Correction to: Bartnik's Mass and Hamilton's Modified Ricci Flow

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Abstract Hyun-Chul Jang observed that we dropped a term in our calculations in Lin and Sormani (Ann Henri Poincaré 17(10):2783–2800, 2016). Tracking the consequences throughout, we see that we need to only slightly change the definition of the asphericity mass, and then, all statements of our theorems, propositions, and lemmas remain the same with slight revisions to the proofs described within. Pengzi Miao then noted that Theorem 1 needs an assumption of nonnegative gauss curvature which we have added as well. Hyun-Chul Jang and Pengzi Miao have agreed to our corrections which we describe herein.

In our original paper [1], we proved

Theorem 1. *Let Σ be the boundary of a closed 3-dimensional region with non-negative scalar curvature. If Σ is a CMC surface diffeomorphic to a sphere, which has area 4π and positive mean curvature, H , then*

$$m_B(\Sigma) \leq m_{aS}(\Sigma) + m_H(\Sigma) \quad (0.1)$$

where $m_{aS}(\Sigma)$ is a nonnegative constant defined using Hamilton's modified Ricci flow that we call the asphericity mass. It depends only upon the restriction of the metric g to the surface Σ . If

$$m_{aS}(\Sigma) = 0 \quad (0.2)$$

then Σ is isometric to a standard sphere, $(\mathbb{S}^2, g_{\mathbb{S}^2})$.

This theorem needs an assumption that Σ has nonnegative Gauss curvature, but is otherwise correct, as are the statements of all other theorems, lemmas, and propositions within the paper, but we must change the definition

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of asphericity mass. Before we provide the corrected definition of asphericity mass, we review Hamilton's modified Ricci flow to establish notation exactly as in the original paper's introduction.

Recall that for (Σ, g_1) of dimension two, Hamilton defined the modified Ricci flow (Σ_t, g_t) satisfying

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = (r - 2K) g_{ij} + 2D_i D_j f = 2M_{ij} \\ g(1, \cdot) = g_1(\cdot), \end{cases} \quad (0.3)$$

where $K = K_t(x)$ is the Gauss curvature of g_t at $x \in \Sigma_t$, and

$$r = r_t = \frac{1}{\text{Area}(\Sigma_t)} \int_{\Sigma_t} 2K_t(x) d\mu = 2 \quad (0.4)$$

is the mean scalar curvature, and $f = f(t, x)$ is the Ricci potential satisfying the equation

$$\Delta f = 2K - r \quad (0.5)$$

with mean value zero. Thus the 2-tensor

$$M_{ij} = (1 - K)g_{ij} + D_i D_j f \quad (0.6)$$

is the trace-free part of Hess(f).

Building upon this work of Hamilton, Chow proved that when Σ is diffeomorphic to a two-dimensional sphere, then the modified Ricci flow exists for all time and (Σ_t, g_t) converges to a standard sphere exponentially fast. In fact M converges to 0 exponentially fast.

The updated definition of asphericity mass is as follows with the new term appearing inside the integral in bold:

Definition 2. The *asphericity mass* of a surface Σ of area 4π and diffeomorphic to a sphere is defined by

$$m_{a\mathbb{S}}(\Sigma) = \lim_{t \rightarrow \infty} m_{a\mathbb{S}}(\Sigma, t), \quad (0.7)$$

where

$$m_{a\mathbb{S}}(\Sigma, t) = \frac{1}{2} \int_1^t 1 - K_*(\tau) E(\tau, t) + \frac{\tau |M|^{*2}}{2} E(\tau, t) d\tau, \quad (0.8)$$

where

$$E(\tau, t) = \exp \left(- \int_\tau^t \frac{s |M|^{*2}(s)}{2} ds \right). \quad (0.9)$$

Here we have the infimum of the Gauss curvature

$$K_*(\tau) = \inf \{ K_\tau(x) : x \in \Sigma_\tau \} \quad (0.10)$$

and the supremum of the norm of the M tensor

$$|M|^{*2}(s) = \sup \{ |M_s(x)|^2 : x \in \Sigma_s \} \quad (0.11)$$

which depend on g_t and $f(t, x)$ of Hamilton's modified Ricci flow. Observe that this mass depends only upon the intrinsic metric on Σ and not on the mean curvature.

All subsequent theorems and lemmas about the asphericity mass are true. We have reposted the paper on the arxiv with all corrections made in bold throughout the paper where they belong so that the paper can be more easily read.

Here we review the changes as in a traditional errata broken into two sections. We are very grateful to Hyun-Chul Jang (U Conn) for his careful reading of this paper, for discovering the error in the publication, and for closely reading our corrections.

1. Finiteness and Rigidity of Asphericity Mass Still Holds

All lemmas and propositions concerning the asphericity mass are correct with the new definition. In particular, Lemma 11 stating the asphericity mass is finite is still true using the same reasoning that $|M|$ converges to 0 exponentially. Proposition 12, which states that the asphericity mass is only 0 when (Σ, g_1) is isometric to a rescaled sphere, is now justified as follows (with corrections in bold):

Proof. Suppose that (Σ, g_1) is isometric to $(\mathbb{S}^2, g_{\mathbb{S}^2})$. $|M| \equiv 0$ and $K \equiv 1$ under the Ricci flow. Thus, $m_{a\mathbb{S}}(\Sigma) = 0$.

Suppose $m_{a\mathbb{S}}(\Sigma) = 0$. Since $m_{a\mathbb{S}}(\Sigma, t)$ is nonnegative and increasing in t , we have that

$$1 - K_*(\tau) \exp \left(- \int_{\tau}^t \frac{s |M|^{*2}(s)}{2} ds \right) + \frac{\tau |M|^{*2}}{2} E(\tau, t) = 0 \quad (1.1)$$

for all t and τ . **Since**

$$1 - K_*(\tau) \exp \left(- \int_{\tau}^t \frac{s |M|^{*2}(s)}{2} ds \right) \geq 0 \quad (1.2)$$

and

$$\frac{\tau |M|^{*2}}{2} E(\tau, t) \geq 0, \quad (1.3)$$

we have

$$1 - K_*(\tau) \exp \left(- \int_{\tau}^t \frac{s |M|^{*2}(s)}{2} ds \right) = 0 \quad (1.4)$$

and

$$\frac{\tau |M|^{*2}}{2} E(\tau, t) = 0 \text{ for all } t \text{ and } \tau. \quad (1.5)$$

It forces that $K = 1$ **and** $|M| = 0$ for all t and that (Σ, g_1) is isometric to a standard sphere by the Uniformization Theorem. $\square$

2. Source of the Error and Repair to the Proof of Theorem 4

The source of the error which lead to our change of the definition of asphericity mass here was in line (5.7) of the proof of Theorem 4 when we applied Lemma 10 of [2]. We lost a term.

Theorem 4, which is applied to prove Theorem 1, states that

$$m_B(\Sigma) \leq m_{aS}(\Sigma) + m_H(\Sigma) + e(M_{\bar{R}}, g_{\bar{R}}) \quad (2.1)$$

under certain hypotheses. As explained correctly in the proof of Theorem 4 in the original paper, these hypotheses implied that we had a unique admissible extension $M_{\bar{R}} = [1, \infty) \times \Sigma$ with prescribed scalar curvature $\bar{R}$. There was a unique $u \in C^{2+\alpha}(M_{\bar{R}})$ with initial condition $u(1, \cdot) = 2/H$ such that the metric $g_{\bar{R}} = u^2 dt^2 + t^2 g_t$ satisfying the asymptotically flat condition and finite ADM mass and

$$m_{\text{ADM}}(M_{\bar{R}}) = \lim_{t \rightarrow \infty} \frac{1}{4\pi} \oint_{\Sigma_t} \frac{t}{2} (1 - u^{-2}) d\sigma. \quad (2.2)$$

Applying the parabolic maximum principle to the parabolic equation of u^{-2} (Lemma 10 in [2]), we should have had the following C^0 bound (but we were missing the second term):

$$u^{-2}(t) \geq \frac{1}{t} \int_1^t \left(K - \frac{\tau^2}{2} \bar{R} \right) E(\tau, t) d\tau + \frac{1}{t} u^{-2}(1) E(1, t). \quad (2.3)$$

To further analyze this new second term, we continue as follows. Using the following two formulas

$$\frac{1}{t} \int_1^t 1 d\tau = 1 - \frac{1}{t} \quad (2.4)$$

and

$$\frac{1}{t} \int_1^t \frac{\tau |M|^{*2}}{2} E(\tau, t) d\tau = \frac{1}{t} \int_1^t \frac{\partial}{\partial \tau} E(\tau, t) d\tau = \frac{1}{t} (1 - E(1, t)),$$

we have

$$1 = \frac{1}{t} \int_1^t 1 d\tau + \frac{1}{t} \int_1^t \frac{\tau |M|^{*2}}{2} E(\tau, t) d\tau + \frac{1}{t} E(1, t). \quad (2.5)$$

With this we correct the remaining computations in the proof of Theorem 4 as follows:

$$\begin{aligned} 1 - u^{-2} &\leq 1 - \frac{1}{t} \int_1^t \left(K - \frac{\tau^2}{2} \bar{R} \right) E(\tau, t) d\tau - \frac{H^2}{4t} E(1, t). \\ &\leq 1 - \frac{1}{t} \int_1^t \left(K_* - \frac{\tau^2}{2} \bar{R}^* \right) E(\tau, t) d\tau - \frac{H^2}{4t} E(1, t). \\ &= \frac{1}{t} \int_1^t 1 - K_* E(\tau, t) + \frac{\tau |M|^{*2}}{2} E(\tau, t) d\tau \\ &\quad + \frac{1}{t} \int_1^t \frac{\tau^2}{2} \bar{R}^* E(\tau, t) d\tau + \frac{1}{t} \left(1 - \frac{H^2}{4} \right) E(1, t). \end{aligned}$$

Also, the Hawking mass of Σ is given by the formula

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 d\sigma \right) = \frac{1}{4\pi} \int_{\Sigma} \frac{1}{2} \left(1 - \frac{H^2}{4} \right) d\sigma. \quad (2.6)$$

Therefore,

$$\begin{aligned} \frac{1}{4\pi} \oint_{\Sigma_t} \frac{t}{2} (1 - u^{-2}) d\sigma &\leq \frac{1}{2} \int_1^t 1 - K_* E(\tau, t) + \frac{\tau |M|^*{}^2}{2} E(\tau, t) d\tau \\ &\quad + \frac{1}{2} \int_1^t \frac{\tau^2}{2} \bar{R}^* E(\tau, t) d\tau + m_H(\Sigma) E(1, t) \\ &= m_{aS}(\Sigma, t) + e_t(M_{\bar{R}}, g_{\bar{R}}) + m_H(\Sigma) E(1, t), \end{aligned}$$

and

$$\begin{aligned} m_{ADM}(M_{\bar{R}}) &\leq m_{aS}(\Sigma) + m_H(\Sigma) \lim_{t \rightarrow \infty} E(1, t) + e(M_{\bar{R}}, g_{\bar{R}}) \\ &\leq m_{aS}(\Sigma) + m_H(\Sigma) + e(M_{\bar{R}}, g_{\bar{R}}) \end{aligned}$$

since $E(1, t) \leq 1$. It follows directly by the definition of the Bartnik mass that

$$m_B(\Sigma) \leq m_{aS}(\Sigma) + m_H(\Sigma) + e(M_{\bar{R}}, g_{\bar{R}}). \quad (2.7)$$

3. Concerning C_0 and the Gauss Curvature of Σ

In Definition 3 of our paper, we define three classes of extensions with prescribed scalar curvature in (1.22)–(1.24). These classes of extensions are defined with mean curvature H satisfying (1.21) so that $H^2/4 > C_0(\bar{R})$. Observe that by taking $t = 1$ in the supremum, we have $C_0(\bar{R}) \geq 0$. Theorems 4 and 5 concern these extension classes satisfying the conditions of Definition 3 and are correct as stated.

In order to apply Theorem 4 to prove Theorem 1, it was suggested that we need positive Gauss curvature to get $C_0 > 0$. Theorem 8 had an hypothesis which appeared too strong implicitly requiring $C_0 > 0$. In fact this theorem holds for any function $\phi \in C^{2,\alpha}(\Sigma)$ satisfying (2.9) as written when $C_0 > 0$ and it holds for any positive $\phi \in C^{2,\alpha}(\Sigma)$ when $C_0 = 0$. Later in Lemma 15, we required positive Gauss curvature for Σ to construct its Ricci flow, but indeed we only need nonnegative Gauss curvature in the hypothesis because the Ricci flow has positive Gauss curvature for all times $t > 0$.

At the top of Section 6, we should state that Lemma 15 and Theorem 4 combined with Proposition 20 complete the proof of Theorem 1 where now Theorem 1 has the additional hypothesis that the Gauss curvature of Σ must be nonnegative.

No other changes to any proofs are needed elsewhere in this paper. In addition to submitting this errata to the publisher, we have reposted the paper on the arxiv with all the necessary corrections made within in their proper

places. We thank Hyun-Chul Jang and Pengzi Miao for their careful reading of our paper.

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- [1] Lin, C.-Y., Sormani, C.: Bartnik's mass and Hamilton's modified Ricci flow. *Ann. Henri Poincaré* **17**(10), 2783–2800 (2016)
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