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On the Stationary Navier–Stokes Equations in the Half-Plane

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Abstract. We consider the stationary incompressible Navier–Stokes equation in the half-plane with inhomogeneous boundary condition. We prove the existence of strong solutions for boundary data close to any Jeffery–Hamel solution with small flux evaluated on the boundary. The perturbation of the Jeffery–Hamel solution on the boundary has to satisfy a nonlinear compatibility condition which corresponds to the integral of the velocity field on the boundary. The first component of this integral is the flux which is an invariant quantity, but the second, called the asymmetry, is not invariant, which leads to one compatibility condition. Finally, we prove the existence of weak solutions, as well as weak–strong uniqueness for small data and provide numerical simulations.

1. Introduction

The stationary and incompressible Navier–Stokes equations in the half-plane

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : y > 1\right\}$$

are

$$\begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{\nabla} p = \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,\\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}^*, \quad \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} = \boldsymbol{0}, \end{aligned} \tag{1.1}$$

where $\boldsymbol{u}^*$ is a boundary condition. Due to the incompressibility of the fluid, the flux is an invariant quantity,

$$\Phi = \int_{\partial\Omega} \boldsymbol{u}^* \cdot \boldsymbol{n} = \int_{\mathbb{R}} v(x, y)\, \mathrm{d}x,$$

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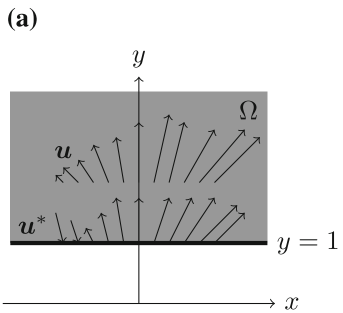

FIGURE 1. **a** The domain Ω we consider is the half plane defined by $x \in \mathbb{R}$ and $y > 1$; **b** an aperture domain; **c** a channel connected to a half-plane

for all $y \geq 1$, where $\boldsymbol{u} = (u, v)$ and $\boldsymbol{n} = (0, 1)$ is the unit vector normal to the boundary of the half-plane. This problem (see Fig. 1a) presents three difficulties: Ω is a two-dimensional unbounded domain, the boundary of $\partial\Omega$ is unbounded, and the boundary data are not zero. There is not much previous work on this problem, but some authors have treated related problems. Concerning the half-plane problem, [19, Section 5] proves the uniqueness of solutions for the steady Stokes equation and the time-dependent Navier–Stokes equation. The so-called Leray's problem, which consists of a finite number of outlets connected to a compact domain, has been studied in detail by Amick [1–3] and several other authors, but the resolvability for large fluxes is still an open problem. Fraenkel [12,13] provides a formal asymptotic expansion of the stream function in case of a curved channel by starting with the Jeffery–Hamel solution [18,22] as the first order. The case of paraboloidal outlets was first treated by Nazarov and Pileckas [30], and more recently by Kaulakytė and Pileckas [24] and Kaulakytė [23]. Another important class of similar problems are the aperture domains, introduced by Heywood [19], as shown in Fig. 1b. The linear approximation was studied in any dimension by Farwig [9] and Farwig and Sohr [10]. The three-dimensional case was treated by Borchers and Pileckas [6], as well as other authors. For the two-dimensional nonlinear problem, Galdi et al. [14] proved that the velocity tends to zero in the L^2-norm for arbitrary values of the flux. For small fluxes, Galdi et al. [16] and Nazarov [29] show that the asymptotic behavior is given by a Jeffery–Hamel solution, but only if the problem is symmetric with respect to the y-axis. The asymptotic behavior of the two-dimensional aperture problem in the nonsymmetric case is still open. Finally, Nazarov et al. [31,32] considered a straight channel connected to a half-plane (see Fig. 1c) and looked under which conditions the asymptotic behavior is given by a Jeffery–Hamel flow in the half-plane and by the Poiseuille flow in the channel. These conditions are described in detail later on. On a more applied side, the bifurcation properties and the stability of Jeffery–Hamel flows have retained the attention of many authors [4,8,28,36,38].

Jeffery–Hamel flows play an important role in the asymptotic behavior of flows carrying flux. They own their name to the work of Jeffery [22] and Hamel [18] and are radial scale invariant solutions of the two-dimensional stationary incompressible Navier–Stokes equations

$$\Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} - \boldsymbol{\nabla} p = \boldsymbol{0}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

in domains

$$D = \left\{ (r\sin(\theta), r\cos(\theta)) \in \mathbb{R}^2 \colon\ r > 0 \text{ and } \theta \in (-\beta, \beta) \right\}$$

with $\beta \in \left(0, \frac{\pi}{2}\right]$, satisfying the boundary condition

$$\boldsymbol{u}\big|_{\partial D \setminus \{\boldsymbol{0}\}} = \boldsymbol{0}.$$

Explicitly, a Jeffery–Hamel solution $(\boldsymbol{u}_{\mathrm{JH}}, p_{\mathrm{JH}})$ is of the form

$$\boldsymbol{u}_{\mathrm{JH}}(r,\theta) = \frac{1}{r} f(\theta)\, \boldsymbol{e}_r, \quad p_{\mathrm{JH}}(r,\theta) = \frac{1}{r^2}\left(2f'(\theta) - C\right),$$

with $C \in \mathbb{R}$ and f a solution of the nonlinear second-order ordinary differential equation

$$f'' + f^2 + 4f = 2C,$$

satisfying the boundary condition $f(\pm\beta) = 0$. The constant C is related to the flux Φ of the flow,

$$\Phi = \int_{-\beta}^{+\beta} f(\theta)\, \mathrm{d}\theta.$$

The Jeffery–Hamel solutions have been intensively studied [4,12,35,37], and, because some basic mathematical questions still remain open, there has been a regain of interest in recent years [7,25,33,34].

In what follows, we are interested in the half-plane case, so we consider $\beta = \frac{\pi}{2}$, i.e., the domain is the upper half plane $D = \mathbb{R} \times (0, \infty)$. In Cartesian coordinates, the two components of the velocity of the Jeffery–Hamel solutions are

$$u_{\mathrm{JH}}(x,y) = \frac{1}{y} f_u\left(\frac{x}{y}\right), \quad v_{\mathrm{JH}}(x,y) = \frac{1}{y} f_v\left(\frac{x}{y}\right), \tag{1.2}$$

with

$$f_u(s) = \frac{s\, f(\arctan s)}{1+s^2}, \quad f_v(s) = \frac{f(\arctan s)}{1+s^2}.$$

The Jeffery–Hamel solutions for $\beta = \frac{\pi}{2}$ have a peculiar property: for small $\Phi < 0$, there is more than one solution. In fact, as shown in Appendix A, $\Phi = 0$ is a tri-critical bifurcation point; see Fig. 2. For small $\Phi > 0$, the Jeffery–Hamel problem has a solution $(\boldsymbol{u}_\Phi^0, p_\Phi^0)$ which is symmetric with respect to the y-axis. The solution $(\boldsymbol{u}_\Phi^0, p_\Phi^0)$ also exists for small values of $\Phi < 0$, but when crossing from $\Phi > 0$ to $\Phi < 0$, an additional pair $(\boldsymbol{u}_\Phi^{\pm 1}, p_\Phi^{\pm 1})$ of asymmetric solutions (related to each other by a reflection with respect to the y-axis) appears. For $\Phi = 0$, the Jeffery–Hamel solution is the zero function and will be ignored in what follows.

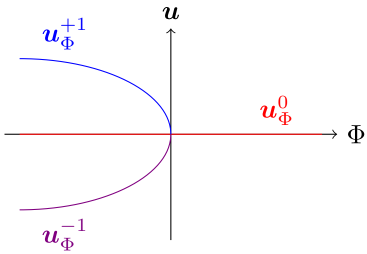

Figure 2. Existence of the Jeffery–Hamel flows for small values of the flux Φ. For $\Phi > 0$, there exists one symmetric solutions, but for $\Phi < 0$, also two additional asymmetric solutions exist

The central idea of the method we use to study (1.1) is to interpret the system as an evolution equation with y playing the role of time. The boundary data of the original problem then become the initial data for the resulting Cauchy problem. This allows discussing the "time" dependence of quantities like the flux $\Phi = \int_{\mathbb{R}} v(x,y)\mathrm{d}x$ and the asymmetry $A = \int_{\mathbb{R}} u(x,y)\mathrm{d}x$ in a natural setting. We assume for the moment sufficient decay for these integrals to make sense. As can be seen from (1.2), the flux and the asymmetry are invariants for a Jeffery–Hamel solution $(\boldsymbol{u}_\Phi^\sigma, p_\Phi^\sigma)$, i.e., they are independent of the time y and therefore

$$\begin{aligned} A &= \int_{\mathbb{R}} u_\Phi^\sigma(x,1)\,\mathrm{d}x = \int_{\mathbb{R}} u_\Phi^\sigma(x,y)\,\mathrm{d}x = \lim_{y\to\infty}\int_{\mathbb{R}} u_\Phi^\sigma(x,y)\,\mathrm{d}x, \\ \Phi &= \int_{\mathbb{R}} v_\Phi^\sigma(x,1)\,\mathrm{d}x = \int_{\mathbb{R}} v_\Phi^\sigma(x,y)\,\mathrm{d}x = \lim_{y\to\infty}\int_{\mathbb{R}} v_\Phi^\sigma(x,y)\,\mathrm{d}x. \end{aligned}$$

The flux is an invariant of the Navier–Stokes equation (1.1), i.e., if $\boldsymbol{u}$ is a solution of the Navier–Stokes equation, then for $y > 1$,

$$\int_{\mathbb{R}} v(x,1)\,\mathrm{d}x = \int_{\mathbb{R}} v(x,y)\,\mathrm{d}x = \lim_{y\to\infty}\int_{\mathbb{R}} v(x,y)\,\mathrm{d}x,$$

but the asymmetry is not an invariant, so typically,

$$\int_{\mathbb{R}} u(x,1)\,\mathrm{d}x \neq \int_{\mathbb{R}} u(x,y)\,\mathrm{d}x \neq \lim_{y\to\infty}\int_{\mathbb{R}} u(x,y)\,\mathrm{d}x.$$

As we will see below, for solutions that are close but not equal to Jeffery–Hamel, the asymmetry is no more an invariant, and this fact is the main source of trouble for the construction of solutions.

Jeffery–Hamel solutions are singular at the origin and to study solutions which are close to Jeffery–Hamel flows, it is necessary to regularize the problem. For this purpose, given a Jeffery–Hamel solution $(\boldsymbol{u}_\Phi^\sigma, p_\Phi^\sigma)$ in the upper half plane $D = \big\{(x,y) \in \mathbb{R}^2 \colon y > 0\big\}$, we restrict it to the domain

$\Omega = \{(x, y) \in \mathbb{R}^2 \colon y > 1\}$ and construct stationary solutions of the Navier–Stokes equations which are close to the Jeffery–Hamel flow by imposing boundary conditions of the form

$$\boldsymbol{u}\big|_{\partial\Omega} = \boldsymbol{u}_\Phi^\sigma\big|_{\partial\Omega} + \boldsymbol{u}_b, \tag{1.3}$$

with $\boldsymbol{u}_b$ small and with zero flux,

$$\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} = 0.$$

Even when considering such boundary conditions, we are not able to perform a fixed point argument on the nonlinearity by inverting the Stokes problem. The main reason is that the flux is determined by the boundary condition, while the asymmetry is not. To adjust the asymmetry, we rewrite the boundary condition as

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_\Phi^\sigma|_{\partial\Omega} + (A, 0)\,\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2} + \boldsymbol{u}_s, \tag{1.4}$$

where

$$A = \int_{\mathbb{R}} u_b,$$

so that $\boldsymbol{u}_s$ has no asymmetry and no flux.

$$\int_{\mathbb{R}} \boldsymbol{u}_s = \boldsymbol{0}.$$

The choice of $\mathrm{e}^{-\frac{1}{2}x^2}$ in (1.4) is for convenience later on, and we could have chosen instead any other smooth function of rapid decay.

We will show the existence of strong solutions to the Navier–Stokes equation (1.1), with the boundary condition (1.4) for Φ small and $\boldsymbol{u}_s$ in a small ball by adjusting the parameter A. The main result is the following:

Theorem 1.1. *For all boundary conditions of the form (1.4) with $\boldsymbol{u}_\Phi^\sigma$ a Jeffery–Hamel solution with small enough flux Φ and all $\boldsymbol{u}_s$ in a small enough neighborhood of zero in some function space, there exists $A \in \mathbb{R}$ (depending continuously on Φ and $\boldsymbol{u}_s$) and a solution $(\boldsymbol{u}, p)$ of the Navier–Stokes equation in Ω satisfying*

$$\lim_{y\to\infty} y\left(\sup_{x\in\mathbb{R}} |\boldsymbol{u} - \boldsymbol{u}_\Phi^\sigma|\right) = 0, \qquad \lim_{y\to\infty} y^2\left(\sup_{x\in\mathbb{R}} |p - p_\Phi^\sigma|\right) = 0,$$

and

$$\begin{aligned} \boldsymbol{\nabla}\boldsymbol{u} &\in L^2(\Omega), & y\boldsymbol{u} &\in L^\infty(\Omega), \\ \boldsymbol{u}/y &\in L^2(\Omega), & y^2\boldsymbol{\nabla}\boldsymbol{u} &\in L^\infty(\Omega). \end{aligned} \tag{1.5}$$

Moreover, if $\boldsymbol{v}$ is a weak solution (defined in (5.1)) of (1.1) and $\boldsymbol{u}$ a strong solution of (1.1) satisfying (1.5), then $\boldsymbol{u} = \boldsymbol{v}$, provided $\boldsymbol{u}^*$ is small enough.

We now discuss more precisely the results of Nazarov et al. [31,32], for the domain shown in Fig. 1c. We note that in this domain, the flux through the channel is not prescribed. They show that by requiring the asymptotic behavior to be an antisymmetric Jeffery–Hamel solution $\boldsymbol{u}_\Phi^{\pm 1}$, there exists a unique solution in some weighted space, and the flux is uniquely determined by the data. Conversely, by requiring that the asymptotic behavior is given by a symmetric Jeffery–Hamel solution $\boldsymbol{u}_\Phi^0$, the Navier–Stokes equation linearized around $\boldsymbol{u}_\Phi^0$ leads to a well-posed problem for $\Phi < 0$ and to an ill-posed one for $\Phi > 0$. So for $\Phi < 0$, there exists a unique solution for all small enough fluxes, but for $\Phi > 0$ the asymptotic behavior is still unknown. So we believe that in case $\Phi < 0$, the Navier–Stokes equation in the half-plane (1.1) has a solution decaying like r^{-1} at infinity whose asymptote is given by a Jeffery–Hamel solution, but in the case $\Phi > 0$, it is still not clear that the solution is in general bounded by r^{-1}.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the function spaces which we use for the mathematical formulation of the problem and prove some basic bounds. In Sect. 3, we rewrite the Stokes equation as a dynamical system, present the associated integral equations, and provide bounds on the solution of the Stokes system, so that in Sect. 4 we can show the existence of strong solutions to the Navier–Stokes system. In Sect. 5, we prove the existence of weak solutions, and, finally, in Sect. 6, we prove the uniqueness of solutions for small data with a weak–strong uniqueness argument. In the last part, we present numerical simulations that show that the asymptotic behavior is most likely not given by a Jeffery–Hamel solution if $\Phi > 0$. In the Appendix, we show the existence of symmetric and asymmetric Jeffery–Hamel solutions with small flux.

2. Function Spaces

As explained in the introduction, our strategy of proof is to rewrite (1.1) as a dynamical system with y playing the role of time. This system is studied by taking the Fourier transform in the variable x, which transforms the system into a set of ordinary differential equations with respect to y. We now define the function spaces for the Fourier transforms of the velocity field, pressure field and the nonlinearity. The choice of spaces is motivated by the scaling property of the equations with respect to x and y when linearized around a Jeffery–Hamel solution. This setup turns out to be natural for the description of the asymptotic behavior of solutions close to Jeffery–Hamel flows. Similar function spaces were already used by Wittwer [39], where the basic operations which are needed for the discussion of the Navier–Stokes equations were discussed. In particular, Wittwer [39] shows basic bounds on the convolution with respect to the variable k, the Fourier conjugate variable of x, which is needed to implement the nonlinearities, and bounds on the convolution with the semigroup $\mathrm{e}^{-|k|y}$ which is associated with the Stokes operator when viewed as a time evolution in y. Further properties and improved bounds have been proved by Hillairet and Wittwer [20] and Boeckle and Wittwer [5].

Definition 2.1. (*Fourier transform and convolution*) For two functions $\hat{f}$ and $\hat{g}$ defined almost everywhere in Ω and which are in $L^1(\mathbb{R})$ for all $y \geq 1$, the inverse Fourier transform of $\hat{f}$ is defined by

$$f(x,y) = \mathcal{F}^{-1}\big[\hat{f}\,\big](x,y) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\hat{f}(k,y)\,\mathrm{d}k,$$

and the convolution by

$$\big(\hat{f} * \hat{g}\big)(k,y) = \int_{\mathbb{R}} \hat{f}(k-\ell,y)\hat{g}(\ell,y)\,\mathrm{d}\ell.$$

We note that with these definitions,

$$fg = \mathcal{F}^{-1}\big[\hat{f} * \hat{g}\big].$$

We now define two families of function spaces: the first one is for functions of k only which will be used for the boundary data, and the second one is for functions of k and y:

Definition 2.2 (*function spaces on $\partial\Omega$*). For $\alpha \geq 0$ and $q \in \mathbb{R}$, let $\mathcal{A}_{\alpha,q}$ be the Banach space of functions $\hat{f} \in C(X,\mathbb{C})$ where

$$X = \begin{cases} \mathbb{R}, & q \geq 0, \\ \mathbb{R}\backslash\{0\}, & q < 0, \end{cases} \tag{2.1}$$

such that $\overline{\hat{f}(k)} = \hat{f}(-k)$ and the norm

$$\big\|\hat{f};\mathcal{A}_{\alpha,q}\big\| = \sup_{X} \frac{|\hat{f}|}{\eta_{\alpha,q}} \qquad \text{with} \qquad \eta_{\alpha,q}(k) = \frac{|k|^q}{1+|k|^{\alpha+q}},$$

is finite. For $\alpha \geq 0$ and $q > 0$, let $\mathcal{T}_{\alpha,q}$ and $\mathcal{W}_{\alpha,q}$ be the Banach space of functions in $\mathcal{A}_{\alpha+1,\min(0,q-1)}$ such that their respective norm

$$\big\|\hat{f};\mathcal{T}_{\alpha,q}\big\| = \sum_{i=0}^{\lceil q-1\rceil} \big\|\partial_k^i\hat{f};\mathcal{A}_{\alpha+1,\min(0,q-1-i)}\big\|,$$

$$\big\|\hat{f};\mathcal{W}_{\alpha,q}\big\| = \sum_{i=0}^{\lceil q-1\rceil} \big\|\partial_k^i\hat{f};\mathcal{A}_{\alpha+1,q-1-i}\big\|,$$

is finite, where $\lceil q-1\rceil$ denotes the ceiling of $q-1$.

Remark 2.3. The parameter α captures the behavior of functions at infinity, which corresponds to the regularity in x in direct space. For example, if $\hat{f} \in \mathcal{A}_{\alpha,0}$ for $\alpha > 1$, then $\hat{f} \in L^1(\mathbb{R})$ and by the dominated convergence theorem, $f \in C(\mathbb{R})$. The index q characterizes the behavior near $k=0$: a function which behaves like $|k|^q$ around zero is in the space $\mathcal{A}_{\alpha,q}$. The space $\mathcal{T}_{\alpha,q}$ includes some characterization of the derivative with respect to k, which is needed to characterize the behavior near $k=0$ as shown in the next lemma.

Lemma 2.4. *For all $\hat{f} \in \mathcal{T}_{\alpha,q}$, the function*

$$\hat{g}(k) = \hat{f}(k) - \left(\sum_{i=0}^{\lceil q-1\rceil} \frac{k^i}{i!}\partial_k^i\hat{f}(0)\right)\chi(|k|),$$

where χ is a smooth cutoff function with

$$\chi\left([0,1]\right)=\{1\}\,,\quad \chi\left([2,\infty)\right)=\{0\}\,,$$

satisfies $\hat{g}\in\mathcal{W}_{\alpha,q}$ and therefore

$$\mathcal{W}_{\alpha,q}=\left\{\hat{f}\in\mathcal{T}_{\alpha,q}\colon \hat{f}^{(i)}(0)=0,\quad \forall i\leq\lceil q-2\rceil\right\}.$$

Proof. Due to the fact that the behavior at large $|k|$ of functions in $\mathcal{T}_{\alpha,q}$ and in $\mathcal{W}_{\alpha,q}$ are the same, we only need to prove the behavior for small $|k|$. In view of the properties of the cutoff function, for $|k|\leq 1$ and $i\leq\lceil q-2\rceil$, we have

$$\begin{aligned}\left|\partial_k^i\hat{g}(k)\right| &= \left|\partial_k^i\hat{f}(k)-\left(\sum_{j=0}^{\lceil q-2\rceil-i}\frac{k^j}{j!}\partial_k^{j+i}\hat{f}(0)\right)\chi(|k|)\right|\\ &\leq\int_0^{|k|}\left|\partial_k^{\lceil q-1\rceil}\hat{f}(\xi)\right|\xi^{\lceil q-2\rceil-i}\mathrm{d}\xi\lesssim\left\|\hat{f};\mathcal{T}_{\alpha,q}\right\||k|^{q-i}\,.\end{aligned}$$

□

For functions of k and y, we define the following spaces with norms reflecting the scaling property of the Jeffery–Hamel solution:

Definition 2.5 (*function spaces on* Ω). For $\alpha\geq 0$ and $q\in\mathbb{R}$, let $\mathcal{B}_{\alpha,q}$ be the Banach space of functions $\hat{f}\in C(X\times[1,\infty)\,,\mathbb{C})$ where X is defined by (2.1), such that $\overline{\hat{f}(k,y)}=\hat{f}(-k,y)$ and the norm

$$\left\|\hat{f};\mathcal{B}_{\alpha,q}\right\|=\sup_{X\times[1;\infty)}\frac{|\hat{f}|}{\mu_{\alpha,q}},$$

is finite, where the weight is given by

$$\mu_{\alpha,q}(k,y)=\begin{cases}\dfrac{1}{y^q}\dfrac{1}{1+(|k|\,y)^{\alpha}}, & q\geq 0,\\[2ex] \dfrac{1}{y^q}\dfrac{1}{1+(|k|\,y)^{\alpha}}\left(1+\dfrac{1}{(|k|\,y)^{-q}}\right), & q<0.\end{cases}$$

For $\alpha>0$ and $q>0$, the space for the velocity field $\mathcal{U}_{\alpha,q}$ is the Banach space of functions in $\mathcal{B}_{\alpha+1,q-1}$ such that the following norm is finite,

$$\left\|\hat{f};\mathcal{U}_{\alpha,q}\right\|=\sum_{i=0}^{\lceil q-1\rceil}\sum_{j=0}^{\lceil\alpha-1\rceil}\left\|\partial_y^j\partial_k^i\hat{f};\mathcal{B}_{\alpha+1-j,q-1-i+j}\right\|.$$

For $\alpha>1$ and $q>1$, the spaces for the pressure $\mathcal{P}_{\alpha,q}$ and for the nonlinearity $\mathcal{R}_{\alpha,q}$ are the Banach spaces of functions in $\mathcal{B}_{\alpha+1,q-1}$ such that the respective norms are finite,

$$\left\|\hat{f};\mathcal{P}_{\alpha,q}\right\|=\sum_{i=0}^{\lceil q-2\rceil}\sum_{j=0}^{\lceil\alpha-1\rceil}\left\|\partial_y^j\partial_k^i\hat{f};\mathcal{B}_{\alpha+1-j,q-1-i+j}\right\|,$$

$$\left\|\hat{f};\mathcal{R}_{\alpha,q}\right\|=\sum_{i=0}^{\lceil q-2\rceil}\sum_{j=0}^{\lceil\alpha-2\rceil}\left\|\partial_y^j\partial_k^i\hat{f};\mathcal{B}_{\alpha+1-j,q-1-i+j}\right\|.$$

Remark 2.6. The parameter α captures the behavior of functions at infinity as a function of $|k|\,y$, which is reminiscent of the scaling properties in x/y of the Jeffery–Hamel solution. By taking the inverse Fourier transform, the parameter α corresponds to the regularity in x in direct space. The index q determines the decay in y at infinity. As we will see below, functions on the boundary which are in $\mathcal{A}_{\alpha,q}$ are in the space $\mathcal{B}_{\alpha,q}$, when evolved in time by $\mathrm{e}^{-|k|y}$. The spaces $\mathcal{U}_{\alpha,q}$, $\mathcal{P}_{\alpha,q}$ and $\mathcal{R}_{\alpha,q}$ include derivatives with respect to k to catch the behavior near $k = 0$ and derivatives with respect to y for the regularity in the y-direction.

Remark 2.7. For $\alpha' \geq \alpha$ and $q' \geq q$, we have the inclusion $\mathcal{X}_{\alpha',q'} \subset \mathcal{X}_{\alpha,q}$ for $\mathcal{X} = \mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$, $\mathcal{W}$, $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{R}$, which will be routinely used without mention.

Remark 2.8. By definition, the restriction of a function $\hat{f} \in \mathcal{B}_{\alpha,q}$ to the boundary $y = 1$ is a function in $\mathcal{A}_{\alpha,q}$. In the same way, the restriction of $\hat{f} \in \mathcal{R}_{\alpha,q}$ is in $\mathcal{T}_{\alpha,q}$.

The above spaces lead to the following regularity in direct space:

Lemma 2.9. *For $\alpha > 1$ and $q \geq 0$, if $\hat{f} \in \mathcal{B}_{\alpha,q}$, we have*

$$f \in C(\Omega), \qquad y^{1+q} f \in L^{\infty}(\Omega), \quad y^{q-\varepsilon} f \in L^2(\Omega),$$

for all $\varepsilon > 0$. For $\alpha > 0$ and $q \geq 0$, if $\hat{f} \in \mathcal{U}_{\alpha,q}$ or $\hat{f} \in \mathcal{P}_{\alpha,q}$ we have

$$f \in C^{\lceil \alpha-1 \rceil}(\Omega), \quad y^{q+i+j} \partial_x^i \partial_y^j f \in L^{\infty}(\Omega), \quad y^{q-1+i+j-\varepsilon} f \in L^2(\Omega),$$

for $i + j \leq \lceil \alpha - 1 \rceil$ and $\varepsilon > 0$.

Proof. We consider $\hat{f} \in \mathcal{B}_{\alpha,q}$. Since for $y \geq 1$ and $\alpha > 1$,

$$\left|\hat{f}(k,y)\right| \leq \frac{1}{y^q} \frac{1}{1 + (|k|\,y)^{\alpha}} \leq \frac{1}{1 + |k|^{\alpha}} \in L^1(\mathbb{R}),$$

by using the dominated convergence theorem, and the fact that $\hat{f}(k,\cdot) \in C([1,\infty))$, we obtain $f \in C(\Omega)$. Since

$$\left|\mathcal{F}^{-1}[\mu_{\alpha,q}]\right| \leq \frac{1}{y^{q+1}} \int_{\mathbb{R}} \frac{1}{1 + z^{\alpha}} \mathrm{d}z \leq \frac{\alpha}{\alpha - 1} \frac{1}{y^{q+1}},$$

$y^{q+1} f \in L^{\infty}$. Finally, by Parseval identity

$$\int_{\mathbb{R}} |f(x,y)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \left|\hat{f}(k,y)\right|^2 \mathrm{d}k \leq \frac{\left\|\hat{f};\mathcal{B}_{\alpha,q}\right\|}{y^{2q+1}} \int_{\mathbb{R}} \frac{1}{1 + z^{2\alpha}} \mathrm{d}z \lesssim \frac{1}{y^{2q+1}},$$

so $y^{q-\varepsilon} f \in L^2(\Omega)$ for all $\varepsilon > 0$.

Finally, we consider $\hat{f} \in \mathcal{U}_{\alpha,q}$ or $\hat{f} \in \mathcal{P}_{\alpha,q}$. Since $\partial_x^i \partial_y^j f = \mathcal{F}^{-1}\left[(\mathrm{i}k)^i\, \partial_y^j \hat{f}\right]$ and $\partial_y^j \hat{f} \in \mathcal{B}_{\alpha-j,q+j}$, we have $|k|^i\, \partial_y^j \hat{f} \in \mathcal{B}_{\alpha-j-i,q+j+i}$, so by applying the previous result, we obtain the claimed properties. $\square$

3. Stokes System

In this section, we consider the following inhomogeneous Stokes system,

$$\Delta \boldsymbol{u} - \boldsymbol{\nabla} p = \boldsymbol{\nabla} \cdot \mathbf{Q}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_b, \tag{3.1}$$

where $\mathbf{Q}$ is a given symmetric tensor. For simplicity, we define

$$R = Q_{12} = Q_{21}, \quad S = \frac{1}{2}\left(Q_{11} - Q_{22}\right).$$

The aim is to determine the compatibility conditions on the boundary data $\boldsymbol{u}_b$ and on the inhomogeneous term $\boldsymbol{R} = (R, S)$, such that (3.1) admits an (α, q)-solution:

Definition 3.1 (*(α, q)-solutions for Stokes*). A pair $(\hat{\boldsymbol{u}}, \hat{p}) \in \mathcal{U}_{\alpha,q} \times \mathcal{P}_{\alpha-1,q+1}$ is called an (α, q)-solution of the Stokes equation, if it satisfies the Fourier transform (with respect to x) of the Stokes equation (3.1).

Lemma 3.2 ((α, q)-solutions are classical solutions). *For $\alpha > 2$ and $q > 0$, if $(\hat{\boldsymbol{u}}, \hat{p})$ is an (α, q)-solution, then its inverse Fourier transform $(\boldsymbol{u}, p)$ has the regularity $(\boldsymbol{u}, p) \in C^2(\Omega) \times C^1(\Omega)$ and satisfies the Stokes equation (3.1) in the classical sense.*

Proof. In view of Lemma 2.9, we obtain that $(\boldsymbol{u}, p) \in C^2(\Omega) \times C^1(\Omega)$, and since $(\hat{\boldsymbol{u}}, \hat{p})$ satisfies the Fourier transform of (3.1), we obtain that $(\boldsymbol{u}, p)$ is a solution of (3.1) in the classical sense. □

Theorem 3.3 (existence of (α, q)-solutions for Stokes). *For all $\alpha > 0$ and $q \geq 1$, if $\hat{\mathbf{Q}} \in \mathcal{R}_{\alpha,q+1}$, $\hat{\boldsymbol{u}}_b \in \mathcal{T}_{\alpha,q}$ and either $\hat{\mathbf{Q}} = \mathbf{0}$ or $q \notin \mathbb{N}$, there exists an (α, q)-solution provided $2\lceil q-1 \rceil$ compatibility conditions hold. In particular, for $q = 1$ there are no compatibility conditions, for $q \in (1, 2]$ there is one compatibility condition*

$$\hat{u}_b(0) + \int_1^\infty \hat{R}(0, y)\,\mathrm{d}y = 0, \quad \hat{v}_b(0) = 0,$$

and for $q \in (2, 3]$, we have the additional conditions,

$$\partial_k \hat{u}_b(0) + \int_1^\infty \partial_k \hat{R}(0, y)\,\mathrm{d}y - 2\mathrm{i}\int_1^\infty (y-1)\,\hat{S}(0, y)\,\mathrm{d}y = 0,$$
$$\partial_k \hat{v}_b(0) + \int_1^\infty \partial_k \hat{R}(0, y)\,\mathrm{d}y + \mathrm{i}\int_1^\infty (y-1)\,\hat{R}(0, y)\,\mathrm{d}y = 0.$$

The rest of this section is devoted to the proof of the existence of an (α, q)-solution for the Stokes system.

Definition 3.4. We define the following operators

$$\left(T_< w\right)(k, y) = \frac{1}{2}\int_1^y \mathrm{e}^{-|k|(y-z)}\left(1 - \chi\left(|k|\,(z-1)\right)\right) w(k, z)\,\mathrm{d}z,$$
$$\left(T_>^{\pm} w\right)(k, y) = \frac{1}{2}\int_y^\infty \left(\mathrm{e}^{-|k|(z-y)} \pm \chi\left(|k|\,(z-1)\right)\mathrm{e}^{|k|(z-y)}\right) w(k, z)\,\mathrm{d}z,$$
$$\left(U_r w\right)(k, y) = (y-1)^r\,\mathrm{e}^{-|k|(y-1)} w(k),$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a smooth cutoff function such that

$$\chi([0,1]) = \{1\}, \quad \chi([2,\infty)) = \{0\}, \tag{3.2}$$

and their combinations

$$\begin{aligned} T^+ &= T_< - T_>^+, \quad T^- = \sigma T_< + \sigma T_>^-, \\ B^+ &= T^+\big|_{y=1}, \quad B^- = T^-\big|_{y=1}, \end{aligned}$$

where

$$\sigma = \mathrm{i} \cdot \mathrm{sign}(k).$$

Remark 3.5. We will see later on that the decay in y of the integral over $1 \le z \le y$ is crucially linked to the behavior of the integrand near $k = 0$. To study the decay of the integral over $1 \le z \le y$ in terms of the decay of the integrand, we use the cutoff function χ in the definition of $T_<$, so that the integrand vanishes at small values of k. To compensate this artificial cutoff, we add the appropriate cutoff in the definition of $T_>^\pm$. The reason for the definition of $T^\pm$ is that these expressions appear often and are regular at $k = 0$.

Proposition 3.6. *Formally, the Fourier transform of the Stokes system is given by*

$$\hat{\boldsymbol{u}} = \mathbf{N}\hat{\boldsymbol{R}} + \mathbf{B}\hat{\boldsymbol{u}}_r, \quad \hat{\boldsymbol{u}}_r = \hat{\boldsymbol{u}}_b - \mathbf{N}\big|_{y=1}\hat{\boldsymbol{R}}, \tag{3.3}$$

$$\hat{p} = -\mathrm{i}kT^+\hat{R} + \mathrm{i}kT^-\hat{S} - \hat{Q}_{12} - U_0\left[2\mathrm{i}k\left(\hat{u}_r + \sigma\hat{v}_r\right)\right], \tag{3.4}$$

where

$$\mathbf{N} = \begin{pmatrix} T^+ - T^-\mathrm{i}k\,(z-y) & -T^- - T^+\mathrm{i}k\,(z-y) \\ T^+\mathrm{i}k\,(z-y) & -T^-\mathrm{i}k\,(z-y) \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} U_0 - |k|\,U_1 & -\mathrm{i}kU_1 \\ -\mathrm{i}kU_1 & U_0 + |k|\,U_1 \end{pmatrix}.$$

Proof. The vorticity is

$$\omega = \partial_x v - \partial_y u, \tag{3.5}$$

and the Stokes equation (3.1) implies the vorticity equation

$$\Delta\omega = \left(\partial_x^2 - \partial_y^2\right)R - 2\partial_x\partial_y S. \tag{3.6}$$

By defining $\gamma = \omega + R$, the divergence-free condition, (3.5), and (3.6) can be rewritten as a first-order differential system in y,

$$\begin{aligned} \partial_y u &= \partial_x v - \gamma + R, \quad \partial_y\gamma = \partial_x\eta - 2\partial_x S, \\ \partial_y v &= -\partial_x u, \quad \partial_y\eta = -\partial_x\gamma + 2\partial_x R. \end{aligned}$$

By taking formally the Fourier transform in the variable x, the divergence-free condition, (3.5) and (3.6) can be rewritten as a dynamical system where y plays the role of time,

$$\partial_y\hat{\boldsymbol{r}} = \mathbf{L}\hat{\boldsymbol{r}} + \hat{\boldsymbol{q}}, \quad \hat{\boldsymbol{u}}(k,1) = \hat{\boldsymbol{u}}_b,$$

where

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\gamma} \\ \hat{\eta} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & \mathrm{i}k & -1 & 0 \\ -\mathrm{i}k & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}k \\ 0 & 0 & -\mathrm{i}k & 0 \end{pmatrix}, \quad \hat{\boldsymbol{q}} = \begin{pmatrix} \hat{R} \\ 0 \\ -2\mathrm{i}k\hat{S} \\ 2\mathrm{i}k\hat{R} \end{pmatrix}.$$

The eigenvalues of L are $\pm|k|$, and since we are interested in solutions with zero velocity at infinity, we have to distinguish between stable and unstable modes, so the solution is given by

$$\begin{aligned} \hat{\boldsymbol{r}}(k,y) &= \int_1^y \mathbf{P}\mathrm{e}^{\mathbf{L}(y-z)}\hat{\boldsymbol{q}}(k,z)\,\mathrm{d}z - \int_y^\infty (\mathbf{1}-\mathbf{P})\,\mathrm{e}^{\mathbf{L}(y-z)}\hat{\boldsymbol{q}}(k,z)\,\mathrm{d}z \\ &\quad + \mathrm{e}^{\mathbf{L}(y-1)}\hat{\boldsymbol{r}}_s(k) \\ \hat{\boldsymbol{r}}_s(k) &= \int_1^\infty (\mathbf{1}-\mathbf{P})\,\mathrm{e}^{\mathbf{L}(1-z)}\hat{\boldsymbol{q}}(k,z)\,\mathrm{d}z + \hat{\boldsymbol{r}}_b(k), \end{aligned}$$

where $\mathbf{P}$ is the projection onto stable modes and $\hat{\boldsymbol{r}}_s$ is such that the boundary condition in (3.1) is satisfied,

$$\mathbf{P} = \frac{1}{2|k|}\begin{pmatrix} |k| & -\mathrm{i}k & 1/2 & 0 \\ \mathrm{i}k & |k| & 0 & -1/2 \\ 0 & 0 & |k| & -\mathrm{i}k \\ 0 & 0 & \mathrm{i}k & |k| \end{pmatrix}, \quad \hat{\boldsymbol{r}}_b = \begin{pmatrix} \hat{u}_b \\ \hat{v}_b \\ 2|k|(\hat{u}_b + \sigma\hat{v}_b) \\ 2\mathrm{i}k(\hat{u}_b + \sigma\hat{v}_b) \end{pmatrix}.$$

Using the Jordan decomposition for $\mathbf{L}$, we can explicitly calculate the exponential and find that

$$\begin{aligned} \hat{u} &= \frac{1}{2}\int_1^y \mathrm{e}^{-|k|(y-z)}\,(1-|k|\,(y-z))\,\hat{R}_-(k,z)\,\mathrm{d}z \\ &\quad - \frac{1}{2}\int_y^\infty \mathrm{e}^{-|k|(z-y)}\,(1+|k|\,(y-z))\,\hat{R}_+(k,z)\,\mathrm{d}z \\ &\quad + \mathrm{e}^{-|k|(y-1)}\left[(1-|k|\,(y-1))\,\hat{u}_s - \mathrm{i}k\,(y-1)\,\hat{v}_s\right], \\ \hat{v} &= -\frac{1}{2}\int_1^y \mathrm{e}^{-|k|(y-z)}\mathrm{i}k\,(y-z)\,\hat{R}_-(k,z)\,\mathrm{d}z \\ &\quad + \frac{1}{2}\int_y^\infty \mathrm{e}^{-|k|(z-y)}\mathrm{i}k\,(y-z)\,\hat{R}_+(k,z)\,\mathrm{d}z \\ &\quad + \mathrm{e}^{-|k|(y-1)}\left[(1+|k|\,(y-1))\,\hat{v}_s - \mathrm{i}k\,(y-1)\,\hat{u}_s\right], \end{aligned}$$

where $\hat{R}_\pm = \hat{R} \pm \sigma\hat{S}$ and

$$\begin{aligned} \hat{u}_s &= \hat{u}_b + \frac{1}{2}\int_1^\infty \mathrm{e}^{-|k|(z-1)}\,(1-|k|\,(z-1))\,\hat{R}_+(k,z)\,\mathrm{d}z, \\ \hat{v}_s &= \hat{v}_b + \frac{1}{2}\int_1^\infty \mathrm{e}^{-|k|(z-1)}\mathrm{i}k\,(z-1)\,\hat{R}_+(k,z)\,\mathrm{d}z. \end{aligned}$$

Using the operators defined in Definition 3.4, we can rewrite the integral equations as

$$\begin{aligned}\hat{u} &= T^{+}\left[\hat{R}-\mathrm{i}k\,(z-y)\,\hat{S}\right]-T^{-}\left[\hat{S}+\mathrm{i}k\,(z-y)\,\hat{R}\right]\\ &\quad+U_0\left[(1-|k|\,(y-1))\,\hat{u}_r-\mathrm{i}k\,(y-1)\,\hat{v}_r\right],\\ \hat{v} &= T^{+}\left[\mathrm{i}k\,(z-y)\,\hat{R}\right]-T^{-}\left[\mathrm{i}k\,(z-y)\,\hat{S}\right]\\ &\quad+U_0\left[(1+|k|\,(y-1))\,\hat{v}_r-\mathrm{i}k\,(y-1)\,\hat{u}_r\right],\end{aligned}$$

where

$$\begin{aligned}\hat{u}_r &= \hat{u}_b-B^{+}\left[\hat{R}-\mathrm{i}k\,(z-1)\,\hat{S}\right]+B^{-}\left[\hat{S}+\mathrm{i}k\,(z-1)\,\hat{R}\right],\\ \hat{v}_r &= \hat{v}_b-B^{+}\left[\mathrm{i}k\,(z-1)\,\hat{R}\right]+B^{-}\left[\mathrm{i}k\,(z-1)\,\hat{S}\right],\end{aligned}$$

which shows (3.3). Finally, from the Fourier transform of the Stokes equation (3.1), we can check that the pressure is effectively given by (3.4). □

To prove the existence of an (α, q)-solution, we have to estimate the operators used in proposition 3.6:

Lemma 3.7. *For $\alpha > 1$, $r \in \mathbb{N}$, and $q \in \mathbb{R}$, the operator $U_r\colon \mathcal{A}_{\alpha,q} \to \mathcal{B}_{\alpha+r,q-r}$ is well defined and continuous.*

Proof. It suffices to prove that

$$\frac{1}{1+|k|^{\alpha}}\left(\frac{y-1}{y}\right)^{r}\mathrm{e}^{-|k|(y-1)} \lesssim \frac{1}{1+(|k|\,y)^{\alpha+r}}.$$

For $|k|\,y \le 1$, the result is trivial and for $|k|\,y > 1$, we distinguish two cases: for $y > 2$, we have

$$\frac{1}{1+|k|^{\alpha}}\left(\frac{y-1}{y}\right)^{r}\mathrm{e}^{-|k|(y-1)} \le \mathrm{e}^{-\frac{1}{2}|k|y} \lesssim \frac{1}{1+(|k|\,y)^{\alpha+r}},$$

and for $1 \le y \le 2$,

$$\begin{aligned}\frac{1}{1+|k|^{\alpha}}\left(\frac{y-1}{y}\right)^{r}\mathrm{e}^{-|k|(y-1)} &\lesssim \frac{1}{1+(|k|\,y)^{\alpha}}\,\frac{1}{(|k|\,y)^{r}}\,(|k|\,(y-1))^{r}\,\mathrm{e}^{-|k|(y-1)}\\ &\lesssim \frac{1}{1+(|k|\,y)^{\alpha+r}}.\end{aligned}$$

□

Lemma 3.8. *For $\alpha > 1$, $r \in \mathbb{N}$ and $q \ge 0$, the operator $U_r\colon \mathcal{W}_{\alpha,q} \to \mathcal{U}_{\alpha+r,q-r}$ is well defined and continuous.*

Proof. First of all, we have

$$\partial_y U_r = U_{r-1} + |k|\,U_r,$$

so $\partial_y U_r : \mathcal{A}_{\alpha+1,q-1} \to \mathcal{B}_{\alpha+r,q-r}$. For $q > 0$, we have

$$\partial_k\left(U_r w\right) = U_r \partial_k w + \mathrm{i}\sigma U_{r+1} w,$$

so by Lemma 3.7 we obtain that $\partial_k\left(U_r w\right) \in \mathcal{B}_{\alpha+r+1,q-r-2}$. The result now follows by a recursion on the number of derivatives. □

Lemma 3.9. *For $\alpha \geq 0$ and $q \geq 0$, the operator $T_{<}\colon \mathcal{B}_{\alpha,q} \to \mathcal{B}_{\alpha+1,q-1}$ is well defined and continuous.*

Proof. Due to the cutoff function, the integral vanishes for $|k|\,(y-1) \leq 1$, and for $|k|\,y \geq 1+|k|$ we split the integral as follows,

$$\begin{aligned}
\left(T_{<}\mu_{\alpha,q}\right)(k,y) &\leq \int_1^{\frac{y+1}{2}} \mathrm{e}^{-|k|(y-z)}\left(1-\chi\left(|k|\,(z-1)\right)\right)\mu_{\alpha,q}(k,z)\,\mathrm{d}z \\
&\quad + \int_{\frac{y+1}{2}}^{y} \mathrm{e}^{-|k|(y-z)}\mu_{\alpha,q}(k,z)\,\mathrm{d}z \\
&\lesssim \mathrm{e}^{-|k|(y-1)/2}\int_1^{\frac{y+1}{2}} \eta_{\alpha,q}(k)\,\mathrm{d}z + \mu_{\alpha,q}(k,y)\int_{\frac{y+1}{2}}^{y} \mathrm{e}^{-|k|(y-z)}\,\mathrm{d}z \\
&\lesssim \mathrm{e}^{-|k|(y-1)/2}\,(y-1)\,\eta_{\alpha,q}(k) + \frac{1}{|k|\,y}\mu_{\alpha,q-1}\,(k,y) \\
&\lesssim \mu_{\alpha+1,q-1}(k,y),
\end{aligned}$$

where for the last step we apply Lemma 3.7. □

Lemma 3.10. *For all $\alpha > 1$ and $q \geq 0$ with $q \neq 1$, the operators $T_{>}^{\pm}\colon \mathcal{B}_{\alpha,q} \to \mathcal{B}_{\alpha+1,q-1}$ are well defined and continuous.*

Proof. For $|k|\,y > 1$, we have

$$\begin{aligned}
\left(T_{>}^{\pm}\mu_{\alpha,q}\right)(k,y) &\lesssim \int_y^\infty \mathrm{e}^{-|k|(z-y)}\mu_{\alpha,q}(k,z)\,\mathrm{d}z \leq \mu_{\alpha,q}\,(k,y)\int_y^\infty \mathrm{e}^{|k|(y-z)}\,\mathrm{d}z \\
&\lesssim \frac{1}{|k|\,y}\mu_{\alpha,q-1}(k,y) \lesssim \mu_{\alpha+1,q-1}(k,y),
\end{aligned}$$

and for $|k|\,y < 1$, we have

$$\begin{aligned}
\left(T_{>}^{\pm}\mu_{\alpha,q}\right)(k,y) &\lesssim \int_y^\infty \mu_{\alpha,q}(k,z)\,\mathrm{d}z \leq |k|^{q-1}\int_{|k|y}^\infty \frac{1}{u^q}\frac{1}{1+u^\alpha}\mathrm{d}u \\
&\lesssim \mu_{\alpha+1,q-1}(k,y).
\end{aligned}$$

□

Lemma 3.11. *For all $\alpha > 0$ and $q \geq 0$ with $q \notin \mathbb{N}$, the operators $T_{\pm}\colon \mathcal{R}_{\alpha,q+1} \to \mathcal{U}_{\alpha+1,q}$ are well defined and continuous.*

Proof. First, by using Lemma 3.10, we have $T_{\pm}\colon \mathcal{B}_{\alpha+1,q} \to \mathcal{B}_{\alpha+2,q-1}$. By taking the derivative with respect to y, we get

$$\begin{aligned}
\partial_y\left(T_{<}q\right)(k,y) &= \frac{1}{2}\left(1-\chi\left(|k|\,(y-1)\right)\right)q(k,y) - |k|\left(T_{<}q\right)(k,y), \\
\partial_y\left(T_{>}^{\pm}q\right)(k,y) &= \frac{-1}{2}\left(1\pm\chi\left(|k|\,(y-1)\right)\right)q(k,y) + |k|\left(T_{>}^{\mp}q\right)(k,y),
\end{aligned}$$

so the time derivative of the operators are

$$\partial_y T^{+} = \frac{1}{2} + \mathrm{i}kT^{-}, \quad \partial_y T^{-} = -\mathrm{i}kT^{+},$$

so $\partial_y T_\pm \colon \mathcal{B}_{\alpha+1,q} \to \mathcal{B}_{\alpha+1,q}$. Since the integrand of $T_<^-$ vanishes at $k=0$, we have

$$\partial_k (T_< w) = T_< \partial_k w + \mathrm{i}\,(y-1)\,\sigma T_< w - \mathrm{i}\sigma \tilde{T}_< (z-1)\,w,$$
$$\partial_k \left(T_>^+ w\right) = T_>^+ \partial_k w - \mathrm{i}\,(y-1)\,\sigma T_>^- w + \mathrm{i}\sigma \tilde{T}_>^- (z-1)\,w,$$
$$\partial_k \left(\sigma T_>^- w\right) = \sigma T_>^- \partial_k w + \mathrm{i}\,(y-1)\,T_>^+ w - \mathrm{i}\tilde{T}_>^+ (z-1)\,w,$$

where a tilde over an operator denotes the same operator where χ is replaced by $\chi + \chi'$, which is also a cutoff function satisfying (3.2). Therefore,

$$\partial_k (T_+ w) = T_+ \partial_k w + \mathrm{i}\,(y-1)\,T_- w - \mathrm{i}\sigma \tilde{T}_- (z-1)\,w,$$
$$\partial_k (T_- w) = T_- \partial_k w - \mathrm{i}\,(y-1)\,T_+ w + \mathrm{i}\tilde{T}_+ (z-1)\,w,$$

and by using the previously shown properties on the operators $T_\pm$, we obtain that $\partial_k (T_\pm w) \in \mathcal{B}_{\alpha+2,q-2}$. By recursion on the number of derivatives, we obtain $T_\pm w \in \mathcal{U}_{\alpha+1,q}$. □

We can now apply these lemmas to prove the existence of an (α, q)-solution:

Proof of Theorem 3.3. By applying Lemma 3.8, we have $\mathbf{B} \colon \mathcal{W}_{\alpha,q} \to \mathcal{U}_{\alpha,q}$. By applying Lemma 3.11, noting that

$$z - y = (z-1) - (y-1)\,,$$

and bounding each resulting term separately, we obtain that $\mathbf{N} \colon \mathcal{R}_{\alpha,q+1} \to \mathcal{U}_{\alpha,q}$ and $\mathbf{N}|_{y=1} : \mathcal{R}_{\alpha,q+1} \to \mathcal{T}_{\alpha,q}$ for $q > 1$ with $q \notin \mathbb{N}$. Therefore, if either $\hat{\mathbf{Q}} = \mathbf{0}$ or $q \notin \mathbb{N}$, the compatibility condition $\hat{\boldsymbol{u}}_r \in \mathcal{W}_{\alpha,q}$ ensures the existence of a solution $(\hat{\boldsymbol{u}}, \hat{p}) \in \mathcal{U}_{\alpha,q} \times \mathcal{P}_{\alpha-1,q+1}$. To deduce the explicit form of the compatibility conditions $\hat{\boldsymbol{u}}_r \in \mathcal{W}_{\alpha,q}$, we use the characterization of $\mathcal{W}_{\alpha,q}$ in terms of elements of $\mathcal{T}_{\alpha,q}$ provided in Lemma 2.4. We obtain that $\hat{\boldsymbol{u}}_r \in \mathcal{W}_{\alpha,q}$ corresponds to $2\lceil q-1 \rceil$ real compatibility conditions. The first compatibility condition is $\hat{\boldsymbol{u}}_r(0) = \mathbf{0}$ and the second $\partial_k \hat{\boldsymbol{u}}_r(0) = \mathbf{0}$. By explicit calculations, we obtain the claimed conditions. □

4. Strong Solutions to the Navier–Stokes Equation

The Navier–Stokes equation in the half-space can be written as the Stokes system (3.1) with $\mathbf{Q} = \boldsymbol{u} \otimes \boldsymbol{u}$, and we are going to look for solutions of the form $\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_\Phi^\sigma + \hat{\boldsymbol{u}}_1$ and perform a fixed point argument on $\hat{\boldsymbol{u}}_1 \in \mathcal{U}_{\alpha,q}$. First of all, the Jeffery–Hamel solution (1.2) at fixed values of y and large values of $\pm s$, where $s = x/y$, satisfies

$$u_\Phi^\sigma(x,y) \approx \frac{-1}{y\,s^2} f'\left(\pm\tfrac{\pi}{2}\right), \quad v_\Phi^\sigma(x,y) \approx \frac{-1}{y\,s^3} f'\left(\pm\tfrac{\pi}{2}\right),$$

so that its Fourier transforms satisfies $\partial_y^i \hat{\boldsymbol{u}}_\Phi^\sigma \in \mathcal{B}_{\alpha,i}$ and

$$\hat{\boldsymbol{u}}_\Phi^\sigma \in \mathcal{U}_{\alpha,0}, \tag{4.1}$$

for arbitrary $\alpha > 1$. In order to treat the nonlinearity $\boldsymbol{u}_\Phi^\sigma \otimes \boldsymbol{u}_1$, we need the following proposition concerning the convolution:

Proposition 4.1. *For $\alpha > 1$ and $q \geq 1$, the convolution $*\colon \mathcal{U}_{\alpha,1} \times \mathcal{U}_{\alpha,q} \to \mathcal{R}_{\alpha,q+1}$ is a continuous bilinear map.*

Proof. First, we show that the map $*\colon \mathcal{B}_{\alpha,p} \times \mathcal{B}_{\alpha,q-1} \to \mathcal{B}_{\alpha,p+q}$ is a continuous bilinear map, for $p, q \geq 0$. If $\hat{f} \in \mathcal{B}_{\alpha,p}$ and $\hat{g} \in \mathcal{B}_{\alpha,q-1}$, $\hat{f}$ is in $L^\infty(\mathbb{R})$ and $\hat{g}$ is in $L^1(\mathbb{R})$ for fixed $y \in [1;\infty)$, so that $\hat{f} * \hat{g} \in C(\mathbb{R})$ (see for example [11, Proposition 8.8]). The dependence of the convolution on the power of y is trivial and it therefore suffices to prove that

$$\mu_{\alpha,0} * \mu_{\alpha,0}^{\nu} \lesssim \mu_{\alpha,1}$$

for $\nu \in (0,1)$, where

$$\mu_{\alpha,0}^{\nu}(k,y) = \frac{1}{(|k|\,y)^{\nu}} \frac{1}{1 + (|k|\,y)^{\alpha-\nu}}.$$

For $|k|\,y \leq 1$, we have

$$\begin{aligned}\int_{\mathbb{R}} \mu_{\alpha,0}\left(\ell,y\right) \mu_{\alpha,0}^{\nu}\left(k-\ell,y\right) \mathrm{d}\ell &\leq \int_{\mathbb{R}} \mu_{\alpha,0}^{\nu}\left(k-\ell,y\right) \mathrm{d}\ell \\ &\leq \frac{1}{y} \int_{\mathbb{R}} \mu_{\alpha,0}^{\nu}\left(\ell,1\right) \mathrm{d}\ell \\ &\lesssim \mu_{\alpha,1}\left(k,y\right),\end{aligned}$$

and, for $|k|\,y > 1$, we have that $\mu_{\alpha,0}^{\nu} \leq \mu_{\alpha,0}$. Therefore, by splitting the integral at $k/2$, we find that

$$\begin{aligned}\int_{\mathbb{R}} \mu_{\alpha,0}\left(\ell,y\right) \mu_{\alpha,0}^{\nu}\left(k-\ell,y\right) \mathrm{d}\ell &\leq \mu_{\alpha,0}\left(k/2,y\right) \int_{\mathbb{R}} \mu_{\alpha,0}\left(\ell,y\right) \mathrm{d}\ell \\ &\leq \mu_{\alpha,0}\left(k/2,y\right) \frac{1}{y} \int_{\mathbb{R}} \mu_{\alpha,0}\left(\ell,1\right) \mathrm{d}\ell \\ &\lesssim \mu_{\alpha,1}\left(k,y\right).\end{aligned}$$

Now, we consider $\hat{f} \in \mathcal{U}_{\alpha,1}$ and $\hat{g} \in \mathcal{U}_{\alpha,q}$. If $\alpha > 0$, we have (see for example [11, Exercise 8.8]) $\partial_k\big(\hat{f} * \hat{g}\big) = \hat{f} * \partial_k \hat{g}$, so by using the previous result, $\hat{f} * \partial_k \hat{g} \in \mathcal{B}_{\alpha+1,q-1}$. By taking the derivative with respect to y, we have $\partial_y\big(\hat{f} * \hat{g}\big) = \partial_y \hat{f} * \hat{g} + \hat{f} * \partial_y \hat{g} \in \mathcal{B}_{\alpha,q+1}$. Finally, by a recursion on the number of derivatives, we obtain that $\hat{f} * \hat{g} \in \mathcal{R}_{\alpha,q+1}$. □

Now, we can state the main theorem:

Theorem 4.2 (existence of (α,q)-solutions for Navier–Stokes). *For $\alpha > 2$ and $q \in (1,2)$, there exists $\nu > 0$ such that for any $\Phi \in \mathbb{R}$ and $\hat{\boldsymbol{u}}_s \in \mathcal{T}_{\alpha,q}$ satisfying*

$$|\Phi| \leq \nu, \quad \|\hat{\boldsymbol{u}}_s; \mathcal{T}_{\alpha,0}\| \leq \nu, \quad \int_{\mathbb{R}} \boldsymbol{u}_s(x)\,\mathrm{d}x = \boldsymbol{0},$$

there exists $A \in \mathbb{R}$ depending continuously on Φ and $\hat{\boldsymbol{u}}_s$ such that there exists $(\boldsymbol{u}, p) \in C^2(\Omega) \times C^1(\Omega)$ satisfying (1.1) with

$$\boldsymbol{u}^*(x) = \boldsymbol{u}_{\Phi}^{\sigma}(x,1) + \frac{(A,0)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} + \boldsymbol{u}_s(x).$$

Moreover, $\hat{\boldsymbol{u}} - \hat{\boldsymbol{u}}_\Phi^\sigma \in \mathcal{U}_{\alpha,q}$ *and* $\hat{p} - \hat{p}_\Phi^\sigma \in \mathcal{P}_{\alpha-1,q+1}$*, so that*

$$\lim_{y\to\infty} y\left(\sup_{x\in\mathbb{R}} |\boldsymbol{u} - \boldsymbol{u}_\Phi^\sigma|\right) = 0, \qquad \lim_{y\to\infty} y^2\left(\sup_{x\in\mathbb{R}} |p - p_\Phi^\sigma|\right) = 0,$$

and

$$\begin{aligned} &\boldsymbol{\nabla u} \in L^2(\Omega), \quad y\boldsymbol{u} \in L^\infty(\Omega), \\ &\boldsymbol{u}/y \in L^2(\Omega), \quad y^2\boldsymbol{\nabla u} \in L^\infty(\Omega). \end{aligned} \tag{4.2}$$

Proof. We look for solutions of the form $\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_\Phi^\sigma + \hat{\boldsymbol{u}}_1$ and perform a fixed point argument on $\hat{\boldsymbol{u}}_1$ in the space $\hat{\boldsymbol{u}}_1 \in \mathcal{U}_{\alpha,q}$. In view of the previous section, the Navier–Stokes equation can be written as the Stokes equation (3.1) for $\boldsymbol{u}_1$, where $\mathbf{Q} = \boldsymbol{u}_\Phi^\sigma \otimes \boldsymbol{u}_1 + \boldsymbol{u}_1 \otimes \boldsymbol{u}_\Phi^\sigma + \boldsymbol{u}_1 \otimes \boldsymbol{u}_1$. The boundary condition is

$$\hat{\boldsymbol{u}}^*(k) = \hat{\boldsymbol{u}}_\Phi^\sigma(k,1) + (A,0)\,\mathrm{e}^{-\frac{1}{2}k^2} + \hat{\boldsymbol{u}}_s(k),$$

and the compatibility conditions of Theorem 3.3 are given by

$$\hat{u}_r(0) = A + \int_1^\infty \hat{R}(0,y)\,\mathrm{d}y, \qquad \hat{v}_r(0) = 0,$$

since by hypothesis $\hat{\boldsymbol{u}}_s(k) = \boldsymbol{0}$. Therefore, by defining $A = -\int_1^\infty \hat{R}(0,y)\,\mathrm{d}y$, the two compatibility conditions are fulfilled. In what follows, $C > 0$ represents a generic constant depending on q, but not on ε. By Proposition 4.1, we have $\hat{\mathbf{Q}} \in \mathcal{R}_{\alpha,q+1}$ and

$$\left\|\hat{\boldsymbol{R}}; \mathcal{R}_{\alpha,q+1}\right\| \le C\left(\|\hat{\boldsymbol{u}}_\Phi^\sigma; \mathcal{U}_{\alpha,0}\| + \|\hat{\boldsymbol{u}}_1; \mathcal{U}_{\alpha,q}\|\right)\|\hat{\boldsymbol{u}}_1; \mathcal{U}_{\alpha,q}\|.$$

Since

$$|A| \le \left|\int_1^\infty \hat{R}(0,y)\,\mathrm{d}y\right| \le \frac{1}{q}\left\|\hat{\boldsymbol{R}}; \mathcal{R}_{\alpha,q+1}\right\|,$$

we have

$$\|\hat{\boldsymbol{u}}^* - \hat{\boldsymbol{u}}_\Phi^\sigma; \mathcal{T}_{\alpha,q}\| \le \|\hat{\boldsymbol{u}}_s; \mathcal{T}_{\alpha,q}\| + C\left\|\hat{\boldsymbol{R}}; \mathcal{R}_{\alpha,q+1}\right\|.$$

By applying Theorem 3.3, we obtain that

$$\|\hat{\boldsymbol{u}}_1; \mathcal{U}_{\alpha,q}\| \le C\,\|\hat{\boldsymbol{u}}_s; \mathcal{T}_{\alpha,q}\| + C\left(\|\hat{\boldsymbol{u}}_\Phi^\sigma; \mathcal{U}_{\alpha,0}\| + \|\hat{\boldsymbol{u}}_1; \mathcal{U}_{\alpha,q}\|\right)\|\hat{\boldsymbol{u}}_1; \mathcal{U}_{\alpha,q}\|.$$

Therefore, for $\nu > 0$ small enough, a fixed point argument shows the existence of a solution $(\hat{\boldsymbol{u}}, \hat{p}) \in \mathcal{U}_{\alpha,q} \times \mathcal{P}_{\alpha-1,q+1}$ of the Fourier transform of the Navier–Stokes equation. Moreover, by the standard argument on the continuity of the fixed point, $\hat{\boldsymbol{u}}$ depends continuously on $\hat{\boldsymbol{u}}_s$ and $\hat{\boldsymbol{u}}_\Phi^\sigma$ so does A. In the same way as in Theorem 3.3, we obtain the claimed regularity and the asymptotic properties by applying Lemma 2.9. □

5. Existence of Weak Solutions

In this section, we define weak solutions for our problem and discuss in particular the technicalities related to the inhomogeneous boundary conditions on an unbounded boundary. To show that our definition of weak solutions is general enough, we then construct such solutions by Leray's method. To study an inhomogeneous boundary problem, it is standard (see for example [26, Chapter 5]) to define weak solutions by using an extension map to write the energy inequality.

We denote by $D^{1,2}_\sigma(\Omega)$ the subspace of the homogeneous Sobolev space of order $(1,2)$ of divergence-free functions on Ω, and by $D^{1,2}_{0,\sigma}(\Omega)$ the completion with respect to the norm of $D^{1,2}_\sigma(\Omega)$ of the vector space of smooth divergence-free functions with compact support in Ω. We refer to [15, Chapter II.6.] for the properties of these spaces. The main tool for studying the existence and uniqueness of weak solutions is the Hardy inequality:

Proposition 5.1 ([27, Section 2.7.1]). *For all $\boldsymbol{u} \in D^{1,2}_{0,\sigma}(\Omega)$, we have*

$$\|\boldsymbol{u}/y\|_2 \leq 2\,\|\boldsymbol{\nabla u}\|_2\,.$$

We define an extension map as follows:

Definition 5.2 (*extension*). Given a boundary condition $\boldsymbol{u}^*$, an extension is a map $\boldsymbol{a} \in D^{1,2}_\sigma(\Omega)$ such that $\boldsymbol{a}/y \in L^2(\Omega)$, $y\boldsymbol{a} \in L^\infty(\Omega)$ and $y^2\boldsymbol{\nabla a} \in L^\infty$, and such that the trace of $\boldsymbol{a}$ on $\partial\Omega$ is $\boldsymbol{u}^*$.

Remark 5.3. If $\hat{\boldsymbol{u}}^* \in \mathcal{A}_{\alpha,0}$ and in view of Theorem 3.3 with $q = 0$ and $\hat{\mathbf{Q}} = 0$, the solution of the Stokes system is an extension using Lemma 2.9.

Definition 5.4 (*weak solution*). A weak solution in the domain Ω with boundary condition $\boldsymbol{u}^*$ is a vector field $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$, where $\boldsymbol{a}$ is an extension of $\boldsymbol{u}^*$ and $\boldsymbol{v} \in D^{1,2}_{0,\sigma}(\Omega)$ which satisfies:

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla \varphi} + \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{\nabla u}) \cdot \boldsymbol{\varphi} = 0, \tag{5.1}$$

for arbitrary smooth divergence-free vector fields $\boldsymbol{\varphi}$ with compact support in Ω.

The main result of this section is the existence of weak solutions:

Theorem 5.5 (existence of weak solutions). *For a small enough boundary condition $\boldsymbol{u}^*$ (more precisely admitting an extension $\boldsymbol{a}$ such that $\|y\boldsymbol{a}\|_\infty + \|\boldsymbol{a}/y\|_2 + \|\boldsymbol{\nabla a}\|_2$ is small enough), there exists a weak solution $\boldsymbol{u}$ in Ω.*

Before proving this theorem, we mention the fact that any weak solution vanishes at infinity in the following sense:

Proposition 5.6. *If $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$ is a weak solution with $\boldsymbol{v} \in D^{1,2}_{0,\sigma}(\Omega)$ and $\boldsymbol{a}/y \in L^2(\Omega)$, then*

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} = \boldsymbol{0},$$

in the following sense

$$\lim_{r\to\infty}\int_{-\pi/2}^{\pi/2} |\boldsymbol{u}(r\sin\theta, 1+r\cos\theta)|^2\,\mathrm{d}\theta = 0.$$

Proof. First of all, by Hardy inequality, we have $\boldsymbol{v}/y \in L^2(\Omega)$, so that $\boldsymbol{u}/y \in L^2(\Omega)$. We define the half-ball Ω_n, the half-shell S_n and the half-circle Γ_n by

$$\begin{aligned}\Omega_n &= B((0,1),n)\cap\Omega, \quad S_n = \Omega_{2n}\backslash\Omega_n,\\ \Gamma_n &= \left\{(n\sin\theta, 1+n\cos\theta), \quad \frac{-\pi}{2}\le\theta\le\frac{\pi}{2}\right\},\end{aligned}$$

with $B((0,1),n)$ the open ball of radius n centered at $(0,1)$. By using the trace theorem in S_1, there exists $C>0$ such that

$$\left\|\boldsymbol{u}; L^2(\Gamma_1)\right\|^2 \le \left\|\boldsymbol{u}; L^2(\partial S_1)\right\|^2 \le C\left\|\boldsymbol{u}; L^2(S_1)\right\|^2 + C\left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(S_1)\right\|^2.$$

By a rescaling argument, we obtain that

$$\frac{1}{n}\left\|\boldsymbol{u}; L^2(\Gamma_n)\right\|^2 \le \frac{C}{n^2}\left\|\boldsymbol{u}; L^2(S_n)\right\|^2 + C\left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(S_n)\right\|^2,$$

and since $y\le n$ in Ω_n, we have

$$\frac{1}{n}\left\|\boldsymbol{u}; L^2(\Gamma_n)\right\|^2 \le C\left\|\boldsymbol{u}/\boldsymbol{y}; L^2(S_n)\right\|^2 + C\left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(S_n)\right\|^2.$$

In the limit $n\to\infty$, the right-hand side converges to zero, because $\boldsymbol{u}/y, \boldsymbol{\nabla}\boldsymbol{u} \in L^2(\Omega)$ and since the integrals over S_n can be written as the difference of integrals over Ω_{2n} and Ω_n. Finally,

$$\int_{-\pi/2}^{\pi/2} |\boldsymbol{u}(n\sin\theta, 1+n\cos\theta)|^2\,\mathrm{d}\theta = \frac{1}{2\pi n}\left\|\boldsymbol{u}; L^2(\Gamma_n)\right\|,$$

and the result is proved. □

As usual, to show the existence of a weak solution in an unbounded domain, we first prove, for arbitrary $n\in\mathbb{N}$, the existence of a weak solution in the domains Ω_n defined in the previous proof. To this end, we introduce the concept of approximate weak solution in Ω_n and then apply the Leray–Schauder theorem to prove the existence of such approximate solutions.

Definition 5.7 (*approximate weak solution*). For $n\in\mathbb{N}$, an approximate weak solution is a vector field $\boldsymbol{u}_n = \boldsymbol{a}+\boldsymbol{v}_n$, where $\boldsymbol{v}_n \in D^{1,2}_{0,\sigma}(\Omega)$ with support in Ω_n, which satisfies

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u}_n : \boldsymbol{\nabla}\boldsymbol{\varphi} + \int_\Omega (\boldsymbol{u}_n\cdot\boldsymbol{\nabla}\boldsymbol{u}_n)\cdot\boldsymbol{\varphi} = 0,$$

for arbitrary smooth divergence-free vector fields $\boldsymbol{\varphi}$ with support in Ω_n.

Lemma 5.8 (existence of approximate weak solution). *Provided $\boldsymbol{u}^*$ is small enough, there exists for all $n\in\mathbb{N}$ an approximate weak solution $\boldsymbol{u}_n = \boldsymbol{a}+\boldsymbol{v}_n$, with $\|\boldsymbol{\nabla}\boldsymbol{v}_n\|\le 1$.*

Proof. First, we note that the trilinear term can be bounded as

$$\left|\int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla}\boldsymbol{u}_n) \cdot \boldsymbol{\varphi}\right| \le \|\boldsymbol{\nabla}\boldsymbol{u}_n\|_2 \, \|\boldsymbol{u}_n\|_4 \, \|\boldsymbol{\varphi}\|_4 \,.$$

Therefore, the map

$$\begin{aligned}
&W^1_{0,\sigma}(\Omega_n) \to \mathbb{R} \\
&\boldsymbol{\varphi} \mapsto -\int_\Omega \boldsymbol{\nabla}\boldsymbol{a} : \boldsymbol{\nabla}\boldsymbol{\varphi} - \int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla}\boldsymbol{u}_n) \cdot \boldsymbol{\varphi},
\end{aligned}$$

is a continuous linear form and by the Riesz representation theorem, there exists a map $F_n : W^{1,2}_{0,\sigma}(\Omega_n) \to W^{1,2}_{0,\sigma}(\Omega_n)$ such that

$$(F_n(\boldsymbol{v}_n), \boldsymbol{\varphi}) = -\int_\Omega \boldsymbol{\nabla}\boldsymbol{a} : \boldsymbol{\nabla}\boldsymbol{\varphi} - \int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla}\boldsymbol{u}_n) \cdot \boldsymbol{\varphi}.$$

The map F_n is continuous on $W^{1,2}_{0,\sigma}(\Omega_n)$ when equipped with the L^4-norm, and, since $W^{1,2}_{0,\sigma}(\Omega_n)$ is compactly embedded in $L^4(\Omega_n)$, F_n is completely continuous.

The problem of finding an approximate solution is equivalent to solving the equation

$$\boldsymbol{v}_n = F_n(\boldsymbol{v}_n)$$

in $W^{1,2}_{0,\sigma}(\Omega_n)$. By the Leray–Schauder fixed point theorem (see for example [17, Theorem 11.6.]), to prove the existence of an approximate weak solution, it is sufficient to prove that the set of all possible solutions of the equation

$$\boldsymbol{v}_n = \lambda F_n(\boldsymbol{v}_n), \tag{5.2}$$

is uniformly bounded in $\lambda \in [0, 1]$.

To this end, we take the scalar product of (5.2) with $\boldsymbol{v}_n$, and after integrations by parts, we get

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{v}_n : \boldsymbol{\nabla}\boldsymbol{v}_n = \lambda \int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla}\boldsymbol{v}_n) \cdot \boldsymbol{a} - \lambda \int_\Omega \boldsymbol{\nabla}\boldsymbol{a} : \boldsymbol{\nabla}\boldsymbol{v}_n. \tag{5.3}$$

Therefore by Hölder inequality, we obtain

$$\|\boldsymbol{\nabla}\boldsymbol{v}_n\|_2^2 \le \lambda \left[(\|\boldsymbol{a}/y\|_2 + \|\boldsymbol{v}_n/y\|_2) \, \|y\boldsymbol{a}\|_\infty + \|\boldsymbol{\nabla}\boldsymbol{a}\|_2 \right] \|\boldsymbol{\nabla}\boldsymbol{v}_n\|_2 \,,$$

and therefore using Hardy inequality,

$$\|\boldsymbol{\nabla}\boldsymbol{v}_n\|_2 \le \lambda \left(\|\boldsymbol{a}/y\|_2 + 2\,\|\boldsymbol{\nabla}\boldsymbol{v}_n\|_2 \right) \|y\boldsymbol{a}\|_\infty + \lambda \, \|\boldsymbol{\nabla}\boldsymbol{a}\|_2 \,.$$

For $\lambda \in [0, 1]$, we finally obtain for n big enough, and $\boldsymbol{a}$ small enough,

$$\|\boldsymbol{\nabla}\boldsymbol{v}_n\|_2 \le \frac{\|\boldsymbol{a}/y\|_2 \, \|y\boldsymbol{a}\|_\infty + \|\boldsymbol{\nabla}\boldsymbol{a}\|_2}{1 - 2\,\|y\boldsymbol{a}\|_\infty},$$

which proves that $\boldsymbol{\nabla}\boldsymbol{v}_n$ is uniformly bounded. □

We are now able to take the limit $n \to \infty$ and prove the existence of a weak solution in Ω:

Proof of Theorem 5.5. By Lemma 5.8, there exists for any $n \in \mathbb{N}$ an approximate weak solution $\boldsymbol{v}_n$ and the sequence $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ is bounded in $D^{1,2}_{0,\sigma}(\Omega)$. Therefore, we can extract a subsequence, denoted also by $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$, which converges weakly to $\boldsymbol{v}$ in $D^{1,2}_{0,\sigma}(\Omega)$. Now, let $\boldsymbol{\varphi}$ be a test function with compact support in Ω. Then, there exists $m \in \mathbb{N}$ such that the support of $\boldsymbol{\varphi}$ is in Ω_m. Therefore, we have for any $n \geq m$,

$$\int_\Omega \boldsymbol{\nabla u}_n : \boldsymbol{\nabla \varphi} + \int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla u}_n) \cdot \boldsymbol{\varphi} = 0.$$

By definition of the weak convergence, we have

$$\lim_{n\to\infty} \int_\Omega \boldsymbol{\nabla v}_n : \boldsymbol{\nabla \varphi} = \int_\Omega \boldsymbol{\nabla v} : \boldsymbol{\nabla \varphi}.$$

Since φ has support in Ω_m,

$$\begin{aligned}
&\left|\int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla u}_n - \boldsymbol{u} \cdot \boldsymbol{\nabla u}) \cdot \boldsymbol{\varphi}\right| \\
&\quad = \left|\int_{\Omega_m} ((\boldsymbol{u}_n - \boldsymbol{u}) \cdot \boldsymbol{\nabla u}_n + \boldsymbol{u} \cdot \boldsymbol{\nabla}\left(\boldsymbol{u}_n - \boldsymbol{u}\right)) \cdot \boldsymbol{\varphi}\right| \\
&\quad \leq \left|\int_{\Omega_m} ((\boldsymbol{v}_n - \boldsymbol{v}) \cdot \boldsymbol{\nabla u}_n) \cdot \boldsymbol{\varphi}\right| \\
&\qquad + \left|\int_{\Omega_m} (\boldsymbol{u} \cdot \boldsymbol{\nabla \varphi}) \cdot (\boldsymbol{v}_n - \boldsymbol{v})\right| \\
&\quad \leq \|\boldsymbol{v}_n - \boldsymbol{v}; L_2(\Omega_m)\| \Big(\|\boldsymbol{\nabla u}_n\|_2 \|\boldsymbol{\varphi}\|_\infty \\
&\qquad + 2 \|\boldsymbol{\nabla u}_n\|_2 \|y \boldsymbol{\nabla \varphi}\|_\infty \Big),
\end{aligned}$$

and therefore since $D^{1,2}_{0,\sigma}(\Omega_m)$ is compactly embedded in $L_2(\Omega_m)$, this proves that

$$\lim_{n\to\infty} \int_\Omega (\boldsymbol{u}_n \cdot \boldsymbol{\nabla u}_n) \cdot \boldsymbol{\varphi} = \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{\nabla u}) \cdot \boldsymbol{\varphi}.$$

□

6. Uniqueness

In this section, we prove a weak–strong uniqueness theorem by exploiting the properties of (α, q)-solutions. Namely, we prove that any weak solution satisfying the decay properties (4.2) of an (α, q)-solution coincides with any weak solutions for the same boundary data. The ideas of the proof that are not specific to the presence of an extension can be found in Hillairet and Wittwer [21] and we refer the reader to this article for some technical details which are omitted here.

Theorem 6.1 (weak–strong uniqueness). *Let $\bar{\boldsymbol{u}}$ be a weak solution that satisfies*

$$\bar{\boldsymbol{u}} \in D^{1,2}_\sigma(\Omega), \quad y\bar{\boldsymbol{u}} \in L^\infty(\Omega), \quad y^2 \boldsymbol{\nabla} \bar{\boldsymbol{u}} \in L^\infty(\Omega), \tag{6.1}$$

and such that $\|y\bar{\boldsymbol{u}}\|_\infty$ *is small enough. Then any weak solution* $\boldsymbol{u}$ *with boundary value* $\boldsymbol{u}^* = \bar{\boldsymbol{u}}|_{\partial\Omega}$ *that satisfies the energy inequality*

$$\int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} \le \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{\nabla v}) \cdot \boldsymbol{a}, \tag{6.2}$$

coincides with $\bar{\boldsymbol{u}}$*.*

Remark 6.2. The (α, q)-solutions found in Theorem 4.2 satisfy the requirement (6.1) on $\bar{\boldsymbol{u}}$.

The remaining part of this section is devoted to the proof of this theorem. To begin with, we prove that the strong solutions can be approximated in the following uniform way:

Lemma 6.3 (approximation of a strong solution). *For any* $\bar{\boldsymbol{u}}$ *that satisfies* (6.1)*, there exists* $C > 0$ *and a sequence* $(\bar{\boldsymbol{u}}_n)_{n\in\mathbb{N}}$ *such that:*

1. $\bar{\boldsymbol{u}}_n$ *has support in* Ω_{2n}*;*
2. $\bar{\boldsymbol{u}}_n = \bar{\boldsymbol{u}}$ *on* Ω_n*;*
3. $\|y\bar{\boldsymbol{u}} - y\bar{\boldsymbol{u}}_n\|_\infty \le C\,\|y\bar{\boldsymbol{u}}\|_\infty$*;*
4. $\left\|y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}} - y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}}_n\right\|_\infty \le C\left(\left\|y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}}\right\|_\infty + \|y\bar{\boldsymbol{u}}\|_\infty\right)$*;*
5. $\|\boldsymbol{\nabla}\bar{\boldsymbol{u}} - \boldsymbol{\nabla}\bar{\boldsymbol{u}}_n\|_2 \le C\left(\|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|_2 + \|y\bar{\boldsymbol{u}}\|_\infty\right)$.

Proof. We define

$$\chi_n(x,y) = \chi\left(\frac{|(x,y)|}{n}\right),$$

where χ is a cutoff function satisfying (3.2), so $\chi_n = 1$ on Ω_n and $\chi_n = 0$ on $\Omega \setminus \Omega_{2n}$. Since $|\boldsymbol{\nabla}\chi_n| = |\chi'|/n$, we obtain that $\|\boldsymbol{\nabla}\chi_n\|_2$ and $\|y\boldsymbol{\nabla}\chi_n\|_\infty$ are uniformly bounded in n. Then we define $\bar{\boldsymbol{u}}_n = \chi_n\bar{\boldsymbol{u}}$, so that the first two properties are valid. We have

$$\|y\bar{\boldsymbol{u}} - y\bar{\boldsymbol{u}}_n\|_\infty \le \|1-\chi_n\|_\infty \|y\bar{\boldsymbol{u}}\|_\infty \le C\,\|y\bar{\boldsymbol{u}}\|_\infty\,,$$

so the third property is proven. Finally, since

$$|\boldsymbol{\nabla}\bar{\boldsymbol{u}} - \boldsymbol{\nabla}\bar{\boldsymbol{u}}_n| \le |1-\chi_n|\,|\boldsymbol{\nabla}\bar{\boldsymbol{u}}| + |\bar{\boldsymbol{u}}|\,|\boldsymbol{\nabla}\chi_n|\,,$$

we obtain

$$\begin{aligned}\left\|y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}} - y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}}_n\right\|_\infty &\le \|1-\chi_n\|_\infty \left\|y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}}\right\|_\infty + \|y\boldsymbol{\nabla}\chi_n\|_\infty \|y\bar{\boldsymbol{u}}\|_\infty \\ &\le C\left(\left\|y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}}\right\|_\infty + \|y\bar{\boldsymbol{u}}\|_\infty\right),\end{aligned}$$

and

$$\begin{aligned}\left\|\boldsymbol{\nabla}\bar{\boldsymbol{u}} - \boldsymbol{\nabla}\bar{\boldsymbol{u}}_n\right\|_2 &\le \|1-\chi_n\|_\infty \|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|_2 + \|\boldsymbol{\nabla}\chi_n\|_2 \|y\bar{\boldsymbol{u}}\|_\infty \\ &\le C\left(\|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|_2 + \|y\bar{\boldsymbol{u}}\|_\infty\right).\end{aligned}$$

□

Then, we prove that the integration by parts with respect to the solution $\bar{\boldsymbol{u}}$ is permitted:

Lemma 6.4 (integration by parts). *For any $\bar{\boldsymbol{u}}$ satisfying (6.1), we have*

$$\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{u} + \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\bar{\boldsymbol{u}} = 0,$$

for all $\boldsymbol{u},\boldsymbol{w} \in D^{1,2}_\sigma(\Omega)$ with $\boldsymbol{u}/y \in L^2(\Omega)$ and $\boldsymbol{w}/y \in L^2(\Omega)$. We note in particular that if $\boldsymbol{u}$ and $\boldsymbol{w}$ are weak solutions, the hypotheses are satisfied.

Proof. Since the support of $\bar{\boldsymbol{u}}_n$ is compact, after an integration by parts, we have

$$\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}}_n)\cdot\boldsymbol{u} + \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\bar{\boldsymbol{u}}_n = 0.$$

By using Hölder inequality and since $\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}_n$ has support only in $\Omega_n^c = \Omega\backslash\Omega_n$,

$$\left|\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}(\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}_n))\cdot\boldsymbol{u}\right| \leq \|\boldsymbol{w}/y\|_2 \left\|y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}} - y^2\boldsymbol{\nabla}\bar{\boldsymbol{u}}_n\right\|_\infty \left\|\boldsymbol{u}/y; L^2(\Omega_n^c)\right\|,$$

so by applying Lemma 6.3, we have

$$\lim_{n\to\infty}\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}}_n)\cdot\boldsymbol{u} = \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{u}.$$

In the same way,

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot(\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}_n)\right| &\leq \|\boldsymbol{w}/y\|_2 \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega_n^c)\right\| \|y\bar{\boldsymbol{u}} - y\bar{\boldsymbol{u}}_n\|_\infty \\ &\leq C\,\|\boldsymbol{w}/y\|_2 \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega_n^c)\right\| \|y\bar{\boldsymbol{u}}\|_\infty,\end{aligned}$$

so we obtain

$$\lim_{n\to\infty}\int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\bar{\boldsymbol{u}}_n = \int_\Omega (\boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\bar{\boldsymbol{u}},$$

and the integration by parts also holds in the limit. Finally, if $\boldsymbol{u} = \boldsymbol{a}+\boldsymbol{v}$ is a weak solution, we have by hypothesis $\boldsymbol{a}/y \in L^2(\Omega)$ and by Hardy inequality $\boldsymbol{v}/y \in L^2(\Omega)$, since $\boldsymbol{v} \in D^1_{0,\sigma}$. □

Next, we prove some results on the extension of the allowed test functions in the definition of weak solutions:

Lemma 6.5. *If $\boldsymbol{u}$ is a weak solution, then*

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} + \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\bar{\boldsymbol{v}} = 0,$$

for any $\bar{\boldsymbol{v}} \in D^1_{0,\sigma}(\Omega)$ such that $y\bar{\boldsymbol{v}} \in L^\infty(\Omega)$.

Proof. Since $\boldsymbol{u}$ is a weak solution and $\bar{\boldsymbol{v}}_n = \chi_n\bar{\boldsymbol{v}}$ is compact,

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\bar{\boldsymbol{v}}_n + \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot\bar{\boldsymbol{v}}_n = 0.$$

We have

$$\left|\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}(\bar{\boldsymbol{v}}-\bar{\boldsymbol{v}}_n)\right| \leq \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega\setminus\Omega_n)\right\| \|\boldsymbol{\nabla}\bar{\boldsymbol{v}} - \boldsymbol{\nabla}\bar{\boldsymbol{v}}_n\|_2,$$

and

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{v}}_n)\right| &\le \left|\int_\Omega (\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{v}}_n)\right| \\ &\quad + \left|\int_\Omega (\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{u})\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{v}}_n)\right| \\ &\le \|y\boldsymbol{a}\|_\infty \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega_n^c)\right\| \left\|(\bar{\boldsymbol{v}}-\bar{\boldsymbol{v}}_n)/y\right\|_2 \\ &\quad + \|\boldsymbol{v}/y\|_2 \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega_n^c)\right\| \|y\bar{\boldsymbol{v}}-y\bar{\boldsymbol{v}}_n\|_\infty \\ &\le 2\|y\boldsymbol{a}\|_\infty \|\boldsymbol{\nabla}\bar{\boldsymbol{v}}-\boldsymbol{\nabla}\bar{\boldsymbol{v}}_n\|_2 \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega_n^c)\right\| \\ &\quad +2\|\boldsymbol{\nabla}\boldsymbol{v}\|_2 \|y\bar{\boldsymbol{v}}-y\bar{\boldsymbol{v}}_n\|_\infty \left\|\boldsymbol{\nabla}\boldsymbol{u}; L^2(\Omega_n^c)\right\|,\end{aligned}$$

so by applying Lemma 6.3, these last two expressions vanish in the limit $n \to \infty$, which proves the lemma. □

Lemma 6.6. *If $\bar{\boldsymbol{u}}$ is a weak solution such that $y\bar{\boldsymbol{u}} \in L^\infty(\Omega)$, then*

$$\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\boldsymbol{v} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{v} = 0,$$

for any $\boldsymbol{v} \in D^1_{0,\sigma}(\Omega)$.

Proof. We have

$$\begin{aligned}\left|\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\boldsymbol{\varphi} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{\varphi}\right| &\le \left|\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\boldsymbol{\varphi}\right| + \left|\int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{\varphi}\right| \\ &\le \|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|_2 \|\boldsymbol{\nabla}\boldsymbol{\varphi}\|_2 + \|y\bar{\boldsymbol{u}}\|_\infty \|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|_2 \|\boldsymbol{\varphi}/y\|_2 \\ &\le \|\boldsymbol{\nabla}\bar{\boldsymbol{u}}\|_2 \left(1+2\|y\bar{\boldsymbol{u}}\|_\infty\right) \|\boldsymbol{\nabla}\boldsymbol{\varphi}\|_2,\end{aligned}$$

and since the form is linear in $\boldsymbol{\varphi}$, the lemma is proved. □

We now prove that the weak solution $\bar{\boldsymbol{u}}$ satisfies an energy equality:

Lemma 6.7. *Any weak solution $\bar{\boldsymbol{u}}$ which satisfies (6.1) verifies the energy equality*

$$\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} = \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{v}})\cdot\boldsymbol{a}. \tag{6.3}$$

Proof. By Lemma 6.6, we have

$$\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\bar{\boldsymbol{v}} = 0,$$

and by Lemma 6.4,

$$\int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{v}})\cdot\bar{\boldsymbol{v}} = 0, \qquad \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{a})\cdot\bar{\boldsymbol{v}} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{v}})\cdot\boldsymbol{a} = 0,$$

so we obtain the energy equality

$$\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} = -\int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{a})\cdot\bar{\boldsymbol{v}} = \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{v}})\cdot\boldsymbol{a}.$$

□

We now have the necessary tools to prove the main theorem of this section:

Proof of Theorem 6.1. Let $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ be two weak solutions with the same boundary conditions, so $\boldsymbol{d} = \boldsymbol{u} - \bar{\boldsymbol{u}} \in D^{1,2}_{0,\sigma}(\Omega)$; see for example [15, Theorem II.7.7]. Then, by using the scalar product on $D^{1,2}_{0,\sigma}(\Omega)$, we have

$$\begin{aligned}\|\boldsymbol{\nabla d}\|_2^2 &= \int_\Omega (\boldsymbol{\nabla u} - \boldsymbol{\nabla}\bar{\boldsymbol{u}}) : (\boldsymbol{\nabla v} - \boldsymbol{\nabla}\bar{\boldsymbol{v}}) \\ &= \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} + \int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} - \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla}\bar{\boldsymbol{v}} - \int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla v}.\end{aligned}$$

By using Lemmas 6.5 and 6.6, the energy equality (6.3) and the energy inequality (6.2), we have

$$\|\boldsymbol{\nabla d}\|_2^2 \leq \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla v})\cdot\boldsymbol{a} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{v}})\cdot\boldsymbol{a} + \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla u})\cdot\bar{\boldsymbol{v}} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{v}.$$

Since $\boldsymbol{u}/y \in L^2(\Omega)$ and $\bar{\boldsymbol{u}}/y \in L^2(\Omega)$ by using Hardy inequality and Lemma 6.4, we have

$$\int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla a})\cdot\boldsymbol{a} = 0, \qquad \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla a})\cdot\boldsymbol{a} = 0,$$

which allows us to rewrite the bound as

$$\begin{aligned}\|\boldsymbol{\nabla d}\|_2^2 &\leq \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla u})\cdot\boldsymbol{a} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{a} \\ &\quad + \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla u})\cdot\bar{\boldsymbol{v}} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{v} \\ &\leq \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla u})\cdot\bar{\boldsymbol{u}} + \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\boldsymbol{u}.\end{aligned}$$

By using Lemma 6.4, we integrate the second term by parts,

$$\|\boldsymbol{\nabla d}\|_2^2 \leq \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{\nabla u})\cdot\bar{\boldsymbol{u}} - \int_\Omega (\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla u})\cdot\bar{\boldsymbol{u}} = \int_\Omega (\boldsymbol{d}\cdot\boldsymbol{\nabla u})\cdot\bar{\boldsymbol{u}}.$$

Again by Lemma 6.4, we have

$$\int_\Omega (\boldsymbol{d}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot\bar{\boldsymbol{u}} = 0,$$

so by Hardy inequality,

$$\|\boldsymbol{\nabla d}\|_2^2 \leq \left|\int_\Omega (\boldsymbol{d}\cdot\boldsymbol{\nabla d})\cdot\bar{\boldsymbol{u}}\right| \leq \|\boldsymbol{d}/y\|_2 \|\boldsymbol{\nabla d}\|_2 \|y\bar{\boldsymbol{u}}\|_\infty \leq 2\|y\bar{\boldsymbol{u}}\|_\infty \|\boldsymbol{\nabla d}\|_2^2.$$

Therefore, if $\|y\bar{\boldsymbol{u}}\|_\infty$ is small enough, we obtain that $\boldsymbol{d} = \boldsymbol{0}$, i.e., $\boldsymbol{u} = \bar{\boldsymbol{u}}$. □

7. Numerical Simulations

To simulate this problem numerically, we truncate the domain Ω to a ball of radius $R = 10^4$, $\Omega_R = \Omega \cap B((0,1),R)$. On the bottom boundary we take an antisymmetric perturbation of a symmetric Jeffery–Hamel,

$$\boldsymbol{u}|_{[-R,R]\times\{1\}} = \boldsymbol{u}_\Phi^0 + \frac{\nu}{r}\sin(2\theta)\boldsymbol{e}_r, \tag{7.1}$$

and on the artificial boundary Γ, which is the upper half circle of radius R, we take

$$\boldsymbol{u}|_\Gamma = \boldsymbol{u}_\Phi^0.$$

In Fig. 3, we represent the velocity field $\boldsymbol{u}$ multiplied by r to see the behavior at large distances. For $\nu = 0$, this corresponds to the Jeffery–Hamel solutions $\boldsymbol{u}_\Phi^0$ which are scale invariant. For negative fluxes $\Phi < 0$, small perturbations have almost no effect on the behavior at large distances, so the asymptotic term is probably given by the Jeffery–Hamel solution $\boldsymbol{u}_\Phi^0$. Conversely for $\Phi > 0$, even a small perturbation drastically changes the behavior of the solution at large distances by somehow rotating the region where the magnitude of the velocity is large. In the case, the asymptotic behavior is very likely not given by the Jeffery–Hamel solution $\boldsymbol{u}_\Phi^0$. This conclusion can also be seen in Fig. 4, where we plot the velocity field $\boldsymbol{u}$ in polar coordinates multiplied by r on the half-circle $\left\{10^3(\sin\theta,\cos\theta), \theta \in \left[\frac{-\pi}{2};\frac{\pi}{2}\right]\right\}$ in terms of ν.

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Appendix A. Jeffery–Hamel Solutions with Small Flux

A Jeffery–Hamel solution in the upper half-plane,

$$\left\{(r\sin\theta, r\cos\theta), \quad r > 0 \text{ and } \theta \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right\},$$

is a radial solution of the Navier–Stokes equations with zero velocity on the boundary and whose velocity norm is $f(\theta)/r$. More explicitly, f has to satisfy the boundary value problem (A.1). Here, we prove an existence theorem:

Theorem A.1. *For every small enough value of the flux Φ, the Jeffery–Hamel equation*

$$f'' + f^2 + 4f = 2C, \quad f\left(\tfrac{\pm\pi}{2}\right) = 0, \tag{A.1}$$

admits a symmetric solution,

$$f_\Phi^0(\theta) = \frac{2\Phi}{\pi}\cos^2(\theta) + O(\Phi^{3/2}),$$

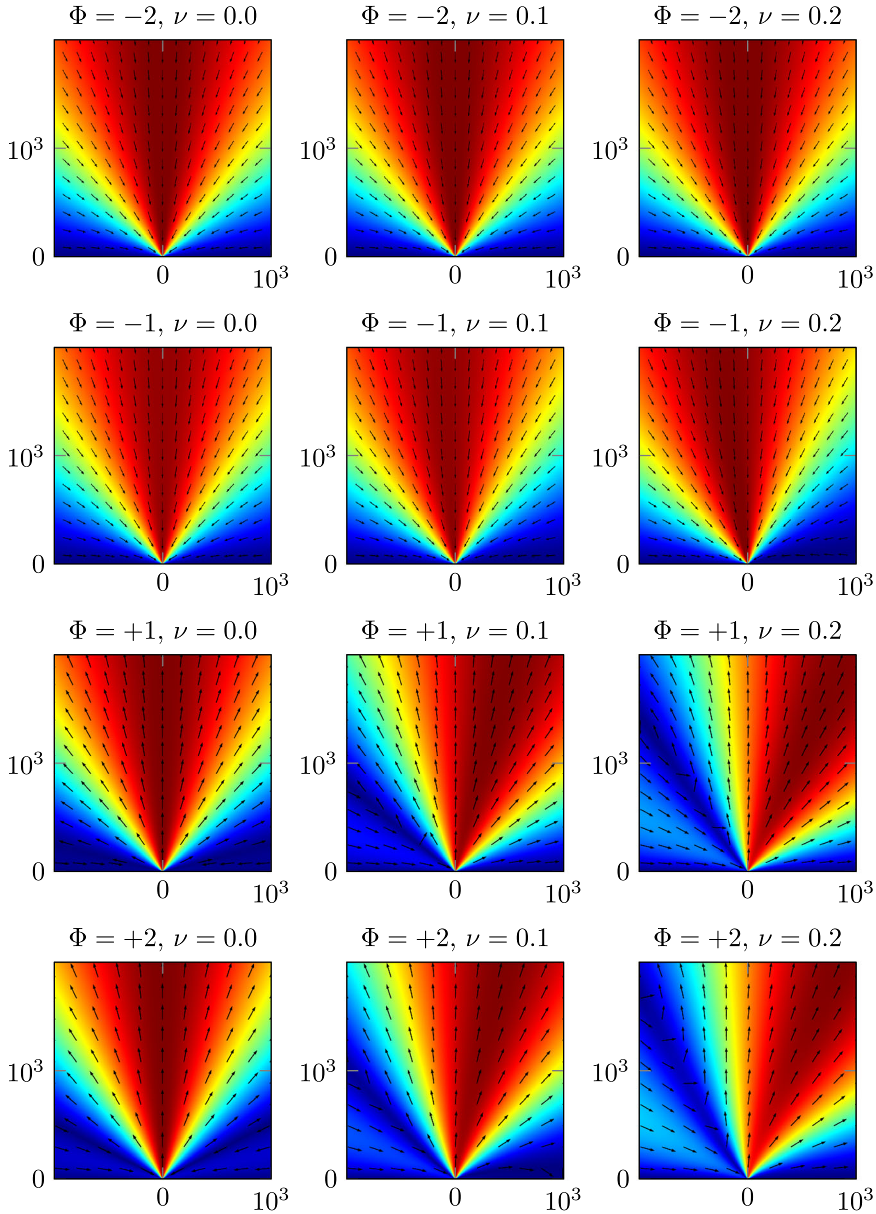

Figure 3. Numerical results for the velocity $\boldsymbol{u}$ multiplied by r, for the boundary condition (7.1) in a domain of size $R = 10^4$, for various values of the flux Φ and of the perturbation ν

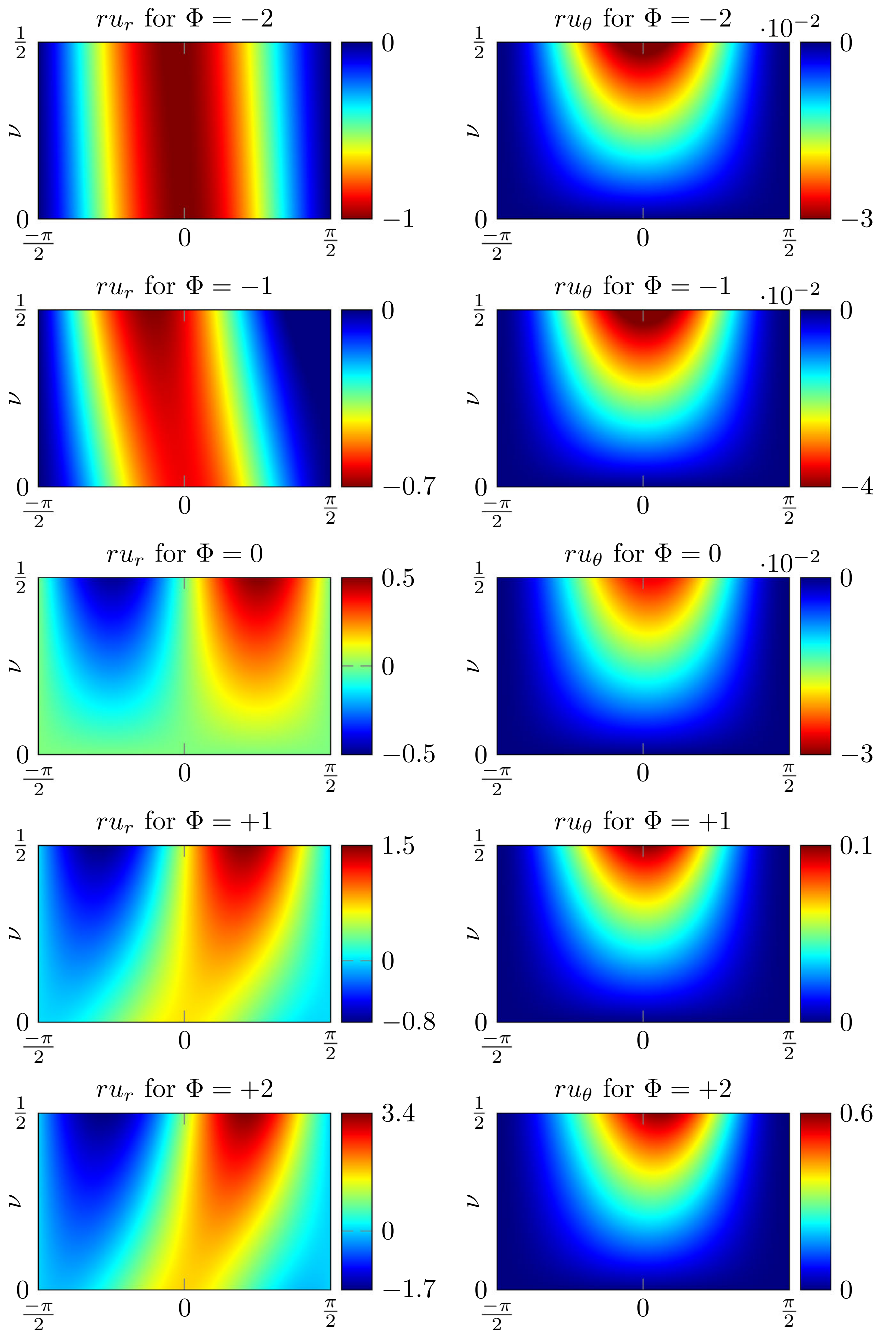

FIGURE 4. Profiles of ru_r and ru_θ on the half-circle of radius 10^3 in term of $\theta \in [-\pi/2, \pi/2]$ and $\nu \in [0, 1]$

and in addition if $\phi < 0$, two quasiantisymmetric solutions,

$$f_\Phi^{\pm 1}(\theta) = \pm\sqrt{\frac{-48\Phi}{\pi}}\sin(2\theta) + O(\Phi).$$

Proof. First of all, the solution of the linear equation

$$f'' + 4f = g,$$

is given by

$$f(\theta) = A\sin(2\theta) + B\cos(2\theta) - L[g](\theta),$$

where

$$L[g](\theta) = \frac{1}{2}\cos(2\theta)\int_{-\pi/2}^{\theta} g(s)\sin(2s)\,\mathrm{d}s - \frac{1}{2}\sin(2\theta)\int_{-\pi/2}^{\theta} g(s)\cos(2s)\,\mathrm{d}s.$$

Therefore the Jeffery–Hamel equation and the boundary condition (A.1) can be rewritten as

$$f(\theta) = A\sin(2\theta) + C\cos^2(\theta) + L[f^2](\theta), \quad \int_{-\pi/2}^{+\pi/2} f^2(\theta)\sin(2\theta)\,\mathrm{d}\theta = 0. \tag{A.2}$$

By defining

$$f_0(\theta) = A\sin(2\theta), \quad f_1(\theta) = C\cos^2(\theta) - \frac{A^2}{3}\cos^4(\theta), \quad f = f_0 + f_1 + \bar{f},$$

the flux condition

$$\Phi = \int_{-\pi/2}^{+\pi/2} f(\theta)\,\mathrm{d}\theta$$

directly gives the definition of C in terms of the flux,

$$C = \frac{2\Phi}{\pi} + \frac{A^2}{4} - \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2}\bar{f}(\theta)\,\mathrm{d}\theta,$$

and the two equations (A.2) can be rewritten by substitution as:

$$\bar{f} = L\left[\left(2f_0 + f_1 + \bar{f}\right)\left(f_1 + \bar{f}\right)\right], \tag{A.3}$$

$$A\left(\Phi + \frac{\pi}{48}A^2\right) = \int_{-\pi/2}^{+\pi/2}\left(A - 2f_0(\theta) - 2f_1(\theta) - \bar{f}(\theta)\right)\bar{f}(\theta)\,\mathrm{d}\theta. \tag{A.4}$$

To find a fixed point of theses equations, we first solve the left-hand side of the second equation

$$A_0\left(\Phi + \frac{\pi}{48}A_0^2\right) = 0$$

for A_0. This equation admits the solution $A_0 = 0$ and in addition, if $\Phi < 0$, the two solutions

$$A_0 = \pm\sqrt{\frac{-48\Phi}{\pi}}.$$

Given one of these three solutions, we define

$$A = A_0 + \bar{A},$$

and (A.4) becomes:

$$\bar{A} = \frac{48}{48\Phi + 3\pi A_0^2} \left[\int_{-\pi/2}^{+\pi/2} \left(A - 2f_0(\theta) - 2f_1(\theta) - \bar{f}(\theta) \right) \bar{f}(\theta) \, \mathrm{d}\theta \right.$$
$$\left. - \frac{\pi}{48} \left(3A_0 + \bar{A} \right) \bar{A}^2 \right]. \tag{A.5}$$

It is easily verified that the maps defined by (A.5) and (A.3) map the ball

$$B_\Phi = \left\{ \left(\bar{A}, \bar{f} \right) \in \mathbb{R} \times C^0\left(\left[-\tfrac{\pi}{2}, \tfrac{\pi}{2} \right] \right) : \ \left| \bar{A} \right| \leq 50 \left| \Phi \right| \ \text{and} \ \left| \bar{f} \right| \leq 600 \left| \Phi \right|^{3/2} \right\}$$

into itself, provided Φ is small enough. Moreover, since the maps (A.5) and (A.3) are multilinear affine maps of $\bar{A}$ and $\bar{f}$, they are contractions from B_Φ into itself, for Φ small enough. The first-order terms which we explicitly computed above prove the claimed leading terms of the symmetric and quasiantisymmetric solutions. □

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