Magnetic-Sparseness and Schrödinger Operators on Graphs

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Abstract. We study magnetic Schrödinger operators on graphs. We extend the notion of sparseness of graphs by including a magnetic quantity called the frustration index. This notion of magnetic-sparseness turns out to be equivalent to the fact that the form domain is an ℓ^2 space. As a consequence, we get criteria of discreteness for the spectrum and eigenvalue asymptotics.

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1. Introduction

Magnetic Schrödinger operators on graphs have been intensively studied in recent years. The topics of research range from essential self-adjointness [6, 11, 21, 22] over Feynman–Kac–Itô formulas [13, 14] to spectral considerations [9, 12, 18, 20].

In this paper, we study estimates on the quadratic forms associated with magnetic Schrödinger operators. These estimates yield spectral bounds as well as asymptotics of the eigenvalues in the case of purely discrete spectrum. The concept we use is called magnetic-sparseness which is composed of various components arising from the magnetic Schrödinger operator. The first ingredient is the geometry of the graph which enters via a combination of sparseness and isoperimetry. The second ingredient is the positive part of the potential of the Schrödinger operator. The third ingredient enters via the magnetic field as the so-called frustration index.

Let us illustrate this concept in the context of the literature in the most basic setting of combinatorial graphs. In Sect. 2, we then introduce the setup of weighted graphs in full detail and generality. For now, let X be a discrete set with an adjacency relation $\sim$ and let

$$Q(\varphi) = \frac{1}{2} \sum_{x,y \in X, x \sim y} |\varphi(x) - \varphi(y)|^2$$

be the quadratic form associated with the Laplacian

$$\Delta\varphi(x) = \sum_{y \in X, y \sim x} (\varphi(x) - \varphi(y))$$

on $\ell^2(X)$ with the scalar product $\langle \cdot, \cdot \rangle$.

A first step to get estimates on Q is the so-called isoperimetric constants. These constants measure the area of the boundary of sets in relation to the volume. For graphs, there are numerous ways to define such a constant. For the context of this work, the area and volume with respect to the number of edges are most relevant. Specifically, an isoperimetric constant is the smallest a such that for every finite vertex set W

$$\#E(W) \leq \frac{a}{2} \#\partial W$$

where $E(W)$ are the directed edges within W and ∂W are the directed edges with one vertex in W and the other in $X \setminus W$. (To count only the edges leaving W , the term on the right-hand side is multiplied by $1/2$.) Building on the seminal work of Alon/Milman [1], Dodziuk [7] and Dodziuk/Kendall [8], it was shown by Fujiwara [10] that for compactly supported φ

$$(1 - \tilde{a})\langle \deg \varphi, \varphi \rangle \leq Q(\varphi) \leq (1 + \tilde{a})\langle \deg \varphi, \varphi \rangle,$$

where $\tilde{a} = \sqrt{1 - (1 + a)^{-2}}$ and $1/(1 + a)$ is the so-called Cheeger constant and the function $\deg$ is the vertex degree. This estimate directly allows to conclude spectral estimates as well as eigenvalue asymptotics by the min-max principle.

A related concept is the one of sparseness. Sparseness measures the number of edges in a set in relation to the number of vertices. Specifically, a sparse graph with constant k satisfies for every finite vertex set W

$$\#E(W) \leq k\#W.$$

This concept was merged with concept of isoperimetric constants in [5] into the so-called (a, k) sparse graphs. For such graphs, one assumes that there are nonnegative a and k such that for all finite W

$$\#E(W) \leq k\#W + \frac{a}{2} \#\partial W.$$

It was shown in [5] that validity of such estimates is equivalent to the existence of $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that

$$(1 - \tilde{a})\langle \deg \varphi, \varphi \rangle - \tilde{k}\langle \varphi, \varphi \rangle \leq Q(\varphi) \leq (1 + \tilde{a})\langle \deg \varphi, \varphi \rangle + \tilde{k}\langle \varphi, \varphi \rangle, \quad (\text{E})$$

for compactly supported φ . Again this directly allows for spectral estimates via the min-max principle. These concepts of isoperimetry and sparseness consolidate the geometric ingredients into the form estimates for the form Q of the Laplacian Δ .

If one now has a Schrödinger operator $\Delta + v$ with positive $v \geq 0$, then one can incorporate v into the sparseness assumption by asking for nonnegative a and k such that for all finite W

$$\#E(W) \leq k\#W + \frac{a}{2}(\#\partial W + v(W)),$$

where $v(W) = \sum_{x \in W} v(x)$. This observation that a positive potential can be thought as weighted boundary edges of a set was already made in [16]. In [13], it was shown that this sparseness assumption is equivalent to form estimates as in (E) for the form $Q + v$. Furthermore, estimates for forms, where v is not necessarily nonnegative, were then achieved by perturbation theory.

Finally, we discuss the last ingredient which is a magnetic field. A magnetic potential enters a Schrödinger operator $\Delta + v$ via an antisymmetric function θ on the edges to result in a magnetic Schrödinger operator of the form $H_\theta = \Delta_\theta + v$ acting as

$$H_\theta \varphi(x) = \sum_{y \in X, y \sim x} (\varphi(x) - e^{i\theta(x,y)} \varphi(y)) + v(x) \varphi(x).$$

In [5], it was shown that (a, k) -sparseness yields a form estimate as in (E) for Q_θ associated with H_θ . However, these estimates are in general no longer equivalent to (a, k) -sparseness.

The purpose of this work is to incorporate the magnetic field into the form estimate. This way the influence of the magnetic field on existence of a spectral gap or on eigenvalue asymptotics is made transparent. Furthermore, we pursue the question whether such a form estimate as above is equivalent to a sparseness assumption which includes the magnetic field. To achieve this, we integrate a concept which is called the frustration index into our notion of sparseness. This will be then called magnetic-sparseness. Specifically, in [18] the following frustration index was considered for a finite set W

$$\iota_\theta(W) = \min \frac{1}{2} \sum_{x, y \in W, x \sim y} |\tau(x) - e^{i\theta(x,y)} \tau(y)|.$$

where the minimum is taken over all functions τ from W into the complex unit circle $\mathbb{T}$ which is attained due to compactness of $\mathbb{T}$. Now, assume the existence of a such that for all finite W

$$\#E(W) \leq (a + 1)\iota_\theta(W) + \frac{a}{2}\# \partial W.$$

Although it is not explicitly stated in [18], one can obtain from their techniques a lower bound for Q_θ for $v = 0$ as it appears in (E) for some $\tilde{a} \in (0, 1)$ depending on a and $k = 0$.

In this work, we merge all these concepts—*isoperimetry*, *sparseness*, the positive part of the potential and the *frustration index*—into one concept called *magnetic-sparseness*. Specifically, we ask for the existence of nonnegative a and k such that

$$\#E(W) \leq (a + 1)\iota_\theta(W) + \frac{a}{2}(\#\partial W + v(W)) + k\#W.$$

In Theorem 3.1, we show that such an inequality is equivalent to a lower bound in the form estimate of (E). However, an upper bound as it appears in (E) is not necessarily valid. In view of the results in [5], this is a surprising new phenomena. Recall that in [5], one shows that (a, k) -sparseness without the frustration index is equivalent to (E) for non-magnetic Schrödinger operators. Furthermore, it is shown in [5] that for the magnetic form Q_θ a two-sided estimate as in (E) holds under the assumption of sparseness.

Here, we improve the lower bound (by including θ into the estimate) but we loose the upper bound. This phenomenon can be explained as follows: In [5], the results for magnetic operators were deduced from the non-magnetic case using Kato's inequality to deduce the bounds for H_θ from H_0 . However in general, Kato's inequality fails to be an equality, so, one can not infer a lower bound for H_0 from a lower bound of H_θ .

In summary, we obtain lower bounds on Q_θ which involves the geometry, the potential as well as the magnetic field. Indeed, our estimates make the influence of any of these components transparent and allow for a very conceptual understanding of the phenomena as well as for various applications. In this sense, our approach not only generalizes all the results above but it also allows for a structural understanding of the recent considerations of [12].

Furthermore, in the literature other variants of the frustration index appear. One prominent example is the following [3]

$$\iota_\theta^{(2)}(W) = \min \frac{1}{2} \sum_{x,y \in W, x \sim y} |\tau(x) - e^{i\theta(x,y)} \tau(y)|^2.$$

Thus, it is natural to ask whether the frustration index from [18] discussed above (which under this perspective has to be denoted by $\iota_\theta^{(1)}$) is special as it characterizes validity of a lower bound in (E). It turns out—and that may come as another surprise—that this is not the case. Indeed, replacing 2 by any $p \in [1, \infty)$ (and Q_0 by Q_0^p) one can show that magnetic-sparseness with respect to p is also equivalent to the lower bound in (E) (of course with $\tilde{a}$ and $\tilde{k}$ now depending on p). However, to prove this one has to open a whole other Pandora's box. Specifically, one now has to consider the functionals of the magnetic p -Schrödinger operator given by

$$Q_\theta^{(p)}(\varphi) = \frac{1}{2} \sum_{x,y \in X, x \sim y} |\varphi(x) - e^{i\theta(x,y)} \varphi(y)|^p + \sum_{x \in X} v(x) |\varphi(x)|^p.$$

So, as a side product one also obtains an ℓ^p -version of the lower bound in (E) for $Q_\theta^{(p)}$. Although our main interest is in the case $p = 2$, these functionals come up naturally in the proof and therefore we included them in the statement of the main theorem, Theorem 3.1, as well. The case $p = 2$ is then again separately discussed in two corollaries, Corollaries 3.5 and 3.7. Let us stress that having the inequality for one $p \in [1, \infty)$, then one obtains it for all p including $p = 2$ (of course with different constants).

The paper is structured as follows. In the next section, we introduce the setup which deals with the more general case of weighted graphs instead of

combinatorial graphs only as considered above. In Sect. 3.1, we present the main results and in Sect. 3.2 we prove a magnetic co-area formula that is the core of our proof. Furthermore, the main theorem is proven in this section. Finally, we present examples in Sect. 4. In particular, in Sect. 4.1, we consider products of graphs and discuss how the results of [12] can be embedded in our context. Furthermore, we discuss magnetic cycles in Sect. 4.2 and give a criterion for magnetic-sparseness in terms of subgraphs. Subsequently, we apply this criterion to tessellations in Sect. 4.3.

2. Set Up and Notation

2.1. Magnetic Forms

Let X be a discrete set. We denote the complex-valued functions with finite support by $C_c(X)$. Any function $m : X \rightarrow (0, \infty)$ extends to a measure of full support on X and we denote by $\ell^p(X, m)$, $1 \leq p < \infty$ the complex Banach space with norm

$$\|f\|_p = \left(\sum_{x \in X} |f(x)|^p m(x) \right)^{1/p}.$$

A *graph* over X is a symmetric function $b : X \times X \rightarrow [0, \infty)$ with zero diagonal such that

$$\sum_{y \in X} b(x, y) < \infty, \quad x \in X.$$

We say $x, y \in X$ are connected by an *edge* if $b(x, y) > 0$. We say a graph b has *standard edge weights* if $b : X \times X \rightarrow \{0, 1\}$.

An *electric potential* is a function $v : X \rightarrow \mathbb{R}$ and a *magnetic potential* is an antisymmetric function $\theta : X \times X \rightarrow \mathbb{R}/2\pi\mathbb{Z}$.

In the subsequent, m always denotes a measure, b denotes a graph, v denotes an electric potential and θ denotes a magnetic potential and we refer to the quadruple (b, θ, v, m) as a *magnetic graph*.

We define $Q_\theta^{(p)} := Q_{b, \theta, v, m}^{(p)} : C_c(X) \rightarrow \mathbb{R}$ via

$$Q_\theta^{(p)}(\varphi) = \frac{1}{2} \sum_{x, y \in X} b(x, y) |\varphi(x) - e^{i\theta(x, y)} \varphi(y)|^p + \sum_{x \in X} v(x) |\varphi(x)|^p m(x),$$

for $\varphi \in C_c(X)$ and $p \in [1, \infty)$. Since the focus of this paper is to study the influence of the magnetic potential, we highlight θ in notation.

We have to bound the negative part v_- of v , where $v_\pm = (\pm v) \vee 0$. To this end, we introduce the dual pairing for functions $f \in C_c(X)$ and $g : X \rightarrow \mathbb{C}$

$$\langle g, f \rangle := \sum_{x \in X} \bar{g}(x) f(x) m(x).$$

Let $p \in [1, \infty)$, b , v_+ , θ and m be given. We say the negative part v_- of an electric potential $v = v_+ - v_-$ is in $\mathcal{K}_\alpha^{p, \theta}$ for $\alpha > 0$ if there is $C_\alpha \geq 0$ such

that

$$\langle v_-, |f|^p \rangle \leq \alpha Q_{b,\theta,v_+,m}^{(p)}(f) + C_\alpha \|f\|_p^p, \quad f \in C_c(X).$$

Clearly, $\mathcal{K}_\alpha^{p,\theta} \subseteq \mathcal{K}_\beta^{p,\theta}$ if $\alpha \leq \beta$ and $Q_{b,\theta,v,m}^{(p)}$ is bounded from below if and only if $v_- \in \mathcal{K}_1^{p,\theta}$. We define

$$\mathcal{K}_{0+}^{p,\theta} = \bigcap_{\alpha>0} \mathcal{K}_\alpha^{p,\theta}.$$

Again we write $\mathcal{K}_\alpha^\theta := \mathcal{K}_\alpha^{2,\theta}$ and $\mathcal{K}_{0+}^\theta := \mathcal{K}_{0+}^{2,\theta}$. In [13, Proposition 2.8], it is shown that in the case $p = 2$ the forms $Q_{b,\theta,v,m}^{(2)}$ are closable in $\ell^2(X, m)$ for any $v_- \in \mathcal{K}_\alpha^\theta$ with $\alpha \in (0, 1)$ and we denote the closure by $Q_\theta = Q_{b,\theta,v,m}$. Furthermore, $D(Q_{b,\theta,v,m}) = D(Q_{b,\theta,v_+,m})$ and

$$Q_\theta(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y) |f(x) - e^{i\theta(x,y)} f(y)|^2 + \sum_{x \in X} v(x) |f(x)|^2 m(x), \quad f \in D(Q_\theta).$$

Note that, in the case $v = 0$, even if the value of $Q_\theta(f)$ does not depend on m for $f \in C_c(X)$, its domain $D(Q_\theta)$ does depend on m . Moreover, by [13, Theorem 2.12] the self-adjoint operator $H_\theta = H_{b,\theta,v,m}$ is a restriction of the formal operator $\mathcal{H}_\theta = \mathcal{H}_{b,\theta,v,m}$

$$\mathcal{H}_\theta f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y) (f(x) - e^{i\theta(x,y)} f(y)) + v(x) f(x)$$

for f in $\mathcal{F}(X) = \{f : X \rightarrow \mathbb{C} \mid \sum_{y \in X} b(x,y) |f(y)| < \infty, x \in X\}$.

We define the *weighted vertex degrees* via

$$\text{Deg} : X \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{m(x)} \sum_{y \in X} b(x,y) + v(x),$$

$$\text{deg} : X \rightarrow \mathbb{R}, \quad x \mapsto \sum_{y \in X} b(x,y) + v(x)m(x),$$

and

$$\text{Deg}^{(+)} : X \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{m(x)} \sum_{y \in X} b(x,y) + v_+(x),$$

$$\text{deg}^{(+)} : X \rightarrow \mathbb{R}, \quad x \mapsto \sum_{y \in X} b(x,y) + v_+(x)m(x).$$

Observe that $\text{Deg}^{(+)} = \text{Deg} + v_-$ and $\text{deg}^{(+)} = \text{deg} + mv_-$.

2.2. Frustration Indices

Physically, two magnetic fields θ_1 and θ_2 act in the same way if H_{θ_1} and H_{θ_2} are equivalent. From the perspective of the magnetic field, this fact can be characterized in several equivalent ways, see, e.g., [6, 12, 15, 18, 19]. In this article, we put forward the notion of frustration index.

The *p-frustration index*, $p \in [1, \infty)$, with respect to θ of a finite set $W \subseteq X$, is defined as

$$\iota_{\theta}^{(p)}(W) := \min_{\tau: W \rightarrow \mathbb{T}} \frac{1}{2} \sum_{x, y \in W} b(x, y) |\tau(x) - e^{i\theta(x, y)} \tau(y)|^p,$$

where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and the minimum is attained by the compactness of $\mathbb{T}^W$. Note that $\iota_{\theta}^{(p)}$ is independent of m . Furthermore, we denote by $H_{\theta, W}$ the magnetic Laplacian associated with $(b|_{W \times W}, \theta|_{W \times W}, v|_W, m|_W)$ on $\ell^2(W, m|_W)$ for finite $W \subseteq X$.

We summarize the basic properties of the frustration indices in the following proposition.

Proposition 2.1. *Let (b, θ, v, m) be a magnetic graph, $p \in [1, \infty)$ and $W \subseteq X$ finite. Then,*

- (a) $\iota_{\theta}^{(p)}(W) \leq 2^{p-1} \iota_{\theta}^{(1)}(W)$.
 - (b) $\iota_{\theta}^{(p)}(V) \leq \iota_{\theta}^{(p)}(W)$ for $V \subseteq W$.
 - (c) $\iota_{\theta}^{(p)}(W) = \sum_{\tilde{W} \in C(W)} \iota_{\theta}^{(p)}(\tilde{W})$, where the sum is taken over the connected components $\tilde{W} \in C(W)$ of W .
 - (d) *The following statements are equivalent:*
 - (i) $\iota_{\theta}^{(1)}(W) = 0$.
 - (ii) $\iota_{\theta}^{(p)}(W) = 0$.
 - (iii) $H_{\theta, W}$ is unitarily equivalent to $H_{0, W}$.
- If additionally $v = 0$, then also the following statement is equivalent:*
- (iv) $\ker(H_{\theta, W}) \neq \{0\}$.

Proof. Statement (a) follows directly from the fact $|\tau(x) - e^{i\theta(x, y)} \tau(y)| \leq 2$ and (b) and (c) are clear from the definition.

Let us turn to the equivalence in (d). The equivalence (i) $\iff$ (ii) is trivial since W is assumed to be finite. The equivalence (i) $\iff$ (iii) can be seen using [18, eq. (3.3)]. To see the equivalence (i) $\iff$ (iv), recall that $\min \sigma(H_{\theta, W}) = \inf_{\|f\|_2=1} \langle f, H_{\theta, W} f \rangle$. We see that

$$0 \leq \min \sigma(H_{\theta, W}) \leq \iota_{\theta}^{(2)}(W)/m(W) \leq 2\iota_{\theta}^{(1)}(W)/m(W).$$

This yields the implication (i) $\implies$ (iv). On the other hand, assume (iv), i.e., $\ker(H_{\theta, W}) \neq \{0\}$. Then, there exists a non-trivial $f : W \rightarrow \mathbb{C}$ such that $H_{\theta, W} f = 0$. In particular, $\langle H_{\theta, W} f, f \rangle = 0$ and $f(x) = e^{i\theta(x, y)} f(y)$ for all $x, y \in W$ such that $b(x, y) > 0$. W.l.o.g., we assume W be connected. Then, there is $\tau : W \rightarrow \mathbb{T}$ such that $\langle H_{\theta, W} \tau, \tau \rangle = 0$. In particular, $\iota_{\theta}^{(1)}(W) = 0$. $\square$

Remark 2.2. A common and direct way to understand the magnetic field is through fluxes which is the sum of the θ 's around the edges of a cycle in a graph. In [19, 2.1 Lemma] (confer [6, Proposition 2.1], [12, Proposition 2.1]), it was shown that if θ_1 and θ_2 have the same fluxes (which is equivalent to the existence of a gauge transformation), then the operators $H_{\theta_1, W}$ and $H_{\theta_2, W}$ are unitarily equivalent. This corresponds to Proposition 2.1 (d) above.

Remark 2.3. The ground state of $H_{0,W}$ is given by the constant function 1_W on W . Then, $\iota_\theta^{(2)}(W)$ is obtained by minimizing the quadratic form of $H_{\theta,W}$ over all gauges. This gives another interpretation of the frustration index for $p = 2$.

Remark 2.4. On cycles the frustration indices for $p = 1$ and $p = 2$ can be explicitly calculated, see Proposition 4.6.

2.3. Magnetic-Sparseness

The boundary of a set $W \subseteq X$ is defined as

$$\partial W := (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

To define quantities like the measure of the boundary or the potential of a set, we will use the convention that a nonnegative function on a discrete set extends to a measure via additivity, i.e., given a set $A \subseteq X$ and $f : X \rightarrow [0, \infty)$ we let $f(A) := \sum_{x \in A} f(x)$.

We turn to the central notion of the paper.

Definition (*Magnetic-sparseness*). Let $a, k \geq 0$ and $p \in [1, \infty)$. We say the magnetic graph (b, θ, v, m) is $(a, k)_p$ -magnetic-sparse if

$$b(W \times W) \leq (1 + a)\iota_\theta^{(p)}(W) + a \left(\frac{1}{2}b(\partial W) + (v_+m)(W) \right) + km(W)$$

for all finite $W \subseteq X$. Furthermore, we say the magnetic graph (b, θ, v, m) is $(a, k)_p$ -bi-magnetic-sparse if both (b, θ, v, m) and $(b, \theta + \pi, v, m)$ are $(a, k)_p$ -magnetic-sparse, i.e.,

$$b(W \times W) \leq (1 + a)(\iota_\theta^{(p)}(W) \wedge \iota_{\theta+\pi}^{(p)}(W)) + a \left(\frac{1}{2}b(\partial W) + (v_+m)(W) \right) + km(W)$$

for all finite $W \subseteq X$, where $\alpha \wedge \beta$ is the minimum of $\alpha, \beta \in \mathbb{R}$.

Remark 2.5. Let us discuss the ingredients which go into the definition of magnetic-sparseness. Fundamentally, it is an assumption that the edge weights within finite sets can be bounded by various other quantities of the graph. The edge weights within W appear as the term $b(W \times W)$ on the left-hand side.

Reading the terms which bound $b(W \times W)$ from the right, the first term that appears is $km(W)$. If everything else on the right-hand side was equal to zero, then the graph is k sparse, i.e., the edge weight of W is bounded by the measure of W .

The next term that appears on the right-hand side is $a(\frac{1}{2}b(\partial W) + (v_+m)(W))$. Assuming for a moment that $v_+ = 0$ as well as $\theta = 0$ and $k = 0$, then this condition relates directly to an isoperimetric inequality. Specifically, having such a bound for some positive $a > 0$ is equivalent to positivity of the classical Cheeger constant

$$h := \inf_{W \subseteq X \text{ finite}} \frac{\frac{1}{2}b(\partial W)}{\deg(W)}$$

since $\deg(W) = \frac{1}{2}b(\partial W) + b(W \times W)$. (Note that the $1/2$ in front of the boundary measure stems from the fact that we only count the edges leaving

W .) In the case of non-trivial v , it was observed already in [16] that the positive part v_+ of v has the effect of boundary edges in isoperimetric considerations. Indeed, one can think of a virtual vertex at infinity which is connected to the graph via v_+ and on which we put Dirichlet boundary conditions, i.e., we ask for all functions to vanish on this virtual vertex.

The remaining term on the right-hand side is the frustration index $\iota_\theta^{(p)}(W)$. It is easily observed that in the case $v = 0$, positivity of the isoperimetric quantity

$$h_\theta^{(p)} := \inf_{W \subseteq X_{\text{finite}}} \frac{\frac{1}{2}b(\partial W) + \iota_\theta^{(p)}(W)}{\deg(W)}$$

is equivalent to $(a, 0)_p$ -magnetic-sparseness with some $a < \infty$, for $p \geq 1$. Moreover, a and $h_\theta^{(p)}$ are related by $a \leq \frac{1}{h_\theta^{(p)}} - 1$. (Whenever the constant a is chosen optimally, then one even has equality.) The constant $h_\theta^{(1)}$ was considered in [18]. The constant $h_\theta^{(2)}$ appears in the work of [3]. Note that for finite and connected X and $p \in [1, \infty)$, we have $h_\theta^{(p)} > 0$ if and only if $\iota_\theta^{(p)}(X) > 0$. The reason to choose the parameter $(1 + a)$ in front of the frustration index is that it arises as the natural choice when proving the functional inequality for the Laplacian (see Theorem 3.1 as follows) which is the main result of the paper.

Remark 2.6. If the graph is $(a, k)_p$ -magnetic-sparse, Proposition 2.1 (a) ensures that the graph is also $(a', k)_1$ -magnetic-sparse with $a' = (1 + a)2^{p-1} - 1$. We will prove below that the converse is also true. Namely, if a graph is $(a, k)_1$ -magnetic-sparse for some $a, k \geq 0$, it is also $(a(p), k(p))_p$ -magnetic-sparse for all $p \in [1, \infty)$ and some $a(p), k(p) \geq 0$, see Theorem 3.1.

Trivially, every graph over a finite set is $(0, k)_p$ -bi-magnetic-sparse for all $p \in [1, \infty)$ and some $k \geq 0$. However, there is also the following simple example which shows that a graph can become magnetic-sparse due to the magnetic field. This example is also discussed in [12].

Example 2.7. We consider the combinatorial graph which arises as the Cayley graph of $X = \mathbb{Z} \times \mathbb{Z} / 3\mathbb{Z}$ with the generators $(\pm 1, 1)$; see Fig. 1. We put standard weights on this graph, i.e., we choose the weights b to take values in $\{0, 1\}$. Now, we put a finite measure m on X , i.e., $m(X) < \infty$, and we let $v = 0$.

For $\theta = 0$, the graph is obviously not magnetic-sparse for any choice of parameters a and k . However, choosing θ to be non-trivial it is not hard to check that this allows us to obtain magnetic-sparse graphs by the virtue of the magnetic potential alone. For details and a more involved example, we refer the reader to Sect. 4.

The next proposition deals with magnetic bi-partite graphs.

Proposition 2.8. *Let (b, θ, v, m) be a magnetic bi-partite graph, $p \in [1, \infty)$ and $W \subseteq X$ finite and $p \geq 1$. Then,*

$$\iota_\theta^{(p)}(W) = \iota_{\theta+\pi}^{(p)}(W).$$

FIGURE 1. A representation of $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, where the decay of size of the triangles of $3\mathbb{Z}$ indicates the decrease in the measure m

In particular, a bi-partite graph (b, θ, v, m) is $(a, k)_p$ -bi-magnetic-sparse if and only if it is $(a, k)_p$ -magnetic-sparse.

Proof. Let $X = X_1 \cup X_2$ be a partition such that $b(x, y) = 0$ if $(x, y) \in X_1 \times X_1$ or $(x, y) \in X_2 \times X_2$. For $\tau : W \rightarrow \mathbb{T}$, define $\tilde{\tau} : W \rightarrow \tau$ by

$$\tilde{\tau}(x) = \begin{cases} \tau(x) & \text{if } x \in X_1 \\ -\tau(x) & \text{if } x \in X_2. \end{cases}$$

The map $\tau \rightarrow \tilde{\tau}$ is clearly a bijection, thus

$$\begin{aligned} \iota_{\theta}^{(p)}(W) &= \min_{\tilde{\tau}: W \rightarrow \mathbb{T}} \frac{1}{2} \sum_{x, y \in W} b(x, y) |\tilde{\tau}(x) - e^{i\theta(x, y)} \tilde{\tau}(y)|^p \\ &= \min_{\tau: W \rightarrow \mathbb{T}} \frac{1}{2} \sum_{x, y \in W} b(x, y) |\tau(x) + e^{i\theta(x, y)} \tau(y)|^p = \iota_{\theta+\pi}^{(p)}(W). \end{aligned}$$

The last point is clear from the definition. $\square$

For further examples, we refer the reader to Sects. 4.2 and 4.3.

3. Functional Inequalities and Magnetic-Sparseness

3.1. Main Results

Before we state the main result of the paper, let us recall the fundamental notions that appear. For a graph b over (X, m) , an electric potential v and a magnetic potential θ a graph is called $(a, k)_p$ -magnetic-sparse for $a, k \geq 0$ and $p \in [1, \infty)$ if for all finite $W \subseteq X$

$$b(W \times W) \leq (1 + a) \iota_{\theta}^{(p)}(W) + a \left(\frac{1}{2} b(\partial W) + (v_+ m)(W) \right) + km(W),$$

where the p -frustration index $\iota_{\theta}^{(p)}(W)$ is given by

$$\iota_{\theta}^{(p)}(W) := \min_{\tau: W \rightarrow \mathbb{T}} \frac{1}{2} \sum_{x, y \in W} b(x, y) |\tau(x) - e^{i\theta(x, y)} \tau(y)|^p.$$

Observe that $(a, k)_p$ -magnetic-sparseness depends only on the positive part v_+ of v . On the other hand, to deal with non-positive potentials as well we introduced the classes $\mathcal{K}_{\alpha}^{p, \theta}$ which put a requirement on the negative part of

a potential. Thus, these two requirements are independent and can be asked for different p as well, i.e., the graph being $(a, k)_p$ -magnetic-sparse but the negative part v_- of v being in $\mathcal{K}^{q, \theta}$ for $p \neq q$.

Furthermore, recall that $\text{Deg}^{(+)} = \text{Deg} + v_-$ and $\deg^{(+)} = \deg + mv_-$.

Theorem 3.1. *Let (b, θ, v, m) be a magnetic graph. The following assertions are equivalent:*

- (i) *There are $a, k \geq 0$ such that the magnetic graph is $(a, k)_1$ -magnetic-sparse.*
- (i') *For all (some) $p \geq 1$, there are $a', k' \geq 0$ such that the magnetic graph is $(a', k')_p$ -magnetic-sparse.*
- (ii) *For all (some) $p \geq 1$, there are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that for $f \in C_c(X)$*

$$(1 - \tilde{a})\langle \text{Deg}^{(+)}, |f|^p \rangle - \tilde{k} \|f\|_p^p \leq Q_{b, \theta, v_+, m}^{(p)}(f).$$

Moreover, given $p_0 \geq 1$, $\alpha \in (0, 1)$ and $v_- \in \mathcal{K}_\alpha^{p_0, \theta}$, the previous points are also equivalent to

- (iii) *There are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that for $f \in C_c(X)$*

$$(1 - \tilde{a})\langle \text{Deg}, |f|^{p_0} \rangle - \tilde{k} \|f\|_{p_0}^{p_0} \leq Q_{b, \theta, v, m}^{(p_0)}(f).$$

Besides that, if $v_- \in \mathcal{K}_\alpha^{2, \theta}$ for some $\alpha \in (0, 1)$, the above assertions are also equivalent to:

- (iv) $D(Q_{b, \theta, v, m}) = \ell^2(X, \deg^{(+)}) \cap \ell^2(X, m)$.

Remark 3.2. All the constants that appear in the different equivalences can be tracked explicitly, see Lemmas 3.13, 3.14, and also 3.15. For instance, in (i') and $p \geq 1$, the constant a' can be chosen depending on a and k given in (i) such that

$$a' = 2^{1-p}(p^2 + 1)^{\frac{p}{2}}(1 + a)^p - 1$$

and k' is a constant such that $k' = 0$ if $k = 0$.

In (ii) and $p \geq 1$, with respect to a and k given in (i), we can take

$$\tilde{a} = 1 - \frac{1}{2^{1-p}(p^2 + 1)^{\frac{p}{2}}(1 + a)^p}$$

and $\tilde{k}$ is a constant such that $\tilde{k} = 0$ if $k = 0$.

In (iii), for $v_- \in \mathcal{K}_\alpha^{p_0, \theta}$, starting with of a and k given in (i),

$$\tilde{a} = 1 - \frac{1 - \alpha}{2^{1-p_0}(p_0^2 + 1)^{\frac{p_0}{2}}(1 + a)^{p_0} - \alpha}.$$

Remark 3.3. In the case $v = 0$ and $k = 0$ in (i) or (i'), the inequality in (ii) and the equality in (iv) hold for all measures m .

Remark 3.4. For $p = 2$, the lower bound in (ii) (and in (iii)) is equivalent to a corresponding upper bound for a different magnetic potential. Via the equality

$$\mathcal{H}_\theta = 2\text{Deg} - \mathcal{H}_{\theta+\pi},$$

which holds on $\mathcal{F}(X)$ and the Green's formula

$$Q_\theta(f, g) = \sum_{x \in X} \mathcal{H}_\theta f(x) g(x) m(x), \quad f, g \in C_c(X),$$

statement (iii) in the theorem above can be seen to be equivalent to

$$(iii') \quad Q_{\theta+\pi}(f) \leq (1 + \tilde{a}) \langle \text{Deg}, |f|^2 \rangle + \tilde{k} \|f\|_2^2.$$

From Proposition 4.8 and Example 4.12 in the next section, it can be seen that there exist magnetic-sparse graphs which are not bi-magnetic-sparse. That is, a corresponding upper bound for Q_θ does not necessarily hold even if Q_θ is lower-bounded. Nevertheless, for bi-magnetic-sparse graphs we can obtain lower and upper bounds in the case $p_0 = 2$.

Corollary 3.5. *Let (b, θ, v, m) be a magnetic graph such that $v_- \in \mathcal{K}_\alpha^{2, \theta}$ for some $\alpha \in (0, 1)$. The following assertions are equivalent:*

- (i) *There are $a, k \geq 0$ such that the magnetic graph is $(a, k)_1$ -bi-magnetic-sparse.*
- (i') *For all (some) $p \geq 1$, there are $a', k' \geq 0$ such that the magnetic graph is $(a', k')_p$ -bi-magnetic-sparse.*
- (ii) *There are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that for $f \in C_c(X)$*

$$(1 - \tilde{a}) \langle \text{Deg}, |f|^2 \rangle - \tilde{k} \|f\|_2^2 \leq Q_\theta(f) \leq (1 + \tilde{a}) \langle \text{Deg}, |f|^2 \rangle + \tilde{k} \|f\|_2^2.$$
- (iii) $D(Q_\theta) = D(Q_{\theta+\pi}) = \ell^2(X, \text{deg}^{(+)}) \cap \ell^2(X, m).$

Proof. Apply Theorem 3.1 with θ and $\theta + \pi$ and condition (iii') of Remark 3.4. $\square$

Remark 3.6. The characterization of bi-magnetic-sparseness via the upper bound only works for $p = 2$ since $Q_\theta^{(p)}(f) + Q_{\theta+\pi}^{(p)}(f) = 2 \langle \text{Deg}, |f|^p \rangle$ for all $f \in C_c(X)$ if and only if $p = 2$. For $p \neq 2$, we only get estimates between $Q_\theta^{(p)}(f) + Q_{\theta+\pi}^{(p)}(f)$ and $\langle \text{Deg}, |f|^p \rangle$ for all $f \in C_c(X)$.

Next, we turn to the corresponding eigenvalue asymptotics in the case $p = 2$. When H_θ has purely discrete spectrum, i.e., the spectrum of H_θ consists of discrete eigenvalues with finite multiplicity, we denote the eigenvalues counted with multiplicity in increasing order by λ_n , $n \geq 0$. Furthermore, set

$$D_\infty := \liminf_{x \rightarrow \infty} \text{Deg}(x),$$

where $x \rightarrow \infty$ means that x converges to the point ∞ in the one-point compactification $X \cup \{\infty\}$ of X , i.e., the $\liminf$ is taken over all sequences of vertices which eventually leave every finite set. In the case $D_\infty = \infty$, we order the vertices $\mathbb{N}_0 \rightarrow X$, $n \mapsto x_n$ bijectively such that

$$D_n := \text{Deg}(x_n) \leq \text{Deg}(x_{n+1}).$$

Corollary 3.7. *Assume the magnetic graph (b, θ, v, m) with $v_- \in \mathcal{K}_\alpha^\theta$ for some $\alpha \in (0, 1)$ is $(a, k)_1$ -magnetic-sparse. Then, the following statements are equivalent:*

- (i) H_θ has purely discrete spectrum.
- (ii) $D_\infty = \infty$.

In this case, we have for the eigenvalues λ_n of H_θ

$$\frac{2(1-\alpha)}{5(1+a)^2-\alpha} \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n}{D_n} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n}{D_n}.$$

Proof. The statement follows directly from the two-sided estimate in Theorem 3.1 (iii) (confer Corollary 3.5 (ii) as well), the formula for $\tilde{a}$ in Remark 3.2 and the min-max principle. $\square$

Remark 3.8. In the case where $v_- \in \mathcal{K}_{0+}$, one can take $\alpha = 0$ in Corollary 3.7 and Remark 3.8.

Remark 3.9. If the case where (b, θ, v, m) is actually $(a, k)_1$ -bi-magnetic-sparse, using Corollary 3.5 one gets:

$$\frac{2(1-\alpha)}{5(1+a)^2-\alpha} \leq \liminf_{n \rightarrow \infty} \frac{\lambda_n}{D_n} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n}{D_n} \leq 2 - \frac{2(1-\alpha)}{5(1+a)^2-\alpha}.$$

Remark 3.10. We introduce $d(x) := \frac{1}{m(x)} \sum_{y \in X} b(x, y)$ for $x \in X$. One has $\text{Deg}(x) = d(x) + v(x)$ and via the inequality $|z+w|^2 \leq 2(|z|^2 + |w|^2)$, $z, w \in \mathbb{C}$, one gets the estimate on $C_c(X) \subseteq \ell^2(X, m)$

$$Q_\theta \leq 2d + v = \left(\frac{2 + \frac{v}{d}}{1 + \frac{v}{d}} \right) \text{Deg}.$$

Setting

$$l := \liminf_{x \rightarrow \infty} \frac{v(x)}{d(x)}$$

we get by the min-max principle

$$\limsup_{n \rightarrow \infty} \frac{\lambda_n}{D_n} \leq \frac{2+l}{1+l}.$$

3.2. Magnetic Isoperimetry

We prove the main theorem via isoperimetric techniques. For a function $f : X \rightarrow \mathbb{R}$, one defines the level sets

$$\Omega_t(f) := \{x \in X \mid f(x) > t\}, \quad t \in \mathbb{R}.$$

In the non-magnetic case, the following area and co-area are well known; see, e.g., [16, Theorems 12 and 13]

$$\begin{aligned} \int_0^\infty m(\Omega_t(f)) dt &= \sum_{x \in X} f(x) m(x), \\ \int_0^\infty b(\partial \Omega_t(f)) dt &= \frac{1}{2} \sum_{x, y \in X} b(x, y) |f(x) - f(y)|, \end{aligned}$$

for all $f : X \rightarrow [0, \infty)$. The key ingredient of the proof of our main theorem is the following magnetic co-area inequality.

Lemma 3.11. For $f \in C_c(X)$ let $\Omega_t = \Omega_t(|f|^p)$, $t \geq 0$. Then,

$$\begin{aligned} & \int_0^\infty \left(\iota_\theta^{(1)}(\Omega_t) + \frac{1}{2}b(\partial\Omega_t) \right) dt \\ & \leq \frac{\sqrt{p^2+1}}{2} \sum_{x,y \in X} b(x,y) |f(x) - e^{i\theta(x,y)} f(y)| \left(\frac{|f(x)|^p + |f(y)|^p}{2} \right)^{\frac{p-1}{p}}. \end{aligned}$$

Proof. For finite graphs and $p = 2$, this formula has been shown in [18, Lemmas 4.3 and 4.7], which can also be extracted from the proof of [3, Lemma 3.2]. The ideas carry over directly to our setting.

For a given function $f : X \rightarrow \mathbb{C}$, we define the following complex-valued function:

$$F_f(x, t) := \begin{cases} f(x)/|f(x)| & : \text{ if } x \in \Omega_t(|f|^p), \\ 0 & : \text{ otherwise.} \end{cases}$$

Then, we can calculate

$$\begin{aligned} & \int_0^\infty \left(\iota_\theta^{(1)}(\Omega_t) + \frac{1}{2}b(\partial\Omega_t) \right) dt \\ & \leq \int_0^\infty \frac{1}{2} \sum_{x,y \in X} b(x,y) |F_f(x, t) - e^{i\theta(x,y)} F_f(y, t)| dt \\ & = \frac{1}{2} \sum_{x,y \in X} b(x,y) \int_0^\infty |F_f(x, t) - e^{i\theta(x,y)} F_f(y, t)| dt. \end{aligned}$$

Note in the last equality above, we used Tonelli's theorem.

For two vertices $x, y \in X$, we assume w.l.o.g. that $|f(x)| \leq |f(y)|$. We calculate

$$\begin{aligned} & \int_0^\infty |F_f(x, t) - e^{i\theta(x,y)} F_f(y, t)| dt \\ & = \int_0^{|f(x)|^p} \left| \frac{f(x)}{|f(x)|} - \frac{e^{i\theta(x,y)} f(y)}{|f(y)|} \right| dt + \int_{|f(x)|^p}^{|f(y)|^p} 1 dt \\ & = \left| \frac{f(x)}{|f(x)|} - \frac{e^{i\theta(x,y)} f(y)}{|f(y)|} \right| |f(x)|^p + |f(y)|^p - |f(x)|^p. \end{aligned}$$

Now, we apply Lemma 3.12 below with

$$v = \frac{f(x)}{|f(x)|}, \quad w = \frac{e^{i\theta(x,y)} f(y)}{|f(y)|}, \quad \text{and} \quad \beta = \frac{|f(y)|}{|f(x)|}$$

and obtain

$$\begin{aligned} & \int_0^\infty |F_f(x, t) - e^{i\theta(x,y)} F_f(y, t)| dt \\ & \leq \sqrt{p^2+1} |f(x) - e^{i\theta(x,y)} f(y)| \left(\frac{|f(x)|^p + |f(y)|^p}{2} \right)^{\frac{p-1}{p}}. \end{aligned}$$

Combining this with the estimate in the beginning, the statement of the lemma follows. $\square$

Lemma 3.12. *For all $v, w \in \mathbb{T}$ and for all $\beta, p \in [1, \infty)$ one has*

$$|v - w| + \beta^p - 1 \leq \sqrt{p^2 + 1}|v - \beta w| \left(\frac{1 + \beta^p}{2} \right)^{\frac{p-1}{p}}.$$

Proof. The claim is obvious for $\beta = 1$. Thus, we can assume $\beta > 1$. Similar to [3, Proposition A.1], we set $t := |v - w|$ and $s := t/(\beta^p - 1)$. We obtain due to $|v| = |w| = 1$

$$|v - \beta w| = \sqrt{(\beta - 1)^2 + \beta t^2} \geq \sqrt{(\beta - 1)^2 + t^2} \geq (\beta - 1)\sqrt{1 + s^2 p^2},$$

where we used $|\beta^p - 1| \geq p|\beta - 1|$ in the last estimate. Moreover,

$$\sqrt{p^2 + 1}\sqrt{1 + s^2 p^2} \geq \sqrt{p^2(1 + s^2) + 2p^2 s} = (1 + s)p$$

which yields together with the estimate above

$$\sqrt{p^2 + 1}|v - \beta w| \geq (1 + s)p(\beta - 1).$$

Furthermore, due to S. Amghibech, [2, Lemma 3], see also [17, Lemma 3.8], we have

$$p(\beta - 1) \left(\frac{\beta^p + 1}{2} \right)^{\frac{p-1}{p}} \geq \beta^p - 1.$$

Hence, taking the last two inequalities together we obtain the desired estimate

$$\sqrt{p^2 + 1}|v - \beta w| \left(\frac{1 + \beta^p}{2} \right)^{\frac{p-1}{p}} \geq (1 + s)(\beta^p - 1) = |v - w| + \beta^p - 1.$$

This finishes the proof. $\square$

Lemma 3.13. *Let $p \in [1, \infty)$. If a magnetic graph is $(a, k)_1$ -magnetic-sparse, $a, k \geq 0$, and $v \geq 0$, then*

$$(1 - \tilde{a})\langle \text{Deg}, |f|^p \rangle - \tilde{k}\|f\|_p^p \leq Q_\theta^{(p)}(f), \quad f \in C_c(X),$$

where

$$\tilde{a} = 1 - \frac{2^{p-1}}{(p^2 + 1)^{\frac{p}{2}}(1 + a)^p}, \quad \tilde{k} = \frac{2^{p-1}pk}{(p^2 + 1)^{\frac{p}{2}}(1 + a)^p}.$$

Proof. Let $f \in C_c(X)$. If

$$\langle \text{Deg}, |f|^p \rangle < k\|f\|_p^p,$$

then the announced values of $\tilde{a}$ and $\tilde{k}$ work in this case since

$$\tilde{k} \geq (1 - \tilde{a})k.$$

So, we assume $\langle \text{Deg}, |f|^p \rangle \geq k\|f\|_p^p$. Let $\Omega_t = \Omega_t(|f|^p)$, $t \geq 0$. Note that the following area formula holds,

$$\int_0^\infty \left(b(\Omega_t \times \Omega_t) + \frac{1}{2}b(\partial\Omega_t) + (vm)(\Omega_t) \right) dt = \int_0^\infty \sum_{x \in \Omega_t} \deg(x) dt = \langle \text{Deg}, |f|^p \rangle.$$

Now, we calculate

$$\langle \text{Deg}, |f|^p \rangle - k\|f\|_p^p = \int_0^\infty \left(b(\Omega_t \times \Omega_t) + \frac{1}{2}b(\partial\Omega_t) + (vm)(\Omega_t) - km(\Omega_t) \right) dt$$

$$\leq \int_0^\infty (1+a) \left(\iota_\theta^{(1)}(\Omega_t) + \frac{1}{2}b(\partial\Omega_t) + (vm)(\Omega_t) \right) dt.$$

In the above inequality, we used the $(a, k)_1$ -magnetic-sparseness. Applying Lemma 3.11 and Hölder inequality, we obtain with $q = p/(p-1)$ that

$$\begin{aligned} & \langle \text{Deg}, |f|^p \rangle - k \|f\|_p^p \\ & \leq (1+a) \left(\frac{\sqrt{p^2+1}}{2} \sum_{x,y \in X} b(x,y) |f(x) - e^{i\theta(x,y)} f(y)| \left(\frac{|f(x)|^p + |f(y)|^p}{2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \sum_{x \in X} (|f|^p vm)(x) \right) \\ & \leq (1+a) \left(\frac{1}{2} \sum_{x,y \in X} b(x,y) |f(x) - e^{i\theta(x,y)} f(y)|^p + \sum_{x \in X} |f(x)|^p (vm)(x) \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{(1+p^2)^{\frac{q}{2}}}{2} \sum_{x,y \in X} b(x,y) \left(\frac{|f(x)|^p + |f(y)|^p}{2} \right)^{\frac{1}{q} \cdot q} + \sum_{x \in X} |f(x)|^p (vm)(x) \right)^{\frac{1}{q}} \\ & \leq 2^{\frac{1}{p}-1} \sqrt{p^2+1} (1+a) Q_\theta^{(p)}(f)^{\frac{1}{p}} \langle \text{Deg}, |f|^p \rangle^{\frac{1}{q}}. \end{aligned}$$

Since the left-hand side of the above inequality is nonnegative by assumption, we can take p th power on both sides. Therefore, we arrive at

$$\begin{aligned} \langle \text{Deg}, |f|^p \rangle^p - kp \langle \text{Deg}, |f|^p \rangle^{p-1} \|f\|_p^p & \leq (\langle \text{Deg}, |f|^p \rangle - k \|f\|_p^p)^p \\ & \leq 2^{1-p} (p^2+1)^{\frac{p}{2}} (1+a)^p Q_\theta^{(p)}(f) \langle \text{Deg}, |f|^p \rangle^{\frac{p}{q}}. \end{aligned}$$

This implies due to $p/q = p-1$ that

$$\frac{2^{p-1}}{(p^2+1)^{\frac{p}{2}}(1+a)^p} \langle \text{Deg}, |f|^p \rangle - \frac{2^{p-1}pk}{(p^2+1)^{\frac{p}{2}}(1+a)^p} \|f\|_p^p \leq Q_\theta^{(p)}(f).$$

This shows the statement with the choice of $(\tilde{a}, \tilde{k})$ in the statement of the lemma. $\square$

Lemma 3.14. *If the magnetic graph is $(a, k)_1$ -magnetic-sparse, $a, k \geq 0$, and $v_- \in \mathcal{K}_\alpha^{p,\theta}$ then*

$$(1 - \tilde{a}) \langle \text{Deg}, |f|^p \rangle - \tilde{k} \|f\|_p^p \leq Q_\theta^{(p)}(f), \quad f \in C_c(X).$$

where

$$\begin{aligned} \tilde{a} &= 1 - \frac{1-\alpha}{2^{1-p}(p^2+1)^{p/2}(1+a)^p - \alpha}, \\ \tilde{k} &= \frac{(1-\alpha)kp + (2^{1-p}(p^2+1)^{p/2}(1+a)^p - 1)C_\alpha}{2^{1-p}(p^2+1)^{p/2}(1+a)^p - \alpha}, \end{aligned}$$

and C_α is the constant from the bound $\langle v_-, |f|^p \rangle \leq \alpha Q_{b,\theta,v_+,m}^{(p)} + C_\alpha \|f\|_p^p$.

Proof. By the lemma above we have, for all $f \in C_c(X)$, the inequality

$$(1 - \tilde{a}_0) \langle (\text{Deg} + v_-), |f|^p \rangle - \tilde{k}_0 \|f\|_p^p \leq Q_{b,\theta,v_+,m}^{(p)}(f)$$

with constants $\tilde{a}_0$ and $\tilde{k}_0$. Hence, together with the bound for v_- we obtain

$$\frac{(1 - \alpha)(1 - \tilde{a}_0)}{1 - \alpha(1 - \tilde{a}_0)} \langle \text{Deg}, |f|^p \rangle - \frac{(1 - \alpha)\tilde{k}_0 + \tilde{a}_0 C_\alpha}{1 - \alpha(1 - \tilde{a}_0)} \|f\|_p^p \leq Q_{b,\theta,v,m}^{(p)}(f).$$

by a straightforward calculation (similar to [5, Lemma A.3]). With the specific constants of Lemma 3.13, the statement follows. $\square$

Lemma 3.15. *Let (b, θ, v, m) be a magnetic graph with $v \geq 0$. Let $p \in [1, \infty)$. If for some $0 < \tilde{a} < 1$ the magnetic graph satisfies*

$$(1 - \tilde{a}) \langle \text{Deg}, |f|^p \rangle - \tilde{k} \|f\|_p^p \leq Q_\theta^{(p)}(f), \quad f \in C_c(X),$$

then the graph is $(a, k)_p$ -magnetic-sparse with

$$a = \frac{\tilde{a}}{1 - \tilde{a}} \quad k = \frac{\tilde{k}}{1 - \tilde{a}}.$$

Proof. Let $W \subseteq X$ be a finite set and let $\tau_0 : W \rightarrow \mathbb{T}$ be the function that attains the minimum in the definition of $\iota_\theta^{(p)}(W)$. We define the following function:

$$f_0(u) := \begin{cases} \tau_0(u) & \text{if } u \in W, \\ 0 & \text{otherwise.} \end{cases}$$

We calculate

$$\begin{aligned} Q_\theta^{(p)}(f_0) &= \frac{1}{2} \sum_{x,y \in X} b(x,y) |f_0(x) - e^{i\theta(x,y)} f_0(y)|^p + \sum_{x \in X} v(x)m(x) |f_0(x)|^p \\ &= \frac{1}{2} \sum_{x,y \in W} b(x,y) |\tau_0(x) - e^{i\theta(x,y)} \tau_0(y)|^p + \frac{1}{2} b(\partial W) + (vm)(W) \\ &= \iota_\theta^{(p)}(W) + \frac{1}{2} b(\partial W) + (vm)(W), \end{aligned}$$

Applying the assumed inequality to the function f_0 , we obtain

$$\begin{aligned} (1 - \tilde{a}) \left(b(W \times W) + \frac{1}{2} b(\partial W) + (vm)(W) \right) - \tilde{k} m(W) \\ \leq \iota_\theta^{(p)}(W) + \frac{1}{2} b(\partial W) + (vm)(W). \end{aligned}$$

Therefore, we have

$$(1 - \tilde{a})b(W \times W) - \tilde{k}m(W) \leq \iota_\theta^{(p)}(W) + \tilde{a} \left(\frac{1}{2}b(\partial W) + (vm)(W) \right).$$

This proves the lemma. $\square$

To prove Theorem 3.1, we apply the lemmas above and the closed graph theorem.

Proof of Theorem 3.1. First, we prove the theorem for nonnegative potential $v \geq 0$ in which case $\deg = \deg^{(+)} := \deg + v_- m$. Let $p \geq 1$ and let us denote by $(i')_p$ and $(ii)_p$ the statements (i') and (ii) with this p , respectively.

$(i) \Rightarrow (ii)_p$: This follows from Lemma 3.13.

$(ii)_p \Rightarrow (i')_p$: This follows from Lemma 3.15.

$(i')_p \Rightarrow (i)$: This follows by Remark 2.6.

We now consider the case $p = 2$ and prove $(ii)_2 \Leftrightarrow (iv)$.

$(ii)_2 \Rightarrow (iv)$: By definition $D(Q_\theta) \subseteq \ell^2(X, m)$. The inequality in $(ii)_2$ implies $D(Q_\theta) \subseteq \ell^2(X, \deg)$ as $\langle \text{Deg} f, f \rangle_m = \sum_X \text{Deg} |f|^2 m = \sum_X \deg |f|^2$. The inclusion $\ell^2(X, \deg) \cap \ell^2(X, m) \subseteq D(Q_\theta)$ follows from the inequality $Q_\theta \leq 2\text{Deg}$ which holds true because $v \geq 0$.

$(iv) \Rightarrow (ii)_2$: This follows from the closed graph theorem applied to the embedding $j : D(Q_\theta) \rightarrow \ell^2(X, \deg^{(+)} + m)$ (cf. [5, Theorem A.1]).

We now turn to the case where v is not necessarily positive but $v_- \in \mathcal{K}_\alpha^{p_0, \theta}$. Since from the definition of magnetic-sparseness v_- appears neither in (i) , (i') nor in (ii) , each of the assertions (i) , $(i')_p$, $(ii)_p$ holds for (b, θ, v, m) if and only if it holds for (b, θ, v_+, m) . The equivalence between (i) , (i') and (ii) follows.

The equivalence between (ii) and (iii) is easy. Indeed, if (iii) holds for $Q_{\theta, v}$, then (ii) holds for Q_{θ, v_+} with the same constants. The converse $(ii) \Rightarrow (iii)$ is true with a change in the constants using the assumption $\langle v_-, |f|^{p_0} \rangle \leq \alpha Q_{b, \theta, v_+, m}^{(p_0)}(f) + C_\alpha \|f\|_{p_0}^{p_0}$, which is the definition of $\mathcal{K}_\alpha^{p_0, \theta}$ and the proof of Lemma 3.14.

In the case $p_0 = 2$, that is $v_- \in \mathcal{K}_\alpha^{2, \theta}$, the domains of $Q_{\theta, v}$ and Q_{θ, v_+} are the same. As a consequence, (iv) in the case $v_- \in \mathcal{K}_\alpha^{2, \theta}$ holds for (b, θ, v, m) if and only if it holds for (b, θ, v_+, m) . $\square$

4. Examples of Magnetic-Sparseness

In this section, we consider products of graphs, magnetic cycles and tessellations. In the section of products of graphs, we provide a structural description of part of the results of [12]. In the section of magnetic cycles, we compute the frustration index for magnetic cycles for $p = 1, 2$. Finally, we use these results to conclude magnetic-sparseness for tessellations under the assumption that the magnetic strength of cycles within the tessellation is large with respect to their length.

4.1. Products of Graphs

In this section, we apply our results to Cartesian products of graphs. Let two magnetic graphs $(b_1, \theta_1, v_1, m_1)$ over X_1 and $(b_2, \theta_2, v_2, m_2)$ over X_2 together

with their magnetic forms $Q_{\theta_1} = Q_{b_1, \theta_1, v_1, m_1}$ and $Q_{\theta_2} = Q_{b_2, \theta_2, v_2, m_2}$ be given. Furthermore, let $\mu : X_1 \times X_2 \rightarrow (0, \infty)$.

We define the product $(b, \theta, v, m_\mu) = (b_1, \theta_1, v_1, m_1) \otimes (b_2, \theta_2, v_2, m_2)$ with respect to μ via

$$\begin{aligned} b(x, y) &= b_1(x_1, y_1)m_2(x_2)1_{\{x_2=y_2\}} + b_2(x_2, y_2)m_1(x_1)1_{\{x_1=y_1\}}, \\ \theta(x, y) &= \theta_1(x_1, y_1)1_{\{x_2=y_2\}} + \theta_2(x_2, y_2)1_{\{x_1=y_1\}}, \\ v(x) &= \frac{1}{\mu(x_1, x_2)}(v_1(x_1) + v_2(x_2)), \\ m_\mu(x) &= \mu(x_1, x_2)m_1(x_1)m_2(x_2); \end{aligned}$$

for $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$. Note that for all $x = (x_1, x_2) \in X_1 \times X_2$,

$$\sum_{y \in X_1 \times X_2} b(x, y) = m_1(x_1) \sum_{y_2 \in Y_2} b_2(x_2, y_2) + m_2(x_2) \sum_{y_1 \in Y_1} b_1(x_1, y_1) < \infty.$$

This product is a natural choice as the following lemma shows.

Lemma 4.1. *The quadratic form $Q_\theta = Q_{b, \theta, v, m_\mu}$ acts as*

$$Q_\theta(f) = \sum_{x_2 \in X_2} Q_{\theta_1}(f(\cdot, x_2))m_2(x_2) + \sum_{x_1 \in X_1} Q_{\theta_2}(f(x_1, \cdot))m_1(x_1)$$

for $f \in D(Q_\theta) \subseteq \ell^2(X_1 \times X_2, m_\mu)$, and the corresponding self-adjoint Laplacian $H_\theta = H_{b, \theta, v, m_\mu}$ is a restriction of the operator $\mathcal{H}_\theta = \mathcal{H}_{b, \theta, v, m_\mu}$ acting as

$$\mathcal{H}_\theta = \frac{1}{\mu}[1_{X_1} \otimes \mathcal{H}_{\theta_2} + \mathcal{H}_{\theta_1} \otimes 1_{X_2}].$$

Proof. This follows by direct calculation. □

There are several ways to prove that the essential spectrum of the magnetic Laplacian is empty, e.g., [6, 12]. Here, we use Lemma 4.1 and the sparseness of only one of the graphs in order to prove the discreteness of the spectrum of H_θ . That is the spirit of the techniques developed in [12] (where the authors use a slightly different product).

Theorem 4.2. *Let $(b_1, \theta_1, v_1, m_1)$ be a magnetic graph over X_1 and $v_1 \geq 0$. Let $(b_2, \theta_2, v_2, m_2)$ be a $(a, 0)_1$ -magnetic-sparse graph over X_2 , with $v_2 \geq 0$ and $\inf \text{Deg}_2 > 0$. Let $\mu : X_1 \times X_2 \rightarrow (0, \infty)$ be such that*

$$\mu(x_1, x_2) \rightarrow 0,$$

as (x_1, x_2) leaves every compact set of $X_1 \times X_2$. Take $(b, \theta, v, m_\mu) := (b_1, \theta_1, v_1, m_1) \otimes (b_2, \theta_2, v_2, m_2)$, constructed as above. Then, H_θ has purely discrete spectrum.

If X_2 is finite and connected and $v_2 \geq 0$, note that $(b_2, \theta_2, v_2, m_2)$ is $(a, 0)_1$ -magnetic-sparse if $\iota_{\theta_2}^{(1)}(X_2) > 0$ or if $v_2(X_2) > 0$.

Proof. Let $0 < D_2 \leq \text{Deg}_2$, where Deg_2 is the weighted vertex degree of $(b_2, \theta_2, v_2, m_2)$. Using Lemma 4.1 and Theorem 3.1 for $(b_2, \theta_2, v_2, m_2)$, we infer for all $f \in C_c(X)$

$$\begin{aligned} Q_\theta(f) &\geq \sum_{x_1 \in X_1} Q_{\theta_2}(f(x_1, \cdot)) m_1(x_1) \\ &\geq (1 - \tilde{a}) \sum_{x_1 \in X_1} \langle \text{Deg}_2, |f(x_1, \cdot)|^2 \rangle_{m_2} m_1(x_1) \\ &\geq (1 - \tilde{a}) D_2 \langle \frac{1}{\mu} f, f \rangle_{m_\mu}, \end{aligned}$$

with $\tilde{a} < 1$. The discreteness of spectrum of H_θ follows from the min-max principle and the fact that μ tends to zero as leaving every compact set. $\square$

We now compute the magnetic-sparseness constants and the frustration indices of products of graphs.

Theorem 4.3. *Let $\mu : X_1 \times X_2 \rightarrow (0, \infty)$. Let $a, k \geq 0$ and $p \in [1, \infty)$. Suppose that $(b_1, \theta_1, v_1, m_1)$ and $(b_2, \theta_2, v_2, m_2)$ are $(a, k)_p$ -magnetic-sparse.*

- (1) *If $k = 0$, then (b, θ, v, m_μ) is $(a, 0)_p$ -magnetic-sparse.*
- (2) *If $\mu \geq c > 0$ for some $c > 0$, then (b, θ, v, m_μ) is $(a, k_\mu)_p$ -magnetic-sparse where $k_\mu := k/c$.*

In order to prove Theorem 4.3, we show a lemma, which is interesting in its own right.

Lemma 4.4. *Let $W \subseteq X_1 \times X_2$ and $p \in [1, \infty)$. For $x_1 \in X_1$, we denote $W_{x_1} = \{y \in X_2 \mid (x_1, y) \in W\}$ and for $x_2 \in X_2$ we denote $W_{x_2} = \{x \in X_1 \mid (x, x_2) \in W\}$. Then, we have*

$$\iota_\theta^{(p)}(W) = \sum_{x_1 \in X_1} m_1(x_1) \iota_{\theta_2}^{(p)}(W_{x_1}) + \sum_{x_2 \in X_2} m_2(x_2) \iota_{\theta_1}^{(p)}(W_{x_2}).$$

Proof. Let $p \in [1, \infty)$ be fixed. Let $\tau_0 : W \rightarrow \mathbb{T}$ be the function that attains the minimum in the definition of $\iota_\theta^{(p)}(W)$. By definition, we have

$$\begin{aligned} \iota_\theta^{(p)}(W) &= \frac{1}{2} \sum_{x_1 \in X_1} m_1(x_1) \sum_{x_2, y_2 \in W_{x_1}} b_2(x_2, y_2) |\tau_0(x_1, x_2) - e^{i\theta_2(x_2, y_2)} \tau_0(x_1, y_2)|^p \\ &\quad + \frac{1}{2} \sum_{x_2 \in X_2} m_2(x_2) \sum_{x_1, y_1 \in W_{x_2}} b_1(x_1, y_1) |\tau_0(x_1, x_2) - e^{i\theta_1(x_1, y_1)} \tau_0(y_1, x_2)|^p \\ &\geq \sum_{x_1 \in X_1} m_1(x_1) \iota_{\theta_2}^{(p)}(W_{x_1}) + \sum_{x_2 \in X_2} m_2(x_2) \iota_{\theta_1}^{(p)}(W_{x_2}). \end{aligned}$$

On the other hand, letting $\tau_1 : W_{x_2} \rightarrow \mathbb{T}$ and $\tau_2 : W_{x_1} \rightarrow \mathbb{T}$ be the function that attains the minima in the definitions of $\iota_{\theta_1}^{(p)}(W_{x_2})$ and $\iota_{\theta_2}^{(p)}(W_{x_1})$, respectively, we have

$$\begin{aligned} & \sum_{x_1 \in X_1} m_1(x_1) \iota_{\theta_2}^{(p)}(W_{x_1}) + \sum_{x_2 \in X_2} m_2(x_2) \iota_{\theta_1}^{(p)}(W_{x_2}) \\ &= \frac{1}{2} \sum_{x_1 \in X_1} m_1(x_1) \sum_{x_2, y_2 \in W_{x_1}} b_2(x_2, y_2) |\tau_1(x_1) \tau_2(x_2) - e^{i\theta_2(x_2, y_2)} \tau_1(x_1) \tau_2(y_2)|^p \\ & \quad + \frac{1}{2} \sum_{x_2 \in X_2} m_2(x_2) \sum_{x_1, y_1 \in W_{x_2}} b_1(x_1, y_1) |\tau_1(x_1) \tau_2(x_2) - e^{i\theta_1(x_1, y_1)} \tau_1(y_1) \tau_2(x_2)|^p \\ & \geq \iota_{\theta}^{(p)}(W), \end{aligned}$$

where we used the fact that $\tau_1 \tau_2 : (x_1, x_2) \mapsto \tau_1(x_1) \tau_2(x_2)$ is a map $W \rightarrow \mathbb{T}$. $\square$

Proof of Theorem 4.3. Let $p \in [1, \infty)$ and let W, W_{x_1}, W_{x_2} be as in Lemma 4.4. Furthermore, for $U \subseteq W_{x_j}$, let

$$R_j(U) := (1 + a) \iota_{\theta_j}^{(p)}(U) + a \left(\frac{1}{2} b_j(\partial U) + (v_{j,+} m_j)(U) \right), \quad j = 1, 2.$$

By direct calculation using the $(a, k)_p$ -magnetic-sparseness, we obtain

$$\begin{aligned} b(W \times W) &= \sum_{x_1 \in X_1} m_1(x_1) b_2(W_{x_1} \times W_{x_1}) + \sum_{x_2 \in X_2} m_2(x_2) b_1(W_{x_2} \times W_{x_2}) \\ &\leq \sum_{x_1 \in X_1} m_1(x_1) (R_2(W_{x_1}) + k m_2(W_{x_1})) \\ & \quad + \sum_{x_2 \in X_2} m_2(x_2) (R_1(W_{x_2}) + k m_1(W_{x_2})) \\ &= (1 + a) \iota_{\theta}^{(p)}(W) + a \left(\frac{1}{2} b(\partial W) + (v_+ m_{\mu})(W) \right) + k m(W). \end{aligned}$$

Invoking $m = m_{\mu}/\mu$, the statement follows by the assumption $k/\mu \leq k_{\mu}$. $\square$

Remark 4.5. Instead of Cartesian products, one can consider also sub-Cartesian products. The considerations are almost identical.

4.2. Magnetic Cycles

We study the notion of frustration indices and of magnetic-sparseness in the case of a cycle. We start with a definition.

Definition (*Magnetic cycle*).

- (a) We call a magnetic graph $\mathcal{C} := (b, \theta, 0, m)$ over a finite set Y a *cycle (graph)* of length

$$l(\mathcal{C}) = n,$$

if there is a bijective map $\Phi : Y \rightarrow \{0, \dots, n-1\}$ such that $b(x, y) > 0$ if and only if $(\Phi(x) - \Phi(y) \bmod n) \in \{1, n-1\}$. We set $x_j := \Phi^{-1}(j)$, $j = 1, \dots, n-1$ and $x_n := x_0$.

(b) The *magnetic flux* of a cycle $\mathcal{C}$ is defined as

$$F_\theta(\mathcal{C}) := \sum_{j=0}^{n-1} \theta(x_j, x_{j+1}) \bmod (2\pi).$$

(c) The *strength of the magnetic field* of a cycle $\mathcal{C}$ is defined as

$$s_\theta(\mathcal{C}) := |1 - \exp(iF_\theta(\mathcal{C}))|.$$

Note that while the sign of $F_\theta(\mathcal{C})$ still depends on the choice of Φ , the value of $s_\theta(\mathcal{C})$ is independent of Φ .

We turn to the computation of the frustration indices.

Proposition 4.6 (Frustration indices of a cycle). *Let $\mathcal{C}$ be a magnetic cycle over Y of length n with standard edge weights and let $W \subseteq Y$ be finite. Then,*

(a)

$$\iota_\theta^{(1)}(W) = \begin{cases} s_\theta(\mathcal{C}) & : W = Y, \\ 0 & : W \neq Y. \end{cases}$$

(b)

$$\iota_\theta^{(2)}(W) = \begin{cases} n|1 - e^{i\delta/n}|^2 & : W = Y, \\ 0 & : W \neq Y, \end{cases}$$

where $\delta := \min_{k \in \mathbb{Z}} |F_\theta(\mathcal{C}) - 2k\pi|$.

Proof. For (a) and (b), it is enough to consider with $W = Y$ by Proposition 2.1 (d).

- (a) This follows easily from [20, Theorem 4.10]. There it is proven that $\iota_\theta^{(1)}(Y)$ is attained for a function τ that is supported on a spanning tree and that satisfies $\tau(x) = e^{i\theta(x,y)}\tau(y)$ for neighbors x and y on this spanning tree.
- (b) In [6, Lemma 2.3], the bottom of $\sigma(H_\theta)$ is computed for $m = 1$ to be $|1 - e^{i\delta/n}|^2$. Since the eigenfunctions have constant absolute value (say, equals to 1), the minimizer of the Rayleigh quotient minimizes also $\iota_\theta^{(2)}(X)$ and, thus,

$$\iota_\theta^{(2)}(Y) = n \inf \sigma(H_\theta) = n|1 - e^{i\delta/n}|^2.$$

□

Remark 4.7. (a) The minimizers of $\iota_\theta^{(1)}$ and of $\iota_\theta^{(2)}$ are very different. For $\iota_\theta^{(2)}$, all edges have the same contribution, whereas the contribution for $\iota_\theta^{(1)}$ is concentrated solely on one edge.

- (b) Note that $\iota_\theta^{(1)}$ is bounded by 2 and is independent of the length of the cycle. It depends only on the magnetic flux.
- (c) In the case of a general magnetic cycle $\mathcal{C}$ over Y (not necessarily with standard edge weights), the same proof as above gives:

$$\iota_\theta^{(1)}(Y) = \min_{1 \leq j \leq n} \{b(x_j, x_{j+1})\} |1 - e^{iF_\theta(\mathcal{C})}|.$$

The exact value of $\iota_\theta^{(2)}(Y)$ in this situation is not clear.

We now turn to magnetic and bi-magnetic-sparseness of cycles.

Proposition 4.8. *Let $\mathcal{C}$ be a magnetic cycle over Y of length $l(\mathcal{C})$ with standard edge weights. Then, $\mathcal{C}$ is $(a, 0)_1$ -magnetic-sparse for some $a > 0$ if and only if $F_\theta(\mathcal{C}) \not\equiv 0 \pmod{(2\pi)}$. In this case, one can take: $a = \frac{2l(\mathcal{C})}{s_\theta(\mathcal{C})} - 1$.*

Moreover, if $l(\mathcal{C})$ is even, then the cycle $\mathcal{C}$ is $(a, 0)_1$ -magnetic-sparse for some $a > 0$ if and only if $F_\theta(\mathcal{C}) \not\equiv 0 \pmod{(2\pi)}$; and if $l(\mathcal{C})$ is odd, then the cycle $\mathcal{C}$ is $(a, 0)_1$ -magnetic-sparse for some $a > 0$ if and only if $F_\theta(\mathcal{C}) \not\equiv 0 \pmod{(\pi)}$.

Proof. First note that $F_\theta(\mathcal{C}) \not\equiv 0 \pmod{(2\pi)}$ if and only if $s_\theta(\mathcal{C}) > 0$. Assume first that $s_\theta(\mathcal{C}) = 0$, then $\iota_\theta^{(1)}(Y) = 0$ and magnetic-sparseness inequality

$$b(W \times W) \leq (1 + a)\iota_\theta^{(p)}(W) + a\left(\frac{1}{2}b(\partial W) + (v_+ m)(W)\right) + km(W)$$

cannot hold for $W = Y$. The cycle $\mathcal{C}$ cannot be $(a, 0)_1$ -magnetic-sparse for some $a > 0$. Assume now $s_\theta(\mathcal{C}) > 0$. Set $a = \frac{2l(\mathcal{C})}{s_\theta(\mathcal{C})} - 1$ and let $W \subseteq Y$. In the case $W = Y$, by Proposition 4.6 (a), we have $b(Y \times Y) = 2l(\mathcal{C})$, $b(\partial W) = 0$, and $\iota_\theta^{(1)}(W) = s_\theta(\mathcal{C})$ and the magnetic-sparseness inequality holds for $W = Y$. For $W \neq Y$, we have $s_\theta(\mathcal{C}) \leq 2$, $b(W \times W) \leq 2l(\mathcal{C})$, $b(\partial W) \geq 4$, and $\iota_\theta^{(1)}(W) = 0$ and magnetic-sparseness inequality also holds for W . Thus, $\mathcal{C}$ is $(a, 0)_1$ -magnetic-sparse.

In the case when $l(\mathcal{C})$ is even, $\mathcal{C}$ is bi-partite and by Proposition 2.8, $\mathcal{C}$ is $(a, 0)$ -bi-magnetic-sparse if and only if it is $(a, 0)$ -magnetic-sparse; the result follows. In the case when $l(\mathcal{C})$ is odd, one has: $F_{\theta+\pi}(\mathcal{C}) \equiv F_\theta(\mathcal{C}) + \pi \pmod{(2\pi)}$ and the result follows. $\square$

Remark 4.9. Using Remark 4.7 (c), it can be shown that a similar statement holds in the case of general magnetic cycles (with non-necessarily standard edge weights).

4.3. Subgraph Criterion and Tessellations

In this section, we give a useful criterion for magnetic-sparseness using subgraphs. Furthermore, we estimate the sparseness-constant for tessellations. At

the end, we show that regular triangulations with $\theta = \pi$ are magnetic-sparse graphs, but not bi-magnetic-sparse.

The results of this section are based on the following proposition, where the subgraphs can be thought of as cycles

Proposition 4.10 (Subgraph criterion for magnetic-sparseness). *Let (b, θ, v, m) be a magnetic graph over X and let $p \in [1, \infty)$. Let J be a set, $a, k \geq 0$, $C > c > 0$, and $M > 0$. Suppose $(b_j, \theta, v_j, m_j)_{j \in J}$ is a family of $(a, k)_p$ -magnetic-sparse-graphs such that for all $x, y \in X$,*

- (a) $c \cdot b(x, y) \leq \sum_{j \in J} b_j(x, y) \leq C \cdot b(x, y)$,
- (b) $\sum_{j \in J} v_{j,+}(x) m(x) \leq C \cdot v_+(x) m(x)$,
- (c) $\sum_{j \in J} m_j(x) \leq M \cdot m(x)$.

Then, (b, θ, v, m) is $(aC/c + C/c - 1, Mk/c)_p$ -magnetic-sparse. Moreover, if $k = 0$, then (a) and (b) are sufficient to conclude $(aC/c + C/c - 1, 0)_p$ -magnetic-sparseness of (b, θ, v, m) .

Proof. Let $W \subset X$ be finite and fix $p \in [1, \infty)$. We write $\iota_{\theta,j}^{(p)}$ for the frustration index of (b_j, θ, v_j, m_j) . Since (b_j, θ, v_j, m_j) is $(a, k)_p$ -magnetic-sparse for all $j \in J$, we obtain

$$\begin{aligned} b(W \times W) &\leq \frac{1}{c} \sum_{j \in J} b_j(W \times W) \\ &\leq \frac{1}{c} \sum_{j \in J} \left((1+a) \iota_{\theta,j}^{(p)}(W) + a \frac{1}{2} b_j(\partial W) + a(v_{j,+} m_j)(W) + k m_j(W) \right) \\ &\leq \frac{C}{c} \left((1+a) \iota_{\theta}^{(p)}(W) + a \frac{1}{2} b(\partial W) + a(v_+ m)(W) \right) + \frac{Mk}{c} m(W). \end{aligned}$$

This finishes the proof of the first the statement. The statement about $k = 0$ is clear. $\square$

Here, a *tessellation* is a planar graph such that there exists a set of subgraphs that are cycles such that every edge belongs to exactly two cycles. (A subgraph is a restriction of the corresponding maps to a subset of the space X .) For a more restrictive notion of planar tessellations, see, e.g., [4].

We will apply Proposition 4.10 to tessellations using the faces as subgraphs. We show that every tessellation is magnetic-sparse whenever the face degree is upper-bounded and the magnetic strength of the faces is lower-bounded from zero.

Let $\mathcal{F}$ be the set of faces of the graph. Let $F \in \mathcal{F}$, we denote by X_F the vertices which belong to F . We define also $b_F := b \cdot 1_{X_F \times X_F}$ and $\theta_F := \theta \cdot 1_{X_F \times X_F}$. The graph $\mathcal{C}_F := (b_F, \theta_F, 0, 1)$ over X_F is a magnetic cycle; see Sect. 4.2.

Corollary 4.11 (Magnetic-sparseness of tessellations). *Let a magnetic tessellation $(b, \theta, 0, 1)$ over X with standard edge weights be given. If*

$$a = \sup_{F \in \mathcal{F}} \frac{2l(\mathcal{C}_F)}{s(\mathcal{C}_F)} - 1 < \infty,$$

then the tessellation is $(a, 0)_1$ -magnetic-sparse.

Proof. Due to Proposition 4.6, the graph over $\mathcal{C}_F$ is $(a_F, 0)_1$ -magnetic-sparse with $a_F = \frac{2l(\mathcal{C}_F)}{s(\mathcal{C}_F)} - 1$. By the tessellation property, every edge belongs to exactly two cycles and, hence,

$$\sum_{F \in \mathcal{F}} b_F(x, y) = 2b(x, y).$$

Thus, by Proposition 4.10 we conclude the statement. $\square$

We give the example of triangulation Cayley graphs (b, θ, v, m) which turn out to be magnetic-sparse, but not bi-magnetic-sparse. These triangulation Cayley graphs can be understood as a generalization of the triangulation of the plane by equilateral triangles.

Example 4.12 (Magnetic-sparse, but not bi-magnetic-sparse). Let $(G, \cdot)$ be an infinite abelian group which is finitely generated by at least two elements with neutral element e and symmetric generating set S such that $e \notin S$. By choosing a possibly bigger S , we can furthermore assume that for every $s \in S$ there exists $r = r(s) \in S$ such that $rs \in S$. Because of the latter condition, we refer to the corresponding Cayley graph as a *triangulation*, i.e., every edge $\{g, sg\}$ in the corresponding Cayley graph belongs to at least one triangle, namely $\{g, sg, rsg\}$ with r chosen as above. Then,

$$1 \leq c := \min_{s \in S} \#\{r \in S \mid rs \in S\} \leq \max_{s \in S} \#\{r \in S \mid rs \in S\} =: C < \infty.$$

We take the magnetic potential $\theta = \pi$ and denote the Cayley graph by $(b, \theta, 0, m)$ for some arbitrary measure m , potential $v = 0$ and we take standard weights $b \in \{0, 1\}$ with $b(g, h) = 1$ if and only if $gh^{-1} \in S$. Due to Proposition 4.6, every triangle is $(2, 0)_1$ magnetic-sparse. Since every edge is contained in at least c and in at most C triangles, Proposition 4.10 yields $(a, 0)_1$ -magnetic-sparseness of $(b, \theta, 0, m)$ with

$$a = \frac{3C}{c} - 1$$

On the other hand, $\theta + \pi = 0 \pmod{2\pi}$ and, hence, $(b, \theta + \pi, 0, m) = (b, 0, 0, m)$. It is well known that abelian Cayley graphs satisfy $\inf_W \frac{b(\partial W)}{\#W} = 0$. By $\deg =$

$\#S$, we infer $\inf_W \frac{b(\partial W)}{b(W \times W)} = 0$ and, therefore, $(b, \theta + \pi, 0, m)$ is not $(a, 0)$ magnetic-sparse for any $a > 0$. If we choose m such that $m(X) < \infty$ and if G is infinite, we also have that $(b, \theta + \pi, 0, m)$ is not (a, k) -magnetic-sparse for any $a, k > 0$.

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