Ann. Henri Poincaré 17 (2016), 2513–2527

1424-0637/16/092513-15
published online March 7, 2016
DOI 10.1007/s00023-016-0467-8

Annales Henri Poincaré

Energy-Time Uncertainty Principle and Lower Bounds on Sojourn Time

Joachim Asch, Olivier Bourget, Victor Cortés and Claudio Fernandez

Abstract. One manifestation of quantum resonances is a large sojourn time, or autocorrelation, for states which are initially localized. We elaborate on Lavine's time-energy uncertainty principle and give an estimate on the sojourn time. For the case of perturbed embedded eigenstates the bound is explicit and involves Fermi's Golden Rule. It is valid for a very general class of systems. We illustrate the theory by applications to resonances for time-dependent systems including the AC Stark effect as well as multistate systems.

1. Introduction

By a state in a Hilbert space $\mathcal{H}$ we understand a normalized vector $\psi \in \mathcal{H}$, respectively, the associated projector $P = |\psi\rangle\langle\psi|$. Given a self-adjoint operator H in $\mathcal{H}$ and the dynamics generated by H, the sojourn time for a state ψ is defined by

$$\mathcal{T} = \mathcal{T}(H, \psi) := \int_{-\infty}^{\infty} |\langle \psi, e^{-iHt}\psi\rangle|^2 \, \mathrm{d}t. \tag{1}$$

$\mathcal{T}$ is a measure of the total amount of time the system spends in its initial state; it equals the autocorrelation $\int_{-\infty}^{\infty} tr\,(PP_t)\,\mathrm{d}t$, where $P_t := e^{-iHt}Pe^{iHt}$; it is infinite if ψ is an eigenstate of H. The main result we shall prove in this paper is a lower bound on $\mathcal{T}$ for a perturbed embedded eigenstate ψ.

More precisely, consider

$$H = H_0 + \kappa V_\kappa,$$

E_0, ψ such that $H_0\psi = E_0\psi$ and E_0 embedded in the continuous spectrum of H_0, κ a small parameter. We assume that the eigenvalue is simple and that the restriction of H_0 to $\mathrm{Ran}P^\perp$ has good propagation properties which persist upon perturbation by κV_κ; it is known that this is implied by a Mourre estimate which we assume in hypothesis (**A**) below. It involves, in particular, multiple commutators of unbounded operators; we now state and discuss our main result

and refer the reader to Sect. 3 for a precise definition of the mathematically more involved objects. Denote $\chi(H \in I)$ the spectral projection of a self-adjoint operator H on a Borel set I, $\mathrm{ad}_A(B) = [A, B] = AB - BA$ the commutator, and by $\mathrm{ad}_A^k(B) = \mathrm{ad}_A^{k-1}(\mathrm{ad}_A(B))$ its iterations.

We assume that the quadruple H_0, V, A, I satisfies

Hypothesis (A). H_0, A are self-adjoint operators, I an open interval and $[0, 1) \ni \kappa \mapsto V_\kappa$ a symmetric operator valued function such that:

1. $e^{-isA} D(H_0) \subset D(H_0)$, $\forall\, s \in \mathbb{R}$, the commutator $[A, H_0]$ is H_0 bounded and a Mourre estimate:
$$\chi(H_0 \in I) i[H_0, A] \chi(H_0 \in I) \geq c\chi(H_0 \in I) + K$$
holds with $c > 0$ and K a compact operator;
2. $\mathrm{ad}_A^k(H_0)$ are H_0 bounded for $k \in \{2, 3\}$;
3. $\mathrm{ad}_A^k(V_\kappa)$ are H_0 bounded uniformly in κ for $k \in \{1, 2\}$;
4. $\kappa \mapsto V_\kappa (H_0 + i)^{-1}$ is differentiable in operator norm.

With $P^\perp = \mathbb{I} - P$, $H^\perp = P^\perp H P^\perp$ and $\left(H^\perp - z\right)^{-1}$ its resolvent reduced to $\mathrm{Ran} P^\perp$ denote

$$F(\kappa, z) := P V_\kappa P^\perp \left(H^\perp - z\right)^{-1} P^\perp V_\kappa P \quad \mathrm{Im} z \neq 0, \kappa \in [0, 1) \tag{2}$$

the weighted reduced resolvent. The explicit assumptions (**A**) imply the existence of the limit to the real axis $F(\kappa, E + i0)$ for $E \in I$ and its continuity in operator norm topology, see Theorem 3.1 below; this is in fact all we need to prove the following lower bound on the sojourn time:

Theorem 1.1. *Assume* (**A**). *Let ψ be an eigenstate of H_0 for a simple eigenvalue $E_0 \in I$. Then it holds for $\kappa \in (0, 1)$ and $H = H_0 + \kappa V_\kappa$:*

$$\mathcal{T}(H, \psi) \geq \frac{1}{\Delta E}$$

where the energy width ΔE, which is defined in (6) below, has the property

$$\Delta E = \kappa^2 \,\mathrm{Im} \left\langle \psi, V_0 P^\perp \left(H_0^\perp - E_0 + i0\right)^{-1} P^\perp V_0 \psi \right\rangle + o(\kappa^2)$$

and $\mathcal{T}(H, \psi) = \infty$ if $\Delta E = 0$.

Here $P = |\psi\rangle \langle\psi|$, $P^\perp = \mathbb{I} - P, H_0^\perp := P^\perp H_0 P^\perp$ and the limit $F(0, E_0 + i0) = P V_0 P^\perp \left(H_0^\perp - E_0 + i0\right)^{-1} P^\perp V_0 P$ exists as a bounded operator.

Without attempting to review the huge amount of literature on quantum resonances and time-energy uncertainty principles we make some remarks.

A well-known time-energy uncertainty principle is

$$\mathcal{T} \geq \frac{6\pi}{5\sqrt{5}} \frac{1}{\Delta H}$$

with the uncertainty $\Delta H := \left(\langle\psi, H^2\psi\rangle - \langle\psi, H\psi\rangle^2\right)^{1/2}$, see [9]. For the situation of Theorem 1.1 one has $\Delta H = \kappa \Delta V_\kappa$ so the denominator of the lower

bound is linear in κ whereas in Theorem 1.1 it is quadratic. In addition, instead of the uncertainty, our bound involves the term

$$\mathrm{Im}\left\langle \psi, V_0 P^\perp \left(H_0^\perp - E_0 + i0\right)^{-1} P^\perp V_0 \psi \right\rangle$$

which is the overlap of ψ with the continuum eigenstates induced by the perturbation; the occurrence of this term is Fermi's Golden rule. See [26], ch XII.6, for more information.

For a general treatment of time-energy principles see Fröhlich–Pfeiffer [25].

Lavine's influence on foundations of the theory was important, see [19]. Our contribution here is to revisit his ideas with state of the art methods and to relate his lower bound to Fermi's Golden Rule thus making it efficient to produce concrete lower bounds which are in accordance to physics folklore and known mathematical results on related aspects. As pointed out below, see Remark 3.4, our theory applies to systems with low regularity meaning that it is sufficient that the extension of F to the real axis is continuous. Furthermore it is flexible enough to accommodate for systems depending periodically on time. We illustrate this with two applications to systems which are important for physics and come as a perturbation of an embedded eigenvalue: the AC Stark effect and multistate systems, see Sects. 4, 5 for additional motivation. With low effort we prove in both cases lower bounds on the sojourn time, Eqs. (13) and (14), which seem to be new.

The reader can trace back the development of general mathematical resonance theory from the papers of Aguilar Combes [1], Simon [29], Orth [24], Cattaneo Graf Hunziker [4]. The main focus of these has been on the location of complex poles of an analytic continuation of the resolvent of H across the absolutely continuous spectrum, respectively, on exponential decay laws of the form

$$\langle \widetilde{\psi}, e^{-itH}\widetilde{\psi}\rangle = a(\kappa)e^{-it\lambda(\kappa)} + b(\kappa, t) \tag{3}$$

where the $\widetilde{\psi}$ is related to the eigenstate ψ, $a(\kappa) = 1 + \mathcal{O}(\kappa^2)$, $\mathrm{Im}(\lambda(\kappa)) = \kappa^2 \,\mathrm{Im}\,\langle\psi, F(0, E_0 + i0)\psi\rangle + o(\kappa^2)$ and $b(\kappa, t)$ can be estimated for small κ.

Our bound on the sojourntime, the square of the $L^2(\mathbb{R}, \mathrm{d}t)$ norm of $\langle \widetilde{\psi}, e^{-itH}\widetilde{\psi}\rangle$, gives complementary information to several points even if the time decay of b is well controlled. It was proven in [4], Theorem 1.2 that $|b(\kappa, t)| \leq \frac{c\kappa^2 |\log|\kappa||}{(1+|t|)}$ under the assumption that six relatively bounded commutators exist. This estimate works for $\widetilde{\psi} = g(H)\psi$ with g a smooth function localizing near E_0, and the assumption $\mathrm{Im}(F(0, E_0 + i0) < 0$ was used in a crucial way. The bound on the error b belonging to $L^2(\mathbb{R}, \mathrm{d}t)$, the exponential decay law implies an asymptotic of the sojourn time of $\widetilde{\psi}$; so in this situation it provides more information than our bound. Here we improve in three points: first we assume only existence of three relatively bounded commutators in (**A**).2 (respectively, two in (**A**).3); second our result concerns the sojourn time of the unperturbed eigenstate ψ, a localization of this state as in [4] is

not needed; this is potentially important for applications, see [18] for a discussion of this point related to open systems. Finally our theory covers the case $\mathrm{Im}(F(0, E_0 + i0)) = 0$; remark that an exponential decay law in this case was proven in estimate (2.19) in [6], which, however, does not provide information on the sojourn time because their error term $b(\kappa, \cdot)$ may not be small in $L^2(\mathbb{R}, \mathrm{d}t)$. The same remark applies to the error bound of [21], Theorem 2.1.

Let us mention that, technically speaking, our lower bound on the sojourn time works for systems for which $F(\kappa, E+i0)$ is continuous whereas the results mentioned above need higher order differentiability, see Remark 3.4 for more information.

Remark that the present method gives lower bounds on the sojourn time. Lower bounds on the resonance width, loosely related to upper bounds on the sojourn time, were discussed, for example, in [2,8].

In the following section we shall discuss the lower bound in an abstract setting; in Sect. 3 Theorem 1.1 is proven in the perturbative situation; in Sect. 4 we apply to the time-periodic case, see Theorems 4.1 and 4.2, and in Sect. 5 to multistate systems, see Theorem 5.1.

2. Abstract Lower Bound on the Sojourn Time and Lavine's Energy Width

We review Lavine's [19] abstract lower bound involving the concept of a suitably defined energy width, Theorem (2.5) below. Then we relate it to Fermi's Golden Rule.

We denote the resolvent of H by $R(z) = (H - z)^{-1}$.

Lemma 2.1. *Let H be a self-adjoint operator on $\mathcal{H}$. Then it holds for any state ψ, any $\lambda \in \mathbb{R}$ and any $\epsilon > 0$*

$$\mathcal{T}(H, \psi) \geq \frac{1}{\epsilon} \left(2\epsilon Im\langle \psi, (H - (\lambda + i\epsilon))^{-1} \psi \rangle \right)^2 . \tag{4}$$

Equality holds for a $\lambda \in \mathbb{R}$ and an $\epsilon > 0$ if and only if $\langle \psi, e^{-iHt}\psi \rangle = e^{-it\lambda}e^{-\epsilon|t|}$.

Proof. By Fourier Transform, the spectral Theorem and unitarity of e^{-itH} one has

$$\left\langle \psi, \frac{1}{2i} \left(R(\lambda + i\epsilon) - R(\lambda - i\epsilon) \right) \psi \right\rangle = \frac{1}{2} \int_{\mathbb{R}} e^{-\epsilon|t|} \left\langle \psi, e^{-it(H-\lambda)}\psi \right\rangle \mathrm{d}t.$$

Now the claim follows from the Cauchy Schwartz in-equality,

$$|\mathrm{Im}\langle \psi, R(\lambda + i\epsilon)\psi \rangle| \leq \frac{1}{2} \left(\int_{\mathbb{R}} e^{-2\epsilon|t|} \right)^{1/2} (\mathcal{T}(H, \psi))^{1/2} = \frac{1}{2} \frac{\sqrt{\mathcal{T}(H, \psi)}}{\sqrt{\epsilon}}, \tag{5}$$

where equality holds if and only if $\left\langle \psi, e^{-it(H-\lambda)}\psi \right\rangle = e^{-\epsilon|t|}$. □

Remarks 2.2. 1. If equality holds in (4) for finite ϵ, then it is a corollary that $\mathcal{T}(H, \psi) = \frac{1}{\epsilon}$ and thus $|2\epsilon \mathrm{Im}\langle \psi, R(\lambda + i\epsilon)\psi \rangle| = 1$.

2. On the other hand, as was remarked in [29], equality in Eq. (5) implies analyticity of the spectral measure of ψ and in particular that its support is the whole real line. Thus, equality cannot hold in general, in particular not for semibounded or gapped Hamiltonians.

Starting from inequality (4) for $\mathcal{T}$, Lavine defines his notion of energy width. Given $\lambda \in \mathbb{R}$, consider the function

$$f(\epsilon) := 2\epsilon \quad \mathrm{Im}\,\langle \psi, R(\lambda + i\epsilon)\psi\rangle \quad (\epsilon > 0)$$

f is non-negative, continuous and monotonous, and $\lim_{\epsilon\to\infty} f(\epsilon) = 2$, $\lim_{\epsilon\to 0} f(\epsilon) = 2\,\langle \psi, \chi\,(H \in \{\lambda\})\,\psi\rangle$. Inspired by Remarks (2.2) one defines

Definition 2.3. The energy width of the state ψ at $\lambda \in \mathbb{R}$ with respect to H is defined as the unique real number

$$\Delta E := \Delta E(H, \psi, \lambda) := \inf\{\epsilon \in (0, \infty) \,:\, 2\epsilon\,\mathrm{Im}\langle \psi, R(\lambda + i\epsilon)\psi\rangle \geq 1\}. \tag{6}$$

Remarks 2.4. 1. $\Delta E(H, \psi, \lambda) \in [0, \infty]$, if $\Delta E(H, \psi, \lambda) = 0$ then the projection of ψ on the eigenspace of H at λ does not vanish.

2. If $\Delta E(H, \psi, \lambda) > 0$ then

$$2\Delta E\,\mathrm{Im}\,\langle \psi, R(\lambda + i\Delta E)\psi\rangle = 1$$

3. An intuition behind the definition of ΔE is provided by spectroscopy: the energy distribution of a resonant state at energy E_r is expected to be Lorentzian (Cauchy). Testing this model, i.e., calculating $f(\epsilon)$ for $d\langle \psi, E_\mu \psi\rangle = \frac{1}{\pi}\frac{\Gamma}{(\mu - E_r)^2 + \Gamma^2}\,d\mu$, we get

$$\int_{\mathbb{R}} \frac{2\epsilon^2}{(\mu - \lambda)^2 + \epsilon^2}\,\frac{1}{\pi}\frac{\Gamma}{(\mu - E_r)^2 + \Gamma^2}\,d\mu = \frac{2\epsilon(\epsilon + g)}{(\lambda - E_r)^2 + (\epsilon + \Gamma)^2}$$

which equals 1 for $\epsilon = \sqrt{\Gamma^2 + (\lambda - E_r)^2}$ which in turn is minimal for $\lambda = E_r$; in this case $\epsilon = \Delta E = \Gamma$, the half-width at half-maximum of the Lorentzian distribution.

With this definition and Lemma (2.1) one gets

Theorem 2.5. *Let H be self-adjoint on $\mathcal{H}$. Then for any state ψ in the domain of H and any $\lambda \in \mathbb{R}$ the following inequalities hold:*

$$\mathcal{T}(H, \psi) \geq \frac{1}{\Delta E(H, \psi, \lambda)} \geq \frac{1}{\|(H - \lambda)\psi\|}. \tag{7}$$

Proof. The first inequality follows from (4) and the definition of ΔE. For the second inequality it suffices to choose $\epsilon = \|(H - \lambda)\psi\|$ and to show that $f\,(\|(H - \lambda)\psi\|) \geq 1$ to conclude that $\|(H - \lambda)\psi\| \geq \Delta E$.

Now, by Hölder's inequality, one has for all $\epsilon > 0$

$$\begin{aligned} 1 = \int_{\mathbb{R}} d\mu_\psi &= \int \sqrt{\frac{(E - \lambda)^2 + \epsilon^2}{\epsilon}}\sqrt{\frac{\epsilon}{(E - \lambda)^2 + \epsilon^2}}\,d\mu_\lambda \\ &\leq \left(\frac{1}{\epsilon}\left(\|(H - \lambda)\psi\|^2 + \epsilon^2\right)\mathrm{Im}\langle \psi, (H - \lambda - i\epsilon)^{-1}\psi\rangle\right)^{1/2}. \end{aligned}$$

Taking the square and choosing $\epsilon = \|(H-\lambda)\psi\|$ we infer $f(\|(H-\lambda)\psi\|) \geq 1$. □

To relate the energy width ΔE to Fermi's Golden Rule in a perturbative situation we prove

Proposition 2.6. *Let H be self-adjoint on $\mathcal{H}$. Let ψ be a state in the domain of H, $\lambda \in \mathbb{R}$, P the orthogonal projector on ψ, $\lambda \in \mathbb{R}$, $R^{\perp}(z) = \left(H^{\perp} - z\right)^{-1}$ the resolvent reduced to the subspace* $\mathrm{Ran}P^{\perp}$.

If $\Delta E(H, \psi, \lambda)$ defined in (6) is non-zero, then it solves the equation

$$\Delta E = \left|\left\langle \psi, \left(H - \lambda - HP^{\perp}R^{\perp}(\lambda + i\Delta E)P^{\perp}H\right)\psi\right\rangle\right|. \tag{8}$$

Proof. Employ "Feshbach type" perturbation theory, that is consider H as perturbation of $\mathbf{D}H := PHP + P^{\perp}HP^{\perp}$, see, for example, [14]. Denote $R_{\mathbf{D}}(z) := (\mathbf{D}H - z)^{-1}$. We have for $PHP^{\perp}$ small enough:

$$\begin{aligned} PR(z)P &= \left(P(H-z)P - PHP^{\perp}R_{\mathbf{D}}(z)P^{\perp}HP\right)^{-1} \quad \text{and thus} \\ \langle \psi, R(z)\psi\rangle^{-1} &= \langle \psi, (H-z)\psi\rangle - F(z) \end{aligned} \tag{9}$$

with

$$F(z) := \left\langle \psi, HP^{\perp}\left(P^{\perp}HP^{\perp} - z\right)^{-1}P^{\perp}H\psi\right\rangle.$$

Denoting $\langle H\rangle := \langle \psi, H\psi\rangle$ it follows that

$$\begin{aligned} &2\Delta E\,\mathrm{Im}\langle \psi, R(\lambda + i\Delta E)\psi\rangle = 1 \\ &\Longleftrightarrow 1 = 2\Delta E \frac{\Delta E + \mathrm{Im}F(\lambda + i\Delta E)}{\left(\langle H\rangle - \lambda - \mathrm{Re}F(\lambda + i\Delta E)\right)^2 + \left(\Delta E + \mathrm{Im}F(\lambda + i\Delta E)\right)^2} \\ &\Longleftrightarrow \Delta E^2 = \left(\langle H\rangle - \lambda - \mathrm{Re}F\right)^2 + \mathrm{Im}F^2 = \left|\langle H\rangle - \lambda - F\right|^2. \end{aligned}$$

□

3. Quadratic Lower Bound for a Perturbed Embedded Eigenvalue

We now prove the lower bound on the sojourn time $\mathcal{T}(H, \psi)$ stated in Theorem 1.1. We make use of a limiting absorption principle and continuity properties of the boundary values of the resolvent on states orthogonal to the unperturbed bound state which are known to hold under hypothesis (**A**), see [4,15,22]; the general idea is to show that a strong Mourre estimate holds for the reduced operator $H^{\perp}$ and then to apply the classical results in the spirit of [16]. We first recall the meaning of the commutators $\mathrm{ad}_A^k(H_0)$ in hypothesis (**A**); this construction, originally developed in [16,23] starts from the sesquilinear form $i(H_0u, Av) - i(Au, H_0v)$ which is well defined for all vectors u, v in the domain $\mathcal{D}(H_0) \cap \mathcal{D}(A)$. We quote the efficient resumé of [4] made in the paragraph following their equation (4): If this form has a bound

$$|i(H_0u, Av) - i(Au, H_0v)| \leqslant C\|u\|\|(H_0 + i)v\|, \tag{10}$$

it extends to the sesquilinear form of a unique self-adjoint operator called $i[H_0, A] = i\mathrm{ad}_A^1(H_0)$ which is bounded relative to H_0. Therefore, the second-order commutator $i[i[H_0, A], A] = -\mathrm{ad}_A^2(H_0)$ is defined as well if the bound (10) with $i[H_0, A]$ instead of H_0 on the left hand side is assumed. For $k \geq 3$ the k-th order commutator, denoted by $\mathrm{ad}_A^k(H_0)$, is then defined recursively in terms of $\mathrm{ad}_A^{k-1}(H_0)$ and A.

Recall the definition of the weighted reduced resolvent (2). We have

Theorem 3.1. *Assume* (**A**) *and let E_0 be a simple eigenvalue of H_0 with eigenprojector P. Then there exists an interval $I \ni E_0$ and a $\kappa_0 > 0$ such that $\forall \kappa, |\kappa| < \kappa_0$, $\forall \lambda \in I$ the norm limit*

$$F(\kappa, \lambda + i0) = \lim_{\epsilon \searrow 0} PV_\kappa P^\perp \left((H_0 + \kappa V_\kappa)^\perp - (\lambda + i\epsilon)\right)^{-1} P^\perp V_\kappa P$$

exists and is bounded uniformly in $|\kappa| < \kappa_0$ and $\lambda \in I$; furthermore for $|\kappa| < \kappa_0$, $\mathrm{Re}(z) \in I$

$$\|F(\kappa, z) - F(\kappa', z')\| \leq c \left(|\kappa - \kappa'| + |z - z'|\right)^{\frac{1}{3}}$$

Proof. The result is proven in [15], Lemma 8.11 under the assumption that $\psi \in D(A^2)$ and that two relatively bounded commutators exist. In [22] it is proven that the relative boundedness of $\mathrm{ad}_A^3(H_0)$ implies $\psi \in D(A^2)$. □

Remark 3.2. Notice that in Theorem 1.1 Assumption **(A)**.2 may be replaced by:

$\mathrm{ad}_A^2(H_0)$ is H_0 bounded and the eigenvector ψ belongs to $D(A^2)$.

Note that [22] constructed an example with two commutators relatively bounded and $\psi \notin D(A^2)$.

We now finish the proof of our main result.

Proof of Theorem (1.1). Suppose $\Delta E > 0$, otherwise $\mathcal{T}$ is infinite, see Remark 2.4.1. Denote the resolvent with respect to $H_0 + \kappa V_\kappa$ by R_κ and $\langle A \rangle_\varphi := \langle \varphi, A\varphi \rangle$. We take account of the Lamb shift, meaning that in Eq. (8) we choose

$$\lambda = \lambda_2(\kappa) := E_0 + \kappa \langle V_\kappa \rangle_\psi - \kappa^2 \mathrm{Re} \left\langle R_0^\perp \left(E_0 + i0\right) \right\rangle_{P^\perp V_\kappa P \psi}.$$

Thus,

$$\Delta E(\lambda_2(\kappa)) = \kappa^2 \Big| \mathrm{Re} \left\langle R_\kappa^\perp \left(\lambda_2 + i\Delta E\right) - R_0^\perp \left(E_0 + i0\right) \right\rangle_{P^\perp V_\kappa P \psi} + i \mathrm{Im} \left\langle R^\perp \left(\lambda_2 + i\Delta E\right) \right\rangle_{P^\perp V_\kappa P \psi} \Big|.$$

From Theorem 3.1 we conclude first that $\Delta E(\lambda_2(\kappa)) = \mathcal{O}(\kappa^2)$ and second that

$$\left| \Delta E(\lambda_2(\kappa)) - \kappa^2 \, \mathrm{Im} \left\langle R_0^\perp \left(E_0 + i0\right) \right\rangle_{P^\perp V_\kappa P \psi} \right| \leq c\kappa^2 \left(\kappa + \Delta E + |\lambda_2 - E_0|\right)^{\frac{1}{3}} = \mathcal{O}\left(\kappa^{\frac{7}{3}}\right).$$

□

Remark 3.3. It is crucial that the a priori estimate (7) is valid for any parameter λ. So we can choose $\lambda_2(\kappa)$ above and thus implement the intuition provided by other theories. For example it was shown in [4], Theorem 1.2, that in the exponential decay law (3) it holds $\mathrm{Re}(\lambda(\kappa)) = \lambda_2(\kappa) + o(\kappa^2)$. The Lamb shift is determined by the perturbations $\mathrm{Re}(E(\kappa)) - E_0$ where E_0 stand for a degenerate eigenvalue of the hydrogen atom and $E(\kappa)$ for the resonance induced from E_0 by the interaction of the atom with a quantized electromagnetic field. By calling the real quadratic correction to E_0 "Lamb shift" in the proof of the theorem above we refer to an important achievement of mathematical physics: the proof that, for a suitable model, the second-order contribution to the Lamb shift is determined by $\kappa^2 \mathrm{Re} F(0, E_0 + i0)$, see [3], Theorem I.3.

Remark 3.4. The proof actually implies the bound $\mathcal{O}(\kappa^{7/3})$ for the error; only $o(\kappa^2)$ is stated in Theorem 1.1. Remark that the continuity of the function $(\kappa, \lambda) \mapsto F(\kappa, \lambda + i0)$ is sufficient to proof of Theorem 1.1. So instead of **(A)** we could have assumed this property; we opted for hypothesis (**A**) because it is simple and explicit. Remark, however, that the (Hölder-) continuity of F, can be inferred by other methods, for example, from singular Mourre theory, see [7], Theorem 3.3. The above mentioned works on exponential decay rules, [4,6,21] assume higher order differentiability.

4. Time-Periodic Perturbations

One feature of the simplicity of our theory is that applies immediately to time-periodic perturbations of Schrödinger operators via Floquet theory. A particular special case is the two body AC Stark effect which is maybe the most simple physically relevant system to which our theory applies. Location and exponential decay for resonances were analyzed in detail by Yajima and Graffi [11,27] using complex deformation methods and by Møller and Skibstedt [21] in great generality using Mourre techniques. Here we aim only at lower bounds on the sojourn time in the two body case for smooth potentials which were not discussed before.

Consider for $t \in \mathbb{R}$

$$H_0 = -\frac{1}{2}\Delta + W \quad \text{and} \quad H(t) = H_0 + \kappa V_\kappa(t) \quad \text{on } H^2(\mathbb{R}^d);$$

We assume that $T-$periodic family $V_\kappa(t)$ and W as a constant function of κ, t satisfy:

Hypothesis (*AT*). Let $(\kappa, t, x) \mapsto g_\kappa(t, x) \in C^\infty([0,1) \times \mathbb{R}/(T\mathbb{Z}) \times \mathbb{R}^d; \mathbb{R})$ be such that for a $\delta > 0$ and all $\alpha \in \mathbb{N}_0^d$

$$\sup_{\kappa \in [0,1), t \in [0,T]} \sup_{x \in \mathbb{R}^d} \|\langle x \rangle^{\delta + \alpha} \partial_x^\alpha g_\kappa(t, x)\| < \infty.$$

With $\omega := \frac{2\pi}{T}$ we denote by

$$\widehat{V_\kappa}(n) = \frac{1}{T} \int_0^T \exp(-\mathrm{in}\omega t) V_\kappa(t)\, \mathrm{d}t$$

the multiplication operator in $L^2(\mathbb{R}^d)$ by the $n-$th Fourier coefficient of V_κ. Remark that $\widehat{V_\kappa}(n)=\overline{\widehat{V_\kappa}(-n)}$.

Under assumption (AT) the propagator U generated by $H(\cdot)$ is well defined; the sojourn time depends on the initial time t_0

$$\mathcal{T}(H(\cdot),\psi,t_0)=\int_{-\infty}^{\infty}|\langle\psi,U(t,t_0)\psi\rangle|^2\ \mathrm{d}t.$$

We now prove a lower bound for its average over t_0:

Theorem 4.1. *Suppose that W and V_κ satisfy assumption (AT). Let $H_0=-\frac{1}{2}\Delta+W$ on $H^2(\mathbb{R}^d)$. Let ψ be a simple eigenstate of H_0 with eigenvalue $E_0\in I\subset\mathbb{R}\backslash\omega\mathbb{Z}$ which is non-resonant, i.e.,: $E_0+\omega\mathbb{Z}\bigcap\sigma_{pp}(H_0)=\{E_0\}$. Then it holds for the lifetime of ψ with respect to $H(\cdot)$ averaged over the initial time t_0:*

$$\frac{1}{T}\int_0^T\mathcal{T}(H(\cdot),\psi,t_0)\ \mathrm{d}t_0\geq\frac{1}{\Delta E}$$

where the energy width ΔE has the property

$$\Delta E=\kappa^2\sum_{n\in\mathbb{Z}}\mathrm{Im}\left\langle\psi,\widehat{V_0}(-n)\widetilde{R}_0\left(E_0+n+i0\right)\widehat{V_0}(n)\psi\right\rangle+o(\kappa^{7/3}).$$

Here $\widetilde{R}_0\left(E_0+n+i0\right)$ is the limit of the resolvent for H_0 for $n\neq0$ and the reduced resolvent for $n=0$, $\widehat{V_0}(n):=\frac{1}{T}\int_0^T e^{\mathrm{i}n\omega t}V_0(t)\,\mathrm{d}t$.

To prove Theorem (4.1) we use the space-time picture advocated by Howland, see [13], and apply our theory to the Floquet Hamiltonian

$$K:=K_0+\kappa V_\kappa(t),\quad K_0:=-i\partial_t-\Delta+W\quad\text{ on }\ H^1(\mathbb{T})\otimes H^2(\mathbb{R}^d)$$

$\mathbb{T}:=\mathbb{R}/(T\mathbb{Z})$. It is known, [13] formula 1.6, that for $\boldsymbol{\varphi}\in L^2(\mathbb{T})\otimes L^2(\mathbb{R}^d)$:

$$\left(e^{-iKs}\boldsymbol{\varphi}\right)(t+s)=U(t+s,t)\boldsymbol{\varphi}(t).\tag{11}$$

Denote for $n\in\mathbb{Z}$ the harmonics $h_n(t):=\frac{1}{\sqrt{T}}\exp\left(in\omega t\right)$ and $P_n:L^2(\mathbb{T})\to L^2(\mathbb{T})$ the projector on span$\{h_n\}$. $-i\partial_t$ and H_0 commute so it holds for $z\in\mathbb{C}\backslash\mathbb{Z}$ strongly:

$$\left(K_0-z\right)^{-1}=\sum_{n\in\mathbb{Z}}P_n\otimes\left(H_0+n-z\right)^{-1}.\tag{12}$$

Proof. (of Theorem (4.1)). In [28], Theorem 1.3 and Lemma 2.4, it was proved that (**AT**) implies self-adjointness of K_0,K and that (**A**) holds in a neighborhood I of E_0 away from $\omega\mathbb{Z}$ for the quadruple $K_0,\mathbb{V}_\kappa,\mathbb{A},I$ with $\mathbb{V}_\kappa$ the operator valued multiplication by $V_\kappa(t)$, $\mathbb{A}:=\mathbb{I}\otimes\frac{1}{2}(xL_D+L_Dx)$ and $L_D:=D(D^2+1)^{-1},D:=-i\nabla$.

The non-resonance condition ensures that E_0 is a simple eigenvalue of K_0 with eigenstate $h_0\otimes\psi$. Application of Theorem (1.1) to the operator K yields

$$\mathcal{T}(K,h_0\otimes\psi)\geq\frac{1}{\Delta E}$$

with

$$\Delta E = \kappa^2 \mathrm{Im} \left\langle h_0 \otimes \psi, \mathbb{V}_0 \mathbb{P}^\perp \left(K_0^\perp - E_0 + i0\right)^{-1} \mathbb{P}^\perp \mathbb{V}_0 h_0 \otimes \psi \right\rangle_{L^2(\mathbb{T}) \otimes L^2(\mathbb{R}^d)} + o(\kappa^2)$$

for $\mathbb{P} = |h_0 \otimes \psi\rangle \langle h_0 \otimes \psi|$. Now $\mathbb{P}^\perp = P_0^\perp \otimes \mathbb{I} + P_0 \otimes P_\psi^\perp$ so by (12):

$$\left(K_0^\perp - z\right)^{-1} = \sum_{n \neq 0} P_n \otimes (H_0 + n - z)^{-1} + P_0 \otimes \left(H_0^\perp - z\right)^{-1}.$$

Furthermore $P_0 \otimes P_\psi \mathbb{V}_0 = \sum_m |h_0\rangle \langle h_m| \otimes P_\psi \widehat{V_0}(0-m)$ which implies

$$\Delta E = \kappa^2 \sum_{n \in \mathbb{Z}} \mathrm{Im} \left\langle \psi, \widehat{V_0}(-n) \widetilde{R}_0 \left(E_0 + n + i0\right) \widehat{V_0}(n) \psi \right\rangle + o(\kappa^{7/3}).$$

For the case $\frac{1}{T} \int_0^T \mathcal{T}(H, \psi, t_0)\ \mathrm{d}t_0 < \infty$ the result now follows from Jensen's inequality and Fubini's Theorem:

$$\begin{aligned} \mathcal{T}(K, h_0 \otimes \psi) &= \int_{\mathbb{R}} \left| \frac{1}{T} \int_0^T \langle \psi, U(t_0 + s, t_0) \psi \rangle\ \mathrm{d}t_0 \right|^2 ds \\ &\leq \int_{\mathbb{R}} \frac{1}{T} \int_0^T |\langle \psi, U(t_0 + s, t_0) \psi \rangle|^2\ \mathrm{d}t_0\ ds \\ &= \frac{1}{T} \int_0^T \mathcal{T}(H(\cdot), \psi, t_0)\ \mathrm{d}t_0 \end{aligned}$$

□

We now apply this general estimate to the AC Stark effect and obtain

Theorem 4.2. *Let $W \in C^\infty(\mathbb{R}^d, \mathbb{R})$ such that for a $\delta > 0$ and $\alpha \in \mathbb{N}_0^d$ $\sup_x \|\langle x \rangle^{\delta + \alpha} \partial_x^\alpha W(x)\| < \infty$. Let $F \in C^\infty([0,T]; \mathbb{R}^d)$, be a T-periodic function with zero mean and Fourier series $F(t) = \sum_{n \neq 0} F_n e^{in\omega t}$.*

Let $q \in C^\infty(\mathbb{R}; \mathbb{R}^d)$, $q(t) = \sum_{n \neq 0} \frac{F_n}{(\mathrm{in}\omega)^2} e^{in\omega t}$ which is a T periodic function such that $\ddot{q} = F$.

Let $E_0 \in \mathbb{R} \backslash \omega\mathbb{Z}$ be a simple eigenvalue of $H_0 := -\frac{1}{2}\Delta + W(x)$ on $H^2(\mathbb{R}^d)$ with eigenvector ψ such that $E_0 + \omega\mathbb{Z} \bigcap \sigma_{pp}(H_0) = \{E_0\}$.

For the sojourn time of ψ with respect to the propagator generated by

$$H(t) := -\frac{1}{2}\Delta + W(x + \kappa q(t)) \quad \text{on } H^2(\mathbb{R}^d), \quad \kappa \in [0,1), t \in \mathbb{R}:$$

it holds:

$$\frac{1}{T} \int_0^T \mathcal{T}(H, \psi, t_0)\, \mathrm{d}t_0 \geq \frac{1}{\Delta E}$$

where the energy width ΔE satisfies,

$$\Delta E = \kappa^2 \sum_{n \in \mathbb{Z}} \mathrm{Im} \sum_{j,k=1}^d \frac{\overline{F_n^j} F_n^k}{\omega^4 n^4} \left\langle \psi, \partial_j W \widetilde{R}_0 (E_0 + n + i0) \partial_k W \psi \right\rangle + o(\kappa^{7/3}) \quad (13)$$

and $\widetilde{R}_0(E_0+n+i0)$ *denotes the limit of the resolvent of* H_0 *if* $n \neq 0$ *and the reduced resolvent if* $n=0$.

Remark 4.3. $H(\cdot)$ is the Hamiltonian of an electron in the potential W and the homogeneous electric field of strength $\kappa F(\cdot)$ in the frame of a freely falling observer. To see this, consider the unitary family of phase space translation operators $S(t)$ on $L^2(\mathbb{R}^d)$

$$S(t) = e^{-i(q(t)D-p(t)x)}e^{i\varphi(t)}$$

with T periodic functions q,p such that $\dot{p}=\kappa F$ and $\dot{q}=p$ and $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\dot{\varphi} = \frac{\dot{q}p-\dot{p}q}{2} - \left(\frac{p^2}{2} - \kappa F q\right)$. Now from the identities

$$S^{-1}xS = x+q, S^{-1}DS = D+p$$

$$i\partial_t e^{-i(q(t)D-p(t)x)}\psi = \left((\dot{q}D-\dot{p}x) + \frac{1}{2}(\dot{p}q-\dot{q}p)\right)e^{-i(q(t)D-p(t)x)}\psi$$

and $\psi \in \mathcal{S}(\mathbb{R}^d)$ it follows that

$$S^{-1}\left(-i\partial_t + \frac{D^2}{2} - \kappa F x + W(x)\right)S\psi = \left(-i\partial_t + \frac{D^2}{2} + W(x+\kappa q(t))\right)\psi.$$

Thus, if $\psi(t)$ solves the Schrödinger equation for $\frac{D^2}{2} + W(x+\kappa q(t))$ then $S(t)\psi(t)$ solves the Schrödinger equation for $\frac{D^2}{2} - \kappa F x + W(x)$ in the laboratory frame.

Proof. (of Theorem (4.2)). Define $V_\kappa(t)$ by

$$\kappa V_\kappa(t,x) := W(x+\kappa q(t)) - W(x).$$

Then V_κ satisfies Hypotheses (AT) and the sojourn time estimate stated in Theorem 4.1 holds. Now $V_0(t,x) = (\partial_\kappa W)(x) = \sum_{j=1}^d q_j(t)\partial_j W(x)$ and

$$\widehat{V_0}(x) = \sum_j \frac{F_n^j}{(\mathrm{in}\omega)^2}\partial_j W(x)$$

from which the formula for ΔE follows. □

5. Sojourn Time of Multistate Systems

Systems with inner degrees of freedom appear in various physical situations. We just mention the Dirac equation and effective Hamiltonians which appear in the Born Oppenheimer approximation, see [5,12,31]. Here we are interested in situations where one channel is binding and the others are propagating; this occurs, for example, in molecular predissociation, [10,17].

Theorem 5.1. *Consider self-adjoint operators two Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *and in* $\mathcal{H}_1 \oplus \mathcal{H}_2$

$$H := \begin{pmatrix} H_1 & \kappa V_\kappa \\ \kappa V_\kappa^* & H_2 \end{pmatrix}$$

such that:

H_1 is self-adjoint in $\mathcal{H}_1$, E_0 is a simple eigenvalue of H_1 with normalized eigenvector ψ_0 and the resolvent if H_1 is compact in a punctured complex neighborhood of E_0;

H_2 is self-adjoint in $\mathcal{H}_2$, there exists a self-adjoint A in $\mathcal{H}_2$ and an interval I around E_0 such that $e^{-isA}D(H_2) \subset D(H_2)$ for $s \in \mathbb{R}$ and $\mathrm{ad}_A^k(H_2)$ is H_2-bounded for $k \in \{1,2\}$ and such that for a positive number c it holds

$$\chi(H_2 \in I)i[H_2, A]\chi(H_2 \in I) \geq c\chi(H_2 \in I);$$

the values of $[0,1) \ni \kappa \to V_\kappa$ are operators from $\mathcal{H}_2$ to $\mathcal{H}_1$ such that

$$\kappa \mapsto V_\kappa \left(H_2 + i\right)^{-1} \quad \text{and} \quad \kappa \mapsto V_\kappa^* \left(H_1 + i\right)^{-1}$$

are norm differentiable and such that

$$A^k V_\kappa^* \left(H_1 + i\right)^{-1} \quad \text{and} \quad V_\kappa A^k \left(H_2 + i\right)^{-1}$$

extend to bounded operators for $k \in \{1,2\}$. Then it holds for H:

$$\mathcal{T}(H, \psi_0 \oplus 0) \geq \frac{1}{\Delta E}$$

where the energy width ΔE has the property

$$\Delta E = \kappa^2 \,\mathrm{Im}\left\langle \psi_0, V_0 \left(H_2 - (E_0 + n) + i0\right)^{-1} V_0^* \psi_0 \right\rangle + o(\kappa^{7/3}). \tag{14}$$

Proof. (of Theorem (5.1)). Define

$$H_0 := H_1 \oplus H_2, \quad \mathbb{A} := 0 \oplus A, \quad \mathbb{V}_\kappa := \begin{pmatrix} 0 & V_\kappa \\ V_\kappa^* & 0 \end{pmatrix}.$$

We show that the quadruple $H_0, \mathbb{V}, \mathbb{A}, I$ satisfies assumption (**A**) and apply Theorem 1.1. Indeed:

$e^{-is\mathbb{A}} = \mathbb{I} \oplus e^{-isA}$ leaves $D(\mathbb{H}_0)$ invariant by the assumption on A; also $\mathrm{ad}_{\mathbb{A}}^k(\mathbb{H}_0) = 0 \oplus \mathrm{ad}_A^k(H_2)$ are $\mathbb{H}_0$ bounded.

$$\begin{aligned} \chi(\mathbb{H}_0 \in I)i[\mathbb{A}, \mathbb{H}_0]\chi(\mathbb{H}_0 \in I) &= 0 \oplus \chi(H_2 \in I)i[A, H_2]\chi(H_2 \in I) \\ &\geq c0 \oplus \chi(H_2 \in I) = c\chi(\mathbb{H}_0 \in I) - c\chi(H_1 \in I) \oplus 0 \end{aligned}$$

by the Mourre estimate for H_2, the second term is compact, upon shrinking I if necessary, because the resolvent of H_1 near E_0 is compact. Furthermore for $k \in \{0,1,2\}$

$$\mathrm{ad}_{\mathbb{A}}^k(\mathbb{V}_\kappa) = \begin{pmatrix} 0 & (-1)^k V_\kappa A^k \\ A^k V_\kappa^* & 0 \end{pmatrix}$$

is relatively $\mathbb{H}_0$ bounded. Remark that $\mathbb{A}^k$ acts on the $\mathcal{H}_2$ component so $\psi_0 \oplus 0$ is in $D(\mathbb{A}^k)$ and we have proven that the assumption (**A**) is satisfied. Now the claim follow from Theorem 1.1 with the observation

$$P^\perp = \begin{pmatrix} \mathbb{I} - |\psi\rangle\langle\psi| & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad P^\perp \mathbb{V}_\kappa P = \begin{pmatrix} 0 & 0 \\ |V_\kappa^*\psi\rangle\langle\psi| & 0 \end{pmatrix}.$$

□

Acknowledgements

We gratefully acknowledge support from the Grants Fondecyt Grants 1120786 and 1141120; Anillo Conicyt PIA-ACT1112, Ecos-Conicyt C10E01, EPLANET. OB, VC, CF thank CPT and JA thanks Facultad de Matemáticas of PUC for hospitality.

References

[1] Aguilar, J., Combes, J.M.: A class of analytic perturbations for one-body Schrödinger Hamiltonians. Comm. Math. Phys. **22**, 269–279 (1971)

[2] Asch, J., Briet, P.: Lower bounds on the width of Stark–Wannier type resonances. Comm. Math. Phys. **179**(3), 725–735 (1996)

[3] Bach, V., Fröhlich, J., Sigal, I.M.: Quantum electrodynamics of confined nonrelativistic particles. Adv. Math. **137**(2), 299–395 (1998)

[4] Cattaneo, L., Graf, G.M., Hunziker, W.: A general resonance theory based on Mourre's inequality. Ann. Henri Poincaré **7**, 583–601 (2006)

[5] Combes, J.M., Seiler, R. : The Born Oppenheimer approximation. In: Velo, G., Wightman, A. (eds.) Rigorous Atomic and Molecular Physics, pp. 185–212. Plenum Press, New York (1981)

[6] Cornean, H.D., Jensen, A., Nenciu, G.: Metastable States When the Fermi Golden Rule Constant Vanishes. Commun. Math. Phys. **334**(3), 1189–1218 (2014)

[7] Faupin, J., Møller, J.S., Skibsted, E.: Second order perturbation theory for embedded eigenvalues. Commun. Math. Phys. **306**(1), 193–228 (2011)

[8] Fernandez, C., Lavine, R.: Lower bounds for resonance widths in potential and obstacle scattering. Comm. Math. Phys. **128**(2), 263–284 (1990)

[9] Gislason, E.A., Sabelli, N.H., Wood, J.W.: New form of the time-energy uncertainty relation. Phys. Rev. A **31**(4), 2078–2081 (1985)

[10] Grigis, A., Martinez, A.: Resonance widths for the molecular predissociation. Anal. PDE **7**(5), 1027–1055 (2014)

[11] Graffi, S., Yajima, K.: Exterior complex scaling and the AC-Stark effect in a Coulomb field. Commun. Math. Phys. **89**(2), 277–301 (1983)

[12] Hagedorn, J., Joye, A.: Mathematical analysis of Born–Oppenheimer approximations. In: Gesztesy, F., Deift, P., Galvez, C., Perry, P., Schlag, W. (eds.) Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. AMS Proceedings of Symposia in Pure mathematics 76, pp. 203–226 (2007)

[13] Howland, J.: Stationary scattering theory for time-dependent Hamiltonian. Math. Ann. **207**, 315–335 (1974)

[14] Howland, J.: The Livsic matrix in perturbation theory. J. Math. Anal. Appl. **50**, 415–437 (1975)

[15] Hunziker, W., Sigal, I.M.: The quantum N-body problem. J. Math. Phys. **41**, 3448–3510 (2000)

[16] Jensen, A., Mourre, E., Perry, P.: Multiple commutator estimates and resolvent smoothness in quantum scattering theory. Annales de l'Institut Henri Poincaré. Section A. Physique Théorique. Nouvelle Série **41**(2), 207–225 (1984)

[17] Klein, M.: On the mathematical theory of predissociation. Ann. Phys. **178**(1), 48–73 (1987)

[18] Könenberg, M., Merkli, M.: On the irreversible dynamics emerging from quantum resonances. http://arxiv.org/pdf/1503.02972 (2015)

[19] Lavine, R.: Spectral density and sojourn times, atomic scattering theory. J. Nutall, ed., University of Western Ontario, London (1978)

[20] Merkli, M., Sigal, I.M.: A time-dependent theory of quantum resonances. Commun. Math. Phys. **201**(3), 549–576 (1999)

[21] Møller, J.S., Skibsted, E.: Spectral theory of time-periodic many-body systems. Adv. Math. **188**(1), 137–221 (2004)

[22] Møller, J.S., Westrich, M.: Regularity of eigenstates in regular Mourre theory. J. Funct. Anal. **260**(3), 852–878 (2011)

[23] Mourre, E.: Absence of singular continuous spectrum for certain selfadjoint operators. Commun. Math. Phys. **78**(3), 391–408 (1981)

[24] Orth, A.: Quantum mechanical resonance and limiting absorption: the many body problem. Commun. Math. Phys. **126**, 559–573 (1990)

[25] Pfeifer, P., Fröhlich, J.: Generalized time-energy uncertainty relations and bounds on lifetimes of resonances. Rev. Modern Phys. **67**(4), 759–779 (1995)

[26] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Vol. 1–4. Academic Press, New York (1975–1979)

[27] Yajima, K.: Resonances for the AC-Stark effect. Commun. Math. Phys. **87**(3), 331–352 (1982)

[28] Yokoyama, K.: Mourre theory for time-periodic systems. Nagoya Math. J. **149**, 193–210 (1998)

[29] Simon, B.: Resonances and complex scaling: a rigorous overview. Int. J. Quantum Chem. **14**(4), 529–542 (1978)

[30] Skibsted, E.: Truncated Gamow functions, α–decay and the exponential Law. Commun. Math. Phys. **104**, 591–604 (1986)

[31] Teufel, S.: Adiabatic perturbation theory in quantum dynamics. Lecture Notes in Mathematics, 1821. Springer, Berlin (2003)

Joachim Asch
Université de Toulon
CNRS, CPT UMR 7332, CS 60584
83041, Toulon cedex 9, France
e-mail: Joachim.Asch@cpt.univ-mrs.fr

and

Aix Marseille Université, CNRS, CPT UMR 7332
13288, Marseille cedex 9, France

Olivier Bourget, Victor Cortés and Claudio Fernandez
Departamento de Matemáticas
Pontificia Universidad Católica de Chile
Av. Vicuña Mackenna 4860
C.P. 690 44 11
Macul, Santiago, Chile
e-mail: bourget@mat.puc.cl;
vcortes@mat.puc.cl;
cfernand@mat.puc.cl

Communicated by Claude Alain Pillet.
Received: July 22, 2015.
Accepted: January 10, 2016.