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Bartnik's Mass and Hamilton's Modified Ricci Flow

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Dedicated to Richard Hamilton on the occasion of his 70th birthday

Abstract. We provide estimates on the Bartnik mass of constant mean curvature surfaces which are diffeomorphic to spheres and have positive mean curvature. We prove that the Bartnik mass is bounded from above by the Hawking mass and a new notion we call the asphericity mass. The asphericity mass is defined by applying Hamilton's modified Ricci flow and depends only upon the restricted metric of the surface and not on its mean curvature. The theorem is proven by studying a class of asymptotically flat Riemannian manifolds foliated by surfaces satisfying Hamilton's modified Ricci flow with prescribed scalar curvature. Such manifolds were first constructed by the first author in her dissertation conducted under the supervision of M. T. Wang. We make a further study of this class of manifolds which we denote Ham_3, bounding the ADM masses of such manifolds and analyzing the rigid case when the Hawking mass of the inner surface of the manifold agrees with its ADM mass.

1. Introduction

Two of the most important quasilocal masses studied in Riemannian General Relativity are the Hawking mass and Bartnik mass of a surface, Σ, which is diffeomorphic to a sphere, has positive mean curvature, and lies in an asymptotically flat three-dimensional Riemannian manifold, M. The manifold, M, has nonnegative scalar curvature and no closed interior minimal surfaces. It may have a boundary, as long as the boundary is a minimal surface and is outward minimizing. We will use $\mathcal{PM}$ to denote the class of such manifolds, M.

In this paper, we relate these two quasilocal masses with a third quantity that we call the "asphericity mass". We prove this new quantity depends only

on the intrinsic geometry of Σ and is 0 if and only if Σ is a standard sphere; thus, it is a measure of "asphericity". We consider it to be a "mass" because it scales like mass and is related to a difference between two quasilocal masses. However, it is not a quasilocal mass.

Before describing our results, we give a very brief review of the key definitions needed to state our theorems. We apologize that we cannot completely survey the results of the many mathematicians and physicists that have contributed to research on mass in general relativity. We review only the results related to our class of three dimensional manifolds $\mathcal{PM}$ that are directly related to the work in this paper. We do not state the full generality of all theorems proven in the papers we review nor related papers that extend these results.

In 1961, Arnowitt–Deser–Misner introduced the ADM mass, which we denote by $m_{\mathrm{ADM}}(M)$, for asymptotically flat three-dimensional manifolds, including $M \in \mathcal{PM}$ [1]. Note that the Riemannian Schwarzschild manifold, $M_{Sch,m}$, for a black hole in a vacuum of mass, m, with metric

$$g = \left(1 - \tfrac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 g_{\mathbb{S}^2} \tag{1.1}$$

has scalar curvature $= 0$ and $m_{\mathrm{ADM}}(M) = m$. In 1968, Hawking [11] introduced the Hawking mass

$$m_{\mathrm{H}}(\Sigma) = \sqrt{\frac{\operatorname{area}(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \oint_{\Sigma} H^2 \mathrm{d}\sigma\right), \tag{1.2}$$

which approaches the ADM mass for large coordinate spheres, Σ_r:

$$m_{\mathrm{ADM}}(M) = \lim_{r \to \infty} m_{\mathrm{H}}(\Sigma_r). \tag{1.3}$$

Note that on the Riemannian Schwarzschild manifold, $M_{Sch,m}$, the Hawking mass of all rotationally symmetric spheres is $m_{\mathrm{H}}(\Sigma) = m \geq 0$. More generally, when $M \in \mathcal{PM}$ is rotationally symmetric,

$$g = (u(r))^2 \mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}, \tag{1.4}$$

the Hawking mass of level sets of r, Σ_r, is nonnegative and increases to $m_{\mathrm{ADM}}(M)$. Even without rotational symmetry, Geroch proved that for $M \in \mathcal{PM}$ and $\Sigma_t \subset M$ evolving by smooth inverse mean curvature flow, the Hawking mass increases (see the appendix to [9]).

Schoen–Yau proved in [18] that for any $M \in \mathcal{PM}$, one has $m_{\mathrm{ADM}}(M) \geq 0$. Huisken–Ilmanen proved the Penrose Inequality that $m_{\mathrm{ADM}}(M) \geq m_{\mathrm{H}}(\partial M) \geq 0$ for $M \in \mathcal{PM}$ [12]. The Hawking mass itself is not necessarily nonnegative, although it is clearly nonnegative for minimal surfaces. Christodoulou and Yau [7] proved that the Hawking mass is nonnegative for a stable 2-sphere with constant mean curvature. However, Huisken–Ilmanen have an example of a $\Sigma \subset M$ where $M \in \mathcal{PM}$ that has $m_{\mathrm{H}}(\Sigma) < 0$ [13].

The Bartnik mass was introduced in [2]. To define it, we first let (Ω^3, g) be the region enclosed by Σ. For any bounded open connected region (Ω, g) with nonnegative scalar curvature, let $\mathcal{PM}(\Omega)$ be the set of "admissible extensions",

$(M, g) \in \mathcal{PM}$, such that Ω embeds isometrically into M. Then the Bartnik's definition for his mass is defined to be

$$\mathrm{m_B}(\Omega) = \inf\{\mathrm{m_{ADM}}(M, g) : (M, g) \in \mathcal{PM}(\Omega)\}. \qquad (1.5)$$

Observe that by the Positive Mass Theorem, we have $\mathrm{m_B}(\Omega) \geq 0$. Using the inverse mean curvature flow, Huisken and Ilmanen [12] proved that if $\mathrm{m_B}(\Omega) = 0$ then Ω is locally isometric to Euclidean space. Recall that Schoen–Yau proved that for any $M \in \mathcal{PM}$, if $\mathrm{m_{ADM}}(M) = 0$ then M is isometric to Euclidean space.

As a quasilocal mass, the Bartnik mass may only depend on Σ and how Σ embeds into M^3 but not on the interior region Ω. Thus it is now standard to define the Bartnik mass as follows:

$$\mathrm{m_B}(\Sigma) = \inf\{\mathrm{m_{ADM}}(M, g) : (M, g) \in \mathcal{PM}'(\Sigma)\}. \qquad (1.6)$$

Here $\mathcal{PM}'(\Sigma)$ is the set of "admissible extensions", $(M, g) \in \mathcal{PM}'$, such that $g|_{\partial(M \backslash \Omega)} = g|_{\partial\Omega}$ and $H_{\partial(M \backslash \Omega)} = H_{\partial\Omega}$. Here $\mathcal{PM}'$ are Lipschitz manifolds, smooth away from Σ, that satisfy the same conditions as manifolds in $\mathcal{PM}$ where nonnegative scalar curvature is defined in the distributional sense across Σ.

Note that the Bartnik mass is nonnegative, $\mathrm{m_B}(\Sigma) \geq 0$. This is an immediate consequence of the fact that the Positive Mass Theorem holds for $M \in \mathcal{PM}'$ as proven by Shi and Tam (see [17, Theorem 3.1]) and Miao [16, Theorem 1].

Suppose that Σ is isometric to a rescaled standard sphere and has constant mean curvature. Then it is well known that $\mathrm{m_B}(\Sigma) \leq \mathrm{m_H}(\Sigma)$. To prove this, one shows that any such Σ includes a Riemannian Schwarzschild manifold among its admissible extensions, and so $\mathrm{m_B}(\Sigma) \leq \mathrm{m_{ADM}}(M_{Sch,m}) = \mathrm{m_H}(\Sigma)$. For completeness of exposition, we include the proof in our Appendix.

In this paper, we consider constant positive mean curvature surfaces which are not isometric to rescaled standard spheres. We construct an admissible extension using Hamilton's modified Ricci flow [10] as in the first author's doctoral dissertation completed under the supervision of Mu-Tao Wang [15]. We prove the following theorem:

Theorem 1. *Let $\Sigma \subset M$ where $M \in \mathcal{PM}$ be a constant mean curvature surface diffeomorphic to a sphere which has area 4π and positive mean curvature. Then*

$$\mathrm{m_B}(\Sigma) \leq m_{a\mathbb{S}}(\Sigma) + \mathrm{m_H}(\Sigma) \qquad (1.7)$$

where $m_{a\mathbb{S}}(\Sigma)$ is a nonnegative constant defined using Hamilton's modified Ricci flow that we call the asphericity mass. It depends only upon the restriction of the metric g to the surface Σ. If

$$m_{a\mathbb{S}}(\Sigma) = 0 \qquad (1.8)$$

then Σ is isometric to a standard sphere, $(\mathbb{S}^2, g_{\mathbb{S}^2})$.

Before we state the definition of the new asphericity mass and our other theorems, we quickly review Hamilton's modified Ricci flow. Recall that for

(Σ, g_1) of dimension two, Hamilton [10] defined the modified Ricci flow (Σ_t, g_t) satisfying

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = (r - 2K) g_{ij} + 2D_i D_j f = 2M_{ij} \\ g(1, \cdot) = g_1(\cdot), \end{cases} \tag{1.9}$$

where $K = K_t(x)$ is the Gauss curvature of g_t at $x \in \Sigma_t$, and

$$r = r_t = \frac{1}{\text{Area}(\Sigma_t)} \int_{\Sigma_t} 2K_t(x) d\mu = 2 \tag{1.10}$$

is the mean scalar curvature, and $f = f(t, x)$ is the Ricci potential satisfying the equation

$$\Delta f = 2K - r \tag{1.11}$$

with mean value zero. Note that we do not use the notation from Hamilton's paper because R in this paper is the scalar curvature of M not of Σ. Observe that by the Gauss Bonnet Theorem, the average Gauss curvature of Σ is 1. Thus the 2-tensor

$$M_{ij} = (1 - K)g_{ij} + D_i D_j f \tag{1.12}$$

is the trace-free part of Hess (f).

Building upon this work of Hamilton, Chow proved in [6] that when Σ is diffeomorphic to a two-dimensional sphere, then the modified Ricci flow exists for all time and (Σ_t, g_t) converges to a standard sphere exponentially fast as $t \to \infty$. In fact M converges to 0 exponentially fast.

Definition 2. The **asphericity mass** of a surface Σ of area 4π and diffeomorphic to a sphere is defined by

$$m_{a\mathbb{S}}(\Sigma) = \lim_{t \to \infty} m_{a\mathbb{S}}(\Sigma, t), \tag{1.13}$$

where

$$m_{a\mathbb{S}}(\Sigma, t) = \frac{1}{2} \int_1^t 1 - K_*(\tau) E(\tau, t) \, d\tau, \tag{1.14}$$

where

$$E(\tau, t) = \exp\left(-\int_\tau^t \frac{s |M|^{*2}(s)}{2} ds\right). \tag{1.15}$$

Here we have the infimum of the Gauss curvature

$$K_*(\tau) = \inf\{K_\tau(x) : x \in \Sigma_\tau\} \tag{1.16}$$

and the supremum of the norm of the M tensor

$$|M|^{*2}(s) = \sup\{|M_s(x)|^2 : x \in \Sigma_s\} \tag{1.17}$$

which depend on g_t and $f(t, x)$ of Hamilton's modified Ricci flow. Observe that this mass depends only upon the intrinsic metric on Σ and not on the mean curvature.

In Sect. 3 we explore this new notion. In Lemma 10 we prove that $m_{a\mathbb{S}}(\Sigma, t)$ is nonnegative and increasing in t. In Lemma 11, we prove that the asphericity mass is finite and the limit exists. In Proposition 12 we show $m_{a\mathbb{S}}(\Sigma) = 0$ if and only if (Σ, g_1) is isometric to a standard sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$.

In Sect. 4 we explore the class of asymptotically flat three-dimensional Riemannian manifolds foliated by Hamilton's modified Ricci flow, denoted Ham_3. These manifolds are later used as the admissible extensions needed to estimate the Bartnik mass and prove Theorem 1. This class includes the class of asymptotically flat rotationally symmetric manifolds with nonnegative scalar curvature, RotSym_3 [Proposition 17]. It also includes admissible extensions of any Σ diffeomorphic to a two sphere with a positive Gauss curvature and arbitrary positive mean curvature that have prescribed 0 scalar curvature [Lemma 15]. In addition, the class includes admissible extensions for Σ with prescribed scalar curvature, $\bar{R}$, not equivalent to 0 as long as $\bar{R}$ satisfies the conditions below (or the hypothesis of Lemma 14).

Definition 3. The class of asymptotically flat three-dimensional Riemannian manifolds foliated by Hamilton's modified Ricci flow, denoted $M \in \mathrm{Ham}_3$, are manifolds $M_{\bar{R}}$ diffeomorphic to $[1, \infty) \times \Sigma$ with metric

$$g'_{\bar{R}} = u^2 \mathrm{d}t^2 + t^2 g \tag{1.18}$$

where $g = g_t$ is defined using the modified Ricci flow and where $u : [1, \infty) \times \Sigma \to (0, \infty)$ depends uniquely upon $(\Sigma, g_1, H, \bar{R})$. Here g_1 is the metric on Σ and $H : \Sigma \to (0, \infty)$ is the mean curvature of $\Sigma = t^{-1}(1)$:

$$u(1, x) = 2/H_x \tag{1.19}$$

and $\bar{R} \in C^\alpha([1, \infty) \times \Sigma)$ is a prescribed scalar curvature function which is asymptotically flat in the sense that

$$\int_1^\infty |\bar{R}|^* t^2 \mathrm{d}t < \infty \quad \text{and} \quad \|\bar{R}t^2\|_{C_{0,\alpha}[t, 4t]} \le \frac{C}{t} \tag{1.20}$$

and which has bounded "scalar energy" with respect to the Ricci flow:

$$C_0(\bar{R}) = \sup_{1 \le t < \infty} \left\{ \int_1^t \left(\frac{\tau^2}{2} \bar{R} - K \right)^* \exp \left(\int_1^\tau \frac{s|M|^{*2}}{2} \mathrm{d}s \right) \mathrm{d}\tau \right\} < H^2/4. \tag{1.21}$$

For fixed (Σ, g_1, H) that encloses a compact region with nonnegative scalar curvature and positive mean curvature, we denote

$$\mathrm{Ham}_3(\Sigma, g_1, H) = \{ M_{\bar{R}} : \ \bar{R} \text{ satisfies } (1.20) \text{ and } (1.21) \}. \tag{1.22}$$

and

$$\mathrm{Ham}_3^0(\Sigma, g_1, H) = \{ M_{\bar{R}} : \ \bar{R} \ge 0 \text{ satisfies } (1.20) \text{ and } (1.21) \}. \tag{1.23}$$

and

$$\mathrm{Ham}_3^0 = \bigcup \mathrm{Ham}_3^0(\Sigma, g_1, H) \tag{1.24}$$

where the union is take over all (Σ, g_1, H).

In Proposition 13, we prove that $\mathrm{Ham}_3^0 \subset \mathcal{PM}$.

In [15], the first author proved that for any (Σ, g_1, H) , $H > 0$, and any prescribed $\bar{R}$ satisfying (1.20) and (1.21), one has a unique $M_{\bar{R}}$. Thus, for (Σ, g_1, H) which is the boundary of a closed 3-dimensional region with nonnegative scalar curvature and positive mean curvature,

$$\mathrm{m_B}(\Sigma) \leq \inf\{\mathrm{m_{ADM}}(M) : \ M \in \mathrm{Ham}_3^0(\Sigma, g_1, H)\}. \tag{1.25}$$

In Sect. 5 we prove the following theorem:

Theorem 4. *If $M \in \mathrm{Ham}_3^0$ with Σ a surface of constant positive mean curvature and area 4π then*

$$\mathrm{m_{ADM}}(M_{\bar{R}}) \leq m_{a\mathbb{S}}(\Sigma) + \mathrm{m_H}(\Sigma) + e(M_{\bar{R}}, g'_{\bar{R}}) \tag{1.26}$$

where the additional term

$$e(M_{\bar{R}}, g'_{\bar{R}}) = \lim_{t\to\infty} e_t(M_{\bar{R}}, g'_{\bar{R}}) \tag{1.27}$$

where

$$e_t(M_{\bar{R}}, g'_{\bar{R}}) = \frac{1}{2}\int_1^t \frac{\tau^2}{2}\bar{R}^*(\tau)E(\tau, t)d\tau \ \ with \ E(\tau, t) \ defined \ as \ in \ (1.15). \tag{1.28}$$

Here $|M|^{*2}(s)$, defined as in (1.17), depends only on the metric g_1 and

$$\bar{R}^*(\tau) = \sup\{\bar{R}(\tau, x) : x \in \Sigma\} \tag{1.29}$$

depends only on the prescribed scalar curvature $\bar{R}$, so that $e(M_{\bar{R}}, g'_{\bar{R}})$ depends only on g_1 and $\bar{R}$.

Before proving this theorem, we first prove that $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is nonnegative and increasing in t [Lemma 18] and the limit in (1.27) exists and is finite [Lemma 19].

In Sect. 6, we apply this theorem to prove our main theorem, Theorem 1. To do so, we prove $e(M_{\bar{R}}, g'_{\bar{R}}) = 0$ if and only if we prescribe zero scalar curvature, $\bar{R} = 0$ [Proposition 20]. Combining this proposition with Theorem 4 then implies Theorem 1.

In Sect. 7, we consider rigidity and monotonicity of the Hawking mass of level sets $t = r$ for $M \in \mathrm{Ham}_3(\Sigma, g_1, H)$. In [15] the first author proved the Hawking mass is monotone under the following hypothesis. Here, we combine the first author's monotonicity result with an analysis of the rigid case:

Theorem 5. *Let (Σ, g_1) be a surface diffeomorphic to a sphere with positive mean curvature (not necessarily constant) and let $M_{\bar{R}} \subset \mathrm{Ham}_3^0$ be its admissible extension with prescribed scalar curvature $\bar{R} \geq 0$ then we have monotonicity as in [15]:*

$$\mathrm{m_H}(\Sigma_r) \ is \ nondecreasing \ where \ \Sigma_r = t^{-1}(r). \tag{1.30}$$

Furthermore, if

$$\mathrm{m_{ADM}}(M_{\bar{R}}) = \mathrm{m_H}(\Sigma) \tag{1.31}$$

then $\bar{R} = 0$ everywhere and Σ is isometric to standard sphere, $(\mathbb{S}^2, g_{\mathbb{S}^2})$ and $M_{\bar{R}}$ is rotationally symmetric. If $\mathrm{m_H}(\Sigma) = 0$ then $M_{\bar{R}}$ is isometric to a rotationally symmetric region in Euclidean space. If $\mathrm{m_H}(\Sigma) = m > 0$ then $M_{\bar{R}}$ is isometric to a rotationally symmetric region in Schwarzschild space of mass m.

Note that we do not assume that Σ is a constant mean curvature surface in the hypothesis of this theorem. This is only a conclusion in the rigid case. This theorem was already known in the rotationally symmetric case to Bartnik [3,5, Section 5].

2. Hamilton's Ricci Flow and Prescribed Scalar Curvature

In this section, we review the first author's construction of asymptotically flat manifolds foliated by Hamilton's modified Ricci flow [15]. Recall that this flow has been defined in the introduction. We first recall Hamilton and Chow's theorems concerning modified Ricci flow first proven in [6,10]. See also Theorems 5.64 and 5.77 from textbook of Chow and Knopf [4].

Theorem 6 (Hamilton) [10]. *Given a surface, Σ diffeomorphic to a sphere, with positive Gauss curvature there exists a unique solution $g(t)$ to Hamilton's modified Ricci flow with $g(1) = g_1$ as defined in (1.9)–(1.12). The solution $g(t)$ converges exponentially in any C^k-norm to a smooth constant-curvature metric g_∞ as $t \to \infty$.*

The Theorem follows from the exponential decay of M. See also [4, Corollary 5.63]. For $k = 0, 1, 2, \ldots$, there are constants $0 < c_k, C_k < \infty$ depending only on g_1 such that

$$|\nabla^k M| \leq C_k e^{-c_k t} \tag{2.1}$$

which proves that the solution $g(t)$ converges exponentially fast in all C^k to a metric g_∞ such that the tensor M_∞ vanishes identically. Therefore, we know that the Gauss curvature has decay rate

$$|K - 1| \leq C e^{-ct} \tag{2.2}$$

where c and C are constants depending on g_1 only.

Theorem 7 (Chow) [6]. *Given a surface, Σ, with arbitrary Gauss curvature, there exists a unique solution to Hamilton's modified Ricci flow as defined in (1.9)–(1.12). Furthermore, the flow eventually has positive Gauss curvature so the Gauss curvature and the tensor M both eventually decay exponentially.*

In [15], the first author constructs asymptotically flat 3-metrics of prescribed scalar curvature using parabolic methods. Given (Σ, g_1) a surface of area 4π which is diffeomorphic to a sphere, an admissible extension is created by taking $M_{\bar{R}} = [1, \infty) \times \Sigma$ equipped with the metric

$$g'_{\bar{R}} = u^2 \mathrm{d}t^2 + t^2 g, \tag{2.3}$$

where $g = g_t$ is the solution of the modified Ricci flow. This metric $g'_{\bar{R}} = u^2 dt^2 + t^2 g_t$ has scalar curvature $= \bar{R}$ if and only if u satisfies the parabolic equation.

$$t\frac{\partial u}{\partial t} = \frac{1}{2}u^2\Delta u + \frac{t^2}{4}|M|^2 u + \frac{1}{2}u - \frac{1}{4}\left(2K - t^2\bar{R}\right)u^3, \qquad (2.4)$$

where $\Delta = \Delta_{g_t}$ is the Laplacian with respect to g_t, $K = K_t$ is the Gauss curvature of g_t, and

$$|M|^2 = M_{ij}M_{kl}g^{ik}g^{jl}. \qquad (2.5)$$

When the manifold is asymptotically flat with suitable prescribed $\bar{R}$, the ADM mass is

$$\mathrm{m_{ADM}}(M_{\bar{R}}, g'_{\bar{R}}) = \lim_{t\to\infty} \mathrm{m_H}(\Sigma_t) = \lim_{t\to\infty} \frac{1}{4\pi}\oint_{\Sigma_t} \frac{t}{2}(1 - u^{-2})\mathrm{d}\sigma \qquad (2.6)$$

as Σ_t are nearly round spheres under the Ricci flow. The fact that $m_{ADM}(M) = \lim_{t\to\infty} m_H(\Sigma_t)$ where Σ_t are nearly round spheres (not just coordinate spheres) was proven by Shi et al. in [19].

Theorems 11–13 of the first author in [15] are combined in the following theorem which provides for the existence and uniqueness of an admissible extension of Σ with prescribed scalar curvature $\bar{R}$:

Theorem 8 [15]. *Assume that $\bar{R} \in C^\alpha(M_{\bar{R}})$ satisfying the decay conditions*

$$\int_1^\infty |\bar{R}|^*t^2 dt < \infty, \quad and \quad ||\bar{R}t^2||_{\alpha, I_t} \le \frac{C}{t} \quad where \quad I_t = [t, 4t], \quad t \ge 1. \qquad (2.7)$$

Let C_0 be the nonnegative constant defined by

$$C_0 = \sup_{1\le t<\infty}\left\{-\int_1^t \left(K - \frac{\tau^2}{2}\bar{R}\right)_* \exp\left(\int_1^\tau \frac{s|M|^{*2}}{2}ds\right)d\tau\right\} < \infty. \qquad (2.8)$$

Then for any function $\phi \in C^{2,\alpha}(\Sigma)$ satisfying

$$0 < \phi < \frac{1}{\sqrt{C_0}} \qquad (2.9)$$

there is a unique positive solution $u \in C^{2+\alpha}(M_{\bar{R}})$ of (2.4) with the initial condition

$$u(1, \cdot) = \phi(\cdot). \qquad (2.10)$$

Moreover, $g'_{\bar{R}}$ satisfies the asymptotically flat condition for $t > t_0$, where t_0 is some fixed constant with finite ADM mass and

$$\mathrm{m_{ADM}}(M_{\bar{R}}) = \lim_{t\to\infty} \frac{1}{4\pi}\oint_{\Sigma_t} \frac{t}{2}(1 - u^{-2})d\sigma. \qquad (2.11)$$

Here, we consider only the special case in which Σ is CMC so ϕ is a constant. Thus

Theorem 9 [15]. *Assume that $\bar{R} \in C^\alpha(M_{\bar{R}})$ satisfying the decay conditions (2.7). Then if the mean curvature H of Σ satisfies*

$$H > 2\sqrt{C_0} \tag{2.12}$$

there is a unique positive solution $u \in C^{2+\alpha}(M_{\bar{R}})$ of (2.4) with the initial condition

$$u(1, \cdot) = 2/H. \tag{2.13}$$

Then $g'_{\bar{R}}$ satisfies the asymptotically flat condition for $t > t_0$, where t_0 is some fixed constant with finite ADM mass and

$$\mathrm{m_{ADM}}(M_{\bar{R}}) = \lim_{t \to \infty} \frac{1}{4\pi} \oint_{\Sigma_t} \frac{t}{2}(1 - u^{-2})d\sigma. \tag{2.14}$$

Theorem 9 immediately implies the existence of a unique $M_{\bar{R}}$ as described in Definition 3.

3. Asphericity Mass

Here we prove Lemmas 10 and 11 which validate the definition of aspherical mass given in Definition 2. We then prove the key Proposition 12 which proves the asphericity mass is 0 if and only if the surface is a rescaled standard sphere.

Lemma 10. *If Σ is diffeomorphic to a sphere, then $m_{a\mathbb{S}}(\Sigma, t)$ is nonnegative and increasing in t.*

Proof. Since Σ is diffeomorphic to a sphere, the Gauss–Bonnet Theorem implies that $\frac{\oint_\Sigma K d\sigma}{\oint_\Sigma d\sigma} = 1$. Thus, $K_* \le 1$. Together with the fact that

$$E(\tau, t) = \exp\left(-\int_\tau^t \frac{s|M|^{*2}(s)}{2}ds\right) < 1, \tag{3.1}$$

we see that the integrant of $m_{a\mathbb{S}}(\Sigma, t)$ is nonnegative:

$$1 - K_* \exp\left(-\int_\tau^t \frac{s|M|^{*2}(s)}{2}ds\right) \ge 1 - \exp\left(-\int_\tau^t \frac{s|M|^{*2}(s)}{2}ds\right) \ge 0. \tag{3.2}$$

Therefore, $m_{a\mathbb{S}}(\Sigma, t)$ is nonnegative and increasing in t. $\qquad\square$

Lemma 11. *The asphericity mass is finite and the limit exists for any (Σ, g_1) such that Σ is diffeomorphic to a sphere.*

Proof. From Lemma 10, we have $m_{a\mathbb{S}}(\Sigma, t)$ is increasing. To show the limit $m_{a\mathbb{S}}(\Sigma) = \lim_{t \to \infty} m_{a\mathbb{S}}(\Sigma, t)$ exists, it suffices to show that $m_{a\mathbb{S}}(\Sigma, t)$ is bounded from above. First observe that

$$m_{a\mathbb{S}}(\Sigma, t) = 1/2 \int_1^t 1 - K_*(\tau)E(\tau, t)d\tau$$

$$= 1/2 \int_1^t 1 - \exp\left(-\int_\tau^t \frac{s|M|^{*2}(s)}{2}ds\right)d\tau + 1/2 \int_1^t (1 - K_*(\tau))E(\tau, t)d\tau.$$

Using the fact that $E(\tau, t) \leq 1$ and

$$|e^x - 1| \leq 2|x| \quad \text{for} \quad |x| \leq 1, \tag{3.3}$$

we see that

$$
\begin{aligned}
m_{a\mathbb{S}}(\Sigma, t) &\leq \int_1^t \int_\tau^t \frac{s\, |M|^{*2}(s)}{2} \mathrm{d}s \mathrm{d}\tau + 1/2 \int_1^t (1 - K_*(\tau))\, \mathrm{d}\tau \\
&\leq 1/2 \int_1^t (s-1)s|M|^{*2}(s)\mathrm{d}s + 1/2 \int_1^t (1 - K_*(\tau))\, \mathrm{d}\tau \\
&\leq C
\end{aligned}
$$

since $|M|$ and $1 - K$ converge to 0 exponentially under the modified Ricci flow [6,10]. Hence, the lemma follows from the monotonic sequence theorem. $\square$

Proposition 12. *We have $m_{a\mathbb{S}}(\Sigma) = 0$ if and only if (Σ, g_1) is isometric to a rescaled standard sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$.*

Proof. Suppose that (Σ, g_1) is isometric to $(\mathbb{S}^2, g_{\mathbb{S}^2})$. $|M| \equiv 0$ and $K \equiv 1$ under the Ricci flow. Thus, $m_{a\mathbb{S}}(\Sigma) = 0$.

Suppose $m_{a\mathbb{S}}(\Sigma) = 0$. Since $m_{a\mathbb{S}}(\Sigma, t)$ is nonnegative and increasing in t, we have that

$$1 - K_*(\tau)\exp\left(-\int_\tau^t \frac{s\, |M|^{*2}(s)}{2}\mathrm{d}s\right) = 0 \tag{3.4}$$

for all t and τ. Thus, $K_*(t) = 1$ which implies the Gauss curvature $K(t,x) \geq 1$ for all t under the Ricci flow. On the other hand, $\frac{1}{4\pi} \oint_\Sigma K \mathrm{d}\sigma = 1$, by the Gauss–Bonnet Theorem. It forces that $K = 1$ for all t and that (Σ, g_1) is isometric to a standard sphere by the Uniformization Theorem. $\square$

4. The Ham₃ Class of Spaces

In this section, we study the class of asymptotically flat three dimensional Riemannian manifolds foliated by Hamilton's modified Ricci flow defined in Definition 3. Recall that the first author has already shown the existence of a unique $M_{\bar{R}}$ as described in Definition 3 (c.f. Theorem 9). We now prove this class of spaces contains many interesting classes of spaces [Lemmas 14, 15] including rotationally symmetric spaces [Proposition 17].

Proposition 13. *Let (Σ, g_1, H), $H > 0$ be the boundary of a compact manifold of dimension three with nonnegative scalar curvature. For any $M_{\bar{R}} \in \mathrm{Ham}_3(\Sigma, g_1, H)$, there is no closed minimal surface in $M_{\bar{R}}$. Moreover, $M_{\bar{R}} \in \mathrm{Ham}_3^0$ is an admissible extension.*

Proof. We apply the tangency principle ([8, Theorem 1.1]) by Fonrtenele and Silva.

Suppose there is a closed minimal surface S. There must exist a smallest t_0 so that Σ_{t_0} is tangent to S at a point p. By the assumption $H > 0$ and Theorem 8, there exists a unique positive solution u and hence mean curvature

$H(p) = 2/t_0 u(p)$ on Σ_{t_0} is positive. By the maximum principle (tangency principle), S and Σ_{t_0} coincide in a neighborhood of p, which is impossible. $\square$

Lemma 14. *Let (Σ, g_1, H), $H > 0$ be the boundary of a compact manifold of dimension three with nonnegative scalar curvature. Let $\bar{R}$ be any prescribed scalar curvature satisfying (1.20) and*

$$\bar{R}_{(x,t)} < 2K_{(x,t)}/t^2 \tag{4.1}$$

where $K_{(x,t)}$ is the Gauss curvature of (Σ, g_t) obtained by Hamilton's modified Ricci flow, then we obtain

$$M_{\bar{R}} \in \mathrm{Ham}_3(\Sigma, g_1, H) \in \mathrm{Ham}_3. \tag{4.2}$$

Proof. This follows immediately because the assumption in (4.1) which implies the integrand in the definition of $C_0(\bar{R})$ is nonpositive. The positive mean curvature implies (1.21) which is equivalent to condition (2.9) in Theorem 8. By Theorem 8, we obtain such a manifold $M_{\bar{R}}$. $\square$

Lemma 15. *If (Σ, g_1) with positive Gauss curvature and prescribed 0 scalar curvature $\bar{R} = 0$ then $M_{\bar{R}}$ is defined and $M_{\bar{R}} \in \mathrm{Ham}_3^0$.*

Proof. Hamilton proved in [10] that (Σ, g_t) has positive Gauss curvature for all t and so (4.1) holds for $\bar{R} = 0$. Since $\bar{R} = 0$ satisfies (1.20), we apply Lemma 14 to complete the proof. $\square$

We next prove that the asymptotically flat rotationally symmetric Riemannian manifolds of dimension 3 including the classical rotationally symmetric gravity wells and black holes lie in Ham_3:

Definition 16. Let RotSym_3 be the class of complete 3-dimensional asymptotically flat rotationally symmetric Riemannian manifolds, (M, g), with

$$g = (f(r))^2 \mathrm{d}r^2 + r^2 g_{S^2} \tag{4.3}$$

of nonnegative scalar curvature $\bar{R} \geq 0$ with no closed interior minimal hypersurfaces which either have no boundary or have a boundary which is a stable minimal hypersurface.

Proposition 17. *If $M \in \mathrm{RotSym}_3$ is asymptotically flat with $r_{min} < 1$ so that*

$$\exists C > 0 \ \ such \ that \ \ |m_H''(r)| < \frac{C}{r^2} \tag{4.4}$$

then $\bar{R}$ satisfies (1.20) and (1.21). Thus for any rotationally symmetric $\Sigma = r^{-1}(t) \in M$ we have

$$r^{-1}[t, \infty) \in \mathrm{Ham}_3. \tag{4.5}$$

Proof. In [14], the second author and Lee proved that there is a one-to-one correspondence between manifolds $M \in \mathrm{RotSym}_3$ and nondecreasing continuous functions, $m_H : [r_{\min}, \infty) \to [0, \infty)$ such that

$$m_H(t) < t/2, \quad \lim_{t \to \infty} m_H(t) = \mathrm{m_{ADM}}(M) < \infty, \quad m_H(r_{\min}) = 0,$$
$$\text{and} \ \ m_H(t) > 0 \quad \text{for} \ \ t > r_{\min} \tag{4.6}$$

where $m_H(t)$ denotes the Hawking mass of the level set $r^{-1}(t)$.

Since (cf. (7.5) and (7.6))

$$m'_H(t) = t^2 \bar{R}(t)/4 \tag{4.7}$$

we have

$$\frac{t}{2} > m_H(t) = m_H(1) + \int_1^t \frac{t^2 \bar{R}(t)}{4} \, \mathrm{d}t. \tag{4.8}$$

Now $K = 1$ and $|M| = 0$ in the rotationally symmetric case. So

$$C_0(\bar{R}) = \sup_{1 \leq t < \infty} \left\{ \int_1^t \left(\frac{\tau^2}{2} \bar{R} - 1 \right) \mathrm{d}\tau \right\} \tag{4.9}$$

$$= \sup_{1 \leq t < \infty} \{2m_H(t) - 2m_H(1) - (t-1)\} \tag{4.10}$$

$$< t - 2(1 - H^2/4) - (t-1) = H^2/4. \tag{4.11}$$

Since $\lim_{t \to \infty} m_H(t) < \infty$, by (4.8) we have

$$\int_1^\infty t^2 \bar{R}(t) \, \mathrm{d}t < \infty. \tag{4.12}$$

So we have the first part of (1.20).

Now consider the weighted Hölder norm:

$$||\bar{R}(t)t^2||_{\alpha, I_r} = \sup \left\{ t_2^\alpha \frac{|\bar{R}(t_1)t_1^2 - \bar{R}(t_2)t_2^2|}{|t_1 - t_2|^\alpha} : t_1 \neq t_2 \in [r, 4r] \right\} \tag{4.13}$$

$$= \sup \left\{ t_2^\alpha \frac{|4m'_H(t_1) - 4m'_H(t_2)|}{|t_1 - t_2|^\alpha} : t_1 \neq t_2 \in [r, 4r] \right\} \tag{4.14}$$

$$\leq 16 \cdot 3^{1-\alpha} r \sup_{[r, 4r]} |m''_H(t)|. \tag{4.15}$$

Assume on the contrary that Hölder part of (1.20) is false, then

$$\lim_{r_j \to \infty} r_j ||\bar{R}(t)t^2||_{\alpha, I_{r_j}} = \infty \tag{4.16}$$

and so

$$\lim_{r_j \to \infty} r_j^2 \sup_{[r_j, 4r_j]} |m''_H(t)| = \infty \tag{4.17}$$

so

$$\lim_{r_j \to \infty} r_j^2 |m''_H(r_j)| = \infty \tag{4.18}$$

which contradicts (4.4). $\square$

5. Estimating and Minimizing the ADM Mass

Here, we prove Theorem 4. First we prove Lemmas 18 and 19.

Lemma 18. *Given $\bar{R} \geq 0$, we see that $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is nonnegative and increasing in t.*

Proof. Recall that

$$e_t(M_{\bar{R}}, g'_{\bar{R}}) = \frac{1}{2} \int_1^t \frac{\tau^2}{2} \bar{R}^*(\tau) E(\tau, t) \mathrm{d}\tau. \tag{5.1}$$

Given $\bar{R} \geq 0$, the integrand of $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is nonnegative. Hence, $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is nonnegative and increasing in t. $\qquad\square$

Lemma 19. *Given $\bar{R} \geq 0$ such that*

$$\int_1^\infty |\bar{R}|^* t^2 dt < \infty, \tag{5.2}$$

we see that the limit $e(M_{\bar{R}}, g'_{\bar{R}})$ in (1.27) exists and is finite.

Proof. Recall that

$$e(M_{\bar{R}}, g'_{\bar{R}}) = \lim_{t \to \infty} e_t(M_{\bar{R}}, g'_{\bar{R}}). \tag{5.3}$$

With $\bar{R} \geq 0$, the integrand of $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is nonnegative, so $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is increasing in t. Moreover, since

$$E(\tau, t) = \exp\left(-\int_\tau^t \frac{s\,|M|^{*2}(s)}{2} \mathrm{d}s\right) \leq 1, \tag{5.4}$$

applying (5.2) we have

$$e_t(M_{\bar{R}}, g'_{\bar{R}}) \leq \frac{1}{2} \int_1^t \frac{\tau^2}{2} \bar{R}^*(\tau) \mathrm{d}\tau < \infty. \tag{5.5}$$

Therefore, $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is increasing and bounded in t, and hence the limit $e(M_{\bar{R}}, g'_{\bar{R}})$ exists and is finite by the monotonic sequence theorem. $\qquad\square$

We now prove Theorem 4:

Proof. By the assumptions, Theorem 9 provides an unique admissible extension $M_{\bar{R}} = [1, \infty) \times \Sigma$ with prescribe scalar curvature $\bar{R}$ is obtained. There exists an unique solution $u \in C^{2+\alpha}(M_{\bar{R}})$ with initial condition $u(1, \cdot) = 2/H$ such that the metric

$$g'_{\bar{R}} = u^2 \mathrm{d}t^2 + t^2 g_t$$

satisfies the asymptotically flat condition and finite ADM mass and

$$\mathrm{m_{ADM}}(M_{\bar{R}}) = \lim_{t \to \infty} \frac{1}{4\pi} \oint_{\Sigma_t} \frac{t}{2} (1 - u^{-2}) \mathrm{d}\sigma. \tag{5.6}$$

Applying the parabolic maximum principle to the parabolic equation of u^{-2} (Lemma 10 in [15]), we have the following C^0 bound:

$$u^{-2}(t) \geq \frac{1}{t} \int_1^t \left(K - \frac{\tau^2}{2}\bar{R}\right)_* E(\tau, t) \mathrm{d}\tau + \frac{1}{t} u^{-2}(1). \tag{5.7}$$

By direction computation,

$$1 - u^{-2} = 1 - \frac{1}{t}\int_1^t \left(K - \frac{\tau^2}{2}\bar{R}\right)_* E(\tau, t)\mathrm{d}\tau - \frac{H^2}{4t}$$

$$\leq 1 - \frac{1}{t}\int_1^t \left(K_* - \frac{\tau^2}{2}\bar{R}^*\right) E(\tau, t)\mathrm{d}\tau - \frac{H^2}{4t}$$

$$= \frac{1}{t}\int_1^t 1 - K_* E(\tau, t)\mathrm{d}\tau + \frac{1}{t}\int_1^t \frac{\tau^2}{2}\bar{R}^* E(\tau, t)\mathrm{d}\tau + \frac{1}{t}\left(1 - \frac{H^2}{4}\right).$$

Also, the Hawking mass of Σ is given by the formula

$$\mathrm{m_H}(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 \mathrm{d}\sigma\right) = \frac{1}{4\pi}\int_\Sigma \frac{1}{2}\left(1 - \frac{H^2}{4}\right)\mathrm{d}\sigma. \quad (5.8)$$

Therefore,

$$\frac{1}{4\pi}\oint_{\Sigma_t} \frac{t}{2}(1 - u^{-2})\mathrm{d}\sigma$$

$$\leq \frac{1}{2}\int_1^t 1 - K_* E(\tau, t)\mathrm{d}\tau + \frac{1}{2}\int_1^t \frac{\tau^2}{2}\bar{R}^* E(\tau, t)\mathrm{d}\tau + \mathrm{m_H}(\Sigma)$$

$$= m_{a\mathbb{S}}(\Sigma, t) + e_t(M_{\bar{R}}, g'_{\bar{R}}) + \mathrm{m_H}(\Sigma),$$

and

$$\mathrm{m_{ADM}}(M_{\bar{R}}) \leq m_{a\mathbb{S}}(\Sigma) + \mathrm{m_H}(\Sigma) + e(M_{\bar{R}}, g'_{\bar{R}}). \quad (5.9)$$

It follows directly by the definition of the Bartnik mass that

$$\mathrm{m_B}(\Sigma) \leq m_{a\mathbb{S}}(\Sigma) + \mathrm{m_H}(\Sigma) + e(M_{\bar{R}}, g'_{\bar{R}}). \quad (5.10)$$

$$\square$$

6. Proving the Main Theorem

To complete the proof of Theorem 1 we need to only prove the following proposition and combine it with Theorem 4:

Proposition 20. *Given $\bar{R} \geq 0$, we have $e(M_{\bar{R}}, g'_{\bar{R}}) = 0$ if and only if prescribed $\bar{R} = 0$.*

Proof. Suppose $\bar{R} = 0$. It is clear that, by the definition,

$$e(M_{\bar{R}}, g'_{\bar{R}}) = \lim_{t\to\infty} \frac{1}{2}\int_1^t \frac{\tau^2}{2}\bar{R}^*(\tau)e^{-\int_\tau^t \frac{s|M|^{*2}(s)}{2}\mathrm{d}s}\mathrm{d}\tau = 0. \quad (6.1)$$

Suppose that $\bar{R} \geq 0$ and $e(M_{\bar{R}}, g'_{\bar{R}}) = 0$. Since $e_t(M_{\bar{R}}, g'_{\bar{R}})$ is nonnegative and increasing in t by Lemma 18. It follows that $\bar{R}^*(\tau) = 0$ for all τ. We, therefore, conclude that $\bar{R} = 0$ since $0 = \bar{R}^* \geq \bar{R} \geq 0$. $\square$

7. Rigidity and Monotonicity of the Hawking Mass

Here we derive a monotonicity formula for Hawking mass which was already known in [15] and then prove Theorem 5.

For $M_{\bar{R}} = [1, \infty) \times \Sigma$ equipped with the metric

$$g'_{\bar{R}} = u^2 \mathrm{d}t^2 + t^2 g, \tag{7.1}$$

where $g = g_t$ is the solution of the modified Ricci flow. Observe that the mean curvature of a level set of t, Σ_t, is

$$H_t(x) = 2/u(t, x). \tag{7.2}$$

Therefore, the Hawking mass (see [15, Theorem 13]) is given by

$$\mathrm{m_H}(\Sigma_t) = \sqrt{\frac{\mathrm{area}\,(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \oint_\Sigma H^2 \mathrm{d}\sigma \right) \tag{7.3}$$

$$= \frac{1}{4\pi} \oint_{\Sigma_t} \frac{t}{2} (1 - u^{-2}(t, x)) \mathrm{d}\sigma. \tag{7.4}$$

From the Gauss–Bonnet Theorem and (2.4), we have the following monotonicity formula provided $\bar{R} \geq 0$.

$$\frac{d}{dt} \mathrm{m_H}(\Sigma_t) = \frac{1}{4\pi} \oint_{\Sigma_t} \frac{1}{2} u^{-1} \Delta u + \frac{t^2}{4} |M|^2 u^{-2} + \frac{t^2}{4} \bar{R} + \left(\frac{1}{2} - \frac{K}{2} \right) d\sigma \tag{7.5}$$

$$= \frac{1}{8\pi} \oint_{\Sigma_t} \frac{|\nabla u|^2}{u^2} + \frac{t^2}{2} |M|^2 u^{-2} + \frac{t^2}{2} \bar{R} d\sigma. \tag{7.6}$$

We now prove Theorem 5:

Proof. By the assumptions and Theorem 8, an admissible extension $M_{\bar{R}}$ exists and the ADM mass can be obtained by

$$\mathrm{m_{ADM}}(M_{\bar{R}}) = \lim_{t \to \infty} \mathrm{m_H}(\Sigma_t). \tag{7.7}$$

Given $\bar{R} \geq 0$, $\mathrm{m_H}(\Sigma_t)$ is increasing by the monotonicity formula (7.6). $\mathrm{m_{ADM}}(M_{\bar{R}}) = \mathrm{m_H}(\Sigma)$ implies that $\frac{d}{dt} \mathrm{m_H}(\Sigma_t) = 0$. Hence $\bar{R} = 0$, $|M| = 0$, and $\nabla u = 0$. Since $|M| = 0$, then Σ is isometric to a standard sphere by [10]. Since $\nabla u = 0$, we have $u(x, t) = u(t)$, so H is constant and $M_{\bar{R}}$ is rotationally symmetric. Since $\bar{R} = 0$ if $\mathrm{m_H} = m \geq 0$ then $M_{\bar{R}}$ is isometric to a rotationally symmetric region in M_{Sch} of mass m or Euclidean space (c.f. Lemma 23). $\square$

8. Open Questions

There are many theorems proven for the rotationally symmetric classes of spaces with nonnegative scalar curvature. It would be interesting to extend these results to the class of spaces Ham_3^0:

Question 21. *What can be said about the vacuum solutions of the Einstein equation which have initial data sets foliated by Hamilton's modified Ricci flow?*

Question 22. *If one fixes (Σ, g_1, H), what can be said about sequences of $M_j \in \mathrm{Ham}_3^0(\Sigma, g_1, H)$ assuming $\mathrm{m_{ADM}}(M_j) \leq m_0$?*

9. Appendix on Rotationally Symmetric Spaces

The following lemma was already basically understood in the rotationally symmetric setting and is proven here using our notation for completeness of exposition:

Lemma 23. *Given* (Σ, g_1) *isometric to a rescaled standard sphere and* $H > 0$ *constant and* $\bar{R} = 0$ *and* $m_H(\Sigma) = m \geq 0$ *and assume* Σ *is the boundary of a region* $\Omega \subset M$ *where* $M \subset \mathcal{PM}$ *, then* $(M_{\bar{R}}, g'_{\bar{R}})$ *is a rotationally symmetric region in a Schwarzschild space or in Euclidean space with metric*:

$$\bar{g} = \frac{1}{1 - 2m/t} dt^2 + t^2 g_{\mathbb{S}^2}. \tag{9.1}$$

Since Hawking mass is constant in a Schwarzschild space we have

$$m_B(\Sigma) \leq m_H(\Sigma). \tag{9.2}$$

In particular $m_H(\Sigma) \geq 0$, *which implies* $H \leq 2$.

Remark 24. Observe that Shi–Tam have proven in [17] that

$$\int_\Sigma H \, d\sigma \leq \int_\Sigma H_0 d\sigma \tag{9.3}$$

which in the constant mean curvature case implies

$$4\pi H \leq \int_\Sigma H_0 d\sigma \tag{9.4}$$

and so

$$m_H(\Sigma) = \sqrt{\frac{1}{4}\left(1 - \frac{1}{4}H^2\right)} \tag{9.5}$$

$$\geq \sqrt{\frac{1}{4}\left(1 - \frac{1}{16\pi}\left(\int_\Sigma H_0 d\sigma\right)^2\right)}. \tag{9.6}$$

Furthermore in the rotationally symmetric case $H_0 = 2$, so $4\pi H \leq 8\pi$ and $H \leq 2$ just as concluded above.

Proof. Consider

$$\bar{g} = u^2(t)\mathrm{d}t^2 + t^2 g_{\mathbb{S}^2}. \tag{9.7}$$

Then $\mathrm{m_H}(t) = \frac{t}{2}(1 - u^{-2}(t))$ equals the Hawking mass of Σ_t.

Observe that $\bar{R} = 0$ implies $\mathrm{m_H'}(t) = 0$ which implies $\mathrm{m_H}(t) = m$. This

$$m = \frac{t}{2}\left(1 - u^{-2}(t)\right) \tag{9.8}$$

and so

$$u^2(t) = \frac{1}{1 - 2m/t}. \tag{9.9}$$

$\square$

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