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Computing Asymptotic Invariants with the Ricci Tensor on Asymptotically Flat and Asymptotically Hyperbolic Manifolds

Marc Herzlich

Abstract. We prove in a simple and coordinate-free way the equivalence between the classical definitions of the mass or of the center of mass of an asymptotically flat manifold and their alternative definitions depending on the Ricci tensor and conformal Killing fields. This enables us to prove an analogous statement in the asymptotically hyperbolic case.

Introduction

Mass is the most fundamental invariant of asymptotically flat manifolds. Originally defined in general relativity, it has since played an important role in Riemannian geometric issues. Other interesting invariants, still motivated by physics, include the energy momentum, the angular momentum, and the center of mass (which will be of interest in this note). Moreover, they have been extended to other types of asymptotic behaviours such as asymptotically hyperbolic manifolds.

Two difficulties occur when handling the mass of an asymptotically flat or hyperbolic manifold (or any of its companion invariants): it is defined as a limit of an integral expression on larger and larger spheres, and it depends on the first derivatives of the metric tensor written in a special chart where the metric coefficients are asymptotic to those of the model (flat, hyperbolic) metric at infinity.

It seems unavoidable that a limiting process is involved in the definitions. But finding expressions that do not depend on the first derivatives but on rather more geometric quantities is an old question that has attracted the attention of many authors. It was suggested by Ashtekar and Hansen [1] (see

also Chruściel [6]) that the mass could be rather defined from the Ricci tensor and a conformal Killing field of the Euclidean space. Equality between the two definitions, as well as a similar identity for the center of mass, has then been proved rigorously by Huang using a density theorem [12], cf. previous work by Corvino and Wu [10] for conformally flat manifolds, and by Miao and Tam [13] through a direct computation in coordinates.

The goal of this short note is twofold: we shall provide first a simple proof of the equality between the classical definitions of the asymptotic invariants and their alternative definitions using the Ricci tensor. Although similar in spirit to Miao–Tam [13], our approach completely avoids computations in coordinates. Moreover, it clearly explains why the equality should hold, by connecting it to a natural integration by parts formula related to the contracted Bianchi identity. A nice corollary of our proof is that it can be naturally extended to other settings where asymptotic invariants have been defined. As an example of this feature, we provide an analogue of our results in the asymptotically hyperbolic setting.

1. Basic Facts

We begin by recalling the classical definitions of the mass and the center of mass of an asymptotically flat manifold, together with their alternative definitions involving the Ricci tensor. In all that follows, the dimension n of the manifolds considered will be taken to be at least 3. We shall restrict ourselves to manifolds with only one end, but the definitions can be straightforwardly extended to the general case.

Definition 1.1. An asymptotically flat manifold is a complete Riemannian manifold (M, g) such that there exists a diffeomorphism Φ (called a chart at infinity) from the complement of a compact set in M into the complement of a ball in $\mathbb{R}^n$, such that, in these coordinates and for some $\tau > 0$,

$$|g_{ij} - \delta_{ij}| = O(r^{-\tau}), \quad |\partial_k g_{ij}| = O(r^{-\tau-1}), \quad |\partial_k \partial_\ell g_{ij}| = O(r^{-\tau-2}),$$

where $r = |x|$ is the Euclidean radius in $\mathbb{R}^n$.

Definition 1.2. If $\tau > \frac{n-2}{2}$ and the scalar curvature of g is integrable, the quantity

$$m(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} (-\delta^e g - d\operatorname{tr}_e g)(\nu^e) \operatorname{dvol}^e_{s_r} \qquad (1.1)$$

exists (where e refers to the Euclidean metric in the given chart at infinity, δ is the divergence defined as the adjoint of the exterior derivative, ν^e denotes the field of Euclidean outer unit normals to the coordinate spheres S_r, and ω_{n-1} is the volume of the unit round sphere of dimension $n-1$) and is independent of the chart chosen around infinity. It is called the mass of the asymptotically flat manifold (M, g).

Definition 1.3. If $\tau > \frac{n-2}{2}$, the scalar curvature Scal^g of g is integrable, $m(g) \neq 0$, and the following so-called Regge–Teitelboim (RT) conditions are satisfied:

$$|g_{ij}^{\mathrm{odd}}| = O(r^{-\tau-1}), \quad |\partial_k\left(g_{ij}^{\mathrm{odd}}\right)| = O(r^{-\tau-2}), \quad (\mathrm{Scal}^g)^{\mathrm{odd}} = O(r^{-2\tau-2})$$

(where $\cdot^{\mathrm{odd}}$ denotes the odd part of a function on the chart at infinity), the quantity

$$c^\alpha(g) = \frac{1}{2(n-1)\omega_{n-1}m(g)} \lim_{r\to\infty} \int_{S_r} \left[x^\alpha(-\delta^e g - \mathrm{d}\,\mathrm{tr}_e\, g) \right.$$

$$\left. - (g-e)(\partial_\alpha, \cdot) + \mathrm{tr}_e(g-e)\, dx^\alpha \right](\nu)\, \mathrm{dvol}_{s_r}^e$$

exists for each α in $\{1, ..., n\}$. Moreover, the vector $\mathbf{C}(g) = (c^1(g), \ldots, c^n(g))$ is independent of the chart chosen around infinity, up to the action of rigid Euclidean isometries. It is called the center of mass of the asymptotically flat manifold (M, g).

The normalization factors may seem somewhat arbitrary in the previous two definitions: they, however, show up naturally if one wants these invariants to be equal to the usual parameters of the standard spacelike slices of the Schwarzschild metrics (in any dimension).

Existence and invariance of the mass have been proved by Bartnik [2] and Chruściel [5]. The center of mass has been introduced by Regge and Teitelboim [15,16], and Beig and Ó Murchadha [3], see also the more recent works of Corvino and Schoen [8,9].

We shall recall here the approach towards existence and well-definedness of these invariants due to Michel [14]. Let g and b be two metrics on a complete manifold M, the latter one being considered as a background metric, hence the notation. Let also $\mathcal{F}^g$ (resp. $\mathcal{F}^b$) be a (scalar) polynomial invariant in the curvature tensor and its subsequent derivatives, V be a function, and $(M_r)_{r \geqslant 0}$ be an exhaustion of M by compact subsets, whose boundaries will be denoted by S_r (later taken as large coordinate spheres in a chart at infinity). One then may compute:

$$\int_{M_r} V\left(\mathcal{F}^g - \mathcal{F}^b\right)\mathrm{dvol}^b = \int_{M_r} V\,(D\mathcal{F})_b(g-b)\,\mathrm{dvol}^b + \int_{M_r} V\,\mathcal{Q}(b,g)\,\mathrm{dvol}^b$$

where $\mathcal{Q}$ denotes the (quadratic) remainder term in the Taylor formula for the functional $\mathcal{F}$. Integrating the linear term by parts leads to:

$$\int_{M_r} V\left(\mathcal{F}^g - \mathcal{F}^b\right)\mathrm{dvol}^b = \int_{M_r} \langle (D\mathcal{F})_b^* V,\, g-b\rangle\,\mathrm{dvol}^b + \int_{S_r} \mathbb{U}(V,g,b)$$

$$+ \int_{M_r} V\,\mathcal{Q}(b,g)\,\mathrm{dvol}^b \qquad (1.2)$$

(where we include here the volume element in the definition of $\mathbb{U}$). This formula shows that

$$\mathcal{H}_{\mathcal{F}}(V, g, b) = \lim_{r \to \infty} \int_{S_r} \mathbb{U}(V, g, b)$$

exists if the following two natural conditions are satisfied:

(1) The metric g is asymptotic to b so that $V\left(\mathcal{F}^g - \mathcal{F}^b\right)$ and $V\,\mathcal{Q}(b,g)$ are integrable;

(2) V belongs to the kernel of $(D\mathcal{F})_b^*$ (the adjoint of the first variation operator of the Riemannian functional $\mathcal{F}$).

Moreover, Michel proves in [14] that $\mathcal{H}_{\mathcal{F}}(V, g, b)$ is an asymptotic invariant, independent of the choice of chart at infinity, if

(3) The background geometry b is rigid enough, in the sense that any two 'charts at infinity' where g is asymptotic to b differ by a diffeomorphism whose leading term is an isometry of b;

(4) $\mathcal{F}^b$ is a constant function.

This last result is a consequence of the diffeomorphism invariance of the integrated scalar invariant $\mathcal{F}^g$.

If one chooses $\mathcal{F}^g = \operatorname{Scal}^g$ on an asymptotically flat manifold (hence $b = e$, the Euclidean metric), one has

$$(D\operatorname{Scal})_e^* V = \operatorname{Hess}^e V + (\Delta^e V)\, e,$$

where Hess^e denotes the Hessian of a function and Δ^e is the Euclidean Laplace operator (defined here as the opposite of the trace of the Hessian, so that it has non-negative spectrum). Its kernel then consists of affine functions. We now let $V \equiv 1$. The scalar curvature of g is integrable and $\tau > \frac{n-2}{2}$ in Definition 1.1, hence

$$\mathcal{H}_{\operatorname{Scal}}(1, g, e) = \lim_{r \to \infty} \int_{M_r} \operatorname{Scal}^g \, \mathrm{dvol}^e \; - \; \lim_{r \to \infty} \int_{M_r} \mathcal{Q}(e, g) \, \mathrm{dvol}^e \qquad (1.3)$$

makes sense since integrability of Scal^g yields convergence of the first term, whereas the integrand in the second term is a combination of terms in $(g-b)\partial^2 g$ and $g^{-1}(\partial g)^2$ which are integrable due to the value of τ. Moreover, an easy computation shows that

$$\mathcal{H}_{\operatorname{Scal}}(1, g, e) = \lim_{r \to \infty} \int_{S_r} \mathbb{U}(1, g, e) = 2(n-1)\omega_{n-1}\, m(g)$$

where $m(g)$ is the classical definition of the mass given in Definition 1.2. Moreover, Michel's analysis recalled above [14] shows that the mass is an asymptotic invariant, independent of the choice of chart at infinity, since Euclidean geometry is a rigid background geometry [2,5] and $\operatorname{Scal}^e \equiv 0$.

If one takes $V = V^{(\alpha)} = x^\alpha$ (the α-th coordinate function in the chart at infinity, for any α in $\{1, ..., n\}$), the same procedure now yields the classical definition of the α-th coordinate of the center of mass, i.e.

$$\mathcal{H}_{\text{Scal}}(V^{(\alpha)}, g, e) = \lim_{r \to \infty} \int_{S_r} \mathbb{U}(V^{(\alpha)}, g, e) = 2(n-1)\omega_{n-1}\, m(g)\, c^\alpha(g)$$

for any $\alpha \in \{1, ..., n\}$. Under the RT conditions, these converge as well and the vector $\mathbf{C}(g)$ is again an asymptotic invariant.

We now recall the alternative definitions of these asymptotic invariants via the Ricci tensor, following the suggestions of Ashtekar and Hansen, Chruściel, etc.:

Definition 1.4. Let X be the radial vector field $X = r\partial_r$ in the chosen chart at infinity. Then, we define the Ricci version of the mass of (M, g) by

$$m_R(g) = -\frac{1}{(n-1)(n-2)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \left(\text{Ric}^g - \frac{1}{2} \text{Scal}^g\, g \right)(X, \nu)\, \text{dvol}^g \quad (1.4)$$

whenever this limit is convergent. For α in $\{1, \ldots, n\}$, let $X^{(\alpha)}$ be the Euclidean conformal Killing field $X^{(\alpha)} = r^2 \partial_\alpha - 2x^\alpha x^i \partial_i$ and define the Ricci version of the center of mass:

$$
\begin{aligned}
c_R^\alpha(g) = \ &\frac{1}{2(n-1)(n-2)\omega_{n-1}m(g)} \\
&\lim_{r \to \infty} \int_{S_r} \left(\text{Ric}^g - \frac{1}{2} \text{Scal}^g\, g \right)(X^{(\alpha)}, \nu)\, \text{dvol}^g \quad (1.5)
\end{aligned}
$$

whenever this limit is convergent. We will call this vector $\mathbf{C}_R(g) = (c_R^1(g), \ldots, c_R^n(g))$.

Notice that these definitions of the asymptotic invariants rely on the Einstein tensor, which seems to be consistent with the physical motivation.

2. Equality in the Asymptotically Flat Case

In this section, we will prove the equality between the classical expressions $m(g)$ and $\mathbf{C}(g)$ of the mass and the center of mass and their Ricci versions $m_R(g)$ and $\mathbf{C}_R(g)$. The proof we will give relies on Michel's approach described above together with two elementary computations in Riemannian geometry.

Lemma 2.1 (The integrated Bianchi identity). *Let h be a C^3 Riemannian metric on a smooth compact domain with boundary Ω and X be a conformal Killing field. Then*

$$\int_{\partial\Omega} \left(\text{Ric}^h - \frac{1}{2} \text{Scal}^h\, h \right)(X, \nu)\, \text{dvol}^h_{\partial\Omega} = \frac{n-2}{2n} \int_\Omega \text{Scal}^h \left(\delta^h X \right)\, \text{dvol}^h_\Omega,$$

where ν is the outer unit normal to $\partial\Omega$.

Proof. This equality is a variation of the well-known Pohozaev identity in conformal geometry, as stated by Schoen [17]. Our version has the advantage that the divergence of X appears in the bulk integral (the classical Pohozaev identity is rather concerned with the derivative of the scalar curvature in the

direction of X). The proof being very simple, we will give it here. From the contracted Bianchi identity $\delta^h(\mathrm{Ric}^h - \frac{1}{2}\mathrm{Scal}^h\,h) = 0$, one deduces that

$$\int_{\partial\Omega}\left(\mathrm{Ric}^h - \frac{1}{2}\mathrm{Scal}^h\,h\right)(X,\nu)\,\mathrm{dvol}^h_{\partial\Omega} = \int_{\Omega}\left\langle\mathrm{Ric}^h - \frac{1}{2}\mathrm{Scal}^h\,h, (\delta^h)^*X\right\rangle_h\,\mathrm{dvol}^h_{\Omega}$$

where $(\delta^h)^*$ in the above computation denotes the adjoint of the divergence on vectors, i.e. the symmetric part of the covariant derivative. Since X is conformal Killing, $(\delta^h)^*X = -\frac{1}{n}(\delta^hX)h$ and

$$\int_{\partial\Omega}\left(\mathrm{Ric}^h - \frac{1}{2}\mathrm{Scal}^h\,h\right)(X,\nu)\,\mathrm{dvol}^h_{\partial\Omega}$$
$$= -\frac{1}{n}\int_{\Omega}\mathrm{tr}_h\left(\mathrm{Ric}^h - \frac{1}{2}\mathrm{Scal}^h\,h\right)(\delta^hX)\,\mathrm{dvol}^h_{\Omega}$$
$$= \frac{n-2}{2n}\int_{\Omega}\mathrm{Scal}^h(\delta^hX)\,\mathrm{dvol}^h_{\Omega}$$

and this concludes the proof.   □

Lemma 2.1 provides a link between the integral expression appearing in the Ricci definition of the asymptotic invariants (see (1.4)) and the bulk integral

$$\int_{\Omega}\mathrm{Scal}^h\left(\delta^hX\right)\mathrm{dvol}^h_{\Omega}.$$

This latter quantity also looks like the one used by Michel to derive the definitions of the asymptotic invariants, provided that some connection can be made between divergences of conformal Killing fields and elements in the kernel of the adjoint of the linearized scalar curvature operator. Such a connection stems from our second lemma:

Lemma 2.2. *Let h be a C^3 Riemannian metric and X a conformal Killing field. If h is Einstein with Einstein constant $\lambda(n-1)$, then $V = \delta^hX$ sits in the kernel of $(D\,\mathrm{Scal})^*_h$. More precisely:*

$$\mathrm{Hess}^h\,V = -\lambda\,V\,h. \qquad (2.1)$$

Proof. Recall that $(D\,\mathrm{Scal})^*_h V = \mathrm{Hess}^h\,V + (\Delta^hV)h - V\,\mathrm{Ric}^h$ [4, 1.159(e)], so that its kernel is precisely the set of solutions of (2.1) if $\mathrm{Ric}^h = \lambda(n-1)h$.

Let ϕ_t be the (local) flow of X, which acts by conformal diffeomorphisms, and e^{2u_t} the conformal factor at time $t \geqslant 0$, with $u_0 = 0$. Hence

$$\mathrm{Ric}^{\phi_t^*h} = \lambda(n-1)\,\phi_t^*h,$$

an equality which can be written equivalently as:

$$\mathrm{Ric}^{e^{2u_t}h} = \lambda(n-1)\,e^{2u_t}h$$

since ϕ_t is conformal. From [4, 1.159(d)],

$$\mathrm{Ric}^{e^{2u_t}h} = \mathrm{Ric}^h - (n-2)\left(\mathrm{Hess}^h u_t - du_t \otimes du_t\right) + \left(\Delta^h u_t - (n-2)|du_t|_h^2\right) h,$$

from which one deduces that

$$-(n-2)\left(\mathrm{Hess}^h u_t - du_t \otimes du_t\right) + \left(\Delta^h u_t - (n-2)|du_t|_h^2\right) h$$
$$= \lambda(n-1)\left(e^{2u_t} - 1\right) h.$$

We now differentiate at $t = 0$. Denoting by $\dot{u}$ the first variation of u_t, the conformal Killing equation yields the following relation between X and $\dot{u}$: $\delta^h X = -n\,\dot{u}$. Taking into account that $u_0 = 0$, one gets:

$$-(n-2)\,\mathrm{Hess}^h \dot{u} + (\Delta^h \dot{u})\,h = 2(n-1)\lambda\,\dot{u}\,h \qquad (2.2)$$

(note that this equation can also be obtained directly from the conformal Killing equation on X but we prefer the proof above as it underlines the relation with the variations of the Ricci curvature under conformal deformations). Tracing this identity yields $2(n-1)\,\Delta^h \dot{u} = 2n(n-1)\,\lambda\,\dot{u}$, so that $\Delta^h \dot{u} = n\lambda\,\dot{u}$. Inserting this in Eq. (2.2) leads to $\mathrm{Hess}^h \dot{u} = -\lambda\,\dot{u}\,h$, which is the desired expression. $\square$

We now have all the necessary elements to prove the equality between the classical expressions of the asymptotic invariants and their Ricci versions in the asymptotically flat case.

Theorem 2.3. *If (M, g) is a C^3 asymptotically flat manifold with integrable scalar curvature and decay rate $\tau > \frac{n-2}{2}$, then the classical and Ricci definitions of the mass agree: $m(g) = m_R(g)$. If $m(g) \neq 0$ and the RT asymptotic conditions are moreover assumed, the same holds for the center of mass, i.e. $c^\alpha(g) = c_R^\alpha(g)$ for any $\alpha \in \{1, ..., n\}$.*

Proof. We shall give the complete proof for the mass only, the case of the center of mass being entirely similar. Fix a chart at infinity on M. As the mass is defined asymptotically, we may freely replace a compact part in M by a (topological) ball, which we shall decide to be the unit ball $B_0(1)$ in the chart at infinity. The manifold is unchanged outside that compact region. For any $R >> 1$, we define a cutoff function χ_R which vanishes inside the sphere of radius $\frac{R}{2}$, equals 1 outside the sphere of radius $\frac{3R}{4}$ and moreover satisfies

$$|\nabla\chi_R| \leqslant C_1 R^{-1}, \quad |\nabla^2\chi_R| \leqslant C_2 R^{-2}, \quad \text{and} \quad |\nabla^3\chi_R| \leqslant C_3 R^{-3}$$

for some universal constants C_i ($i = 1, 2, 3$) not depending on R. We shall now denote $\chi = \chi_R$ unless some confusion is about to occur. We then define a metric on the annulus $\Omega_R = A(\frac{R}{4}, R)$:

$$h = \chi g + (1 - \chi)e,$$

and we shall also denote by h the complete metric obtained by gluing the Euclidean metric inside the ball $B_0(\frac{R}{4})$ and the original metric g outside the ball $B_0(R)$.

Let now X be a conformal Killing field for the Euclidean metric. Lemma 2.2 tells us that $V = \delta^e X$ sits in the kernel of the adjoint of the

linearized scalar curvature operator, i.e. $(D\,\mathrm{Scal})_e^* V = 0$. We now compute as in Lemma 2.1 over the annulus $\Omega_R = A(\frac{R}{4}, R)$:

$$\int_{S_R} \left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\right)(X, \nu^h) = \int_{\Omega_R} \langle \mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\,, (\delta^h)^* X\rangle\,,$$

where the volume forms and scalar products are all relative to h but have been removed for clarity. (Notice that the boundary contribution at $\frac{R}{4}$ vanishes since h is flat there). We now split $(\delta^h)^* X$ into its trace part $-\frac{1}{n}(\delta^h X)h$ and its tracefree part $(\delta^h)_0^* X$ (where $(\delta^h)_0$ is as above the conformal Killing operator), so that

$$\int_{S_R} \left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\right)(X, \nu^h) = -\frac{1}{n}\int_{\Omega_R} \mathrm{tr}_h\left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\right)\delta^h X$$

$$+ \int_{\Omega_R} \langle \mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\,, (\delta^h)_0^* X\rangle.$$

Hence,

$$\int_{S_R} \left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\right)(X, \nu^h) = \frac{n-2}{2n}\int_{\Omega_R} (\delta^h X)\,\mathrm{Scal}^h$$

$$+ \int_{\Omega_R} \langle \mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h, (\delta^h)_0^* X\rangle. \quad (2.3)$$

We now choose $X = r\partial_r$ (the radial dilation vector field), so that $\delta^e X = -n$ in this case. We can now replace the volume form dvol^h, the divergence δ^h, and the conformal Killing operator $(\delta^h)_0^*$ by their Euclidean counterparts dvol^e, δ^e, and $(\delta^e)_0^*$: indeed, from our asymptotic decay conditions, our choice of cutoff function χ, and the facts that $\tau > \frac{n-2}{2}$ and $|X| = r$, one has for the first term in the right-hand side of (2.3):

$$\int_{\Omega_R} (\delta^h X)\,\mathrm{Scal}^h\,\mathrm{dvol}^h - \int_{\Omega_R} (\delta^e X)\,\mathrm{Scal}^h\,\mathrm{dvol}^e = O\left(R^{n-2\tau-2}\right) = o(1)$$

as R tends to infinity (note that the second term in the left-hand side does not tend alone to zero at infinity as the scalar curvature of h may not be uniformly integrable). As $(\delta^e)_0^* X = 0$, the last term in (2.3) can be treated in the same way and it is $o(1)$, too. One concludes that, in the case X is the radial field,

$$\int_{S_R} \left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h\right)(X, \nu^h)\,\mathrm{dvol}_{S_R}^e = \frac{n-2}{2n}\int_{\Omega_R} (\delta^e X)\,\mathrm{Scal}^h\,\mathrm{dvol}^e + o(1).$$

$$(2.4)$$

We now argue as in Michel's analysis (see Eq. (1.2)) but over the annulus Ω_R:

$$
\int_{\Omega_R} (\delta^e X)\,\mathrm{Scal}^h\,\mathrm{dvol}^e \;=\; \int_{S_R} \mathbb{U}(\delta^e X, g, e) + \int_{S_{\frac{R}{4}}} \mathbb{U}(\delta^e X, h, e)
$$

$$
+ \int_{\Omega_R} (\delta^e X)\mathcal{Q}(e, h)\,\mathrm{dvol}^e .
$$

But the boundary contribution at $r = \frac{R}{4}$ vanishes since $h = e$ there and, moreover, taking into account $\delta^e X = -n$, the assumptions on χ, our asymptotic decay conditions, and $\tau > \frac{n-2}{2}$, the integral containing the $\mathcal{Q}$-term tends to 0 when R goes to infinity, for the very same reason that made it integrable in Michel's analysis. Thus, one gets eventually:

$$
\frac{1}{2(n-1)\omega_{n-1}} \int_{S_R} \left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h \right)(r\partial_r, \nu^h)\,\mathrm{dvol}_{S_r} = \frac{2-n}{2}\,m(g) + o(1),
$$

and this proves the expected result.

If one now chooses $X = X^{(\alpha)} = r^2\partial_\alpha - 2x^\alpha x^i \partial_i$, i.e. X is the essential conformal Killing field of $\mathbb{R}^n$ obtained by conjugating a translation by the inversion map, one has $\delta^e X^{(\alpha)} = 2nx^\alpha = 2nV^{(\alpha)}$ and one can use the same argument. Some careful bookkeeping shows that all appropriate terms are $o(1)$ due to the Regge–Teitelboim conditions and one concludes that

$$
\frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \left(\mathrm{Ric}^h - \frac{1}{2}\,\mathrm{Scal}^h\,h \right)(X^{(\alpha)}, \nu^h)\,\mathrm{dvol}_{S_r}
$$

$$
= (n-2)\,m(g)\,c^\alpha(g) + o(1)
$$

as expected. $\qquad\square$

3. Asymptotically Hyperbolic Manifolds

We now show that the same approach can be used to get analogous expressions in other settings where asymptotic invariants have been defined. Looking back at what has been done in the previous sections, we see that the proofs relied on the following two crucial facts:

(1) The definition of the invariant should come (through Michel's analysis) from a Riemannian functional, which should in turn be related with some version of the Bianchi identity;

(2) There should exist some link between conformal Killing vectors and functions in the kernel of the adjoint linearized operator of the relevant Riemannian functional.

In the presence of these two features, the equality between the classical definition of the invariants (à la Michel) and their Ricci versions follows almost immediately, as the estimates necessary to cancel out all irrelevant terms are exactly the same as those used in the definition of the invariants, see for instance Eq. (2.4) and the arguments surrounding it.

We insist on the fact that this idea is completely general and might be applied to a number of different geometric settings. As an example of this, we shall study the case of asymptotically hyperbolic manifolds. The mass was defined there by Chruściel and the author [7] and independently by Wang [18], see [11] for a comparison.

Definition 3.1. An asymptotically hyperbolic manifold (with one end) is a complete Riemannian manifold (M, g) such that there exists a diffeomorphism Φ (chart at infinity) from the complement of a compact set in M into the complement of a ball in $\mathbb{H}^n$ (equipped with the background hyperbolic metric $b = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}}$), satisfying the following condition: if $(\epsilon_0 = \partial_r, \epsilon_1, ..., \epsilon_n)$ is a b-orthonormal basis, and $g_{ij} = g(\epsilon_i, \epsilon_j)$, there exists some $\tau > 0$ such that,

$$|g_{ij} - \delta_{ij}| = O(e^{-\tau r}), \quad |\epsilon_k \cdot g_{ij}| = O(e^{-\tau r}), \quad |\epsilon_k \cdot \epsilon_\ell \cdot g_{ij}| = O(e^{-\tau r}).$$

Definition 3.2. If $\tau > \frac{n}{2}$ and $(\mathrm{Scal}^g + n(n-1))$ is integrable in $L^1(e^r \mathrm{dvol}_b)$, the linear map $\mathbf{M}(g)$ defined by

$$V \longmapsto \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to \infty} \int_{S_r} \left[V \left(-\delta^b g - d \, \mathrm{tr}_b \, g \right) \right.$$
$$\left. + \mathrm{tr}_b (g-b) dV - (g-b)(\nabla^b V, \cdot) \right] (\nu) \, \mathrm{dvol}_{s_r}$$

is well defined on the kernel of $(D \, \mathrm{Scal})^*_b$ and is independent of the chart at infinity. It is called the mass of the asymptotically hyperbolic manifold (M, g).

As in the asymptotically flat case, the normalization factor comes from the computation for a reference family of metrics, which are the generalized Kottler metrics in the asymptotically hyperbolic case.

Existence and invariance of the mass can be proven using Michel's approach [14]. The kernel $\mathcal{K} = \ker(D \, \mathrm{Scal})^*_b$ is the space of functions V solutions of

$$\mathrm{Hess}^b \, V = V \, b.$$

It is $(n+1)$-dimensional and is generated, in the coordinates above, by the functions $V^{(0)} = \cosh r$, $V^{(\alpha)} = x^\alpha \sinh r$ (for $\alpha \in \{1, \ldots, n\}$), where $(x^\alpha) = (x^1, \ldots, x^n)$ are the Euclidean coordinates on the unit sphere induced by the standard embedding $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. (An alternative definition of $\mathcal{K}$ is provided as follows: when the hyperbolic space is seen as the upper hyperboloid in Minkowski spacetime, then $\mathcal{K}$ is the set of functions generated by the restrictions of the Minkowskian coordinate functions to the hyperboloid.)

Contrarily to the asymptotically flat case, the center of mass is already included here and does not need to be defined independently. Indeed, the space $\mathcal{K}$ is an irreducible representation of $O_0(n, 1)$ (the isometry group of the hyperbolic space), so that all functions V contribute to a single vector-valued invariant $\mathbf{M}(g)$. In the asymptotically flat case, the kernel $\mathcal{K}$ splits into a trivial 1-dimensional representation (the constant functions) which gives rise to the mass, and the standard representation of $\mathbb{R}^n \rtimes O(n)$ on $\mathbb{R}^n$ (the linear functions), which gives birth to the (vector-valued) center of mass.

The hyperbolic conformal Killing fields are the same as those of the Euclidean space, but their divergences must now be explicited with respect to the hyperbolic metric. In the ball model of the hyperbolic space, one computes that $\delta^b X^{(0)} = -nV^{(0)}$ for the radial dilation vector field $X^{(0)}$, whereas $\delta^b X^{(\alpha)} = -nV^{(\alpha)}$ for the (inverted) translation fields.

We can now argue as above, but starting with the modified Einstein tensor

$$\tilde{G}^g = \mathrm{Ric}^g - \frac{1}{2}\,\mathrm{Scal}^g\, g - \frac{(n-1)(n-2)}{2}\,g.$$

The Bianchi-like formula analogous to that of Lemma 2.1 reads, for any conformal Killing field X,

$$\int\limits_{\partial\Omega} \tilde{G}^g(X,\nu)\,\mathrm{dvol}^g_{\partial\Omega} = \frac{n-2}{2n}\int\limits_{\Omega}\left(\mathrm{Scal}^g + n(n-1)\right)\,\delta^g X\,\mathrm{dvol}^g_\Omega\ ,$$

and we note that the right-hand side is the expected expression to apply Michel's approach for the definition of the mass [7,14]. The sequel of the proof is now completely similar to the one given above. The very same arguments that provide convergence of the mass in Michel's approach show that all irrelevant contributions at infinity cancel out, so that, keeping the same notation as in the previous sections (the only difference being that polynomial decay estimates must be changed to exponential ones),

$$\int\limits_{\Omega_R} (\delta^h X)\left(\mathrm{Scal}^h + n(n-1)\right)\mathrm{dvol}^h$$

$$= \int\limits_{\Omega_R} (\delta^b X)\left(\mathrm{Scal}^h + n(n-1)\right)\mathrm{dvol}^b + o(1)$$

$$= \int\limits_{S_R} \mathbb{U}(\delta^b X, g, b) + o(1).$$

The relation between the divergences of the conformal Killing vectors and the elements in the kernel of the adjoint linearized operator comes again from Lemma 2.2, and one concludes as above with the following alternative definition of the mass involving the Ricci tensor:

Theorem 3.3. *For any $i \in \{0, ..., n\}$,*

$$\mathbf{M}(g)\left[V^{(i)}\right] = -\frac{1}{n}\,\mathbf{M}(g)\left[\delta^b X^{(i)}\right]$$

$$= -\frac{1}{(n-1)(n-2)\omega_{n-1}}\lim_{r\to\infty}\int\limits_{S_r} \tilde{G}^g(X^{(i)},\nu)\,\mathrm{dvol}_{S_r}.$$

Acknowledgements

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Marc Herzlich
Institut montpelliérain Alexander Grothendieck
UMR 5149 CNRS-UM
Université de Montpellier
Montpellier, France
e-mail: `marc.herzlich@umontpellier.fr`