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Homogenized Diffusion Limit of a Vlasov–Poisson–Fokker–Planck Model

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Abstract. The approximation by diffusion and homogenization of the initial-boundary value problem of the Vlasov–Poisson–Fokker–Planck model is studied for a given velocity field with spatial macroscopic and microscopic variations. The L^1-contraction property of the Fokker–Planck operator and a two-scale Hybrid-Hilbert expansion are used to prove the convergence towards a homogenized Drift–Diffusion equation and to exhibit a rate of convergence.

1. Introduction

The paper is intended to study the approximation by diffusion and homogenization of the initial-boundary value problem for the Vlasov–Poisson–Fokker–Planck system. The Fokker–Planck equation at the diffusion scale reads as

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\left[v \cdot \nabla_x f^\varepsilon + \nabla_v \cdot \left(\left(\frac{1}{\varepsilon}u^\varepsilon(x) - \nabla_x \Phi^\varepsilon\right) f^\varepsilon\right)\right] = \frac{1}{\varepsilon^2}\nabla_v \cdot \left[v\, f^\varepsilon + \nabla_v f^\varepsilon\right]. \tag{1}$$

The distribution $f^\varepsilon = f^\varepsilon(t, x, v)$ is a positive function depending on the time $t \geq 0$, the position x which belongs to a bounded subset $\omega \subset \mathbb{R}^d$ and the velocity $v \in \mathbb{R}^d$, where $d = 1, 2$ or 3 is the dimension. The parameter ε is a positive number related to the scaled thermal mean free path. The potential $\Phi^\varepsilon = \Phi^\varepsilon(t, x)$ describes the self-variations of the charge density [27]. It solves the homogeneous Poisson equation

$$\begin{cases} -\Delta_x \Phi^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon \, \mathrm{d}v, \\ \Phi^\varepsilon{}_{|\partial\omega} = 0. \end{cases} \tag{2}$$

The vector field u^ε is oscillating spatially with both macroscopic and microscopic oscillations. For simplicity, we shall denote the cell period by the unit d-dimensional cube $Y = (0,1)^d$ and consider the sequence u^ε:

$$u^\varepsilon(x) := u\left(x, \frac{x}{\varepsilon}\right) - \varepsilon \nabla_x \Phi_b(t,x) \tag{3}$$

where u is a given Y-periodic function with respect to the second variable, with values in $\mathbb{R}^d$ and Φ_b is a harmonic extension on $\bar{\omega}$ of a given potential boundary data, also denoted Φ_b $(\Delta_x \Phi_b = 0, \;\; x \in \omega)$.

Our aim is to analyze the convergence, as ε goes to zero, of the solution of the system (1)–(3) subject to a given initial data which might depend on ε:

$$f^\varepsilon(t=0) = f_I^\varepsilon \tag{4}$$

and a specular reflection boundary condition: $\forall\, t \geq 0$ and $(x,v) \in \partial\omega \times \mathbb{R}^d$

$$f^\varepsilon(t,x,v) = f^\varepsilon(t,x,v - 2(v \cdot n(x))\, n(x)\,), \tag{5}$$

where $n(x)$ is the outward unit vector in the position $x \in \partial\omega$.

In kinetic transport theory (gas dynamics, neutron transport, plasmas,...), the diffusion approximation is, since many years, a large field of research. Earlier works go back to transport equations associated with force-free case [2,3]. Progressively, different fluid models have been obtained as a hydrodynamic limit of kinetic equations for prescribed macroscopic potential [25,26]. Nowadays, more attention is paid to the variations of the potential by coupling the distribution with the Poisson equation [4,5,11,15,16,18] and recently by adding a potential with microscopic variations in the same order of the mean free path of the diffusive operator, leading to phenomena of homogenization [10,17,19,22,30]. The concept of relative entropy dissipation is useful to approximate the solution of the kinetic model to its corresponding equilibrium state [6,7,12,21,24,28,29]. Various models are approximated using ideas like Hilbert expansion method and Chapman–Enskog development. When some coefficients have both macroscopic and microscopic variations, the multi-scale Hilbert development and the multi-scale convergence [1] are well adapted to extract the homogenized effects (see for example [10]).

The present paper is devoted to the analysis of the diffusion limit of a Vlasov–Fokker–Planck model where a general form for the velocity field (not necessarily a gradient) and microscopic variations are considered. We will also take into account the self variation of the potential in the one dimension by coupling the density with the Poisson equation. In this case ($d = 1$), we establish uniform a priori estimates for hyper well-prepared initial data. Note that, with the Poisson coupling and for $d > 1$, we can deal with solution for the Vlasov–Poisson–Fokker–Planck system in a renormalized sense. We present (for instance) the analysis of the convergence in a linear and multi-dimensional setting. The case of adding the Poisson coupling and $d > 1$ will be the goal of a forthcoming paper.

The paper is organized as follows. In Sect. 2, we give some notations and preliminaries on the properties of the Fokker–Planck operator and the two-scale convergence. Section 3 is devoted to the formal derivation of the homogenized fluid model for a force-free case, letting the expression of the cell operator. In Sect. 3, we present the assumptions we require throughout the analysis and establish the main result (Theorem 4.1). In Sect. 5 we prove rigorously the convergence (ε goes to zero) in the one dimension ($d = 1$) where we take into account the Poisson coupling. The proof is based on a two-scale Hybrid-Hilbert expansion. In Sect. 6, we extend the two-scale method to the linear setting with a general form of the velocity field and without restriction on the dimension. We finish by analyzing the case of ballistic motion for a quasi-periodic vector field.

2. Preliminaries

We denote by ω the position space and by $\Omega := \omega \times \mathbb{R}^d$ the phase space. For all time $T > 0$, ω_T and Ω_T stand as:

$$\omega_T := (0,T) \times \omega \quad \text{and} \quad \Omega_T := (0,T) \times \Omega.$$

The incoming and the outgoing parts of the boundary, $\partial\Omega := \partial\omega \times \mathbb{R}^d$, are

$$\partial\Omega^{\pm} = \{(x,v) \in \partial\Omega,\ \pm(v \cdot n(x)) > 0\}.$$

The charge and current densities are given by

$$\varrho^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon(t,x,v)\,\mathrm{d}v \quad \text{and} \quad j^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} f^\varepsilon(t,x,v)\,v\,\mathrm{d}v. \tag{6}$$

The total mass and the kinetic energy are defined by

$$\mathcal{M}^\varepsilon(t) = \int_\Omega f^\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \quad \text{and} \quad \mathcal{K}^\varepsilon(t) = \int_\Omega f^\varepsilon(t,x,v)\,\frac{|v|^2}{2}\,\mathrm{d}x\,\mathrm{d}v. \tag{7}$$

For convenience, the Fokker–Planck operator is denoted by

$$L^*(f) = \Delta_v f + \nabla_v \cdot (v\,f) \tag{8}$$

It is the adjoint (in $\mathcal{D}'$) of

$$L(f) = \Delta_v f + v \cdot \nabla_v f \tag{9}$$

We remark that

$$L^*(f) = \nabla_v \cdot \left[M(v)\nabla_v \frac{f}{M(v)}\right]$$

where M is the normalized *Maxwellian* with mean velocity equal to zero:

$$M(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}.$$

The Fokker–Planck operator, acting on the Hilbert space $L^2(M^{-1}\,\mathrm{d}v) := L^2(\mathbb{R}^d,\ M^{-1}\,\mathrm{d}v)$, is an unbounded operator with domain

$$D(L^*) := \left\{ f \in L^2(M^{-1}\,\mathrm{d}v) \,/\, \nabla_v\left(\frac{f}{M}\right) \in [L^2(M\,\mathrm{d}v)]^d \right\}.$$

It follows that,

Lemma 2.1. [6]

1. *The null space of* L^* *is spanned by the Maxwellian* M:
$$L^*(f) = 0 \Longleftrightarrow f = \varrho(t,x)\, M(v).$$
2. *Entropy dissipation: for all non-decreasing function* $\mathcal{H}$ *on* $\mathbb{R}^+$,
$$Diss_{\mathcal{H}}(f) := \int_{\mathbb{R}^d} L^*(f)\,\mathcal{H}\left(\frac{f}{M}\right)\,\mathrm{d}v = -\int_{\mathbb{R}^d} M\left|\nabla_v \frac{f}{M}\right|^2 \mathcal{H}'\left(\frac{f}{M}\right) \le 0.$$
In particular,
$$Diss_{log}(f) := \int_{\mathbb{R}^d} L^*(f)\,log\left(\frac{f}{M}\right)\,\mathrm{d}v = -\int_{\mathbb{R}^d}\left|2\nabla_v\sqrt{f} + v\sqrt{f}\right|^2\mathrm{d}v, \quad (10)$$
therefore,
$$Diss_{log}(f) = 0 \Leftrightarrow L^*(f) = 0 \Leftrightarrow f = \varrho\, M(v).$$

In all the sequel, we will use the subscript $(\ldots)_\#$ to mean that we consider functions defined on the whole space in y and Y-periodic with respect to y. Indeed, the following spaces $\mathcal{C}_\#$, $\mathcal{C}^\infty_\#$ and $L^p_\#$ correspond to continuous, indefinitely differentiable and L^p_{loc} functions respectively, defined on $\mathbb{R}^d$. For two-scale oscillating function $\psi \equiv \psi(x, \frac{x}{\varepsilon})$, the notation $(\psi)_\varepsilon$ refers to the value of ψ at $(x, x/\varepsilon)$.

Due to the presence of two-scale sequence u^ε, it is more convenient to use the two-scale convergence [1] to highlight the effect of the fast variation. To do this, we identify each bounded sequence $\eta^\varepsilon := \eta^\varepsilon(t,x,v)$ of $L^2_{\mathrm{loc}}(\Omega_T)$ to its two-scale Riesz's representation [22]. We denote this two-scale function by $\tilde{\eta^\varepsilon} := \tilde{\eta^\varepsilon}(t,x,y,v) \in L^2_{\mathrm{loc}}(\Omega_T;\ L^2_\#(Y))$ in the sense that for all $\psi \in \mathcal{D}(\Omega_T;\ \mathcal{C}_\#(Y))$,
$$\int_{\Omega_T} \eta^\varepsilon(t,x,v)\psi\left(t,x,\frac{x}{\varepsilon},v\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$
$$= \int_{\Omega_T}\int_Y \tilde{\eta^\varepsilon}(t,x,y,v)\psi(t,x,y,v)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v. \quad (11)$$

Definition 2.2. Let η^ε be a bounded sequence $L^2(\Omega_T)$. We say that η^ε two-scale converges towards $\tilde{\eta} \equiv \eta(t,x,y,v)$ if $\tilde{\eta^\varepsilon}$ converges weakly (towards $\tilde{\eta}$) in $L^2(\Omega_T \times Y)$. Equivalently, for all $\psi \in \mathcal{D}(\Omega_T;\ \mathcal{C}_\#(Y))$,
$$\int_{\Omega_T} \eta^\varepsilon(t,x,v)\psi\left(t,x,\frac{x}{\varepsilon},v\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$
$$\to \int_{\Omega_T}\int_Y \tilde{\eta}(t,x,y,v)\psi(t,x,y,v)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v.$$
As a consequence, for all bounded sequence η^ε in $L^2(\Omega_T)$, there exists a subsequence which two-scale converges to $\tilde{\eta}$.

Lemma 2.3. [1] *Let* η^ε *be a sequence that two-scale converges to* $\tilde{\eta} := \eta(t,x,y,v)$ *in* $L^2(\Omega_T)$. *Then,* η^ε *weakly converges in* $L^2(\Omega_T)$ *towards* η, *satisfying*
$$\eta(t,x,v) := \int_Y \tilde{\eta}(t,x,y,v)\,\mathrm{d}y$$

and

$$\lim_{\varepsilon\to 0}\|\eta^\varepsilon\|_{L^2(\Omega_T)} \geq \|\tilde\eta\|_{L^2(\Omega_T\times Y)} \geq \|\eta\|_{L^2(\Omega_T)}.$$

Furthermore, if

$$\lim_{\varepsilon\to 0}\|\eta^\varepsilon\|_{L^2(\Omega_T)} = \|\tilde\eta\|_{L^2(\Omega_T\times Y)},$$

we say that η^ε *converges in two-scale strongly and if* $\tilde\eta \in L^2(\Omega_T;\, \mathcal{C}_\#(Y))$, *then*

$$\lim_{\varepsilon\to 0}\left\|\eta^\varepsilon - \tilde\eta\left(t, x, \frac{x}{\varepsilon}, v\right)\right\|_{L^2(\Omega_T)} = 0.$$

Remark 2.4. 1. We notice that the concept of two-scale convergence is also defined in L^p space for all $p \in]1,\infty]$. Further extensions on two-scale convergence in $L\log L$ can be found in [14].

2. If a sequence converges in two-scale strongly towards a y-independent function, then it converges in L^p-strong.
3. If u^ε converges in two-scale strongly to $\tilde u$ and v^ε converges in two-scale to $\tilde v$ (in L^2 for example). Then,

$$\int_\omega u^\varepsilon(x)\, v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \to \int_\omega\int_Y \tilde u(x,y)\tilde v(x,y)\psi(x,y), \quad \forall\, \psi \in \mathcal{D}(\omega;\, \mathcal{C}_\#(Y)).$$

3. Formal Asymptotics: Two-Scale Hilbert Expansion

Let us consider the linear force-free case (without potential):

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\, v\cdot\nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon(x)\cdot\nabla_v f^\varepsilon = \frac{1}{\varepsilon^2}(\Delta_v f^\varepsilon + \nabla_v\cdot(v\, f^\varepsilon))$$

and assume that f^ε is a bounded sequence in L^2_{loc}. According to the representation (11), there exists $\tilde f^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}^+\times\omega\times Y\times\mathbb{R}^d)$ such that for all $\psi\in\mathcal{C}(\Omega_T;\mathcal{C}_\#(Y))$, we have

$$\int_\Omega f^\varepsilon(t,x,v)\psi\left(t,x,\frac{x}{\varepsilon},v\right) = \int_\Omega\int_Y \tilde f^\varepsilon(t,x,y,v)\psi(t,x,y,v). \tag{12}$$

According to this representation, $\nabla_x f^\varepsilon$ becomes $\nabla_x\tilde f^\varepsilon + \frac{1}{\varepsilon}\nabla_y\tilde f^\varepsilon$ and $\tilde f^\varepsilon$ solves the following equivalent two-scale Fokker–Planck equation:

$$\partial_t\tilde f^\varepsilon + \frac{1}{\varepsilon}\, v\cdot\nabla_x\tilde f^\varepsilon + \frac{1}{\varepsilon^2}\left[v\cdot\nabla_y + u(x,y)\cdot\nabla_v - L^*\right](\tilde f^\varepsilon) = 0 \tag{13}$$

where L^* is given by (8). We denote by

$$\mathcal{L}_x = -v\cdot\nabla_y - \Delta_v + (v - u(x,y))\cdot\nabla_v$$

and its distributional adjoint (appearing in (14)) stands as

$$\mathcal{L}_x^* := v\cdot\nabla_y + u(x,y)\cdot\nabla_v - L^*.$$

So, that (13) is equivalent with

$$\partial_t\tilde f^\varepsilon + \frac{1}{\varepsilon}\, v\cdot\nabla_x\tilde f^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}_x^*(\tilde f^\varepsilon) = 0 \tag{14}$$

Now, let us derive formally the limit fluid model. Assume that $\tilde{f}^\varepsilon$ behaves (as ε goes to zero) like

$$\tilde{f}^\varepsilon(t,x,y,v) \sim \tilde{f}(t,x,y,v) + \varepsilon\,\tilde{f}_1(t,x,y,v) + \varepsilon^2\,\tilde{f}_2(t,x,y,v) + \cdots \tag{15}$$

where the coefficients of the development $\tilde{f}, \tilde{f}_i$, $i = 1, 2, \ldots$ are Y-periodic with respect to the fast variable $y = x/\varepsilon$. By plugging this development in (14) we obtain

$$\partial_t \left[\tilde{f} + \varepsilon\,\tilde{f}_1 + \varepsilon^2\,\tilde{f}_2 + \cdots\right] + \frac{1}{\varepsilon}\left[v\cdot\nabla_x \tilde{f} + \varepsilon\, v\cdot\nabla_x\tilde{f} + \varepsilon^2\, v\cdot\nabla_x \tilde{f}_2 + \cdots\right]$$
$$\frac{1}{\varepsilon^2}\mathcal{L}_x^*(\tilde{f}) + \frac{1}{\varepsilon}\mathcal{L}_x^*(\tilde{f}_1) + \mathcal{L}_x^*(\tilde{f}_2) + \cdots = 0$$

Identifying the coefficients of the same power of ε, we get,

$$\mathcal{L}_x^*(\tilde{f}) := v\cdot\nabla_y\tilde{f} + u(x,y)\cdot\nabla_v\tilde{f} - L^*\tilde{f} = 0, \tag{16}$$

$$\mathcal{L}_x^*(\tilde{f}_1) = -v\cdot\nabla_x\tilde{f} \tag{17}$$

and

$$\mathcal{L}_x^*(\tilde{f}_2) = -\partial_t\tilde{f} - v\cdot\nabla_x\tilde{f}_1. \tag{18}$$

Now, we shall study the spectral properties of the cell operator $\mathcal{L}_x^*$ in order to expect the leading profile $\tilde{f}$ and obtain the homogenized fluid model. We define the weighted Hilbert space:

$$\mathbb{L}_M^2 = \left\{\tilde{f} \in L^2\left(\mathrm{d}y\mathrm{d}v/M\right)/\tilde{f} : \text{Y-periodic with respect to}\, y\right\}.$$

Proposition 3.1. [20] *The operator $\mathcal{L}_x^*$ is an unbounded operator on $\mathbb{L}_M^2$ with domain*

$$D(\mathcal{L}_x^*) = \left\{\tilde{f} \in \mathbb{L}_M^2/\ \nabla_v(\tilde{f}/M) \in [L^2(M(v)\mathrm{d}y\mathrm{d}v)]^d\right\}$$

and satisfying:

1. *There exists a non negative and normalized function $\varphi = \varphi(x,y,v) \in D(\mathcal{L}_x^*)$ such that*
$$\mathcal{N}(\mathcal{L}_x^*) = \mathbb{R}\,\varphi.$$
2. *The range $\mathcal{R}(\mathcal{L}_x^*)$ is characterized by*:
$$\mathcal{R}(\mathcal{L}_x^*) = \left\{\tilde{g} \in \mathbb{L}_M^2 \Big/ \int_{\mathbb{R}^d}\int_Y \tilde{g}(y,v)\,\mathrm{d}y\,\mathrm{d}v = 0\right\}.$$
3. *Let $\tilde{g} \in \mathcal{R}(\mathcal{L}_x^*)$. Then, there exists $\tilde{f} \in \mathcal{D}(\mathcal{L}_x^*)$ such that $\mathcal{L}_x^*\tilde{f} = \tilde{g}$ which is uniquely defined under the condition $\int_{\mathbb{R}^d}\int_Y \tilde{f}(y,v)\,\mathrm{d}y\,\mathrm{d}v = 0$.*
4. *The equilibrium state φ has an exponential decay as $|v|$ goes to infinity*:
$$\exists\,\beta < 1\ /\ \forall p \in [1,k], \quad |\partial_{x_i}^p\varphi| + |\partial_{y_i}^p\varphi| + |\partial_{v_i}^p\varphi| \le C_k\, e^{-\beta\,|v|^2/2},$$
where the constant C_k depends only on k and $\|u\|_{W^{k,\infty}(\omega\times Y)}$.

According to this proposition, the leading function, $\tilde{f}$ has the following profile:

$$\tilde{f}(t,x,y,v) = \varrho(t,x)\varphi(x,y,v), \tag{19}$$

where

$$\varrho = \int_{\mathbb{R}^d}\int_Y \tilde{f}(y,v)\,\mathrm{d}y\,\mathrm{d}v$$

is the homogenized density associated with $\tilde{f}$ and φ is the defined cell function. Replacing $\tilde{f}$ in (17) by this expression, we infer

$$\mathcal{L}_x^*\tilde{f}_1 = -(v\cdot\nabla\varrho)\,\varphi - \varrho\, v\cdot\nabla_x\varphi.$$

Using the assumption **A4**, the function φ satisfies

$$\int_{\mathbb{R}^d}\int_Y v_i\varphi\,\mathrm{d}y\,\mathrm{d}v = 0 \tag{20}$$

which implies also that $\int_{\mathbb{R}^d}\int_Y v\cdot\nabla_x\varphi\,\mathrm{d}y\,\mathrm{d}v = 0$ and then, $v_i\,\varphi$ and $v\cdot\nabla_x\varphi = \nabla_x\cdot(v\,\varphi)$ belong to $[\mathcal{R}(\mathcal{L}_x^*)]$. Choosing $\tilde{f}_1 \in \mathcal{R}(\mathcal{L}_x^*)$, we get

$$\tilde{f}_1(t,x,y,v) = -\nabla_x\varrho(t,x)\cdot{\mathcal{L}_x^*}^{-1}(v\varphi) - \varrho\,{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi). \tag{21}$$

Equation (18) becomes

$$-\mathcal{L}_x^*(\tilde{f}_2) = \partial_t\varrho\,\varphi - \nabla_x\cdot\left[v\otimes{\mathcal{L}_x^*}^{-1}(v\varphi)\;\nabla_x\varrho\right] - \nabla_x\cdot\left[\varrho\,\left(v\,{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi)\right)\right]. \tag{22}$$

We can define the diffusion matrix $\mathbb{D}(x)$ and the coefficient $\xi(x)$ by

$$\begin{cases}\mathbb{D}(x) = \int_{\mathbb{R}^d}\int_Y v\otimes{\mathcal{L}_x^*}^{-1}(v\varphi)\,\mathrm{d}y\,\mathrm{d}v\ \in\mathbb{R}^{d\times d},\\[2ex] \xi(x) = \int_{\mathbb{R}^d}\int_Y v\,{\mathcal{L}_x^*}^{-1}\left[v\cdot\nabla_x\varphi\right]\in\mathbb{R}^d.\end{cases} \tag{23}$$

The fact that the cell function φ is normalized (in (y,v)) and the solvability condition

$$\int_{\mathbb{R}^d}\int_Y \mathcal{L}_x^*(\tilde{f}_2)\,\mathrm{d}y\,\mathrm{d}v = 0$$

yield the following *homogenized Drift–Diffusion equation*:

$$\partial_t\varrho + \nabla_x\cdot\left[-\mathbb{D}(x)\nabla_x\varrho - \xi(x)\,\varrho\right] = 0. \tag{24}$$

Lemma 3.2. *The diffusion matrix* $\mathbb{D}(x)$*, given by* (23)*, is positive.*

Proof. The function $\varphi\in\mathcal{N}(\mathcal{L}_x^*)$ is a non negative function and satisfies

$$\int_{\mathbb{R}^d}\int_Y v\,\varphi\,\mathrm{d}y\,\mathrm{d}v = 0.$$

Therefore, there exists a vector function $\Psi = \Psi(x,y,v)\in\mathbb{R}^d$ such that

$$\mathcal{L}_x\Psi = -v.$$

$$\begin{aligned}\mathbb{D}(x) &:= \int_{\mathbb{R}^d}\int_Y v\otimes{\mathcal{L}^*}_x^{-1}(v\varphi)\,\mathrm{d}y\,\mathrm{d}v = -\int_{\mathbb{R}^d}\int_Y \mathcal{L}_x\Psi\otimes{\mathcal{L}^*}_x^{-1}(v\varphi)\,\mathrm{d}y\,\mathrm{d}v\\ &= -\int_{\mathbb{R}^d}\int_Y (\Psi\otimes v)\,\varphi\,\mathrm{d}y\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_Y (\Psi\otimes\mathcal{L}_x\Psi)\,\varphi\,\mathrm{d}y\,\mathrm{d}v\end{aligned}$$

Let, for $i,\, j \in \{1,\ldots,d\}$, $D_{ij}(x)$ the general coefficient of the matrix $\mathbb{D}(x)$. Then,

$$\begin{aligned}(D_{ij}+D_{ji})(x) &= \int_{\mathbb{R}^d}\int_Y \left(\Psi_i\,\mathcal{L}_x\Psi_j + \Psi_j\,\mathcal{L}_x\Psi_i\right)\,\varphi\,\mathrm{d}y\,\mathrm{d}v\\ &= \int_{\mathbb{R}^d}\int_Y \left(\mathcal{L}_x(\Psi_i\Psi_j) + 2\nabla\Psi_i\cdot\nabla\Psi_j\right)\,\varphi\,\mathrm{d}y\,\mathrm{d}v\\ &= 2\int_{\mathbb{R}^d}\int_Y \nabla_v\Psi_i\cdot\nabla_v\Psi_j\;\varphi\,\mathrm{d}y\,\mathrm{d}v\end{aligned}$$

Let $\xi\in\mathbb{R}^d$ and $\Xi = (\Psi_1\xi_1,\ldots,\Psi_d\xi_d)$. Then,

$$\langle (D+D^T)\xi,\xi\rangle = \int_{\mathbb{R}^d}\int_Y \left(\|\nabla_v\Xi\|^2\right)\varphi\mathrm{d}y\,\mathrm{d}v \geq 0.$$

□

4. Assumptions and Main Result

Throughout the analysis, we shall assume

A1. *Smoothness of the velocity field.* The velocity field u is smooth, bounded and Y-periodic: $\exists\, k\geq 2\;/\;u\in W^{k,\infty}(\omega\times Y)$ and

$$\int_Y u(x,y)\,\mathrm{d}y = \bar{u}$$

where $\bar{u}$ is independent of x.

A2. *Smoothness of the potential.* The potential and its time derivative satisfy:

$$(\Phi_b,\partial_t\Phi_b)\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+;\; W^{2,\infty}\times W^{1,\infty}).$$

A3. *Positive and hyper well-prepared initial data.* The initial distribution satisfies

$$f_I^\varepsilon(x,v) = \varrho_I(x)\varphi\left(x,\frac{x}{\varepsilon},v\right)\geq 0$$

where φ is given in Proposition 3.1 and the sequence $\varrho_I\in L^\infty(\omega)$.

A4. *No ballistic motion.* The cell function φ, given by Proposition 3.1 satisfies the condition

$$\int_{\mathbb{R}^d}\int_Y v\,\varphi\mathrm{d}y\,\mathrm{d}v = 0.$$

Our main result is the following

Theorem 4.1. *Assume that Assumptions* **A1**–**A4** *are satisfied. Then,*

1. *In the one-dimensional case. Let* $(f^\varepsilon,\Phi^\varepsilon)$ *be a weak solution of the Fokker–Planck–Poisson system* (1)–(5) *in one dimension. Then,* $\forall\, T>0$, $\exists\, C_T > 0/$

$$\sup_{t\leq T}\int_\Omega \left|f^\varepsilon(t,x,v) - \varrho(t,x)\,\varphi\left(x,\frac{x}{\varepsilon},v\right)\right|\;\mathrm{d}x\,\mathrm{d}v \leq C_T\varepsilon.$$

In particular,

$$\sup_{t\le T}\int_\omega \left|\varrho^\varepsilon(t,x)-\varrho(t,x)\int_{\mathbb{R}^d}\varphi\left(x,\frac{x}{\varepsilon},v\right)\mathrm{d}v\right|\le C_T\,\varepsilon$$

and

$$\Phi^\varepsilon\to\Phi \quad in \quad L^2((0,T)\times\omega)$$

where (ϱ,Φ) *is the solution of the following homogenized Drift–Diffusion-Poisson system:*

$$\begin{cases}\partial_t\varrho+\partial_x j=0,\\ j=-\mathbb{D}(x)\,\partial_x\varrho-\xi(x)\,\varrho+\varrho\,\lambda(x)\,\partial_x(\Phi+\Phi_b),\\ -\partial^2_{xx}\Phi=\varrho_I,\\ (j\cdot n(x),\Phi)_{|x\in\partial\omega}=0,\\ \varrho_{|\,t=0}=\varrho_I\end{cases}\tag{25}$$

The diffusion coefficient $\mathbb{D}(x)$ *and* $\xi(x)$ *are given by* (23), $\lambda(x)$ *is given in* (36) *and the cell function* φ *is given by Proposition* 3.1.

2. *In the linear multi-dimensional case. Let* f^ε *be a weak solution of the linear Fokker–Planck* (1)–(5) *(with* $\Phi^\varepsilon=\Phi_b=0$*). Then,* $\forall\,T>0$, $\exists\,C_T>0/$

$$\sup_{t\le T}\int_\Omega\left|f^\varepsilon(t,x,v)-\varrho(t,x)\,\varphi\left(x,\frac{x}{\varepsilon},v\right)\right|\,\mathrm{d}x\,\mathrm{d}v\le C_T\,\varepsilon$$

where ϱ *is the solution of the homogenized Drift–Diffusion equation*

$$\begin{cases}\partial_t\varrho+\nabla_x\cdot j=0,\\ j=-\mathbb{D}(x)\nabla_x\varrho-\xi(x)\,\varrho,\\ \varrho_{\,|t=0}=\varrho_I,\\ j\cdot n(x)=0,\quad x\in\partial\omega.\end{cases}\tag{26}$$

The matrix $\mathbb{D}(x)$ *and the coefficient* $\xi(x)$ *are given by* (23) *and* φ *is the cell function.* □

Remark 4.2. When we deal with hydrodynamic limits of Boltzmann–Poisson system, two ideas are developed in the literature, the moment method [19,22] and the Hilbert expansion method [9,25,26]. Here, we would like to investigate the Hilbert method based on the contraction property [8] of the collision operator which requires a lot of regularity on the limit system. A Hybrid-Hilbert expansion is used to analyze the diffusion limit of the BGK model (coupled to Poisson) with inflow boundary data [9]. The same expansion is also used to study the behavior of the Fermi–Dirac statistics without detailed balance principle assumption [17,23]. In these previous examples, a correction of the linear Hilbert expansion is introduced to approximate the singularity due to the Poisson coupling and to control the time derivative of electrostatic field [23]. In the present context, we deal with a *two-scale variations* setting and also in one dimension we take into account the Poisson coupling. We notice that we are not able to establish an entropy dissipation for the system and use compactness method which requires uniform L^p-estimates. Such an estimate seems to be difficult to obtain due to the general form of the field (not necessary a gradient) $u^\varepsilon = u(x, \frac{x}{\varepsilon})$. We will use in the present analysis "the robust tool" based on the *contraction property* of the Fokker–Planck operator. In addition, we carefully analyze the Poisson coupling in one dimension by adding the correction introduced in [9] using a two-scale Hybrid-Hilbert expansion.

5. The One-Dimensional Case

The subject of this section is to analyze the one-dimensional case where we take into account the self-variation of the potential. In a first step we construct a relative Maxwellian which will be useful to obtain uniform L^p-bound on some momenta of the solution of coupled system. Then, in a second step, we prove the convergence result given in Theorem 4.1 for $d = 1$.

Step 1. Construction of a relative Maxwellian.

let $Y = (0, 1)$ be the cell period. The field u is smooth and its average is a constant $\bar{u}$:

$$\int_0^1 u(x,\tau)\mathrm{d}\tau = \bar{u}.$$

With such assumption, there exists a smooth function

$$\psi_0 = \psi_0(x,y)/u = \partial_y\psi_0(x,y) + \bar{u}.$$

Choosing ψ_0 with null average with respect to the second variable, $\int_0^1 \psi_0(x,\tau)$ $\mathrm{d}\tau = 0$, we get

$$\varepsilon\,\partial_x\left[x \mapsto \psi_0\left(x, \frac{x}{\varepsilon}\right)\right] = (u + \varepsilon\,\partial_x\psi_0)\left(x, \frac{x}{\varepsilon}\right) - \bar{u}.$$

The fact that the integral $\int_0^1 \partial_x\psi_0(x,\tau)\mathrm{d}\tau = 0$ implies that: there exists a smooth function

$$\psi_1 = \psi_1(x,y)/ -\partial_x\psi_0(x,y) = \partial_y\psi_1(x,y).$$

Now, we define

$$\theta^\varepsilon(x) = -\int_0^x (\partial_x\psi_1)\left(x', \frac{x'}{\varepsilon}\right)\mathrm{d}x'.$$

We have,

$$(\partial_x\theta^\varepsilon)(x) = -\partial_x\psi_1\left(x, \frac{x}{\varepsilon}\right)$$

and if we consider the function

$$\Psi^\varepsilon(x) := \psi_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\,\psi_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\theta^\varepsilon(x)$$

then, its derivative satisfies

$$\partial_x\Psi^\varepsilon = \frac{1}{\varepsilon}\,u\left(x, \frac{x}{\varepsilon}\right).$$

The scaled Fokker–Planck equation (1) becomes

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v\cdot\nabla_x f^\varepsilon - \nabla_x(\Phi^\varepsilon + \Phi_b - \Psi^\varepsilon)\cdot\nabla_v f^\varepsilon) = \frac{1}{\varepsilon^2}(\Delta_v f^\varepsilon + \nabla_v\cdot(v\, f^\varepsilon)). \tag{27}$$

We define the *total relative Maxwellian* as

$$\begin{aligned} M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon}(t,x,v) &= M(v)e^{-\Phi^\varepsilon(t,x)-\Phi_b(t,x)+\Psi^\varepsilon(x)} \\ &:= \frac{1}{\sqrt{2\pi}}e^{-v^2/2-\Phi^\varepsilon(t,x)-\Phi_b(t,x)+\Psi^\varepsilon(x)}. \end{aligned}$$

It follows that $M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon}$ satisfies the following useful relation:

$$\begin{aligned} &\left[v\cdot\nabla_x - \nabla_x(\Phi^\varepsilon+\Phi_b)\cdot\nabla_v + \frac{1}{\varepsilon}u\left(x,\frac{x}{\varepsilon}\right)\cdot\nabla_v\right] M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon} \\ &\quad = L^*(M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon}) = 0. \end{aligned} \tag{28}$$

Step 2. A priori estimates: relative entropy

Lemma 5.1. *Let ε be a non negative parameter. Then, the scaled Fokker–Planck–Poisson system has a weak solution satisfying*

$$t\mapsto \mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \frac{1}{2}\|\partial_x\Phi^\varepsilon\|^2_{L^2}(t) + \int_0^t \|j^\varepsilon\|^2_{L^2}(s)\,ds \in L^\infty_{loc}(\mathbb{R}^+)$$

where j^ε, $\mathcal{M}^\varepsilon(t)$ and $\mathcal{K}^\varepsilon(t)$ are given in (6) and (7). Moreover, the distribution function has an exponential decay:

$$f^\varepsilon \le C_T\exp(-v^2/2) \tag{29}$$

and then Φ^ε is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^+; W^{2,\infty}(\omega))$. □

Proof of Lemma 5.1. The proof of the existence of a weak solution is well known in one-dimensional setting. We refer to [9] for a similar case. The uniform L^1-norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f^\varepsilon(t)\|_{L^1(\Omega)} = 0$$

can be obtained by integrating the scaled equation (1) with respect to $\mathrm{d}x\,\mathrm{d}v$, using the conservation property of the Fokker–Planck operator:

$$\int_{\mathbb{R}^d} L^*(f)\mathrm{d}v = 0$$

and the fact that the equation is subject to specular reflection boundary assumption which does not induce boundary fluxes. Multiplying the Eq. (1) by

$$\log\left(\frac{f^\varepsilon}{M_{\Phi_\varepsilon+\Phi_b-\Psi^\varepsilon}}\right) = \log f^\varepsilon + v^2/2 + \Phi^\varepsilon + \Phi_b - \Psi^\varepsilon$$

and integrating by parts we get, thanks to (28),

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^\varepsilon(\log f^\varepsilon + v^2/2 + \Phi^\varepsilon + \Phi_b - \Psi^\varepsilon) - \int_\omega \varrho^\varepsilon\partial_t\Phi^\varepsilon \\ &\quad = \int_\Omega L^*(f^\varepsilon)\log\frac{f^\varepsilon}{M(v)} + \int_\omega \partial_t\Phi_b\,\varrho^\varepsilon. \end{aligned}$$

Using the fact that Φ^ε solves the homogeneous Poisson equation, we infer

$$\int_\omega \varrho^\varepsilon \Phi^\varepsilon = \|\partial_x \Phi^\varepsilon\|^2_{L^2(\omega)} \quad \text{and} \quad \int_\omega \varrho^\varepsilon \partial_t \Phi^\varepsilon = \frac{\mathrm{d}}{2dt}\|\partial_x \Phi^\varepsilon\|^2_{L^2(\omega)}$$

Using the entropy inequality (10), and remarking that we do not have entropy production terms due the boundary, we get

$$\left[\int_{x,v} f^\varepsilon(\log f^\varepsilon + \frac{v^2}{2} + \Phi_b - \Psi^\varepsilon) + \frac{1}{2}\|\partial_x \Phi^\varepsilon\|^2_{L^2}\right]_0^t$$
$$+ \frac{1}{\varepsilon^2}\int_0^t\int_\Omega \left|2\partial_v \sqrt{f^\varepsilon} + v\sqrt{f^\varepsilon}\right|^2 \mathrm{d}v \le \int_0^t\int_\omega \partial_t \Phi_b\, \varrho^\varepsilon.$$

Remarking the following inequality

$$f^\varepsilon(\log f^\varepsilon + v^2/4) = f^\varepsilon \log\left(\frac{f^\varepsilon}{e^{-v^2/4}}\right) \ge f^\varepsilon - e^{-v^2/4}$$

and using Assumptions **A1**–**A4**, we infer that for all $T > 0$ and $t \in (0, T)$

$$\mathcal{K}^\varepsilon(t) + \frac{1}{2}\|\partial_x \Phi^\varepsilon\|^2_{L^2}(t) + \frac{1}{\varepsilon^2}\int_0^t\int_\Omega \left|2\partial_v \sqrt{f^\varepsilon} + v\sqrt{f^\varepsilon}\right|^2$$
$$\le C_T\left(1 + \int_0^t \mathcal{M}^\varepsilon(s)\,\mathrm{d}s\right) \le C_T.$$

Moreover, the current density can be bounded in a first step in L^1 using the entropy dissipation and the finite bound of the mass and the kinetic energy. Indeed,

$$j^\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{R}} v\, f^\varepsilon\,\mathrm{d}v = \frac{1}{\varepsilon}\int_{\mathbb{R}} v\sqrt{f^\varepsilon}(\sqrt{f^\varepsilon} + 2\partial_v\sqrt{f^\varepsilon}\,)\,\mathrm{d}v$$

which implies that

$$\int_0^T \|j^\varepsilon(t)\|^2_{L^1}\mathrm{d}t$$
$$\le \frac{1}{\varepsilon^2}\int_0^T\int_\Omega \left|2\partial_v \sqrt{f^\varepsilon} + v\sqrt{f^\varepsilon}\right|^2 + \sup_{t\le T}\mathcal{M}^\varepsilon(t) + \sup_{t\le T}\mathcal{K}^\varepsilon(t) \le C_T.$$

Note that (at this stage) the previous estimate gives a uniform bound for the charge and current densities. Indeed ϱ^ε and j^ε are uniformly bounded in $L^\infty(0,T;\,L^1(\omega))$ and $L^2(0,T;\,L^1(\omega))$ respectively and,

$$\partial_t \varrho^\varepsilon + \partial_x j^\varepsilon = 0.$$

This equation and the Poisson equation imply that:

$$-\partial^2_{xx}\partial_t \Phi^\varepsilon = \partial_t \varrho^\varepsilon := -\partial_x j^\varepsilon$$

which gives a uniform bound for $\partial_t \Phi^\varepsilon$ in $L^2(0,T;\,L^\infty(\omega))$ in the one-dimensional case. To establish the L^2-control for j^ε and the exponential decay (29), we shall use the weak maximum principle which is enough to conclude the proof. This is the subject of the following paragraph.

Weak maximum principle: L^∞-estimate.

Let $f^\varepsilon(t=0) = \varrho_I(x)\varphi(x, \frac{x}{\varepsilon}, v)$ a hyper well-prepared initial data. We define the scalar coefficient α^ε by

$$\alpha^\varepsilon = \|\varrho_I\|_{L^\infty} \exp\left(\int_0^t \|\partial_t(\Phi^\varepsilon + \Phi_b)\|_{L^\infty}(s)\,\mathrm{d}s\right).$$

Then, the function $g^\varepsilon = f^\varepsilon - \alpha^\varepsilon\, M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon}(t,x,v)$ satisfies the following system

$$\begin{cases} \partial_t g^\varepsilon + \frac{1}{\varepsilon}(v\,\nabla_x \cdot g^\varepsilon - \partial_x(\Phi^\varepsilon + \Phi_b - \Psi^\varepsilon)\cdot\nabla_v g^\varepsilon) - \frac{L^*(g^\varepsilon)}{\varepsilon^2} \\ \quad = (\alpha^\varepsilon\,\partial_t(\Phi^\varepsilon+\Phi_b) - \alpha^{\varepsilon\prime})(t)\, M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon} \le 0 \\ \\ g^\varepsilon(t=0) = (\varrho_I - \|\varrho_I\|_{L^\infty})M_{\Phi^\varepsilon+\Phi_b-\Psi^\varepsilon} \le 0. \end{cases}$$

Applying the weak maximum principle for transport equations, we infer that $g^\varepsilon \le 0$ a. e. As a consequence, we get

$$f^\varepsilon \le \|\varrho_I\|_{L^\infty} \exp\left(-\frac{|v|^2}{2} - \Phi^\varepsilon - \Phi_b + \Psi^\varepsilon + \int_0^t \|\partial_t(\Phi^\varepsilon+\Phi_b)\|_{L^\infty}(s)\mathrm{d}s\right)$$

which gives the uniform upper bound for f^ε, appearing in (29), using the fact that $\partial_t\Phi^\varepsilon$ is bounded in $L^1(0,T;\, L^\infty(\omega))$ and Assumptions **A2** and **A3**. From the decay (29), one can show that the current density belongs to $L^2((0,T)\times\omega)$. Indeed,

$$\begin{aligned} &\int_0^T \|j^\varepsilon(t)\|_{L^2}^2 \mathrm{d}t \\ &\quad \le \frac{1}{\varepsilon^2}\int_0^T\int_\Omega \left|2\partial_v\sqrt{f^\varepsilon} + v\sqrt{f^\varepsilon}\right|^2 \times \sup_{t\le T}\sup_{x\in\omega}\left(\int_{\mathbb{R}} v^2 f^\varepsilon\,\mathrm{d}v\right) \le C_T \end{aligned}$$

and this ends the proof of Lemma 5.1. □

5.1. Convergence in 1-D: Two-Scale Hybrid-Hilbert Expansion

This section is devoted to the rigorous analysis of the convergence ($\varepsilon \to 0$), in the one-dimensional case. Note that we would like to prove the convergence of $(f^\varepsilon, \Phi^\varepsilon)$ satisfying (1)–(5) towards to $(\varrho\,\varphi(x, \frac{x}{\varepsilon}, v),\, \Phi)$, where $(\varrho,\, \Phi)$ is solution of the homogenized Drift–Diffusion-Poisson system (25) stated in the main theorem:

$$\begin{cases} \partial_t\varrho + \partial_x j = 0, \\ j = -\mathbb{D}(x)\,\partial_x\varrho - \xi(x)\,\varrho + \varrho\,\lambda(x)\,\partial_x(\Phi+\Phi_b), \\ -\partial_{xx}^2\Phi = \varrho_I, \\ (j\cdot n(x), \Phi)_{|x\in\partial\omega} = 0, \\ \varrho_{|\,t=0} = \varrho_I \end{cases}$$

where Φ_b is the harmonic extension of the boundary data for the potential. The coefficients $\mathbb{D}(x)$ and ξ are given by (23) and $\lambda(x)$ is defined in (36). In all this section, the notations $\tilde{f}^\varepsilon$, $\tilde{f}$, $\tilde{f}_1^\varepsilon, \ldots$ refer to the two-scale representation of

f^ε, f, $f_1^\varepsilon, \ldots$ according to the relation (12). The proof of convergence is based on the contraction property of $\mathcal{L}_x^*$: For all $\tilde{f} = \tilde{f}(y,v) \in L^2_{loc}(\Omega \times Y)$, we have

$$\int_{\mathbb{R}^d}\int_Y \mathcal{L}_x^*(\tilde{f}) sign(\tilde{f})\,\mathrm{d}y\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_Y L^*(\tilde{f}) sign(\tilde{f})\mathrm{d}y\,\mathrm{d}v \le 0. \qquad (30)$$

We go back to the two-scale expansion of f^ε, introduced in Sect. 3, and make a correction at the $\varepsilon-$order:

$$\tilde{f}^\varepsilon(t,x,v) = \tilde{f} + \varepsilon \tilde{f}_1^\varepsilon + \varepsilon^2 \tilde{f}_2 + r^\varepsilon.$$

The leading term $\tilde{f}$ belongs to the null space of $\mathcal{L}_x^*$:

$$\tilde{f} = \varrho(t,x)\,\varphi\,(x,y,v)$$

where the macroscopic unknown $\varrho = \int_Y \int_{\mathbb{R}} \tilde{f}\,\mathrm{d}y\mathrm{d}v$ is the homogenized charge density of $\tilde{f}$. The second term $\tilde{f}_1^\varepsilon$ depends on the self-consistent potential in order to avoid the singularity created by the Poisson coupling: we take $\tilde{f}_1^\varepsilon \in \mathcal{R}(\mathcal{L}_x^*)$, satisfying

$$\begin{aligned}\mathcal{L}_x^*(\tilde{f}^\varepsilon{}_1) &= -v\cdot\nabla_x \tilde{f} + \nabla_x(\Phi^\varepsilon + \Phi_b)\cdot\nabla_v \tilde{f},\\ &= -\nabla_x\varrho\cdot v\varphi - \varrho\, v\cdot\nabla_x\varphi + \varrho\nabla_x(\Phi^\varepsilon+\Phi_b)\cdot\nabla_v\varphi.\end{aligned}$$

The function $v\varphi$, $v\cdot\nabla_x\varphi$ and $\nabla_v\varphi$ belong to the range of $\mathcal{L}_x^*$. Then, one can choose $\tilde{f}_1^\varepsilon$ as

$$\begin{aligned}&\tilde{f}_1^\varepsilon(t,x,y,v)\\ &\quad= -\nabla_x\varrho(t,x)\cdot{\mathcal{L}_x^*}^{-1}(v\,\varphi) - \varrho\,{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi) + \varrho\,\nabla_x(\Phi^\varepsilon+\Phi_b)\cdot{\mathcal{L}_x^*}^{-1}(\nabla_v\varphi).\end{aligned} \qquad (31)$$

We define

$$\begin{aligned}\tilde{f}_1(t,x,y,v) = &-\nabla_x\varrho(t,x)\cdot{\mathcal{L}_x^*}^{-1}(v\,\varphi) - \varrho\,{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi)\\ &+\varrho\nabla_x(\Phi+\Phi_b)\cdot{\mathcal{L}_x^*}^{-1}(\nabla_v\varphi)\end{aligned} \qquad (32)$$

where (ϱ,Φ) is the solution of the homogenized Drift–Diffusion-Poisson model (25). Notice that $\tilde{f}_1$ is such that

$$\mathcal{L}_x^*(\tilde{f}_1) = -v\cdot\nabla_x\tilde{f} + \nabla_x(\Phi+\Phi_b)\cdot\nabla_v\tilde{f}$$

and the difference $\tilde{f}_1^\varepsilon - \tilde{f}_1$ satisfies

$$\tilde{f}_1^\varepsilon - \tilde{f}_1 = \varrho\,\nabla_x(\Phi-\Phi^\varepsilon)\cdot{\mathcal{L}_x^*}^{-1}(\nabla_v\varphi) \qquad (33)$$

The function $\tilde{f}_2$ is the solution in $\mathcal{R}(\mathcal{L}_x^*)$ of

$$-\,\mathcal{L}_x^*(\tilde{f}_2) = \partial_t\tilde{f} + v\cdot\nabla_x\tilde{f}_1. \qquad (34)$$

Integrating (34) with respect to $\mathrm{d}y\,\mathrm{d}v$, the solvability condition gives the Drift–Diffusion equation

$$\partial_t\varrho + \nabla_x\cdot\left(-\mathbb{D}(x)\nabla_x\varrho - \xi(x)\,\varrho + \varrho\,\lambda(x)\nabla_x(\Phi+\Phi_b)\right) = 0, \qquad (35)$$

where $\mathbb{D}(x)$ and $\xi(x)$ are given by (23) and

$$\lambda(x) := \int_{Y\times\mathbb{R}^d} v\otimes{\mathcal{L}_x^*}^{-1}(\nabla_v\varphi)\,\mathrm{d}y\,\mathrm{d}v \quad \in \mathbb{R}^{d\times d}. \qquad (36)$$

If we replace $\tilde{f}$ and $\tilde{f}_1$ by their expressions in (34), we get

$$-\mathcal{L}_x^*(\tilde{f}_2) = \nabla_x \cdot \Bigg[\Big(\mathbb{D}(x)\,\varphi - v \otimes \mathcal{L}_x^{*\,-1}(v\varphi)\Big)\,\nabla_x \varrho + \varrho\,\Big(\lambda\,\varphi - v \otimes \mathcal{L}_x^{*\,-1}(\nabla_v \varphi)\Big) \times \nabla_x(\Phi + \Phi_b)\Bigg] - \nabla_x \cdot \Big[\xi\varphi - v \otimes \mathcal{L}_x^{*\,-1}(v \cdot \nabla_x \varphi)\varrho\Big].$$

Now, let us approximate the remainder r^ε:

$$r^\varepsilon(t,x,v) := f^\varepsilon(t,x,v) - \Big[\tilde{f} - \varepsilon \tilde{f}_1^\varepsilon - \varepsilon^2 \tilde{f}_2\Big]\Big(t, x, \frac{x}{\varepsilon}, v\Big).$$

It satisfies the following scaled transport equation:

$$\begin{cases} \partial_t r^\varepsilon + \frac{1}{\varepsilon}(v \cdot \nabla_x r^\varepsilon + \nabla_x(\Phi^\varepsilon + \Phi_b) \cdot \nabla_v\, r^\varepsilon) + \frac{1}{\varepsilon^2} u(x, \frac{x}{\varepsilon}) \\ \quad \cdot \nabla_v r^\varepsilon - \frac{L^*(r^\varepsilon)}{\varepsilon^2} = S^\varepsilon \\ r^\varepsilon(t=0) = -\varepsilon(\tilde{f}_1(t=0) - \varepsilon\, \tilde{f}_2)_\varepsilon(t=0), \end{cases} \tag{37}$$

where the source term stands as

$$S^\varepsilon = -\Big[\varepsilon \partial_t \tilde{f}_1^\varepsilon + \varepsilon^2 \partial_t \tilde{f}_2 + \varepsilon v \cdot \nabla_x \tilde{f}_2 + v \cdot \nabla_x(\tilde{f}_1^\varepsilon - \tilde{f}_1) - \varepsilon \nabla_x(\Phi^\varepsilon + \Phi_b) \cdot \nabla_v \tilde{f}_2\Big]_\varepsilon + \Big[\nabla_x \Phi^\varepsilon \cdot \nabla_v(\tilde{f}_1^\varepsilon - \tilde{f}_1) + \nabla_x(\Phi^\varepsilon - \Phi) \cdot \nabla_v \tilde{f}_1\Big]_\varepsilon$$

and the subscript $[...]_\varepsilon$ means that we take the value at $y = \frac{x}{\varepsilon}$.

Multiplying (37) by $sign(r^\varepsilon)$ and integrating with respect to all the variables, we get

$$\begin{aligned} \|r^\varepsilon(t)\|_{L^1(\Omega)} &\le \|r^\varepsilon(0)\|_{L^1(\Omega)} + \frac{1}{\varepsilon^2}\int_0^t \int_{x,v} \Delta_v r^\varepsilon\, sign(r^\varepsilon)\,\mathrm{d}\tau + \int_0^t \|S^\varepsilon(t)\|_{L^1(\Omega)} \\ &\le \|r^\varepsilon(0)\|_{L^1(\Omega)} + \int_0^t \|S^\varepsilon(\tau)\|_{L^1(\Omega)}\mathrm{d}\tau. \end{aligned}$$

Using the fact f_1 and f_2 are smooth enough, we can deduce that

$$\|S^\varepsilon\|_{L^1(\Omega)} \le C_T\left(\varepsilon + \varepsilon\|\partial_t \nabla_x(\Phi^\varepsilon - \Phi)\|_{L^1\omega)} + \|\nabla_x(\Phi^\varepsilon - \Phi)\|_{L^1\Omega)}\right).$$

The Poisson coupling, the continuity equation and the uniform bound of $\|j^\varepsilon\|_{L^1}$ imply that (*in one dimension*) we can establish

$$\|S^\varepsilon(t)\| \le C_T\,(\varepsilon + \|(\varrho^\varepsilon - \varrho)(t)\|_{L^1}) \le C_T\,(\varepsilon + \|r^\varepsilon(t)\|_{L^1}).$$

The initial value of $\|r^\varepsilon\|_{L^1} = \mathcal{O}(\varepsilon)$. Then, the Gronwall lemma yields

$$\|r^\varepsilon(t)\|_{L^1} \le C_T\left(\varepsilon + \int_0^t \|r^\varepsilon(s)\|_{L^1}\mathrm{d}s\right) \le C_T\varepsilon, \qquad \forall\, t \le T.$$

From the fact that the field $\partial_x \Phi^\varepsilon$ belongs to $L^\infty(0, T, W^{1,1})$ and the solution (ϱ, Φ) has a good regularity (in the one-dimensional setting), we deduce that $\tilde{f}_1^\varepsilon$ and $\tilde{f}_2$ are bounded in $L^1((0,T) \times \Omega)$ and

$$\sup_{t \le T} \int_\Omega \Big| f^\varepsilon(t,x,v) - \varrho(t,x)\,\varphi\Big(x, \frac{x}{\varepsilon}, v\Big)\Big|\,\mathrm{d}x\,\mathrm{d}v \le C_T\varepsilon.$$

Moreover, using the L^∞-uniform bound of f^ε we deduce

$$\sup_{t\le T}\left\| f^\varepsilon - \varrho\,\varphi\left(x,\frac{x}{\varepsilon},v\right)\right\|_{L^p} \le C_{T,p}\,\varepsilon^{1/p}, \qquad \forall p\in[1,\infty[$$

which implies that ϱ^ε converges in two-scale strongly to $\varrho\int_{\mathbb{R}}\varphi(x,y,v)\mathrm{d}v$ in all L^p for $p\in(1,\infty)$ and weakly to ϱ for $p\ge 1$. In turns, we obtain the strong convergence of the potential Φ^ε to Φ in all L^p for $p<\infty$. This ends the proof of convergence for the one-dimensional case.

6. The Linear Multi-dimensional Case

The subject of this section is to extend the analysis to the multi-dimensional case. We restrict ourself (for instance) to the linear setting without restriction on the dimension. So, one can incorporate the potential in the expression of the vector field u by considering the following scaled Fokker–Planck equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x f^\varepsilon + \frac{1}{\varepsilon^2}u\left(x,\frac{x}{\varepsilon}\right)\cdot\nabla_v f^\varepsilon = \frac{L^*(f^\varepsilon)}{\varepsilon^2} \tag{38}$$

where L^* is the Fokker–Planck operator given by (8). As explained in Sect. 3, we can identify f^ε by $\tilde f^\varepsilon = \tilde f^\varepsilon(t,x,y,v)$ in the sense

$$\int_\Omega f^\varepsilon(t,x,v)\psi\left(t,x,\frac{x}{\varepsilon},v\right) = \int_\Omega\int_Y \tilde f^\varepsilon(t,x,y,v)\psi(t,x,y,v)$$

for all $\psi\in\mathcal{D}(\mathbb{R}^+\times\bar\omega;\,\mathcal{C}_\#(Y))$. The two-scale function $\tilde f^\varepsilon = \tilde f^\varepsilon(t,x,y,v)$ is the solution of the equivalent two-scale transport equation:

$$\partial_t\tilde f^\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x\tilde f^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}_x^*\tilde f^\varepsilon = 0$$

where,

$$\mathcal{L}_x^* = v\cdot\nabla_y + u(x,y)\cdot\nabla_v - L^*$$

and its distributional adjoint is

$$\mathcal{L}_x = -v\cdot\nabla_y - \Delta_v + (v-u(x,y))\cdot\nabla_v.$$

Remark 6.1. 1. Let us comment the case of a field u^ε coming from a potential with macroscopic and microscopic variations:

$$u^\varepsilon(x) = \nabla_x\Phi^\varepsilon$$

where

$$\Phi^\varepsilon(x) = \Phi\left(x,\frac{x}{\varepsilon}\right)$$

so that, u^ε has the form:

$$u^\varepsilon(x) = \nabla_x\Phi\left(x,\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\nabla_y\Phi\left(x,\frac{x}{\varepsilon}\right).$$

We remark that with this case, we can proceed like in [10] (see also [22] when we added the Poisson coupling), by defining the relative maxwellians:

$$M_\Phi(x,y,v) = \exp(-|v|^2/2 - \Phi(x,y))$$

and

$$M_{\Phi^\varepsilon}(x,v) = M_\Phi\left(x, \frac{x}{\varepsilon}, v\right)$$

and rewrite the Fokker-Planck operator as

$$L^*(f) = \nabla_v \cdot \left[M_\Phi \nabla_v \frac{f}{M_\Phi}\right].$$

The main point of the analysis is related on the fact that the cell function, defined in Proposition 3.1, is here the relative Maxwellian M_Φ. Indeed, if we define the effective potential

$$\Phi_e(x) = \log\left(\int_Y e^{-\Phi(x,y)} \mathrm{d}y\right),$$

then $M_{\Phi+\Phi_e}$ is normalized with respect to $\mathrm{d}v\mathrm{d}y$ and we have

$$\mathcal{L}_x^*(\varphi) = 0 \Leftrightarrow \exists\ \varrho(t,x)/\varphi = \varrho(t,x) M_\Phi(x,y,v).$$

Moreover,

$$[v \cdot \nabla_y + u(x,y) \cdot \nabla_v] M_\Phi = L^*(M_\Phi) = 0.$$

We remark also that we can construct an upper solution for the scaled Boltzmann equation, leading to a uniform L^∞-bound for the distribution f^ε. Then, using the relative entropy, by multiplying the scaled Boltzmann equation by $f^\varepsilon/M_{\Phi^\varepsilon}$, we can deduce that f^ε behaves like its local equilibrium $\varrho^\varepsilon M_{\Phi^\varepsilon}$. This is enough, using compactness arguments like velocity averaging lemma and/or div-curl lemma [19], to prove compactness properties and derive rigorously the homogenized fluid model. The details of this analysis is the subject of a forthcoming paper [13].

2. Due to the fact that the vector field u has a general form (not necessarily a gradient), we are not be able to construct a relative equilibrium solution which solves the transport part of the equation and belongs to null space of the Fokker–Planck operator.
3. A first attempt to justify the limit (ε goes to zero) is to use the duality method [23]. This method is well adapted for linear equations but in the present setting, we need a uniform L^p-estimate to deal with the duality in two-scale. Nevertheless, the duality method yields the form of the two scale limit of f^ε which is bounded in $\mathcal{D}'_\#$. Indeed, writing (38) in its two-scale form, we get

$$\varepsilon^2 \partial_t \tilde{f}^\varepsilon + \varepsilon v \cdot \nabla_x \tilde{f}^\varepsilon + \mathcal{L}_x^*(\tilde{f}^\varepsilon) = 0. \tag{39}$$

We deduce, as ε goes to 0, that $\tilde{f}^\varepsilon$ converges in $\mathcal{D}'_\#$ to $\tilde{f}$ satisfying

$$\mathcal{L}_x^*(\tilde{f}) = 0.$$

This implies that: there exists a function $\varrho = \varrho(t,x)$ such that,

$$\tilde{f}(t,x,y,v) = \varrho(t,x)\varphi(x,y,v)$$

where the unknown $\varrho(t,x) = \int_Y \int_{\mathbb{R}^d} \tilde{f}(t,x,y,v)\,\mathrm{d}y\,\mathrm{d}v$.

4. By taking reflection boundary assumption, the weak solution of the scaled Boltzmann equation, satisfies
$$\frac{\mathrm{d}}{\mathrm{d}t}\|f^\varepsilon(t)\|_{L^1} = 0$$
and using Assumption **A3**, the total mass is uniformly bounded.
5. The idea of our proof is to analyze the behavior limit of the difference $f^\varepsilon - \varrho(t,x)\varphi(x,\frac{x}{\varepsilon},v)$ in L^1 which does not require a uniform L^p-estimate to deal with two-scale convergence. We remark that also in this linear case, we should in some sense use the Hilbert expansion and the contraction property of the Fokker–Planck operator and the key point is that the property is satisfied by both L^* and the cell operator $\mathcal{L}_x^*$:
$$\int_{\mathbb{R}^d}\int_Y \mathcal{L}_x^*(\tilde f) sign(\tilde f)\,\mathrm{d}y\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_Y L^*(\tilde f) sign(\tilde f)\mathrm{d}y\,\mathrm{d}v \le 0. \tag{40}$$

According the the previous remark, and since the equation is linear, we do not have the singularity coming from the Poisson coupling, we can use the original Hilbert expansion introduced in Sect. 3. Indeed, we denote by $\tilde f$, $\tilde f_1$ and $\tilde f_2$, respectively, the solution of (16), (17) and (18):
$$\begin{aligned}\tilde f(t,x,y,v) &= \varrho(t,x)\varphi(x,y,v),\\ \tilde f_1(t,x,y,v) &= -\nabla_x\varrho(t,x)\cdot {\mathcal{L}_x^*}^{-1}(v\,\varphi) - \varrho\,{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi)\end{aligned}$$
and $\tilde f_2$ is such that
$$\begin{aligned}-\mathcal{L}_x^*(\tilde f_2) = \nabla_x\cdot &\left[(v\otimes {\mathcal{L}_x^*}^{-1}(v\varphi) - \mathbb{D}(x)\varphi)\;\nabla_x\varrho\right]\\ &-\nabla_x\cdot\left[\varrho\left(\left(v{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi)\right) - \xi(x)\right)\right]\end{aligned}$$
where $\mathbb{D}(x)$ and $\xi(x)$ are defined in (23). Then,
$$\tilde f_2 = \nabla_x\cdot(\chi_1\nabla_x\varrho) + \nabla_x(\varrho\chi_2) \tag{41}$$
where
$$-\mathcal{L}_x^*(\chi_1) = \left[(v\otimes{\mathcal{L}_x^*}^{-1}(v\varphi) - \mathbb{D}\varphi)\right]$$
and
$$-\mathcal{L}_x^*(\chi_2) = \left[v\,{\mathcal{L}_x^*}^{-1}(v\cdot\nabla_x\varphi) - \xi(x))\right].$$

Now, let us assume that the initial data are hyper well prepared:
$$f^\varepsilon(t=0) = \varrho_I\,\varphi\left(x,\frac{x}{\varepsilon},v\right)$$
and for all $t\ge 0$, f^ε satisfies the specular reflection boundary condition:
$$f^\varepsilon(t,x,v) = f^\varepsilon(t,x,v-2(v\cdot n(x))n(x)),\quad \text{for } (x,v)\in\partial\Omega.$$

Let $\tilde{f}$, $\tilde{f}_1$ and $\tilde{f}_2$ satisfying, respectively, (19), (21) and (41) where ϱ is the solution of the homogenized Drift–Diffusion equation (43):

$$\begin{cases} \partial_t \varrho + \nabla_x \cdot j = 0, \\ j = -\mathbb{D}(x)\nabla_x \varrho - \xi(x)\,\varrho, \\ \varrho_{\,|t=0} = \varrho_I, \\ j \cdot n(x) = 0, \quad x \in \partial\omega \end{cases}$$

where the matrix $\mathbb{D}(x)$ and the coefficient $\xi(x)$ are given by (23). Writing the equation satisfied by

$$r^\varepsilon = f^\varepsilon(t,x,v) - \left[\tilde{f} - \varepsilon \tilde{f}_1 - \varepsilon^2 \tilde{f}_2\right]\left(t, x, \frac{x}{\varepsilon}, v\right)$$

we get

$$\partial_t r^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x r^\varepsilon + \frac{1}{\varepsilon^2} u\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla_v r^\varepsilon - \frac{L^*(r^\varepsilon)}{\varepsilon^2} = S^\varepsilon$$

where the source term is

$$S^\varepsilon(t,x,v) = -\varepsilon\left[\partial_t \tilde{f}_1 + \varepsilon\, \partial_t \tilde{f}_2 + v \cdot \nabla_x \tilde{f}_2\right]\left(t, x, \frac{x}{\varepsilon}, v\right)$$

and the initial data

$$\|r^\varepsilon(t=0)\|_{L^1} = \mathcal{O}(\varepsilon).$$

The L^1-contraction (40) gives

$$\|r^\varepsilon(t)\|_{L^1(\Omega)} \le \|r^\varepsilon(0)\|_{L^1(\Omega)} + \int_0^t \|S^\varepsilon(\tau)\|_{L^1(\Omega)} d\tau \le C_T \varepsilon, \quad \forall\, t \le T$$

which in turns implies, thanks to the good decay of f_1 and f_2, to

$$\left\| f^\varepsilon - \varrho(t,x)\varphi\left(x, \frac{x}{\varepsilon}, v\right)\right\|_{L^\infty(0,T;\, L^1(\Omega))} \le C_T \varepsilon.$$

where the constant C_T depends only on T and $\|u\|_{W^{k,\infty}}$. This completes the proof of the main result for the linear case without restriction on the dimension. The case involving the ballistic case will be the subject of the next section.

6.1. The Case of Ballistic Motion

We remark that the condition $\int_Y \int_{\mathbb{R}^d} v\,\varphi\,\mathrm{d}y\,\mathrm{d}v = 0$ stated in assumption **A4** ensures that no ballistic motion is involved. Here, we would like to extend our analysis for a case of a ballistic dynamics by considering a quasi-periodic vector field:

$$u^\varepsilon(x) = u\left(x, \frac{x}{\varepsilon}\right) \tag{42}$$

So that, the cell function is x-dependent, satisfying the condition

$$\int_{\mathbb{R}^d} \int_Y v\,\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}v = \int_{\mathbb{R}^d} \int_Y v\,\mathcal{L}_x^{*-1}(0)\,\mathrm{d}y\,\mathrm{d}v = C \ne 0.$$

We notice that, the vector field satisfies the assumption **A1** and the constant C is x-independent.

Proposition 6.2. *Assume that* **A1**–**A3** *are satisfied and the vector field* u^ε *has the profile given in* (42). *Let* f^ε *be a weak solution of the linear Fokker–Planck* (1)–(5) (*with* $\Phi^\varepsilon = \Phi_b = 0$). *Then,* $\forall\, T > 0,\ \exists\, C_T > 0/$

$$\sup_{t\le T}\int_\Omega \left| f^\varepsilon(t,x,v) - \varrho(t,x-Ct/\varepsilon)\,\varphi\left(x,\frac{x}{\varepsilon},v\right)\right| \,\mathrm{d}x\,\mathrm{d}v \le C_T\,\varepsilon$$

where ϱ *is the solution of the homogenized Drift–Diffusion equation*

$$\begin{cases} \partial_t\varrho + \nabla_x\cdot j = 0,\\ j = -\mathbb{D}\nabla_x\varrho - \xi\,\varrho,\\ \varrho_{\,|t=0} = \varrho_I,\\ j\cdot n(x) = 0, \quad x\in\partial\omega. \end{cases} \tag{43}$$

The matrix $\mathbb{D}$ *and the coefficient* ξ *are given by* (49) *and* φ *is the cell-function.*

Proof of Proposition 6.2. The idea of derivation is based on Remark 2.2 of [20] (see also [16]). Indeed, let us start by the deriving the homogenized fluid model in this case of ballistic motion.

First step: transformation of coordinates. We go back to the scaled Boltzmann equation, written in its two-scale form (14) by considering the two-scale distribution function: $\tilde f^\varepsilon = \tilde f^\varepsilon(t,x,y,v)$. We introduce the transformation of coordinates $(x,v)\mapsto(z,w)$:

$$z = x - C\,t/\varepsilon \quad \text{and} \quad w = v - C.$$

We denote by

$$\tilde{\tilde f}^\varepsilon(t,z,y,w) := \tilde f^\varepsilon(t, z + C\,t/\varepsilon, y, w + C)$$

where $\tilde f^\varepsilon$ is the solution of (14). Then, the function $\tilde{\tilde f}^\varepsilon$ solves the scaled transport equation:

$$\partial_t\tilde{\tilde f}^\varepsilon + \frac{1}{\varepsilon}w\cdot\nabla_z\tilde{\tilde f}^\varepsilon + \frac{1}{\varepsilon^2}\Big[(w+C)\cdot\nabla_y\tilde{\tilde f}^\varepsilon + (u(z+C\,t/\varepsilon,y)-C)\\ \cdot\nabla_w\tilde{\tilde f}^\varepsilon - \nabla_w\cdot\left(\nabla_w\tilde{\tilde f}^\varepsilon + w\tilde{\tilde f}^\varepsilon\right)\Big] = 0$$

which is equivalent to,

$$\partial_t\tilde{\tilde f}^\varepsilon + \frac{1}{\varepsilon}w\cdot\nabla_z\tilde{\tilde f}^\varepsilon + \frac{1}{\varepsilon^2}\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}(\tilde{\tilde f}^\varepsilon) = 0$$

and, for all $\xi\in\mathbb{R}^d$ and $f\equiv f(t,z,y,w)$,

$$\tilde{\mathcal{L}}^*_\xi(f) = (w+C)\cdot\nabla_y f + (u(\xi,y)-C)\cdot\nabla_w f - \nabla_w\cdot[\nabla_w f + wf].$$

Second step: derivation of the homogenized Drift–Diffusion model. We remark that, with these new coordinates, we have a two-scale Boltzmann equation associated with an ε-dependent vector field. With such property, we cannot apply directly the Hilbert development used previously. We point out, that this property is related to the fact that the velocity has macroscopic variations (u

is x-dependent). Our idea is to use a Chapman–Enskog expansion of two-scale solution $\tilde{\tilde{f}}^{\varepsilon}$, by assuming that it behaves like

$$\tilde{\tilde{f}}^{\varepsilon}(t,z,y,w) \sim \tilde{\tilde{f}}_0(t,z,y,w) + \varepsilon\, \tilde{\tilde{f}}_1(t,z,y,w) + \varepsilon\, \tilde{\tilde{f}}_2(t,z,y,w) + \cdots, \tag{44}$$

where $\tilde{\tilde{f}}_0$, $\tilde{\tilde{f}}_1, \ldots$ are Y-periodic with respect to y. Plugging this development in the previous equation, one can choose

$$\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}(\tilde{\tilde{f}}_0) := 0, \tag{45}$$

$$\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}(\tilde{\tilde{f}}_1) = -w \cdot \nabla_z \tilde{\tilde{f}}_0 \tag{46}$$

and

$$\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}(\tilde{\tilde{f}}_2) = -\partial_t \tilde{\tilde{f}}_0 - w \cdot \nabla_z \tilde{\tilde{f}}_1. \tag{47}$$

Now, we remark, using Proposition 3.1, that the null space of the cell operator $\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}$ is spanned by the shifted cell function $\tilde{\varphi}(.,.)$ parameterized by $z + C\,t/\varepsilon$:

$$\tilde{\varphi}(z,y,w) := \varphi(z + C\,t/\varepsilon, y, w + C).$$

From Proposition 3.1, the function $\tilde{\varphi} = \tilde{\mathcal{L}}^{*-1}_{z+C\,t/\varepsilon}(0)$ is normalized with respect with (y,w):

$$\int_{\mathbb{R}^d} \int_Y \tilde{\varphi}(z + C\,t/\varepsilon, y, w)\, \mathrm{d}y\, \mathrm{d}w = 1.$$

As a consequence, one can choose the leading term in the previous development in the form

$$\tilde{\tilde{f}}^{\varepsilon}_0(t,z,y,w) := \varrho(t,z)\, \tilde{\varphi}(z + C\,t/\varepsilon, y, w) \tag{48}$$

We notice that, the ε-dependence $\tilde{\tilde{f}}_0$ is incorporated in the cell function and the homogenized charge density depends only on (t,z). Moreover,

$$\begin{aligned}\int_{\mathbb{R}^d} \int_Y w\, \tilde{\varphi}(z + C\,t/\varepsilon, y, w)\mathrm{d}y\mathrm{d}w &= \int_{\mathbb{R}^d} \int_Y w\varphi(z + C\,t/\varepsilon, y, w + C)\mathrm{d}y\mathrm{d}w \\ &= \int_{\mathbb{R}^d} \int_Y (v - C)\varphi(z + C\,t/\varepsilon, y, v)\mathrm{d}y\mathrm{d}v = 0.\end{aligned}$$

The Eq. (46) becomes

$$-\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}(\tilde{\tilde{f}}_1) = \nabla_z \varrho \cdot w\tilde{\varphi} + \varrho\, w \cdot \nabla_z \tilde{\varphi}$$

The functions $w_i\tilde{\varphi}$, for all $i \in \{1,\ldots,d\}$ and $w \cdot \nabla_z \tilde{\varphi}$ belong to the range of $\tilde{\mathcal{L}}^*_{z+C\,t/\varepsilon}$. Let $\tilde{\theta}(z,.,.) := \tilde{\mathcal{L}}^{*-1}_z(w\tilde{\varphi}(z,.,.))$ and $\tilde{\theta}_1(z,.,.) := \tilde{\mathcal{L}}^{*-1}_z$ $(w \cdot \nabla_z \tilde{\varphi}(z,.,.))$. Then,

$$\begin{aligned}\tilde{\tilde{f}}_1(t,z,y,w) &= (\nabla_z \varrho \cdot \tilde{\theta}) + \varrho\, \tilde{\theta}_1 \\ &= \nabla_z \varrho(t,z) \cdot \tilde{\theta}(z,y,w) + \varrho(t,z)\, \tilde{\theta}_1(z,y,w)\end{aligned}$$

Denoting by

$$\begin{cases} \mathbb{D}(z) = \int_{\mathbb{R}^d} \int_Y w \otimes \mathcal{L}_z^{*-1}(w\tilde{\varphi}(z,y,w))\, \mathrm{d}y\, \mathrm{d}w \\ \xi(z) = \int_{\mathbb{R}^d} \int_Y w\, \mathcal{L}_z^{*-1}(w \cdot \nabla_z \varphi(z,y,w))\, \mathrm{d}y\, \mathrm{d}w \end{cases} \tag{49}$$

Integrating the Eq. (47) with respect $\mathrm{d}y\,\mathrm{d}w$, we obtain the homogenized Drift–Diffusion model:

$$\partial_t \varrho + \nabla_z \cdot [-\mathbb{D}(z + C\,t/\varepsilon)\nabla_z \varrho - \xi(z + C\,t/\varepsilon)\varrho] = 0$$

For the rigorous proof of convergence, we proceed like in the previous case using the two-scale Hilbert expansion and the contraction property:

$$\int_{\mathbb{R}^d} \int_Y \mathcal{L}^*_{z+C\,t/\varepsilon}(r) sign(r) \mathrm{d}y\, \mathrm{d}w \leq 0$$

Let

$$\tilde{\tilde{f}}_1^\varepsilon(t,z,y,w) = \nabla_z \varrho(t,z) \cdot \tilde{\theta}(z + Ct/\varepsilon, y, w) + \varrho(t,z)\, \tilde{\theta}_1(z + Ct/\varepsilon, y, w) \tag{50}$$

and $\tilde{\tilde{f}}_2^\varepsilon$ is a solution of (47). Then, writing the two-scale transport equation satisfied by the remainder

$$r^\varepsilon := r^\varepsilon(t,z,y,w) := \tilde{\tilde{f}}^\varepsilon - \tilde{\tilde{f}}_0^\varepsilon - \varepsilon\, \tilde{\tilde{f}}_1^\varepsilon - \varepsilon^2\, \tilde{\tilde{f}}_2^\varepsilon$$

satisfying

$$\sup_{t \leq T} \int_{\Omega \times Y} |r^\varepsilon(t,z,y,w)|\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}w \leq C_T\, \varepsilon$$

which is equivalent, using the fact that $\tilde{\tilde{f}}_1$ and $\tilde{\tilde{f}}_2$ have a good decay, that

$$\sup_{t \leq T} \int_\Omega |f^\varepsilon(t,x,v) - \varrho(t, x - Ct/\varepsilon)\varphi(x, x/\varepsilon, v)|\, \mathrm{d}x\, \mathrm{d}v \leq C_T\, \varepsilon$$

and this ends the proof of Proposition 6.2. □

7. Concluding Remarks

We remark that in the multi-dimensional setting, we have two difficulties. The first one is the fact that the vector u^ε is not a gradient. With this assumption, it seems that we can not use directly the entropy dissipation. The second point, if we take into account the Poisson coupling, is the fact that the solution has only a renormalized sense. Indeed,

1. For fixed $\varepsilon > 0$. we are able (for instance) to prove the existence of solutions only in a renormalized sense: $f^\varepsilon \in L\log L$ and $\nabla_x \Phi^\varepsilon \in L^2$. We refer to [16] for a similar case.
2. Only the L^1-norm is uniform (in ε) due to the presence of the singular term $\varepsilon^{-2} u^\varepsilon \nabla_v f^\varepsilon$ for the case of reflection boundary condition. It seems that it is difficult to use the relative entropy method to obtain a uniform L^p-estimate. Moreover, the L^1-norm is limited to the case of specular

reflection assumptions. In the inflow boundary condition setting, we do not have directly the conservation of mass due to singular fluxes:

$$\frac{d}{\mathrm{d}t}\|f^\varepsilon\|_{L^1}(\Omega) + \frac{1}{\varepsilon}\int_{\partial\Omega^+} f_b^\varepsilon |v\cdot n(x)|\,\mathrm{d}\sigma\,\mathrm{d}v = \frac{1}{\varepsilon}\int_{\partial\Omega^-} f_b^\varepsilon |v\cdot n(x)|\,\mathrm{d}\sigma\,\mathrm{d}v$$

The multi-dimensional and self-consistent setting is a very interesting task.

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References

[1] Allaire, G.: Homogenization and two-scale convergence. SIAM J. Math. Anal. **23**(6), 1482–1518 (1992)

[2] Banasiak, J., Mika, J.R.: Diffusion limit for the linear Boltzmann equation of the neutron transport theory. M2AS **17**, 1071–1087 (1994)

[3] Bardos, C., Sentos, R., Sentis, R.: Diffusion approximation and computation of the critical size. Trans. Am. Math. Soc. **284**(2), 617–649 (1984)

[4] Ben Abdallah, N.: Weak solutions of the initial-boundary value problem for the Vlasov–Poisson system. Math. Methods Appl. Sci. **17**, 451–476 (1994)

[5] Ben Abdallah, N., Chaker, H., Schmeiser, C.: The high field asymptotics for a fermionic Boltzmann equation: entropy solutions and kinetic shock profiles. J. Hyperbolic Differ. Equ. **4**, 679–704 (2007)

[6] Ben Abdallah, N., Degond, P.: On a hierarchy of macroscopic models for semiconductors. J. Math. Phys. **37**(7), 3306–3333 (1996)

[7] Ben Abdallah, N., Dolbeault, J.: Relative entropies for kinetic equations in bounded domains (irreversibility, stationary solutions, uniqueness). Arch. Ration. Mech. Anal. **168**(4), 253–298 (2003)

[8] Ben Abdallah, N., Escobedo, M., Mischler, S.: Convergence to the equilibrium for the Pauli equation without detailed balance condition. C. R. Math. Acad. Sci. Paris **341**(1), 5–10 (2005)

[9] Ben Abdallah, N., Tayeb, M.-L.: Diffusion approximation for the one dimensional Boltzmann–Poisson system. DCDS-B **4**, 1129–1142 (2004)

[10] Ben Abdallah, N., Tayeb, M.-L.: Diffusion Approximation and homogenization of the semiconductor Boltzmann equation. Multiscale Model Simul. **4**(3), 896–914 (2005)

[11] Dolbeault, J.: Free energy and solutions of the Vlasov–Poisson–Fokker–Planck system: external potential and confinement (large time behavior and steady states). J. Math. Pures Appl. **78**(2), 121–157 (1999)

[12] Dolbeault, J.: Kinetic models and quantum effects: a modified Boltzmann equation for Fermi–Dirac particles. Arch. Ration. Mech. Anal. **127**(2), 101–131 (1994)

[13] El-Ghani, N., Tayeb, M.-L.: Diffusion approximation and homogenization of a Vlasov–Poisson–Fokker–Planck system: a relative Entropy approach (in progress)

[14] Frenod, E., Raviart, P.A., Sonnendrücker, E.: Two scale expansion of a singularly perturbed convection equation. J. Maths. Pures Appl. **80**(8), 815–843 (2001)

[15] Golse, F., Poupaud, F.: Limite fluide des équations de Boltzmann des semi-conducteurs pour une statistique de Fermi–Dirac. Asymptot. Anal. **6**, 135–169 (1992)

[16] Goudon, T.: Hydrodynamic limit for the Vlasov–Poisson–Fokker–Planck system: analysis of two dimensional case. Math. Models Methods Appl. Sci. **15**, 737–752 (2005)

[17] Goudon, T., Mellet, A.: Homogenization and diffusion asymptotics of the linear Boltzmann. ESAIM Control Optim. Calc. Var. **9**, 371–398 (2003)

[18] Goudon, T., Nieto, J., Poupaud, F., Soler, J.: Multidimensional high-field limit of the electrostatic Vlasov–Poisson–Fokker–Planck system. J. Differ. Equ. **213**, 418–442 (2005)

[19] Goudon, T., Poupaud, F.: Approximation by homogeneization and diffusion of kinetic equations. Commun. Partial Differ. Equ. **26**, 537–569 (2001)

[20] Hairer, M., Pavliotis, G.A.: Periodic homogenization for hypoelliptic diffusions. J. Stat. Phys. **117**(1–2), 261–279 (2004)

[21] Masmoudi, N., Tayeb, M.-L.: Diffusion limit of a semiconductor Boltzmann–Poisson system. SIAM J. Math. Anal. **38**, 1788–1807 (2007)

[22] Masmoudi, N., Tayeb, M.-L.: Diffusion and homogenization approximation for semiconductor Boltzmann–Poisson system. J. Hyperbolic Differ. Equ. **5**(1), 65–84 (2008)

[23] Masmoudi, N., Tayeb, M.-L.: On the diffusion limit of a semiconductor Boltzmann–Poisson system without micro-reversible process. Commun. Partial Differ. Equ. **35**(7), 1163–1175 (2010)

[24] Michel, P., Mischler, S., Perthame, B.: General relative entropy inequality: an illustration on growth models. J. Math. Pures Appl. (9) **84**(9), 1235–1260 (2005)

[25] Poupaud, F.: Diffusion approximation of the linear semiconductor Boltzmann equation: analysis of boundary layers. Asymptot. Anal. **4**(4), 293–317 (1991)

[26] Poupaud, F., Schmeiser, S.: Charge transport in semiconductors with degeneracy effects. Math. Methods Appl. Sci. **14**(5), 301–318 (1991)

[27] Poupaud, F., Soler, J.: Parabolic limit and stability of the Vlasov–Fokker–Planck system. M3AS **10**(7), 1027–1045 (2000)

[28] Villani, C.: Entropy Production and Convergence to Equilibrium, Entropy Methods for the Boltzmann Equation. Lecture Notes in Mathematics, vol. 1916, pp. 1–70 (2008)

[29] Villani, C.: Hypocoercivity. Mem. Am. Math. Soc. **202**(950) (2009)

[30] Weinan, E.: Homogenization of linear and nonlinear transport equations. Commun. Pure Appl. Math. **45**(3), 301–326 (1992)

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