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The Infrared Problem in QED: A Lesson from a Model with Coulomb Interaction and Realistic Photon Emission

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Abstract. The scattering of photons and heavy classical Coulomb interacting particles, with realistic particle–photon interaction (without particle recoil) is studied adopting the Koopman formulation for the particles. The model is translation invariant and allows for a complete control of the Dollard strategy devised by Kulish–Faddeev and Rohrlich (KFR) for QED: in the adiabatic formulation, the Møller operators exist as strong limits and interpolate between the dynamics and a non-free asymptotic dynamics, which is a unitary group; the S-matrix is non-trivial and exhibits the factorization of all the infrared divergences. The implications of the KFR strategy on the open questions of the LSZ asymptotic limits in QED are derived in the field theory version of the model, with the charged particles described by second quantized fields: i) asymptotic limits of the charged fields, $\Psi_{\mathrm{out/in}}(x)$, are obtained as strong limits of modified LSZ formulas, with corrections given by a Coulomb phase operator and an exponential of the photon field; ii) free asymptotic electromagnetic fields, $B_{\mathrm{out/in}}(x)$, are given by the massless LSZ formula, as in Buchholz approach; iii) the asymptotic field algebras are a semidirect product of the canonical algebras generated by $B_{\mathrm{out/in}}$, $\Psi_{\mathrm{out/in}}$; iv) on the asymptotic spaces, the Hamiltonian is the sum of the free (commuting) Hamiltonians of $B_{\mathrm{out/in}}$, $\Psi_{\mathrm{out/in}}$ and the same holds for the generators of the space translations.

1. Introduction

For the solution of the infrared problem in quantum electrodynamics (QED) two strategies have been adopted:
A) Exploit the fact that all experiments involve limitations on the detection of soft photons.
In the perturbative approach, this has led to the introduction of an infrared cutoff ΔE corresponding to the energy resolution of the photon detectors [26]. A Lorentz invariant formulation of such a program has been advocated by

Buchholz [6] and further developed in [4,8]. A similar philosophy is at the basis of Steinmann's notion of particle detection [25].
B) Pursue the program of the construction of an S-matrix by quantum field theory (QFT) methods.
A crucial difficulty in this direction is the infraparticle spectrum of the charged particles, which seems to preclude the existence of asymptotic Lehmann–Symanzik–Zimmermann (LSZ), or Haag–Ruelle limits of the charged fields. In fact, as a consequence of Gauss law, the energy–momentum spectrum of the charged states cannot have a sharp mass [7], so that Dybalski's extension of the Haag–Ruelle theory [13] does not apply; it has been argued that such a mass spectrum can be explained in terms of the infinite asymptotic photon content of the charged states [14,15].

Chung has proposed [10] that the infrared divergencies disappear in perturbative QED if a proper description of the asymptotic states is adopted, based on non-Fock coherent factors for the electromagnetic field, indexed by the asymptotic momenta of the charged particles. The effectiveness of the Chung ansatz for the cancelation of the infrared divergencies has been controlled by Kibble [20]; however, Chung ansatz raises many problems in relation with general structures in QFT:
i) the space–time translation covariance of Chung's asymptotic charged states (depending on non-translation invariant coherent factors)
ii) the possibility of obtaining the Chung charged states from appropriate asymptotic charged fields
iii) the space–time covariance and canonical structure of such fields
iv) the existence of modified LSZ formulas for them.

A simple source of information has been provided by infrared models in which a semi-classical treatment is made possible by the use of external currents or of a dipole approximation [1,2,20]. Historically, such models have played a crucial role for supporting the Chung picture; however, the absence of a dynamical description of charged particles and the dipole approximation in the photon interaction prevent space–time translation invariance and do not allow for the discussion of i)–iv).

The action of the space–time translations on asymptotic states of Chung type has been discussed in [14,15] in terms of a splitting of the total Hamiltonian H and momentum P as a sum, $H = H_{0\,\mathrm{ph}} + H_{\mathrm{charge}}$, of a free photon term and a "particle" contribution, with definite mass. In this picture, the infraparticle spectrum is explained by the infinite photon content of the charged states.

A non-perturbative control of the validity of Chung ansatz has been obtained in non-relativistic QED [9]. The main difficulty in non-relativistic models is given by the electromagnetic corrections to the energy–momentum relation $E(p)$ of the asymptotic charged particles, a problem which plays a minor role in QED, where, by the Lorentz invariance of the energy–momentum spectrum (which persists for charged states [3] in spite of the non implementability of the Lorentz symmetry), only a mass renormalization is admitted. An LSZ modified formula for one charged particle state has been derived in [9]; the discussion of asymptotic particle observables has been restricted to the velocity and Coulomb effects have not been discussed.

A concrete approach to the complete construction of an S-matrix in QED in terms of Møller operators has been discussed by Kulish and Faddeev [21] and by Rohrlich [19], who realized the crucial role of Dollard strategy of using a modified large time dynamics [11,12], rather than the standard free dynamics.

Dollard strategy has been exploited and rigorously controlled only for non-relativistic Coulomb scattering and its extension to QED faces substantial problems, due to the infinite photon emission, the related persistent effects and the lack of covariance under space–time translations of the photon interaction in Dollard's dynamics (beyond the choice of the initial time in Dollard treatment of Coulomb interaction).

On the basis of the Kulish–Faddeev identification of a Dollard reference dynamics in QED, Rohrlich has proposed [19] to use it with the same role as the free dynamics in the standard interaction picture. In this way, he introduces (non-free) fields and states associated with a "modified interaction picture" and describes the scattering in terms of transitions between such states; he also argues that, for large times, such fields coincide with the asymptotic fields proposed by Zwanziger [27]. The time dependence of such asymptotic fields, reflecting asymptotic effects of photon emission and Coulomb distorsions, does not satisfy the group property and space–time covariance is problematic.

As before, soluble infrared models do not provide instructive information on such issues: in particular, the Pauli–Fierz–Blanchard model [1] leads to a Dollard dynamics which is a group and substantially includes the full photon interaction.

The aim of this work is to shed light on the above problems by discussing a model describing Coulomb scattering with photon emission by N heavy charged particles, treated as classical particles, with no recoil induced by the photon interaction. Even if the particle dynamics is not affected by the photon interaction, the model is not explicitely solvable, the photon dynamics depends on the non-trivial particle trajectories and detailed estimates are needed for the control of infrared effects arising from Coulomb asymptotic distorsions.

Such distorsions are of the same form ($\sim\log t$) as those induced by photon recoil, which should not, therefore, lead to substantial changes for the control of the asymptotic dynamics. Moreover, the results of [9] confirm that photon recoil does not change the (coherent) characterization of asymptotic photons in terms of the asymptotic charged particle velocity.

As a consequence of the no-recoil approximation, there is no scattering in the one charged particle sector; however, in the many particle sectors the S-matrix is non-trivial and for its construction the Coulomb asymptotic distortions play a substantial role. The overall picture qualifies as a realistic description of scattering of heavy charged particles, with strong effects for particles of high charge.

The fact that the interaction is translation invariant and the electromagnetic current is not pre-assigned is a distinctive feature of the model with respect to the external current and Pauli–Fierz–Blanchard models. In fact, the model allows for the discussion of the asymptotic limit of charged particle variables and charged fields, with non-trivial electromagnetic effects even in the one charged particle sector and a complete control of Coulomb effects. In this way, asymptotic field algebras are constructed, with a full control of the problems raised in i)–iv).

The model is defined by the Hamiltonian

$$H = h_0 + h_I + H_0 + H_{I,\,r}\,,$$

where h_0 is the (non-relativistic or relativistic) free Hamiltonian of N classical particles in the Koopman formulation, h_I the Coulomb interaction, H_0 the free photon Hamiltonian, $H_{I,\,r}$ the (renormalized) particle–photon interaction. H is invariant under space translations, with generator denoted by $\mathcal{P}$.

The first aim is to check the effectiveness of the Dollard–Kulish–Faddeev–Rohrlich strategy for the mathematical control of the asymptotic limit. In accord with the general approach and analysis of [22], we obtain, for the N charged particle channel (see Theorem 5.4):

1) *Existence of the Møller operators.* The Møller operators $\Omega_\pm$ are obtained by introducing an adiabatic switching, $e^{-\varepsilon|t|}$, of the electromagnetic coupling, and a Dollard reference dynamics $U^\varepsilon_D(t)$,

$$\Omega_\pm = \lim_{\varepsilon\to 0,\, t\to\pm\infty} U^{\varepsilon\,*}(t)\, U^\varepsilon_D(t). \tag{1.1}$$

2) *The group of asymptotic space–time translations.* There is a unique family of unitary operators $U_{\rm as}(a,t)$, $a \in \mathbb{R}^3$, $t \in \mathbb{R}$, satisfying the interpolation formulas, Eq. (5.20), and therefore the group property and strong continuity, given by

$$U_{\rm as}(a,s) \equiv U_+(a,s) = U_-(a,s) = \lim_{\varepsilon\to 0\,,\, t\to\pm\infty} U^{\varepsilon\,*}_D(t)\, U_0(a,s)\, U^\varepsilon_D(t)\,, \tag{1.2}$$

with $U_0(a,t)$ the free space–time translations. The corresponding generators are

$$H_{\rm as} = h_0 + \alpha_{\rm as}(H_0(a^*,a))\;,\;\; \mathcal{P}_{\rm as} = -i\sum_i^N \partial/\partial q_i + \alpha_{\rm as}(P_{\rm ph}(a^*,a))\,,$$

with $P_{\rm ph}$ the photon momentum and $\alpha_{\rm as}$ the standard non-Fock coherent shift, $a^* \to a^* + \sum_i J(p_i)$, associated to the momenta p_i of the charged particles, Eqs. (4.17) and (5.18). $U_{\rm as}$ is determined by U_D, but the two notions are basically different, contrary to the discussion of the asymptotic fields adopted in Refs. [19,27].

3) The S-*matrix*, $S = \Omega_+^*\,\Omega_-$ is *invariant under the asymptotic space–time translations* $U_{\rm as}(a,t)$.

4) *The infrared divergences* due to the Coulomb interaction and to the soft photon emission *factorize*, Eq. (5.27).

The second aim of this work is to shed light on the asymptotic limits of the Heisenberg fields, especially on the still open problem of the existence of LSZ limits of charged fields and of their space–time covariance properties.

This can be done in the (second quantized) field theory version of the model, with the introduction of charged fields $\Psi^*(f,t)$, f a test function of the

Koopman variables. In spite of the no-recoil approximation, such charged fields are space–time covariant, with non-trivial and not pre-assigned dynamics; their asymptotic limit exists as a strong limit, providing information which is not available in soluble infrared models [1,2,20] and is only partially given by the analysis of the one charged particle sector in non-relativistic QED [9].

On one side, the Møller operators automatically provide Heisenberg asymptotic fields, defined on the scattering spaces $\mathcal{H}_\pm$, see Proposition 5.1, by

$$\Psi^*_{\rm out/in}(f) = \Omega_\pm\, \Psi^*(f)\, \Omega^*_\pm, \quad a_{\rm out/in}(g) = \Omega_\pm\, a(g)\, \Omega^*_\pm. \tag{1.3}$$

The fields $\Psi^*_{\rm out/in}$ and $a_{\rm out/in}$ obey (equal times) canonical commutation relations, but their (Heisenberg) time evolution is not free, being explicitly given by $H = H_{\rm as}(\Psi_{\rm out/in}, a_{\rm out/in})$. This shows the effectiveness of the Kulish–Faddeev–Rohrlich–Zwanziger strategy also for the construction of asymptotic fields; their dynamics is not free, but no Coulomb distortion appears (in contrast with the properties of the asymptotic fields proposed by Zwanziger and Rohrlich).

For the existence of the asymptotic limit of charged fields, the Dollard corrections are essential; they give rise to *modified LSZ (Haag–Ruelle) formulas*. For the charged Heisenberg fields in the variables $P = -i\partial/\partial q$, p (see Eq. (6.1)), one has

$$\Psi^*_{\rm out/in}(f) = \lim_{\varepsilon\to 0} \lim_{t\to\pm\infty} \int \mathrm{d}P\, \mathrm{d}p\, f_{-t}(P,p)\, \Psi^{\varepsilon\,*}_t(P,p)\, e^{i\rho^\varepsilon_t\,(\chi^\varepsilon_t(P,p))}$$
$$\times \exp -i \int_0^t \mathrm{d}s\, A^\varepsilon_t(\overleftrightarrow{\partial_t}\, D_{t-s} * j^\varepsilon(v(p);s)), \tag{1.4}$$

where A is the electromagnetic potential, $D_t(x)$ the massless commutator function, $*$ denotes the convolution in the space variable x, f_t denotes the free time evolution of f, $\Psi^\varepsilon_t, \rho^\varepsilon_t, A^\varepsilon_t$ the Heisenberg (adiabatically switched) time evolution of the corresponding variables, $j^\varepsilon_\mu(v(p);x,x_0) \equiv e\, v_\mu(p)\tilde\eta(x - vx_0)\, e^{-\varepsilon|x_0|}$, $v_\mu \equiv (1, v)$, a function of p corresponding to the (free) asymptotic particle current (with an ultraviolet cutoff $\eta(k)$), ρ is the charge density, χ is a Coulomb phase.

As derived in general by Buchholz [5] on the basis of locality and of the Huyghens principle, LSZ (Haag–Ruelle) asymptotic limits of the electromagnetic fields exist, without any Dollard correction, and define massless fields. In our case, the ordinary LSZ procedure converges in all sectors and yields massless asymptotic fields $b_{\rm out/in}$; they are related to $a_{\rm out/in}$ by

$$b_{\rm out/in}(g) = a_{\rm out/in}(g) + (\rho_{\rm out/in}(J))(g), \tag{1.5}$$

for any $g(k,\lambda) \in \mathcal{S}(\mathbb{R}^3)$. All the above asymptotic limits exist in the strong operator sense on the scattering space (on an invariant dense domain, for e.m. fields), yielding the usual Haag–Ruelle limits of products of operators.

The fields $\Psi_{\mathrm{out/in}}$ and $b_{\mathrm{out/in}}$ define a *semi-direct product of canonical algebras*, with the non-standard commutation relations ($b^{\#} = b,\, b^*$)

$$[b^{\#}_{\mathrm{out/in}}(k,\lambda),\, \Psi^*_{\mathrm{out/in}}(P,p)] = J(k,\lambda,p)\, \Psi^*_{\mathrm{out/in}}(P,p) \tag{1.6}$$

and $H = h_0(\Psi_{\mathrm{out/in}}) + H_0(b_{\mathrm{out/in}})$ on the scattering space; while $b_{\mathrm{out/in}}$ are free fields, $\Psi_{\mathrm{out/in}}$ are not.

The LSZ (Haag–Ruelle) formula for the charged fields can also be written with the e.m. factor replaced by *the exponential of a string-like integral of the asymptotic photon field*, Eq. (6.24).

Equation (1.4) corresponds to the following LSZ formula for QED, which automatically arises from the Dollard dynamics of the charged fields introduced by Kulish–Faddeev and Rohrlich (KFR), through the same steps as in Sects. 6.1, 6.2a:

$$\psi^*_{\mathrm{out/in}}(f) = \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} \int d^3p\; f_{-t}(p)\, \psi^{\varepsilon\,*}_t(p)\; e^{i\frac{e^2}{4\pi}\,\mathrm{sign}\, t\,\ln|t|\,\int d^3q\,\frac{\rho(q)}{v(p,q)}}$$
$$\times \exp -i\int_0^t \mathrm{d}s\, A^{\varepsilon}_t(\overleftrightarrow{\partial}_t\, D_{t-s} * j^{\varepsilon}(v(p);s)), \tag{1.7}$$

with $v(p,q)$ the Lorentz invariant relative velocity and $j^{\varepsilon}_{\mu}(v(p);s,\mathbf{x}) \equiv\; e\, v_{\mu}(p)\, \delta(\mathbf{x}-\mathbf{v}s)$, see Eqs. (S4-10,11) and (S4-21,22) in [19].

The effectiveness of the KFR strategy in QED has been controlled by Zwanziger [28], with the cancelation of the infrared divergencies in the perturbative expansion of the reduction formulas. Our model gives non-perturbative support to that strategy; it also shows that the same strategy leads to asymptotic fields as (modified) LSZ (HR) strong limits, with a complete control of the ensuing structure and covariance properties of the asymptotic field algebras. The commutation relations of the asymptotic fields, Eq. (1.6), directly follow from the last (e.m. field) term in Eqs. (1.4) and (1.7); their space–time transformations are parametrized by the choice of the space–time origin, implicit in such a term, giving rise to asymptotic charged fields $\psi_{\mathrm{as}}(p,x)$, as in Eqs. (6.20) and (6.21).

In both Eqs. (1.4) and (1.7) the Dollard modifications are parametrized by $v(p)$, p the momentum variable of the interacting Heisenberg field, smeared with f_{-t}; convergence of the LSZ formula implies that $v(p)$ can be identified with the asymptotic "particle" velocity, independently of recoil assumptions.

The use of an adiabatic cutoff, while technically important, does not seem to be essential for the results since, in the model, one may avoid it by adopting a modified (Dollard) correction to the LSZ formula for the charged fields [23].

The lesson for the infrared problem in QED is manifold, briefly:

(i) the Kulish–Faddeev–Rohrlich approach to the infrared problem in QED [19,21], based on Dollard's strategy, allows for a systematic control of the asymptotic limit;

(ii) in particular, it allows for the reexamination of the open problem of the asymptotic condition for charged fields; modified LSZ (HR) formulas can be written, yielding asymptotic limits of charged fields;

(iii) the asymptotic field algebra, generated by a free photon field (given by Buchholz asymptotic limit) and by the asymptotic charged fields, has the structure of a semidirect product, reproducing Chung's ansatz;
(iv) the Hamiltonian is the sum of the free Hamiltonians of the asymptotic fields, Eq. (6.17); the time evolution of the asymptotic charged fields is not free only for the presence of infrared photons, with no residue of the Coulomb interaction.

2. The Model

The model describes N classical charged particles of charge e_i and mass m_i, $i = 1, \ldots, N$, with mutual interaction given by a Coulomb potential $\mathcal{V}$, regularized at the origin, and interacting with (transverse) photons.

The classical particles configurations are described by wave functions on the phase space Γ, $\psi(\mathbf{q}_1, \ldots, \mathbf{q}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N) \in L^2(d^{3N}q, d^{3N}p)$, $|\psi|^2$ representing the density in phase space governed by the Liouville time evolution; the time evolution of ψ is given by the Koopman Hamiltonian

$$h = -i \sum_i \left(v_i \frac{\partial}{\partial q_i} - \frac{\partial \mathcal{V}}{\partial q_i} \frac{\partial}{\partial p_i} \right) \equiv h_0 + h_I, \tag{2.1}$$

$$\mathcal{V}(q) = \sum_{j \neq i} \frac{e_i e_j}{8\pi(|q_i - q_j|^2 + a^2)^{1/2}}. \tag{2.2}$$

We consider both the non-relativistic and the relativistic case, given by

$$v_i(p) \equiv p_i/m, \text{ respectively } v_i(p) \equiv \frac{p_i}{(p_i^2 + m^2)^{1/2}}. \tag{2.3}$$

For simplicity, we have omitted the vector notation for the operators $\mathbf{q}_i$, $\mathbf{p}_i$, $\partial/\partial \mathbf{q}_i$, $\partial/\partial \mathbf{p}_i$ and for their scalar products; this will also be done in the following. $q_t(q,p)$, $p_t(q,p)$ will denote the solutions of the classical equations, with initial data q, p; a is a fixed ultraviolet cutoff.

The Hilbert space $\mathcal{H} = L^2(\Gamma) \times \mathcal{H}_F$ can be identified with the space of L^2 functions $\psi(q,p)$, $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots, p_N)$, taking values in the Fock Hilbert space $\mathcal{H}_F$, where the transverse photons are described by the standard canonical destruction and creation operators $a(k, \lambda)$, $a^*(k, \lambda)$ ($\lambda = \pm 1$ denoting the helicity), with commutation relations $[\, a(k, \lambda),\, a^*(k', \lambda')\,] = \delta(k - k')\, \delta_{\lambda\, \lambda'}$, etc.
The total Hamiltonian H is (always omitting the vector notation and denoting by $\epsilon(k, \lambda)$ the polarization vectors)

$$H = h + H_0 + H_{I,\,r}, \quad H_0 = \sum_{\lambda = \pm 1} \int \mathrm{d}^3k\, |k|\, a^*(k, \lambda)\, a(k, \lambda), \tag{2.4}$$

$$H_{I,\,r} = H_I - \Delta E(p), \quad H_I = a(f(q,p)) + a(f(q,p))^* \equiv H_I(a, a^*, q, p), \tag{2.5}$$

$$f(k, \lambda; q, p) = \frac{1}{(2\pi)^{3/2}} \sum_i \frac{1}{\sqrt{2|k|}} e_i \eta(k)\, e^{ikq_i}\, \epsilon(k, \lambda)\, v_i(p),$$

$$a(f(q,p)) \equiv \sum_{\lambda=\pm 1} \int \mathrm{d}^3k\, a(k,\lambda) f(k,\lambda;q,p), \tag{2.6}$$

where we have introduced a (real rotationally invariant) ultraviolet cutoff $\eta(k)$, $\eta(0) = 1$, with Fourier transform $\tilde{\eta} \in \mathcal{D}(\mathbb{R}^3)$; $\Delta E(p)$ is a C^1 function of the particle momenta, to be determined below (see Sect. 5), playing the role of a mass counter-term, subtracting persistent effects of H_I.

The Hamiltonian H is invariant under space translations, $T(a)$, $a \in \mathbb{R}^3$ and the corresponding generator

$$\mathcal{P} = -i \sum_i \partial/\partial q_i + P_{\mathrm{ph}}, \quad P_{\mathrm{ph}} = \sum_{\lambda=\pm 1} \int \mathrm{d}^3k\, k\, a^*(k,\lambda)\, a(k,\lambda), \tag{2.7}$$

is conserved. $P_c = \sum_i p_i$ is also conserved, but $P_c + P_{\mathrm{ph}} \neq \mathcal{P}$ is not, corresponding to the absence of particle recoil in the photon emission.

The particle total energy

$$E(q,p) = K(p) + \mathcal{V}(q)\,,$$

$K(p)$ the kinetic energy, commutes with H, and is, therefore, a constant of motion; however, $E + H_0 + H_{I,\,r}$ is not.

For the treatment of photon emission, it is important to limit the particle velocities to be smaller than the velocity of light (which is 1 in our units). To this purpose, in the non-relativistic case, we shall restrict our discussion to a suitable subspace $\mathcal{H}_{\mathrm{nr}} \subset \mathcal{H}$.

Assuming that

$$\kappa_0 \equiv \frac{N^2 e_{\max}^2}{4\pi a\, m_{\min}} < \tfrac{1}{2}\ ,$$

it is enough to take

$$\mathcal{H}_{\mathrm{nr}} = P_{\mathrm{nr}}\mathcal{H} \equiv L^2(\Gamma_{\mathrm{nr}}) \times \mathcal{H}_F, \tag{2.8}$$

with P_{nr} the projector on $K \leq K_{\max} \equiv \frac{1}{2}\kappa\, m_{\min}$, with $\kappa < 1 - 2\kappa_0$. In fact, the conservation of E implies

$$v_{i,\,t}^2 \leq (\kappa + 2\kappa_0) < 1, \quad \forall t, \quad \forall i. \tag{2.9}$$

Independent of the infrared problem, the convergence of the Møller operators requires some "time smearing" and for this purpose we adopt an adiabatic regularization given by an adiabatic switching $e^{-\varepsilon|t|}$. It can be combined with the Dollard strategy, as discussed in [22]. It is enough to use it only for the particle–photon interaction; its introduction also for the Coulomb interaction will only be convenient for displaying a complete factorization of the infrared divergences.

As needed in all QFT models with persistent effects, mass counter-terms will be introduced, both in the Hamiltonian and in the Dollard correction to the free dynamics.

In the N charged particle sector, we always consider the N particle channel, corresponding to asymptotic configurations excluding non-trivial charged particle clusters. All the following results hold for all values of the charges and of the masses; for simplicity, in the following, we shall omit the particle indices for charges and masses, putting $e_i = e$, $m_i = m$.

We start with the particle dynamics and scattering, providing the Møller operators in the Koopman formulation and the estimates on the particle trajectories which are needed for the control of photon emission (Sect. 3); for both purposes, the main point is the control of the Coulomb effects (the analysis becoming much simpler, but still instructive, for short-range potentials).

Then we introduce the (renormalized) electromagnetic interaction and a Dollard reference dynamics for the full time evolution, both with an adiabatic regularization, and prove the existence of the corresponding Møller operators (Sect. 4, Proposition 4.1).

In Sect. 5 we discuss the removal of the adiabatic switching, after the specification of the mass counter-term, obtaining the existence of the Møller operators (Sect. 5.1), the existence and characterization of the asymptotic dynamics (Sect. 5.2), the interpolation formulas and the explicit factorization of the infrared divergences in the S-matrix (Theorem 5.4).

In Sect. 6, we discuss the second quantized version of the model and prove the existence of LSZ asymptotic limits of the Heisenberg fields (Sect. 6.2a,b); the asymptotic algebra and its covariance under space–time translations are analyzed in Sect. 6.2c. Finally, in Sect. 6.2d we derive an asymptotic form of the corrections to the standard LSZ formulas for the charged fields.

3. Particle Dynamics and Scattering

For the definition of the Dollard reference dynamics for the particles, it is convenient to introduce the following operators:

$$Q_i \equiv i\partial/\partial p_i, \qquad P_i \equiv -i\partial/\partial q_i, \tag{3.1}$$

which satisfy

$$[\, q_i,\, P_i\,] = i\delta_{i\,j}, \quad [\, p_i,\, Q_i\,] = -i\delta_{i\,j}. \tag{3.2}$$

Then, in a notation covering both the non-relativistic and the relativistic case,

$$h_0 = \sum_i v_i P_i\,, \qquad h_I = \sum_{i\,j;j\neq i} w_{i\,j}(q;a) Q_i \equiv h_I(q,Q;a)\,,$$
$$w_{i\,j}(q;a) = \frac{-e_i e_j\,(q_i - q_j)}{4\pi(|q_i - q_j|^2 + a^2)^{3/2}}. \tag{3.3}$$

The free Heisenberg evolution is

$$q_i(t) = q_i + v_i t, \quad p_i(t) = p_i, \quad Q_i(t) = Q_i + V_i t, \quad P_i(t) = P_i,$$

with ($\alpha, \beta = 1, 2, 3$ the vector components)

$$V_i = (V_i^\alpha(p,P),\, \alpha = 1,2,3), \quad V_i^\alpha(p,P) \equiv \sum_{\beta=1}^{3} \frac{\partial v_i^\beta(p)}{\partial p_i^\alpha} P_i^\beta\,, \tag{3.4}$$

reducing to $V_i = P_i/m$ in the non-relativistic case.

With the same motivations as in Dollard treatment of Coulomb scattering [11,12,22], a reference large time dynamics for particle scattering may be identified by putting $q = vt$, $Q = Vt$ in h_I, where for simplicity we take $a = 0$:

$$h_D(t) \equiv h_0 + h_I(vt, Vt; 0). \tag{3.5}$$

No adiabatic switching is necessary for the particle scattering and for simplicity it will not be introduced until the end of Sect. 5, where it will be used to display the explicit dependence on the "infrared cutoff" ε in the S-matrix.

Proposition 3.1. 1) *The Hamiltonian h is essentially self-adjoint on the domain $C_0^1 \subset L^2(\Gamma)$ of differentiable functions $\psi(q,p)$ of compact support and its exponential $u(t) = e^{-iht}$ leaves C_0^1 invariant;*
2) *the equation*

$$idu_D(t)/dt = h_D(t)\, u_D(t), \qquad u_D(\pm 1) = u_0(\pm 1)\,, \tag{3.6}$$

with $u_0(t) \equiv e^{-ih_0 t}$ and $h_D(t)$ given by Eq. (3.5), has a unique solution for $|t| \geq 1$ leaving C_0^1 invariant, given by

$$u_D(t) = u_0(t) \exp\left(i \frac{e^2}{4\pi} \operatorname{sign} t \, \ln|t| \sum_{i<j} \frac{v_i - v_j}{|v_i - v_j|^3} (V_i - V_j) \right), \tag{3.7}$$

satisfying

$$[\, u_0(s),\, u_D(t)\,] = 0, \quad [\, p,\, u_D(t)\,] = 0, \quad [\, P,\, u_D(t)\,] = 0; \tag{3.8}$$

3) *the following strong limits exist*

$$strong - \lim_{t\to\pm\infty} u(t)^*\, u_D(t) = \omega_\pm; \tag{3.9}$$

moreover,

$$\begin{aligned} &strong - \lim_{t\to\pm\infty} u_D(t)^*\, u_D(t+s) = u_0(s), \\ &u(t)\,\omega_\pm = \omega_\pm u_0(t); \end{aligned} \tag{3.10}$$

4) *let $C_0^1(\delta, K)$ denote the set of $\psi(q,p)$ with support in a compact set K and such that $\psi(q,p) = 0$ if $|v_i - v_j| < \delta$, for some $i \neq j$, then, $\forall \psi \in D_\pm \equiv \omega_\pm D_0^1$, $D_0^1 \equiv \cup_{\delta\, K}\, C_0^1(\delta, K)$ dense in $L^2(\Gamma)$,*

$$||(p_t - p_\pm)\,\psi||_{L^2} = O(|t|^{-1} \ln|t|), \tag{3.11}$$

where $p_\pm$ is defined, on $D_\pm$, by

$$p_\pm\, \omega_\pm = \omega_\pm\, p\,; \tag{3.12}$$

5) *let $\Gamma_\pm$ be the complements of the sets*

$$\{(q,p) : \omega(t)\psi(q,p) \to 0,\, t \to \pm\infty,\, \forall \psi \in D_0^1\}, \quad \omega(t) \equiv u^*(t) u_D(t)\,;$$

then, $\omega_\pm L^2(\Gamma) = L^2(\Gamma_\pm)$ and $(\omega_\pm\psi)(q,p) = \psi(\gamma_\pm(q,p))$, $(q,p) \in \Gamma_\pm$, with $\gamma_\pm : \Gamma_\pm \mapsto \Gamma = L^2(d^{3N}q, d^{3N}p)$ measure preserving transformations. $p_\pm$ are multiplication operators $p_\pm(q,p)$ on $L^2(\Gamma_\pm)$, essentially self-adjoint on $D_\pm$ and, almost everywhere in $\Gamma_\pm$ (with $v_\pm \equiv v(p_\pm)$),

$$p_t(q,p) = p_\pm + \delta p_\pm\, t^{-1} + O(t^{-2}\ln t), \quad \delta p_{\pm\, i} \equiv -\sum_{j\neq i} w_{ij}(v_\pm, 0); \tag{3.13}$$

6) $\omega_\pm K = (K + \mathcal{V})\omega_\pm$ *and, therefore,* $s \equiv \omega_+^* \omega_-$ *commutes with the particle kinetic energy* K*; thus, in the non-relativistic case, it leaves* $\mathcal{H}_{\rm nr}$ *invariant.*

Proof. 1) The solutions of the classical equations $q_t(q, p)$, $p_t(q, p)$ are C^1 functions of q, p and t, actually p_i are uniformly bounded in t, as a consequence of the energy conservation ($E = K + \mathcal{V}(q)$, $\mathcal{V}$ bounded below); hence, they define a one-parameter unitary group

$$u(t)\psi(q, p) \equiv \psi(q_{-t}(q, p), p_{-t}(q, p)) , \tag{3.14}$$

with locally finite propagation speed, leaving C_0^1 invariant. By Stone theorem, its generator is e.s.a on an invariant domain.

2) Since the argument of the exponential in Eq. (3.7) has a dense invariant domain of analytic vectors, given, e.g., by functions $\psi(q,\, p)$ analytic in q and of compact support in p, with $|v_i - v_j| > \delta$, $\forall i \neq j$, the right-hand side of Eq. (3.7) is well defined and

$$\begin{aligned} &u_D(t)\psi(q, p) \equiv \psi(q^D_{-t}, p^D_{-t}) , \\ &\qquad (q_t^{D\alpha})_i \equiv q_i^\alpha + v_i^\alpha t - \frac{e^2}{4\pi} \operatorname{sign} t \ln|t| \sum_{j\neq i} \frac{(v_i - v_j)^\beta}{|v_i - v_j|^3} \frac{\partial v_i^\alpha}{\partial p_i^\beta}, \quad (p_t^D)_i \equiv p_i, \end{aligned} \tag{3.15}$$

leaves C_0^1 invariant and satisfies Eq. (3.6).

By hermiticity of $h_D(t)$ on C_0^1, for any two solutions u_D^1, u_D^2, one has

$$(d/dt)(u_D^1(t)\psi,\, u_D^2(t)\psi) = 0$$

and uniqueness follows.

3) For the existence of $\omega_\pm$, we note that $\forall \psi \in C_0^1(\delta, K)$, using

$$u_D(t)^*\, Q\, u_D(t) = Q + Vt \equiv Q_t^D$$

and Eq. (3.8), one has

$$\begin{aligned} i(d/dt)(u^*(t)\, u_D(t))\psi &= u^*(t)\, [-h_I(q, Q) + h_I(vt, Vt; 0)] u_D(t)\, \psi \\ &= u^*(t)\, u_D(t) [-h_I(q_t^D, Q_t^D) + h_I(vt, Vt; 0)]\, \psi \\ &= u^*(t)\, u_D(t) \sum_{i\neq j} [-w_{ij}(q_t^D)\, Q_i \\ &\qquad + \big(-w_{ij}(q_t^D) + w_{ij}(vt; 0)\big)\, V_i t\,]\psi. \end{aligned}$$

Now, using $||\partial v_i^\beta / \partial p_i^\alpha|| \leq m^{-1}$,

$$|q_{i,t}^D - q_{j,t}^D| \geq \delta |t| - 2(e^2/4\pi m)(N-1)\delta^{-2}\, \ln|t| - \sup_{K,i,j} |q_i - q_j| , \tag{3.16}$$

so that the norm of first term is bounded by $O(|t|^{-2})$. The same holds for the Sup norm, since $u^*(t)\, u_D(t)$ amounts to a change of variables.

The difference in round brackets consists of a term which can be estimated by $O(|t|^{-2}\, \ln|t|)$ and a term of the form $(v_i - v_j)t\, \big(a_t^{-3} - b_t^{-3}\big)$, with, for large $|t|$,

$$a_t = O(|t|), \quad b_t = O(|t|), \quad |a_t - b_t| = O(\ln|t|).$$

Therefore, the term is bounded by $O(|t|^{-2} \ln|t|)$.

In conclusion, one has

$$||(d/dt)u^*(t)\, u_D(t)\psi|| \leq O(|t|^{-2}\, \ln|t|). \tag{3.17}$$

and the same holds for the Sup norm. The first of Eq. (3.10) follows immediately from the explicit form of $u_D(t)$; then the second follows as in Propositions 2.1, 2.2 of [22].

4) The above estimates imply, $\forall \psi \in C_0^1(\delta, K)$, pointwise convergence of $(\omega(t)\psi)(q, p)$, for $|t| \to \infty$, and

$$||(\omega_\pm - \omega(t))\, \psi|| = O(|t|^{-1}\, \ln|t|).$$

Therefore, on $C_0^1(\delta, K)$, where the multiplication operators $p_t = p_t(q, p)$ are bounded uniformly in t, using $p_t = \omega(t)\, p\, \omega^*(t)$, one has

$$\begin{aligned} p_t\, \omega_\pm &= p_t\, (\omega(t) + O(|t|^{-1}\, \ln|t|)) = \omega(t)\, p + O(|t|^{-1}\, \ln|t|) \\ &= \omega_\pm\, p + O(|t|^{-1}\, \ln|t|) = p_\pm\, \omega_\pm + O(|t|^{-1}\, \ln|t|). \end{aligned}$$

5) Lemma A.1 applies with $\omega_t \to \omega(t)$ and $D \to \cup_n C_0^1(1/n, K_n)$, for any sequence K_n covering Γ and $\omega_\pm L^2(\Gamma) = L^2(\Gamma_\pm)$ follows. In the notation of Lemma A.1, $p_t(q, p) = p(\gamma_t(q, p))$, which converges to $p(\gamma_\pm(q, p)) = p_\pm(q, p)$ for almost all $(q, p) \in \Gamma_\pm$; then, by Eq. (3.11), $\forall \psi \in D_0^1$,

$$\begin{aligned} (p_\pm \omega_\pm \psi)(q, p) &= \lim_{t \to \pm\infty} (\omega(t) p \psi)(q, p) = \lim_{t \to \pm\infty} p(\gamma_t(q, p))\psi(\gamma_t(q, p)) \\ &= p_\pm(q, p)(\omega_\pm \psi)(q, p). \end{aligned}$$

Therefore, $p_\pm$ coincides with the multiplication operators $p_\pm(q, p)$ on $D_\pm$, where $p_\pm$ are e.s.a. by Eq. (3.12). Convergence of $p_t(q, p)$ to $p_\pm(q, p)$ on $\Gamma_\pm$ and

$$q_t(q, p) - q = \int_0^t ds\, v(p_s) \tag{3.18}$$

imply, for $(q, p) \in \Gamma_\pm$, $\forall \varepsilon > 0$, for large $|s|$,

$$|q_{i,s}(q, p) - q_{j,s}(q, p)| \geq (1 - \varepsilon)|v_{i\,\pm}(q, p) - v_{j\,\pm}(q, p)|\, |s|, \quad \forall i \neq j$$

with $v_\pm(q, p) \equiv v(p_\pm)$. By Eq. (3.12), $|v_{i\,\pm} - v_{j\,\pm}| \neq 0$ a.e. in $\Gamma_\pm$; hence, a.e. in $\Gamma_\pm$,

$$\begin{aligned} t^2\, dp_{i\,t}/dt &= -t^2 \sum_{j \neq i} w_{i\,j}(q_t, a) = -\sum_{j \neq i} w_{i\,j}(q_t/t, a/t) \\ &\to -\sum_{j \neq i} w_{i\,j}(v_\pm, 0) \equiv \delta p_{i,\pm}. \end{aligned} \tag{3.19}$$

This implies $v(p_s) = v_\pm + O(1/s)$ and, by Eq. (3.18),

$$q_t/t - v_\pm = O(\ln t/t),$$

which gives

$$t^2\, d\, p_{i\,t}/dt = -\delta p_{i\,\pm}(q, p) + O(\ln t/t),$$

a.e. in $\Gamma_\pm$; Eq. (3.13) then follows from

$$p_{i\,\pm} - p_{i\,t} = \int_t^\infty ds\; dp_{i\,s}/ds.$$

6) $u_D^*(t)\,\mathcal{V}(q)\,u_D(t) = \mathcal{V}(q_t^D)$ is bounded uniformly in t and converges strongly to zero by Eq. (3.16). Therefore, since $[K+\mathcal{V},\, u(t)^*\,] = 0$ and by Eq. (3.8), $[K,\, u_D(t)] = 0$,

$$\omega_\pm K = (K+\mathcal{V})\omega_\pm - \lim_{t\to\pm\infty} u(t)^* u_D(t)\mathcal{V}(q_t^D) = (K+\mathcal{V})\omega_\pm .$$

□

In the case of repulsive Coulomb potential, the unitarity of $\omega_\pm$ easily follows for $N=2$; a proof is given in Appendix B.

4. Dynamics and Scattering with an Adiabatic Regularization

We shall construct a regularized dynamics $U^\varepsilon(t)$ and a regularized Dollard reference dynamics $U_D^\varepsilon(t)$, corresponding to the substitution: $q_i \to v_i t$, in $H_{I.\,r}$; we choose an ε regularization corresponding to the replacement $e \to e^{-\varepsilon|t|}e$:

$$H_{I,\,r} \to H^\varepsilon_{I,\,r}(t) \equiv e^{-\varepsilon|t|} H_I - e^{-2\varepsilon|t|}\Delta E(p)\ , \tag{4.1}$$

$$H^\varepsilon_{ID}(t) \equiv e^{-\varepsilon|t|}\, H_I(a, a^*, vt, p) - e^{-2\varepsilon|t|}\Delta E_D(p). \tag{4.2}$$

The counter-term $\Delta E(p)$ is needed for the convergence of the Møller operators for $\varepsilon\to 0$, which is obtained for $\Delta E_D(p) = \Delta E(p)$, Eq. (5.6); it has to cancel the photon contribution to the particle energy, which is of the second order in e (Sect. 5).

Proposition 4.1. 1) *The Hamiltonian H, Eq. (2.4), is essentially self-adjoint on the dense domain D of C_0^1 functions $\psi(q,p)$ with values in $D(H_0)$;*
2) *its adiabatic version for $\varepsilon > 0$*

$$H^\varepsilon(t) = h + H_0 + H^\varepsilon_{I,\,r}(t) \tag{4.3}$$

defines a family $U^\varepsilon(t) = u(t)\,\mathcal{U}_0(t)\,\mathcal{U}^\varepsilon(t)$, $\mathcal{U}_0(t) \equiv e^{-iH_0\,t}$, of unitary operators as the unique solution, leaving D invariant, of

$$i(d/dt)\, U^\varepsilon(t)\,\psi = H^\varepsilon(t)\, U^\varepsilon(t)\,\psi, \quad \forall\psi\in D\,, \tag{4.4}$$

given by (with $f(k;q,p)$ defined by Eq. (2.6))

$$\mathcal{U}^\varepsilon(t) = \exp\left(-i\int_0^t ds\, e^{-\varepsilon|s|}\, H_I(s)\right) e^{i\Phi^\varepsilon_t - i\varphi^\varepsilon_t}\,, \tag{4.5}$$

$$H_I(s) = a(f_s) + a(f_s)^*\,,\ \ f_s(k;q,p) \equiv e^{-i|k|s}\, f(k; q_s(q,p), p_s(q,p)), \tag{4.6}$$

$$\Phi^\varepsilon_t \equiv \Phi^\varepsilon_t(q,p) \equiv i/2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s'\, e^{-\varepsilon(|s|+|s'|)}[\,H_I(s),\, H_I(s')\,], \tag{4.7}$$

$$\varphi^\varepsilon_t \equiv \varphi^\varepsilon_t(q,p) \equiv -\int_0^t ds\, e^{-2\varepsilon|s|}\Delta E(p_s(q,p))\,; \tag{4.8}$$

3) *the operators*

$$H^\varepsilon_D(t) \equiv h_D(t) + H_0 + H^\varepsilon_{ID}(t) \tag{4.9}$$

are hermitean on D and the equation

$$i(\mathrm{d}/\mathrm{d}t) U^\varepsilon_D(t) = H^\varepsilon_D(t)\, U^\varepsilon_D(t) \tag{4.10}$$

defines a family $U^\varepsilon_D(t) = u_D(t)\,\mathcal{U}_0(t)\,\mathcal{U}^\varepsilon_D(t)$*, of unitary operators as its unique solution leaving* D *invariant, with* $\mathcal{U}^\varepsilon_D(0) = 1$*, given by*

$$\mathcal{U}^\varepsilon_D(t) = \exp\left(-i\int_0^t \mathrm{d}s\, e^{-\varepsilon|s|}\, H_{ID}(s)\right) e^{i\Phi_t^{D\,\varepsilon} - i\varphi_t^{D\,\varepsilon}}\,, \tag{4.11}$$

$$H_{ID}(s) = a(f_s^D) + a(f_s^D)^*, \quad f_s^D(k;p) \equiv e^{-i|k|s}\, f(k; vs, p)\ , \tag{4.12}$$

$$\Phi_t^{D\,\varepsilon} \equiv \Phi_t^{D\,\varepsilon}(p) \equiv i/2 \int_0^t ds \int_0^s \mathrm{d}s' e^{-\varepsilon(|s|+|s'|)}[\, H_{ID}(s),\, H_{ID}(s')\,], \tag{4.13}$$

$$\varphi_t^{D\,\varepsilon} \equiv \varphi_t^{D\,\varepsilon}(p) \equiv -\int_0^t \mathrm{d}s\, e^{-2\varepsilon|s|}\,\Delta E(p); \tag{4.14}$$

4) $\forall \varepsilon > 0$, *the following strong limits exist:*

$$\lim_{t\to\pm\infty} U^\varepsilon(t)^*\, u_D(t)\,\mathcal{U}_0(t) \equiv W^\varepsilon_{0\pm}\,\omega_\pm\,,$$

$$U^\varepsilon_D(t)^* u_D(t)\,\mathcal{U}_0(t) = \mathcal{U}^{\varepsilon\,*}_D(t) \to_{t\to\pm\infty} \tag{4.15}$$

$$\to \exp(i[a(F_\pm^{D\,\varepsilon}(p)) + a(F_\pm^{D\,\varepsilon}(p))^*)])\ \exp(-i\Phi_\pm^{D\,\varepsilon}(p) + i\varphi_\pm^{D\,\varepsilon}(p)) \equiv W^\varepsilon_{D\,\pm}\,,$$

$$F_\pm^{D\,\varepsilon}(k,\lambda;p) = -i\sum_{i=1}^N J^\varepsilon_\pm(k,\lambda,p_i)\,, \tag{4.16}$$

$$J^\varepsilon_\pm(k,\lambda,p) = \frac{e}{(2\pi)^{3/2}}\,\frac{\epsilon(k,\lambda)\,\eta(k)\,v(p)}{(2|k|)^{1/2}(|k| - v(p)k \mp i\varepsilon)}\,, \tag{4.17}$$

$$\lim_{t\to\pm\infty} U^\varepsilon(t)^* U^\varepsilon_D(t) = W^\varepsilon_{0\pm}\,\omega_\pm\, W^{\varepsilon\,*}_{D\pm} = W^\varepsilon_{0\pm}(q,p)\, W^{\varepsilon\,*}_{D\pm}(p_\pm)\,\omega_\pm \equiv \Omega^\varepsilon_\pm; \tag{4.18}$$

$W^\varepsilon_{0\pm}(q,p)$, $W^\varepsilon_{D\pm}(p)$ *are "time ordered Weyl exponentials", acting on* $\mathcal{H}_F$ *and indexed by the particle variables* q, p*; they are given by Eqs.* (4.5) *and* (4.11) *with* $t = \pm\infty$*, and similarly for* $\Phi_\pm^{D\,\varepsilon}$, $\varphi_\pm^{D\,\varepsilon}$*.*

Remark. Equations (4.5) and (4.11) provide the existence and the explicit expression of the standard formula for $\mathcal{U}^\varepsilon(t)$ and $\mathcal{U}^\varepsilon_D(t)$ in terms of time-ordered exponentials,

$$\mathcal{U}^\varepsilon(t) = T(e^{-i\int_0^t \mathrm{d}s\, H^\varepsilon_{int}(s)})\,,$$

where T denotes the chronological ordering, according to the free photon dynamics and

$$\begin{aligned} H^\varepsilon_{int}(s) &\equiv \mathcal{U}_0^*(s)u^*(s)H^\varepsilon_{I,\,r}(s)u(s)\mathcal{U}_0(s)\,; \\ \mathcal{U}^\varepsilon_D(t) &= T(e^{-i\int_0^t \mathrm{d}s\, H^\varepsilon_{int\,D}(s)})\,, \\ H^\varepsilon_{int\,D}(s) &\equiv \mathcal{U}_0^*(s)u_D^*(s)H^\varepsilon_{I,\,r}(s)u_D(s)\mathcal{U}_0(s). \end{aligned}$$

Apart from the mass renormalization counter-term, $\mathcal{U}^\varepsilon_D(t)$ is the same operator used in [19,21] for the identification of the Dollard reference dynamics, with their asymptotic current taken as a function of our classical variables p_i.

For the proof of Proposition 4.1 we need the following Lemma, proved in Appendix C.

Lemma 4.2. *Let $f_\alpha(k), |k|^{-1/2} f_\alpha(k) \in L^2(d^3k)$, $\alpha \in \mathbb{R}$; if they are differentiable with respect to α in $L^2(d^3k)$, then,*

$$U(f_\alpha) \equiv e^{i(a(f_\alpha)+a(f_\alpha)^*)}, \tag{4.19}$$

is strongly differentiable on $D(H_0)$ and

$$\frac{\mathrm{d}U(f_\alpha)}{d\alpha} = [i(a(f'_\alpha) + a(f'_\alpha)^*) + C_\alpha]\, U(f_\alpha) \tag{4.20}$$

with $f'_\alpha \equiv \partial_\alpha f_\alpha$, $C_\alpha \equiv \frac{1}{2}\int d^3k\,(f_\alpha \overline{f'_\alpha} - \overline{f_\alpha} f'_\alpha)$. Moreover, if $|k| f_\alpha \in L^2(d^3k)$, then $U(f_\alpha)$ leaves $D(H_0)$ invariant.

Proof (of Proposition 4.1)
1)–3). For fixed q, p, the argument of the exponential which defines $\mathcal{U}^\varepsilon(t)$ is of the form $-i(a(F^\varepsilon_t(q,\, p)) + a(F^\varepsilon_t(q,\, p))^*)$, with

$$F^\varepsilon_t(k; q,\, p) = \int_0^t \mathrm{d}s\, e^{-\varepsilon|s|} f_s(k; q,\, p) \equiv \sum_i F^\varepsilon_t(k, q_i, p_i). \tag{4.21}$$

A similar form holds for $\mathcal{U}^\varepsilon_D(t)$, with $f_s(k; q, p)$ replaced by $f^D_s(k; p)$. Both $F^\varepsilon_t(k; q,\, p)$ and $F^{D\,\varepsilon}_t(k; p)$ satisfy the conditions of Lemma 4.2, with respect to t, q, p. Therefore, $\forall \varepsilon \geq 0$ the right-hand sides of Eqs. (4.5) and (4.11) define unitary operators $\mathcal{U}^\varepsilon(t)$, $\mathcal{U}^\varepsilon_D(t)$ in the Fock space $\mathcal{H}_F$, indexed by the particle coordinates q, p.
Such unitary operators leave $D(H_0)$ invariant and are strongly differentiable with respect to t, q, p, on $D(H_0)$. Hence, in $\mathcal{H}$ the unitary operators $U^\varepsilon(t)$, $U^\varepsilon_D(t)$ leave D invariant.
Again by Lemma 4.2, $U^\varepsilon(t)$, $U^\varepsilon_D(t)$ are strongly differentiable with respect to t on D and satisfy Eqs. (4.4) and (4.10), respectively. Hermiticity of $H^\varepsilon(t)$ implies $d/dt\, V^*(t)\, U^\varepsilon(t) = 0$ for any solution $V(t)$ of Eq. (4.4) leaving D invariant and, therefore, uniqueness; the same for $H^\varepsilon_D(t)$.
For $\varepsilon > 0$, this implies 2), 3). For $\varepsilon = 0$, the uniqueness of the solution of Eq. (4.4) implies that $U(t)$ is a one-parameter group; Eq. (4.4) and the invariance of D imply the self-adjointness of H.
4). The left-hand side of Eq. (4.15) reads

$$\mathcal{U}^{\varepsilon\,*}(t)\, \mathcal{U}_0(t)^*\, u(t)^*\, u_D(t)\, \mathcal{U}_0(t) = \mathcal{U}^{\varepsilon\,*}(t)\, u(t)^* u_D(t).$$

Using Proposition 3.1 and Eqs. (4.7), (4.8) and (4.21), it converges to

$$\begin{aligned} &e^{i\int_0^{\pm\infty} \mathrm{d}s\, e^{-\varepsilon|s|} H_I(s)}\, e^{-i\Phi^\varepsilon_{\pm\infty} + i\varphi^\varepsilon_{\pm\infty}}\, \omega_\pm \\ &\quad \equiv e^{i(a(F^\varepsilon_\pm(q,\,p)) + a(F^\varepsilon_\pm(q,\,p))^*)}\, e^{-i\Phi^\varepsilon_\pm + i\varphi^\varepsilon_\pm}\, \omega_\pm. \end{aligned} \tag{4.22}$$

In fact,
i) for $|t| \to \infty$, $F^\varepsilon_t(k; q, p)$ converges in $L^2(d^3k)$ uniformly in q, p on compact sets K, so that $\exp i\,(a(F^\varepsilon_t(q,\, p)) + a(F^\varepsilon_t(q,\, p))^*)$ converges strongly on $L^2(K, dq\, \mathrm{d}p) \otimes D_{fin}$ and, therefore, everywhere;
ii) by similar arguments, $||f_s(k; q, p)||_{L^2(d^3k)}$ is uniformly bounded in s and in q, p on compact sets, so that

$$< f_s,\, f_r > \equiv \int \mathrm{d}^3k\, (f_s\, f^*_r - f^*_s\, f_r) \tag{4.23}$$

is uniformly bounded in s, r and in q, p on compact sets. Then

$$\Phi_t^\varepsilon(q,p) = i/2 \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\, e^{-\varepsilon(|s|+|r|)} \,< f_s,\, f_r >$$

converges for $|t| \to \infty$.
The left-hand side of Eq. (4.16) reads

$$e^{i\,(a(F_t^{D\,\varepsilon}(p))+a(F_t^{D\,\varepsilon}(p))^*)}\, e^{-i\Phi_t^{D\,\varepsilon}+i\varphi_t^{D\,\varepsilon}}, \quad F_t^{D\,\varepsilon}(k;p) \equiv \int_0^t \mathrm{d}s\, e^{-\varepsilon|s|}\, f_s^D(k;p) \tag{4.24}$$

and converges as $|t| \to \infty$, by the above argument applied to $f_s^D(k;p)$. An explicit calculation gives Eqs. (4.16) and (4.17) and the unitarity of $W_{D\pm}^\varepsilon$, as in Lemma 4.2.
The first equality in Eq. (4.18) follows from Eqs. (4.15) and (4.16) and the unitarity of $W_{D\pm}^\varepsilon$, which implies the convergence of the adjoint of Eq. (4.16). The second equality follows from Eq. (3.12).

5. Removal of the Adiabatic Switching

In this Section, we perform the limit $\varepsilon \to 0$; the crucial ingredient is the use of the Dollard reference dynamics $U_D^\varepsilon(t)$, but, as anticipated in Sect. 2, the introduction of suitable counter-terms will be required. We have to consider the behavior of the operators $\Omega_\pm^\varepsilon$, Eq. (4.18), as $\varepsilon \to 0$. Using Eqs. (4.21), (4.22) and (4.16), one has

$$\begin{aligned}\Omega_\pm^\varepsilon &= e^{i(a(F_\pm^\varepsilon(q,p))+a(F_\pm^\varepsilon(q,p))^*)}\, e^{-i(a(F_\pm^{D\,\varepsilon}(p_\pm))+a(F_\pm^{D\,\varepsilon}(p_\pm))^*)} \\ &\quad\times e^{i(-\Phi_\pm^\varepsilon+\Phi_\pm^{D\,\varepsilon}(p_\pm))} e^{i(\varphi_\pm^\varepsilon-\varphi_\pm^{D\,\varepsilon}(p_\pm))}\,\omega_\pm \equiv \Omega(\Delta F_\pm^\varepsilon)\, e^{-i\Delta\Phi_\pm^\varepsilon+i\Delta\varphi_\pm^\varepsilon}\,\omega_\pm,\end{aligned} \tag{5.1}$$

where $\Omega(\Delta F_\pm^\varepsilon)$ denotes the product of the two Weyl exponentials in the l.h.s.

The convergence of the term $\Omega(\Delta F_\pm^\varepsilon)$ amounts to the cancelation of the infrared divergences associated to infinite photon emission, thanks to the Dollard subtraction given by the coherent factors, Eqs. (4.16) and (4.17).

For the convergence of the phases, we shall use the fact that $\Phi_\pm^\varepsilon$ and $\Phi_\pm^{D\,\varepsilon}$ involve the commutators

$$[A_i(x)\,,\, A_j(y)] \equiv i D_{i\,j}(x-y), \quad x, y \in \mathrm{I\!R}^4, \quad i,j = 1,2,3.$$

The fields $A_i(x)$ are free because their time dependence is given by the interaction representation and one has ($x^2 = x_0^2 - \mathbf{x}^2$)

$$D_{i\,j}(x) = \delta_{i\,j} D(x) + \partial_i\partial_j(\mathrm{sign}(\mathrm{x}_0)\,\theta(\mathrm{x}^2) + \mathrm{x}_0/|\mathbf{x}|\,\theta(-\mathrm{x}^2))/4\pi, \tag{5.2}$$

with $D(x)$ the standard commutator function. $D_{i\,j}(x)$ has spacelike support and it is homogeneous of degree -2.

We denote by $D_{i\,j}^\eta(x)$ the double convolution in the space variables

$$D_{i\,j}^\eta(x) \equiv \int \mathrm{d}^3\xi\, \mathrm{d}^3\eta\, D_{i\,j}(x-(\xi-\eta))\,\tilde\eta(\xi)\,\tilde\eta(\eta).$$

Considering the case of $t > 0$, we have to control the $\varepsilon \to 0$ limit of

$$\Phi^{\varepsilon}_{+}(q,p) = \tfrac{1}{2} \sum_{m,n=1}^{N} \int_0^\infty G^{\varepsilon}_{m\,n}(x_0,q,p), \mathrm{d}x_0 \equiv \tfrac{1}{2} \sum_{m,n=1}^{N} (\Phi^{\varepsilon}_{+})_{mn}, \tag{5.3}$$

where omitting as before the vector notation for q and v,

$$\begin{aligned} G^{\varepsilon}_{m\,n}(x_0,q,p) \; &\equiv e^2 \int_{0 \le y_0 \le x_0} \mathrm{d}^3x\, \mathrm{d}^4y\, e^{-\varepsilon(x_0+y_0)}\, v_m(x_0)\, v_n(y_0)\, D^\eta(x-y) \\ &\quad \times \delta(\mathbf{x} - q_m(x_0))\, \delta(\mathbf{y} - q_n(y_0)), \end{aligned} \tag{5.4}$$

with $v_n(x_0)$, $q_n(x_0)$ given by the solution at time x_0 of the equations of motion with initial data (q,p) and $v_m\, v_n\, D^\eta \equiv \sum_{i\,j=1}^{3} v_{m\,i}\, v_{n\,j}\, D^\eta_{i\,j}$.
$\Phi^{D\,\varepsilon}_{+}(p_+)$ is given by Eq. (5.3) with $G^{\varepsilon}_{m\,n}$ replaced by $G^{D\,\varepsilon}_{m\,n}$,

$$\begin{aligned} G^{D\,\varepsilon}_{m\,n}(x_0,q,p) \; &\equiv e^2 \int_{0 \le y_0 \le x_0} \mathrm{d}^3x\, \mathrm{d}^4y\, e^{-\varepsilon(x_0+y_0)}\, v_{m\,+}\, v_{n\,+}\, D^\eta(x-y) \\ &\quad \times \delta(\mathbf{x} - v_{m\,+}x_0)\, \delta(\mathbf{y} - v_{n\,+}y_0) \equiv G^{\varepsilon}(x_0, p_{m\,+}, p_{n\,+}) \end{aligned} \tag{5.5}$$

(reproducing Eq. (S4.21) in [19]).

The convergence of the off-diagonal terms in the phase difference $\Phi^{\varepsilon}_{\pm} - \Phi^{D\,\varepsilon}_{\pm}(p_\pm)$, Eq. (5.7) below with $n \neq m$, amounts to the Dollard cancelation of the Lienard–Wiechert corrections to the Coulomb phases arising from "photon exchanges". The corresponding diagonal terms, $n = m$, are logarithmically divergent, Eq. (5.7) with $n = m$, even if their $1/\varepsilon$ divergent terms are canceled by Dollard's subtraction; they correspond to a logarithmically divergent "mass renormalization" effect produced by the Coulomb asymptotic distortion of the trajectories (a point which is not discussed in [19]). The problem is solved by the introduction of the same "mass renormalization" counter-term $\Delta E(p)$ in H_I and in H_D,

$$\Delta E(p) = \sum_n \delta E(p_n) \equiv \tfrac{1}{2} \sum_n e^2 \int_{y_0 \le 0} \mathrm{d}^4y\, v_n^2 D^\eta(-y) \delta(\mathbf{y} - v_n y_0); \tag{5.6}$$

it corresponds to the linearization, with parameters p_n, of the particle trajectories in Eq. (5.4), $q_n(y_0) = q_n(x_0) + (y_0 - x_0)v_n$, $v_n = v(p_n)$. We denote by $(\varphi^{\varepsilon}_{\pm})_n$ the contribution of $\delta E(p_n)$ to $\varphi^{\varepsilon}_{\pm}$, see Eq. (4.8).

Actually, the introduction of the above counter-term in the Dollard dynamics cancels the divergence of the Dollard phases $(\Phi^{D\,\varepsilon}_{\pm})_{nn}$ and, therefore, the convergence of the diagonal terms reduces to the convergence of $(\Phi^{\varepsilon}_{\pm})_{nn} - (\varphi^{\varepsilon}_{\pm})_n$, i.e., to the effect of the renormalization counter-term.

In the following, for simplicity, in the non-relativistic case, we shall write $\mathcal{H}$ for $\mathcal{H}_{\mathrm{nr}}$ and $\Gamma_\pm$ for $\gamma_\pm^{-1}(\Gamma_{\mathrm{nr}})$, Γ_{nr} the set of non-relativistic particle configurations, Eq. (2.8). We also introduce

$$\mathcal{H}_\pm \equiv \omega_\pm L^2(\Gamma) \times \mathcal{H}_F = L^2(\Gamma_\pm) \times \mathcal{H}_F.$$

5.1. The Møller Operators

Proposition 5.1. *As $\varepsilon \to 0$,*
1) $\Omega(\Delta F^{\varepsilon}_{\pm})$, *Eq.* (5.1) *converge strongly, to unitary operators, on* $\mathcal{H}_{\pm}$*;*
2) *for almost all* (q, p) *in* $\Gamma_{\pm}$,

$$(\Phi^{\varepsilon}_{\pm} - \Phi^{D\,\varepsilon}_{\pm}(p_{\pm}))_{mn} = \delta_{m\,n} R(p_{\pm}, \delta p_{\pm}) \ln \varepsilon + O(1), \tag{5.7}$$

with $R(p_{\pm}, \delta p_{\pm})$ *defined below, see Eq.* (5.11)*;*
3) *with the choice of the counter-term given by Eq.* (5.6)*, we have*

$$(\Phi^{\varepsilon}_{\pm})_{nn} - (\varphi^{\varepsilon}_{\pm})_n = O(1)\,, \ \textit{a.e. in } \Gamma_{\pm}$$

and

$$(\Phi^{D\,\varepsilon}_{\pm}(p_{\pm}))_{nn} - (\varphi^{D\,\varepsilon}_{\pm}(p_{\pm}))_n \equiv \delta\varphi^{D\,\varepsilon}_{\pm}(p_{n\,\pm}) = O(1)\,;$$

the phase $\delta\varphi^{D\,\varepsilon}_{\pm}(p_{n\,\pm})$ *vanishes after the redefinition*

$$\mathcal{U}^{\varepsilon}_{D} \to \mathcal{U}^{\varepsilon}_{D}\, e^{i \sum_i \delta\varphi^{D\,\varepsilon}_{\pm}(p_i)}$$

for $\pm t > 0$*, which shall be understood in the following.*
Therefore, with the above choice of the counter-terms, the following strong limits exist:

$$\lim_{\varepsilon \to 0} \Omega^{\varepsilon}_{\pm} \equiv \Omega_{\pm} = W_{\pm}\, \omega_{\pm} \quad \textit{on } \mathcal{H}, \quad W_{\pm} \textit{ unitary in } \mathcal{H}_{\pm} = \omega_{\pm}\, \mathcal{H},$$
$$W_{\pm} = \lim_{\varepsilon \to 0} W^{\varepsilon}_{0\pm}(q, p)\, W^{\varepsilon\,*}_{D\pm}(p_{\pm}) \quad \textit{on } \mathcal{H}_{\pm}. \tag{5.8}$$

Proof. 1) Omitting the dependence on polarization vectors, we put

$$\Delta F^{\varepsilon}_{\pm}(k) \equiv F^{\varepsilon}_{\pm}(k; q, p) - F^{D\,\varepsilon}_{\pm}(k; p_{\pm}(q, p))$$

and we have (with the notation of Lemma 4.2 and of Eq. (4.23))

$$\Omega(\Delta F^{\varepsilon}_{\pm})) = U(\Delta F^{\varepsilon}_{\pm})\, e^{\frac{1}{2} < F^{\varepsilon}_{\pm}\,,\, F^{D\,\varepsilon}_{\pm} >}.$$

We shall prove convergence, in $L^2(d^3k)$, almost everywhere in $\Gamma_{\pm}$, of both i) $\Delta F^{\varepsilon}_{\pm}(k, q, p)$ and ii) $|k|^{-1/2}\, \Delta F^{\varepsilon}_{\pm}(k, q, p)$. i) Implies strong convergence of $U(\Delta F^{\varepsilon}_{\pm})$ in Fock space, for fixed q, p, and their strong convergence, as multiplication operators in q, p, on $\mathcal{H}_{\pm}$, by a Lebesgue dominated convergence argument, to unitary operators.
Moreover, by Eq. (4.17), for fixed p in $\Gamma_{\pm}$, $|k|^{1/2}\, F^{D\,\varepsilon}_{\pm}(k, \lambda; p)$ converges in $L^2(\mathrm{d}^3k)$ as $\varepsilon \to 0$. Then, since

$$\langle F^{\varepsilon}_{\pm}\,,\, F^{D\,\varepsilon}_{\pm} \rangle \;=\; \langle \Delta F^{\varepsilon}_{\pm}\,,\, F^{D\,\varepsilon}_{\pm} \rangle,$$

ii) implies convergence of the phase factors, pointwise and, therefore, strongly on $\mathcal{H}_{\pm}$.
It is enough to prove ii), which implies i) thanks to the ultraviolet cutoff $\eta(k)$. To this purpose, we note that Eqs. (3.13) and (3.18) imply that, for almost all (q, p) in $\Gamma_{\pm}$, for large $|s|$,

$$|k| s - k\, q_{i\,s} \equiv (|k| - k\, v_{i\,\pm})\, s'(s, k/|k|) \equiv \theta(k/|k|)\, |k|\, s'$$

defines a function $s'(s, k/|k|) \equiv s - \Delta s$, satisfying $J(s) \equiv \partial \Delta s / \partial s = O(s^{-1})$ and $\Delta s = O(\ln |s|)$, uniformly in $k/|k|$; $s(s')$ will denote the inverse function, for large s', at fixed $k/|k|$. Then, considering for simplicity positive times, apart from an integral over a finite time interval, $0 < s < c$, giving rise to a convergent term, $|k|^{-1/2}\,\Delta F^{\varepsilon}_{+}$ may be written as $\eta(k)/|k|$ times

$$\int_c^\infty \mathrm{d}s\, e^{-\varepsilon s} e^{-i\,|k|\,s}\,(e^{ikq_s(q,p)}\,v_s - e^{ikv_+(q,p)s}\,v_+)$$
$$= \int_c^\infty \mathrm{d}s\, e^{-\varepsilon s}\,(e^{-i\theta\,|k|\,s'} - e^{-i\theta\,|k|\,s})\,v_+$$
$$+ \int_c^\infty \mathrm{d}s'\, e^{-\varepsilon s(s')} e^{-i\theta\,|k|\,s'}\, g(s', k/|k|),$$

with $g(s', k/|k|)$ of order $1/s'$ and, therefore, in $L^2(\mathrm{d}s')$, with norm bounded uniformly in $k/|k|$. Since, for bounded momenta, θ is bounded away from 0, the last term converges in $L^2(d|k|)$, uniformly in $k/|k|$, as $\varepsilon \to 0$. Therefore, it gives a contribution to $|k|^{-1/2}\,\Delta F^{\varepsilon}_{+}$ which converges in $L^2(d^3k)$. By an obvious change of variables, and omitting as before the integration over a finite interval, the first term can be written

$$\int_c^\infty \mathrm{d}s\, e^{-i\theta\,|k|\,s}\, e^{-\varepsilon s}(e^{-\varepsilon\Delta s}/(1+J) - 1)\, p_+.$$

Therefore, it is the Fourier transform of a function which converges, as $\varepsilon \to 0$, in $L^2(\mathrm{d}s)$, since $1/(1+J) = 1 + O(s^{-1})$ and, for $s \geq 1$,

$$e^{\varepsilon \ln s} - 1 \leq \varepsilon \ln s\, e^{\varepsilon \ln s}.$$

Hence, its contribution to $|k|^{-1/2}\,\Delta F^{\varepsilon}_{+}$ converges in $L^2(d^3k)$, with convergence rate $O(\varepsilon^{1/2-\delta})$, $\forall \delta > 0$. Similarly for $\Delta F^{\varepsilon}_{-}$.
2) First, we consider the terms corresponding to $m \neq n$, with non-collinear $v_{m\,+}$, $v_{n\,+}$, a condition which holds almost everywhere in Γ_+, by Eq. (3.12). For their contribution to $\Delta\Phi^{\varepsilon}_{+}$ we exploit Lemma 5.2 below. In fact, as a consequence of Eqs. (3.13) and (3.18), the particle trajectories satisfy Eqs. (5.12)–(5.14), for any $\alpha < 1$, a.e. in Γ_+.

Then, for $m \neq n$, uniformly in ε,

$$G^{\varepsilon}_{m\,n}(s) - G^{D\,\varepsilon}_{m\,n}(s) \leq O(s^{-(1+\alpha)})$$

and the corresponding contribution to $\Delta\Phi^{\varepsilon}_{+}$ is convergent with rate $O(\varepsilon^{\alpha})$.
For $n = m$, we compute the logarithmic divergences, which arise from the subleading terms in the asymptotic estimates of the trajectories.
In this case, for q, p in Γ_+, the velocities $|v_i|$ have a bound less than 1 and, therefore, by the support properties of D and η, $x_0 - y_0$ is bounded uniformly in x_0, i.e., $x_0 - y_0 < T$, in the integration in Eq. (5.4). The diagonal term G^{ε}_{nn} is a functional $\mathcal{G}^{\varepsilon}$ of the nth particle trajectory, $\{q_n(\tau),\, p_n(\tau)\} \equiv \{(q_{n\,\tau}(q,p),\, p_{n\,\tau}(q,p)),\, \tau \in \mathbb{R}\}$,

$$G^{\varepsilon}_{nn}(x_0, q, p) = \mathcal{G}^{\varepsilon}(x_0; \{q_n(\tau),\, p_n(\tau)\}); \tag{5.9}$$

the expression for the corresponding Dollard term is given by the Dollard trajectories, $G^{D\,\varepsilon}_{nn}(x_0, q, p) = \mathcal{G}^{\varepsilon}(x_0; \{v_{n\,+}\tau,\, p_{n\,+}\})$. We introduce

$$G^{\varepsilon}_{n\,+}(x_0, q, p) \equiv \mathcal{G}^{\varepsilon}(x_0; \{v_{n\,+}\tau + v'(p_{n\,+})\,\delta p_{n\,+} \ln\tau,\, p_{n\,+} + \delta p_{n\,+}/\tau\}),$$

$v'(p) \equiv \partial v(p)/\partial p$, which satisfies, a.e. in Γ_+,

$$|G^\varepsilon_{nn}(x_0,q,p) - G^\varepsilon_{n\,+}(x_0,q,p)| \le C(q,p)\ln|x_0|/x_0^2, \tag{5.10}$$

uniformly in ε. In fact, by Eq. (3.13), omitting the index n,

$$\begin{aligned} &q(x_0) - q(y_0) - v_+(x_0-y_0) - v'(p_+)\,\delta p_+ \ln(x_0/y_0) \\ &= \int_{y_0}^{x_0} \mathrm{d}s\,(v(s) - v_+ - v'(p_+)\delta p_+/s) \\ &= O(\ln x_0/x_0^2)(x_0-y_0) = O(\ln x_0/x_0^2)T; \end{aligned}$$

since $D^\eta(x)$ is a C^∞ function of $\mathbf{x}$ and $p_n(t)$ satisfies the estimate (3.13), Eq. (5.10) follows, uniformly in ε since $x_0 - y_0 \le T$. In conclusion, the contribution to $\Delta\Phi^\varepsilon_+$ is

$$\begin{aligned} (\Delta\Phi^\varepsilon_+)_{nn} &= \int \mathrm{d}x_0\,(G^\varepsilon_{nn}(x_0,q,p) - G^{D\,\varepsilon}_{nn}(x_0,q,p)) \\ &= \int \mathrm{d}x_0\,(G^\varepsilon_{n\,+}(x_0,q,p) - G^{D\,\varepsilon}_{nn}(x_0,q,p)) + O(1). \end{aligned} \tag{5.11}$$

$G^\varepsilon_{n\,+} - G^{D\,\varepsilon}_{nn}$ is a function of $p_{n\,+}, \delta p_{n\,+}$ and x_0; by a Taylor expansion in τ around $\tau = x_0$, it is of the form $((R(p_{n\,+},\delta p_{n\,+}) + O(\varepsilon))\,x_0^{-1}\,e^{-2\varepsilon x_0} + O(x_0^{-2})$ and 2) follows. Similarly for $(\Delta\Phi^\varepsilon_-)_{nn}$.

3) $(\Phi^{D\,\varepsilon}_+(p_+))_{nn}$ is given by Eq. (5.5) and $(\varphi^{D\,\varepsilon}_+(p))_n$, see Eqs. (4.14) and (5.6), is given by the same expression, without the factor $e^{-\varepsilon(x_0+y_0)}$ and the restriction $0 \le y_0$. Hence, they only depend on $p_{n\,+}$ and their difference converges as $\varepsilon \to 0$.

With a change of variables, $\mathbf{x}' \equiv \mathbf{x} - q_n(x_0)$, $\mathbf{y}' \equiv \mathbf{y} - q_n(x_0)$, $y_0' = y_0 - x_0$ in Eq. (5.4), putting $G^\varepsilon_{nn} = \hat{G}^\varepsilon_{nn}\,e^{-2\varepsilon x_0}$, one has, for (q,p) in Γ_+,

$$\begin{aligned} &\tfrac{1}{2}\hat{G}^\varepsilon_{nn}(x_0,q,p) - \delta E(p_{n\,x_0}(q,p)) = -\tfrac{1}{2}e^2 \int_{y_0'\le 0} \mathrm{d}^4y'\,D^\eta(-y')\,v_n(x_0) \\ &\quad\times[v_n(x_0+y_0')\,\delta(\mathbf{y}' - q_n(x_0+y_0') + q_n(x_0))\,e^{\varepsilon\,y_0'} - v_n(x_0)\,\delta(\mathbf{y}' - v_n(x_0)\,y_0')] \end{aligned}$$

As before, $|y_0'| \le T$ and, therefore, $v_n(x_0+y_0') = v_n(x_0) + O(x_0^{-2})$, $q_n(x_0 + y_0') + q_n(x_0) = v_n(x_0)\,y_0'/m + O(x_0^{-2})$. Hence, $\hat{G}^\varepsilon_{nn}(x_0,q,p) - \delta E(p_{n\,x_0}(q,p)) \in L^1(dx_0)$ uniformly in ε and $(\Phi^\varepsilon_+)_{nn} - (\varphi^\varepsilon_+)_n$ converge for $\varepsilon \to 0$, with rate $O(\varepsilon \ln \varepsilon)$.

Convergence of the regularized Møller operators follows, with rate of convergence $O(\varepsilon^{1/2-\delta})$, $\forall\delta > 0$, on $\mathcal{H}_+ = L^2(\Gamma_+) \times \mathcal{H}_F$; their limit, W_+, is a product of phases and Weyl operators, acting as multiplication operators on $L^2(\Gamma_+)$, and therefore unitary operators in $\mathcal{H}_+$. The same applies for $t \to -\infty$. □

Lemma 5.2. *Let $q_n(x_0), v_n(x_0) \in C^1(\mathbb{R})$ satisfy, for $x_0 \to \infty$,*

$$|\dot{q}_n(x_0)| < 1 - \delta, \tag{5.12}$$

$$q_n(x_0)/x_0 = v_{n\,+} + O(x_0^{-\alpha}), \tag{5.13}$$

$$p_n(x_0) = p_{n\,+} + O(x_0^{-\alpha}), \tag{5.14}$$

with $\delta, \alpha > 0$*. Then, for* $v_{m\,+}, v_{n\,+}$ *non-collinear,* $\forall \varepsilon \geq 0$*, denoting* $j_m(x) \equiv v_m(x_0)\delta(\mathbf{x} - q_m(x_0))$*,* $j_{m\,+}(x) \equiv v_{m\,+}\delta(\mathbf{x} - v_{m\,+}x_0)$*, omitting the vector notation as in Eqs.* (5.4) *and* (5.5)*,*

$$\int_{0<y_0<x_0} \mathrm{d}^3x\, \mathrm{d}^4y\, e^{-\varepsilon y_0}\, j_m(x)\, D^{\eta}(x-y)\, j_n(y)$$
$$= \int_{0<y_0<x_0} \mathrm{d}^3x\, \mathrm{d}^4y\, e^{-\varepsilon y_0}\, j_{m\,+}(x)\, D^{\eta}(x-y)\, j_{n\,+}(y) \,+\, O(x_0^{-1-\alpha}), \qquad (5.15)$$

uniformly in ε*. By homogeneity of* $D_{i\,j}$*, for* $\varepsilon = 0$*, the first term in the r. h. s. is of the form* $v_{n\,+}v_{m\,+}C(v_{m\,+}, v_{n\,+})\, x_0^{-1}$*.*

Proof. The l.h.s. of Eq. (5.15) is well defined, for all x_0, thanks to the regularization given by $\tilde{\eta}$ and, by a change of variables, $\tau \equiv y_0/x_0$, $\mathbf{x}' \equiv \mathbf{x}/x_0$, $\mathbf{y}' \equiv \mathbf{y}/x_0$, it becomes

$$\frac{1}{x_0}\int_{0<\tau<1} d\tau\, d^3x'\, d^3y'\; v_m(x_0)\, v_n(\tau x_0)\, e^{-\varepsilon x_0 \tau}$$
$$D^{\tilde{\eta}_{x_0}}(\mathbf{x}' - \mathbf{y}', 1-\tau)\, \delta(\mathbf{x}' - q_m(x_0)/x_0)\, \delta(\mathbf{y}' - q_n(\tau x_0)/x_0), \quad (5.16)$$

where $\tilde{\eta}_{x_0}(\xi) \equiv \tilde{\eta}(x_0\, \xi)x_0^3$ and the homogeneity of $D_{i\,j}$ has been used. For large x_0, the integrand of Eq. (5.16) vanishes for $\tau < \delta/2$. In fact, $D^{\tilde{\eta}_{x_0}}$ has spacelike support, apart from a correction of order x_0^{-1}; moreover, $\tau < \delta/2$ and Eq. (5.12) imply

$$|q_m(x_0) - q_n(\tau x_0)|/x_0 \leq x_0^{-1}(|q_m(0)| + |q_n(0)|) + (1-\delta)(1+\tau) < 1 - \tau,$$

the last inequality following, for large x_0, from

$$(1-\tau)/(1+\tau) \geq (1-\tau)^2 > 1 - \delta.$$

Therefore, by Eq. (5.13), $q_n(\tau x_0)/x_0 = \tau v_{n\,+} + O(x_0^{-\alpha})$ in Eq. (5.15). Moreover, for non-collinear asymptotic velocities and x_0 large, the support of integrand in Eq. (5.16) excludes a neighborhood of $x' - y' = 0$.
Outside a neighborhood of $\mathbf{x} = 0$, the second term in the representation of $D_{i\,j}(x)$, Eq. (5.2), is a bounded function with spacelike support and bounded derivatives inside the spacelike region, and the same applies to its convolution with $\tilde{\eta}_{x_0}$, with bounds uniform in x_0, apart from the addition of a uniformly bounded function with support within a distance of order x_0^{-1} from the light cone; therefore, one may replace $q_m(x_0)/x_0 \mapsto \tau v_{m\,+}$ and $q_n(\tau x_0)/x_0 \mapsto \tau v_{n\,+}$ in Eq. (5.15), with an error of order $x_0^{-\alpha}$.
The first term, $\delta_{i\,j}D(x)$, only involves δ functions and may therefore be treated explicitly; the result follows, for $v_{m\,+}$, $v_{n\,+}$ non-collinear, from Eqs. (5.13) and (5.14), the convolution with η_{x_0} giving rise to corrections of order x_0^{-2}. □

5.2. The Asymptotic Dynamics

As shown in [22], any Dollard reference dynamics allowing for the existence of Møller operators defines asymptotic dynamics, for $t \to \pm\infty$, $U_{\pm}(t)$, which need not coincide with the free dynamics, but are always one-parameter groups, satisfying the Møller intertwining relations. In presence of an adiabatic procedure, the latter property involves the recovery of the dynamics from its adiabatic regularization [22].

Therefore, the next step in the analysis of the model is the determination of the asymptotic dynamics, and the verification of the intertwining relations.

The resulting asymptotic dynamics, $U_+(t) = U_-(t) \equiv U_{\mathrm{as}}(t)$ is uniquely determined by the Dollard dynamics $U_D(t)$ but cannot be identified with it, as implicit in Rohrlich and Zwanziger notions of asymptotic fields and dynamics.

Proposition 5.3. *With the counter-term given by Eq. (5.6), one has*
1) *the existence of the following strong limits*

$$U_\pm(s) \equiv \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} U_D^{\varepsilon\,*}(t)U_D^\varepsilon(t+s)\,, \tag{5.17}$$

which define the asymptotic dynamics

$$U_+(s) = U_-(s) = u_0(s)\,\alpha_{\mathrm{as}}(\mathcal{U}_0(s)) \equiv U_{\mathrm{as}}(s)\,,$$

with α_{as} the coherent automorphism of the photon algebra

$$\alpha_{\mathrm{as}}(a^*(k,\lambda)) = a^*(k,\lambda) + J(k,\lambda;p)\ , \tag{5.18}$$

$J(k,\lambda;p) = \sum_i J(k,\lambda,p_i)$, $J = J_\pm^{\varepsilon=0}$, see Eq. (4.17); in the non-relativistic case, $U_{\mathrm{as}}(s)$ leaves $\mathcal{H}_{\mathrm{nr}}$ invariant;
2) *the recovering of $U(t)$ from the regularized dynamics $U^\varepsilon(t)$:*

$$\lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} U^{\varepsilon\,*}(t)U^\varepsilon(t+s) \equiv \lim_{\varepsilon\to 0}\tilde{U}^\varepsilon(s) = U(s)\,, \tag{5.19}$$

all the limits being strong;
3) *the interpolation formula*

$$U(t)\,\Omega_\pm = \Omega_\pm\,U_{\mathrm{as}}(t)\,; \tag{5.20}$$

4) *covariance under space translations,*

$$\mathcal{P}\,\Omega_\pm = \Omega_\pm\,\mathcal{P}_{\mathrm{as}} \quad \mathcal{P}_{\mathrm{as}} \equiv i\sum_i \partial/\partial q_i + \alpha_{\mathrm{as}}(P_{\mathrm{ph}}). \tag{5.21}$$

Proof. 1) By definition,

$$U_D^{\varepsilon\,*}(t)U_D^\varepsilon(t+s) = \mathcal{U}_D^{\varepsilon\,*}(t)\,\mathcal{U}_0^*(t)\,u_D^*(t)\,u_D(t+s)\,\mathcal{U}_0(t+s)\,\mathcal{U}_D^\varepsilon(t+s)\,;$$

by Eq. (4.16), $[u_D, \mathcal{U}_0] = 0$ and Eq. (3.10), the above expression converges strongly, for $t \to \pm\infty$, to

$$W_{D\,\pm}^\varepsilon\,u_0(s)\,\mathcal{U}_0(s)\,W_{D\,\pm}^{\varepsilon\,*} \to_{\varepsilon\to 0} u_0(s)\,e^{-iH_0(J)s} \equiv U_{\mathrm{as}}(s)\,,$$
$$H_0(J) \equiv \sum_\lambda \int d^3k\,|\mathbf{k}|(a^*(k,\lambda) + J(k,\lambda;p))\,(a(k,\lambda) + J(k,\lambda;p)); \tag{5.22}$$

$U_{\mathrm{as}}(s)$ leaves $\mathcal{H}_{\mathrm{nr}}$ invariant since so does u_0 and $H_0(J)$ acts as a multiplication operator on the particle space.

2) From Eq. (4.5) one has (for large positive t)

$$\begin{aligned} U^{\varepsilon\,*}(t)U^{\varepsilon}(t+s) &= e^{-i\Phi^{\varepsilon}_t + i\varphi^{\varepsilon}_t} e^{i\int_0^t ds'\, e^{-\varepsilon s'}\, H_I(s')} \\ &\quad \times u(s)\,\mathcal{U}_0(s) e^{-i\int_0^{t+s} \mathrm{d}s'\, e^{-\varepsilon s'}\, H_I(s')} e^{i\Phi^{\varepsilon}_{t+s} - i\varphi^{\varepsilon}_{t+s}} \\ &= u(s)\,\mathcal{U}_0(s) e^{i\int_s^{t+s} ds'\, e^{-\varepsilon s'}\, H_I(s')\, e^{\varepsilon s}} e^{-i\int_0^{t+s} \mathrm{d}s' e^{-\varepsilon s'} H_I(s')} \\ &\quad \times e^{(-i\Phi^{\varepsilon}_{s,t+s} + i\varphi^{\varepsilon}_{s,t+s})e^{2\varepsilon s}} e^{i\Phi^{\varepsilon}_{t+s} - i\varphi^{\varepsilon}_{t+s}}. \end{aligned}$$

Now, by Proposition 5.1, both the phase factors on the r.h.s. converge as $t \to \infty$ and $\varepsilon \to 0$, as multiplication operators on $\Omega_+(\mathcal{H})$, which coincides with $\mathcal{H}_+$ since $\Omega_+ = W_+\omega_+$, (Eq. (5.8)), with W_+ unitary operators in $\mathcal{H}_F$, indexed by q, p. Therefore, the factor $e^{2\varepsilon s}$ can be substituted by 1 and

$$\begin{aligned} &\varphi^{\varepsilon}_{s,t+s} - \varphi^{\varepsilon}_{t+s} \to -\varphi_s\ , \\ &-\Phi^{\varepsilon}_{s,t+s} + \Phi^{\varepsilon}_{t+s} \to \Phi_s + i/2 \int_s^{\infty} \mathrm{d}r_1 \int_0^s \mathrm{d}r_2\, [H_I(r_1),\, H_I(r_2)] \end{aligned} \tag{5.23}$$

(with a compact integration range, by locality). The product of the exponentials involving H_I can be written as the exponential of

$$\begin{aligned} &-i \int_0^s \mathrm{d}s'\, e^{-\varepsilon s'}\, H_I(s') - i \int_s^{t+s} \mathrm{d}s'\, e^{-\varepsilon s'}\, H_I(s')(1 - e^{\varepsilon s}) \\ &\quad + \tfrac{1}{2} \int_s^{t+s} \mathrm{d}r_1 \int_0^{t+s} \mathrm{d}r_2\, [H_I(r_1),\, H_I(r_2)]\, e^{-\varepsilon(r_1+r_2)} e^{\varepsilon s}. \end{aligned}$$

In the limit $t \to \infty$, $\varepsilon \to 0$, as a consequence of the antisymmetry of the integrand, the last term exactly cancels the last term on the r.h.s. of Eq. (5.23). With the notation of Eq. (4.19), the exponential of the first two terms is of the form $U(G^{\varepsilon}(s,t))$, with

$$G^{\varepsilon}(s,t) \equiv F^{\varepsilon}_s + (F^{\varepsilon}_{t+s} - F^{\varepsilon}_s)(1 - e^{\varepsilon s}).$$

For $t \to \infty$, with the notation of Eqs. (4.17), (4.22) and (5.1), one has that $F^{\varepsilon}_{t+s} \to F^{\varepsilon}_+ = F^{D\,\varepsilon}_+ + \Delta F^{\varepsilon}_+$, a.e. in Γ_+; for $\varepsilon \to 0$, ΔF^{ε}_+ converges in $L^2(\mathrm{d}^3k)$ and $||F^{D\varepsilon}_+||_{L^2(\mathrm{d}^3k)}$ is bounded by $O(\ln \varepsilon)$, uniformly in $q, p \in \Gamma_+$, apart from sets of arbitrarily small measure. Therefore, for $t \to \infty$ and then $\varepsilon \to 0$, $G^{\varepsilon}(s,t)(k,q,p)$ converges to F^0_s in $L^2(\mathrm{d}^3k)$, uniformly in $q, p \in \Gamma_+$, apart from sets of arbitrarily small measure. Then, strongly on $\Omega_+\,\mathcal{H}_{\mathrm{nr}}$,

$$U(G^{\varepsilon}(s,t)) \to U(F^0_s) = \exp\left(-i \int_0^s \mathrm{d}s\, H_I(s')\right);$$

(the phases and the operator W_+ act as multiplication operators in the variables $q, p \in \Gamma_+$ and, therefore, leave the support of $\psi(q,p) \in \mathcal{H}_F$, $(q,p) \in \Gamma_+$, invariant). The same applies for $t \to -\infty$ and Eq. (5.19) follows.

3) Using Eqs. (5.19) and (5.17), one has

$$U(s)\,\Omega_\pm = \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} U^{\varepsilon\,*}(t)\,U^\varepsilon(t+s)\,U^{\varepsilon\,*}(t+s)\,U^\varepsilon_D(t+s)$$
$$= \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} U^{\varepsilon\,*}(t)\,U^\varepsilon_D(t)\,U^{\varepsilon\,*}_D(t)\,U^\varepsilon_D(t+s) = \Omega_\pm\,U_{\rm as}(t).$$

4) By Eqs. (5.8) and (3.12),

$$\Omega_\pm = \lim_{\varepsilon\to 0} W^\varepsilon_{0\pm}(q,p)\,\omega_\pm\,W^{\varepsilon\,*}_{D\pm}(p)$$

and, since the space translations $T(a)$ commute with $\omega_\pm$ and $W^{\varepsilon\,*}_{0\pm}(p)$,

$$T(a)\,\Omega_\pm = \lim_{\varepsilon\to 0} T(a)\,W^\varepsilon_{0\pm}(q,p)\,\omega_\pm\,W^{\varepsilon\,*}_{D\pm}(p)$$
$$= \Omega_\pm\,\lim_{\varepsilon\to 0} W^\varepsilon_{D\pm}(q,p)\,T(a)\,W^{\varepsilon\,*}_{D\pm}(p) = \Omega_\pm\,\alpha_{\rm as}(T(a)) \equiv \Omega_\pm\,(T_{\rm as}(a)).$$

□

Summarizing, for the above model, describing classical particles with Coulomb interaction and with realistic, translation invariant, coupling to the quantized electromagnetic field, the introduction of an asymptotic reference dynamics $U_D(t)$ a la Dollard, an adiabatic switching and a particle energy renormalization term, we obtain the existence of the Møller operators and of the scattering matrix, describing infinite photon emission.

The Møller operators interpolate between the dynamics, $U(t)$, and the asymptotic dynamics, $U_{\rm as} \equiv U_+(t) = U_-(t)$, uniquely associated to $U_D(t)$ by Eq. (5.17); the S-matrix is invariant under $U_{\rm as}(t)$ and $T_{\rm as}(a)$.

The Møller operators and the S-matrix exhibit a factorization of the infrared divergences which may also be displayed for the particle scattering. In fact, the same Møller operators are obtained if an adiabatic switching is adopted also for the particle Coulomb interaction, as discussed in Appendix D.

We have, therefore,

Theorem 5.4. *For the model defined by Eqs.* (2.1)–(2.6) *and* (5.6)*, with the adiabatic regularization given by Eqs.* (4.1) *and* (4.2) *and (Dollard) reference dynamics* $U^\varepsilon_D(t)$ *given by Eqs.* (3.5), (3.7), (4.9) *and* (4.10)*, or with* h_D *replaced by* h^ε_D*, Eq.* (D.1)*, one has:*

i) *the Møller operators exist as strong limits*

$$\Omega_\pm = \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty}\Omega^\varepsilon_t, \quad \Omega^\varepsilon_t \equiv U^{\varepsilon\,*}(t)\,U^\varepsilon_D(t), \tag{5.24}$$

on the whole Hilbert space (*with* $L^2(\mathbb{R}^{6N})$ *replaced by* $L^2(\Gamma_{\rm nr})$*, Eq.* (2.8)*, in the n.r. case);*

ii) *the asymptotic dynamics* $U_{\rm as}(t)$ *associated to the Dollard reference dynamics, Eq.* (5.17)*, is a one-parameter continuous group; it is the product of the free particle dynamics and a non-Fock coherent transformation of the free photon dynamics, Eq.* (5.22)*, indexed by the moments of the particles; the Møller operators interpolate between the dynamics* $U(t)$ *and the asymptotic dynamics* $U_{\rm as}(t)$

$$H\,\Omega_\pm = \Omega_\pm\,H_{\rm as}, \quad H_{\rm as} \equiv h_0 + \alpha_{\rm as}(H_0)\,; \tag{5.25}$$

iii) *the scattering matrix, $S = \Omega_+^* \, \Omega_-$, commutes with the asymptotic dynamics $U_{\rm as}$ and with the asymptotic space translations $T_{\rm as}$.*
For two particles with repulsive Coulomb interaction, $\Omega_\pm$ and S are unitary operators;
iv) *the Møller operators explicitly display the photon infrared divergences,*

$$\Omega_\pm = \lim_{\varepsilon\to 0} W^\varepsilon_{0\pm}(q,p)\, W^{\varepsilon\,*}_{D\pm}(p_\pm)\,\omega_\pm = \lim_{\varepsilon\to 0} W^\varepsilon_{0\pm}(q,p)\,\omega_\pm\, W^{\varepsilon\,*}_{D\pm}(p)\,; \quad (5.26)$$

in fact, $W^\varepsilon_{0\pm}(q,p)\,\omega_\pm$ are the Møller operators relative to the free photon dynamics, Eq. (4.15), and their infrared divergences are canceled by the (time-ordered) non-Fock coherent factors $W^{\varepsilon\,*}_{D\pm}(p)$;
v) *the explicit factorization of all the infrared divergences is displayed in the following form of the S-matrix:*

$$S = \lim_{\varepsilon\to 0} W^\varepsilon_{D+}(p)\, e^{-il^\varepsilon\,\mathcal{V}_D}\, S^\varepsilon_0\, e^{-il^\varepsilon\,\mathcal{V}_D}\, W^{\varepsilon\,*}_{D-}(p)\,, \quad (5.27)$$

with S^ε_0 the standard adiabatic S-matrix, corresponding to the Hamiltonian

$$H^\varepsilon(t) = h_0 + e^{-\varepsilon|t|}h_I + H_0 + H^\varepsilon_{I,r}(t),$$
$$l^\varepsilon \equiv \int_1^\infty \mathrm{d}s e^{-\varepsilon s}/s, \quad \mathcal{V}_D \equiv \frac{e^2}{4\pi}\sum_{i<j}\frac{v_i - v_j}{|v_i - v_j|^3}(V_i - V_J). \quad (5.28)$$

6. LSZ Asymptotic Limits

The model sheds light also on LSZ asymptotic limits in the presence of Coulomb interactions and infinite photon emission. For definiteness, we consider a system of identical charged (fermionic) particles, so that the Hilbert space is of the form $\mathcal{H} = \sum_n \mathcal{H}^n$, $\mathcal{H}^n = L^2_{\rm ant}(\mathrm{I\!R}^{6n}) \oplus \mathcal{H}_F$, $L^2_{\rm ant}$ the space of L^2 functions of n positions and momenta, antisymmetric under odd permutations. The Hamiltonian is given on $\mathcal{H}^n$ by Eqs. (2.4)–(2.6) (the dynamics leaving invariant the antisymmetric wave functions).

In the following, we adopt the relativistic form of the velocity, the second of Eq. (2.3); the same results hold in the non-relativistic case with suitable domain and momentum space restrictions on the charged fields.

6.1. Charged Fields and their Dollard Dynamics

The charged fields $\Phi(q,p)$ are defined, on all $\psi^n \in \mathcal{H}^n$, $f \in \mathcal{S}(\mathrm{I\!R}^6)$, by

$$(\Phi(f)^*\psi^n)(q, q_1 \ldots p \ldots p_n) = \sqrt{(n+1)}\,(f(q,p)\psi^n(q_1 \ldots p_n))_{\rm ant},$$

the index *ant* denoting the projection on the antisymmetric subspace, and satisfy the anti-commutation relations

$$\{\Phi(q,p),\, \Phi^*(q',p')\} = \delta(q-q')\,\delta(p-p')\,, \quad \{\Phi(q,p),\, \Phi(q',p')\} = 0.$$

In the following, it will be convenient to work with the partial Fourier transform $\Psi^*(P,p)$ of $\Phi^*(q,p)$,

$$\Psi^*(P,p) = (2\pi)^{-3/2}\int \mathrm{d}q\, e^{iqP}\,\Phi^*(q,p)$$

and, correspondingly, use states defined by antisymmetric L^2 functions $\psi(P_1, p_1 \dots P_n, p_n)$. As usual,

$$\Psi^*(f) \equiv \int \mathrm{d}P\,\mathrm{d}p\,\Psi^*(P,p)f(P,p), \quad f \in \mathcal{S}(\mathbb{R}^6),$$

$$\rho(P,p) \equiv \Psi^*(P,p)\Psi(P,p), \quad \rho(p) \equiv \int \mathrm{d}P\,\rho(P,p).$$

Then,

$$\rho(g) \equiv \int \mathrm{d}P\,\mathrm{d}p\,\rho(P,p)\,g(P,p),$$

and

$$\rho\rho(F) \equiv \int \mathrm{d}r\,\mathrm{d}r'\,\rho(r)\rho(r')F(r,r'), \quad r \equiv (P,p)$$

are (unbounded) multiplication operators in Fock space, for all measurable g, F, with $F(r,r)$ measurable, and satisfy, for $F(r,r') = F(r',r)$,

$$\begin{aligned}
&e^{i\rho(f)}\Psi^*(P,p)e^{-i\rho(f)} = e^{if(P,p)}\Psi^*(P,p)\,, \\
&e^{\frac{i}{2}\rho\rho(F)}\Psi^*(P,p)e^{-\frac{i}{2}\rho\rho(F)} \\
&\quad = \Psi^*(P,p)\,e^{i\int \mathrm{d}P'\,\mathrm{d}p'\,\rho(P',p')F(P,p,P',p') + \frac{i}{2}F(P,p,P,p)}\,,
\end{aligned}$$

The use of the Wick-ordered product $:\rho\rho:$ leads to the same equation with the omission of $e^{\frac{i}{2}F(P,p,P,p)}$.

The free particle Hamiltonian takes the form

$$h_0 = \int \mathrm{d}P\,\mathrm{d}p\,\Psi^*(P,p)Pv(p)\Psi(P,p)$$

and

$$u_0(t)\Psi^*(f)u_0^*(t) = \Psi^*(f_{-t}), \quad f_{-t}(P,p) = e^{iPvt}f(P,p)\,; \tag{6.1}$$

clearly, $\rho(P,p)$ is invariant under the free evolution.

The Dollard evolution operator $U_D^\varepsilon(t) = u_D(t)\mathcal{U}_0(t)\mathcal{U}_D^\varepsilon(t)$, with $\mathcal{U}_0$ the free electromagnetic evolution, is given by (see Eqs. (3.4), (4.11)–(4.14) and the redefinition in Proposition 5.1),

$$u_D(t) = u_0(t)e^{\frac{i}{2}:\rho\rho:(C_t)}, \tag{6.2}$$

$$C_t \equiv \frac{e^2}{4\pi}\,\mathrm{sign}\,t\,\ln|t|\frac{v-v'}{|v-v'|^3}\,(V-V') \equiv \mathrm{sign}\,t\,\ln|t|\,C(P,\,p,\,P',\,p')\,,$$

$$\begin{aligned}
\mathcal{U}_D^\varepsilon(t) &= e^{-i\int \mathrm{d}p\,[a(F_t^{D\varepsilon}(p))+h.c.]\rho(p)}\,e^{\frac{i}{2}\int \mathrm{d}p\,\mathrm{d}p'\,L_t^\varepsilon(p,p')\rho(p)\rho(p')} \\
&\quad \times e^{i\int \mathrm{d}p\,\rho(p)\,(\delta E(p)\,\int_0^t \mathrm{d}s\,e^{-2\varepsilon|s|} + \delta\varphi_\pm^{D\,\varepsilon}(p))}\,,
\end{aligned}$$

$$F_t^{D\varepsilon}(p) = F_t^{D\varepsilon}(k,\lambda,p) \equiv F_\pm^{D\varepsilon}(k,\lambda,p)(1-e^{-\varepsilon|t|}e^{-i(|k|-vk)t})\,, \tag{6.3}$$

with $F_\pm^{D\varepsilon}(k,\lambda,p)$ ($\pm = \mathrm{sign}\,\mathrm{t}$) the one-particle coherent factor given by Eq. (4.17),

$$L_t^\varepsilon(p,p') \equiv \tfrac{1}{2}\int_0^t \mathrm{d}s\,[G^\varepsilon(s,p,p') + G^\varepsilon(s,p',p)]\,, \tag{6.4}$$

$G(s,p,p')$ given by Eq. (5.5), $\delta E(p)$ by Eq. (5.6).

Since u_D and $\mathcal{U}_D^\varepsilon$ are multiplication operators in the P, p representation, they commute with $\rho(P,p)$ and, therefore,

$$[U_D^\varepsilon(t),\ \rho(P,p)] = 0.$$

The Dollard dynamics of the field $\Psi(f)$ is, therefore, given by

$$\begin{aligned}\Psi_D^{\varepsilon\,*}(f,t) &\equiv U_D^\varepsilon(t)\,\Psi^*(f)\,U_D^{\varepsilon\,*}(t)\\ &= \int \mathrm{d}P\,\mathrm{d}p\, f_{-t}(P,p)\,\Psi^*(P,p)\,e^{-i[a(F_t^{D\varepsilon}(p),t)+h.c.]}\,e^{i\rho\,(\chi_t^\varepsilon(P,p))}\,, \end{aligned}\tag{6.5}$$

where

$$\begin{aligned}a(F_t^{D\varepsilon}(p),t) &\equiv \sum_\lambda \int \mathrm{d}k\, a(k,\lambda)F_t^{D\varepsilon}(k,\lambda,p)\,e^{ikt}\,,\\ \rho\,(\chi_t^\varepsilon(P,p)) &= \int \mathrm{d}P'\,\mathrm{d}p'\,\rho(P',p')(L_t^\varepsilon(p,p') + C_t(P,p,P',p') + c_t^\varepsilon(p,p'))\,,\end{aligned}\tag{6.6}$$

$c_t^\varepsilon(p,p') \equiv \mathrm{Im}(F_t^{D\varepsilon}(p), F_t^{D\varepsilon}(p'))$, having a finite limit for $t\to\infty$ and then for $\varepsilon\to 0$; a "diagonal phase" has been omitted since it vanishes for $t\to\pm\infty$, see Proposition (5.1).

The integration in Eq. (6.5) is well defined since the exponential is strongly continuous in P, p by Lemma 4.2 and Eqs. (6.2) and (5.5) (on a dense domain and, therefore, everywhere). $\rho\,(\chi_t^\varepsilon)$ describes the (logarithmically divergent) Coulomb phases, with C_t arising from the (classical) Coulomb interactions of the particles and L_t^ε representing the Lienard–Wiechert corrections produced by the interaction with the photons.

For the electromagnetic field we have

$$\begin{aligned}a_D^{\varepsilon\,*}(k,\lambda,t) &\equiv U^{D\varepsilon}(t)\,a^*(k,\lambda)\,U^{D\varepsilon\,*}(t)\\ &= e^{-i|k|t}\,a^*(k,\lambda) - i\int \mathrm{d}p\,F_t^{D\varepsilon}(k,\lambda,p)\,\rho(p)\,,\end{aligned}\tag{6.7}$$

on the sum of the N particle domains D introduced before (C^1 wave functions of compact support with values in $D(H_0)$, see Prop. 4.1), still denoted by D. D is also stable under $U_{\mathrm{as}}(t)$, as a consequence of Lemma 4.2.

6.2. LSZ Asymptotic Limits of Heisenberg Fields

a. Charged fields

The Heisenberg asymptotic charged fields $\Psi_{\mathrm{out/in}}(P,p)$ are defined by

$$\Psi^*_{\mathrm{out/in}}(f) = \Omega_\pm\,\Psi^*(f)\,\Omega_\pm^* = \lim_{\varepsilon\to 0}\Omega_\pm^\varepsilon\,\Psi^*(f)\,\Omega_\pm^{\varepsilon *}$$

on $\mathcal{H}_\pm = \Omega_\pm\,\mathcal{H}$, see Eq. (5.8). By Eq.(6.5),

$$\Psi^*_{\mathrm{out/in}}(f) = \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} U^{\varepsilon\,*}(t)\,\Psi_D^{\varepsilon\,*}(f,t)\,U^\varepsilon(t). \tag{6.8}$$

Thus, for the construction of the Heisenberg asymptotic field, the rôle of the Dollard dynamics is to provide explicit corrections to the free evolution, which allow for the existence of the asymptotic limits; the main virtue of the Dollard correction is to subtract the infrared divergent terms which arise in the standard formulation.

It should be stressed that, while in the standard case the interaction picture free fields are isomorphic to the asymptotic Heisenberg fields, the "interaction picture" Dollard fields $\Psi^{\varepsilon}_{D}(f,t)$, $a^{\varepsilon}_{D}(k,\lambda,t)$, which strictly correspond to the fields introduced by Rohrlich for QED [19], have little to do with the asymptotic fields; in particular, their time evolution is substantially different, see below.

Equation (6.8) can be written in an ("adiabatic") LSZ form. To this purpose, we note that $a(F^{D\varepsilon}_t(p),t)+h.c.$ can be written in terms of the usual invariant smearing (in the space variables) of the electromagnetic potential A with the Green function $D(x)\,\theta(x_0)$ of the wave equation,

$$a(F^{D\varepsilon}_t(p),t)+h.c. = e\,v(p)\int_0^t \mathrm{d}s\,e^{-\varepsilon|s|}\,A(\overleftrightarrow{\partial_t}\,D_{t-s}*\tilde{\eta}_{v(p)s})\,, \qquad (6.9)$$

where $A(\overleftrightarrow{\partial_t}\,D)\equiv -A(\dot{D})+\dot{A}(D)$ and $\tilde{\eta}_{v(p)s}(x)\equiv\tilde{\eta}(x-v(p)s)$.

The r.h.s. of Eq. (6.9) has a simple physical interpretation since $Y^{\varepsilon}_{\mu}(x,t;v)\equiv\int_0^t \mathrm{d}s\,e\,v_{\mu}\,e^{-\varepsilon|s|}\,(D_{t-s}*\tilde{\eta}_{vs})(x)$ is the Lienard–Wiechert potential generated at time t, with vanishing Cauchy data at $t=0$, by the current $j^{\varepsilon}_{\mu}(v;x,s)\equiv e\,v_{\mu}\tilde{\eta}(x-vs)\,e^{-\varepsilon|s|}$, $v_{\mu}\equiv(1,v)$.

Then, denoting by Ψ^{ε}_t, A^{ε}_t, ρ^{ε}_t the Heisenberg time evolution of Ψ, A and ρ, under $U^{\varepsilon}(t)$, we have, on $\mathcal{H}_{\pm}$,

$$\Psi^*_{\mathrm{out/in}}(f)=\lim_{\varepsilon\to 0}\lim_{t\to\pm\infty}\int \mathrm{d}P\,\mathrm{d}p\,f_{-t}(P,p)\,\Psi^{\varepsilon\,*}_t(P,p)\,e^{i\rho^{\varepsilon}_t\,(\chi^{\varepsilon}_t(P,p))}$$
$$\times\exp -i\int_0^t \mathrm{d}s\,A^{\varepsilon}_t(\overleftrightarrow{\partial_t}\,D_{t-s}*j^{\varepsilon}(v(p);s)). \qquad (6.10)$$

Equation (6.10) provides an explicit modification of the standard LSZ prescription for the asymptotic limit of the charged fields. It amounts to the insertion of Coulomb phases and of the exponential of an electromagnetic operator, both given by fields at time t, smeared with explicitly given test functions.

The effect of the electromagnetic factor is to provide a shift of the electromagnetic potential at time t by the Lienard–Wiechert potential $Y^{\varepsilon}(x,t;v)$ produced in a Huyghens cone by the above current j^{ε}_{μ}. The Coulomb phases and the exponential of A commute, since so do the corresponding terms in Eq. (6.5).

We stress that the main achievement of the LSZ asymptotic limit, with respect to the interaction picture approach, is fully reproduced in Eq. (6.10), which has a well-defined meaning (within the adiabatic approach) independent of the existence of the Møller operators.

The strong convergence of the density operators on $\Omega_{\pm}\mathcal{H}^n$, $\forall n$, for $t\to\infty$ and then $\varepsilon\to 0$,

$$\rho^{\varepsilon}_t(F)\equiv U^{\varepsilon\,*}(t)\,\rho(F)\,U^{\varepsilon}(t)\to\Omega_{\pm}\,\rho(F)\,\Omega^*_{\pm}\equiv\rho_{\mathrm{out/in}}(F)\,,$$

$F(P,p)$ bounded, follows from the invariance of $\rho(P,p)$ under the Dollard evolution and the norm boundedness of $\rho(F)$ on $\mathcal{H}^n$. Clearly, $\rho_{\rm out/in}(P,p) = \Psi^*_{\rm out/in}(P,p)\Psi_{\rm out/in}(P,p)$; both $\rho_{out}(F)$ and $\rho_{in}(F)$ define commutative algebras.

Furthermore, by the same argument, if F^ε_t converges uniformly to F_∞, then

$$\rho^\varepsilon_t(F^\varepsilon_t) \to \rho_{\rm out/in}(F_\infty), \quad e^{i\rho^\varepsilon_t(F^\varepsilon_t)} \to e^{i\rho_{\rm out/in}(F_\infty)}, \tag{6.11}$$

strongly on $\Omega_\pm \mathcal{H}^n$, $\forall n$; by a density argument, the second of Eqs. (6.11) only requires the uniform convergence of F^ε_t on compact sets.

b. Electromagnetic fields

For the asymptotic limit of the electromagnetic field, it is convenient to work with their Weyl exponentials,

$$W(f,\lambda) \equiv e^{-i(a(f,\lambda)+h.c.)}.$$

Then, for (complex) f and $|k|^{-1/2}f$ in L^2, on $\mathcal{H}_\pm$,

$$\begin{aligned} W_{\rm out/in}(f,\lambda) &\equiv \Omega_\pm\, W(f,\lambda)\, \Omega^*_\pm \equiv e^{-i(a_{\rm out/in}(f,\lambda)+h.c.)} \\ &= \lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} U^{\varepsilon\,*}(t)\, U^\varepsilon_D(t)\, W(f,\lambda)\, U^{\varepsilon\,*}_D(t)\, U^\varepsilon(t) \equiv \lim_{\varepsilon,\,t} W^\varepsilon_t(f,\lambda). \end{aligned} \tag{6.12}$$

By Eq. (6.7),

$$W^\varepsilon_t(f,\lambda) = U^{\varepsilon\,*}(t)\, W(f_{-t},\lambda)\, e^{-i[\int {\rm d}p\, {\rm d}k\ i\, \overline{F^{D\varepsilon}_t}(k,\lambda,p)\, \rho(p)\, f(k)+h.c.]}\, U^\varepsilon(t)\,,$$

with $f_t(k) \equiv f(k)\, e^{-i\,|k|\,t}$. As in Eq. (6.11), the second factor converges for $t\to\pm\infty$ and $\varepsilon\to 0$, to

$$\exp(i\rho_{\rm out/in}(J)((f,\lambda)+h.c.))\,,$$

where $J(k,\lambda,p)$, given by Eq. (4.17) with $\varepsilon=0$, is integrated with $\rho_{\rm out/in}(p)$ and $f(k)$. In fact, $k^{1/2}J \in L^2(d^3k)$, with norm bounded uniformly in p, so that, after integration in k, $F^{D\,\varepsilon}_\pm(p)$ converges uniformly for bounded p.
Hence, also the first factor converges. Its limit is a unitary operator which is strongly continuous in α for $f\to\alpha f$, since so are $W_{\rm out/in}(\alpha f,\lambda)$ (by definition) and the limit of the second factor, by the above estimate. Therefore,

$$U^{\varepsilon\,*}(t)\, W(f_{-t},\lambda)\, U^\varepsilon(t) \to e^{-i(b_{\rm out/in}(f,\lambda)+h.c.)}. \tag{6.13}$$

Since $a_{\rm out/in}$, $a^*_{\rm out/in}$, briefly $a^\#_{\rm out/in}$ and $\rho_{\rm out/in}(J)(f,\lambda)$ are well defined on $\Omega_\pm D$, on such a domain one has

$$a^\#_{\rm out/in}(f,\lambda) = b^\#_{\rm out/in}(f,\lambda) - \rho_{\rm out/in}(J)(f,\lambda). \tag{6.14}$$

Equation (6.14) states that $b^\#_{\rm out/in}$ are related to $a^\#_{\rm out/in}$ by the second quantized version of the transformation of Eq. (5.18).
$b^\#_{\rm out/in}(f,\lambda)$ define *free massless fields*; in fact, by Eq. (5.19), for $t\to\infty$ and then $\varepsilon\to 0$,

$$U^*(\tau)\, U^{\varepsilon\,*}(t)\, U^\varepsilon(t+\tau) \equiv T^\varepsilon(t+\tau,\tau) \to 1$$

strongly and the same holds for its adjoint; therefore,

$$U^*(\tau)\, b^{\#}_{\mathrm{out/in}}(f,\lambda)\, U(\tau) = \lim_{\varepsilon,\, t} U^*(\tau)\, U^{\varepsilon\,*}(t)\, a^{\#}(f_{-t},\lambda)\, U^{\varepsilon}(t)\, U(\tau)$$
$$= \lim_{\varepsilon,\, s} T^{\varepsilon}(s,\tau) U^{\varepsilon\,*}(s)\, a^{\#}(f_{\tau-s},\lambda)\, U^{\varepsilon}(s) T^{\varepsilon\,*}(s,\tau) = b^{\#}_{\mathrm{out/in}}(f_\tau,\lambda).$$

We shall denote by $B_{\mathrm{out/in}}(x,t)$ the corresponding fields in Minkowski space. They are the result of LSZ formulas for massless asymptotic fields, on the whole scattering spaces, with no need of Dollard corrections, in agreement with the general analysis by [5] on the asymptotic limit in the massless case.

c. Asymptotic algebras and space–time translations

In the following, when no confusion arises, the indexes *out/in* shall be replaced by the single index *as*; we only recall that J and H_{as} are independent of the two alternatives. Since

$$[\rho_{\mathrm{as}}(F),\, \rho_{\mathrm{as}}(G)] = [\rho_{\mathrm{as}}(F),\, a^{\#}_{\mathrm{as}}(f,\lambda)] = 0\,,$$

both B_{out} and B_{in} satisfy the CCR and commute with $\rho_{\mathrm{out/in}}(F)$, respectively. As asymptotic field algebras we take the polynomial algebra $\mathcal{F}_{\mathrm{as}}$ generated by the free photon fields $B_{\mathrm{as}}(x,0)$, their time derivative and by the asymptotic charged fields $\Psi^{\#}_{\mathrm{as}}$, smeared, e.g., with test functions in $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}(\mathbb{R}^6)$, respectively.

By construction, Eqs. (6.8) and (6.12), $a^{\#}_{\mathrm{as}}$ and $\Psi^{\#}_{\mathrm{as}}$ are canonical independent fields (at equal times); $a^{\#}_{\mathrm{as}}$ is well defined on $\Omega_\pm D$, which is stable under $\Psi^{\#}_{\mathrm{as}}(f)$, for

$$\hat f(q,p) = (2\pi)^{-3/2} \int \mathrm{d}P\, e^{-iqP} f(P,p)$$

of compact support. Equation (6.14) implies, therefore, the following commutation relations between B_{as} and Ψ_{as}:

$$[b^*_{\mathrm{as}}(k,\lambda),\, \Psi^*_{\mathrm{as}}(P,p)] = J(k,\lambda,p)\, \Psi^*_{\mathrm{as}}(P,p), \tag{6.15}$$
$$[b_{\mathrm{as}}(k,\lambda),\, \Psi^*_{\mathrm{as}}(P,p)] = J(k,\lambda,p)\, \Psi^*_{\mathrm{as}}(P,p). \tag{6.16}$$

They hold, together with the equations for their h.c., on $\Omega_\pm D$, for the operators obtained by smearing $b^{\#}_{\mathrm{as}}$ with f, $k^{-1/2}f(k), k^{1/2}f(k) \in L^2$, and Ψ^*_{as} with g, $\hat g$ of compact support.

They extend, by closure of the corresponding operators, to $g \in \mathcal{S}(\mathbb{R}^6)$, since $||k^{1/2}J(k,\lambda,p)||_{L^2(\mathrm{d}^3k)}$, is of order $|p|$. In particular, the commutation relations, Eqs. (6.15) and (6.16), hold for the fields which generate $\mathcal{F}_{\mathrm{as}}$, all smeared with test functions in $\mathcal{S}$, on $D_{\mathrm{as}} \equiv \mathcal{F}_{\mathrm{as}}\psi_0$, ψ_0 the vacuum vector; $D_{\mathrm{out/in}}$ are dense in $\mathcal{H}_\pm$ by Eq. (6.14) and cyclicity of the vacuum for the fields at $t = 0$.

With respect to the standard case, the above non-standard commutation relations are the only modification, produced in the asymptotic algebras by the LSZ asymptotic formula for the charged fields, Eq. (6.10).

By Proposition 5.3, the Hamiltonian is given, on $\Omega_\pm D(h_0 + H_0)$, by

$$H = \Omega_\pm H_{\rm as}\Omega_\pm^* = H_{\rm as}(\Psi_{\rm as},\, a_{\rm as}) = h_0(\Psi_{\rm as}) + H_0(b_{\rm as}) \\ = \int \mathrm{d}P\,\mathrm{d}p\, v(p)\, P\, \Psi_{\rm as}^*(P,p)\Psi_{\rm as}(P,p) + \sum_\lambda \int d^3k\, |k|\, b_{\rm as}^*(k,\lambda)\, b_{\rm as}(k,\lambda). \tag{6.17}$$

Similarly, from Eq. (5.21) one has, for the generator of space translations, Eq. (2.7),

$$\mathcal{P} = \int \mathrm{d}P\,\mathrm{d}p\, P\, \Psi_{\rm as}^*(P,p)\Psi_{\rm as}(P,p) + \sum_\lambda \int \mathrm{d}^3k\, k\, b_{\rm as}^*(k,\lambda)\, b_{\rm as}(k,\lambda) \\ \equiv P_c(\Psi_{\rm as}) + P_{\rm ph}(b_{\rm as}) \tag{6.18}$$

In the above decompositions, Eqs. (6.17) and (6.18), the two terms commute, due to Eqs. (6.15) and (6.16). The commutativity of the two terms is also implied by the fact that $H_0(b_{\rm as})$ and $P_{\rm ph}(b_{\rm as})$ implement the space–time translations of the free massless field $b_{\rm as}$; this reproduces the structure advocated in [15] in terms of explicit functions of the asymptotic fields, with $H_{\rm charge} \equiv h_0(\Psi_{\rm as})$, $P_{\rm charge} \equiv P_c(\Psi_{\rm as})$.

The space–time evolution of $\Psi_{\rm as}^*(P,p)$ follows from Eqs. (6.15)–(6.18):

$$U(a,t)^*\, \Psi_{\rm as}^*(P,p)\, U(a,t) \\ = e^{-b_{\rm as}^*(J^{a,t}(p)-J(p))}\, e^{iPv(p)t}\, e^{-iPa}\, \Psi_{\rm as}^*(P,p)\, e^{b_{\rm as}(\bar{J}^{a,t}(p)-J(p))}, \tag{6.19}$$

with $J^{a,t}(k,\lambda,p) \equiv e^{i(kt-ka)} J(k,\lambda,p)$. Hence, even if the Hamiltonian is the sum of two free Hamiltonians, *the time evolution of* $\Psi_{\rm as}^*$ *is not free as a consequence of the commutation relations, Eqs.* (6.15) *and* (6.16). Thus, on the N charged particle states Ψ_N obtained by applying $\Psi_{\rm as}^*$ to the vacuum,

$$H\,\Psi_N \neq h_0(\Psi_{\rm as})\,\Psi_N.$$

Given Eqs. (6.17) and (6.18), the non-commutativity of $\Psi_{\rm as}$ and $b_{\rm as}$, Eqs. (6.15) and (6.16), is crucial for the absence of an eigenvalue at the bottom of the spectrum of the Hamiltonian in the one-particle sector, at given $\mathcal{P} = P_{\rm charge} + P_{\rm ph}(b_{\rm as})$ and given particle momentum p. In fact, in general $\mathcal{P}$ and $\sum_i p_i$ commute with H; by Eqs. (6.17) and (6.18), on one-particle states ψ,

$$H\psi = (h_0(\psi_{\rm as}) + H_0(b_{\rm as}))\psi = (\mathcal{P}\, v(p) + [H_0(b_{\rm as}) - P_{\rm ph}(b_{\rm as})\, v(p)])\psi.$$

The operator in square brackets is positive since $|v(p)| < 1$ and the bottom of the spectrum of H is an eigenvalue iff ψ is the vacuum vector for $b_{\rm as}$, which is admitted iff $J(k,\lambda,p) \in L^2(\mathrm{d}^3k)$, for a set of p of positive measure.

It is also important to stress that neither the commutation relations nor the time evolution of the asymptotic fields are affected by the Coulomb and Lienard–Wiechert corrections in the LSZ procedure.

One may introduce the fields

$$\Psi_{\rm as}^*(P,p,x) \equiv e^{-b_{\rm as}^*(J^x(p)-J(p))}\, \Psi_{\rm as}^*(P,p)\, e^{b_{\rm as}(\bar{J}^x(p)-J(p))}, \tag{6.20}$$

which transform covariantly under space–time translations $U(a)$, $a = (a_i, a_0)$:

$$U(a)^* \, \Psi^*_{\rm as}(P, p, x) \, U(a) = e^{iPv(p)a_0} \, e^{-iPa} \, \Psi^*_{\rm as}(P, p, x + a). \tag{6.21}$$

Their commutation relations with $b^{\#}_{\rm as}$ are

$$[b^{\#}_{\rm as}(k, \lambda), \, \Psi^*_{\rm as}(P, p, x)] = e^{\pm ikx} \, J(k, \lambda, p) \, \Psi^*_{\rm as}(P, p, x). \tag{6.22}$$

The field algebra $\mathcal{F}_{\rm as}$ identifies a unique C^* algebra $\mathcal{A}_{\rm as}$, generated by $\Psi_{\rm as}$ and the Weyl exponentials of $b_{\rm as}$,

$$\mathcal{W}_{as}(f) = e^{-i(b_{as}(f) + h.c.)},$$

with $f = f(k, \lambda)$ a C^∞ complex function of fast decrease. Eqs. (6.15) and (6.16) become

$$\mathcal{W}_{\rm out/in}(f) \, \Psi^*_{\rm out/in}(P, p) \, \mathcal{W}_{\rm out/in}(f)^* = \Psi^*_{\rm out/in}(P, p) e^{2i \, {\rm Re} J(f, p)} \, ,$$
$$J(f, p) = \sum_\lambda \int d^3k \, J(k, \lambda, p) \, f(k, \lambda). \tag{6.23}$$

The algebras $\mathcal{A}_{\rm out/in}$ have the structure of a semidirect product of fermion and Weyl algebras, of the same form as that discussed by Herdegen [17, 18]. However, the time evolution is very different, since, in Herdegen algebra, $\Psi_{\rm as}$ is a free field (of definite mass). In fact, the time evolution of the Herdegen variables is given by the sum of the free Hamiltonian for $\Psi_{\rm as}$ and a free (e.m.) Hamiltonian commuting with $\Psi_{\rm as}$; in our algebra, this would amount to replace, in Eq. (6.17), $H_0(b_{\rm as})$ with $H_0(a_{\rm as})$. Moreover, the representation of the semidirect product algebra adopted by Herdegen differs from ours by the absence of the vacuum state, which would give rise, in his case, to charged states of definite mass.

d. Asymptotic form of the corrections to the LSZ formula

The modification of the standard LSZ formulas for the charged fields arising from the electromagnetic interaction can be written in terms of asymptotic e.m. fields and asymptotic currents.

Proposition 6.1. *The asymptotic charged fields, Eq. (6.10), are also given by the following LSZ formula*

$$\Psi^*_{\rm out/in}(f) = \lim_{\varepsilon \to 0} \lim_{t \to \pm\infty} \int {\rm d}P \, {\rm d}p \, f_{-t}(P, p) \, \Psi^{\varepsilon \, *}_t(P, p) \, e^{i\rho^\varepsilon_t \, (C_t(P, p))}$$
$$e^{i\rho_{\rm out/in} \, (L^\varepsilon_\pm(p) + c(p))} \exp -iB_{\rm out/in}(j^\varepsilon_\pm(v(p))) \, , \tag{6.24}$$

on $\mathcal{H}_\pm$, with the notation of Eq. (6.6), C_t and L^ε_t given by Eqs. (6.3) and (6.4), $j^\varepsilon_{\pm \, j}(v; x) \equiv j^\varepsilon_j(v; x) \, \theta(\pm x^0)$, so that

$$B_{\rm out/in}(j^\varepsilon_\pm(v)) = e \int_0^{\pm\infty} {\rm d}s \, e^{-\varepsilon|s|} \int d^3x \, B_{\rm out/in}(x + vs, s) \, v \, \tilde{\eta}(x) \, ;$$

$$c(p, p') \equiv \lim_{\varepsilon \to 0} \int d^4x \, d^4y \, D_{ij}(x - y)(-j_{\pm \, i}(v', x) + 1/2 \, j^\varepsilon_{\pm \, i}(v', x)) \, j^\varepsilon_{\pm \, j}(v, y) \, ,$$

with $j_{\pm \, j} \equiv j^{\varepsilon = 0}_{\pm \, j}$, $v' \equiv v(p')$.

Proof. The basic content of Eq. (6.24) is that, in Eq. (6.10) one may take first the asymptotic limits of the fields ρ_t^ε and A_t^ε, keeping ε and t fixed in the test functions. The resulting procedure for the limits will be shown to give the same result as the diagonal procedure, Eq. (6.10), apart from the correction given by $c(p)$.

1. We first control the ε, t limits of the fields in the phase and in the electromagnetic factor, keeping ε, t fixed in the smearing functions.
Eqs. (6.13) and (6.14) imply that

$$V_A(p, \varepsilon', \tau, \varepsilon, t) \equiv e^{-i\int_0^t dx_0\, A_\tau^{\varepsilon'}(\overleftrightarrow{\partial}_\tau D_{\tau-x_0} * j_\pm^\varepsilon(v(p), x_0))}$$

converges strongly to

$$e^{-i\int_0^t \mathrm{d}x_0 B_{\mathrm{out/in}}(j_\pm^\varepsilon(v(p),x_0))} \\ = e^{-i(a_{\mathrm{out/in}}(F_t^{D\,\varepsilon}(p))+h.c.)} e^{-(\rho_{\mathrm{out/in}}(i\,J)(F_t^{D\,\varepsilon}(p))-h.c.)} \tag{6.25}$$

as $\tau \to \pm\infty$ and then $\varepsilon' \to 0$, since $\int_0^t \mathrm{d}s\, e^{-\varepsilon|s|}\, D_{\tau-s} * \tilde{\eta}_{vs}$ is a regular solution of the wave equation, corresponding to $f_{-\tau}$ in Eq. (6.13).
Similarly, $V_\rho(p, \varepsilon', \tau, \varepsilon, t) \equiv \exp i\rho_\tau^{\varepsilon'}(L_t^\varepsilon(p))$ converges strongly, as $\tau \to \pm\infty$ and then $\varepsilon' \to 0$, ε, t fixed, to

$$V_\rho(p, 0, \pm\infty, \varepsilon, t) \equiv \exp i\rho_{\mathrm{out/in}}(L_t^\varepsilon(p)).$$

2. We must prove that, in Eq. (6.10), the phase and electromagnetic factor can be replaced by their asymptotic version, apart from a phase, i.e., on $\mathcal{H}_\pm$, omitting the p dependence and using $[\rho_{\mathrm{out/in}},\, b_{\mathrm{out/in}}] = 0$,

$$V_\rho(\varepsilon, t, \varepsilon, t) V_A(\varepsilon, t, \varepsilon, t) - e^{i\rho_{\mathrm{out/in}}(L_t^\varepsilon)} V_A(0, \pm\infty, \varepsilon, t)\, e^{+i\rho_{\mathrm{out/in}}(c - c_t^\varepsilon)} \to 0 \tag{6.26}$$

strongly as $t \to \pm\infty$ and then $\varepsilon \to 0$.
Using $[U_D^\varepsilon(\tau)\,,\, \rho(p)] = 0$ and (see Eq. (6.7))

$$U_D^{\varepsilon'}{}^*(\tau)\, a(k,\lambda)\, U_D^{\varepsilon'}(\tau) = e^{-ik\tau}\left(a(k,\lambda) - \rho(i\overline{F_\tau^{D\varepsilon'}}))(k,\lambda)\right),$$

the first term in Eq. (6.26) can be written as

$$\Omega_t^\varepsilon\, e^{i\rho(L_t^\varepsilon(p))}\, e^{-i(a(F_t^{D\,\varepsilon}(p))+h.c.)}\, \Omega_t^{\varepsilon\,*}\, e^{-[\rho_t(\overline{F_t^{D\varepsilon}})(F_t^{D\,\varepsilon}(p))\,-\,h.c.]}. \tag{6.27}$$

2a. The last factor in Eq. (6.27) involves the smearing of $\rho(p')$ with $-2i$ times the imaginary part of the scalar product $(F_t^{D\,\varepsilon}(p'),\, F_t^{D\,\varepsilon}(p))$ as functions of k and λ. Similarly for the last factor in Eq. (6.25). By an explicit control, $\mathrm{Im}\,(F_t^{D\,\varepsilon}(p'),\, F_t^{D\,\varepsilon}(p))$ converges, for $t \to \pm\infty$ and then $\varepsilon \to 0$, to

$$\lim_{\varepsilon\to 0}\lim_{t\to\pm\infty} \mathrm{Im}\,(-iJ(p'),\, F_t^{D\,\varepsilon}(p)) - \tfrac{1}{2}c_t^\varepsilon(p,p') + \tfrac{1}{2}c(p,p')\,,$$

uniformly for p, p' bounded. Both terms are non-vanishing due to the presence of ε singularities in the corresponding integrals.
Then, by the second of Eqs. (6.11), the last factor in Eqs. (6.27) and (6.25) converge, for $t \to \pm\infty$ and then $\varepsilon \to 0$, and their limits differ by the factor $e^{i\rho_{\mathrm{out/in}}(c'(p))}$, $c' = c - \lim_{\varepsilon,t} c_t^\varepsilon$. Both limits leave $\mathcal{H}_\pm$ invariant.

2b. We have to discuss the convergence of the remaining factors in Eq. (6.27), on $\mathcal{H}_\pm$. Since $\Omega_t^{\varepsilon\,*}$ converges on $\mathcal{H}_\pm$, its limit inverts $\Omega_\pm$; Eq. (6.26) reduces, therefore, to

$$(\Omega_t^\varepsilon - \Omega_\pm)\, e^{i\rho(L_t^\varepsilon)}\, e^{-i(a(F_t^{D\,\varepsilon}(p))+h.c.)} \to 0\,, \text{ strongly on } \mathcal{H}. \qquad (6.28)$$

Since, for $t \to \pm\infty$, $L_t^\varepsilon(p')$ converges to $L_\pm^\varepsilon(p')$ uniformly for bounded p', $F_t^{D\,\varepsilon}(k,\lambda,p) \to F_\pm^{D\,\varepsilon}(k,\lambda,p)$ in $L^2(d^3k)$ and $\Omega_t^\varepsilon \to \Omega_\pm^\varepsilon$, we are left with the limit in ε. Since $\Omega_\pm^\varepsilon = W_\pm^\varepsilon\,\omega_\pm\,$, $[W_\pm^\varepsilon, \rho(p')] = 0 = [W_\pm, \rho(p')]\,$, $\omega_\pm\,\rho = \rho_{\mathrm{out/in}}\,\omega_\pm$, the exponential of ρ can be moved to the left, becoming $e^{i\rho_{\mathrm{out/in}}(L_\pm^\varepsilon)}$.
Since $\omega_\pm\, a(F_\pm^{D\,\varepsilon}(p)) = a(F_\pm^{D\,\varepsilon}(p_\pm))\,\omega_\pm$ and $\omega_\pm L^2(\Gamma)\times\mathcal{H}_F = \mathcal{H}_\pm$, we are reduced to

$$(W_\pm^\varepsilon - W_\pm)\, e^{-i(a(F_\pm^{D\,\varepsilon}(p_\pm))+h.c.)} \to 0 \qquad (6.29)$$

strongly on $\mathcal{H}_\pm$. Using Eq. (5.1) and the fact that, for any n-particle subspace $L^2(\Gamma_\pm^{(n)})$, $W_\pm^\varepsilon$ are multiplication operators $W_\pm^\varepsilon(q',p')$, one has

$$W_\pm^\varepsilon\, e^{-i(a(F_\pm^{D\,\varepsilon}(p_\pm))+h.c.)} = e^{-<\Delta F_\pm^\varepsilon(q',p'),\, F_\pm^{D\,\varepsilon}(p_\pm)>}\, e^{-i(a(F_\pm^{D\,\varepsilon}(p_\pm))+h.c.)}\, W_\pm^\varepsilon.$$

By the proof of Proposition 5.1, $|k|^{1/2}F_\pm^{D\,\varepsilon}(p_\pm)$ and $|k|^{-1/2}\Delta F_\pm^\varepsilon(q',p')$ converge in $L^2(\mathrm{d}^3k)$, for almost all (q',p') in $\Gamma_\pm$; therefore, the above phases converge to

$$\lim_{\varepsilon\to 0}\lim_{\varepsilon'\to 0}\langle \Delta F_\pm^{\varepsilon'}(q',p'),\, F_\pm^{D\,\varepsilon}(p_\pm)\rangle\,,$$

a.e. in $\Gamma_\pm$. Eq. (6.27) then follows from

$$(W_\pm^\varepsilon - W_\pm) \to 0 \quad \text{on } \mathcal{H}_\pm, \quad W_\pm\mathcal{H}_\pm = \mathcal{H}_\pm.$$

□

In Eq. (6.24), the electromagnetic correction required for the LSZ asymptotic limit of the charged field is replaced by a string-like factor involving the massless asymptotic photon field $B_{\mathrm{out/in}}$; the string is a straight line, with direction given by the momentum variable p of the test function f.

From this point of view, a convenient strategy for the asymptotic limit of the charged fields is to first obtain the massless asymptotic photon field and then use the corresponding string as the e.m. correction in the LSZ formula for the charged field.

In Eq. (6.24), also the Lienard–Wiechert modification involves the asymptotic density $\rho_{\mathrm{out/in}}$ of the particle momentum p, whose construction does not require the LSZ limit of the charged field (see Eq. (6.11)). Since $L_\pm^\varepsilon(p,p') = L_\pm^\varepsilon(p',p)$, the Lienard–Wiechert factor has no effect on the canonical structure of the asymptotic charged fields.

On the contrary, the finite phase factor involving $\rho_{\mathrm{out/in}}(c(p))$ (not necessary for the existence of the LSZ limit) has an antisymmetric part, which cancels the changes induced by the exponential of the asymptotic e.m. field on the charged field anti-commutation relations.

To write the Coulomb correction $\rho_{\mathrm{out/in}}(C_t(P,p))$ in terms of asymptotic variables, one should adiabatically switch also the Coulomb potential, since

$\exp i\rho_{\mathrm{as}}(C_t) - \exp i\rho_t^{\varepsilon}(C_t)$ does not converge for $t \to \pm\infty$. The asymptotic form of the Coulomb correction will be more conveniently discussed in a further work, within a framework which does not use at all the adiabatic switching, on the basis of an improved Dollard subtraction.

7. Conclusions

The strategy advocated by Kulish–Faddeev [21] and Rohrlich [19] for QED, supported by the cancelation of the infrared divergences in the perturbation expansion, has been rigorously controlled in a translationally invariant model reproducing basic infrared problems of QED.

Technically, this has been obtained by introducing an adiabatic procedure and mass renormalization counter-terms; both ingredients are characteristic of the Feynman–Dyson approach, in its non-perturbative version discussed by Hepp [16].

The field theory version of the model provides a strategy for the control of the asymptotic limit of the Heisenberg charged fields, through an explicit modification of the LSZ (HR) formulas. The resulting asymptotic fields are very different from those advocated by Zwanziger [27], Schweber [24] and Rohrlich [19].

The modifications of the LSZ formula may be written in two alternative forms: one is given in terms of the photon field at the same time t of the charged field, the other by a string-like factor involving the massless asymptotic photon field. The formulas only involve Heisenberg fields and geometrical factors; they represent the transcription of the KFR strategy into the LSZ (Haag–Ruelle) approach and are, therefore, good candidates for asymptotic formulas in QED (see Eq. (1.7)). Making explicit their dependence on the origin x of the string required for the LSZ limit, they produce asymptotic charged fields $\psi_{\mathrm{out/in}}(p,x)$, space–time covariant in the momentum variable p lying on the mass shell and in the space–time point x.

For the one-particle sector, our first LSZ formula is close to that proved by Chen et al. [9], Eqs. (III.29) and (III.30), for one-particle states in non-relativistic QED. In fact, their LSZ modification factor $W_{k,\sigma_t}(v,t)$ acts on a (previously constructed) one-electron state which requires an infrared dressing by a Weyl operator W of the same form, at $t = 0$, to mimic the action of an interpolating field. Then, modulo different infrared regularizations and other technical points, their correction factor $W_{k,\sigma_t}(v,t)\,W^{-1}$ corresponds to our e.m. correction for the one-particle case.

For the asymptotic limit of the electromagnetic field, the ordinary LSZ (HR) limit (with no Dollard correction) applies, in accord with Buchholz result [5], and defines massless fields $B_{\mathrm{out/in}}$. The canonical fields $\Psi_{\mathrm{out/in}}$ and $B_{\mathrm{out/in}}$ generate asymptotic algebras with a semidirect product structure, their commutation relations, Eqs. (6.15) and (6.16), being determined by the electromagnetic field corrections to the LSZ formula for the charged fields. Their time evolution is generated by the sum of the free Hamiltonians of $\Psi_{\mathrm{out/in}}$ and $B_{\mathrm{out/in}}$ Eq. (6.17).

Such a decomposition, which also holds for the momentum, reproduces the splitting $P^\mu = P^\mu_{\mathrm{charge}} + P^\mu_{\mathrm{ph}}$, advocated in [15]. The resulting structure is substantially different from the Herdegen proposal of a semidirect product of asymptotic algebras [17,18], which involves a different Hamiltonian, giving rise to free charged asymptotic fields.

The absence of charged states of definite mass, which in Herdegen analysis requires the absence of the vacuum, follows here from the above decomposition of the Hamiltonian and the non-trivial commutation relations between Ψ_{as} and B_{as}. On the other hand, the mass shell appears in the spectrum of P^μ_{charge}, which acts on $\psi_{\mathrm{as}}(p,x)$ leaving x fixed.

The space–time transformations of Ψ_{as}, B_{as} are also different from those of the asymptotic fields proposed by Zwanziger [27], mainly because his charged fields include (Coulomb–Lienard–Wiechert) phase operators which spoil the group property of their time dependence.

Appendix A. Asymptotic Limits of Classical Configurations

Lemma A.1. *Let γ_t, $t \in \mathbb{R}$, be invertible measure preserving transformations of $\Gamma = \mathbb{R}^{6N}$, with the Lebesgue measure $\mathrm{d}x$, defining, therefore, unitary operators ω_t, $\omega_t\psi(x) \equiv \psi(\gamma_t x)$, in $L^2(\Gamma, \mathrm{d}x)$. If ω_t converge strongly, for $t \to \pm\infty$, to $\omega_\pm$, and $\psi(\gamma_t x)$ converge pointwise, $\forall\psi \in D$, $D \equiv \cup_n C^1(A_n)$, A_n open bounded sets covering Γ apart from a set of zero measure, then there exist measurable subsets $\Gamma_\pm$ and measure preserving transformations $\gamma_\pm : \Gamma_\pm \to \Gamma$ such that:*

$$\gamma_t x \to_{t\to\pm\infty} \gamma_\pm x \quad \forall x \in \Gamma_\pm, \tag{A.1}$$

$(\omega_\pm\psi)(x)$ vanishes (a.e.) in the complement of $\Gamma_\pm$ and

$$(\omega_\pm\psi)(x) = \psi(\gamma_\pm x), \quad \forall\psi \in L^2(\Gamma, \mathrm{d}x)\ , \quad x \in \Gamma_\pm, \tag{A.2}$$
$$\omega_\pm L^2(\Gamma, \mathrm{d}x) = L^2(\Gamma_\pm, \mathrm{d}x). \tag{A.3}$$

Proof. Let $\Gamma_\pm$ be the complements of the sets

$$\{x : \psi(\gamma_t\, x) \to_{t\to\pm\infty} 0, \quad \forall\psi \in D\}.$$

$\Gamma_\pm$ are measurable since D is separable in the Sup norm. Let $x \in \Gamma_+$; then there exists $\psi_x \in D$ such that $\gamma_t\, x \in \mathrm{supp}(\psi_x)$, $\forall t > t_x$. $\forall\epsilon > 0$, the (compact) support of ψ_x can be covered by a finite number of balls B_i^ϵ, of radius ϵ, and a partition of unity argument shows that for some index i, $\gamma_t\, x \in B_i^\epsilon$ for all large t, so that $\gamma_t\, x$ has the Cauchy property; we denote by $(\gamma_+\, x)$ its limit. For $\psi \in D$, Eq. (A.2) follows by the identification of L^2 limits with pointwise limits. Since D is dense in L^2 and ω_+ is an isometry, γ_+ preserves the measure, so that Eq. (A.2) extends to L^2 and the image of ω_+ can be identified with $L^2(\Gamma_\pm, \mathrm{d}x)$. □

Appendix B. Completeness of the Møller Operators in the Repulsive Two-Particle Case

We consider the non-relativistic case, the relativistic case being very similar. For $N = 2$, in the reference frame where $p_1 = -p_2 \equiv p$, all trajectories have non-zero relative asymptotic velocity $v_\pm$ and, therefore,

$$|x_t| \equiv |q_{1\,t} - q_{2\,t}| \geq (1-\epsilon)\,|v_\pm|\,|t| \quad \text{for } |t| \quad \text{large.} \tag{B.1}$$

This allows for the existence of the limit of $u_D^*(t)\,u(t)\,\psi$ as $t \to \pm\infty$, which implies the unitarity of $\omega_\pm$. In fact,

$$(\mathrm{d}/\mathrm{d}t)\,(u_D^*(t)\,u(t))\,\psi = u_D^*(t)\,u(t)\,(w(x_t)X(t) - w(v_t t;0)V(t)\,t)\,\psi, \tag{B.2}$$

where

$$\begin{aligned} v \equiv \dot{q}_1 - \dot{q}_2\,, \quad V \equiv V_1 - V_2\,, \quad X \equiv (Q_1 - Q_2)\,, \\ X(t) = u^*(t)Xu(t)\;, \quad V(t) = u^*(t)Vu(t). \end{aligned} \tag{B.3}$$

Now, by Eq. (3.14), $u(t)$ induces a linear transformation on X, V, with coefficients given by matrices $A(t) = A_{\alpha\,\beta}(q,p,t)$, etc., $\alpha, \beta = 1, 2, 3$,

$$X(t) = A(t)X + B(t)V, \quad V(t) = C(t)X + D(t)V, \tag{B.4}$$

which (in the non-relativistic case, with reduced mass = 1) satisfy

$$\begin{aligned} \dot{A} \; &= C, \quad \dot{B} = D, \\ \dot{C}_{\alpha\beta}(t) \; &= -\mathcal{V}''_{\alpha\gamma}(t)\,A_{\gamma\,\beta}(t)\;, \quad \dot{D}_{\alpha\,\beta}(t) = -\,-\,\mathcal{V}''_{\alpha\gamma}(t)\,B_{\gamma\,\beta}(t)\;, \end{aligned}$$

with $\mathcal{V}''_{\alpha\gamma} \equiv \partial^2\mathcal{V}/\partial x_\alpha \partial x_\gamma$. Then,

$$C(t) = C(t_0) - \int_{t_0}^{t} \mathcal{V}''(s)\,\mathrm{d}s A(t_0) - \int_{t_0}^{t} \mathcal{V}''(s)\,\mathrm{d}s \int_{t_0}^{s} C(s')\,\mathrm{d}s'.$$

By Eq. (B.1), $\mathcal{V}''(t) = O(t^{-3})$, so that

$$\sup_{t\geq t_0} ||C(t)|| \leq ||C(t_0)|| + ||A(t_0)||\,O(t_0^{-2}) + \sup_{t\geq t_0} ||C(t)||\,O(t_0^{-1})$$

and, therefore,

$$\sup_{t\geq t_0} ||C(t)|| \leq ||C(t_0)||\,(1 + O(t_0^{-1})). \tag{B.5}$$

This implies $||A(t)|| = O(t)$ and $||\dot{C}(t)|| = O(t^{-2})$.
Then, $||A(t) - C(t)t|| = O(\ln t)$. The same conclusion holds for $B(t)$ and for $D(t)$. This yields the estimate, $\forall\psi \in D_0^1$ (see Proposition 3.1)

$$||(w(x_t)(X(t) - V(t)\,t)\,\psi|| = O(t^{-2}\ln t). \tag{B.6}$$

On the other hand,

$$||(w(x_t) - w(v_t t;0))\,V(t)\,t)\,\psi|| = O(t^{-2}\ln t) \tag{B.7}$$

since $||V(t)\psi||$ is bounded by Eq. (B.5) for $C(t)$ and D and Eq. (B.1) implies

$$|x_t - v_t t| = O(\ln t)$$

on the support of ψ, which is left invariant by $V(t)$. Then, the argument following Eq. (3.17) applies.

Appendix C. Proof of Lemma 4.2

Existence and unitarity of $U(f)$, $\forall f \in L^2(d^3k)$, follows from the essential self-adjointness of $a(f) + a(f)^*$ on the domain D_{fin} of vectors describing finite numbers of particles. If $f, |k|^{-1/2} f \in L^2(d^3k)$, then $a(f) + a(f)^*$ is well defined on $D(H_0)$ as a consequence of the following estimates for $a^\#$, $a^\# = a, a^*$, on D_{fin} and, therefore, on $D(H_0)$:

$$\begin{aligned} ||a^\#(f)\Psi||^2 &\le ||k^{-1/2} f||^2 (\Psi,\, H_0\, \Psi) + ||f||^2\, ||\Psi||^2 \\ &\le ||k^{-1/2} f||^2 (a\, ||H_0\Psi||^2 + (1/4a)||\Psi||^2) + ||f||^2 ||\Psi||^2, \quad a > 0. \end{aligned} \tag{C.1}$$

Furthermore, under the above assumptions of differentiability, one has

$$U(f_{\alpha+\epsilon}) - U(f_\alpha) = U(f_\alpha + \epsilon f'_\alpha + \epsilon g_\alpha(\epsilon)) - U(f_\alpha),$$

with $g(\epsilon)$, $|k|^{-1/2}\, g(\epsilon) \to 0$ in $L^2(d^3k)$ as $\epsilon \to 0$, which can also be written as

$$U(f_\alpha)\, [U(\epsilon f'_\alpha) U(\epsilon g_\alpha(\epsilon))\, e^{(\epsilon\, C_\alpha + \epsilon\, o(\epsilon))} - \mathbf{1}].$$

Now, by Eq. (C.1), $\forall \psi \in D(H_0)$, $h, |k|^{-1/2} h \in L^2(d^3k)$

$$||(\mathrm{d}/\mathrm{d}\lambda)\, U(\lambda h)\psi||^2 \le 2\, ||h|k|^{-1/2}||^2\, (||H_0\psi||^2 + ||\psi||^2) + 4||h||^2\, ||\psi||^2,$$

which implies

$$||(U(h) - \mathbf{1})\, \psi|| \le ||h|k|^{-1/2}||\, c_\psi + ||h||\, d_\psi. \tag{C.2}$$

Then, for $h = \epsilon\, g_\alpha(\epsilon)$, $\forall \psi \in D(H_0)$,

$$\epsilon^{-1}\, (U(\epsilon g_\alpha(\epsilon)) - \mathbf{1})\psi \to 0, \quad \text{as } \epsilon \to 0.$$

On the other hand, by Stone theorem, on $D(H_0)$

$$\epsilon^{-1}\, (U(\epsilon f'_\alpha) - \mathbf{1}) \to i\, (a(f'_\alpha) + a(f'_\alpha)^*)$$

so that, on $D(H_0)$,

$$\mathrm{d}U(f_\alpha)/\mathrm{d}\alpha = U(f_\alpha)\, [i(a(f'_\alpha) + a(f'_\alpha)^*) + C_\alpha]$$

and Eq. (4.20) follows.

Finally, $f_t(k) = f(k)\, e^{-i|k|t}$ satisfies the above conditions for $f_\alpha(k)$, if $f(k)$, $|k|^{-1/2} f(k), |k| f(k) \in L^2(d^3k)$ and, therefore, $\forall \psi \in D(H_0)$

$$e^{iH_0 t}\, U(f)\, \psi = U(f_{-t})\, e^{iH_0 t}\, \psi \tag{C.3}$$

is differentiable in L^2 with respect to t, and this implies that $U(f)\, \psi \in D(H_0)$.

Appendix D. Adiabatic Switching of the Coulomb Interaction

To display the complete factorization of the infrared divergences, we introduce the Hamiltonian

$$H^{\varepsilon,\varepsilon'} = h_0 + H_0 + e^{-\varepsilon'|t|} h_I + H^\varepsilon_{I,\,r}(t). \tag{D.1}$$

We proceed as in Sect. (3.4), with $H^\varepsilon_D(t)$ replaced by

$$H^{\varepsilon,\varepsilon'}_D(t) = h_0 + H_0 + e^{-\varepsilon'|t|} h_I(vt, Vt; 0) + H^\varepsilon_{I,\,D}(t)\,,$$

with $H^{\varepsilon}_{I,\,D}(t)$ still given by Eq. (4.2). For $t \to \pm\infty$, the result are the Møller operators

$$\Omega^{\varepsilon,\varepsilon'}_{\pm} = W^{\varepsilon,\varepsilon'}_{0\pm}\,\omega^{\varepsilon'}_{\pm}\,W^{\varepsilon\,*}_{D\pm} = W^{\varepsilon,\varepsilon'}_{0\pm}(q,p)\,W^{\varepsilon\,*}_{D\pm}(p^{\varepsilon'}_{\pm})\,\omega^{\varepsilon'}_{\pm}$$

with $p^{\varepsilon'}_{\pm}\,\omega^{\varepsilon'}_{\pm} = \omega^{\varepsilon'}_{\pm}\,p$, as in Eq. (3.12).
In fact, the estimate Eq. (3.16) holds uniformly in ε' and, therefore, the limits in Eqs. (3.9)–(3.11) are uniform in ε'. This also implies that $\omega^{\varepsilon'}_{\pm}$ converges to $\omega_{\pm}$ as $\varepsilon' \to 0$. Moreover, with an obvious extension of the notation of Appendix A, as a consequence of the adiabatic cutoff, both $\gamma^{\varepsilon'}_{t}(q,p)$ and its inverse converge pointwise a.e. in Γ as $t \to \pm\infty$. The limit of the first, $\gamma^{\varepsilon'}_{\pm}(q,p)$, satisfies

$$(\omega^{\varepsilon'}_{\pm}\psi)(q,p) = \psi(\gamma^{\varepsilon'}_{\pm}(q,p)).$$

Convergence of $\gamma^{\varepsilon'}_{t}(q,p)^{-1}$ a.e. in Γ implies $\Gamma^{\varepsilon'}_{\pm} = \Gamma$ and then Eq. (A.3), applied to $\omega^{\varepsilon'}_{\pm}$, implies that $\omega^{\varepsilon'}_{\pm}$ are unitary operators. By the above uniformity argument, Eq. (A.1) applies to $\gamma^{\varepsilon'}_{\pm}$, i.e., for $\varepsilon' \to 0$,

$$\gamma^{\varepsilon'}_{\pm}(q,p) \to \gamma_{\pm}(q,p)$$

a.e. in $\Gamma_{\pm}$. In particular,

$$p^{\varepsilon'}_{\pm}(q,p) \to p_{\pm}(q,p), \quad \forall (q,p) \in \Gamma_{\pm}$$

and Eq. (3.19) holds uniformly in ε'. This allows for the control of the convergence of the W operators as in Sect.(5.1), see Proposition (5.1):

$$\lim_{\varepsilon\to 0} W^{\varepsilon,\varepsilon}_{0\pm}(q,p)\,W^{\varepsilon\,*}_{D\pm}(p^{\varepsilon}_{\pm}) = \lim_{\varepsilon\to 0}\lim_{\varepsilon'\to 0} W^{\varepsilon,\varepsilon'}_{0\pm}(q,p)\,W^{\varepsilon\,*}_{D\pm}(p^{\varepsilon'}_{\pm}) = W_{\pm}.$$

In fact, the estimates for the convergence of $\Delta F^{\varepsilon,\varepsilon'}_{\pm}$ in the proof of Proposition 5.1 only rely on the limit in Eq. (3.19), which is uniform in ε'.
Such uniformity also implies that the estimates of Eqs. (5.13) and (5.14), and therefore the convergence of the phases, are uniform in ε'. In conclusion,

$$\lim_{\varepsilon\to 0}\Omega^{\varepsilon,\varepsilon}_{\pm} = \lim_{\varepsilon\to 0}\Omega^{\varepsilon}_{\pm} = \Omega_{\pm}.$$

By an explicit calculation, the Møller operators $\omega^{\varepsilon}_{\pm}$ are related to the standard adiabatic Møller operators $\omega^{\varepsilon}_{0\,\pm}$, defined solely by an adiabatic switching of the Coulomb interaction with no Dollard correction, by

$$\omega_{\pm} = \lim_{\varepsilon\to 0}\omega^{\varepsilon}_{0\,\pm}\,e^{\pm i l^{\varepsilon}\,\mathcal{V}_D}, \quad l^{\varepsilon} = \lim_{t\to\infty} l^{\varepsilon}_{t}$$
$$l^{\varepsilon}_{t} \equiv \int_1^t e^{-\varepsilon s}\,1/s\,\mathrm{d}s, \quad \mathcal{V}_D = \frac{e^2}{4\pi}\sum_{i<j}\frac{v_i - v_j}{|v_i - v_j|^3}\,(V_i - V_j). \tag{D.2}$$

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