Asymptotic Analysis of Dotsenko–Fateev Integrals

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Abstract. We develop a method for evaluating asymptotics of certain contour integrals that appear in conformal field theory under the name of Dotsenko–Fateev integrals and that are natural generalizations of the classical hypergeometric functions. We illustrate the method by establishing a number of estimates that are useful in the context of martingale observables for multiple Schramm–Loewner evolution processes.

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1. Introduction

In this paper, we will be interested in integrals of the form

$$\int_A^{(z_1+, z_2+, z_1-, z_2-)} \prod_{j=1}^N (u - z_j)^{a_j} du, \quad (1.1)$$

where $N \geq 2$ is an integer, $\{a_j\}_{j=1}^N \subset \mathbb{R}$ is a set of exponents, $\{z_j\}_{j=1}^N \subset \mathbb{C}$ is a finite collection of points, and the Pochhammer integration contour encloses two of these points, say z_1 and z_2 . More precisely, the Pochhammer contour in (1.1) begins at the base point $A \in \mathbb{C}$, encircles z_1 once in the positive (counterclockwise) direction, returns to A , encircles z_2 once in the positive direction, returns to A , encircles z_1 once in the negative direction, returns to A , and finally encircles z_2 once in the negative direction before returning to A , see Fig. 1. The remaining points $\{z_j\}_{j=3}^N$ are assumed to lie exterior to the contour, and the powers in the integrand are assumed to take their principal values at the starting point and are then analytically continued along the contour.

By employing what is sometimes called the screening method, Dotsenko and Fateev obtained representations of correlation functions in conformal field theory (CFT) in terms of such integrals [3, 4] (see also [2]). Integrals of the form (1.1) are therefore often referred to as Dotsenko–Fateev integrals. Various aspects of such integrals and related special functions have been studied in a number of places, e.g., [7, 8]. For $N = 2$ and $N = 3$, the integral (1.1) can be expressed in terms of gamma functions and the classical hypergeometric function ${}_2F_1$, respectively. In the case of $N = 4$, a comparison with the representation of Appell’s F_1 function as an Euler integral (cf. [7]) shows that (1.1) can be viewed as an analytic continuation of Appell’s function. We

FIGURE 1. The Pochhammer integration contour in (1.1) consists of four loops based at A encircling z_1 and z_2 . The loop denoted by 1 is traversed first, then the loop denoted by 2 is traversed, and so on

also mention [5, 9, 10] which discuss related questions in the context of the Schramm–Loewner evolution (SLE).

The basic problem we are interested in here is how to compute the asymptotics of integral (1.1) as one (or several) of the points z_k , $k = 3, \dots, N$, approaches z_1 or z_2 . The computation of such asymptotics presents difficulties, because if the point z_k , $k = 3, \dots, N$, approaches z_1 , say, then the contour gets squeezed between z_k and z_1 which, in general, gives rise to a singular behavior. For $N = 3$, the relevant asymptotics can be obtained from well-known identities and expansions of hypergeometric functions. However, for $N \geq 4$, the analysis is more intricate.

In this paper, we propose a method which makes it possible to compute the asymptotics of (1.1) to all orders for any $N \geq 3$ as one (or many) of the points z_k , $k \geq 3$, approaches z_1 or z_2 . The obtained expansions are power series in the relevant small quantities with coefficients explicitly given in terms of gamma functions and integrals of the form (1.1) of a lower order $\leq N - 1$. In the case of $N = 4$, it is conceivable that the asymptotic formulas we derive can be obtained also from known properties of Appell's function. However, as far as we are aware, formulas of this type have not appeared in the literature even in this simplest case.

By applying a linear fractional transformation, we may assume that $z_1 = 0$ and $z_2 = 1$ in (1.1); this yields an expression of the form

$$\int_A^{(0+, 1+, 0-, 1-)} v^{a_1} (1-v)^{a_2} \prod_{j=3}^N (v-w_j)^{a_j} dv. \quad (1.2)$$

For simplicity, we will present the results in this paper for the class of integrals corresponding to $N = 4$. Although the proposed method works for arbitrary values of N and an arbitrary number $M \leq N$ of merging points, its application involves a large number of steps when M is large (the number of steps typically grows like M , because, roughly speaking, each time two points merge one of the series contributing to the asymptotics splits into two).

1.1. Brief Description of Method

Let $F(w_1, w_2) \equiv F(a, b, c, d; w_1, w_2)$ denote the integral in (1.2) for $N = 4$, that is, $F(w_1, w_2)$ is defined for $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$ by

$$F(w_1, w_2) = \int_A^{(0+, 1+, 0-, 1-)} v^a (v-w_1)^b (v-w_2)^c (1-v)^d dv, \quad (1.3)$$

where $A \in (0, 1)$ is a base point, $a, b, c, d \in \mathbb{R}$ are real exponents, and w_1, w_2 are assumed to lie outside the contour. In order to make F single-valued, we have restricted the domain of definition in (1.3) to $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$. We will assume that $a, d \in \mathbb{R} \setminus \mathbb{Z}$ are not integers, because otherwise the integral in (1.3) vanishes identically or can be computed by a residue calculation. Note that F can be analytically continued to a multiple-valued analytic function of $w_1, w_2 \in \mathbb{C} \setminus \{0, 1\}$. Our goal is to compute the asymptotic behavior of F to all orders as one or both of the points w_1 and w_2 approach 0 or 1.

The basic idea is the following: If we want to consider the limit $w_1 \rightarrow 0$ say, then we rewrite F as a sum of two terms (see Eq. (4.13)). One term which is defined by the same integral as F except that w_1 is now assumed to lie inside the contour in the same component as 0 (see Eq. (4.11)), and a second term which is defined by a similar expression but with the Pochhammer contour enclosing $\{0, w_1\}$ instead of $\{0, 1\}$ (see (4.12) and (4.15)). The asymptotics of both of these terms can easily be computed to all orders by simply replacing the factors in the integrands by their asymptotic expansions as $w_1 \rightarrow 0$ such as

$$(v - w_1)^b \sim \sum_{k=0}^{\infty} \frac{\Gamma(b+1)}{\Gamma(b+1-k)} (-1)^k v^{b-k} \frac{w_1^k}{k!}, \quad w_1 \rightarrow 0. \quad (1.4)$$

We emphasize that it is, in general, not possible to compute the asymptotics of F as $w_1 \rightarrow 0$ by substituting expansion (1.4) directly into (1.3). Indeed, such a procedure gives the correct contribution from the first term, but completely ignores the contribution from the second term.

1.2. Two Examples

Our initial motivation for studying asymptotics of integrals of the form (1.1) was that they appear naturally when constructing observables for multiple SLE curves via the screening method within CFT. Thus, in the second part of the paper, in order to illustrate our method, we consider two concrete examples of such observables where integrals of the form (1.1) are important. By applying the techniques developed in the first part of the paper, we are able to derive asymptotic estimates for the integrals in these examples. The estimates we establish are used in the derivation of the Green's function and Schramm's formula for multiple SLE presented in [11]. We expect similar asymptotic estimates to be needed also in the context of other SLE observables derived via the screening method.

1.3. Organization of the Paper

In Sect. 2, we present four different examples of asymptotic expansions to all orders which can be derived by our method. Before turning to the full description of the method and the proofs of the above expansions in Sect. 4, we consider the hypergeometric case of $N = 3$ in Sect. 3 as motivation. The two examples from SLE theory are introduced in Sect. 5. In Sect. 6, we derive the estimates relevant for the first example corresponding to the Green's function. In Sect. 7, we derive the estimates relevant for the second example corresponding to Schramm's formula.

2. Four Asymptotic Theorems

The purpose of this paper is to propose a method which makes it possible to compute the asymptotics of (1.2) as one (or several) of the points w_k , $k = 3, \dots, N$, approaches 0 or 1. There are clearly many different cases to consider depending on the value of N , on the number of points w_k involved in

the limiting process, and on whether each w_k approaches 0 or 1. For the sake of presentation, we have chosen to discuss four cases in detail. The presented cases all have $N = 4$ and correspond to the following limits: (a) $w_2 \rightarrow 1$, (b) $w_1 \rightarrow 0$, (c) $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$, and (d) $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$. These four cases, which are treated one by one in Theorems 2.1–2.4, illustrate the different situations that may arise, and it will be clear from the analysis of these cases how to apply the method also in other cases.

2.1. Notation

We let $F(w_1, w_2) \equiv F(a, b, c, d; w_1, w_2)$, $G(a, c, d; w)$, and $H(a, d)$ denote the integral given in (1.2) for $N = 4$, $N = 3$, and $N = 2$, respectively. That is, $F(w_1, w_2)$ is defined by (1.3), $G(a, c, d; w)$ is defined by

$$G(a, c, d; w) = \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w)^c (1 - v)^d dv, \quad w \in \mathbb{C} \setminus [0, \infty), \quad (2.1)$$

where w is assumed to lie outside the contour, and $H(a, d)$ is defined by

$$H(a, d) = \int_A^{(0+, 1+, 0-, 1-)} v^a (1 - v)^d dv. \quad (2.2)$$

The Pochhammer integration contour in (2.1) begins at the base point $A \in (0, 1)$, encircles the point 0 once in the positive (counterclockwise) direction, returns to A , encircles 1 once in the positive direction, returns to A , and so on. The point w lies exterior to all loops; the factors in the integrand take their principal values at the starting point and are then analytically continued along the contour. A similar interpretation of Pochhammer contours applies to definition (1.3) of $F(w_1, w_2)$ and elsewhere. Throughout the paper we adopt the convention that unless stated otherwise, the principal branch is used for all complex powers and logarithms, i.e., $\ln w = \ln |w| + i \arg w$ with $\arg w \in (-\pi, \pi]$. For $c \in \mathbb{C}$, we define $\rho_c(w)$ by

$$\rho_c(w) = \begin{cases} e^{-i\pi c}, & \operatorname{Im} w \geq 0, \\ e^{i\pi c}, & \operatorname{Im} w < 0. \end{cases} \quad (2.3)$$

2.2. Asymptotic Theorems

We can now state the asymptotic results. The proofs are given in Sect. 4.

Theorem 2.1 (Asymptotics as $w_2 \rightarrow 1$). *Let $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ be such that $c + d \notin \mathbb{Z}$. Then, F satisfies the following asymptotic expansion to all orders as $w_2 \rightarrow 1$ with $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$:*

$$F(w_1, w_2) \sim \sum_{k=0}^{\infty} \left\{ A_k^{(1)}(w_1)(w_2 - 1)^k + A_k^{(2)}(w_1)(w_2 - 1)^{c+d+1+k} \right\} \rho_c(w_2),$$

where the coefficients $A_k^{(j)}(w_1) \equiv A_k^{(j)}(a, b, c, d; w_1)$, $j = 1, 2$, are given by

$$A_k^{(1)}(w_1) = \frac{e^{2\pi i d} - 1}{e^{2\pi i(c+d)} - 1} \frac{\Gamma(c+1)}{\Gamma(c+1-k)k!} G(a, b, c+d-k; w_1),$$

$$A_k^{(2)}(w_1) = \frac{(e^{2\pi i a} - 1)e^{\pi i d}}{1 - e^{2\pi i(c+d)}} \sum_{l=0}^k \frac{\Gamma(a+1)\Gamma(b+1)(1-w_1)^{b-l}}{\Gamma(a+1-k+l)\Gamma(b+1-l)(k-l)!l!} H(c, d+k).$$

Theorem 2.2 (Asymptotics as $w_1 \rightarrow 0$). *Let $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ be such that $a+b \notin \mathbb{Z}$. Then, F satisfies the following asymptotic expansion to all orders as $w_1 \rightarrow 0$ with $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$:*

$$F(w_1, w_2) \sim \sum_{k=0}^{\infty} \left\{ B_k^{(1)}(w_2) w_1^k + \rho_{a+b}(w_1) B_k^{(2)}(w_2) w_1^{a+b+1+k} \right\}$$

where the coefficients $B_k^{(j)}(w_2) \equiv B_k^{(j)}(a, b, c, d; w_2)$, $j = 1, 2$, are given by

$$B_k^{(1)}(w_2) = \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b)} - 1} \frac{\Gamma(b+1)(-1)^k}{\Gamma(b+1-k)k!} G(a+b-k, c, d; w_2),$$

$$B_k^{(2)}(w_2) = \frac{(e^{2\pi i d} - 1)e^{\pi i a}}{e^{2\pi i(a+b)} - 1} \sum_{l=0}^k \frac{\Gamma(c+1)\Gamma(d+1)(-1)^l (-w_2)^{c-k+l}}{\Gamma(c+1-k+l)\Gamma(d+1-l)(k-l)!l!} H(a+k, b).$$

Theorem 2.3 (Asymptotics as $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$ with $|w_1/w_2| < 1 - \delta$). *Let $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ be such that $a+b, a+b+c \notin \mathbb{Z}$. Then, F satisfies the following asymptotic expansion to all orders as $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$ with $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$ such that $|w_1/w_2| < 1 - \delta$ for some $\delta > 0$:*

$$F(w_1, w_2) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \left\{ C_{kl}^{(1)} w_1^k w_2^l + C_{kl}^{(2)} \rho_{a+b+c}(w_2) w_2^{a+b+c+1+l} \left(\frac{w_1}{w_2} \right)^k \right. \\ \left. + C_{kl}^{(3)} \rho_{a+b}(w_1) \rho_c(w_2) w_1^{a+b+1+l} w_2^c \left(\frac{w_1}{w_2} \right)^k \right\},$$

where the coefficients $C_{kl}^{(j)} \equiv C_{kl}^{(j)}(a, b, c, d)$, $j = 1, 2, 3$, are given by

$$C_{kl}^{(1)} = \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b+c)} - 1} \frac{\Gamma(b+1)\Gamma(c+1)(-1)^{k+l}}{\Gamma(b+1-k)\Gamma(c+1-l)} H(a+b+c-k-l, d),$$

$$C_{kl}^{(2)} = \frac{(e^{2\pi i a} - 1)(e^{2\pi i d} - 1)e^{\pi i(a+b)}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \frac{\Gamma(b+1)\Gamma(d+1)(-1)^{k+l}}{\Gamma(b+1-k)\Gamma(d+1-l)} H(a+b-k+l, c),$$

$$C_{kl}^{(3)} = \frac{(e^{2\pi i d} - 1)e^{\pi i a}}{e^{2\pi i(a+b)} - 1} \frac{\Gamma(c+1)\Gamma(d+1)(-1)^{k+l}}{\Gamma(c+1-k)\Gamma(d+1-l)} H(a+k+l, b).$$

Theorem 2.4 (Asymptotics as $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$). *Let $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ be such that $a+b, c+d \notin \mathbb{Z}$. Then, F satisfies the following asymptotic expansion to all orders as $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$ with $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$:*

$$F(w_1, w_2) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\rho_c(w_2)}{k!l!} \left\{ D_{kl}^{(1)} w_1^k (w_2 - 1)^l + D_{kl}^{(2)} w_1^k (w_2 - 1)^{c+d+1+l} \right. \\ \left. + \rho_{a+b}(w_1) D_{kl}^{(3)} w_1^{a+b+1+k} (w_2 - 1)^l \right\}$$

where the coefficients $D_{kl}^{(j)} \equiv D_{kl}^{(j)}(a, b, c, d)$, $j = 1, 2, 3$, are given by

$$\begin{aligned} D_{kl}^{(1)} &= \frac{(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(c+d)} - 1)} \frac{\Gamma(b+1)\Gamma(c+1)(-1)^k}{\Gamma(b+1-k)\Gamma(c+1-l)} H(a+b-k, c+d-l), \\ D_{kl}^{(2)} &= \frac{(e^{2\pi ia} - 1)e^{\pi id}}{1 - e^{2\pi i(c+d)}} \frac{\Gamma(b+1)\Gamma(a+b+1-k)(-1)^k}{\Gamma(b+1-k)\Gamma(a+b+1-k-l)} H(c, d+l), \\ D_{kl}^{(3)} &= \frac{(e^{2\pi id} - 1)e^{\pi ia}}{e^{2\pi i(a+b)} - 1} \frac{\Gamma(c+1)\Gamma(-c-d+k-l)}{\Gamma(c+1-l)\Gamma(-c-d+l)} H(a+k, b). \end{aligned}$$

Remark 2.5. For definiteness, we have stated Theorem 2.3 under the assumption that $|w_1/w_2| < 1 - \delta$. It will be clear from the full description of the method in Sect. 4 that the case $|w_2/w_1| < 1 - \delta$ can be treated similarly. The case $|w_1| \asymp |w_2|$ can also be handled by similar steps, but in this case the coefficients depend on the quotient $\alpha := w_2/w_1$. In fact, a slight modification of the proof of Theorem 2.3 yields the following result (see Remark 4.4): If $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ satisfy $a+b, a+b+c \notin \mathbb{Z}$ and $w_2 = \alpha w_1$, then F satisfies the following asymptotic expansion to all orders as $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$ with $w_1, w_2 \in \mathbb{C} \setminus [0, \infty)$ such that $\alpha \in \mathbb{C} \setminus [0, \infty)$ and $\delta < |\alpha| < \delta^{-1}$ for some $\delta > 0$:

$$\begin{aligned} F(w_1, w_2) \sim \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{E_{kl}^{(1)}}{k!l!} w_1^k w_2^l + \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ E_l^{(2)}(\alpha) \rho_{a+b+c}(w_2) w_2^{a+b+c+1+l} \right. \\ \left. + E_l^{(3)}(\alpha) \rho_{a+b+c}(w_1) w_1^{a+b+c+1+l} \right\}, \end{aligned} \quad (2.4)$$

where $E_{kl}^{(1)} = C_{kl}^{(1)}$ and the coefficients $E_{kl}^{(j)}(\alpha) \equiv E_{kl}^{(j)}(a, b, c, d; \alpha)$, $j = 2, 3$, are given by

$$\begin{aligned} E_{kl}^{(2)}(\alpha) &= \frac{(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{\pi i(a+b)}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \frac{\Gamma(d+1)(-1)^l}{\Gamma(d+1-l)} G_{\text{inside}}(a+l, b, c; \alpha^{-1}), \\ E_{kl}^{(3)}(\alpha) &= \frac{(e^{2\pi id} - 1)e^{\pi i(a+c)}}{e^{2\pi i(a+b)} - 1} \frac{\Gamma(d+1)(-1)^l}{\Gamma(d+1-l)} G(a+l, c, b; \alpha). \end{aligned}$$

Here $G_{\text{inside}}(a, b, c; w)$ is defined by the same formula (2.1) as $G(a, b, c; w)$ except that w is assumed to lie inside the contour in the same component as 0.

Remark 2.6. The function $H(a, d)$ defined in (2.2) can be expressed in terms of Gamma functions as follows:

$$H(a, d) = (-1 + e^{2\pi ia} - e^{2\pi i(a+d)} + e^{2\pi id}) \frac{\Gamma(a+1)\Gamma(d+1)}{\Gamma(a+d+2)}.$$

This formula is easily derived by collapsing the Pochhammer contour onto the interval $[0, 1]$ and comparing the resulting expression with the definition of the Euler beta function.

Remark 2.7. The function $G(a, d)$ defined in (2.1) can be expressed in terms of the hypergeometric function ${}_2F_1$ as follows:

$$G(a, c, d; w) = \rho_c(w) \frac{4\pi^2 w^c {}_2F_1(-c, a+1, a+d+2; 1/w)}{e^{-\pi i(a+d+2)} \Gamma(a+d+2) \Gamma(-a) \Gamma(-d)}, \quad w \in \mathbb{C} \setminus [0, \infty). \quad (2.5)$$

Indeed, if $\tilde{G}$ denotes the function

$$\tilde{G}(a, c, d; w) = \int_A^{(0+, 1+, 0-, 1-)} v^a (w-v)^c (1-v)^d dv, \quad w \in \mathbb{C} \setminus (-\infty, 1],$$

where w lies exterior to the contour, then G and $\tilde{G}$ are related by

$$G(a, c, d; w) = \rho_c(w) \tilde{G}(a, c, d; w), \quad w \in \mathbb{C} \setminus \mathbb{R}. \quad (2.6)$$

Expression (2.5) follows because the Pochhammer integral expression for the hypergeometric function ${}_2F_1$ (see, e.g., [12, Eq. (15.6.5)]) implies that

$$\tilde{G}(a, c, d; w) = \frac{4\pi^2 w^c {}_2F_1(-c, a+1, a+d+2; 1/w)}{e^{-(a+d+2)\pi i} \Gamma(a+d+2) \Gamma(-a) \Gamma(-d)}, \quad w \in \mathbb{C} \setminus (-\infty, 1]. \quad (2.7)$$

3. The Hypergeometric Case of $N = 3$

Before turning to the full description of the method and the proofs of Theorems 2.1–2.4, it is helpful to consider, as motivation, the case $N = 3$ in which the integral in (1.2) reduces to a hypergeometric function.

Let $G(a, c, d; w)$ be the function defined in (2.1) and corresponding to (1.2) with $N = 3$. Equation (2.5) expresses $G(a, c, d; w)$ in terms of the hypergeometric function ${}_2F_1$, and we can use known properties of this function to derive the asymptotics of G as $w \rightarrow 0$ or $w \rightarrow 1$. For definiteness, we consider the limit $w \rightarrow 1$. We will show the following analog of Theorem 2.1.

Proposition 3.1 (Asymptotics of G as $w \rightarrow 1$). *Let $a, c, d \in \mathbb{R} \setminus \mathbb{Z}$ be such that $c + d \notin \mathbb{Z}$. Then, G satisfies the following asymptotic expansion to all orders as $w \rightarrow 1$ with $w \in \mathbb{C} \setminus [0, \infty)$:*

$$G(a, c, d; w) \sim \sum_{k=0}^{\infty} \left\{ \hat{A}_k^{(1)} (w-1)^k + \hat{A}_k^{(2)} (w-1)^{c+d+1+k} \right\} \rho_c(w), \quad (3.1)$$

where the coefficients $\hat{A}_k^{(j)} \equiv \hat{A}_k^{(j)}(a, c, d)$, $j = 1, 2$, are given by

$$\hat{A}_k^{(1)} = \frac{e^{2\pi i d} - 1}{e^{2\pi i(c+d)} - 1} \frac{\Gamma(c+1)}{\Gamma(c+1-k)k!} H(a, c+d-k), \quad (3.2a)$$

$$\hat{A}_k^{(2)} = \frac{(e^{2\pi i a} - 1)e^{\pi i d}}{1 - e^{2\pi i(c+d)}} \frac{\Gamma(a+1)}{\Gamma(a+1-k)k!} H(c, d+k). \quad (3.2b)$$

Proof. The function ${}_2F_1(a, c, d; z)$ is an analytic function of z with a branch cut along $[1, \infty)$; in particular, it is not analytic at $z = 1$. In order to find the asymptotics of G as $w \rightarrow 1$, we therefore first use the hypergeometric identity (see [13, Eq. (10.12)])

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, 1+a+b-c; 1-z) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, 1+c-a-b; 1-z), \\ &\quad z \in \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty)), \end{aligned} \quad (3.3)$$

to rewrite Eq. (2.7) as

$$\begin{aligned} \tilde{G}(a, c, d; w) &= - \frac{4\pi e^{i\pi(a+d+2)} \sin(\pi a) \Gamma(a+1) \sin(\pi d) \csc(\pi(c+d))}{\Gamma(-c-d) \Gamma(a+c+d+2)} w^c \\ &\quad \times {}_2F_1\left(1+a, -c, -c-d; 1-\frac{1}{w}\right) \\ &\quad + \frac{4\pi e^{i\pi(a+d+2)} \sin(\pi a) \sin(\pi d) \Gamma(d+1)}{\Gamma(-c) \Gamma(c+d+2) \sin(\pi(c+d))} w^c \left(\frac{w-1}{w}\right)^{c+d+1} \\ &\quad \times {}_2F_1\left(d+1, a+c+d+2; c+d+2; 1-\frac{1}{w}\right), \quad w \in \mathbb{C} \setminus (-\infty, 1]. \end{aligned} \quad (3.4)$$

The hypergeometric functions in (3.4) are analytic at $w = 1$. Hence, we can write (3.4) as

$$\tilde{G}(a, c, d; w) = \hat{P}_1(w) + (w-1)^{c+d+1} \hat{P}_2(w), \quad (3.5)$$

where $\hat{P}_j(w) \equiv \hat{P}_j(a, c, d; w)$, $j = 1, 2$, are analytic at $w = 1$. Recalling (2.6), it follows that G admits an expansion of the form (3.1) for some complex coefficients $\hat{A}_k^{(j)}$. It is possible to derive expressions (3.2) for these coefficients from (3.4) by employing the expansions

$$w^c \sim \sum_{j=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c+1-j)} \frac{(w-1)^j}{j!}, \quad \left(\frac{w-1}{w}\right)^a \sim \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(a+l)}{\Gamma(a)} \frac{(w-1)^{a+l}}{l!},$$

which are valid as $w \rightarrow 1$, together with the identity

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where the Pochhammer symbol $(a)_k$ is defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2) \cdots (a+k-1).$$

However, in order to arrive at the simple expressions in (3.2), this approach requires a somewhat elaborate resummation of the coefficients and it is actually more convenient to derive (3.2) by proceeding as in the proof of Theorem 2.1. $\square$

4. Description of Method

In this section, we describe our method by considering, in turn, the following four asymptotic sectors for the function $F(w_1, w_2)$ defined in (1.3): (a) w_2 near 1, (b) w_1 near 0, (c) w_1 and w_2 both near 0, and (d) w_1 near 0 and w_2 near 1. The limits considered in Theorems 2.1–2.4 belong to these four sectors, respectively, and the proofs of these theorems will also be given.

The basic idea of our method is to show that it is possible, for each asymptotic sector under consideration, to derive a generalization of the hypergeometric identity (3.5) from which asymptotics to arbitrary order can be obtained by simply replacing the factors in the integrand with their asymptotic expansions. Actually, we will see in Sect. 6 that it is often convenient in applications to work with the generalizations of the hypergeometric identity (3.5) themselves. These generalizations are presented in Propositions 4.1–4.5, respectively. Throughout the discussion, $b, c \in \mathbb{R}$ and $a, d \in \mathbb{R} \setminus \mathbb{Z}$ denote some given parameters and we write $F(w_1, w_2)$ for $F(a, b, c, d; w_1, w_2)$. Furthermore, we let $\mathcal{D}_0 \subset \mathbb{C}^2$ and $\mathcal{D}_1 \subset \mathbb{C}^2$ denote the domains

$$\mathcal{D}_0 = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_2 \in \mathbb{C} \setminus [0, \infty), w_1 \in \mathbb{C} \setminus ([0, \infty) \cup \gamma_{(w_2, \infty)})\}$$

and

$$\mathcal{D}_1 = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_1 \in \mathbb{C} \setminus [0, \infty), w_2 \in \mathbb{C} \setminus ((-\infty, 1] \cup \gamma_{(w_1, \infty)})\}, \quad (4.1)$$

where $\gamma_{(w_j, \infty)} \subset \mathbb{C}$ denotes a branch cut from w_j to ∞ . These branch cuts will be needed to make certain functions below single-valued; to be specific, we henceforth choose $\gamma_{(w_j, \infty)} = \{rw_j \mid r \geq 1\}$.

4.1. The Sector $w_2 \rightarrow 1$

We will determine the behavior of $F(w_1, w_2)$ for w_2 close to 1 by deriving a generalization of (3.5).

Define $\tilde{F}(w_1, w_2) \equiv \tilde{F}(a, b, c, d; w_1, w_2)$ by

$$\tilde{F}(w_1, w_2) = \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv, \quad (w_1, w_2) \in \mathcal{D}_1, \quad (4.2)$$

where w_1 and w_2 lie exterior to the contour. Then,

$$F(w_1, w_2) = \rho_c(w_2) \tilde{F}(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0 \cap \mathcal{D}_1, \quad (4.3)$$

where ρ_c is the function in (2.3).

Assuming that $c + d \notin \mathbb{Z}$, we define two functions $P_j : \mathcal{D}_1 \rightarrow \mathbb{C}$, $j = 1, 2$, as follows. The function P_1 is defined (up to a constant) by the same formula as $\tilde{F}$ except that the point w_2 is assumed to lie inside the contour in the same component as 1; more precisely, for $(w_1, w_2) \in \mathcal{D}_1$,

$$P_1(w_1, w_2) = \frac{e^{2\pi id} - 1}{e^{2\pi i(c+d)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv, \quad (4.4)$$

FIGURE 2. In the definition of the function P_1 , the point w_1 lies exterior to the contour, whereas w_2 lies inside the contour in the same component as 1

where w_1 lies outside the contour and w_2 lies inside the contour in the same component as 1, see Fig. 2.

The function $P_2 : \mathcal{D}_1 \rightarrow \mathbb{C}$ is defined as follows. First, given $w_1 \in \mathbb{C} \setminus [0, \infty)$, we define $P_2(w_1, w_2)$ for $\operatorname{Re} w_2 \in (0, 1)$ with $\operatorname{Im} w_2 > 0$ sufficiently small by

$$P_2(w_1, w_2) = \frac{(e^{2\pi i a} - 1)e^{\pi i d}}{1 - e^{2\pi i(c+d)}} \times \int_A^{(0+, 1+, 0-, 1-)} (w_2 + s(1 - w_2))^a (w_2 - w_1 + s(1 - w_2))^b s^c (1 - s)^d ds, \quad (4.5)$$

where $A \in (0, 1)$ and the points $\frac{w_2}{w_2 - 1}$ and $\frac{w_1 - w_2}{1 - w_2}$ are assumed to lie exterior to the contour. Then, for each $w_1 \in \mathbb{C} \setminus [0, \infty)$, we use analytic continuation to extend P_2 to a (single-valued) analytic function of $w_2 \in \mathbb{C} \setminus ((-\infty, 1] \cup \gamma_{(w_1, \infty)})$. The latter step is permissible because the function P_2 can be analytically continued as long as the points $\frac{w_2}{w_2 - 1}$ and $\frac{w_1 - w_2}{1 - w_2}$ stay away from the set $\{0, 1, \infty\}$, i.e., as long as $w_2 \notin \{0, 1, w_1, \infty\}$.

Let $f(w_1, w_2 \pm i0)$ denote the boundary values of a function $f(w_1, w_2)$ as w_2 approaches the real axis from above and below, respectively. The following lemma provides the desired generalization of the hypergeometric identity (3.5).

Proposition 4.1. *Suppose $a, b, c, d \in \mathbb{R}$ and $a, d, c + d \notin \mathbb{Z}$. Then, the function F obeys the identity*

$$F(w_1, w_2) = \rho_c(w_2) [P_1(w_1, w_2) + (w_2 - 1)^{c+d+1} P_2(w_1, w_2)], \quad (w_1, w_2) \in \mathcal{D}_0. \quad (4.6)$$

Proof. By (4.3), it is enough to show that

$$\tilde{F}(w_1, w_2) = P_1(w_1, w_2) + (w_2 - 1)^{c+d+1} P_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_1. \quad (4.7)$$

It is actually enough to show that

$$\tilde{F}(w_1, w_2 + i0) = P_1(w_1, w_2 + i0) + |1 - w_2|^{c+d+1} e^{i\pi(c+d+1)} P_2(w_1, w_2 + i0), \quad w_1 \in \mathbb{C} \setminus [0, \infty), \quad w_2 \in (0, 1). \quad (4.8)$$

Indeed, for each $w_1 \in \mathbb{C} \setminus [0, \infty)$, both sides of Eq. (4.7) are analytic functions of $w_2 \in \mathbb{C} \setminus ((-\infty, 1] \cup \gamma_{(w_1, \infty)})$ which can be extended to multiple-valued

FIGURE 3. The semicircles $S_w^\pm$ FIGURE 4. The contours L_A^j , $j = 1, \dots, 4$

analytic functions of $w_2 \in \mathbb{C} \setminus \{0, 1, w_1\}$. Hence, (4.7) follows from (4.8) by analytic continuation.

Let us prove (4.8). Let $\epsilon > 0$ be small. Let $w_1 \in \mathbb{C} \setminus [0, \infty)$ and $w_2 \in (0, 1)$. Given $w \in \mathbb{C}$, let

$$S_w^+ = \{w + \epsilon e^{i\phi} \mid 0 \leq \phi \leq \pi\}, \quad S_w^- = \{w + \epsilon e^{i\phi} \mid -\pi \leq \phi \leq 0\},$$

denote counterclockwise semicircles of radius ϵ centered at w , see Fig. 3. Furthermore, for $A \in (0, 1)$, let L_A^j , $j = 1, \dots, 4$, denote the contours

$$\begin{aligned} L_A^1 &= [A, i\epsilon] \cup [i\epsilon, -\epsilon], L_A^2 = [-\epsilon, -i\epsilon] \cup [-i\epsilon, A], \\ L_A^3 &= [A, 1 - i\epsilon] \cup [1 - i\epsilon, 1 + \epsilon], L_A^4 = [1 + \epsilon, 1 + i\epsilon] \cup [1 + i\epsilon, A], \end{aligned}$$

oriented so that $\sum_1^4 L_A^j$ is a counterclockwise contour enclosing 0 and 1, see Fig. 4.

Then, we can write

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) &= \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} + e^{2\pi i(a+c+d)} \int_{L_{w_2+\epsilon}^4} \right. \\ &\quad \left. - e^{2\pi i(a+d)} \int_{S_{w_2}^- + L_{w_2-\epsilon}^2} + e^{2\pi id} \int_{-L_{w_2-\epsilon}^1 + S_{w_2}^-} - e^{2\pi i(c+d)} \int_{L_{w_2+\epsilon}^4} \right. \\ &\quad \left. - \int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv \end{aligned}$$

and

$$\begin{aligned}
P_1(w_1, w_2 + i0) = & \frac{e^{2\pi id} - 1}{e^{2\pi i(c+d)} - 1} \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} \right. \\
& + e^{2\pi i(a+d+c)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+ - L_{w_2-\epsilon}^2} - e^{2\pi i(d+c)} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} \\
& \left. - \int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv.
\end{aligned}$$

Simplification gives

$$\begin{aligned}
\tilde{F}(w_1, w_2 + i0) = & \left\{ (1 - e^{2\pi id}) \left(\int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2} \right) \right. \\
& + (e^{2\pi ia} - 1) \left(\int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} + e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4} - e^{2\pi id} \int_{S_{w_2}^-} \right) \left. \right\} \\
& \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv
\end{aligned}$$

and

$$\begin{aligned}
P_1(w_1, w_2 + i0) = & \frac{e^{2\pi id} - 1}{e^{2\pi i(d+c)} - 1} \left\{ (1 - e^{2\pi i(d+c)}) \left(\int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2} \right) \right. \\
& + (e^{2\pi ia} - 1) \left(\int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} + e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+} \right) \left. \right\} \\
& \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) = & \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi d}}{\sin(\pi(d+c))} \left\{ \int_{L_{w_2+\epsilon}^3} + e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4} \right. \\
& + \frac{e^{2\pi ic}(e^{2\pi id} - 1)}{1 - e^{2\pi ic}} \int_{S_{w_2}^+ + S_{w_2}^-} \left. \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv.
\end{aligned}$$

Using the identity

$$(w_2 - v)^c = \begin{cases} e^{-i\pi c} (v - w_2)^c, & v \in S_{w_2}^+ \cup L_{w_2+\epsilon}^4, \\ e^{i\pi c} (v - w_2)^c, & v \in S_{w_2}^- \cup L_{w_2+\epsilon}^3, \end{cases}$$

we can write this as

$$\begin{aligned}
\tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) = & \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi d}}{\sin(\pi(d+c))} \left\{ \int_{L_{w_2+\epsilon}^3} + e^{2\pi id} \int_{L_{w_2+\epsilon}^4} \right. \\
& + \frac{e^{2\pi id} - 1}{1 - e^{2\pi ic}} \int_{S_{w_2}^+} + \frac{e^{2\pi ic}(e^{2\pi id} - 1)}{1 - e^{2\pi ic}} \int_{S_{w_2}^-} \left. \right\} \\
& \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \quad w_1 \in \mathbb{C} \setminus [0, \infty), \quad w_2 \in (0, 1).
\end{aligned}$$

Factoring out $\frac{1}{e^{2\pi ic}-1}$, we obtain

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi(d+c)}}{\sin(\pi(d+c))(e^{2\pi ic} - 1)} \left\{ (e^{2\pi ic} - 1) \int_{L_{w_2+\epsilon}^3} \right. \\ &\quad \left. + e^{2\pi id} (e^{2\pi ic} - 1) \int_{L_{w_2+\epsilon}^4} + (1 - e^{2\pi id}) \int_{S_{w_2}^+} + e^{2\pi ic} (1 - e^{2\pi id}) \int_{S_{w_2}^-} \right\} \\ &\quad \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv. \end{aligned}$$

That is,

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi(d+c)}}{\sin(\pi(d+c))(e^{2\pi ic} - 1)} \int_{w_2+\epsilon}^{(w_2+, 1+, w_2-, 1-)} \\ &\quad \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv. \end{aligned}$$

Performing the change of variables $s = \frac{v-w_2}{1-w_2}$, which maps the interval $(w_2, 1)$ to the interval $(0, 1)$, this yields

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{e^{i(a+d)\pi} \sin(a\pi)}{\sin(\pi(d+c))} \int_A^{(0+, 1+, 0-, 1-)} (w_2 + s(1 - w_2))^a \\ &\quad \times (w_2 + s(1 - w_2) - w_1)^b (s(1 - w_2))^c ((1 - w_2)(1 - s))^d (1 - w_2) ds. \end{aligned}$$

Comparing this expression with definition (4.5) of P_2 , Eq. (4.8) follows. $\square$

Since both terms on the right-hand side of identity (4.6) are well-behaved for w_2 close to 1, the behavior of F as $w_2 \rightarrow 1$ can easily be extracted from this identity.

Proof of Theorem 2.1. Suppose $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ and $c + d \notin \mathbb{Z}$. Using identity (4.6), we can easily prove Theorem 2.1. Indeed, the functions P_1 and P_2 in (4.6) admit asymptotic expansions to all orders as follows. Substituting the expansion

$$(w_2 - v)^c \sim \sum_{k=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c+1-k)} \frac{(w_2 - 1)^k}{k!} (1 - v)^{c-k}, \quad w_2 \rightarrow 1,$$

into the definition of $P_1(w_1, w_2)$ and recalling that w_2 and 1 lie in the same component inside the contour, we find

$$\begin{aligned} P_1(w_1, w_2) &\sim \frac{e^{2\pi id} - 1}{e^{2\pi i(c+d)} - 1} \sum_{k=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c+1-k)} \frac{(w_2 - 1)^k}{k!} \\ &\quad \times \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (1 - v)^{c+d-k} dv, \quad w_2 \rightarrow 1, \end{aligned} \tag{4.9}$$

where the integral on the right-hand side is exactly $G(a, b, c + d - k; w_1)$. Similarly, substituting the expansions

$$(w_2 + s(1 - w_2))^a \sim \sum_{m=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a+1-m)} (1 - s)^m \frac{(w_2 - 1)^m}{m!}, \quad w_2 \rightarrow 1,$$

and

$$(w_2 + s(1 - w_2) - w_1)^b \sim \sum_{l=0}^{\infty} \frac{\Gamma(b+1)}{\Gamma(b+1-l)} (1-s)^l (1-w_1)^{b-l} \frac{(w_2-1)^l}{l!}, \quad w_2 \rightarrow 1,$$

into definition (4.5) of $P_2(w_1, w_2)$, we find, as $w_2 \rightarrow 1$,

$$\begin{aligned} P_2(w_1, w_2) &\sim \frac{(e^{2\pi ia} - 1)e^{\pi id}}{1 - e^{2\pi i(c+d)}} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(a+1)\Gamma(b+1)(1-w_1)^{b-l}}{\Gamma(a+1-m)\Gamma(b+1-l)} \\ &\quad \times \frac{(w_2-1)^{m+l}}{m!l!} H(c, d+m+l). \end{aligned} \quad (4.10)$$

Substituting (4.9) and (4.10) (with the summation variable m replaced by $k = m + l$) into (4.6), we arrive at the expansion given in Theorem 2.1. $\square$

4.2. The Sector $w_1 \rightarrow 0$

In order to determine the behavior of $F(w_1, w_2)$ as $w_1 \rightarrow 0$, we define two functions $Q_j : \mathcal{D}_0 \rightarrow \mathbb{C}$, $j = 1, 2$, as follows. The function $Q_1(w_1, w_2)$ is defined for $(w_1, w_2) \in \mathcal{D}_0$ by

$$Q_1(w_1, w_2) = \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \quad (4.11)$$

where $A \in (0, 1)$, w_1 lies inside the contour in the same component as 0, and w_2 lies outside the contour. Given $w_2 \in \mathbb{C} \setminus [0, \infty)$, we define $Q_2(w_1, w_2)$ for $\operatorname{Re} w_1 \in (0, 1)$ with $\operatorname{Im} w_1 < 0$ sufficiently small by

$$Q_2(w_1, w_2) = \frac{(e^{2\pi id} - 1)e^{-\pi ib}}{1 - e^{-2\pi i(a+b)}} \int_A^{(0+, 1+, 0-, 1-)} s^a (1 - s)^b (sw_1 - w_2)^c (1 - sw_1)^d ds, \quad (4.12)$$

where $A \in (0, 1)$ and the points $\frac{w_2}{w_1}$ and $\frac{1}{w_1}$ lie exterior to the contour. For each $w_2 \in \mathbb{C} \setminus [0, \infty)$, we then use analytic continuation to extend Q_2 to a function of $w_1 \in \mathbb{C} \setminus ([0, \infty) \cup \gamma_{(w_2, \infty)})$. We have the following analog of Proposition 4.1.

Proposition 4.2. *Suppose $a, b, c, d \in \mathbb{R}$ and $a, d, a + b \notin \mathbb{Z}$. Then, the function F obeys the identity*

$$F(w_1, w_2) = Q_1(w_1, w_2) + w_1^{a+b+1} Q_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0. \quad (4.13)$$

Proof. By analyticity, it is enough to show that

$$F(w_1 - i0, w_2) = Q_1(w_1 - i0, w_2) + w_1^{a+b+1} Q_2(w_1 - i0, w_2) \quad (4.14)$$

for $w_1 \in (0, 1)$ and $w_2 \in \mathbb{C} \setminus [0, \infty)$.

Let $\epsilon > 0$ be small. Let $w_1 \in (0, 1)$ and $w_2 \in \mathbb{C} \setminus [0, \infty)$. Then,

$$\begin{aligned}
F(w_1 - i0, w_2) = & \left\{ \int_{L^1_{w_1 - \epsilon}} + e^{2\pi i(a+b)} \int_{L^2_{w_1 - \epsilon}} - e^{2\pi i a} \int_{S^+_{w_1}} + e^{2\pi i a} \int_{L^3_{w_1 + \epsilon}} + e^{2\pi i(a+d)} \int_{L^4_{w_1 + \epsilon}} \right. \\
& + e^{2\pi i(a+d)} \int_{S^+_{w_1}} - e^{2\pi i(a+b+d)} \int_{L^2_{w_1 - \epsilon}} - e^{2\pi i d} \int_{L^1_{w_1 - \epsilon} + S^+_{w_1} + L^4_{w_1 + \epsilon}} \\
& \left. - \int_{L^3_{w_1 + \epsilon}} + \int_{S^+_{w_1}} \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv,
\end{aligned}$$

where w_2 lies exterior to the contours. Moreover,

$$\begin{aligned}
Q_1(w_1 - i0, w_2) = & \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b)} - 1} \left\{ \int_{S^+_{w_1} + L^1_{w_1 - \epsilon}} + e^{2\pi i(a+b)} \int_{L^2_{w_1 - \epsilon} + S^-_{w_1} + L^3_{w_1 + \epsilon}} \right. \\
& + e^{2\pi i(a+b+d)} \int_{L^4_{w_1 + \epsilon} - S^-_{w_1} - L^2_{w_1 - \epsilon}} - e^{2\pi i d} \int_{L^1_{w_1 - \epsilon} + S^+_{w_1} + L^4_{w_1 + \epsilon}} + \left. \int_{-L^3_{w_1 + \epsilon}} \right\} \\
& \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv.
\end{aligned}$$

Simplification gives

$$\begin{aligned}
F(w_1 - i0, w_2) = & \left\{ (1 - e^{2\pi i d}) \int_{L^1_{w_1 - \epsilon}} + e^{2\pi i(a+b)} (1 - e^{2\pi i d}) \int_{L^2_{w_1 - \epsilon}} \right. \\
& + (e^{2\pi i a} - 1) \left((e^{2\pi i d} - 1) \int_{S^+_{w_1}} + \int_{L^3_{w_1 + \epsilon}} + e^{2\pi i d} \int_{L^4_{w_1 + \epsilon}} \right) \left. \right\} \\
& \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv
\end{aligned}$$

and

$$\begin{aligned}
Q_1(w_1 - i0, w_2) = & \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b)} - 1} \left\{ (1 - e^{2\pi i d}) \int_{L^1_{w_1 - \epsilon}} + e^{2\pi i(a+b)} (1 - e^{2\pi i d}) \int_{L^2_{w_1 - \epsilon}} \right. \\
& + e^{2\pi i(a+b)} (1 - e^{2\pi i d}) \int_{S^-_{w_1}} + (e^{2\pi i(a+b)} - 1) \int_{L^3_{w_1 + \epsilon}} \\
& + e^{2\pi i d} (e^{2\pi i(a+b)} - 1) \int_{L^4_{w_1 + \epsilon}} + (1 - e^{2\pi i d}) \int_{S^+_{w_1}} \left. \right\} \\
& \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) = & \frac{(e^{2\pi i d} - 1)e^{-2\pi i b}}{1 - e^{-2\pi i(a+b)}} \left\{ (1 - e^{2\pi i b}) \int_{L^1_{w_1 - \epsilon}} \right. \\
& + e^{2\pi i(a+b)} (1 - e^{2\pi i b}) \int_{L^2_{w_1 - \epsilon}} + e^{2\pi i b} (e^{2\pi i a} - 1) \int_{S^+_{w_1} + S^-_{w_1}} \left. \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv.
\end{aligned}$$

Using the identity

$$(v - w_1)^b = \begin{cases} e^{i\pi b} (w_1 - v)^b, & v \in S^+_{w_1} \cup L^1_{w_1 + \epsilon}, \\ e^{-i\pi b} (w_1 - v)^b, & v \in S^-_{w_1} \cup L^2_{w_1 + \epsilon}, \end{cases}$$

we can write this as

$$\begin{aligned}
F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) &= \frac{(e^{2\pi id} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} \left\{ (1 - e^{2i\pi b}) \int_{L_{w_1 - \epsilon}^1} \right. \\
&\quad \left. + e^{2\pi ia}(1 - e^{2i\pi b}) \int_{L_{w_1 - \epsilon}^2} + e^{2\pi ib}(e^{2\pi ia} - 1) \int_{S_{w_1}^+} + (e^{2\pi ia} - 1) \int_{S_{w_1}^-} \right\} \\
&\quad \times v^a(w_1 - v)^b(v - w_2)^c(1 - v)^d dv, \quad w_1 \in (0, 1), \quad w_2 \in \mathbb{C} \setminus [0, \infty).
\end{aligned}$$

That is,

$$\begin{aligned}
F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) &= \frac{(e^{2\pi id} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} \int_{w_1 - \epsilon}^{(0+, w_1 +, 0-, w_1 -)} \\
&\quad \times v^a(w_1 - v)^b(v - w_2)^c(1 - v)^d dv.
\end{aligned} \tag{4.15}$$

Performing the change of variables $s = \frac{v}{w_1}$, which maps the interval $(0, w_1)$ to the interval $(0, 1)$, we obtain

$$\begin{aligned}
F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) &= \frac{(e^{2\pi id} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} w_1^{a+b+1} \int_A^{(0+, 1+, 0-, 1-)} \\
&\quad \times s^a(1 - s)^b(sw_1 - w_2)^c(1 - sw_1)^d ds.
\end{aligned}$$

The lemma follows. $\square$

Proof of Theorem 2.2. Suppose $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ and $a + b \notin \mathbb{Z}$. In the same way that (4.6) can be used to determine the asymptotics of F as $w_2 \rightarrow 1$, identity (4.13) can be used to determine the asymptotics of $F(w_1, w_2)$ as $w_1 \rightarrow 0$. Indeed, the expansion given in Theorem 2.2 follows from (4.13) after substituting the following asymptotic expansions as $w_1 \rightarrow 0$ into the definitions of Q_1 and Q_2 :

$$(v - w_1)^b \sim \sum_{k=0}^{\infty} \frac{\Gamma(b+1)}{\Gamma(b+1-k)} (-1)^k v^{b-k} \frac{w_1^k}{k!}, \tag{4.16a}$$

$$(sw_1 - w_2)^c \sim \sum_{k=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c+1-k)} \frac{s^k (-w_2)^{c-k} w_1^k}{k!}, \tag{4.16b}$$

$$(1 - sw_1)^d \sim \sum_{k=0}^{\infty} \frac{\Gamma(d+1)}{\Gamma(d+1-k)} \frac{(-1)^k s^k w_1^k}{k!}. \tag{4.16c}$$

$\square$

4.3. The Sector $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$

We next determine the asymptotics of $F(w_1, w_2)$ in the regime where both w_1 and w_2 approach zero. Assuming $a + b, a + b + c \notin \mathbb{Z}$, we define two functions $R_j : \mathcal{D}_0 \rightarrow \mathbb{C}$, $j = 1, 2$, as follows. The function $R_1(w_1, w_2)$ is defined for $(w_1, w_2) \in \mathcal{D}_0$ by

$$R_1(w_1, w_2) = \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b+c)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv,$$

where $A \in (0, 1)$ and both points w_1 and w_2 are assumed to lie inside the contour in the same component as 0. For $0 < \operatorname{Re} w_1 < \operatorname{Re} w_2 < 1$ with $\operatorname{Im} w_1 < 0$ and $\operatorname{Im} w_2 < 0$, we define $R_2(w_1, w_2)$ by

$$R_2(w_1, w_2) = \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \times \int_A^{(0+, 1+, 0-, 1-)} s^a (sw_2 - w_1)^b (1-s)^c (1-sw_2)^d ds, \quad (4.17)$$

where we assume $A \in (0, 1)$ is so large that $\operatorname{Re}(Aw_2 - w_1) > 0$, that the point $\frac{w_1}{w_2}$ lies inside the contour in the same component as 0, and that $\frac{1}{w_2}$ lies outside the contour. We then use analytic continuation to extend R_2 to all of $\mathcal{D}_0$. We have the following analog of Proposition 4.1.

Proposition 4.3. *Suppose $a, b, c, d \in \mathbb{R}$ and $a, d, a+b, a+b+c \notin \mathbb{Z}$. Then, the function F obeys the following identity for $(w_1, w_2) \in \mathcal{D}_0$:*

$$F(w_1, w_2) = R_1(w_1, w_2) + w_2^{a+c+1} R_2(w_1, w_2) + w_1^{a+b+1} Q_2(w_1, w_2). \quad (4.18)$$

Proof. Both sides of (4.18) are analytic functions of $(w_1, w_2) \in \mathcal{D}_0$ which extend to multiple-valued analytic functions of

$$(w_1, w_2) \in \mathbb{C}^2 \setminus (\{w_1 = 0\} \cup \{w_1 = 1\} \cup \{w_2 = 0\} \cup \{w_2 = 1\} \cup \{w_1 = w_2\}).$$

Hence, by Proposition 4.2, it is enough to show that

$$Q_{1-}(w_1, w_2) = R_{1-}(w_1, w_2) + w_2^{a+c+1} R_{2-}(w_1, w_2), \quad 0 < w_1 < w_2 < 1, \quad (4.19)$$

where, for a function f , we use the short-hand notation $f_-(w_1, w_2) := f(w_1 - i0, w_2 - i0)$.

Let $0 < w_1 < w_2 < 1$ and suppose $0 < \epsilon < \frac{1}{2} \min\{w_1, w_2 - w_1, 1 - w_2\}$. Then,

$$Q_{1-}(w_1, w_2) = \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi i(a+b+c)} \int_{L_{w_2-\epsilon}^2} - e^{2\pi i(a+b)} \int_{S_{w_2}^+} \right. \\ \left. + e^{2\pi i(a+b)} \int_{L_{w_2+\epsilon}^3} + e^{2\pi i(a+b+d)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+} - e^{2\pi i(a+b+c+d)} \int_{L_{w_2-\epsilon}^2} - e^{2\pi id} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} \right. \\ \left. - \int_{L_{w_2+\epsilon}^3} + \int_{S_{w_2}^+} \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv$$

and

$$R_{1-}(w_1, w_2) = \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b+c)} - 1} \left\{ \int_{S_{w_2}^+ + L_{w_2-\epsilon}^1} + e^{2\pi i(a+b+c)} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} \right. \\ \left. + e^{2\pi i(a+b+c+d)} \int_{L_{w_2+\epsilon}^4 - S_{w_2}^- - L_{w_2-\epsilon}^2} - e^{2\pi id} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} \right. \\ \left. - \int_{L_{w_2+\epsilon}^3} \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv,$$

where the principal branch is used for all powers. A computation gives

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \\ &\times \left\{ (1 - e^{2i\pi c}) \int_{L^1_{w_2-\epsilon}} + e^{2\pi i(a+b+c)}(1 - e^{2i\pi c}) \int_{L^2_{w_2-\epsilon}} + e^{2\pi ic}(e^{2\pi i(a+b)} - 1) \int_{S^+_{w_2} + S^-_{w_2}} \right\} \\ &\times v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv, \quad 0 < w_1 < w_2 < 1. \end{aligned}$$

Using the identity

$$(v - w_2)^c = \begin{cases} e^{i\pi c}(w_2 - v)^c, & v \in S^+_{w_2} \cup L^1_{w_2-\epsilon}, \\ e^{-i\pi c}(w_2 - v)^c, & v \in S^-_{w_2} \cup L^2_{w_2-\epsilon}, \end{cases}$$

we can write this as

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \left\{ (1 - e^{2i\pi c}) \int_{L^1_{w_2-\epsilon}} \right. \\ &+ e^{2\pi i(a+b)}(1 - e^{2i\pi c}) \int_{L^2_{w_2-\epsilon}} + e^{2\pi ic}(e^{2\pi i(a+b)} - 1) \int_{S^+_{w_2}} + (e^{2\pi i(a+b)} - 1) \int_{S^-_{w_2}} \left. \right\} \\ &\times v^a(v - w_1)^b(w_2 - v)^c(1 - v)^d dv, \quad 0 < w_1 < w_2 < 1. \end{aligned}$$

That is,

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \\ &\times \int_{w_2-\epsilon}^{(0+, w_2+, 0-, w_2-)} v^a(v - w_1)^b(w_2 - v)^c(1 - v)^d dv, \quad 0 < w_1 < w_2 < 1, \end{aligned}$$

where w_1 lies inside the contour in the same component as 0. Applying the change of variables $s = \frac{v}{w_2}$, which maps the interval $(0, w_2)$ to the interval $(0, 1)$, we obtain

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} w_2^{a+c+1} \\ &\times \int_A^{(0+, 1+, 0-, 1-)} s^a(sw_2 - w_1)^b(1 - s)^c(1 - sw_2)^d ds, \quad 0 < w_1 < w_2 < 1, \end{aligned}$$

where $A \in (0, 1)$ is so large that $Aw_2 - w_1 > 0$. Equation (4.19) follows. $\square$

Proof of Theorem 2.3. Suppose $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ and $a + b, a + b + c \notin \mathbb{Z}$. Theorem 2.3 follows by expanding the integrands in the definitions of R_1, R_2, Q_2 as $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$ and substituting the resulting expressions into (4.18). We have stated Theorem 2.3 under the assumption that $|w_1/w_2| < 1 - \delta$; hence, we use expansion (4.16b) of $(sw_1 - w_2)^c$ and the expansion

$$(sw_2 - w_1)^b \sim \sum_{k=0}^{\infty} \frac{\Gamma(b+1)(-1)^k}{\Gamma(b+1-k)} \frac{(sw_2)^{b-k} w_1^k}{k!} \quad (4.20)$$

of the factor $(sw_2 - w_1)^b$. $\square$

Remark 4.4. To derive expansion (2.4) of F as $w_1, w_2 \rightarrow 0$ with $|w_1| \asymp |w_2|$, we proceed in the same way as the proof of Theorem 2.3, i.e., we expand R_1, R_2, Q_2 as $w_1, w_2 \rightarrow 0$ and substitute the resulting expressions into (4.18). However, in this case, since $|w_1/w_2|$ is not necessarily smaller than 1, we do not use expansions (4.16b) and (4.20) of $(sw_1 - w_2)^c$ and $(sw_2 - w_1)^b$; instead, we simplify the expressions for Q_2 and R_2 using the identities $sw_1 - w_2 = w_1(s - \alpha)$ and $sw_2 - w_1 = w_2(s - \alpha^{-1})$ where $\alpha = w_2/w_1$; then, the remaining factors are expanded as in the proof of Theorem 2.3.

4.4. The Sector $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$

We finally consider the behavior of $F(w_1, w_2)$ when w_1 is near 0 and w_2 is near 1. Assuming that $a + b, c + d \notin \mathbb{Z}$, we define two functions $\tilde{Q}_1 : \mathcal{D}_1 \rightarrow \mathbb{C}$ and $T_1 : \mathcal{D}_0 \rightarrow \mathbb{C}$ as follows. We define $\tilde{Q}_1$ for $(w_1, w_2) \in \mathcal{D}_1$ by

$$\tilde{Q}_1(w_1, w_2) = \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv,$$

where $A \in (0, 1)$, w_1 lies inside the contour in the same component as 0, and w_2 lies outside the contour. Then,

$$Q_1(w_1, w_2) = \rho_c(w_2) \tilde{Q}_1(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0 \cap \mathcal{D}_1,$$

where ρ_c is given by (2.3). For $0 < \operatorname{Re} w_1 < \operatorname{Re} w_2 < 1$ with $\operatorname{Im} w_1 < 0$ and $\operatorname{Im} w_2 > 0$, we define $T_1(w_1, w_2)$ by

$$T_1(w_1, w_2) = \frac{(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(c+d)} - 1)} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv,$$

where w_1 lies inside the contour in the same component as 0, w_2 lies inside the contour in the same component as 1, and we assume that $\operatorname{Re} w_1 < A < \operatorname{Re} w_2$. We then use analytic continuation to extend T_1 to all of $\mathcal{D}_1$.

Proposition 4.5. *Suppose $a, b, c, d \in \mathbb{R}$ and $a, d, a + b, c + d \notin \mathbb{Z}$. Then, the function F obeys the identity*

$$\begin{aligned} F(w_1, w_2) &= \rho_c(w_2) [T_1(w_1, w_2) + (w_2 - 1)^{c+d+1} P_2(w_1, w_2)] \\ &\quad + w_1^{a+b+1} Q_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0 \cap \mathcal{D}_1. \end{aligned} \quad (4.21)$$

Proof. In view of (4.13), it is enough to show that

$$\tilde{Q}_1(w_1, w_2) = T_1(w_1, w_2) + (w_2 - 1)^{c+d+1} P_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_1. \quad (4.22)$$

If $f(w_1, w_2)$ is a function of w_1 and w_2 , we use the notation $f_*(w_1, w_2) := f(w_1 - i0, w_2 + i0)$. By analyticity, Eq. (4.22) will follow if we can show that the following identity holds for $0 < w_1 < w_2 < 1$:

$$\tilde{Q}_{1*}(w_1, w_2) = T_{1*}(w_1, w_2) + |1 - w_2|^{c+d+1} e^{i\pi(c+d+1)} P_{2*}(w_1, w_2). \quad (4.23)$$

Let $0 < w_1 < w_2 < 1$ and let $0 < \epsilon < \frac{1}{2} \min\{w_1, w_2 - w_1, 1 - w_2\}$. Then,

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) = & \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \left\{ \int_{L^1_{w_2-\epsilon}} + e^{2\pi i(a+b)} \int_{L^2_{w_2-\epsilon} + S^-_{w_2} + L^3_{w_2+\epsilon}} + e^{2\pi i(a+b+c+d)} \int_{L^4_{w_2+\epsilon}} \right. \\ & - e^{2\pi i(a+b+d)} \int_{S^-_{w_2} + L^2_{w_2-\epsilon}} + e^{2\pi id} \int_{-L^1_{w_2-\epsilon} + S^-_{w_2}} - e^{2\pi i(c+d)} \int_{L^4_{w_2+\epsilon}} \\ & \left. - \int_{L^3_{w_2+\epsilon}} - \int_{S^-_{w_2}} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv, \end{aligned}$$

and

$$\begin{aligned} T_{1*}(w_1, w_2) = & \frac{(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(c+d)} - 1)} \left\{ \int_{L^1_{w_2-\epsilon}} + e^{2\pi i(a+b)} \int_{L^2_{w_2-\epsilon} + S^-_{w_2} + L^3_{w_2+\epsilon}} \right. \\ & + e^{2\pi i(a+b+c+d)} \int_{L^4_{w_2+\epsilon} + S^+_{w_2} - L^2_{w_2-\epsilon}} - e^{2\pi i(c+d)} \int_{L^1_{w_2-\epsilon} + S^+_{w_2} + L^4_{w_2+\epsilon}} \\ & \left. + \int_{-L^3_{w_2+\epsilon} - S^-_{w_2}} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) - T_{1*}(w_1, w_2) = & \frac{e^{2\pi ia} - 1}{1 - e^{-2i\pi(c+d)}} \left\{ (1 - e^{-2i\pi c}) \int_{L^3_{w_2+\epsilon}} \right. \\ & + e^{2\pi id} (e^{2i\pi c} - 1) \int_{L^4_{w_2+\epsilon}} + (1 - e^{2\pi id}) \int_{S^+_{w_2} + S^-_{w_2}} \left. \right\} \\ & \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv. \end{aligned}$$

Using the identity

$$(w_2 - v)^c = \begin{cases} e^{-i\pi c} (v - w_2)^c, & v \in S^+_{w_2} \cup L^4_{w_2+\epsilon}, \\ e^{i\pi c} (v - w_2)^c, & v \in S^-_{w_2} \cup L^3_{w_2+\epsilon}, \end{cases}$$

we can write this as

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) - T_{1*}(w_1, w_2) = & \frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} \left\{ (e^{2i\pi c} - 1) \int_{L^3_{w_2+\epsilon}} + e^{2\pi id} (e^{2i\pi c} - 1) \int_{L^4_{w_2+\epsilon}} \right. \\ & + (1 - e^{2\pi id}) \int_{S^+_{w_2}} + e^{2i\pi c} (1 - e^{2\pi id}) \int_{S^-_{w_2}} \left. \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv \\ = & \frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} \int_{w_2+\epsilon}^{(w_2+, 1+, w_2-, 1-)} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \end{aligned}$$

where $0 < w_1 < w_2 < 1$ and w_1 lies exterior to the Pochhammer contour. Performing the change of variables $s = \frac{v-w_2}{1-w_2}$, which maps the interval $(w_2, 1)$ to the interval $(0, 1)$, we obtain, for $0 < w_1 < w_2 < 1$ and $A \in (0, 1)$,

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) - T_{1*}(w_1, w_2) = & \frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} (1 - w_2)^{c+d+1} \\ & \times \int_A^{(0+, 1+, 0-, 1-)} (w_2 + s(1 - w_2))^a (w_2 + s(1 - w_2) - w_1)^b s^c (1 - s)^d ds. \end{aligned}$$

Equation (4.23) now follows from definition (4.5) of P_2 . $\square$

Proof of Theorem 2.4. By expanding the integrands in the definitions of T_1 , P_2 , Q_2 as $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$, and substituting the resulting expressions into (4.21), Theorem 2.4 is obtained after a lengthy computation. $\square$

5. Two Examples from SLE Theory

In this section, in an effort to illustrate the method described above, we present two examples from SLE theory which involve Dotsenko–Fateev integrals of the form (1.1). The first example is related to the Green’s function observable for two commuting SLE curves and involves an integral of the form (1.1) with $N = 4$ and (see (5.2))

$$a_1 = a_2 = \alpha - 1, \quad a_3 = a_4 = -\frac{\alpha}{2},$$

where $\alpha > 1$ is a parameter. The second example is related to Schramm’s formula for the same SLE system and involves an integral of the form (1.1) with $N = 4$ and (see (5.12))

$$a_1 = \alpha, \quad a_2 = \alpha - 2, \quad a_3 = a_4 = -\frac{\alpha}{2}.$$

For each example, we derive the asymptotic estimates which are needed in order to establish that the relevant integral describes the given observable.

5.1. Multiple SLE and Dotsenko–Fateev Integrals

Let us briefly recall the definition of multiple SLE systems and describe how Dotsenko–Fateev integrals arise when constructing observables for such systems via the screening method, see [11] for a more complete discussion. SLE_κ curves are constructed by solving Loewner’s differential equation

$$\partial_t g_t(z) = \frac{2/\kappa}{g_t(z) - \xi_t^1}, \quad g_0(z) = z, \quad (5.1)$$

where the driving term ξ_t^1 is standard Brownian motion. The curve itself is defined by $\lim_{y \rightarrow 0+} g_t^{-1}(\xi_t^1 + iy)$; this is a random continuous curve growing from 0 to ∞ in the upper half-plane $\mathbb{H} = \{\text{Im } z > 0\}$. If $\kappa \leq 4$, the curve is simple and stays in $\mathbb{H}$ for $t > 0$. SLE curves appear as scaling limits of interfaces in various critical lattice models. It is natural to consider scaling limits of multiple interfaces simultaneously, and this leads to multiple SLE. We are interested in multiple SLE with two curves started from $\xi^1, \xi^2 \in \mathbb{R}$, respectively, and growing toward ∞ in $\mathbb{H}$, see [6]. The marginal law of the SLE started from ξ^1 is that of a variant of SLE with an additional marked boundary point at ξ^2 . For this variant, which is called $\text{SLE}_\kappa(2)$, the dynamics of the driving term is given by the system $d\xi_t^1 = dB_t + (2/\kappa)/(\xi_t^1 - \xi_t^2)$, where B_t is standard Brownian motion, $\xi_t^2 := g_t(\xi^2)$, and g_t solves (5.1) with ξ_t^1 as driving term. An important feature is that the system can be grown in a commutative way [6]. The extra drift term entails serious difficulties when constructing observables for such SLE systems.

In [11], the screening method is used to derive explicit formulas for two of the most natural SLE observables: the renormalized probability that the

system passes infinitesimally near a given point in $\mathbb{H}$ (the Green's function) and the probability that the system passes to the right of a given point in $\mathbb{H}$ (Schramm's formula). The derivation in [11] proceeds as follows: First, using a CFT description of the multiple SLE system, the screening method is employed to generate explicit “guesses” for the given observables in terms of Dotsenko–Fateev integrals. The “guesses” are then shown to indeed describe the desired probabilities via a sequence of probabilistic arguments. The latter arguments rely heavily on appropriate asymptotic estimates for the relevant Dotsenko–Fateev integrals. In the remainder of this paper, we derive the estimates needed in [11].

5.2. Example 1: Green's Function

Set $\alpha = 8/\kappa$ and define the function $I(z, \xi^1, \xi^2)$ for $z \in \mathbb{H}$ and $-\infty < \xi^1 < \xi^2 < \infty$ by

$$I(z, \xi^1, \xi^2) = \int_A^{(z+, \xi^2+, z-, \xi^2-)} (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} (u - \xi^1)^{-\frac{\alpha}{2}} (\xi^2 - u)^{-\frac{\alpha}{2}} du, \quad (5.2)$$

where $A = (z + \xi^2)/2$ is a base point and the Pochhammer integration contour is displayed in Fig. 5. Moreover, define the function $\mathcal{G}(z, \xi^1, \xi^2)$ for $\alpha \in (1, \infty) \setminus \mathbb{Z}$, $z = x + iy \in \mathbb{H}$, and $\xi^1 < \xi^2$ by

$$\mathcal{G}(z, \xi^1, \xi^2) = \frac{1}{\hat{c}} y^{\alpha + \frac{1}{\alpha} - 2} |z - \xi^1|^{1-\alpha} |z - \xi^2|^{1-\alpha} \operatorname{Im} (e^{-i\pi\alpha} I(z, \xi^1, \xi^2)), \quad (5.3)$$

where the constant $\hat{c} \equiv \hat{c}(\kappa)$ is given by

$$\hat{c} = \frac{4 \sin^2 \left(\frac{\pi\alpha}{2} \right) \sin(\pi\alpha) \Gamma \left(1 - \frac{\alpha}{2} \right) \Gamma \left(\frac{3\alpha}{2} - 1 \right)}{\Gamma(\alpha)} \quad \text{with} \quad \alpha = \frac{8}{\kappa}. \quad (5.4)$$

FIGURE 5. The Pochhammer integration contour in (5.2) is the composition of four loops based at the point $A = (z + \xi^2)/2$

This definition of $\mathcal{G}(z, \xi^1, \xi^2)$ can be extended to all $\alpha > 1$ by continuity, see [11]. It follows from (5.2) and (5.3) that the product $y^{1-\frac{1}{\alpha}}\mathcal{G}(z, \xi^1, \xi^2)$ only depends on the two angles θ^1 and θ^2 defined by $\theta^j := \arg(z - \xi^j)$, $j = 1, 2$, see [11, Section 6.2.1]. Hence, we may define the function h by

$$\mathcal{G}(z, \xi^1, \xi^2) = y^{\frac{1}{\alpha}-1} h(\theta^1, \theta^2), \quad z \in \mathbb{H}, \quad -\infty < \xi^1 < \xi^2 < \infty. \quad (5.5)$$

Let $\Delta \subset \mathbb{R}^2$ denote the triangular domain

$$\Delta = \{(\theta^1, \theta^2) \in \mathbb{R}^2 \mid 0 < \theta^1 < \theta^2 < \pi\}.$$

By applying the method of Sects. 2–4, we can prove the following proposition which is used in [11] to derive a formula for the Green's function.

Proposition 5.1 (Estimates for Green's function). *Let $\alpha \geq 2$. Then, the function $h(\theta^1, \theta^2)$ defined in (5.5) is a smooth function of $(\theta^1, \theta^2) \in \Delta$ and has a continuous extension to the closure $\bar{\Delta}$ of Δ . This extension satisfies*

$$h(\theta^1, \pi) = \sin^{\alpha-1} \theta^1, \quad \theta^1 \in [0, \pi], \quad (5.6)$$

$$h(\theta, \theta) = h_f(\theta), \quad \theta \in (0, \pi), \quad (5.7)$$

where $h_f(\theta)$ is defined by

$$\begin{aligned} h_f(\theta) &= \frac{2^{\alpha+1}\pi}{\hat{c}} \sin\left(\frac{\pi\alpha}{2}\right) \sin^{2\alpha-2}(\theta) \\ &\quad \times \operatorname{Re} \left[e^{-\frac{1}{2}i\pi\alpha} {}_2F_1\left(1-\alpha, \alpha, 1; \frac{1}{2}(1-i\cot(\theta))\right) \right]. \end{aligned} \quad (5.8)$$

Moreover, there exists a constant $C > 0$ such that

$$0 \leq h(\theta^1, \theta^2) \leq C \sin^{\alpha-1} \theta^1, \quad (\theta^1, \theta^2) \in \bar{\Delta}, \quad (5.9)$$

and

$$\frac{|h(\theta^1, \theta^2) - h(\theta^1, \pi)|}{\sin^{\alpha-1} \theta^1} \leq C \frac{|\pi - \theta^2|}{\sin \theta^1}, \quad (\theta^1, \theta^2) \in \Delta. \quad (5.10)$$

Proof. See Sect. 6. □

5.3. Example 2: Schramm's Formula

Let $\alpha > 1$ and define the function $\mathcal{M}(z, \xi)$ by

$$\mathcal{M}(z, \xi) = y^{\alpha-2} z^{-\frac{\alpha}{2}} (z - \xi)^{-\frac{\alpha}{2}} \bar{z}^{1-\frac{\alpha}{2}} (\bar{z} - \xi)^{1-\frac{\alpha}{2}} J(z, \xi), \quad z \in \mathbb{H}, \quad \xi > 0, \quad (5.11)$$

where $J(z, \xi)$ is the integral defined by

$$J(z, \xi) = \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (5.12)$$

with the integration contour from $\bar{z}$ to z passing to the right of ξ , see Fig. 6. Moreover, define the function $P(z, \xi)$ by

$$P(z, \xi) = \frac{1}{c_{\alpha}} \int_x^{\infty} \operatorname{Re} \mathcal{M}(x' + iy, \xi) dx', \quad z \in \mathbb{H}, \quad \xi > 0. \quad (5.13)$$

FIGURE 6. The integration contour used in definition (5.11) of $\mathcal{M}(z, \xi)$ is a path from $\bar{z}$ to z which passes to the right of ξ

where the normalization constant $c_\alpha \in \mathbb{R}$ is given by

$$c_\alpha = -\frac{2\pi^{3/2}\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{3\alpha}{2}-1\right)}{\Gamma\left(\frac{\alpha}{2}\right)^2\Gamma(\alpha)}. \quad (5.14)$$

We will prove the following proposition which establishes the properties of P needed for the proofs in [11].

Proposition 5.2 (Estimates for Schramm’s formula). *For each $\alpha > 1$, the function $P(z, \xi)$ defined in (5.13) is a well-defined smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$ which satisfies*

$$|P(z, \xi)| \leq C(\arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (5.15a)$$

$$|P(z, \xi) - 1| \leq C(\pi - \arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0. \quad (5.15b)$$

Proof. See Sect. 7. □

6. Proof of Proposition 5.1

By applying the method developed in Sects. 2–4, we can determine the behavior of the Dotsenko–Fateev integral in (5.2). This will lead to asymptotic formulas for the behavior of $h(\theta^1, \theta^2)$ near the boundary of Δ from which Proposition 5.1 will follow.

Let $F(w_1, w_2) \equiv F(a, b, c, d; w_1, w_2)$ be the function defined in (1.3) with a, b, c, d given by

$$a = b = \alpha - 1, \quad c = d = -\frac{\alpha}{2}, \quad (6.1)$$

i.e., for $w_1, w_2 \in \mathcal{D}_0$,

$$F(w_1, w_2) = \int_A^{(0+, 1+, 0-, 1-)} v^{\alpha-1} (v - w_1)^{\alpha-1} (v - w_2)^{-\frac{\alpha}{2}} (1 - v)^{-\frac{\alpha}{2}} dv, \quad (6.2)$$

where $A \in (0, 1)$ is a base point and w_1, w_2 are assumed to lie outside the contour. The Pochhammer contour in (5.2) encloses the variable points z and ξ^2 . In order to easily apply the results from Sects. 2–4, we first need to express h in terms of the integral F whose contour encloses the fixed points 0 and 1. This can be achieved by applying a linear fractional transformation which maps z and ξ^2 to 0 and 1, respectively.

Lemma 6.1 (Representation for h). *For each non-integer $\alpha > 1$, the function h defined in (5.5) admits the representation*

$$h(\theta^1, \theta^2; \alpha) = \frac{\sin^{\alpha-1} \theta^1}{\hat{c}} \operatorname{Im} \left[\sigma(\theta_2) (-e^{i\theta^2})^{\alpha-1} F(w_1, w_2) \right], \quad (\theta^1, \theta^2) \in \Delta, \quad (6.3)$$

where $w_1 \equiv w_1(\theta^1, \theta^2)$ and $w_2 \equiv w_2(\theta^1, \theta^2)$ are given by

$$w_1 = 1 - e^{-2i\theta^2}, \quad w_2 = \frac{1 - e^{-2i\theta^2}}{1 - e^{-2i\theta^1}} = \frac{\sin \theta^2}{\sin \theta^1} e^{-i(\theta^2 - \theta^1)}, \quad (6.4)$$

the constant $\hat{c}$ is defined in (5.4), and

$$\sigma(\theta^2) = \begin{cases} e^{-i\pi\alpha}, & \theta^2 \geq \frac{\pi}{2}, \\ e^{i\pi\alpha}, & \theta^2 < \frac{\pi}{2}. \end{cases} \quad (6.5)$$

Proof. Introducing the new variable $v = \frac{u-z}{\xi^2-z}$ in (5.2), we find

$$\begin{aligned} I(z, \xi^1, \xi^2) &= \int_A^{(0+, 1+, 0-, 1-)} (v(\xi^2 - z))^{\alpha-1} (z - \bar{z} + v(\xi^2 - z))^{\alpha-1} \\ &\quad \times (z + v(\xi^2 - z) - \xi^1)^{-\frac{\alpha}{2}} ((\xi^2 - z)(1 - v))^{-\frac{\alpha}{2}} (\xi^2 - z) dv \\ &= (\xi^2 - z)^{\alpha-1} F(w_1, w_2) \times \begin{cases} 1, & x \leq \xi^2, \\ e^{2i\pi(\alpha-1)}, & x > \xi^2, \end{cases} \end{aligned} \quad (6.6)$$

where $\frac{\xi+z}{z-\xi}$ and $\frac{z-\bar{z}}{z-\xi}$ are not enclosed by the contour, and the variables

$$w_1 = \frac{z - \bar{z}}{z - \xi^2} = \frac{2i}{\cot \theta^2 + i}, \quad w_2 = \frac{\xi^1 - z}{\xi^2 - z} = \frac{\cot \theta^1 + i}{\cot \theta^2 + i},$$

can be expressed as in (6.4). The extra factor of $e^{2i\pi(\alpha-1)}$ in (6.6) which is present for $x > \xi^2$ arises from the factor $(z - \bar{z} + v(\xi^2 - z))^{\alpha-1}$ as follows. Let v belong to the contour in (6.6). Then, the complex number $z - \bar{z} + v(\xi^2 - z)$ lies in the upper half-plane. If $\frac{\pi}{2} \leq \theta^2 < \pi$ (i.e., if $x \leq \xi^2$), then $v - w_1$ also lies in the upper half-plane, but if $0 < \theta^2 < \pi/2$ (i.e., if $x > \xi^2$), then $v - w_1$ has crossed the negative real axis into the lower half-plane. The factor $e^{2\pi i(\alpha-1)}$

FIGURE 7. The complex numbers $w_1 = 1 - e^{-2i\theta^2}$ and $w_1/w_2 = 1 - e^{-2i\theta^1}$ lie on the circle of radius one centered at 1

is inserted to compensate for this crossing of the branch cut. Equations (5.3), (5.5), and (6.6) give

$$\begin{aligned} h(\theta^1, \theta^2) &= \frac{1}{\tilde{c}} y^{\alpha-1} |z - \xi^1|^{1-\alpha} |z - \xi^2|^{1-\alpha} |z - \xi^1|^{1-\alpha} |z - \xi^2|^{1-\alpha} \operatorname{Im} [e^{-i\pi\alpha} I(z, \xi^1, \xi^2)] \\ &= \frac{1}{\tilde{c}} \sin^{\alpha-1}(\theta^1) \sin^{\alpha-1}(\theta^2) \operatorname{Im} [\sigma(\theta^2)(-\cot \theta^2 - i)^{\alpha-1} F(w_1, w_2)], \end{aligned} \quad (6.7)$$

where σ is given by (6.5). The representation (6.3) follows. $\square$

Remark 6.2. For $(\theta^1, \theta^2) \in \Delta$, w_1 and w_1/w_2 lie on the circle of radius one centered at 1 (see Fig. 7), while w_2 lies in the open lower half-plane, i.e., $\operatorname{Im} w_2 < 0$.

Remark 6.3. The value of $F(w_1, w_2)$ in (6.3) is, strictly speaking, not well defined by (6.2) for $\theta^2 = \pi/2$, because in this case $w_1 = 2 \notin \mathcal{D}_0$. However, by analytic continuation, the function F in (6.3) extends to a multiple-valued function of $w_1, w_2 \in \mathbb{C} \setminus \{0, 1\}$. Equation (6.3) then extends continuously across the line $\theta^2 = \pi/2$.

Given $\delta > 0$ and $c > 0$, we define the open subsets S_j , $j = 1, \dots, 5$, of Δ by (see Figs. 8, 9, 10)

$$\begin{aligned} S_1 &= \{(\theta^1, \theta^2) \in \Delta \mid 0 < \theta^2 - \theta^1 < c\sqrt{2}, \theta^1 > \delta, \theta^2 < \pi - \delta\}, \\ S_2 &= \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^2 < c, \arctan \frac{\theta^1}{\theta^2} < \frac{\pi}{4} - \delta\right\} \\ &\cup \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^2 > \pi - c, \arctan \frac{\theta^1}{\pi - \theta^2} < \frac{\pi}{4} - \delta\right\}, \end{aligned}$$

FIGURE 8. The asymptotic sectors S_1 and S_2 FIGURE 9. The asymptotic sectors S_3 and S_4

$$S_3 = \left\{ (\theta^1, \theta^2) \in \Delta \mid \theta^2 > \pi - c, \arctan \frac{\pi - \theta^2}{\theta^1} < \frac{\pi}{4} - \delta, \arctan \frac{\pi - \theta^2}{\pi - \theta^1} < \frac{\pi}{4} - \delta \right\},$$

$$S_4 = \left\{ (\theta^1, \theta^2) \in \Delta \mid \theta^1 + \theta^2 < c\sqrt{2}, \delta < \arctan \frac{\theta^1}{\theta^2} < \frac{\pi}{4} \right\}$$

$$\cup \left\{ (\theta^1, \theta^2) \in \Delta \mid \theta^2 - \theta^1 > \pi - c\sqrt{2}, \delta < \arctan \frac{\pi - \theta^2}{\theta^1} < \frac{\pi}{2} - \delta \right\}$$

$$\cup \left\{ (\theta^1, \theta^2) \in \Delta \mid \theta^1 + \theta^2 > 2\pi - c\sqrt{2}, \delta < \arctan \frac{\pi - \theta^2}{\pi - \theta^1} < \frac{\pi}{4} \right\},$$

$$S_5 = \left\{ (\theta^1, \theta^2) \in \Delta \mid \theta^1 < c, \theta^2 - \theta^1 > \delta, \theta^1 + \theta^2 < \pi - \delta \right\}.$$

FIGURE 10. The asymptotic sector S_5

The asymptotics of $h(\theta^1, \theta^2)$ as (θ^1, θ^2) approaches the boundary of the triangle Δ , can be described in terms of the five asymptotic sectors $\{S_j\}_1^5$. Indeed, as the next lemma shows, the first four sectors S_j , $j = 1, \dots, 4$, correspond to the asymptotic regions treated in Propositions 4.1–4.5, respectively, while the sector S_5 corresponds to the region where $w_2 \rightarrow \infty$.

If $w \in \mathbb{C}$ and A is a subset of $\mathbb{C}$, we write $\text{dist}(w, A)$ for the Euclidean distance from w to A ; we write $\text{dist}(w, A \cup \{\infty\}) > \epsilon$ to indicate that $\text{dist}(w, A) > \epsilon$ and $|w| < 1/\epsilon$.

Lemma 6.4. *Let $\delta > 0$ and let w_1, w_2 be given by (6.4). Then, there exist constants $c > 0$ and $\epsilon > 0$ such that the following estimates hold:*

- (1) $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2 - 1| < 1 - \epsilon$ for all $(\theta^1, \theta^2) \in S_1$,
- (2) $|w_1| < 1 - \epsilon$ and $|w_2| > 1 + \epsilon$ for all $(\theta^1, \theta^2) \in S_2$,
- (3) $|w_1| < 1 - \epsilon$ and $|w_2| < 1 - \epsilon$ for all $(\theta^1, \theta^2) \in S_3$,
- (4) $|w_1| < 1 - \epsilon$ and $\text{dist}(w_2, \{0, \infty\}) > \epsilon$ for all $(\theta^1, \theta^2) \in S_4$,
- (5) $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2| > 1 + \epsilon$ for all $(\theta^1, \theta^2) \in S_5$.

Proof. The proof follows easily from definition (6.4) of w_1 and w_2 . □

Assume that $\alpha \geq 2$ is such that $\frac{3\alpha}{2}, 2\alpha \notin \mathbb{Z}$; the cases when $\frac{3\alpha}{2}$ and/or 2α is an integer will be considered separately. Then, $a, b, c, d \in \mathbb{R} \setminus \mathbb{Z}$ and $a + b, c + d, a + b + c \notin \mathbb{Z}$. In fact, since $a + b = 2\alpha - 2 \geq 0$, it can be seen from Lemma 6.4 and Theorem 2.2 that the function $F(w_1, w_2)$ in (6.3) is bounded in the sector S_2 . On the other hand, since $c + d + 1 = 1 - \alpha < -1$, the sum in Theorem 2.1 which involves the coefficients $A_k^{(2)}$ is, in general, singular as $w_2 \rightarrow 1$. However, it turns out that the contribution from this sum to h vanishes identically because of the taking of the imaginary part in (6.3). In order to see this, we need to consider the function P_2 of Proposition 4.1 from which the coefficients $A_k^{(2)}$ originated in more detail.

Lemma 6.5. Let $P_2(w_1, w_2)$ denote the function defined in (4.5) with a, b, c, d given by (6.1) and define $X : \Delta \rightarrow \mathbb{C}$ by

$$X(\theta^1, \theta^2) = \sigma(\theta^2)(-e^{i\theta^2})^{\alpha-1} e^{-\frac{i\pi\alpha}{2}} (w_2 - 1)^{1-\alpha} P_2(w_1, w_2), \quad (6.8)$$

where the variables w_1 and w_2 are given by (6.4). Then,

$$\operatorname{Im} X(\theta^1, \theta^2) = 0 \quad \text{in } \Delta. \quad (6.9)$$

Proof. From definition (4.5) of P_2 , we see that for $w_1 \in \mathbb{C} \setminus [0, \infty)$ and $w_2 \in (0, 1)$ we have

$$\begin{aligned} P_2(w_1, w_2 + i0) &= \frac{(e^{2\pi ia} - 1)e^{\pi id}}{1 - e^{2\pi i(c+d)}} |1 - w_2|^{a+b} \\ &\times \int_A^{(0+, 1+, 0-, 1-)} \left(s - 1 - \frac{1}{w_2 - 1}\right)^a \left(s - 1 + \frac{w_1 - 1}{w_2 - 1}\right)^b s^c (1 - s)^d ds. \end{aligned} \quad (6.10)$$

By definition, the value of $P_2(w_1, w_2)$ at a general point $(w_1, w_2) \in \mathcal{D}_1$ is determined by analytic continuation of (6.10) within the connected set $\mathcal{D}_1 \subset \mathbb{C}^2$. The branches of the complex powers in (6.10) are fixed by requiring that the principal branch is used initially at the base point $s = A \in (0, 1)$; for definiteness, let us choose $A = 1/2$. This means that whenever the points

$$A - 1 - \frac{1}{w_2 - 1} \quad \text{and} \quad A - 1 + \frac{w_1 - 1}{w_2 - 1}, \quad (6.11)$$

cross the negative real axis during the analytic continuation, extra factors of $e^{\pm 2\pi ia}$ and $e^{\pm 2\pi ib}$, respectively, have to be inserted in (6.10).

In order to evaluate the function X in (6.8), we need the value of P_2 at points $(w_1, w_2) \in \mathcal{E}$, where $\mathcal{E}$ denotes the subset of $\mathbb{C}^2$ characterized by (6.4), i.e.,

$$\mathcal{E} = \left\{ (w_1, w_2) = \left(1 - e^{-2i\theta^2}, \frac{\sin \theta^2}{\sin \theta^1} e^{-i(\theta^2 - \theta^1)} \right) \mid (\theta^1, \theta^2) \subset \Delta \right\}.$$

If w_1 and w_2 are given by (6.4), then

$$\cot \theta^2 < \cot \theta^1, \quad \frac{1}{w_2 - 1} = \frac{\cot \theta^2 + i}{\cot \theta^1 - \cot \theta^2}, \quad -\frac{w_1 - 1}{w_2 - 1} = \frac{\cot \theta^2 - i}{\cot \theta^1 - \cot \theta^2}.$$

Hence, we have, for all $(w_1, w_2) \in \mathcal{E}$,

$$\operatorname{Im} \left(A - 1 - \frac{1}{w_2 - 1} \right) < 0, \quad \operatorname{Im} \left(A - 1 + \frac{w_1 - 1}{w_2 - 1} \right) > 0. \quad (6.12)$$

This shows that neither of the points in (6.11) crosses the negative real axis as long as (w_1, w_2) remains within $\mathcal{E}$. We can therefore find a formula for P_2 valid in $\mathcal{E}$ as follows.

Let (w_1, w_2) be a point in $\mathcal{E}$ corresponding to (θ^1, θ^2) via (6.4). Then,

$$w_2 = 1 + \frac{\sin(\theta^2 - \theta^1)}{\sin \theta^1} e^{-i\theta^2}. \quad (6.13)$$

Let $0 < \epsilon < |w_1 - 1|$ be small and let $(\tilde{w}_1(t), \tilde{w}_2(t))$, $t \in [0, 1]$, be the path in $\mathcal{D}_1$ defined by $\tilde{w}_1(t) = w_1$ for all t , while the path $\tilde{w}_2(t)$ starts at $1 - \epsilon + i0$,

proceeds clockwise around the small circle of radius ϵ centered at 1 until it reaches the point $1 + \epsilon e^{-i\theta^2}$, and then proceeds along the straight line segment $[1 + \epsilon e^{-i\theta^2}, w_2]$ until it reaches w_2 .

As $\tilde{w}_2$ moves along the arc from $1 - \epsilon + i0$ to $1 + \epsilon e^{-i\theta^2}$, the point $A - 1 - \frac{1}{\tilde{w}_2(t)-1}$ crosses the negative real axis from the upper into the lower half-plane once (this adds a factor of $e^{2\pi ia}$ to (6.10)), and, provided that $\text{Im } w_1 \leq 0$ (i.e., $\theta^2 \geq \pi/2$), $A - 1 + \frac{\tilde{w}_1(t)-1}{\tilde{w}_2(t)-1}$ also crosses the negative real axis from the upper into the lower half-plane once (this adds a factor of $e^{2\pi ib}$ to (6.10)). If $\text{Im } w_1 > 0$, then $A - 1 + \frac{\tilde{w}_1(t)-1}{\tilde{w}_2(t)-1}$ does not cross the negative real axis. By varying θ^1 in (6.13), we see that the part of the path for which $\tilde{w}_2$ belongs to the segment $[1 + \epsilon e^{-i\theta^2}, w_2]$ lies in $\mathcal{E}$; hence, the analytic continuation along this part adds no more factors to (6.10). We end up with the following formula for P_2 on $\mathcal{E}$:

$$\begin{aligned} P_2(w_1, w_2) &= \frac{(e^{2\pi ia} - 1)e^{\pi id}}{1 - e^{2\pi i(c+d)}} e^{\pi i(a-b)} (w_2 - 1)^{a+b} \\ &\quad \times \int_A^{(0+, 1+, 0-, 1-)} \left(s - 1 - \frac{1}{w_2 - 1}\right)^a \left(s - 1 + \frac{w_1 - 1}{w_2 - 1}\right)^b s^c (1-s)^d ds \\ &\quad \times \begin{cases} e^{2\pi ib}, & \text{Im } w_1 \leq 0, \\ 1, & \text{Im } w_1 > 0, \end{cases} \quad (w_1, w_2) \in \mathcal{E}, \end{aligned}$$

where $1 + \frac{1}{w_2-1}$ and $1 - \frac{w_1-1}{w_2-1}$ lie exterior to the contour. Substituting this formula into (6.8) and simplifying, we find

$$\begin{aligned} X(\theta^2, \theta^2) &= (-e^{i\theta^2})^{\alpha-1} (w_2 - 1)^{\alpha-1} e^{2\pi i\alpha} \\ &\quad \times \int_A^{(0+, 1+, 0-, 1-)} \left(s - 1 - \frac{1}{w_2 - 1}\right)^{\alpha-1} \left(s - 1 + \frac{w_1 - 1}{w_2 - 1}\right)^{\alpha-1} s^{-\frac{\alpha}{2}} (1-s)^{-\frac{\alpha}{2}} ds, \end{aligned}$$

where w_1, w_2 are given by (6.4). But

$$(-e^{i\theta^2})^{\alpha-1} (w_2 - 1)^{\alpha-1} = (\sin^{\alpha-1} \theta^2) (\cot \theta^1 - \cot \theta^2)^{\alpha-1} e^{-\pi i(\alpha-1)},$$

and, by (6.12),

$$\left(s - 1 - \frac{1}{w_2 - 1}\right)^{\alpha-1} \left(s - 1 + \frac{w_1 - 1}{w_2 - 1}\right)^{\alpha-1} = \left(\frac{1 + ((1-s)\cot \theta^1 + s\cot \theta^2)^2}{(\cot \theta^1 - \cot \theta^2)^2}\right)^{\alpha-1}.$$

Hence,

$$\begin{aligned} X(\theta^2, \theta^2) &= -\sin^{\alpha-1}(\theta^2) (\cot \theta^2 - \cot \theta^1)^{\alpha-1} e^{\pi i\alpha} \\ &\quad \times \int_A^{(0+, 1+, 0-, 1-)} \left(\frac{1 + ((1-s)\cot \theta^1 + s\cot \theta^2)^2}{(\cot \theta^1 - \cot \theta^2)^2}\right)^{\alpha-1} s^{-\frac{\alpha}{2}} (1-s)^{-\frac{\alpha}{2}} ds. \end{aligned} \quad (6.14)$$

If $g(s)$ is an analytic function, then the general identity $\int_{\gamma} g(w)dw = \int_{\bar{\gamma}} \overline{g(\bar{v})}d\bar{v}$ implies

$$\begin{aligned}
& \overline{\int_A^{(0+,1+,0-,1-)} g(s) s^c (1-s)^d ds} = \int_A^{(0-,1-,0+,1+)} \overline{g(\bar{s})} s^c (1-s)^d ds \\
&= \frac{-1 + e^{-2\pi ic} - e^{-2\pi i(c+d)} + e^{-2\pi id}}{-1 + e^{2\pi ic} - e^{2\pi i(c+d)} + e^{2\pi id}} \int_A^{(0+,1+,0-,1-)} \overline{g(\bar{s})} s^c (1-s)^d ds \\
&= e^{-2\pi i(c+d)} \int_A^{(0+,1+,0-,1-)} \overline{g(\bar{s})} s^c (1-s)^d ds.
\end{aligned}$$

Using this identity to compute the imaginary part of (6.14), we arrive at

$$\begin{aligned}
\operatorname{Im} X(\theta^1, \theta^2) &= -\sin^{\alpha-1}(\theta^2)(\cot \theta^2 - \cot \theta^1)^{\alpha-1} \int_A^{(0+,1+,0-,1-)} (e^{\pi i \alpha} - e^{-\pi i \alpha} e^{-2i\pi(-\frac{\alpha}{2}-\frac{\alpha}{2})}) \\
&\quad \times \left(\frac{1 + ((1-s)\cot \theta^1 + s\cot \theta^2)^2}{(\cot \theta^1 - \cot \theta^2)^2} \right)^{\alpha-1} s^{-\frac{\alpha}{2}} (1-s)^{-\frac{\alpha}{2}} ds = 0.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Using the identities of Propositions 4.1–4.5 to replace F in the expression (6.3) for h , and using that the contribution from P_2 vanishes due to Lemma 6.5, we arrive at the next lemma, which provides four representations for h which are suitable for determining the behavior of h for $(\theta^1, \theta^2) \in S_j$, $j = 1, \dots, 4$, respectively. For $(\theta^1, \theta^2) \in S_5$, we will use the original representation (6.3).

Lemma 6.6. *Suppose $\alpha \geq 2$ satisfies $\frac{3\alpha}{2}, 2\alpha \notin \mathbb{Z}$. Then, for all $(\theta^1, \theta^2) \in \Delta$,*

$$h(\theta^1, \theta^2) = \frac{1}{\tilde{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} e^{-\frac{i\pi\alpha}{2}} P_1(w_1, w_2) \right], \quad (6.15)$$

$$h(\theta^1, \theta^2) = \frac{1}{\tilde{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} (Q_1(w_1, w_2) + w_1^{2\alpha-1} Q_2(w_1, w_2)) \right], \quad (6.16)$$

$$\begin{aligned}
h(\theta^1, \theta^2) &= \frac{1}{\tilde{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta_2) (-e^{i\theta^2})^{\alpha-1} (R_1(w_1, w_2) + w_2^{\frac{\alpha}{2}} R_2(w_1, w_2) \right. \\
&\quad \left. + w_1^{2\alpha-1} Q_2(w_1, w_2)) \right], \quad (6.17)
\end{aligned}$$

$$\begin{aligned}
h(\theta^1, \theta^2) &= \frac{1}{\tilde{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta_2) (-e^{i\theta^2})^{\alpha-1} (e^{-\frac{i\pi\alpha}{2}} T_1(w_1, w_2) \right. \\
&\quad \left. + w_1^{2\alpha-1} Q_2(w_1, w_2)) \right], \quad (6.18)
\end{aligned}$$

where w_1, w_2 are given by (6.4).

In order to prove Proposition 5.1, we only need leading and subleading estimates on F , so we shall be content with this level of precision. The required bounds on the functions P_1, Q_j, R_j, T_1, F are then collected in the next lemma.

Lemma 6.7. *Suppose $a, b \geq 0$ and $c, d \leq 0$ satisfy $a, d, a+b, c+d, a+b+c \notin \mathbb{Z}$. Let $\epsilon > 0$. Then, the following estimates hold:*

- (a) $|P_1(w_1, w_2)| \leq C$ and $|P_1(w_1, w_2) - P_1(w_1, 1)| \leq C|w_2 - 1|$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $\operatorname{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2 - 1| < 1 - \epsilon$.
- (b) $|Q_1(w_1, w_2)| \leq C|w_2|^c$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $|w_1| < 1 - \epsilon$ and $\operatorname{dist}(w_2, \{0, 1\}) > \epsilon$.

- (c) $|Q_2(w_1, w_2)| \leq C|w_1|^c$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $|w_1| < 1 - \epsilon$ and $\text{dist}(\frac{w_2}{w_1}, \{0, 1\}) > \epsilon$.
- (d) $|R_1(w_1, w_2)| \leq C$ and $|R_1(w_1, w_2) - R_1(0, 0)| \leq C(|w_1| + |w_2|)$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $|w_1| < 1 - \epsilon$ and $|w_2| < 1 - \epsilon$.
- (e) $|R_2(w_1, w_2)| \leq C|w_2|^b$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $\text{dist}(\frac{w_2}{w_1}, \{0, 1\}) > \epsilon$ and $|w_2| < 1 - \epsilon$.
- (f) $|T_1(w_1, w_2)| \leq C$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $|w_1| < 1 - \epsilon$ and $\text{dist}(w_2, \{0, \infty\}) > \epsilon$.
- (g) $|F(w_1, w_2)| \leq C|w_2|^c$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2| > 1 + \epsilon$.

Proof. The estimates follow directly from the definitions of the functions P_1, Q_j, R_j, T_1 , and F . $\square$

Remark 6.8. If (w_1, w_2) lies on a branch cut, the bounds in Lemma 6.7 should be interpreted as saying that both the left and right boundary values obey the bounds.

We are now in a position to prove Proposition 5.1. Indeed, since h clearly is smooth in the interior of Δ and the parameter $\delta > 0$ which defines the sectors S_j is arbitrary, Proposition 5.1 follows from the following result.

Lemma 6.9. *Let $\alpha \geq 2$. Then, the function $h(\theta^1, \theta^2)$ defined in (5.5) satisfies the following estimates:*

$$|h(\theta^1, \theta^2)| \leq C \sin^{\alpha-1} \theta^1, \quad (\theta^1, \theta^2) \in \cup_{j=1}^5 S_j, \quad (6.19)$$

$$\frac{|h(\theta^1, \theta^2) - h_f(\theta^2)|}{\sin^{\alpha-1} \theta^1} \leq C \frac{|\theta^2 - \theta^1|}{\sin \theta^1}, \quad (\theta^1, \theta^2) \in S_1, \quad (6.20)$$

$$\frac{|h(\theta^1, \theta^2) - \sin^{\alpha-1} \theta^1|}{\sin^{\alpha-1} \theta^1} \leq C \frac{\sin \theta^2}{\sin \theta^1}, \quad (\theta^1, \theta^2) \in S_3, \quad (6.21)$$

where $h_f(\theta)$ is defined in (5.8).

Proof. Let us first assume that $\alpha \geq 2$ satisfies $\frac{3\alpha}{2}, 2\alpha \notin \mathbb{Z}$. Equation (6.15), Lemma 6.4 (1), and Lemma 6.7(a) show that (6.19) holds in S_1 . Also, by Lemma 6.7(a), the following estimate is valid in S_1 :

$$\frac{|h(\theta^1, \theta^2) - h(\theta^2, \theta^2)|}{\sin^{\alpha-1} \theta^1} \leq C|w_2 - 1| = C \frac{\sin(\theta^2 - \theta^1)}{\sin \theta^1} \leq C \frac{|\theta^2 - \theta^1|}{\sin \theta^1}, \quad (6.22)$$

where

$$h(\theta^2, \theta^2) = \frac{1}{c} \sin^{\alpha-1}(\theta^2) \text{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} e^{-\frac{i\pi\alpha}{2}} P_1(w_1, 1) \right].$$

For a, b, c, d given by (6.1), we have (cf. (2.7))

$$\begin{aligned} P_1(w_1, 1) &= \frac{e^{2\pi i d} - 1}{e^{2\pi i(c+d)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (1 - v)^{c+d} dv \\ &= 2i\pi (-1 + e^{i\pi\alpha}) (-w_1)^{\alpha-1} {}_2F_1 \left(1 - \alpha, \alpha; 1; \frac{1}{w_1} \right). \end{aligned}$$

It follows that $h(\theta^1, \theta^1) = h_f(\theta^1)$ where h_f is given by (5.8). Equation (6.20) then follows from (6.22).

Using the fact that $\text{dist}(\frac{w_2}{w_1}, \{0, 1\}) > \epsilon$ for all $(\theta^1, \theta^2) \in \Delta$, Lemmas 6.4(2) and 6.7(b) and (c) imply

$$|Q_1 + w_1^{2\alpha-1}Q_2| \leq C|w_2|^{-\frac{\alpha}{2}} + C|w_1|^{\frac{3\alpha}{2}-1} \leq C, \quad (\theta^1, \theta^2) \in S_2.$$

Hence, Eq. (6.16) shows that (6.19) holds in S_2 .

Similarly, Lemma 6.4 (3) and Lemma 6.7 (c), (d), and (e) show that

$$|R_1 + w_2^{\frac{\alpha}{2}}R_2 + w_1^{2\alpha-1}Q_2| \leq C + C|w_2|^{\frac{3\alpha}{2}-1} + C|w_1|^{\frac{3\alpha}{2}-1} \leq C, \quad (\theta^1, \theta^2) \in S_3.$$

Hence, Eq. (6.17) implies that (6.19) holds in S_3 . Also, by Lemma 6.7 (d), since $\alpha \geq 2$, the following estimate is valid in S_3 :

$$\begin{aligned} \frac{|h(\theta^1, \theta^2) - h(\theta^1, \pi)|}{\sin^{\alpha-1}\theta^1} &\leq C(|w_1| + |w_2|) + C|w_2|^{\frac{3\alpha}{2}-1} + C|w_1|^{\frac{3\alpha}{2}-1} \leq C(|w_1| + |w_2|) \\ &\leq C|\pi - \theta^2| + C\left|\frac{\sin \theta^2}{\sin \theta^1}\right| \leq C\left|\frac{\sin \theta^2}{\sin \theta^1}\right|, \end{aligned}$$

where

$$h(\theta^1, \pi) = \frac{\sin^{\alpha-1}\theta^1}{\hat{c}} \text{Im} [e^{-i\pi\alpha}R_1(0, 0)].$$

For a, b, c, d given by (6.1), we have

$$\begin{aligned} R_1(0, 0) &= \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b+c)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^{a+b+c}(1-v)^d dv \\ &= - \frac{(e^{2i\pi a} - 1)(e^{2i\pi d} - 1)\Gamma(d+1)\Gamma(a+b+c+1)}{\Gamma(a+b+c+d+2)} \\ &= - \frac{(e^{2i\pi\alpha} - 1)(e^{-i\pi\alpha} - 1)\Gamma(1 - \frac{\alpha}{2})\Gamma(\frac{3\alpha}{2} - 1)}{\Gamma(\alpha)}. \end{aligned}$$

Taking definition (5.4) of $\hat{c}$ into account, it follows that $h(\theta^1, \pi) = \sin^{\alpha-1}\theta^1$. This proves (6.21).

Lemma 6.4 (4) and Lemma 6.7(c) and (f) show that

$$|e^{-\frac{i\pi\alpha}{2}}T_1 + w_1^{2\alpha-1}Q_2| \leq C + C|w_1|^{-\frac{\alpha}{2}} \leq C, \quad (\theta^1, \theta^2) \in S_4.$$

Hence, Eq. (6.18) implies that (6.19) holds in S_4 .

Lemma 6.4 (5) and Lemma 6.7(g) show that

$$|F| \leq C|w_2|^{\frac{3\alpha}{2}-1} \leq C, \quad (\theta^1, \theta^2) \in S_5.$$

Hence, Eq. (6.3) shows that (6.19) holds in S_5 . This completes the proof of the lemma in the case when $\frac{3\alpha}{2}$ and 2α are not integers.

Assume now that $\frac{3\alpha}{2}$ and/or 2α is an integer. Then, some of the functions in Lemma 6.7 degenerate, so a slightly different argument is required. We do not give complete details, but outline the relevant steps.

Suppose first that $\alpha \notin \mathbb{Z}$ but $\frac{3\alpha}{2}$ or 2α is an integer. Then, the limit $w_2 \rightarrow 1$ can still be treated as before, because $c + d = -\alpha$ is not an integer. However, the limits involving $w_1 \rightarrow 0$ or $w_2 \rightarrow 0$ cannot be treated in the same way in general, because $a + b = 2\alpha - 2$ and/or $a + b + c = \frac{3\alpha}{2} - 2$ is an

integer. However, since $\alpha \geq 2$, we have $a + b > 0$ and $a + b + c > 0$. Hence, the integral (6.2) defining F is nonsingular at $v = 0$ (also in the limit as w_1 and w_2 approach zero). Hence, we can derive the leading behavior of F in these regimes using the following alternative approach: First, we collapse the two loops of the Pochhammer contour enclosing the origin down to the interval $[0, A]$. Then, we find the leading-order asymptotics by Taylor expanding the integrand as w_1 and/or w_2 approaches zero.

Assume finally that $\alpha = n \geq 2$ is an integer. This case is considered in [11], where an expression for $h(\theta^1, \theta^2; n)$ is derived by taking the limit of the defining Eq. (6.2) for F as $\alpha \rightarrow n$. In order to prove (6.19)–(6.21) in this case, we compute the limits as $\alpha \rightarrow n$ of each of the four equations in Lemma 6.6. This gives four analogous equations valid for $\alpha = n$. As above, it follows from these equations that h satisfies (6.19)–(6.21). The crucial point is that the singular contribution from P_2 vanishes as a consequence of (6.9). $\square$

7. Proof of Proposition 5.2

Lemma 7.1. *The function $J(z, \xi)$ defined in (5.12) is a well-defined smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$.*

Proof. Since $\alpha > 1$, the integral defining $J(z, \xi)$ is convergent for each $z \in \mathbb{H}$ and each $\xi > 0$. To prove the smoothness of J , we first assume that $\alpha > 1$ is an integer. In this case, the integral in (5.12) can be computed explicitly in terms of logarithms and powers of z , $\bar{z}$, $z - \xi$, and $\bar{z} - \xi$ (see [11] for the case $\alpha = 2$). Hence, $J(z, \xi)$ is smooth for $(z, \xi) \in \mathbb{H} \times (0, \infty)$.

Assume $\alpha > 1$ is not an integer. Then, fixing a base point $A > \xi$, we can rewrite expression (5.12) for $J(z, \xi)$ as

$$J(z, \xi) = \frac{1}{(1 - e^{2i\pi\alpha})^2} \int_A^{(z+, \bar{z}+, z-, \bar{z}-)} (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \\ z \in \mathbb{H}, \quad \xi > 0, \quad (7.1)$$

where the integration contour is the composition of four loops $\{l_j\}_1^4$ based at A (see Fig. 11) and the integrand is evaluated using analytic continuation along the contour. More precisely, the loop l_1 encircles z once in the counterclockwise direction, l_2 encircles $\bar{z}$ once in the counterclockwise direction, l_3 encircles z once in the clockwise direction, and l_4 encircles $\bar{z}$ once in the clockwise direction. On the first half of l_1 , the principal branch is used, but as the contour l_1 encircles z in the counterclockwise direction, the power $(u - z)^\alpha$ in the integrand picks up an additional factor of $e^{2i\pi\alpha}$ with respect to the principal branch; then, as l_2 encircles $\bar{z}$ in the counterclockwise direction, the power $(u - \bar{z})^{\alpha-2}$ in the integrand picks up the factor $e^{2i\pi(\alpha-2)}$ and so on. Collapsing the contour onto a single path from $\bar{z}$ to z and collecting the exponential factors, we see that (7.1) reduces to (5.12). Since the contour in (7.1) avoids the branch points, the integral in (7.1) can be differentiated an unlimited number of times with respect to z , $\bar{z}$, and ξ . This completes the proof of the lemma. $\square$

FIGURE 11. The integration contour in (7.1) is the composition of the four loops l_j , $j = 1, \dots, 4$, based at the point $A > \xi$

Remark 7.2 (A main difference between the two examples). The integrals relevant for Example 2 have better convergence properties than those of Example 1. Indeed, the Pochhammer integral expression (7.1) for the function $J(z, \xi)$ relevant for Example 2 is well defined for any $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. If $\alpha > 1$ is not an integer, expression (5.12) for J can be recovered from (7.1) by collapsing the contour down to a simple path from $\bar{z}$ to z . The integral in (5.12) converges because the exponents of the factors $(u - z)^\alpha$ and $(u - \bar{z})^{\alpha-2}$ are both > -1 when $\alpha > 1$.

On the other hand, the Pochhammer contour appearing in definition (5.2) of the function $I(z, \xi^1, \xi^2)$ relevant for Example 1 encircles the points z and ξ^2 and the corresponding integrand involves the factors $(u - z)^{\alpha-1}$ and $(\xi^2 - u)^{-\alpha/2}$. Thus, this contour can only be collapsed down to simple path from z to ξ^2 if $0 < \alpha < 2$. Since Proposition 5.1 is stated under the assumption that $\alpha \geq 2$, this means that the analysis of Example 1 requires the use of a Pochhammer contour (in fact, the contour can be collapsed at z , but not at ξ^2).

Remark 7.2 suggests that it should be easier to prove Proposition 5.2 than Proposition 5.1. This is indeed the case. It is of course still possible to proceed as in Example 1 and use the full machinery of Sects. 2–4 to prove Proposition 5.2. However, in order to complement the discussion of Example 1, we will in what follows present a more elementary proof of Proposition 5.2 which relies on direct estimates. The takeaway is that in some simpler cases

FIGURE 12. The contour from $\bar{z} = \xi + re^{-i\theta}$ to $z = \xi + re^{i\theta}$ defined in Eq. (7.3)

one has the choice of either using the machinery of Sects. 2–4 or a more naive approach based on direct estimates.

Lemma 7.3. *The function $J(z, \xi)$ defined in (5.12) satisfies the following estimates:*

$$|J(z, \xi)| \leq C|z - \xi|^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (7.2a)$$

$$|J(z, \xi)| \leq C|x|^{-\frac{\alpha}{2}}|x - \xi|^{-\frac{\alpha}{2}}y^{2\alpha-1}, \quad x > \xi, \quad y > 0, \quad \xi > 0. \quad (7.2b)$$

$$|Re J(z, \xi)| \leq Cy|z|^{\alpha-2}, \quad |z| \geq 2\xi, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (7.2c)$$

where $z = x + iy$.

Proof. To prove (7.2a), we let $z = \xi + re^{i\theta}$ and choose the following parametrization of the integration contour in (5.12) (see Fig. 12):

$$u = \xi + re^{i\varphi}, \quad -\theta \leq \varphi \leq \theta. \quad (7.3)$$

This yields after simplification

$$J(\xi + re^{i\theta}, \xi) = ir^{\frac{3\alpha}{2}-1} \int_{-\theta}^{\theta} (e^{i\varphi} - e^{i\theta})^{\alpha} (e^{i\varphi} - e^{-i\theta})^{\alpha-2} (\xi + re^{i\varphi})^{-\frac{\alpha}{2}} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi, \\ r > 0, \quad \theta \in (0, \pi), \quad \xi > 0. \quad (7.4)$$

It follows that

$$|J(\xi + re^{i\theta}, \xi)| \leq r^{\frac{3\alpha}{2}-1} \int_{-\theta}^{\theta} |e^{i\varphi} - e^{i\theta}|^{\alpha} |e^{i\varphi} - e^{-i\theta}|^{\alpha-2} |\xi + re^{i\varphi}|^{-\frac{\alpha}{2}} d\varphi.$$

Since

$$|\xi + re^{i\varphi}| \geq \frac{|re^{i\varphi} + r|}{2},$$

for all $r > 0$, $\varphi \in (-\pi, \pi)$, and $\xi > 0$, we obtain the estimate

$$|J(\xi + re^{i\theta}, \xi)| \leq Cr^{\alpha-1} \int_{-\theta}^{\theta} |e^{i\varphi} - e^{i\theta}|^{\alpha} |e^{i\varphi} - e^{-i\theta}|^{\alpha-2} |e^{i\varphi} + 1|^{-\frac{\alpha}{2}} d\varphi.$$

The integral remains bounded as $\theta \uparrow \pi$, because $\frac{3\alpha}{2} - 2 > -1$. Thus, we arrive at

$$|J(\xi + re^{i\theta}, \xi)| \leq Cr^{\alpha-1}, \quad r > 0, \quad \theta \in (0, \pi), \quad \xi > 0,$$

which is (7.2a).

To prove (7.2b), we let $z = x + iy$ in (5.12) and use the parametrization $u = x + is$, $-y \leq s \leq y$, of the contour from $\bar{z}$ to z . Assuming that $x > \xi$, this yields

$$J(z, \xi) = \int_{-y}^y (is - iy)^{\alpha} (is + iy)^{\alpha-2} (x + is)^{-\frac{\alpha}{2}} (x - \xi + is)^{-\frac{\alpha}{2}} ids. \quad (7.5)$$

It follows that

$$\begin{aligned} |J(z, \xi)| &\leq \int_{-y}^y |s - y|^{\alpha} |s + y|^{\alpha-2} |x + is|^{-\frac{\alpha}{2}} |x - \xi + is|^{-\frac{\alpha}{2}} ds \\ &\leq |x|^{-\frac{\alpha}{2}} |x - \xi|^{-\frac{\alpha}{2}} (2y)^{\alpha} \int_{-y}^y |s + y|^{\alpha-2} ds \\ &\leq C|x|^{-\frac{\alpha}{2}} |x - \xi|^{-\frac{\alpha}{2}} y^{2\alpha-1}, \quad x > \xi, \quad y > 0, \quad \xi > 0. \end{aligned}$$

This proves (7.2b).

To prove (7.2c), we note that if $f(u)$ is an analytic function, then

$$\overline{\int_{\bar{z}}^z f(u) du} = - \int_{\bar{z}}^z \overline{f(\bar{u})} du. \quad (7.6)$$

Hence,

$$\begin{aligned} 2\operatorname{Re} J(z, \xi) &= \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du \\ &\quad - \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du. \end{aligned}$$

Since

$$(u - z)^2 - (u - \bar{z})^2 = -4iy(u - x),$$

this can be written as

$$\operatorname{Re} J(z, \xi) = -2iy \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} (u - x) du. \quad (7.7)$$

Assuming that $z = re^{i\theta}$ satisfies $|z| > \xi$ and adopting the parametrization

$$u = re^{i\varphi}, \quad -\theta \leq \varphi \leq \theta,$$

of the integration contour from $\bar{z}$ to z , we arrive at

$$\begin{aligned} \operatorname{Re} J(re^{i\theta}, \xi) &= 2yr^{\frac{3\alpha}{2}-2} \int_{-\theta}^{\theta} (e^{i\varphi} - e^{i\theta})^{\alpha-2} (e^{i\varphi} - e^{-i\theta})^{\alpha-2} (re^{i\varphi} - \xi)^{-\frac{\alpha}{2}} \\ &\quad \times (e^{i\varphi} - \cos \theta) e^{i\varphi(1-\frac{\alpha}{2})} d\varphi. \end{aligned}$$

In view of the estimate

$$|re^{i\varphi} - \xi| \geq \frac{r}{2}$$

valid for all $r \geq 2\xi$, $\varphi \in (0, \pi)$, and $\xi > 0$, this implies

$$|\operatorname{Re} J(re^{i\theta}, \xi)| \leq C y r^{\alpha-2} \int_{-\theta}^{\theta} |e^{i\varphi} - e^{i\theta}|^{\alpha-2} |e^{i\varphi} - e^{-i\theta}|^{\alpha-2} |e^{i\varphi} - \cos \theta| d\varphi$$

$$r \geq 2\xi, \theta \in (0, \pi), \xi > 0.$$

Since $2\alpha - 3 > -1$, the integral remains bounded as $\theta \uparrow \pi$. This proves (7.2c). $\square$

We next extend the function $J(z, \xi)$ continuously to all of $\bar{\mathbb{H}} \times (0, \infty)$. As suggested by (5.12), we define $J(x, \xi)$ for $x \in \mathbb{R}$ by

$$J(x, \xi) = \begin{cases} 0, & x \geq \xi, \\ \int_{L_x} (u-x)^{2\alpha-2} u^{-\frac{\alpha}{2}} (u-\xi)^{-\frac{\alpha}{2}} du, & x < \xi, \end{cases} \quad (7.8)$$

where the integration contour L_x is a loop starting and ending at x which avoids the branch cut along $(-\infty, \xi)$ and which encloses ξ in the counterclockwise direction, see Fig. 13.

Lemma 7.4. *For each $\xi > 0$, the function $J(z, \xi)$ defined by (5.12) and (7.8) is a continuous function of $z \in \bar{\mathbb{H}}$.*

Proof. Fix $\xi > 0$. Since $\alpha > 1$, the integral in (7.8) converges for every $x < \xi$ (including $x = 0$). Equation (7.2b) implies that $z \mapsto J(z, \xi)$ is continuous

FIGURE 13. The integration contour L_x in (7.8) is a loop which encloses ξ in the counterclockwise direction

at each point in (ξ, ∞) . Moreover, Eq. (7.2a) implies that J is continuous at $z = \xi$.

We next show that J is continuous at each point in $(-\infty, 0) \cup (0, \xi)$. Letting $s = \frac{\pi}{\theta}\varphi$ and simplifying, we can write (7.4) as

$$J(\xi + re^{i\theta}, \xi) = \int_{-\pi}^{\pi} g_{r,\theta}(s) ds,$$

where

$$g_{r,\theta}(s) = ir^{\frac{3\alpha}{2}-1} \frac{\theta}{\pi} (e^{\frac{i\theta s}{\pi}} - e^{i\theta})^\alpha (e^{\frac{i\theta s}{\pi}} - e^{-i\theta})^{\alpha-2} (\xi + re^{\frac{i\theta s}{\pi}})^{-\frac{\alpha}{2}} e^{\frac{i\theta s}{\pi}(1-\frac{\alpha}{2})}.$$

Let $\epsilon > 0$. Then,

$$\begin{aligned} |g_{r,\theta}(s)| &\leq C |e^{\frac{i\theta s}{\pi}} - e^{i\theta}|^\alpha |e^{\frac{i\theta s}{\pi}} - e^{-i\theta}|^{\alpha-2} \\ &\leq C 2^\alpha \max \left(\left| \theta + \frac{\theta s}{\pi} \right|^{\alpha-2}, \left| \theta + \frac{\theta s}{\pi} - 2\pi \right|^{\alpha-2} \right) \\ &\leq C \max (|\pi + s|^{\alpha-2}, |\pi - s|^{\alpha-2}), \quad s \in (-\pi, \pi), \end{aligned}$$

for all $r \in (\epsilon, \epsilon^{-1})$ with $|r - \xi| > \epsilon$ and all $\theta \in [\frac{\pi}{2}, \pi]$. Since $|\pi \pm s|^{\alpha-2} \in L^1([-\pi, \pi])$, this shows that there exists a function $G(s)$ in $L^1([-\pi, \pi])$ such that

$$|g_{r,\theta}(s)| \leq G(s), \quad s \in (-\pi, \pi), \quad (7.9)$$

for all $r \in (\epsilon, \epsilon^{-1})$ with $|r - \xi| > \epsilon$ and all $\theta \in [\frac{\pi}{2}, \pi]$. Since $\epsilon > 0$ was arbitrary, if the point $x_0 = \xi + r_0 e^{i\pi}$ belongs to $(-\infty, 0) \cup (0, \xi)$, dominated convergence gives

$$\lim_{z \rightarrow x_0} J(z, \xi) = \lim_{\substack{\theta \uparrow \pi \\ r \rightarrow r_0}} \int_{-\pi}^{\pi} g_{r,\theta}(s) ds = \int_{-\pi}^{\pi} g_{r_0,\pi}(s) ds = J(x_0, \xi).$$

This shows that $J(\cdot, \xi)$ is continuous at each point in $(-\infty, 0) \cup (0, \xi)$.

It only remains to show that $J(\cdot, \xi) : \mathbb{H} \rightarrow \mathbb{C}$ is continuous at $z = 0$. To prove this, let $c = \frac{\xi}{2}$ and let $z = re^{i\theta}$ with $\theta \in [0, \pi]$ and $r \in (0, c)$. Let γ denote the contour from $\bar{z}$ to z used in definition (5.12) of $J(z, \xi)$. We write γ as the union of five subcontours as follows (see Fig. 14):

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,$$

where

$$\begin{aligned} \gamma_1 : \{re^{i\varphi} \mid -\theta \leq \varphi \leq 0\}, & \quad \gamma_2 : \{u - i0 \mid r \leq u \leq \xi - c\}, \\ \gamma_3 : \{\xi + ce^{i\varphi} \mid -\pi \leq \varphi \leq \pi\}, & \quad \gamma_4 : \{u + i0 \mid r \leq u \leq \xi - c\}, \\ \gamma_5 : \{re^{i\varphi} \mid 0 \leq \varphi \leq \theta\}. & \end{aligned} \quad (7.10)$$

Then,

$$J(z, \xi) = \sum_{j=1}^5 J_j(z),$$

FIGURE 14. The integration contour γ is the composition of the five subcontours γ_j , $j = 1, \dots, 5$, defined in (7.10)

where

$$J_j(z) = \int_{\gamma_j} (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \quad j = 1, \dots, 5.$$

We claim that

$$|J_j(z)| \leq Cr^{\frac{3\alpha}{2}-1}, \quad |z| < c, \quad z \in \mathbb{H}, \quad j = 1, 5. \quad (7.11)$$

Indeed, let us consider the case of $J_1(z)$. Since $|u - \xi| \geq c$ for $u \in \gamma_1$, we have

$$|J_1(z)| \leq \int_{\gamma_1} |u - z|^\alpha |u - \bar{z}|^{\alpha-2} r^{-\frac{\alpha}{2}} c^{-\frac{\alpha}{2}} |du|.$$

Moreover, the inequalities

$$|u - \bar{z}| \leq |u - z| \leq 2r$$

are valid for $u \in \gamma_1$. Hence, if $\alpha \geq 2$, then

$$|J_1(z)| \leq C \int_{\gamma_1} |u - z|^{2\alpha-2} r^{-\frac{\alpha}{2}} |du| \leq C \int_{\gamma_1} r^{\frac{3\alpha}{2}-2} |du| \leq Cr^{\frac{3\alpha}{2}-1}.$$

On the other hand, if $1 < \alpha < 2$, then

$$\begin{aligned} |J_1(z)| &\leq C \int_{\gamma_1} (2r)^\alpha |u - \bar{z}|^{\alpha-2} r^{-\frac{\alpha}{2}} |du| \leq Cr^{\frac{\alpha}{2}} \int_0^\theta |re^{-i\varphi} - re^{-i\theta}|^{\alpha-2} r d\varphi \\ &\leq Cr^{\frac{3\alpha}{2}-1} \int_0^\theta \left| \frac{2}{\pi}(\theta - \varphi) \right|^{\alpha-2} d\varphi \leq Cr^{\frac{3\alpha}{2}-1} \frac{\theta^{\alpha-1}}{\alpha-1} \leq Cr^{\frac{3\alpha}{2}-1}. \end{aligned}$$

This proves (7.11) for $J_1(z)$; the proof for $J_5(z)$ is similar.

We next show that

$$\lim_{z \rightarrow 0} J_j(z) = J_j(0), \quad j = 2, 4. \quad (7.12)$$

To establish (7.12), we let $u = r + s$ and write

$$J_2(z) = \int_0^c f_{r,\theta}(s) ds,$$

where

$$f_{r,\theta}(s) = \chi_{[0,c-r]}(s)(s+r-re^{i\theta})^\alpha(s+r-re^{-i\theta})^{\alpha-2}(s+r)^{-\frac{\alpha}{2}}|s+r-\xi|^{-\frac{\alpha}{2}}e^{\frac{\alpha i\pi}{2}}$$

and $\chi_{[0,c-r]}$ denotes the characteristic function of the interval $[0, c-r]$. We will show that there exists a function $F(s)$ in $L^1((0, c))$ such that

$$|f_{r,\theta}(s)| \leq F(s), \quad s \in (0, c), \quad (7.13)$$

for all $r \in (0, c)$ and all $\theta \in [0, \pi]$. Dominated convergence then gives

$$\lim_{z \rightarrow 0} J_2(z) = \lim_{r \rightarrow 0} \int_0^c f_{r,\theta}(s) ds = \int_0^c f_{0,\theta}(s) ds = J_2(0),$$

showing that $J_2(z)$ satisfies (7.12).

In order to prove (7.13), we note that

$$|s+r-re^{i\theta}| = |s+r-re^{-i\theta}| \quad \text{and} \quad |s+r-\xi| \geq c$$

for $s \in (0, c-r)$. This gives

$$|f_{r,\theta}(s)| \leq |s+r-re^{i\theta}|^{2\alpha-2}(s+r)^{-\frac{\alpha}{2}}c^{-\frac{\alpha}{2}}.$$

Using the inequalities

$$|s+r-re^{i\theta}| \leq s+2r, \quad s+r \geq \frac{s+2r}{2},$$

we find

$$|f_{r,\theta}(s)| \leq |s+2r|^{\frac{3\alpha}{2}-2}(2/c)^{\frac{\alpha}{2}}, \quad s \in (0, c), \quad r \in (0, c), \quad \theta \in [0, \pi].$$

Since

$$|s+2r|^{\frac{3\alpha}{2}-2} \leq \begin{cases} (3c)^{\frac{3\alpha}{2}-2}, & \alpha \geq \frac{4}{3}, \\ s^{\frac{3\alpha}{2}-2}, & 1 < \alpha < \frac{4}{3}, \end{cases}$$

we deduce that (7.13) holds with

$$F(s) = (2/c)^{\frac{\alpha}{2}} \max \left((3c)^{\frac{3\alpha}{2}-2}, s^{\frac{3\alpha}{2}-2} \right), \quad 0 < s < c.$$

This proves (7.12) for $j = 2$; the proof when $j = 4$ is similar.

Finally, since the integration contour is independent of z , it is easy to see that

$$\lim_{z \rightarrow 0} J_3(z) = J_3(0). \quad (7.14)$$

Since $J(z, \xi) = \sum_{j=1}^5 J_j(z)$, the continuity of $J(z, \xi)$ at $z = 0$ follows from Eqs. (7.11), (7.12), and (7.14). This completes the proof of the lemma. $\square$

Lemma 7.5. *For each $\xi > 0$, the partial derivatives $\partial_x J(z, \xi)$ and $\partial_y J(z, \xi)$ have continuous extensions to $\mathbb{H} \setminus \{0, \xi\}$.*

Proof. Fix $\xi > 0$. Defining $W(u)$ by

$$W(u) = u^{-\frac{\alpha}{2}}(u - \xi)^{-\frac{\alpha}{2}}, \quad (7.15)$$

we can write

$$J(z, \xi) = \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} W(u) du.$$

An integration by parts gives

$$\begin{aligned} J(z, \xi) &= -\frac{\alpha}{\alpha-1} \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W(u) du \\ &\quad - \frac{1}{\alpha-1} \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-1} W'(u) du. \end{aligned} \quad (7.16)$$

Differentiating with respect to z and $\bar{z}$, we find

$$\begin{aligned} J_z(z, \xi) &= \alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du \\ &\quad + \frac{\alpha}{\alpha-1} \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W'(u) du \end{aligned}$$

and

$$\begin{aligned} J_{\bar{z}}(z, \xi) &= \alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-2} W(u) du \\ &\quad + \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} W'(u) du. \end{aligned}$$

Since $\alpha > 1$, these expressions for J_z and $J_{\bar{z}}$ are well defined for each $z \in \mathbb{R} \setminus \{0, \xi\}$. Repeating the above arguments that led to the continuity of $J(z, \xi)$ at each point $z \in \mathbb{R} \setminus \{0, \xi\}$, we infer that this provides continuous extensions of J_z and $J_{\bar{z}}$ to $\mathbb{H} \setminus \{0, \xi\}$. $\square$

Lemma 7.6. *For each fixed $\xi > 0$, the function $J(z, \xi)$ satisfies*

$$\operatorname{Re} J(x, \xi) = 0, \quad x \in \mathbb{R}, \quad \xi > 0. \quad (7.17)$$

Moreover,

$$\begin{cases} J(x + iy, \xi) = O(1), \\ \operatorname{Re} J(x + iy, \xi) = O(y), \end{cases} \quad y \downarrow 0, \quad x \in \mathbb{R} \setminus \{0, \xi\}, \quad (7.18a)$$

and

$$J(x + iy, \xi) = O(y^{2\alpha-1}), \quad y \downarrow 0, \quad x > \xi, \quad (7.18b)$$

where the error terms are uniform with respect to x in compact subsets of $\mathbb{R} \setminus \{0, \xi\}$.

Proof. Equation (7.17) follows by letting $y \rightarrow 0$ in (7.7). The asymptotic formulas (7.18a) are then a direct consequence of Lemma 7.4 and Lemma 7.5. Equation (7.18b) follows from (7.2b). $\square$

In the above lemmas, we have established several properties of the function J . We next turn to the analysis of the functions $\mathcal{M}$ and P defined in (5.11) and (5.13), respectively.

Lemma 7.7. *The function $\mathcal{M}(z, \xi)$ defined in (5.11) satisfies the following estimates:*

$$|\mathcal{M}(z, \xi)| \leq Cy^{\alpha-2}|z|^{1-\alpha}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (7.19a)$$

$$|\mathcal{M}(z, \xi)| \leq Cy^{3\alpha-3}|z|^{1-\alpha}|z - \xi|^{1-\alpha}|x|^{-\frac{\alpha}{2}}|x - \xi|^{-\frac{\alpha}{2}}, \quad x > \xi, \quad y > 0, \quad \xi > 0, \quad (7.19b)$$

$$\begin{aligned} |\operatorname{Re} \mathcal{M}(z, \xi)| &\leq Cy^{\alpha-1}|z|^{-\alpha}|z - \xi|^{-\alpha} \left[(|x||x - \xi| + y^2)|z|^{\alpha-2} \right. \\ &\quad \left. + (|x| + |\xi - x|)|z - \xi|^{\alpha-1} \right], \quad |z| \geq 2\xi, \quad z \in \mathbb{H}, \quad \xi > 0, \end{aligned} \quad (7.19c)$$

where $z = x + iy$. In particular, for each fixed $\xi > 0$,

$$\mathcal{M}(z, \xi) = O(|x|^{1-\alpha}), \quad x \rightarrow \pm\infty, \quad y > 0, \quad (7.20a)$$

$$\operatorname{Re} \mathcal{M}(z, \xi) = O(|x|^{-\alpha}), \quad x \rightarrow \pm\infty, \quad y > 0, \quad (7.20b)$$

where the error terms are uniform with respect to y in compact subsets of $(0, \infty)$.

Proof. The estimates (7.19a) and (7.19b) follow immediately from the definition of $\mathcal{M}$ together with the estimates (7.2a) and (7.2b) of Lemma 7.3. The estimate (7.19c) follows by applying (7.2a) and (7.2c) to the identity

$$\begin{aligned} \operatorname{Re} \mathcal{M}(z, \xi) &= y^{\alpha-2}|z|^{-\alpha}|z - \xi|^{-\alpha} \operatorname{Re} [\bar{z}(\bar{z} - \xi)J(z, \xi)] \\ &= y^{\alpha-2}|z|^{-\alpha}|z - \xi|^{-\alpha} [(x^2 - x\xi - y^2)\operatorname{Re} J(z, \xi) - y(\xi - 2x)\operatorname{Im} J(z, \xi)]. \end{aligned} \quad (7.21)$$

Finally, the asymptotic equations in (7.20) are an immediate consequence of (7.19a) and (7.19c). $\square$

Lemma 7.8. *The function $\mathcal{M} = \mathcal{M}_1 + i\mathcal{M}_2$ satisfies*

$$\partial_y \mathcal{M}_1(z, \xi) = -\partial_x \mathcal{M}_2(z, \xi), \quad z \in \mathbb{H}, \quad \xi > 0.$$

Proof. Since the statement only involves derivatives with respect to x and y , we can assume that $\xi > 0$ is fixed. We need to prove that $\operatorname{Im} \bar{\partial} \mathcal{M} = 0$. In terms of the function $W(u)$ defined in (7.15), we can write

$$\mathcal{M}(z, \xi) = y^{\alpha-2}z^{-\frac{\alpha}{2}}(z - \xi)^{-\frac{\alpha}{2}}\bar{z}^{1-\frac{\alpha}{2}}(\bar{z} - \xi)^{1-\frac{\alpha}{2}} \int_{\bar{z}}^z (u - z)^{\alpha}(u - \bar{z})^{\alpha-2}W(u)du.$$

An integration by parts gives

$$\begin{aligned} \mathcal{M}(z, \xi) &= -y^{\alpha-2}z^{-\frac{\alpha}{2}}(z - \xi)^{-\frac{\alpha}{2}}\bar{z}^{1-\frac{\alpha}{2}}(\bar{z} - \xi)^{1-\frac{\alpha}{2}} \frac{1}{\alpha - 1} \\ &\quad \times \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1}(u - \bar{z})^{\alpha-1}W(u)du + \int_{\bar{z}}^z (u - z)^{\alpha}(u - \bar{z})^{\alpha-1}W'(u)du \right). \end{aligned} \quad (7.22)$$

Differentiating with respect to $\bar{z}$, we find

$$\begin{aligned}\bar{\partial}\mathcal{M}(z, \xi) &= \left(\frac{i}{2} \frac{\alpha-2}{y} + \frac{1-\frac{\alpha}{2}}{\bar{z}} + \frac{1-\frac{\alpha}{2}}{\bar{z}-\xi} \right) \mathcal{M}(z, \xi) \\ &\quad + y^{\alpha-2} z^{-\frac{\alpha}{2}} (z-\xi)^{-\frac{\alpha}{2}} \bar{z}^{1-\frac{\alpha}{2}} (\bar{z}-\xi)^{1-\frac{\alpha}{2}} \\ &\quad \times \left(\alpha \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha-2} W(u) du + \int_{\bar{z}}^z (u-z)^{\alpha} (u-\bar{z})^{\alpha-2} W'(u) du \right).\end{aligned}\tag{7.23}$$

The integrals in (7.23) are convergent at the endpoints z and $\bar{z}$ because $\alpha > 1$. Substituting expression (7.22) for $\mathcal{M}$ into (7.23) and simplifying, we obtain

$$\begin{aligned}\bar{\partial}\mathcal{M}(z, \xi) &= R(z, \xi) \left\{ -i(\alpha-2)[x(x-\xi) + y^2] \right. \\ &\quad \times \left(\alpha \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha-1} W(u) du + \int_{\bar{z}}^z (u-z)^{\alpha} (u-\bar{z})^{\alpha-1} W'(u) du \right) \\ &\quad + 2(\alpha-1)y\bar{z}(\bar{z}-\xi) \\ &\quad \times \left. \left(\alpha \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha-2} W(u) du + \int_{\bar{z}}^z (u-z)^{\alpha} (u-\bar{z})^{\alpha-2} W'(u) du \right) \right\},\end{aligned}\tag{7.24}$$

where the real-valued function $R(z, \xi)$ is given by

$$R(z, \xi) = \frac{1}{2(\alpha-1)} y^{\alpha-3} |z|^{-\alpha} |z-\xi|^{-\alpha}.$$

To establish the identity $\text{Im } \bar{\partial}\mathcal{M} = 0$, it is enough to show that

$$\frac{\bar{\partial}\mathcal{M} - \overline{\bar{\partial}\mathcal{M}}}{R} = 0.\tag{7.25}$$

Using (7.6) and (7.24), we can write the left-hand side of (7.25) as

$$\begin{aligned}&-i(\alpha-2)[x(x-\xi) + y^2] \\ &\quad \times \left(\alpha \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha-1} W(u) du + \int_{\bar{z}}^z (u-z)^{\alpha} (u-\bar{z})^{\alpha-1} W'(u) du \right) \\ &\quad + 2(\alpha-1)y\bar{z}(\bar{z}-\xi) \\ &\quad \times \left(\alpha \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha-2} W(u) du + \int_{\bar{z}}^z (u-z)^{\alpha} (u-\bar{z})^{\alpha-2} W'(u) du \right) \\ &\quad - i(\alpha-2)[x(x-\xi) + y^2] \\ &\quad \times \left(-\alpha \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha-1} W(u) du - \int_{\bar{z}}^z (u-z)^{\alpha-1} (u-\bar{z})^{\alpha} W'(u) du \right) \\ &\quad - 2(\alpha-1)yz(z-\xi) \\ &\quad \times \left(-\alpha \int_{\bar{z}}^z (u-z)^{\alpha-2} (u-\bar{z})^{\alpha-1} W(u) du - \int_{\bar{z}}^z (u-z)^{\alpha-2} (u-\bar{z})^{\alpha} W'(u) du \right).\end{aligned}\tag{7.26}$$

The right-hand side of (7.26) involves eight integrals. Integrating by parts in the third, fourth, and eighth of these integrals and using that the first and fifth integrals cancel, we see that the expression in (7.26) can be written as

$$\begin{aligned}
 & i(\alpha - 2)[x(x - \xi) + y^2] \\
 & \times \left(- \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-1} W'(u) du + \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^\alpha W'(u) du \right) \\
 & + 2(\alpha - 1)y\bar{z}(\bar{z} - \xi) \\
 & \times \left(- \alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du - \alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} \frac{(u - \bar{z})^{\alpha-1}}{\alpha - 1} W'(u) du \right. \\
 & - \alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} \frac{(u - \bar{z})^{\alpha-1}}{\alpha - 1} W'(u) du - \int_{\bar{z}}^z (u - z)^\alpha \frac{(u - \bar{z})^{\alpha-1}}{\alpha - 1} W''(u) du \Big) \\
 & - 2(\alpha - 1)yz(z - \xi) \\
 & \times \left(- \alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du \right. \\
 & + \alpha \int_{\bar{z}}^z \frac{(u - z)^{\alpha-1}}{\alpha - 1} (u - \bar{z})^{\alpha-1} W'(u) du + \int_{\bar{z}}^z \frac{(u - z)^{\alpha-1}}{\alpha - 1} (u - \bar{z})^\alpha W''(u) du \Big). \tag{7.27}
 \end{aligned}$$

A long but straightforward computation shows that the expression in (7.27) equals

$$2\alpha y \int_{\bar{z}}^z \frac{d}{du} \left[(u - z)^\alpha (u - \bar{z})^\alpha u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} \left(\frac{2x - \xi}{u - z} - \frac{x - \xi}{u} - \frac{x}{u - \xi} \right) \right] du.$$

Since $\alpha > 1$, the fundamental theorem of calculus implies that the integral vanishes. This proves (7.25) and completes the proof of the lemma. $\square$

Lemma 7.9. *The function $P(z, \xi)$ defined in (5.13) is a well-defined smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$.*

Proof. By Lemma 7.1, $\mathcal{M} = \mathcal{M}_1 + i\mathcal{M}_2$ is a smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$. Moreover, by Eq. (7.20), there exists an $\epsilon > 0$ such that $\mathcal{M}_1(x + iy) = O(|x|^{-1-\epsilon})$ and $\mathcal{M}_2(x + iy) = O(|x|^{-\epsilon})$ as $|x| \rightarrow \infty$ uniformly with respect to y in compact subsets of $(0, \infty)$. It follows that the integral in definition (5.13) of P converges. Furthermore, Lemma 7.8 shows that the integral $\int (\mathcal{M}_1 dx - \mathcal{M}_2 dy)$ is independent of the path. We infer that $P(z, \xi)$ can be written as

$$\begin{aligned}
 P(z, \xi) &= -\frac{1}{c_\alpha} \int_{\infty}^z \left(\mathcal{M}_1(z', \xi) dx' - \mathcal{M}_2(z', \xi) dy' \right) \\
 &= -\frac{1}{c_\alpha} \int_{\infty}^z \operatorname{Re} [\mathcal{M}(z', \xi) dz'], \quad z \in \mathbb{H}, \tag{7.28}
 \end{aligned}$$

where $z' = x' + iy'$ and the contour of integration runs from $\infty + ic$, $c > 0$, to z . Since $\mathcal{M}(z, \xi)$ is a smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$, so is $P(z, \xi)$. $\square$

Lemma 7.10. *For each $\xi > 0$, the function $\mathcal{M}(z, \xi)$ satisfies*

$$\begin{cases} \mathcal{M}(x + iy, \xi) = O(y^{\alpha-2}), \\ \operatorname{Re} \mathcal{M}(x + iy, \xi) = O(y^{\alpha-1}), \end{cases} \quad y \downarrow 0, \ x \in \mathbb{R} \setminus \{0, \xi\}, \tag{7.29}$$

and

$$\mathcal{M}(x + iy, \xi) = O(y^{3\alpha-3}), \quad y \downarrow 0, \quad x > \xi,$$

where the error terms are uniform with respect to x in compact subsets of $\mathbb{R} \setminus \{0, \xi\}$.

Proof. This follows from Lemma 7.6 and definition (5.11) of $\mathcal{M}$. $\square$

Lemma 7.11. *For each $\xi > 0$, the function $z \mapsto P(z, \xi)$ defined in (5.13) has a continuous extension to $\mathbb{H} \setminus \{0\}$ which satisfies*

$$P(x, \xi) = \begin{cases} 1, & -\infty < x < 0, \\ 0, & 0 < x < \infty, \end{cases} \quad \xi > 0. \quad (7.30)$$

Proof. Fix $\xi > 0$. Expression (7.28) for P together with Lemma 7.10 implies that there exist real constants $\{P_j\}_1^3$ such that the function

$$z \mapsto \begin{cases} P(z, \xi), & z \in \mathbb{H}, \\ P_1, & z \in (-\infty, 0), \\ P_2, & z \in (0, \xi), \\ P_3, & z \in (\xi, \infty), \end{cases}$$

is continuous $\mathbb{H} \setminus \{0, \xi\} \rightarrow \mathbb{R}$. Letting z approach $\infty + i0$ in (7.28), we deduce that $P_3 = 0$. It follows that

$$P(\xi + re^{i\theta}, \xi) = -\frac{1}{c_\alpha} \operatorname{Re} \left(ir \int_0^\theta \mathcal{M}(\xi + re^{i\varphi}, \xi) e^{i\varphi} d\varphi \right)$$

for $r > 0$ and $\theta \in (0, \pi)$. In view of estimate (7.19a), this yields

$$\begin{aligned} |P(\xi + re^{i\theta}, \xi)| &\leq Cr \int_0^\theta |\mathcal{M}(\xi + re^{i\varphi}, \xi)| d\varphi \\ &\leq Cr^{\alpha-1} \int_0^\theta (\sin^{\alpha-2} \varphi) |\xi + re^{i\varphi}|^{1-\alpha} d\varphi, \quad r > 0, \quad \xi > 0. \end{aligned} \quad (7.31)$$

Letting $r \downarrow 0$, we infer that if we set $P(\xi, \xi) = 0$, then $P(z, \xi)$ is continuous at $z = \xi$. Moreover, taking the limit $r \downarrow 0$ in (7.31) with $\theta = \pi$, it follows that $P_2 = 0$.

It only remains to prove that $P_3 = 1$. This will follow if we can show that the normalization constant c_α defined in (5.14) satisfies

$$c_\alpha = -\operatorname{Re} \int_{S_r} \mathcal{M}(z, \xi) dz, \quad r > 0, \quad (7.32)$$

where S_r is a counterclockwise semicircle of radius r centered at 0:

$$S_r : re^{i\varphi}, \quad 0 \leq \varphi \leq \pi.$$

Lemmas 7.8 and 7.10 show that the right-hand side of (7.32) is independent of $r > 0$. We can therefore evaluate it in the limit as $r \downarrow 0$. Recalling the definition of $\mathcal{M}$, we write

$$\int_{S_r} \mathcal{M}(z, \xi) dz = \int_0^\pi f_r(\varphi) d\varphi,$$

where

$$f_r(\varphi) = i(\sin^{\alpha-2} \varphi)(re^{i\varphi} - \xi)^{-\frac{\alpha}{2}}(re^{-i\varphi} - \xi)^{1-\frac{\alpha}{2}} J(re^{i\varphi}, \xi).$$

The function J is bounded on each compact subset of $\bar{\mathbb{H}}$ by Lemma 7.4. Hence, $f_r(\varphi)$ obeys the estimate

$$|f_r(\varphi)| \leq C \sin^{\alpha-2} \varphi, \quad r < \xi/2, \quad \varphi \in [0, \pi].$$

Since the function $\sin^{\alpha-2} \varphi$ belongs to $L^1((0, \pi))$, dominated convergence yields

$$\begin{aligned} -\operatorname{Re} \int_{S_r} \mathcal{M}(z, \xi) dz &= -\operatorname{Re} \lim_{r \rightarrow 0} \int_0^\pi f_r(\varphi) d\varphi = -\operatorname{Re} \int_0^\pi \lim_{r \rightarrow 0} f_r(\varphi) d\varphi \\ &= -\operatorname{Re} \int_0^\pi i(\sin^{\alpha-2} \varphi)(-\xi + i0)^{-\frac{\alpha}{2}}(-\xi - i0)^{1-\frac{\alpha}{2}} J(0, \xi) d\varphi \\ &= -(\operatorname{Im} J(0, \xi)) \xi^{1-\alpha} \int_0^\pi \sin^{\alpha-2} \varphi d\varphi. \end{aligned}$$

But setting $r = \xi$ and $\theta = \pi$ in (7.4), we find that the value of $J(z, \xi)$ at $z = 0$ is given by

$$J(0, \xi) = i\xi^{\alpha-1} \int_{-\pi}^\pi (1 + e^{i\varphi})^{\frac{3\alpha}{2}-2} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi, \quad \xi > 0.$$

It follows that

$$-\operatorname{Re} \int_{S_r} \mathcal{M}(z, \xi) dz = -\left(\int_{-\pi}^\pi (1 + e^{i\varphi})^{\frac{3\alpha}{2}-2} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi \right) \left(\int_0^\pi \sin^{\alpha-2} \varphi d\varphi \right).$$

Since

$$\int_{-\pi}^\pi (1 + e^{i\varphi})^{\frac{3\alpha}{2}-2} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi = \frac{2\pi\Gamma(\frac{3\alpha}{2}-1)}{\Gamma(\frac{\alpha}{2})\Gamma(\alpha)}, \quad \int_0^\pi \sin^{\alpha-2} \varphi d\varphi = \frac{\sqrt{\pi}\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})},$$

Equation (7.32) follows. $\square$

Remark 7.12. The proof of Lemma 7.11 shows that the constant c_α can be alternatively expressed as

$$c_\alpha = \int_{-\infty}^\infty \operatorname{Re} \mathcal{M}(x, y, \xi) dx,$$

where the right-hand side is independent of the choice of $y > 0$ and $\xi > 0$.

In view of Lemma 7.9, the next result completes the proof of Proposition 5.2.

Lemma 7.13. *We have*

$$|P(z, \xi)| \leq C(\arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (7.33a)$$

$$|P(z, \xi) - 1| \leq C(\pi - \arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0. \quad (7.33b)$$

Proof. Using that $P(x, \xi) = 0$ for $x > 0$, we can write

$$P(re^{i\theta}, \xi) = -\frac{r}{c_\alpha} \operatorname{Re} \int_0^\theta \mathcal{M}(re^{i\varphi}, \xi) i e^{i\varphi} d\varphi, \quad r > 0, \theta \in (0, \pi), \xi > 0.$$

Estimate (7.19a) now yields

$$\begin{aligned} |P(re^{i\theta}, \xi)| &\leq Cr \int_0^\theta |\mathcal{M}(re^{i\varphi}, \xi)| d\varphi \leq C \int_0^\theta (\sin^{\alpha-2} \varphi) d\varphi \\ &\leq C\theta^{\alpha-1}, \quad r > 0, \theta \in (0, \pi), \xi > 0, \end{aligned}$$

which is (7.33a). Similarly, since $P(x, \xi) = 1$ for $x < 0$, we can write

$$P(re^{i\theta}, \xi) = 1 + \frac{r}{c_\alpha} \operatorname{Re} \int_\theta^\pi \mathcal{M}(re^{i\varphi}, \xi) i e^{i\varphi} d\varphi, \quad r > 0, \theta \in (0, \pi), \xi > 0.$$

Estimate (7.19a) now yields

$$\begin{aligned} |P(re^{i\theta}, \xi) - 1| &\leq Cr \int_\theta^\pi |\mathcal{M}(re^{i\varphi}, \xi)| d\varphi \leq C \int_\theta^\pi (\sin^{\alpha-2} \varphi) d\varphi \\ &\leq C(\pi - \theta)^{\alpha-1}, \quad r > 0, \theta \in (0, \pi), \xi > 0, \end{aligned}$$

which is (7.33b). $\square$

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