one cycle, one can find eigenfunctions thereon for which $\nu_n \not\sim n$; but in this case even the set of points of accumulation may depend on the choice of eigenbasis.

Mathematics Subject Classification. 34B45, 34L10, 81Q35.

The work of M.H. and J.B.K. was supported by the Fundação para a Ciência e a Tecnologia, Portugal, via the program “Investigador FCT”, reference IF/01461/2015 (J.B.K.), and “Bolsheiro de Investigação”, reference PD/BD/128412/2017 (M.H.), and via projects PTDC/MAT-CAL/4334/2014 and PTDC/MAT-PUR/1788/2020. The work of D.M. and M.P. was supported by the Deutsche Forschungsgemeinschaft (Grant 397230547). This article is based upon work from COST Action 18232 MAT-DYN-NET, supported by COST (European Cooperation in Science and Technology), www.cost.eu. The authors would like to thank Jonathan Rohleder for bringing an improvement of Theorem 4.2 to their attention, and both anonymous referees for their careful reading of an earlier version of this paper and their thoughtful comments, which greatly helped to improve it.

1. Introduction

Given a differential operator with a real eigenfunction ψ , its *nodal domains* are the connected components in the support of its positive part along with the connected components in the support of its negative part.

The classical oscillation theorem, first proved in Sturm's classical paper [40], states that the n th eigenfunction ψ_k of a Sturm–Liouville operator with continuous coefficients and separated boundary conditions on a compact interval has $n - 1$ zeros in the interior of the interval, that is, $\nu_n = n$ nodal domains. We refer to [23] for a historical overview of the generalisations of this result, including more general coefficients and boundary conditions.

The counterpart in higher dimensions, Courant's Nodal Domain Theorem [17], states that the number ν_n of nodal domains of the eigenfunction ψ_n associated with the n th eigenvalue of the Dirichlet Laplacian on a bounded domain in $\mathbb{R}^d$ is no larger than n . Pleijel's theorem [36], which establishes an asymptotic bound on the quotient ν_n/n , sharpens Courant's result by stating that the number of eigenvalues for which equality may hold is finite if $d = 2$.

The main goal of the present note is to discuss the behaviour of the sequence ν_n/n , and thus explore the validity, or lack thereof, of Pleijel's theorem, in an intermediate setting between *intervals* on the one hand and higher-dimensional *domains* on the other: compact quantum graphs—i.e. Laplacians or similar operators on metric graphs with finitely many, finite, edges [10]—somehow lie in between. This is essentially due to the topological configurations that can be taken by metric graphs.

Over the last century, Courant's theorem has been extended to various settings; quoting [1], its proof boils down to three points:

- (a) *the variational characterisation of eigenvalues,*
- (b) *the maximum principle,*
- (c) *the Unique Continuation Property.*

And this is where troubles arise. Indeed, the first two points (a), (b) can be enforced by assuming the relevant operator to be associated with a quadratic form that satisfies the first Beurling–Deny condition and whose domain is compactly embedded in the Hilbert space: this includes, for example, the cases of Laplacians with Neumann or Robin boundary conditions; of discrete Laplacians on combinatorial graphs; or Laplacians on metric graphs with natural vertex conditions (i.e. continuity on the whole metric graph along with a Kirchhoff condition on the normal derivatives at each vertex); this is the setting of the most general currently available versions of Courant's theorem [27].

But the third one (c) has a geometric flavour and is known to fail—in general—for both combinatorial and metric graphs.

Given an eigenfunction ψ , failure of the Unique Continuation Property on a (combinatorial or metric) graph implies that classical nodal domains may not exhaust the graph: this suggests to study both *strong* and *weak* nodal domains, i.e. the (closures of the) connected components of both $\{\psi > 0\}$ and $\{\psi \geq 0\}$, and likewise of both $\{\psi < 0\}$ and $\{\psi \leq 0\}$. Courant-type bounds on the nodal count of eigenvectors associated with the n th (possibly non-simple)

eigenvalue of a combinatorial graph were derived in the seminal paper [18]; see also [41] for an overview and refinement of later results inspired by [18]. A further complication, both on combinatorial and metric graphs, arises from the possibility that ν_n may depend on the specific choice of eigenbasis if the eigenvalues are not simple.

However, on metric graphs we still have the property that if an eigenfunction vanishes on an open subset of an edge, then it must vanish on the whole edge; so a way to enforce the Unique Continuation Property, and also to remove any ambiguities regarding non-simple eigenvalues, is simply to assume that all eigenvalues are simple and no eigenfunctions vanish on any vertices, which is known to be true for the usual realisations of the Laplacian under certain topological assumptions on the graph (no loops) and then genericity assumptions on the edge lengths. Under these assumptions, it is known that equality of ν_n and n holds for metric trees ([37]) and only for them ([4]).

In the case of quantum graphs which may have cycles, the Courant–Pleijel theory was first obtained—again, only under the aforementioned genericity assumptions—by Gnutzmann, Smilansky and Weber in [22]; their proof mirrors the original one by Pleijel but, unlike in Pleijel's result, it only yields that the number ν_n of nodal domains associated with the n th eigenfunction is *generically* bounded from above by n . A complementary lower bound $\nu_n \geq n - \beta$, likewise for generic eigenfunctions, was established in [7]; here, β denotes the number of independent cycles in the graph (i.e. the first Betti number). In particular, this shows that for such graphs and eigenfunctions, $\frac{\nu_n}{n} \rightarrow 1$ as $n \rightarrow \infty$. Under the same assumptions, the nodal deficiency $n - \nu_n$ or more precisely the number of “surplus” zeros $\phi_n - (n - 1) \in \{0, \dots, \beta\}$ of the n th eigenfunction ψ_n (ϕ_n being the number of its zeros) has since been studied by Band, Berkolaiko and their co-authors in several papers since [5] (see, for example, [2, 12] and the references therein). For example, it is known [2] that the sequence $\phi_n - (n - 1)$ admits a well-defined probability distribution, which in the case of graphs with only disjoint cycles is even binomial. However, such results only make sense in the abovementioned case of eigenfunctions which do not vanish on any edges.

One of the goals of this note is to describe the asymptotic behaviour of the sequence ν_n as $n \rightarrow \infty$, removing this genericity assumption, for any given choice of orthonormal basis of eigenfunctions, where far richer behaviour is possible. To this end, we employ an alternative method based on an elementary isoperimetric inequality due to Nicaise [35] which seems to have been little known at the time of [22]. This way we can drop any condition on the eigenfunctions, including the usual genericity conditions, which seem hard to check on graphs with complicated topologies and typically fail on graphs with non-trivial automorphism group. More interestingly, we present classes of graphs (including all graphs whose shortest edge is a loop) for which the number of nodal domains—counted along a suitable sequence of mutually orthogonal eigenfunctions—is actually strictly smaller than the index of the corresponding eigenvalue and remains so in the asymptotic limit. We will also demonstrate the strength of our approach by applying it to two classes of

operators not previously considered in this context: quite general Schrödinger operators with (nonnegative) L^1 -potentials and a large variety of vertex conditions (more precisely, those local vertex conditions that characterise the class of metric graph Laplacians associated with positive, irreducible semigroups) and the p -Laplacian with natural vertex conditions.

Let us now describe our main results, and the structure of the paper, more precisely. After recalling the standard definition of metric graphs and introducing the general class of Schrödinger operator we are going to work with in Sect. 2, our first main results, Theorem 3.4 and Proposition 3.5, are presented in Sect. 3; these describe the set of accumulation points of the sequence $(\frac{\nu_n}{n})_{n \in \mathbb{N}}$ for the aforementioned Schrödinger operators; in particular, these points of accumulation always form a finite subset of $(0, 1]$ and can be described explicitly in terms of the edge lengths of the graph. The proofs rely upon a series of lemmata that are discussed in Sect. 3.1.

Our results can be significantly refined if we restrict to the case of Laplacians with natural vertex conditions. We delve into this setting in Sect. 4 where among other things we show that on graphs that contain a cycle and have pairwise commensurable edge lengths, i.e. the ratio of any pair of the graph's edge lengths is a rational number, $(\frac{\nu_n}{n})_{n \in \mathbb{N}}$ will always have at least two points of accumulation, one strictly smaller than 1. (Here, we are also excluding the trivial case that the graph is itself a cycle.) In this section, we also give a simple example, Example 4.5 demonstrating that even the set of points of accumulation of $(\frac{\nu_n}{n})_{n \in \mathbb{N}}$ itself may actually depend on the choice of eigenbasis, even for the Laplacian with natural conditions, reinforcing the principle that many features of interest are lost if we restrict to the “generic” setting where all eigenvalues are simple.

With the aim of showing the flexibility of our approach based on isoperimetric inequalities applied locally to the nodal domains, rather than global linear algebraic manipulations as in [22], or ergodicity properties of the *secular manifold* as in [2], in the last part of the paper we turn to an important *nonlinear* operator. Section 5 is devoted to the theory of p -Laplacian, and to obtaining a Pleijel-type theorem in this context, Theorem 5.2. On metric graphs, these nonlinear operators were introduced in [33, Section 6.7] and their spectral properties were studied in [8, 19]; their theory is significantly less developed than in the case of intervals, though. In turn, the theory of general p -Laplace operators is not as well understood as in the linear case of $p = 2$: even the existence of infinitely many eigenvalues or the validity of the Unique Continuation Property seems to be unknown in general. A Courant-type Nodal Domain Theorem on domains was proved in [21], though; it yields an upper bound $\nu_{n,p} \leq 2n - 2$, which can be refined to $\nu_{n,p} \leq n$ under the additional assumption that the unique continuation principle prevails. Results of Sturm oscillation type for p -Laplacians with potential in one dimension were obtained in [13, 38]. It is all the more interesting that our Theorem 5.2 implies in particular the sharper bound $\nu_{n,p} \leq n$ even in an environment where the Unique Continuation Property clearly fails.

We include a number of auxiliary results about the p -Laplacian on metric graphs and its eigenvalues in “Appendix A”, where, in particular, we give the result that its variational eigenvalues satisfy the same Weyl asymptotics as the p -Laplacian on an interval. Finally, in “Appendix B”, we give a bound on the first eigenvalue of a general (linear) Schrödinger operator in terms of the L^1 -norm of its potential and the average edge length of $\mathcal{G}$ only, reminiscent of [28], and needed for the proofs in Sect. 3; despite being rather elementary, it may be of some independent interest.

2. General Setting

Let $\mathcal{G}$ be a *compact metric graph*, i.e. a finite combinatorial graph $G = (V, E)$ each of whose edges $e \in E$ is identified with an interval $(0, \ell_e)$ of finite length; we write $|\mathcal{G}| = \sum_{e \in E} \ell_e$ for the total length of the graph. See [34] for a precise definition of this metric measure space and the function spaces $C(\mathcal{G})$ and $L^2(\mathcal{G})$, as well as the Sobolev space $H^1(\mathcal{G})$ of all continuous functions over $\mathcal{G}$ that are edgewise weakly differentiable with weak derivative in $L^2(\mathcal{G})$. For a vertex $v \in V$, we let E_v denote the set of edges in E incident to v , and $\deg(v) := \#E_v$ its degree.

In what follows, we will give a description of the operators we will be considering. Note that all we will need for the results there are certain more or less abstract properties which these operators satisfy; in particular, the reader unfamiliar with the theory presented in this section may imagine the important special case of a Schrödinger operator with smooth (or even zero) potential and (possibly) delta couplings, or else any of the usual vertex conditions, at the vertices.

We first consider a possible relaxation of the continuity condition at the vertices to allow for *weighted* continuity encoded in a nonnegative vector of edge weights $w_v \in \mathbb{R}_{>0}^{\deg(v)}$, $v \in V$, i.e.

$$w_{e,v} f_e(v) = w_{f,v} f_f(v) \quad \text{if } v \in e \cap f. \quad (2.1)$$

Indeed, in this case we can define, in a natural way, a space $H_w^1(\mathcal{G})$ of edgewise H^1 -functions that satisfy (2.1) at the vertices. Note that while functions in $H_w^1(\mathcal{G})$ may be discontinuous at the vertices, they can only change sign at a vertex, i.e. take on positive and negative values in any neighbourhood of a vertex, if they are zero at that vertex.

We then define, for $q \in L^1(\mathcal{G})$ and a matrix $\mathcal{B} \in M_{2|E| \times 2|E|}(\mathbb{C})$ consisting of block matrices $\mathcal{B}_{vw} \in \mathbb{C}^{\deg(v) \times \deg(w)}$ for all $v, w \in V$ the sesquilinear form

$$\begin{aligned} a(f) &:= \int_{\mathcal{G}} (|f'(x)|^2 + q(x)|f(x)|^2) dx + \sum_{v,w \in V} (\mathcal{B}_{vw} f(w), f(v))_{\mathbb{C}^{\deg(v)}}, \\ D(a) &:= H_w^1(\mathcal{G}), \end{aligned} \quad (2.2)$$

where for each $v, w \in V$, $\mathcal{B}_{vw}$ is a $\deg(v) \times \deg(w)$ -matrix, and for $v \in V$, $f(v) = (f_e(v))_{e \in E_v}$. If we want to emphasise the dependence on the potential

and the vertex conditions, then we will also write

$$a_{q,\mathcal{B},w} \quad \text{in place of just } a.$$

At any rate, it follows from the theory presented in [33, Section 6.5] that this form is bounded and elliptic; hence, the associated operator $A = A(q, \mathcal{B}, w)$ is (minus) the generator of an analytic, strongly continuous semigroup on $L^2(\mathcal{G})$. This semigroup is of trace class, and therefore, A has pure point spectrum.

If in particular q is real-valued and $\mathcal{B}$ is Hermitian for all $v \in V$, then a is a closed quadratic form; hence, $A(q, \mathcal{B}, w)$ is a self-adjoint operator that is bounded from below. This setting includes, as special cases, realisations of the Laplacian on $\mathcal{G}$ with so-called *natural vertex conditions* (continuity across vertices, all normal derivatives sum up to 0 at each vertex), corresponding to $q \equiv 0$, $\mathcal{B} = 0$ and $w \equiv 1$, as well as (standard) delta couplings (continuity across vertices, at each vertex the sum of all normal derivatives equals a multiple of the point evaluation at the same vertex), where $q \equiv 0$, $w \equiv 1$, and $\mathcal{B}$ is a diagonal matrix with respect to the canonical basis of $\mathbb{C}^{2|E|}$.

Now, because $u^+ \in H_w^1(\mathcal{G})$ for all $u \in H_w^1(\mathcal{G})$ due to positivity of the edge weights w , it is known, cf. [33, Theorem 6.85] that the semigroup is positive if and only if so is the semigroup generated by each $-\mathcal{B}$. (This is in particular the case if $\mathcal{B}$ is diagonal, which covers delta couplings, including weighted versions thereof.) In this context, we refer to the condition (2.1) and the weighted Kirchhoff–Robin-type condition associated with the matrix $\mathcal{B}$ collectively as *positivity preserving* vertex conditions. Finally, all these assertions remain valid if, for some $V_0 \subset V$ (where possibly, trivially, $V_0 = \emptyset$), we consider the operator $A(q, \mathcal{B}, w, V_0)$ associated with the restriction of the form a to $H_{0,w}^1(\mathcal{G}; V_0)$, the space of all functions in $H_w^1(\mathcal{G})$ that vanish on the vertices in V_0 . (In this case, of course, we only require that $\mathcal{B}_{vw}$ be defined for $v, w \in V \setminus V_0$.) Finally, let

$$H_0^1(\mathcal{G}) := H_0^1(\mathcal{G}; V) \tag{2.3}$$

denote the space of globally H^1 -functions which vanish at *all* vertices, which is clearly contained in the domain of the considered forms.

The Schrödinger operators associated with these classes of forms were thoroughly studied in [29].

In all these cases, the discrete spectrum of $A(q, \mathcal{B}, w, V_0)$ consists of real eigenvalues $\lambda_n(q, A, w, V_0)$ repeated according to their finite multiplicities, characterised by the usual Courant–Fischer max-min and min-max principles, which diverge to $+\infty$ as $n \rightarrow \infty$, and whose eigenfunctions may be chosen to be real and to form an orthonormal basis $(\psi_n)_{n \in \mathbb{N}}$ of $L^2(\mathcal{G})$. (We mostly avoid this heavy notation and simply write $\lambda_n(\mathcal{G}) := \lambda_n(q, \mathcal{B}, w, V_0)$.)

Suitably adapting the proof of [32, Proposition 3.7.(1)] to the case of $w \not\equiv 1$ and $q \not\equiv 0$, one can easily prove that the associated semigroup, if positive, is additionally irreducible if there is no point in V_0 whose removal would disconnect $\mathcal{G}$. Hence, by the Kreĭn–Rutman theorem, we deduce that the first eigenspace is one-dimensional and spanned by a positive function (the Perron eigenfunction) ψ_1 : i.e. $\psi_1(x) > 0$ for a.e. $x \in \mathcal{G}$. Indeed, more holds: it was proved in [29] that a strong maximum principle holds; namely, the Perron

eigenfunction vanishes only at the vertices in V_0 . This was proved in [29] for the case of block-diagonal $\mathcal{B}$ (corresponding to the case of local vertex conditions) only; in this paper, for simplicity we also restrict ourselves to this case.

We will denote by $\lambda_n^D = \lambda_n^D(\mathcal{G})$ the n th lowest eigenvalue (counting multiplicities) of the Schrödinger operator with potential q and Dirichlet conditions at *all* vertices of $\mathcal{G}$, that is, whose form domain is $H_0^1(\mathcal{G})$; in this case, the graph decomposes into a disjoint collection of intervals; moreover, the associated sesquilinear form is exactly (2.2) restricted to $H_0^1(\mathcal{G})$. We note the following eigenvalue interlacing result for future reference. This is an immediate variant of interlacing results stated in [10, Chapter 3.1.6] (cf. also [9, Section 4.1]).

Lemma 2.1 *With the above assumptions and notation, for all $n \geq |V| + 1$ we have*

$$\lambda_{n-|V|}^D(\mathcal{G}) \leq \lambda_n(\mathcal{G}) \leq \lambda_n^D(\mathcal{G}).$$

Proof. Both inequalities are an immediate consequence of the min-max characterisation of the respective eigenvalues and the fact that the forms agree on $H_0^1(\mathcal{G})$, the latter in conjunction with the inclusion of the form domains $H_0^1(\mathcal{G}) \subset D(a)$, the former in conjunction with the fact that the quotient space $D(a)/H_0^1(\mathcal{G})$ is at most $|V|$ -dimensional. $\square$

3. Pleijel's Theorem for Schrödinger Operators on Metric Graphs

Our main result is a variation of Pleijel's theorem for metric graphs. We impose the following assumptions throughout this section.

Assumption 3.1 $\mathcal{G}$ is a compact, connected metric graph with underlying combinatorial graph $G = (V, E)$ and edge lengths ℓ_e , $e \in E$; we set $\ell_{\min} := \min_{e \in E} \ell_e$. We also fix a (possibly empty) set $V_0 \subset V$ and a potential $q \in L^1(\mathcal{G})$ with $q \geq q_{\min}$ for some $q_{\min} \in \mathbb{R}$ and suppose $\mathcal{B}$ is a Hermitian block-diagonal $2|E| \times 2|E|$ -matrix, i.e. $\mathcal{B}_{vw} = 0$ for $v \neq w$, such that the semigroup $(e^{-t\mathcal{B}})_{t \geq 0}$ is positive and $(w_v)_{v \in V} \in \mathbb{R}^{2|E|}$ is a vector such that $w_v \in \mathbb{R}_+^{\deg(v)}$ for all $v \in V$.

We recall that $(e^{-t\mathcal{B}})_{t \geq 0}$ is positive if and only if all entries of $\mathcal{B}$ are real and all off-diagonal entries are non-positive; this includes all (real) delta coupling conditions.

Under these assumptions, we will consider the operator associated with the form $a_{q,A,w}$ introduced in Sect. 2. In this section, we fix once and for all an (*a priori* arbitrary) eigenbasis of this operator.

Definition 3.2 Let $(\psi_n)_{n \in \mathbb{N}}$ be an orthonormal sequence of eigenfunctions $(\psi_n)_{n \in \mathbb{N}}$ with associated eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ of the Schrödinger operator $A(q, \mathcal{B}, w, V_0)$ associated with the form $a_{q,\mathcal{B},w}$. As already mentioned in the introduction, the *nodal domains* of any eigenfunction ψ_k are the respective closures in the metric space $\mathcal{G}$ of connected components of the sets $\{\psi_k \neq 0\}$. We occasionally denote by $\mathcal{G}_1, \dots, \mathcal{G}_{\nu_k}$ the nodal domains themselves, and by $\partial\mathcal{G}_i$ the

topological boundary of $\mathcal{G}_i$ in $\mathcal{G}$. We denote the nodal count of this sequence by $(\nu_n)_{n \in \mathbb{N}}$.

The following simple example demonstrates that, contrary to the previously mentioned generic case, the nodal domains of an eigenfunction might not exhaust the whole graph, and an eigenfunction might have the same sign on two adjacent nodal domains.

Example 3.3 We consider the equilateral 4-star; more precisely, we take $\mathcal{G}$ to consist of four edges $\mathbf{e}_1, \dots, \mathbf{e}_4$, each of length 1 and identified with the interval $[0, 1]$, joined at a common vertex of degree four (identified with 0 on each edge), and with the other four vertices each being of degree one. An eigenfunction φ with respect to the eigenvalue $\frac{\pi^2}{4}$ of the Laplacian on $\mathcal{G}$ with standard vertex conditions is given by $\varphi(x) = \sin(\frac{\pi}{2}x)$ on $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, $\varphi(x) = -2\sin(\frac{\pi}{2}x)$ on $\mathbf{e}_3$ and $\varphi(x) = 0$ on $\mathbf{e}_4$. The three nodal domains of φ are the (closed) edges $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Clearly, they do not cover the whole graph $\mathcal{G}$. Moreover, although the nodal domains $\mathbf{e}_1$ and $\mathbf{e}_2$ are adjacent, φ has the same sign on both.

A priori the sequence $\nu_k \in \mathbb{N}$, including the points of accumulation, can depend on the precise choice of basis, see Example 4.5, unless suitable assumptions on the edge lengths $(\ell_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}}$ and the graph topology are imposed that force all eigenvalues to be simple.

Furthermore, here and throughout, given a sequence $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, we will write

$$\text{acc} \{a_n : n \in \mathbb{N}\}$$

to denote its set of points of accumulation. With this we are now ready to formulate our first main theorem.

Theorem 3.4 *For all quantum graphs satisfying Assumption 3.1, the nodal count $(\nu_n)_{n \in \mathbb{N}}$ satisfies*

$$\text{acc} \left\{ \frac{\nu_n}{n} : n \in \mathbb{N} \right\} \subset \left\{ \frac{\sum_{\mathbf{e} \in \mathbf{E}_0} \ell_{\mathbf{e}}}{|\mathcal{G}|} : \mathbf{E} \supset \mathbf{E}_0 \text{ is a non-empty set of edges} \right\}. \quad (3.1)$$

In particular, $\text{acc} \left\{ \frac{\nu_n}{n} : n \in \mathbb{N} \right\}$ is a finite set, and

$$0 < \frac{\ell_{\min}}{|\mathcal{G}|} \leq \liminf_{n \rightarrow \infty} \frac{\nu_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{\nu_n}{n} \leq 1. \quad (3.2)$$

While the right-hand side of (3.1) does not depend on the parameters $q, \mathcal{B}, w$, the set inclusion in (3.1) is sharp in the case of a graph consisting of just one interval. Indeed, recall that on intervals, in the case of Sturm–Liouville problems, $\nu_n = n$ for all $n \in \mathbb{N}$ (see [23]).

As mentioned in the introduction, the key driving force behind the potential appearance of a non-trivial set of points of accumulation of $\frac{\nu_n}{n}$ between 0 and 1 here is the failure of the unique continuation principle, as evidenced by the following characterisation.

Proposition 3.5 *Under Assumption 3.1, we have*

$$\operatorname{acc} \left\{ \frac{\nu_n}{n} : n \in \mathbb{N} \right\} = \operatorname{acc} \left\{ \frac{|\operatorname{supp} \psi_n|}{|\mathcal{G}|} : n \in \mathbb{N} \right\}. \quad (3.3)$$

The proof of Theorem 3.4 and Proposition 3.5 is based on the following principles, the proofs of which, in turn, are postponed to Sect. 3.1.

Lemma 3.6 (Weyl asymptotics). *We have*

$$\lambda_n(\mathcal{G}) = \frac{\pi^2}{|\mathcal{G}|^2} n^2 + o(n^2) \quad \text{as } n \rightarrow \infty. \quad (3.4)$$

Lemma 3.7 (Relationship between ν_n and λ_n). *Suppose in addition to Assumption 3.1 that $q_{\min} = 0$. Then, there exists $n_0 \in \mathbb{N}$ depending only on the metric graph $\mathcal{G}$ and the potential $0 \leq q \in L^1(\mathcal{G})$ such that, for all $n \geq n_0$,*

$$\begin{aligned} |\operatorname{supp} \psi_n| \cdot \frac{\lambda_n(\mathcal{G})^{1/2} - \|q\|_1}{\pi} - (2|E| - 1)|V| &\leq \nu_n \\ &\leq |\operatorname{supp} \psi_n| \cdot \frac{\lambda_n(\mathcal{G})^{1/2}}{\pi} + |V|. \end{aligned} \quad (3.5)$$

In particular,

$$\nu_n = |\operatorname{supp} \psi_n| \cdot \frac{\lambda_n(\mathcal{G})^{1/2}}{\pi} + O(1) \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

These two lemmata are logically independent of each other; in particular, in (3.5) we explicitly do not use the Weyl asymptotics to estimate λ_n . Lemma 3.6 in particular can be refined significantly for specific types of vertex conditions and potentials; for example, in the case of the Laplacian with natural vertex conditions if $\mathcal{G}$ is not a cycle, then we may strengthen (3.4) to

$$\left(n - \frac{|N| + \beta}{2} \right)^2 \frac{\pi^2}{|\mathcal{G}|^2} \leq \lambda_n(\mathcal{G}) \leq \left(n - 2 + \beta + \frac{|N| + \beta}{2} \right)^2 \frac{\pi^2}{|\mathcal{G}|^2},$$

where $|N|$ is the number of degree one vertices and β is the first Betti number (number of independent cycles) of the graph, as follows from [8, Theorems 4.7 and 4.9]. More generally, if $q \in L^\infty(\mathcal{G})$, then we may obtain the two-sided estimate

$$(n - c_1)^2 \frac{\pi^2}{|\mathcal{G}|^2} \leq \lambda_n(\mathcal{G}) \leq (n + c_2)^2 \frac{\pi^2}{|\mathcal{G}|^2} \quad \text{for all } n \in \mathbb{N} \quad (3.7)$$

for constants $c_1, c_2 > 0$ depending only on $\mathcal{G}$ and $\|q\|_\infty$; this is a consequence of Lemma 2.1 and a simple variational argument bounding q in terms of the constant potential $\|q\|_\infty$, and zero.

Let us now show how Lemmata 3.6 and 3.7 lead to the proofs of the main results. To prove Proposition 3.5, if we combine (3.6) and (3.4), then we obtain the asymptotic behaviour

$$\nu_n = \frac{|\operatorname{supp} \psi_n|}{|\mathcal{G}|} n + o(n) \quad \text{as } n \rightarrow \infty, \quad (3.8)$$

if $q \geq 0$. This in turn immediately yields the result claimed in Proposition 3.5 for arbitrary lower bounded potentials, since the eigenvalue problem $Au = \lambda u$ is equivalent to the shifted eigenvalue problem $(A - q_{\min})u = (\lambda - q_{\min})u$, where the potential $q - q_{\min} \in L^1(\mathcal{G})$ associated with the shifted Schrödinger operator $A - q_{\min}$ is nonnegative.

The other ingredient in the proof of Theorem 3.4 is the following “weak” unique continuation principle, whose proof will also be given in Sect. 3.1; Theorem 3.4 is a direct consequence of (3.8) and (3.9).

Lemma 3.8 (Possible values of $|\text{supp } \psi_n|$). *Under Assumption 3.1, we have, for all $n \in \mathbb{N}$:*

- (i) *There exists some non-empty subset $E_0 = E_0(n) \subset E$ such that $\text{supp } \psi_n = \bigcup_{e \in E_0} e$; in particular,*

$$\{|\text{supp } \psi_n| : n \in \mathbb{N}\} \subset \left\{ \sum_{e \in E_0} \ell_e : E_0 \subset E \text{ is a non-empty set of edges} \right\}. \quad (3.9)$$

- (ii) *If e is some edge of $\mathcal{G}$ with $e \subset \text{supp } \psi_n$ and $v \in V \setminus V_0$ is a vertex incident to e , then e is a loop or there is a second edge $f \neq e$ incident to v with $f \subset \text{supp } \psi$.*

Remark 3.9 Note that Lemma 3.8 gives additional information on the geometric structure of the supports of the eigenfunctions ψ_n . For instance, part (ii) implies that—for sufficiently large n — $\text{supp } \psi_n$ contains a cycle or a path that connects two vertices that are in V_0 or of degree one. In particular, we find that if n is sufficiently large and e is an edge of $\mathcal{G}$ with $\text{supp } \psi_n = e$, then $|E| = 1$ or e is a loop.

As a consequence of the observations in Remark 3.9, part (i) of Lemma 3.8 and Proposition 3.5 we obtain the following

Corollary 3.10 *Under Assumption 3.1, if*

$$\frac{\ell_{\min}}{|\mathcal{G}|} = \liminf_{n \rightarrow \infty} \frac{\nu_n}{n} \quad (3.10)$$

holds, then $|E| = 1$ or there is a loop in $\mathcal{G}$ of length $\ell_{\min}$.

3.1. Proofs of the Lemmata

Here, we give the proofs of the three main auxiliary results, Lemmata 3.6, 3.7 and 3.8 which, combined, yield Theorem 3.4. We suppose throughout, without further comment, that Assumption 3.1 holds.

Proof of Lemma 3.6 This is an immediate consequence of Lemma 2.1, together with the fact that the eigenvalues of the operator associated with the restriction of the form $a_{q,\mathcal{B},w}$ to $H_0^1(\mathcal{G})$, that is, with Dirichlet boundary conditions everywhere, satisfy the usual Weyl asymptotics on any bounded interval and thus any finite union of disjoint intervals, see, for example, [3, Lemma 2.1]. $\square$

The second, Lemma 3.7, is in turn based on the principle that $\lambda_n(\mathcal{G})$ is always the first eigenvalue of any nodal domain of ψ_n , and as a consequence, that the maximal size of any nodal domain converges to zero as $n \rightarrow \infty$.

Lemma 3.11 *Given $n \in \mathbb{N}$, the eigenvalue $\lambda_n(\mathcal{G})$ and the associated eigenfunction ψ_n , with nodal domains $\mathcal{G}_1, \dots, \mathcal{G}_{\nu_n}$, for each $j = 1, \dots, \nu_n$ we have*

$$\lambda_n(\mathcal{G}) = \lambda_1(\mathcal{G}_j),$$

where the operator associated with the latter eigenvalue has Dirichlet conditions at all the boundary points of $\mathcal{G}_j$ corresponding to zeros of ψ_n (but the same vertex conditions as before at the interior vertices of $\mathcal{G}_j$, and the same potential q restricted to $\mathcal{G}_j$).

Proof Suppose without loss of generality that $\psi_n \geq 0$ in $\mathcal{G}_j$, with strict inequality except at the Dirichlet vertices of $\mathcal{G}_j$, and set $\phi_n := \psi_n \chi_{\mathcal{G}_j}$; in a slight abuse of notation, we will identify ϕ_n with its restriction to $\mathcal{G}_j$ in $L^2(\mathcal{G}_j)$. We observe that ϕ_n is an edgewise solution of the eigenvalue equation on the edges of $\mathcal{G}_j$ that satisfies Dirichlet conditions at the boundary points of $\mathcal{G}_j$ and the same vertex conditions as ψ_n at the interior vertices of $\mathcal{G}_j$; moreover, the corresponding eigenvalue, which we can read off the eigenvalue equation, is $\lambda_n(\mathcal{G})$.

That is, $\lambda_n(\mathcal{G})$ is an eigenvalue of $\mathcal{G}_j$, i.e. $\lambda_n(\mathcal{G}) = \lambda_k(\mathcal{G}_j)$ for some $k \geq 1$; moreover, its eigenfunction ϕ_n is, by construction, strictly positive in $\mathcal{G}_j$ except at boundary points of $\mathcal{G}_j$ and any interior Dirichlet vertices. By [29, Theorem 3], it is possible to choose the first eigenfunction φ_1 of $\lambda_1(\mathcal{G}_j)$ to have this property. Hence, the L^2 -scalar product of ϕ_n and φ_1 is strictly positive. Orthogonality of eigenfunctions on $\mathcal{G}_j$ belonging to different eigenspaces implies that $\lambda_n(\mathcal{G}) = \lambda_1(\mathcal{G}_j)$. (In fact, we could even infer $\phi_n = c\varphi_1$ for some $c > 0$, since the eigenspace corresponding to $\lambda_1(\mathcal{G}_j)$ has dimension 1, but we do not need this.) $\square$

The other ingredient we need for the proof of Lemma 3.7 is an estimate on the first eigenvalue of any operator $A(q, \mathcal{B}, w, \mathbf{V}_0)$ on any compact, connected graph (which in practice will be one of the nodal domains of ψ_n), which is given in Proposition B.1 in “Appendix B”. This proposition, when applied to the nodal domains $\mathcal{G}_j$ of ψ_n upon invoking Lemma 3.11, leads to the following estimate on the size of $\mathcal{G}_j$.

Lemma 3.12 *For all $n \in \mathbb{N}$, for all nodal domains $\mathcal{G}_j$, $j = 1, \dots, \nu_n$, we have*

$$|\mathcal{G}_j| \leq \frac{2\pi|\mathbf{E}|}{\sqrt{\lambda_n(\mathcal{G}) + \|q\|_1^2} - \|q\|_1}. \quad (3.11)$$

In particular, if $\lambda_n(\mathcal{G})$ is sufficiently large; explicitly, if

$$\lambda_n(\mathcal{G}) > \left(\frac{2\pi|\mathbf{E}|}{\ell_{\min}} + \|q\|_1 \right)^2 - \|q\|_1^2,$$

then no nodal domain can contain more than one vertex of $\mathcal{G}$.

Proof Fix a nodal domain $\mathcal{G}_j$; then $\mathcal{G}_j$ certainly cannot have more than $2|\mathbf{E}|$ edges. (Note that it could contain both ends of a given edge in $\mathcal{G}$ without containing the whole edge.) Now, by Lemma 3.11, we have $\lambda_n(\mathcal{G}) = \lambda_1(\mathcal{G}_j)$; combining this with the estimate (B.1) applied to $\mathcal{G}_j$ yields

$$\lambda_n(\mathcal{G}) \leq \left(\frac{2\pi|\mathbf{E}|}{|\mathcal{G}_j|} + \|q\|_1 \right)^2 - \|q\|_1^2.$$

Rearranging yields (3.11). If $\lambda_n(\mathcal{G})$ is sufficiently large as stated, then $|\mathcal{G}_j| < \ell_{\min}$ for all j , meaning no nodal domain can contain an entire edge. $\square$

Remark 3.13 The proof shows that if $\mathcal{G}_j$ is an interval, then since we may take $|\mathbf{E}(\mathcal{G}_j)| = 1$ in (B.1), (3.11) may be improved to

$$|\mathcal{G}_j| \leq \frac{\pi}{\sqrt{\lambda_n(\mathcal{G}) + \|q\|_1^2} - \|q\|_1}. \quad (3.12)$$

Proof of Lemma 3.7 Firstly, observe that by definition of the nodal domains, for any $n \in \mathbb{N}$, we have

$$|\text{supp } \psi_n| = \sum_{j=1}^{\nu_n} |\mathcal{G}_j|. \quad (3.13)$$

Now, note that $\lambda_n(\mathcal{G}) \rightarrow \infty$ (this follows from the compactness of the resolvent and the semi-boundedness of the form $a_{q,\mathcal{B},w}$, but can also be obtained as a consequence of Lemma 3.6). Hence, by Lemma 3.12 there exists some $n_0 \in \mathbb{N}$, which may be chosen to depend only on the metric graph $\mathcal{G}$ and q , such that the interior of each nodal domain $\mathcal{G}_j$ contains at most one vertex of $\mathcal{G}$, for all $n \geq n_0$. For such n , we suppose the nodal domains are ordered in such a way that, for some $m \leq |\mathbf{V}|$, $\mathcal{G}_1, \dots, \mathcal{G}_m$ each contain exactly one vertex in their respective interiors, while $\mathcal{G}_{m+1}, \dots, \mathcal{G}_{\nu_n}$ are all intervals; in particular, for all $j \geq m+1$, by Lemma 3.11, $\lambda_n(\mathcal{G}) = \lambda_1(\mathcal{G}_j)$. Now, on the one hand, since $q \geq 0$, $\lambda_1(\mathcal{G}_j) \geq \pi^2/|\mathcal{G}_j|^2$, whence

$$|\mathcal{G}_j| \geq \frac{\pi}{\lambda_n(\mathcal{G})^{1/2}}.$$

On the other hand, using (3.12), for such nodal domains we also have, supposing without loss of generality that $\lambda_{n_0} > \|q\|_1^2$,

$$|\mathcal{G}_j| \leq \frac{\pi}{\sqrt{\lambda_n(\mathcal{G}) + \|q\|_1^2} - \|q\|_1} \leq \frac{\pi}{\lambda_n(\mathcal{G})^{1/2} - \|q\|_1}.$$

Summing over j , we obtain the two-sided estimate

$$\begin{aligned} (\nu_n - m) \cdot \frac{\pi}{\lambda_n(\mathcal{G})^{1/2}} + \sum_{j=1}^m |\mathcal{G}_j| \\ \leq |\text{supp } \psi_n| \leq (\nu_n - m) \cdot \frac{\pi}{\lambda_n(\mathcal{G})^{1/2} - \|q\|_1} + \sum_{j=1}^m |\mathcal{G}_j|. \end{aligned} \quad (3.14)$$

Invoking (3.11), we may estimate the size of the first m nodal domains by

$$0 \leq \sum_{j=1}^m |\mathcal{G}_j| \leq \frac{2\pi|\mathbf{E}|m}{\lambda_n(\mathcal{G})^{1/2} - \|q\|_1}.$$

Using this in (3.14) and rearranging yield

$$|\operatorname{supp} \psi_n| \cdot \frac{\lambda_n(\mathcal{G})^{1/2} - \|q\|_1}{\pi} - (2|\mathbf{E}| - 1)m \leq \nu_n \leq |\operatorname{supp} \psi_n| \cdot \frac{\lambda_n(\mathcal{G})^{1/2}}{\pi} + m. \quad (3.15)$$

Observing that (3.15) is monotonic in m and using $m \leq |\mathbf{V}|$ yields (3.5). $\square$

Let us finally turn to Lemma 3.8. The main tool in its proof is a result in Sturm–Liouville theory that is likely to be already known; we provide a proof, since we could not find an appropriate reference in the literature.

Lemma 3.14 *If $I \subset \mathbb{R}$ is an open interval containing 0, $q \in L^1(I)$, and $u \in W_{loc}^{2,1}(I) \hookrightarrow C^1(I)$ is a distributional solution of $-u'' + qu = 0$ such that $u(0) = u'(0) = 0$, then $u = 0$ in I .*

Proof For any $x \in [0, \infty) \cap I$, we have

$$|u'(x)| = \left| \int_0^x u''(s) \, ds \right| \leq \int_0^x |q(s)u(s)| \, ds$$

and

$$|u(x)| = \left| \int_0^x u'(s) \, ds \right| \leq \int_0^x |u'(s)| \, ds.$$

Summing the two inequalities yields

$$\begin{aligned} |u(x)| + |u'(x)| &\leq \int_0^x |u'(s)| + |q(s)u(s)| \, ds \\ &\leq \int_0^x (1 + |q(s)|)(|u(s)| + |u'(s)|) \, ds; \end{aligned}$$

it now follows from Grönwall's lemma that $|u(x)| + |u'(x)| = 0$; we thus conclude that $u(x) = 0$ for all $x \in [0, \infty) \cap I$ and hence all $x \in I$. $\square$

Proof of Lemma 3.8 For part (i), it suffices to prove the following unique continuation statement: if any eigenfunction ψ_n has a zero at some point x_0 in the interior of an edge, then either $\psi_n \equiv 0$ in a neighbourhood of x_0 or $\psi'_n(x_0) \neq 0$. This, in turn, follows from Lemma 3.14, applied to ψ_n with a suitably adjusted q .

We prove part (ii) by contradiction using similar arguments: suppose that $\mathbf{e}$ is not a loop and that ψ_n vanishes on all edges incident to $\mathbf{v} \in \mathbf{V} \setminus \mathbf{V}_0$ with $\mathbf{f} \neq \mathbf{e}$. The vertex conditions associated with the operator $A(q, \mathcal{B}, w, \mathbf{V}_0)$ yield $\psi_{\mathbf{e},n}(\mathbf{v}) = 0$ and $\psi'_{\mathbf{e},n}(\mathbf{v}) = 0$ where $\psi_{\mathbf{e},n}$ denotes the restriction of ψ_n to $\mathbf{e}$. But then, as in the proof of part (i), Grönwall's lemma yields that $\psi_{n,\mathbf{e}}$ vanishes in a neighbourhood about $\mathbf{v}$ if $\lambda_n - q_{\min} \geq 0$ and therefore $\mathbf{e}$ is not a subset of $\operatorname{supp} \psi_n$. $\square$

4. A Stronger Pleijel's Theorem for the Laplacian with Natural Vertex Conditions

We consider the free Laplacian with natural conditions at all vertices, i.e. throughout this section, we suppose, in addition to Assumption 3.1, that $q \equiv 0$, $\mathcal{B} = 0$, $\mathbf{w} \equiv 1$, $V_0 = \emptyset$. In this case, we can say somewhat more.

The principal result of [11] (see Theorem 3.6 and Remark 3.7 there) states that, given a fixed graph topology without loops, the set of edge length vectors for which all eigenvalues of the corresponding graph are simple and all eigenfunctions do not vanish in the vertices is of the second Baire category (i.e. it is a countable intersection of open dense sets). As a consequence of Proposition 3.5, we obtain $\lim_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$ in this case; however, this could also be concluded using the main result in [22] which states that $\lim_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$ holds in the generic case where no eigenfunctions vanish at any vertices of the loop-free graph. Nevertheless, for future reference, we state our observation in the following

Theorem 4.1 ([11, 22]). *If $\mathcal{G}$ does not contain any loops, then the set of edge length vectors in $\mathbb{R}_+^{|\mathcal{E}|}$ for which, for the corresponding graph with the given topology and these edge lengths, all eigenvalues are simple and $\lim_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$, is of the second Baire category (i.e. it is a countable intersection of open dense sets).*

Put differently, in the case of natural vertex conditions and no potential, “almost all” graphs (in the usual sense of holding generically and being loop-free) have all eigenvalues simple and satisfy $\lim_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$.

Here, we wish to say more about the “non-generic” cases. The following theorem states that for graphs with pairwise commensurable edge lengths, at least $\limsup_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$ holds and thus the upper bound in Theorem 3.4 is sharp.¹

Theorem 4.2 *If the edge lengths of $\mathcal{G}$ are pairwise commensurable, then for every choice of orthonormal basis $(\psi_n)_{n \in \mathbb{N}}$ of the Laplacian with natural vertex conditions we have $\limsup_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$.*

Actually, we expect that on *any* graph $\mathcal{G}$, there exists a choice of (natural Laplacian) eigenfunctions for which $\limsup_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$. This would be an immediate consequence of the following conjecture together with Proposition 3.5.

Conjecture 4.3 Let, as usual, $\mathcal{G}$ be a compact, connected metric graph, and let $(\psi_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathcal{G})$ consisting of eigenfunction of the Laplacian with natural vertex conditions on $\mathcal{G}$. Then, there exists a subsequence $(\psi_{n_k})_{k \in \mathbb{N}}$ such that no eigenfunction ψ_{n_k} vanishes identically on any edge of $\mathcal{G}$.

Remark 4.4 (a) It follows from Theorem 4.1 that the conjecture is true generically; it is also true in the case where all edge lengths are pairwise commensurable, by Theorem 4.2. Moreover, it was shown in [2, Proposition

¹We thank Jonathan Rohleder for pointing out that Theorem 4.2 holds for *every* choice of orthonormal basis; in an earlier version, we had merely claimed the existence of such a basis.

A.1] that Conjecture 4.3 is also true if the edge lengths are rationally independent. A counterexample would hence require a graph to have at least two, but at most $|E| - 1$ rationally independent edge lengths. Additionally, topological constraints exist, too: it follows from [39, Lemma 2.7 and Corollary 2.8] that certain so-called *lasso trees* (i.e. graphs that can be constructed by attaching at most one loop to any leaf of a tree) cannot be counterexamples, either.

- (b) An alternative viewpoint which might occur naturally when thinking about Conjecture 4.3 is taken in [16], where the authors consider, for a compact graph with a fixed topology (denote by G the underlying discrete graph) and a distinguished set of vertices V_0 , the possible existence of eigenfunctions vanishing at V_0 . By adjusting the edge lengths to make the eigenvalue of interest 1, they reduce the object of interest to a subset of the $|E|$ -dimensional torus (the so-called *secular manifold* W_G) and in the case of pairwise incommensurable edge lengths obtain bounds on the possible size of the submanifold $W_G(V_0)$ consisting of those eigenfunctions vanishing at the given vertices, in dependence on the edge lengths $[\ell]_{\text{mod } 2\pi}$. Now, it can be shown that if Conjecture 4.3 does not hold for some graph G , then there exists a subtorus whose intersection with the secular manifold is contained in the union of the submanifolds $W_G(V_0)$ over $\emptyset \neq V_0 \subset V$. Here, the dimension of said subtorus equals the dimension of the $\mathbb{Q}$ -span of the entries of ℓ . Therefore, the analysis on the dimension of the manifolds $W_G(V_0)$ in [16, Theorem 1.3] may suggest an approach to deal with Conjecture 4.3. However, we emphasise that the two problems are not equivalent, since in Conjecture 4.3 the eigenfunctions are still allowed to vanish at the vertices, as long as they do not vanish identically on the edges.

Before giving the proof of Theorem 4.2, we will give a simple example which shows that the sequence $\frac{\psi_n}{n}$, and even its set of points of accumulation, can depend on the choice of the basis of eigenfunctions ψ_n .

Example 4.5 Consider again the equilateral 4-star from Example 3.3. Then, there are two families of eigenfunctions (and corresponding eigenvalues):

- Eigenfunctions which are invariant under permutation of the edges; up to scalar multiples these are of the form $\varphi_k(x) = \cos(\pi k x)$, $k \in \mathbb{N}$, on each edge $e_j \simeq [0, 1]$, with corresponding eigenvalues $\pi^2 k^2$, each of which has multiplicity one.
- Eigenfunctions which vanish at the central vertex: the corresponding eigenvalues, $\pi^2(k - \frac{1}{2})^2$, $k \in \mathbb{N}$, all have multiplicity three. Any function ϕ in the eigenspace has the form $c_j \sin(\pi(k - \frac{1}{2})x)$ on each edge e_j , where the coefficients $c_j = c_j(\phi) \in \mathbb{R}$ are chosen in such a way that the Kirchhoff condition is satisfied at the vertex.

We present two different choices for the c_j , which give rise to two different families of orthogonal bases with different nodal counts. To keep the presentation more compact and easier to read, we present these choices in table form:

	c_1	c_2	c_3	c_4		c_1	c_2	c_3	c_4
ϕ_1	1	-1	0	0	ϕ_1	1	-1	0	0
ϕ_2	0	0	1	-1	ϕ_2	1	1	-2	0
ϕ_3	1	1	-1	-1	ϕ_3	1	1	1	-3

Thus, for example, in the second case, for each $k \in \mathbb{N}$ there is an eigenfunction $\phi_3 = \phi_3(k)$ which takes the form $\sin(2\pi(k-1)x)$ on each of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and $-3\sin(2\pi(k-1)x)$ on $\mathbf{e}_4$. The orthogonality of ϕ_1, ϕ_2, ϕ_3 within each family is easy to check, as we simply require that the respective row vectors have inner product zero with each other, while the Kirchhoff condition is satisfied as long as the sum of the entries in each vector is zero. (The eigenfunctions will not have norm one, but this is obviously just a question of rescaling.) Now, in the first family, there are two eigenfunctions each supported on two different edges and one supported on all four; in the second family, the second eigenfunction is supported on three edges rather than two. It follows from Proposition 3.5 (also taking into account the nature of the eigenfunctions not vanishing on the central vertex) that in the first case the set of points of accumulation of the sequence $\frac{\nu_n}{n}$ is $\{\frac{1}{2}, 1\}$ and in the second case, it is $\{\frac{1}{2}, \frac{3}{4}, 1\}$.

Proof of Theorem 4.2 By inserting dummy vertices as necessary, we may assume that the graph is in fact equilateral; after rescaling if necessary, we may also assume without loss of generality that each edge has length 1. The following proof is essentially based on the possibility of considering all eigenfunctions as linear combinations of full frequency eigenfunctions on each edge; more precisely, for each $k \in \mathbb{N}$ it is known that $4\pi^2 k^2$ is an eigenvalue of multiplicity $\beta + 1$, where β is the first Betti number of $\mathcal{G}$; we refer to [6] for details. A basis of the corresponding eigenspace is obtained by choosing the following functions:

- the function $\phi_k \in H^1(\mathcal{G})$ given by $\phi_k(x) = \cos(2\pi kx)$ on each edge $\mathbf{e} \simeq [0, 1]$;
- and, given a specific choice of independent cycles $\mathcal{C}_1, \dots, \mathcal{C}_\beta$ with associated edge sets $\mathbf{E}_1, \dots, \mathbf{E}_\beta$ (i.e. $\mathcal{C}_j = \bigcup_{\mathbf{e} \in \mathbf{E}_j} \mathbf{e}$), the functions $\phi_{k,j} \in H^1(\mathcal{G})$ given by $\phi_{k,j}(x) = \sin(2\pi kx)$ on each edge $\mathbf{e} \simeq [0, 1]$ in $\mathbf{E}_j$ and $\phi_{k,j} = 0$ on each edge $\mathbf{e} \in \mathbf{E} \setminus \mathbf{E}_j$.

We point out that these linearly independent eigenfunctions are not necessarily orthogonal, but that will not be needed for the following argument. We also observe that while the eigenfunctions $\phi_{k,1}, \dots, \phi_{k,\beta}$ vanish at all the vertices of $\mathcal{G}$, the eigenfunction ϕ_k is nonzero at all vertices.

Now, let $\psi_{n_0}, \dots, \psi_{n_0+\beta}$ denote the eigenfunctions appearing in the given orthonormal basis associated with the eigenvalue $4\pi^2 k^2$ for some $n_0 = n_0(k)$. Then, we may write each of these eigenfunctions as a linear combination of $\phi_k, \phi_{k,1}, \dots, \phi_{k,\beta}$, and since $\psi_{n_0}, \dots, \psi_{n_0+\beta}$ are linearly independent, the coefficient corresponding to ϕ_k appearing in these linear combinations has to be nonzero for at least one ψ_{n_0+l} , $l = l(k) \in \{0, 1, \dots, \beta\}$. By our previous observations on the vanishing and non-vanishing behaviour of $\phi_k, \phi_{k,1}, \dots, \phi_{k,\beta}$ at

the vertices of $\mathcal{G}$, we conclude that ψ_{n_0+l} is nonzero in the vertices of $\mathcal{G}$. Putting $n_k = n_0(k) + l(k)$, we obtain a subsequence $(\psi_{n_k})_{k \in \mathbb{N}}$ with $\text{supp } \psi_{n_k} = \mathcal{G}$ for all k . That $\limsup_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$ now follows immediately from Theorem 3.4 and Proposition 3.5. $\square$

In the proof of Theorem 4.2, given a cycle with pairwise commensurable edge lengths, we constructed a sequence of eigenfunctions whose support was said cycle. Using Proposition 3.5, we can obtain the following result as a by-product:

Proposition 4.6 *If $\mathcal{G}$ contains a cycle $\mathcal{C}$ with corresponding edge set E_0 , so that the lengths of the edges in E_0 are pairwise commensurable, then the orthonormal basis of eigenfunctions of $\mathcal{G}$ may be chosen so that $\frac{\sum_{e \in E_0} \ell_e}{|\mathcal{G}|}$ is a point of accumulation of $\frac{\nu_n}{n}$. In particular, if $\mathcal{G}$ contains a loop of length ℓ , then ℓ may be chosen so that $\frac{\ell}{|\mathcal{G}|}$ is a point of accumulation of $\frac{\nu_n}{n}$. In particular, the lower estimate of (3.2) is sharp whenever $\ell_{\min}$ is realised by a loop of $\mathcal{G}$.*

Proposition 4.6 has two obvious consequences which are nevertheless worth stating explicitly. First given any $\varepsilon > 0$, there exists a graph $\mathcal{G}$ such that for this graph $\liminf_{n \rightarrow \infty} \frac{\nu_n}{n} < \varepsilon$. Secondly, if $\mathcal{G}$ is neither a tree nor a itself a cycle and has pairwise commensurable edge lengths, then there exists an orthonormal basis so that $\liminf_{n \rightarrow \infty} \frac{\nu_n}{n} < 1$ holds. This is not necessarily true if the graph is a tree: indeed, the following example shows that there are trees with pairwise commensurable edge lengths where any eigenfunction of the Laplacian with natural conditions is supported on the whole tree, which in turn yields, by Proposition 3.5, that $\lim_{n \rightarrow \infty} \frac{\nu_n}{n} = 1$ holds for any orthonormal basis of eigenfunctions.

Example 4.7 Consider the 3-star $\mathcal{G}$ consisting of three edges e_1, e_2, e_3 of edge lengths ℓ_1, ℓ_2, ℓ_3 , respectively. An eigenfunction $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ corresponding to some eigenvalue $\lambda > 0$ is of the form $\varphi_j(x) = c_j \cos(\sqrt{\lambda}x)$ on the edge $e_j \simeq [0, \ell_j]$ where ℓ_j corresponds to the centre vertex of the star. If φ vanished on some edge e_i , we would obtain $c_i = 0$ and $c_j \neq 0$ for $j \neq i$. Then, continuity in the centre vertex yields $0 = \varphi_j(\ell_j)$ for $j \neq i$ and thus $0 = \cos(\ell_j \sqrt{\lambda})$. Therefore, there is some $m_j \in \mathbb{N}$ such that $\ell_j \sqrt{\lambda} = \pi(m_j - \frac{1}{2})$. This yields

$$\frac{\ell_k}{\ell_j} (2m_k - 1) = 2m_j - 1 \quad (4.1)$$

for $k, j \neq i$. Now, we choose $\ell_1 = 1$, $\ell_2 = 2$ and $\ell_3 = 4$. Suppose without loss of generality that $\ell_k > \ell_j$ in (4.1). Then, with our choice of the edge lengths, (4.1) clearly leads to a contradiction, since the left-hand side is an even integer, whereas the right-hand side is odd. Therefore, all eigenfunctions on the 3-star with edge lengths 1, 2 and 4 must be supported on the whole graph.

Remark 4.8 Theorem 4.1 and Proposition 4.6 also hold if any mix of delta couplings and Dirichlet conditions is imposed at some vertices, although for the former we still need a certain additional genericity assumption (coming

FIGURE 1. The $[3, 2]$ -pumpkin chain $\mathcal{G}$

from [11, Theorem 3.6]) on the delta couplings. In the former case, the proof is essentially identical; in the latter case, we may directly construct eigenfunctions supported on the cycle out of suitably adjusted sine curves, which in particular vanish at all vertices and thus satisfy all possible delta couplings there. We expect Proposition 4.6 to hold for many tree graphs as well, although here the situation is more complicated, as Example 4.7 shows.

We finish this section with a discussion of the case of equality in (3.1). Recall that a metric graph $\mathcal{G}$ is called a *flower graph* if all its edges are loops. If, in addition, $\mathcal{G}$ has only two edges, then we call it a *figure eight graph*.

Proposition 4.9 *In addition to Assumption 3.1, suppose that $q \equiv 0$, $\mathcal{B} = 0$, $\mathbf{w} \equiv 1$, $\mathbf{V}_0 = \emptyset$. Then, there exists an orthonormal basis $(\psi_n)_{n \in \mathbb{N}}$ of eigenfunctions so that for each non-empty subset $E_0 \subset E$, there is a subsequence $(\psi_{n_k})_{k \in \mathbb{N}}$ with*

$$\text{supp } \psi_{n_k} = \bigcup_{e \in E_0} e, \quad k \in \mathbb{N}$$

if and only if one the following cases occurs:

- (i) $\mathcal{G}$ is an interval or a cycle;
- (ii) $\mathcal{G}$ is a figure eight graph;
- (iii) $\mathcal{G}$ is a flower graph with pairwise commensurable edge lengths.

In particular, for any of these graphs, for this choice of an orthonormal basis, equality holds in (3.1).

Before we give a proof of Proposition 4.9, we point out there may be other graphs for which there is equality (3.1), as the following example will show. In particular, it will demonstrate that a full characterisation of equality in (3.1) would have to incorporate both the local and global combinatorial and metric structure of the graph and would therefore be too technically complicated to be treated here.

Example 4.10 Consider the $[3, 2]$ -pumpkin chain $\mathcal{G}$ that consists of two vertices $\mathbf{v}_1$ and $\mathbf{v}_2$, three parallel edges $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of equal length 1 each connecting the two vertices, and one pendant loop $\mathbf{e}_4$ of length 1 attached to the vertex $\mathbf{v}_2$; see Fig. 1.

In the spirit of [9, Lemma 5.4], we choose an orthonormal basis of eigenfunctions consisting of (i) *longitudinal functions*, functions that are radially symmetric with respect to the vertex $\mathbf{v}_1$, and (ii) *transversal functions*, functions whose support is contained in one of the pumpkins $\mathbf{e}_1 \cup \mathbf{e}_2 \cup \mathbf{e}_3$ or $\mathbf{e}_4$. The

infinite orthonormal subsequence of longitudinal eigenfunctions corresponds to a 1-dimensional Sturm–Liouville problem; these eigenfunctions are therefore supported on the whole graph (see [28, Section 5.2] for details). For $k \in \mathbb{N}$, the transversal eigenfunctions supported on e_4 are given by $\phi(x) = \sin(2\pi kx)$ for $x \in e_4 \simeq [0, 1]$. The transversal eigenfunctions ϕ with support in $e_1 \cup e_2 \cup e_3$ are given by $\phi(x) = c_j \sin(\pi kx)$ for $x \in e_j \simeq [0, 1]$, $j = 1, 2, 3$ and constants $c_j \in \mathbb{R}$ with $c_1 + c_2 + c_3 = 0$. By choosing $(c_1, c_2, c_3) = (1, 1, -2)$ and $(c_1, c_2, c_3) = (1, -1, 0)$, respectively, we obtain—after normalisation—an orthonormal basis of eigenfunctions, and by Proposition 3.5, the set of points of accumulation of the sequence $\frac{\nu_n}{n}$ is $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, which coincides with set of values $\frac{\sum_{e \in E_0} \ell_e}{|\mathcal{G}|}$ for non-empty subsets $E_0 \subset E$. Note, however, that there is no possible choice of eigenfunctions supported on, say, $e_1 \cup e_4$; thus, there is no contradiction to Proposition 4.9.

Proof of Proposition 4.9 Clearly, the statement is true if $\mathcal{G}$ has only one edge, so we may assume that $\mathcal{G}$ has at least two edges.

Suppose first that there exists an orthonormal basis of eigenfunctions as stated in Proposition 4.9. Then, in particular, for each edge of $\mathcal{G}$ there is a sequence of eigenfunctions supported exactly on that edge. From Lemma 3.8 and Remark 3.9, we infer that each edge is a loop and therefore $\mathcal{G}$ is a flower graph. If $\mathcal{G}$ has two edges, then we are clearly in case (ii).

It remains to show that the edge lengths of $\mathcal{G}$ are pairwise commensurable if $\mathcal{G}$ has at least three edges. Let $e_1 \neq e_2$ be any two edges of $\mathcal{G}$. Then, by assumption, there is an eigenfunction ϕ with associated eigenvalue $\lambda > 0$ for which $\text{supp } \phi = e_1 \cup e_2$. Since φ vanishes on all edges different from e_1 and e_2 and is continuous in the centre vertex v of the flower, we obtain $\phi(v) = 0$. Therefore, there exist $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ such that $\phi(x) = a_j \sin(\sqrt{\lambda}x)$ for $j = 1, 2$ and $x \in e_j \simeq [0, \ell_{e_j}]$. Then, $\phi(v) = 0$ yields $\sin(\sqrt{\lambda}\ell_{e_j}) = 0$, and thus, for $j = 1, 2$, there exists some $k_j \in \mathbb{N}$ such that $\sqrt{\lambda}\ell_{m_j} = \pi k_j$. We obtain $\frac{\ell_{e_1}}{\ell_{e_2}} = \frac{k_1}{k_2} \in \mathbb{Q}$; hence, ℓ_{e_1} and ℓ_{e_2} are commensurable.

Finally, we show that for each of the graphs in (i), (ii) and (iii), there does indeed exist an orthonormal basis of eigenfunctions with the claimed properties. Such a basis obviously exists for an interval or cycle. For a figure eight graph, such a basis can be constructed following arguments similar to the ones used in Example 4.10. So suppose $\mathcal{G}$ is a flower graph with pairwise commensurable edge lengths. Then, there exist $a > 0$ and $m_e \in \mathbb{N}$ with $\ell_e = am_e$ for all $e \in E$. It is sufficient to show that for each subset $E_0 \subset E$, there is an infinite sequence of eigenfunctions associated with pairwise different eigenvalues that are supported on $\bigcup_{e \in E_0} e$. Indeed such a sequence exists: for $k \in \mathbb{N}$, there is an eigenfunction ϕ_k associated with the eigenvalue $\frac{4\pi^2 k^2}{a^2}$ given by $\phi_k(x) = \sin(\frac{2\pi k}{a}x)$ for $x \in e \simeq [0, \ell_e]$ on $e \in E_0$ and $\phi_k = 0$ on $e \in E \setminus E_0$. $\square$

5. Pleijel's Theorem for the p -Laplacian

In this last section, we are going to turn to a different class of operators. For a compact and connected metric graph $\mathcal{G}$ and $p \in (1, \infty)$, we let $W^{1,p}(\mathcal{G})$ denote the space of edgewise $W^{1,p}$ -functions that are continuous across the vertices. The p -Laplacian on metric graphs can be generally introduced by considering the Fréchet differentiable energy functional

$$\mathfrak{E}_p : u \mapsto \int_{\mathcal{G}} |u'|^p \, dx, \quad u \in D(\mathfrak{E}_p) := W^{1,p}(\mathcal{G}),$$

and taking its Fréchet derivative in the real Hilbert space $L^2(\mathcal{G})$; this returns natural vertex conditions, i.e. continuity across the vertices along with a nonlinear analogue of Kirchhoff's condition. Unlike in the linear case of $p = 2$, different notions of eigenvalues for the p -Laplacian may *a priori* coexist, see “Appendix A”, with *Carathéodory* eigenvalues being more general than *variational* ones. Given a general compact metric graph, it seems to be unknown how large the set of Carathéodory eigenvalues of this operator is, but its subset that is most relevant for our purposes—the set of variational eigenvalues—is certainly countably infinite; such variational eigenvalues can be characterised by the Ljusternik–Schnirelmann principle, a nonlinear counterpart of the linear min-max principle.

Here, we will denote by $(\lambda_{n,p}(\mathcal{G}))_{n \in \mathbb{N}}$ the sequence of variational eigenvalues, along with a sequence of associated (Carathéodory) eigenfunctions $(\psi_{n,p})_{n \in \mathbb{N}}$, which we fix throughout; each eigenfunction has $\nu_{n,p}$ corresponding nodal domains $\mathcal{G}_1, \dots, \mathcal{G}_{\nu_{n,p}}$.

Actually, in view of the nonlinear versions of the Beurling–Dény conditions in [15], as in (2.2), different vertex conditions inducing (nonlinear) positive semigroups can be obtained upon considering the above energy on spaces of the form $W_w^{1,p}(\mathcal{G})$ and/or adding boundary terms; we expect our results to continue to hold for these. However, owing to a lack of background theory available for such nonlinear operators on metric graphs, we will not pursue such generalisations here.

In this section, we will always impose the following

Assumption 5.1 $\mathcal{G}$ is a compact, connected metric graph with underlying combinatorial graph $G = (V, E)$ and edge lengths ℓ_e , $e \in E$; we set $\ell_{\min} := \min_{e \in E} \ell_e$. We also fix $p \in (1, \infty)$ and let $q = \frac{p}{p-1}$ be its Hölder conjugate.

Our third main result, a version of Pleijel's theorem for the p -Laplacian with natural vertex conditions, is a direct analogue of Theorem 3.4.

Theorem 5.2 *Under Assumption 5.1, and with the notation on the nodal count introduced above, we have*

$$\text{acc} \left\{ \frac{\nu_{n,p}}{n} : n \in \mathbb{N} \right\} \subset \left\{ \frac{\sum_{e \in E_0} \ell_e}{|\mathcal{G}|} : E \supset E_0 \text{ is a non-empty set of edges} \right\}. \quad (5.1)$$

In particular, $\text{acc} \left\{ \frac{\nu_{n,p}}{n} : n \in \mathbb{N} \right\}$ is a finite set, and

$$0 < \frac{\ell_{\min}}{|\mathcal{G}|} \leq \liminf_{n \rightarrow \infty} \frac{\nu_{n,p}}{n} \leq \limsup_{n \rightarrow \infty} \frac{\nu_{n,p}}{n} \leq 1, \quad (5.2)$$

where $\ell_{\min} := \min\{\ell_e : e \in E\}$.

We also observe that Proposition 3.5 holds verbatim with $\nu_{n,p}$ and $\psi_{n,p}$ in place of ν_n and ψ_n , respectively. The proof of Theorem 5.2 (and Proposition 3.5 in this case) follows exactly the same lines as above.

In this case, we give a short proof of the Weyl asymptotics for $\lambda_{n,p}(\mathcal{G})$ in “Appendix” (see Theorem A.3), as it does not previously seem to have been established for the p -Laplacian on metric graphs. We next state p -versions of unique continuation (cf. Lemma 3.8), the fact that $\lambda_{n,p}$ is the first Dirichlet eigenvalue restricted to each nodal domain of $\psi_{n,j}$ (cf. Lemma 3.11) and a basic upper bound on the first Dirichlet eigenvalue (cf. Proposition B.1), respectively.

The following lemma on unique continuation is actually valid for any vertex conditions enforced in the (real) Sobolev space $W^{1,p}(\mathcal{G})$, the domain of $\mathfrak{E}_p$, since they necessarily result in real eigenvalues and eigenfunctions.

Lemma 5.3 (Possible values of $|\text{supp } \psi_{n,p}|$). *Under Assumption 5.1,*

$$\left\{ |\text{supp } \psi_{n,p}| : n \in \mathbb{N} \right\} \subset \left\{ \sum_{e \in E_0} \ell_e : E \supset E_0 \text{ is a non-empty set of edges} \right\}. \quad (5.3)$$

Proof This follows immediately from the assertion that if $\psi_{n,p}(x) = 0$ for some x in the interior of an edge e , then either $\psi_{n,p}$ changes sign in any open neighbourhood of x , or $\psi_{n,p}$ vanishes identically on that edge. Suppose that $\psi_{n,p}(x) = 0$ at some interior point $x \in e$, and that $\psi_{n,p}$ does *not* change sign at x . Then, by the smoothness properties of $\psi_{n,p}$ stated in Lemma A.1, we also have $\psi'_{n,p}(x) = 0$. That is, $\psi_{n,p}$ is a solution of

$$\begin{aligned} u' &= |v|^{\frac{1}{p-1}} \text{sgn } v \\ v' &= -\lambda |u|^{p-1} \text{sgn } u \end{aligned} \quad \text{in a neighbourhood of } x$$

with boundary conditions

$$u(x) = v(x) = 0.$$

By [30, Theorem 3.1], this equation has exactly one smooth solution, which in this case is clearly the zero function. Hence, $\psi_{n,p}$ vanishes identically in a neighbourhood of x and so, extending the argument, on the whole metric edge $e \simeq (0, \ell_e)$. $\square$

Lemma 5.4 *Under Assumption 5.1, for all $n \in \mathbb{N}$*

$$\lambda_{n,p}(\mathcal{G}) = \lambda_1(\mathcal{G}_j),$$

where the latter is the smallest variational eigenvalue of the p -Laplacian on $\mathcal{G}_j$ with Dirichlet conditions at all the boundary points of $\mathcal{G}_j$ corresponding to zeros of $\psi_{n,p}$ and natural conditions at all other vertices of $\mathcal{G}_j$.

Proof In analogy with (2.3), denote by $W_0^{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j)$ the domain of the functional associated with the eigenvalue problem on $\mathcal{G}_j$ as described in the assertion; then by choice of $\mathcal{G}_j$, $\psi_{n,p}|_{\mathcal{G}_j} \in W_0^{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j)$. As usual, in a slight abuse of notation we will identify $W_0^{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j)$ with a closed subspace of $W^{1,p}(\mathcal{G})$ and in particular simply write $\psi_{n,p} \in W_0^{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j)$. We start by observing that $\psi_{n,p}$ is clearly an eigenfunction on $\mathcal{G}_j$, for the eigenvalue $\lambda_{n,p}(\mathcal{G})$, as follows from the fact that

$$\int_{\mathcal{G}} |\psi'_{n,p}(x)|^{p-2} \psi'_{n,p}(x) \varphi'(x) \, dx = \lambda_{n,p}(\mathcal{G}) \int_{\mathcal{G}} |\psi_{n,p}(x)|^{p-2} \psi_{n,p}(x) \varphi(x) \, dx$$

for all $\varphi \in W^{1,p}(\mathcal{G})$ and hence, in particular, for all $\varphi \in W_0^{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j)$. Moreover, $\psi_{n,p}$ is either strictly positive or strictly negative in (the connected set) $\mathcal{G}_j \setminus \partial\mathcal{G}_j$, as is an immediate consequence of the definition of nodal domains. The proof of [25, Theorem 1.1] may now be repeated verbatim to show that $\lambda_{n,p}(\mathcal{G})$ is in fact the first eigenvalue of the p -Laplacian on $\mathcal{G}_j$ with the desired vertex conditions. $\square$

The following upper bound was proved in [19, Theorem 3.8]. Again, this bound extends to the lowest variational eigenvalue of all realisations of the p -Laplacian induced by the functional $\mathfrak{E}_p$ defined on a superset of $W_0^{1,p}(\mathcal{G})$.

Lemma 5.5 *Under Assumption 5.1, let V_0 be a (finite) non-empty set of points of $\mathcal{G}$, such that $\mathcal{G} \setminus V_0$ is connected, and, for $p \in (1, \infty)$, let $\lambda_{1,p}(\mathcal{G}; V_0)$ be the first eigenvalue of the p -Laplacian with Dirichlet conditions at V_0 and natural conditions at all other vertices. Then,*

$$\lambda_{1,p}(\mathcal{G}; V_0) \leq \frac{p}{q} \left(\frac{\pi_p |E|}{|\mathcal{G}|} \right)^p. \quad (5.4)$$

Here, as usual, π_p is the constant defined via $\pi_p = \frac{2\pi}{p \sin(\frac{\pi}{p})}$. The final auxiliary result we need is an analogue of Lemma 3.12, an estimate from above on the size of the nodal domains (equivalently, a lower bound on $\lambda_{n,p}$), which is itself a direct consequence of the preceding two lemmata. This establishes in particular (together with Lemma 5.3) that the number of nodal domains does in fact diverge to infinity as $n \rightarrow \infty$.

Lemma 5.6 *Fix $n \in \mathbb{N}$ and let $\mathcal{G}_1, \dots, \mathcal{G}_{\nu_{n,p}}$ be the nodal domains of $\psi_{n,p}$. Then, for all $j = 1, \dots, \nu_{n,p}$ we have*

$$|\mathcal{G}_j| \leq \frac{2\pi_p |E| p^{1/p}}{(q \lambda_{n,p}(\mathcal{G}))^{1/p}}. \quad (5.5)$$

In particular, if $n \in \mathbb{N}$ is large enough, specifically, if $\lambda_{n,p}(\mathcal{G}) > \frac{p}{q} \left(\frac{2\pi_p |E|}{|\mathcal{G}|} \right)^p$, then no nodal domain can contain more than one vertex.

Proof Fix a nodal domain $\mathcal{G}_j$, then since $\mathcal{G}_j$ cannot have more than $2|E|$ edges, by Lemma 5.4 and Lemma 5.5, the latter applied to $\mathcal{G}_j$, we have

$$\lambda_{n,p}(\mathcal{G}) = \lambda_{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j) [= \lambda_{1,p}^D(\mathcal{G}_j)] \leq \frac{p}{q} \left(\frac{\pi_p |E|}{|\mathcal{G}|} \right)^p.$$

Rearranging yields (5.5). The other assertion is clear. $\square$

We can now formulate a version of the central Lemma 3.7 for the p -Laplacian.

Lemma 5.7 *For all sufficiently large $n \in \mathbb{N}$, we have*

$$\begin{aligned} & \frac{|\operatorname{supp} \psi_{n,p}|}{\pi_p} \cdot \left(\frac{q\lambda_{n,p}(\mathcal{G})}{p} \right)^{1/p} - (2|E| - 1)|V| \\ & \leq \nu_{n,p} \leq \frac{|\operatorname{supp} \psi_{n,p}|}{\pi_p} \cdot \left(\frac{q\lambda_{n,p}(\mathcal{G})}{p} \right)^{1/p} + |V| \end{aligned} \quad (5.6)$$

Concretely, the condition on $\lambda_{n,p}(\mathcal{G})$ from Lemma 5.6 is enough to ensure that (5.6) holds.

Proof We suppose n is large enough that there are in fact $|V|$ nodal domains containing exactly one vertex of $\mathcal{G}$, while the rest contain no vertices; that this is possible is guaranteed by Lemma 5.6. Let $\mathcal{G}_1, \dots, \mathcal{G}_{\nu_{n,p}}$ be the nodal domains of $\psi_{n,p}$. We assume that $\mathcal{G}_1, \dots, \mathcal{G}_{|V|}$ each contain a vertex, while the rest do not; then, each $\mathcal{G}_j$ is an interval with Dirichlet conditions at its endpoints if $j > |V|$, and in this case

$$\lambda_{n,p}(\mathcal{G}) = \lambda_{1,p}(\mathcal{G}_j; \partial\mathcal{G}_j) = \frac{p}{q} \left(\frac{\pi_p}{|\mathcal{G}_j|} \right)^p,$$

i.e. $|\mathcal{G}_j| = \pi_p \left(\frac{p}{q\lambda_{n,p}(\mathcal{G})} \right)^{1/p}$. Hence, as in the proof of Lemma 3.7, using the definition of the nodal domains,

$$|\operatorname{supp} \psi_{n,p}| = \sum_{j=1}^{\nu_{n,p}} |\mathcal{G}_j| = \sum_{j=1}^{|V|} |\mathcal{G}_j| + (\nu_{n,p} - |V|)\pi_p \left(\frac{p}{q\lambda_{n,p}(\mathcal{G})} \right)^{1/p}.$$

The sum on the right-hand side is nonnegative and may be controlled from above using Lemma 5.6; this yields

$$\begin{aligned} (\nu_{n,p} - |V|)\pi_p \left(\frac{p}{q\lambda_{n,p}(\mathcal{G})} \right)^{1/p} & \leq |\operatorname{supp} \psi_{n,p}| \leq (\nu_{n,p} - |V|)\pi_p \left(\frac{p}{q\lambda_{n,p}(\mathcal{G})} \right)^{1/p} \\ & + 2\pi_p |V| |E| \left(\frac{p}{q\lambda_{n,p}(\mathcal{G})} \right)^{1/p}. \end{aligned}$$

Rearranging yields (5.6). $\square$

Proof of Theorem 5.2 and of Proposition 3.5 for the p -Laplacian Upon combining the result of Lemma 5.7 with the Weyl asymptotics of Theorem A.3, we obtain

$$\nu_{n,p} = \frac{|\operatorname{supp} \psi_{n,p}|}{|\mathcal{G}|} n + o(n) \quad \text{as } n \rightarrow \infty,$$

which in particular proves Proposition 3.5 for the p -Laplacian. Lemma 5.3 now yields (5.1); the other assertions of Theorem 5.2 follow immediately. $\square$

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. Weyl's Law for the p -Laplacian on Metric Graphs

The goal of this section is, firstly, to recall briefly the construction of the variational eigenvalues of the p -Laplacian (with natural vertex conditions, that is, continuity and an appropriate p -version of the Kirchhoff condition); this is well known on intervals and domains, and nothing changes in the case of metric graphs (see also [19]); secondly, we will show that the Weyl asymptotics known for the p -Laplacian eigenvalues on the interval also holds on metric graphs. This is a simple application of Dirichlet–Neumann bracketing.

We recall that the n th variational eigenvalue of the p -Laplacian on a graph $\mathcal{G}$ with natural vertex conditions, $p \in (1, \infty)$, may be characterised variationally in terms of the Krasnosel'skii genus. More precisely, analogously to [13, Section 5], see also [21, Section 3], we consider the manifold

$$\mathcal{S} := \left\{ f \in W^{1,p}(\mathcal{G}) : \|f\|_{L^p(\mathcal{G})}^p = 1 \right\},$$

and for a closed, symmetric, non-empty set $\mathcal{A} \subset \mathcal{S}$ its Krasnosel'skii genus $\gamma(\mathcal{A}) \in \mathbb{N}$ by

$$\gamma(\mathcal{A}) := \inf\{k \in \mathbb{N} : \text{there exists } \Phi : \mathcal{A} \rightarrow \mathbb{S}^k \text{ continuous and odd}\},$$

(or $\gamma(\mathcal{A}) = \infty$ if this infimum is infinite). Here, $\mathbb{S}^k$ denotes the unit sphere in $\mathbb{R}^k$ for $k \in \mathbb{N}$ and a map $\Phi : \mathcal{A} \rightarrow \mathbb{S}^k$ is called *odd* if $\Phi(-f) = -\Phi(f)$ holds for all $f \in \mathcal{A}$. Finally, for every $n \in \mathbb{N}$ we set $\mathcal{F}_n := \{\mathcal{A} \subset \mathcal{S} : \gamma(\mathcal{A}) \geq n\}$. Then, we may define the n th variational eigenvalue $\lambda_{n,p}(\mathcal{G})$ of the p -Laplacian on $\mathcal{G}$ with natural vertex conditions by

$$\lambda_{n,p}(\mathcal{G}) = \inf_{\mathcal{A} \in \mathcal{F}_n} \sup_{f \in \mathcal{A}} \int_{\mathcal{G}} |f'(x)|^p dx. \quad (\text{A.1})$$

That this does indeed give rise to an infinite sequence of eigenvalues on any compact metric graph $\mathcal{G}$ follows from the same argument as the one used in [13], see also [20, 21]. While *a priori* $(\lambda_{n,p}(\mathcal{G}))_{n \in \mathbb{N}}$ is just a sequence of critical points of a certain functional, mimicking the proof of [14, Theorem 2.1] one can show

by known methods that each such *variational eigenvalue* is actually associated with an eigenfunction in the following weak sense.

Lemma A.1 *For each $n \in \mathbb{N}$, there exists a (so-called Carathéodory) eigenfunction associated with $\lambda = \lambda_{n,p}(\mathcal{G})$, i.e. a nonzero solution $\psi_{n,p} = u$ of the system*

$$\begin{aligned} u' &= |v|^{\frac{1}{p-1}} \operatorname{sgn} v \\ v' &= -\lambda |u|^{p-1} \operatorname{sgn} u. \end{aligned}$$

such that u and v satisfy the continuity and Kirchhoff-type vertex conditions, respectively. In particular, $\psi_{n,p}$ is a real, absolutely continuous function, and so is $|\psi_{n,p}|^{p-1} \operatorname{sgn} \psi_{n,p}$.

In particular, and with the terminology of [14]: like on intervals with Dirichlet or Neumann boundary conditions, each variational eigenvalue is a Carathéodory eigenvalue, too.

We also define the corresponding eigenvalues in the case that all vertices of $\mathcal{G}$ are equipped with either a Dirichlet or a Neumann condition, in which case $\mathcal{G}$ decomposes into the disjoint union of $|\mathbf{E}|$ edges, or intervals; this obviously includes the case $|\mathbf{E}| = 1$ where $\mathcal{G}$ is just a (bounded) interval itself. We define the natural analogues of $\mathcal{S}$, namely

$$\begin{aligned} \mathcal{S}^D &:= \left\{ f \in W_0^{1,p}(\mathcal{G}) : \|f\|_{L^p(\mathcal{G})}^p = 1 \right\}, \\ \mathcal{S}^N &:= \left\{ f \in \bigoplus_{e \in \mathbf{E}} W^{1,p}(0, \ell_e) : \|f\|_{L^p(\mathcal{G})}^p = 1 \right\}, \end{aligned}$$

where $W_0^{1,p}(\mathcal{G})$ is, analogously to $H_0^1(\mathcal{G}) := H^1(\mathcal{G}; \mathbf{V})$ in (2.3), the space of all functions in $W^{1,p}(\mathcal{G})$ vanishing at *all* vertices, and $\bigoplus_{e \in \mathbf{E}} W^{1,p}(0, \ell_e)$ is to be identified with a superset of $W^{1,p}(\mathcal{G})$ in the obvious way. Then, defining the Krasnosel'skii genus in the same way as above, and finally

$$\mathcal{F}_n^{D,N} := \{ \mathcal{A} \subset \mathcal{S}^{D,N} : \gamma(\mathcal{A}) \geq n \},$$

we define the respective n th variational eigenvalues by

$$\lambda_{n,p}^{D,N}(\mathcal{G}) = \inf_{\mathcal{A} \in \mathcal{F}_n^{D,N}} \sup_{f \in \mathcal{A}} \int_{\mathcal{G}} |f'(x)|^p dx. \quad (\text{A.2})$$

Again, it is easy to see that in both cases, there is a sequence of eigenvalues; this is proved explicitly in [30, Theorems 3.3 and 3.4] for the p -Laplacian on intervals (but it makes no difference if we consider a disjoint union of intervals). We may also consider eigenvalues $\lambda_{n,p}^D(\mathcal{G}; \mathbf{V}_0)$ with a Dirichlet condition imposed at some subset $\mathbf{V}_0$ of the vertices and natural conditions at the rest; all the definitions are analogous, and we do not go into details.

The following Dirichlet–Neumann bracketing principle is an immediate consequence of the respective eigenvalue definitions.

Lemma A.2 Fix $p \in (1, \infty)$ and $\mathcal{G}$. With the notation introduced above, we have

$$\lambda_{n,p}^N(\mathcal{G}) \leq \lambda_{n,p}(\mathcal{G}) \leq \lambda_{n,p}^D(\mathcal{G})$$

for all $n \geq 1$.

Proof We observe that $\mathcal{S}^D \subset \mathcal{S} \subset \mathcal{S}^N$, whence $\mathcal{F}_n^D \subset \mathcal{F}_n \subset \mathcal{F}_n^N$. The statement is now an immediate consequence of the characterisations (A.1) and (A.2). $\square$

Theorem A.3 (Weyl asymptotics). Fix $p \in (1, \infty)$ and suppose the graph $\mathcal{G}$ has total length $|\mathcal{G}|$. Then, the n th variational eigenvalue $\lambda_n(\mathcal{G})$ satisfies

$$\lambda_{n,p}(\mathcal{G}) = (p-1) \left(\frac{\pi_p}{|\mathcal{G}|} \right)^p n^p + o(n^p) \quad \text{as } n \rightarrow \infty, \quad (\text{A.3})$$

where we recall $\pi_p = \frac{2\pi}{p \sin(\frac{\pi}{p})}$.

A corresponding Weyl asymptotics for the Dirichlet p -Laplacian on general domains in $\mathbb{R}^n$ was established only very recently, see [31].

Proof We first observe that the Weyl asymptotics (A.3) holds for the p -Laplacian on an interval with both Dirichlet and Neumann boundary conditions (see [30, Theorems 3.3 and 3.4]). Hence, it also holds in the case that $\mathcal{G}$ is a disjoint collection of intervals; equivalently, for any graph $\mathcal{G}$ it holds for $\lambda_{n,p}^N(\mathcal{G})$ and $\lambda_{n,p}^D(\mathcal{G})$. The conclusion of the theorem now follows immediately from Lemma A.2. $\square$

Appendix B. An Estimate on the First Eigenvalue of General Schrödinger Operators

In this appendix, we give an estimate on the first eigenvalue $\lambda_1(\mathcal{G})$ of any Schrödinger operator $A = A(q, \mathcal{B}, w, V_0)$ of the form introduced in Sect. 2, on any compact metric graph. Estimates of this level seem to be new at this level of generality and may be of some independent interest, although there is considerable room for improvement. In practice, we will take Assumption 3.1; however, the following statement and proof are also valid for general $q \in L^1(\mathcal{G})$, not necessarily bounded from below. The proof also shows that the norm $\|q\|_1$ may be replaced by $\|q_+\|_1$, the norm of the positive part of q . (This is a trivial consequence of the variational characterisation of λ_1 .)

Proposition B.1 Keeping the notation of Sects. 2 and 3, under Assumption 3.1, we have

$$\lambda_1(\mathcal{G}) \leq \left(\frac{\pi |\mathcal{E}|}{|\mathcal{G}|} + \|q\|_1 \right)^2 - \|q\|_1^2. \quad (\text{B.1})$$

Note that $\frac{|\mathcal{G}|}{|\mathcal{E}|}$ is exactly the average edge length of $\mathcal{G}$. If $q \equiv 0$, a similar but stronger inequality was obtained in [28, Theorem 4.2].

Proof We first observe that the inequality

$$\|f\|_\infty^2 \leq 2\|f'\|_2\|f\|_2 \quad (\text{B.2})$$

is valid for all $f \in H_0^1(\mathcal{G})$: indeed, fixing any edge e , identified with the interval $[0, \ell_e]$, and any $x \in (0, \ell_e)$, by the fundamental theorem of calculus and the Cauchy–Schwarz inequality, since $f(0) = 0$, we have

$$|f(x)|^2 = \int_0^x (|f(t)|^2)' dt \leq 2 \int_0^x |f'(t)||f(t)| dt \leq 2\|f'\|_2\|f\|_2.$$

Now, suppose that $f \in H_0^1(\mathcal{G})$ satisfies $\|f\|_2 = 1$, then by (B.2)

$$\lambda_1(\mathcal{G}) \leq \int_{\mathcal{G}} |f'(x)|^2 + q(x)|f(x)|^2 dx \leq \|f'\|_2^2 + 2\|q\|_1\|f'\|_2.$$

Taking the infimum over all such functions f yields

$$\lambda_1(\mathcal{G}) \leq \lambda_1^D(0) + 2\lambda_1^D(0)^{1/2}\|q\|_1,$$

where $\lambda_1^D(0)$ is the Dirichlet Laplacian on $\mathcal{G}$, i.e. with zero potential and Dirichlet conditions at all vertices of $\mathcal{G}$ (that is, the Dirichlet Laplacian on the collection of $|E|$ disjoint intervals comprising the edges of $\mathcal{G}$). Now, at least one edge of $\mathcal{G}$ has length at least $|\mathcal{G}|/|E|$; and so $\lambda_1^D(0) \leq \pi^2|E|^2/|\mathcal{G}|^2$. This yields (B.1). $\square$

References

- [1] Alessandrini, G.: On Courant's nodal domain theorem. *Forum Math.* **10**, 521–532 (1998)
- [2] Alon, L., Band, R., Berkolaiko, G.: Nodal statistics on quantum graphs. *Commun. Math. Phys.* **362**, 909–948 (2018)
- [3] Atkinson, F.V., Mingarelli, A.B.: Asymptotics of the number of zeros and the eigenvalues of general weighted Sturm–Liouville problems. *J. Reine Angew. Math.* **375**(376), 380–393 (1987)
- [4] Band, R.: The nodal count $\{0, 1, 2, 3, \dots\}$ implies the graph is a tree. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372**, 20120504 (2014)
- [5] Band, R., Berkolaiko, G., Raz, H., Smilansky, U.: The number of nodal domains on quantum graphs as a stability index of graph partitions. *Commun. Math. Phys.* **311**, 815–838 (2012)
- [6] von Below, J.: A characteristic equation associated with an eigenvalue problem on c^2 -networks. *Lin. Algebra Appl.* **71**, 309–325 (1985)
- [7] Berkolaiko, G.: A lower bound for nodal count on discrete and metric graphs. *Commun. Math. Phys.* **278**, 803–819 (2008)
- [8] Berkolaiko, G., Kennedy, J.B., Kurasov, P., Mugnolo, D.: Edge connectivity and the spectral gap of combinatorial and quantum graphs. *J. Phys. A Math. Theor.* **50**, 365201 (2017)
- [9] Berkolaiko, G., Kennedy, J.B., Kurasov, P., Mugnolo, D.: Surgery principles for the spectral analysis of quantum graphs. *Trans. Am. Math. Soc.* **37**, 5153–5197 (2019)

- [10] Berkolaiko, G., Kuchment, P.: *Introduction to Quantum Graphs, Volume 186 of Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI (2013)
- [11] Berkolaiko, G., Liu, W.: Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph. *J. Math. Anal. Appl.* **445**, 803–818 (2017)
- [12] Berkolaiko, G., Weyand, T.: Stability of eigenvalues of quantum graphs with respect to magnetic perturbation and the nodal count of the eigenfunctions. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372**, 20120522 (2014)
- [13] Binding, P., Drábek, P.: Sturm–Liouville theory for the p -Laplacian. *Stud. Sci. Math. Hung.* **40**, 375–396 (2003)
- [14] Binding, P.A., Rynne, P.B.: Variational and non-variational eigenvalues of the p -Laplacian. *J. Differ. Equ.* **244**, 24–39 (2008)
- [15] Cipriani, F., Grillo, G.: Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces. *J. Reine Ang. Math.* **562**, 201–235 (2003)
- [16] Colin de Verdière, Y., Truc, F.: Topological resonances on quantum graphs. *Ann. Henri Poincaré* **19**, 1419–1438 (2018)
- [17] Courant, R.: Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke. *Nachr. Ges. Wiss. Göttingen. Math Phys.* 81–84 (1923)
- [18] Davies, E.B., Gladwell, G.M.L., Leydold, J., Stadler, P.F.: Discrete nodal domain theorems. *Linear Algebra Appl.* **336**, 51–60 (2001)
- [19] Del Pezzo, L.M., Rossi, J.D.: The first eigenvalue of the p -Laplacian on quantum graphs. *Anal. Math. Phys.* **6**, 365–391 (2016)
- [20] Drábek, P., Robinson, S.B.: Resonance problems for the p -Laplacian. *J. Funct. Anal.* **169**, 189–200 (1999)
- [21] Drábek, P., Robinson, S.B.: On the generalization of the Courant nodal domain theorem. *J. Differ. Equ.* **181**, 58–71 (2002)
- [22] Gnutzmann, S., Smilansky, U., Weber, J.: Nodal counting on quantum graphs. Special section quantum graphs. *Waves Random Media* **14**, S61–S73 (2004)
- [23] Hinton, D.: Sturm’s 1836 oscillation results—evolution of the theory. In: Amrein, W.O., Hinz, A.M., Pearson, D.B. (eds.) *Sturm–Liouville Theory—Past and Present*, pp. 1–27. Birkhäuser, Basel (2005)
- [24] Karreskog, G., Kurasov, P., Trygg Kupersmidt, I.: Schrödinger operators on graphs: symmetrization and Eulerian cycles. *Proc. Am. Math. Soc.* **144**, 1197–1207 (2016)
- [25] Kawohl, B., Lindqvist, P.: Positive eigenfunctions for the p -Laplace operator revisited. *Analysis (Munich)* **26**, 545–550 (2006)
- [26] Kennedy, J.B., Kurasov, P., Léna, C., Mugnolo, D.: A theory of spectral partitions on metric graphs. *Calc. Var.* **60**, 61 (2021)
- [27] Keller, M., Schwarz, M.: Courant’s nodal domain theorem for positivity preserving forms. *J. Spectr. Theory* **10**, 271–309 (2020)
- [28] Kennedy, J.B., Kurasov, P., Malenová, G., Mugnolo, D.: On the spectral gap of a quantum graph. *Ann. Henri Poincaré* **17**, 2439–2473 (2016)
- [29] Kurasov, P.: On the ground state for quantum graphs. *Lett. Math. Phys.* **109**, 2491–2512 (2019)

- [30] Lang, J., Edmunds, D.: Eigenvalues, Embeddings and Generalised Trigonometric Functions. Lecture Notes in Mathematics, vol. 2016. Springer-Verlag, Heidelberg (2011)
- [31] Mazurowski, L.: A Weyl law for the p -Laplacian, preprint (2019). [arXiv:1910.11855](https://arxiv.org/abs/1910.11855)
- [32] Mugnolo, D.: Gaussian estimates for a heat equation on a network. *Networks Het. Media* **2**, 55–79 (2007)
- [33] Mugnolo, D.: Semigroup Methods for Evolution Equations on Networks. Springer, Berlin (2014)
- [34] Mugnolo, D.: What is actually a quantum graph?, preprint (2019). [arXiv:1912.07549](https://arxiv.org/abs/1912.07549)
- [35] Nicaise, S.: Spectre des réseaux topologiques finis. *Bull. Sci. Math.* **111**, 401–413 (1987)
- [36] Pleijel, Å.: Remarks on Courant's nodal line theorem. *Commun. Pure Appl. Math.* **9**, 543–550 (1956)
- [37] Pokorný, Yu.V., Pryadiev, V.L., Al'Obeid, A.: On the oscillation of the spectrum of a boundary value problem on a graph. *Math. Notes* **60**, 351–353 (1996)
- [38] Reichel, W., Walter, W.: Sturm–Liouville type problems for the p -Laplacian under asymptotic non-resonance conditions. *J. Differ. Equ.* **156**, 50–70 (1999)
- [39] Serio, A.: On extremal eigenvalues of the graph Laplacian. *J. Phys. A Math. Theor.* **54**, 015202 (2021)
- [40] Sturm, C.: Mémoire sur les Équations différentielles linéaires du second ordre. *J. Math. Pures Appl.* **1**, 106–186 (1836)
- [41] Urschel, J.C.: Nodal decompositions of graphs. *Linear Algebra Appl.* **53**, 60–71 (2018)

Matthias Hofmann
 Grupo de Física Matemática,
 Faculdade de Ciências
 Universidade de Lisboa
 Campo Grande, Edifício C6
 1749-016 Lisboa
 Portugal
 e-mail: mhofmann@fc.ul.pt

James B. Kennedy
 Grupo de Física Matemática and Departamento de Matemática,
 Faculdade de Ciências
 Universidade de Lisboa
 Campo Grande,
 Edifício C6
 1749-016 Lisboa
 Portugal
 e-mail: jbkennedy@fc.ul.pt

Delio Mugnolo and Marvin Plümer
Lehrgebiet Analysis, Fakultät Mathematik und Informatik
FernUniversität in Hagen
58084 Hagen
Germany
e-mail: delio.mugnolo@fernuni-hagen.de;
marvin.pluemer@fernuni-hagen.de

Communicated by Alain Joye.

Received: December 11, 2020.

Accepted: June 2, 2021.