



Aharonov–Bohm Effect and High-Momenta Inverse Scattering for the Klein–Gordon Equation

Miguel Ballesteros and Ricardo Weder

Abstract. We analyze spin-0 relativistic scattering of charged particles propagating in the exterior, $\Lambda \subset \mathbb{R}^3$, of a compact obstacle $K \subset \mathbb{R}^3$. The connected components of the obstacle are handlebodies. The particles interact with an electromagnetic field in Λ and an inaccessible magnetic field localized in the interior of the obstacle (through the Aharonov–Bohm effect). We obtain high-momenta estimates, with error bounds, for the scattering operator that we use to recover physical information: we give a reconstruction method for the electric potential and the exterior magnetic field and prove that, if the electric potential vanishes, circulations of the magnetic potential around handles (or equivalently, by Stokes’ theorem, magnetic fluxes over transverse sections of handles) of the obstacle can be recovered, modulo 2π . We additionally give a simple formula for the high-momenta limit of the scattering operator in terms of certain magnetic fluxes, in the absence of electric potential. If the electric potential does not vanish, the magnetic fluxes on the handles above referred can be only recovered modulo π and the simple expression of the high-momenta limit of the scattering operator does not hold true.

1. Introduction

We analyze spin-0 relativistic scattering of charged particles propagating in the exterior, $\Lambda \subset \mathbb{R}^3$, of a compact obstacle $K \subset \mathbb{R}^3$. The connected components of the obstacle are handlebodies. In particular, they can be the union of a finite number of bodies diffeomorphic to tori or to balls. Some of them can be patched through the boundary. We assume that the particle interacts with a short-range magnetic field B and a short-range electric potential A_0 , both of them defined

Research partially supported by the project PAPIIT-DGAPA UNAM IN102215.
Miguel Ballesteros and Ricardo Weder: Fellows of the Sistema Nacional de Investigadores.

in Λ . The obstacle is shielded and contains an inaccessible magnetic field. The only information, from the magnetic field inside the obstacle, we may have access to is through circulations of the magnetic potential around the handles of the obstacle. The aim of this paper is proving that the electromagnetic field can be reconstructed from the high-momenta limit of the scattering operator, as well as some information from the circulations of the magnetic potentials around the handles of the obstacle. The latter is the Aharonov–Bohm effect [2, 15, 16], a purely quantum phenomenon.

There are many related results in the literature dealing with a similar setting (obstacle magnetic-scattering and Aharonov–Bohm effect in three dimensions) in the non-relativistic case, see [4–6, 12, 13], and the references quoted there. In this paper, we analyze the relativistic case, which is physically and mathematically relevant because all the previous works referred above use a high-velocity limit for their reconstruction formulas. It is, thus, evident the necessity to take into consideration special-relativity if high energies are to be addressed. Regarding the Aharonov–Bohm effect (if the magnetic field vanishes in Λ), we actually find important differences and similarities between the non-relativistic and the relativistic cases: in the non-relativistic case the leading order of the scattering operator as the velocity v goes to infinity contains only the magnetic potential, and the contribution of the electric potential appears in the next order term that is of order $1/v$. However, in the relativistic case, the magnetic and the electric potentials appear in the leading term as the momentum goes to infinity. This means that, in contrast to the non-relativistic case, in the relativistic model the electric and the magnetic potentials have the same order of contribution, which produces some differences in the information one can recover from high-momenta scattering, between both cases, concerning the magnetic potential. Actually, if the electric potential vanishes, we prove that what can be recovered from scattering is pretty much the same in both cases (namely, fluxes of the magnetic potential, modulo 2π , around the handles) and we find a similar (very simple) expression for the high-momenta limit of the scattering operator, in terms of certain magnetic fluxes. This, however, is not valid anymore if the electric potential does not vanish. In this circumstance, in the relativistic case, the leading term of the high-momenta limit of the scattering operator depends non-trivially on the electric potential and the fluxes around the handles can be recovered only modulo π , while in the non-relativistic case having a non-vanishing electric potentials does not change the matter.

In our work we, study inverse scattering for the Klein–Gordon equation in the case that the sesquilinear form associated with the classical field energy is positive definite. The direct scattering problem in case where it is not positive definite is studied in [18].

Our main results are presented in Sect. 2.2. Specifically they are stated in Theorems 2.7, 2.9, 2.11 and 2.12. Theorem 2.7 gives the high-momenta limit of the scattering operator. It is the main input from which we recover information from the scattering operator. This is the most laborious result and the core of our proof. The proof of it is presented in Sect. 3.4.2, which is based in the

results of Sect. 3.4.1. As a matter of fact, Sects. 3.4 and 3.7 are devoted to the proof of Theorem 2.7. In Sect. 3.4 we give the main arguments, while some technical results are deferred to Sect. 3.7. In Theorem 2.9 we prove that the electric potential and the magnetic field can be recovered, in a certain region, from the high-momenta limit of the scattering operator. Theorem 2.11 gives the specific information from the magnetic potential that we can recover from the high-momenta limit of the scattering operator, namely, certain circulations of the magnetic potential around handles of the obstacle. In Theorem 2.12 we provide a very simple expression of the high-momenta limit of the scattering operator in terms of some magnetic fluxes. Theorems 2.11 and 2.12 require the electric potential to vanish, since our main interest is to present the Aharonov–Bohm effect. However, similar results are valid in the presence of a non-trivial electric potential, as it is presented in the body of Sect. 2.2 and proved in Sect. 3.6.3.

Related results, in two dimensions, are proved in the non-relativistic case in [7, 12] and [13] (where the long-range behavior of the magnetic potentials is the main issue) and in [31] and [39]. The magnetic Schrödinger equation, in the whole space, is studied in [3]. In [3] and [31] one can see that the circulation of the magnetic potential appears on the leading order term of the high velocity asymptotic of the scattering operator, whereas the contribution of the electric potential appears only on the second order term (see [31] for related issues). For relativistic equations in the whole space, see [8] and [28]. The time dependent methods for inverse scattering that we use are introduced in [9], for the Schrödinger equation. A survey about many different applications of the time dependent method for inverse scattering can be found in [42]. The direct scattering problem for the Klein–Gordon is studied in [18, 40, 41], and the references quoted there. Inverse scattering problems in classical relativistic mechanics are addressed in [25–27].

The Aharonov–Bohm effect has been studied from different perspectives. The high velocity regime in scattering theory was analyzed by Ballesteros and Weder (see [4–7]), Nicoleau [31] and Weder [39]. Semi-classical analysis was carried out by Ito and Tamura (see [22–24]). Roux [34], Roux and Yafaev ([35, 36]) and Yafaev ([43, 44]) examined the singularities on the diagonal of the scattering amplitude. The Aharonov–Bohm effect for boundary value problems was studied by Eskin (see [10, 11]). A mixture of time-dependent, stationary and boundary value techniques was used in [12, 13]. For the Aharonov–Bohm effect in spectral theory see, for example, [14] and [21].

Our paper is organized as follows: Sect. 2 presents the model and the main results. In Sect. 3 we give all details of the proofs of our results. It is divided in several subsections: Sect. 3.2 deals with the self-adjointness of the Hamiltonians; Sect. 3.3 proves the existence of the wave operators and presents some properties of the wave and scattering operators; Sect. 3.4 is devoted to the proof of Theorem 2.7, for the regular case in Sect. 3.4.1 and the general case in Sect. 3.4.2. Theorems 2.9, 2.11 and 2.12 are proved, respectively, in Sects. 3.5, 3.6.1 and 3.6.2. In Sect. 3.7 we prove some technical results that

are used in Sects. 3.3 and 3.4. Section 3.7.1 is entirely dedicated to deal with laborious and technical computations used in Sect. 3.4.

2. Description of the Model and Main Results

2.1. Description of the Model

We study the propagation of a relativistic particle outside a bounded magnet, K , in three dimensions, i.e. the particle propagates in the exterior domain $\Lambda := \mathbb{R}^3 \setminus K$. We assume that inside K there is a magnetic field that produces a magnetic flux. We suppose, furthermore, that in Λ there are an electric potential A_0 and a magnetic field B . This is a more general situation than the one of the Aharonov–Bohm effect. The obstacle K is, of course, a classical macroscopic object defined in \mathbb{R}^3 . The electromagnetic field is also a classical quantity defined in Λ , the space where the particles propagate. However, the position of the particle is a quantum quantity which is not represented by the multiplication operator by the variable $x \in \mathbb{R}^3$, where the obstacle lives. As a matter of fact, the components of this operator are not self-adjoint in the Hilbert space at stake and, therefore, they cannot represent a quantum mechanical observable. The free Hamiltonian operator H_0 that we describe below [see Definition 2.5] is diagonalized by a unitary operator U [see (2.26)] in such a way that the positive and negative energy subspaces are separated as a direct sum [see (2.27)]. Following [40,41], in the diagonal representation that we just described, we define the position operator as a multiplication operator by the variable $x \in \mathbb{R}^3$. See Sect. 2.1.3 for a discussion of these issues.

2.1.1. The Magnet K . We assume that the magnet K is a compact submanifold of \mathbb{R}^3 . Moreover, $K = \cup_{j=1}^L K_j$, where the sets $K_j, 1 \leq j \leq L$, are the connected components of K . We suppose that the K_j 's are handle bodies. For a precise definition of handlebodies see [4], where we study in detail the homology and the cohomology of K and Λ . In intuitive terms, K is the union of a finite number of bodies diffeomorphic to tori or to balls. Some of them can be patched through the boundary. See Fig. 1.

2.1.2. The Magnetic Field and the Electric Potential. In the following assumptions, we summarize the conditions on the magnetic field and the electric potential that we use. We denote by Δ the self-adjoint realization of the Laplacian in $L^2(\mathbb{R}^3)$ with domain $\mathbf{H}^2(\mathbb{R}^3)$, the Sobolev space of function with distributional derivatives up to order 2 square integrable.

Assumption 2.1. We assume that the magnetic field, B , is a real-valued, bounded 2-form in $\overline{\Lambda}$, that is two times continuously differentiable in $\overline{\Lambda}$, and, furthermore,

1. B is closed : $dB|_{\Lambda} \equiv \text{div } B|_{\Lambda} = 0$.
2. There are no magnetic monopoles in K :

$$\int_{\partial K_j} B = 0, \quad j \in \{1, 2, \dots, L\}. \tag{2.1}$$

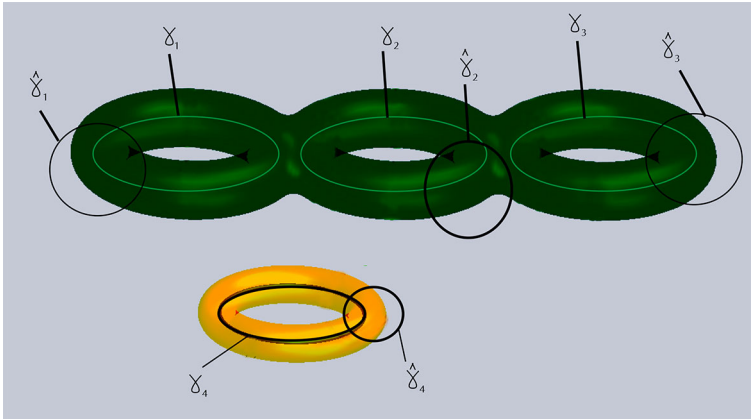


FIGURE 1. The magnet $K = \cup_{j=1}^L K_j \subset \mathbb{R}^3$ where K_j are handlebodies, for every $j \in \{1, \dots, L\}$. The exterior domain, $\Lambda := \mathbb{R}^3 \setminus K$. The curves $\gamma_k, k = 1, 2, \dots, m$, are a basis of the first singular homology group of K and the curves $\hat{\gamma}_k, k = 1, 2, \dots, m$, are a basis of the first singular homology group of Λ

3.

$$\left| \left(\frac{\partial}{\partial x_1} \right)^a \left(\frac{\partial}{\partial x_2} \right)^b \left(\frac{\partial}{\partial x_3} \right)^c B(x) \right| \leq C(1 + |x|)^{-\mu},$$

for some $\mu > 2$, and every $a, b, c \in \{0, 1, 2\}$ with $a + b + c \leq 2$, (2.2)

where C is a general, not specified, constant.

4. The electric potential, A_0 , is a real-valued function defined in Λ . We suppose that for some $\varepsilon > 0$

$$\langle A_0^2 \phi, \phi \rangle \leq \langle \mathbf{p}\phi, \mathbf{p}\phi \rangle + (m^2 - \varepsilon)\langle \phi, \phi \rangle, \tag{2.3}$$

where $\mathbf{p} := -i\nabla$ is the momentum operator, for every $\phi \in \mathbf{H}_0^1(\Lambda)$, the closure of $C_0^\infty(\Lambda)$ in $\mathbf{H}^1(\Lambda)$ (see [37] and [40] for explicit conditions on A_0 implying (2.3)). The latter being the Sobolev space of functions with distributional derivatives up to order 1 square integrable. In (2.3) we use the inner product in $L^2(\Lambda)$. We, furthermore, assume there exists a constant C_δ such that

$$\langle A_0^2 \phi, \phi \rangle \leq \delta \langle \mathbf{p}\phi, \mathbf{p}\phi \rangle + C_\delta \langle \phi, \phi \rangle, \tag{2.4}$$

for some $\delta < 1/5$ and for every $\phi \in \mathbf{H}_0^1(\Lambda)$. We suppose additionally that for some C^∞ function κ defined in \mathbb{R}^3 such that $\kappa(x) = 0$ for x in a neighborhood of K , and with $1 - \kappa$ compactly supported, κA_0 is two times continuously differentiable and

$$\left| \left(\frac{\partial}{\partial x_1} \right)^a \left(\frac{\partial}{\partial x_2} \right)^b \left(\frac{\partial}{\partial x_3} \right)^c \kappa A_0(x) \right| \leq C(1 + |x|)^{-\zeta},$$

for some $\zeta > 1$, and every $a, b, c \in \{0, 1, 2\}$ with $a + b + c \leq 2$.

(2.5)

We notice that the properties of A_0 above permit it to have a finite or even an infinite number of singularities.

The Magnetic Potentials. Let $\{\hat{\gamma}_j\}_{j=1}^m$ be the closed curves defined in equation (2.6) of [4] (see Fig. 1). We prove in Corollary 2.4 of [4] that the equivalence classes of these curves are a basis of the first singular homology group of Λ . We introduce below a function that gives the magnetic flux across surfaces that have $\{\hat{\gamma}_j\}_{j=1}^m$ as their boundaries.

Definition 2.2. The flux, Φ , is a function $\Phi : \{\hat{\gamma}_j\}_{j=1}^m \rightarrow \mathbb{R}$.

We now define a class of magnetic potentials with a given flux.

Definition 2.3. Let B be a closed 2– form that satisfies Assumption 2.1. We denote by $\mathcal{A}_\Phi(B)$ the set of all continuous 1– forms, A , in $\bar{\Lambda}$ that satisfy

1.

$$|A(x)| \leq C(1 + |x|)^{-\zeta}, \quad \zeta > 1, \tag{2.6}$$

2.

$$\int_{\hat{\gamma}_j} A = \Phi(\hat{\gamma}_j), \quad j \in \{1, 2, \dots, m\}, \tag{2.7}$$

3.

$$dA|_\Lambda \equiv \text{curl } A|_\Lambda = B|_\Lambda. \tag{2.8}$$

Furthermore, we denote by $\mathcal{A}_\Phi^{(\text{reg})}(B)$ the set of 2-times continuously differentiable functions $A \in \mathcal{A}_\Phi(B)$ such that

$$\left| \left(\frac{\partial}{\partial x_1} \right)^a \left(\frac{\partial}{\partial x_2} \right)^b \left(\frac{\partial}{\partial x_3} \right)^c A(x) \right| \leq C(1 + |x|)^{-\zeta},$$

for every $x \in \bar{\Lambda}$ and every $a, b, c \in \{0, 1, 2\}$ with $a + b + c \leq 2$. (2.9)

Here the superscript (reg) stands for regular.

Remark 2.4. In Theorem 3.7 of [4] we construct the Coulomb potential, A_C , that belongs to $\mathcal{A}_\Phi(B)$ with a $\zeta > 1$ that depends on μ . For this purpose condition (2.1) is essential. The same proof applies to see that (2.2) implies that $A_C \in \mathcal{A}_\Phi^{(\text{reg})}(B)$. Actually, the Coulomb magnetic potential has a regularizing effect, in the sense that it is one time more differentiable than B . However, this subtlety is not relevant for the purposes of this paper. Notice that we use the same quantity ζ in (2.5), (2.6) and (2.9). We do it for convenience to keep as simple as possible our notation.

In Lemma 3.8 of [4] we prove that for any $A, \tilde{A} \in \mathcal{A}_\Phi(B)$ there is a C^1 0– form λ in $\bar{\Lambda}$ such that

$$\tilde{A} - A = d\lambda. \tag{2.10}$$

Moreover, we can take

$$\lambda(x) := \int_{C(x_0, x)} (\tilde{A} - A), \tag{2.11}$$

where x_0 is any fixed point in Λ and $C(x_0, x)$ is any simple differentiable curve in Λ with starting point x_0 and ending point x . Furthermore,

$$\lambda_\infty(x) := \lim_{r \rightarrow \infty} \lambda(rx) \tag{2.12}$$

exists and it is continuous in $\mathbb{R}^3 \setminus \{0\}$ and homogeneous of order zero, i.e. $\lambda_\infty(rx) = \lambda_\infty(x), r > 0, x \in \mathbb{R}^3 \setminus \{0\}$. Moreover,

$$|\lambda_\infty(x) - \lambda(x)| \leq \int_{|x|}^\infty b(r) dr, \text{ for some } b(r) \in L^1(0, \infty). \tag{2.13}$$

Actually, (2.6) and (2.11) imply that λ_∞ is a constant function. We denote this constant by $\lambda_\infty \equiv \lambda_\infty(x)$. In Lemma 3.8 of [4] we consider a more general case where λ_∞ is not necessarily constant.

2.1.3. The Free Klein–Gordon Equation. The free Klein–Gordon equation is given by

$$\left(i \frac{\partial}{\partial t}\right)^2 \phi = (\mathbf{p}^2 + m^2)\phi, \tag{2.14}$$

where $m > 0$ is the mass of the particle. We do not include the obstacle in the free evolution. This mathematically means that we are looking for solutions $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$. To analyze (2.14) we proceed as in [40, 41] and we study an equivalent system of differential equations that has the advantage of being of order one in $\frac{\partial}{\partial t}$. For this purpose we define:

Definition 2.5. (*Free Hamiltonian*) Let B_0 be the operator

$$B_0 := \left(\mathbf{p}^2 + m^2\right)^{1/2}, \tag{2.15}$$

with domain $\mathbf{H}^1(\mathbb{R}^3)$.

We denote by \mathcal{H}_0 the Hilbert space

$$\mathcal{H}_0 := \text{dom}(B_0) \oplus L^2(\mathbb{R}^3) \tag{2.16}$$

with inner product

$$\langle \phi, \psi \rangle_{\mathcal{H}_0} := \langle B_0 \phi_1, B_0 \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \tag{2.17}$$

for $\phi = (\phi_1, \phi_2), \psi = (\psi_1, \psi_2)$. Note that under the identification $\phi_1 = \phi, \phi_2 = \frac{\partial}{\partial t} \phi, \psi_1 = \psi, \psi_2 = \frac{\partial}{\partial t} \psi$, with ϕ, ψ solutions to (2.14), (2.17) is the sesquilinear form associated to the classical field energy of the free Klein–Gordon equation.

The free Hamiltonian H_0 is given by

$$H_0 := \begin{pmatrix} 0 & i \\ -iB_0^2 & 0 \end{pmatrix},$$

with $\text{dom}(H_0) := \text{dom}(B_0^2) \oplus \text{dom}(B_0) = \mathbf{H}^2(\mathbb{R}^3) \oplus \mathbf{H}^1(\mathbb{R}^3)$. (2.18)

Notice that H_0 is self-adjoint in \mathcal{H}_0 .

The free Klein–Gordon Eq. (2.14) is equivalent to the system of differential equations

$$i \frac{\partial}{\partial t} \psi = H_0 \psi, \tag{2.19}$$

where $\psi : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{C}^2$ with $\psi_1 = \phi, \psi_2 = \frac{\partial}{\partial t} \phi$. Note the slight difference with [40, 41] where the reduction to a system is made with $\psi_1 = \phi, \psi_2 = i \frac{\partial}{\partial t} \phi$.

Let us denote,

$$L^2(\mathbb{R}^3)^2 := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3),$$

and consider the following unitary transformation :

$$F_W : \mathcal{H}_0 \mapsto L^2(\mathbb{R}^3)^2 \tag{2.20}$$

given by

$$F_W := \begin{pmatrix} B_0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.21}$$

It follows that

$$F_W H_0 F_W^{-1} = B_0 \beta, \tag{2.22}$$

where

$$\beta := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{2.23}$$

We define the matrices

$$Q := 2^{-1/2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad Q^{-1} := 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{2.24}$$

that diagonalize β :

$$Q \beta Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1. \end{pmatrix}, \tag{2.25}$$

Let U , [40, 41] be the unitary operator

$$U := Q F_W, \quad \mathcal{H}_0 \mapsto L^2(\mathbb{R}^3)^2. \tag{2.26}$$

It follows that

$$\hat{H}_0 := U H_0 U^{-1} = \begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix}. \tag{2.27}$$

In this representation the free Klein–Gordon equation is equivalent to the system,

$$i \frac{\partial}{\partial t} \psi = \hat{H}_0 \psi, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in L^2(\mathbb{R}^3)^2. \tag{2.28}$$

The functions ψ_+, ψ_- are, respectively, the positive and negative energy components of the solution. The negative energy solutions are interpreted as antiparticles, in the usual way. In the physics literature, the representation (2.28), but with a scalar product that is not positive definite, is called the free particle or Feshbach–Villars representation, see [17, 20]. We define the position operator (see the conclusions in [40]), \hat{x}_P as multiplication by x in this representation,

$$\hat{x}_P \psi(x) = x \psi(x), \tag{2.29}$$

and then, $|\psi_{\pm}|^2$ are, respectively, the probability densities for particles with positive and negative energy. Note that this is possible because the scalar product in $L^2(\mathbb{R}^3)^2$ is positive definite.

In the representation in \mathcal{H}_0 the position operator is given by,

$$x_P := U^{-1} \hat{x}_P U = \begin{pmatrix} x + i \frac{\mathbf{P}}{B_0^2} & 0 \\ 0 & x \end{pmatrix}. \tag{2.30}$$

Note that $x + i \frac{\mathbf{P}}{B_0^2}$ is different from the Newton–Wigner position operator [29].

Observe that multiplication by x in the representation \mathcal{H}_0 can not be a position operator. In fact, it is not a selfadjoint operator in \mathcal{H}_0 and, hence, it is not a quantum mechanical observable. Actually in the representation \mathcal{H}_0 , x is a classical parameter that is used to parametrize the classical, macroscopic, objects like the magnet K and the electric and magnetic fields, but, as mentioned above, the operator that gives the position of the quantum particle is x_P .

High-Momentum States We designate by \mathbb{S}^2 the unit sphere in \mathbb{R}^3 . We need consider high-momentum states that under the free evolution have negligible interaction with the magnet. For this purpose, for every $\nu \in \mathbb{S}^2$ we denote by [see Eq. (2.5)]

$$\Lambda_\nu := \left\{ x \in \Lambda : x + \tau\nu \in \Lambda, \forall \tau \in \mathbb{R} \right\}, \tag{2.31}$$

and

$$\Lambda_{\kappa,\nu} := \left\{ x \in \Lambda_\nu : \kappa(x + \tau\nu) = 1, \forall \tau \in \mathbb{R} \right\}, \tag{2.32}$$

where we recall that κ is introduced in the lines above Eq. (2.5). Since the classical free evolution of a relativistic particle is given by $x + \tau\nu$ for some $\nu \in \mathbb{S}^1$, a state that in the representation $L^2(\mathbb{R}^3)^2$ is given by a function ϕ with support in Λ_ν has no interaction with the magnet under the classical evolution since,

$$\begin{aligned} & (\chi_K(x_P) U^{-1} \phi(\cdot + \nu\tau), U^{-1} \phi(\cdot + \nu\tau))_{\mathcal{H}_0} \\ &= (\chi_K(x) \phi(x + \nu\tau), \phi(x + \nu\tau))_{L^2(\mathbb{R}^3)^2} = 0, \end{aligned}$$

where for any set O we denote by χ_O the characteristic function of O . Our high-momenta states are defined in the representation $L^2(\mathbb{R}^3)^2$ as,

$$e^{ix \cdot v\nu} \phi(x), \quad \phi \in L^2(\mathbb{R}^3)^2, \quad \text{with } \phi \text{ supported in } \Lambda_\nu. \tag{2.33}$$

In the representation \mathcal{H}_0 they are given by,

$$e^{ix_P \cdot v\nu} U^{-1} \phi(x), \quad \phi \in L^2(\mathbb{R}^3)^2, \quad \text{with } \phi \text{ supported in } \Lambda_\nu.$$

In Eq. (2.33) the operators $e^{ix \cdot v\nu}$ represent a momentum shift corresponding to $v\nu$. Then $v\nu$ symbolize momentum. We use the notation v to represent the norm of the shifted momentum. We proceed in this way to keep a notation similar to the one we used in previous papers ([4–7]), where the non-relativistic case is addressed. The high-momentum limit amounts to take v to infinity. The

physical intuition is that for high momentum the free quantum evolution is close to the classical free evolution and then, our high-momenta states will have negligible interaction with the magnet K .

2.1.4. The Interacting Klein–Gordon Equation. The Klein–Gordon equation for a particle in Λ with electric potential A_0 and magnetic field B is given by

$$\left(i\frac{\partial}{\partial t} - A_0\right)^2 \phi = \left((\mathbf{p} - A)^2 + m^2\right)\phi, \tag{2.34}$$

where the magnetic potential $A \in \mathcal{A}_\Phi(B)$ satisfies $dA = B$ and $\phi : \mathbb{R} \times \Lambda \mapsto \mathbb{C}$ is the wave function. As in the free case, to analyze (2.34), we trade it by an equivalent system of differential equations of order 1 in the time derivative [39, 40]:

$$i\frac{\partial}{\partial t}\psi = H(\underline{A})\psi, \tag{2.35}$$

where $\psi : \mathbb{R} \times \Lambda \mapsto \mathbb{C}^2$,

$$\underline{A} := (A_0, A) \tag{2.36}$$

and

$$H(\underline{A}) := \begin{pmatrix} 0 & i \\ -iB(\underline{A})^2 & 2A_0 \end{pmatrix}, \quad \text{with } B(\underline{A})^2 := (\mathbf{p} - A)^2 + m^2 - A_0^2. \tag{2.37}$$

Equation (2.34) is equivalent to (2.35) with $\psi_1 = \phi, \psi_2 = \frac{\partial}{\partial t}\phi$.

In Sect. 3.2 we prove that $B(\underline{A})^2$ is a strictly positive operator and that the interacting Hamiltonian, $H(\underline{A})$, whose domain is described below, is a self-adjoint operator.

Definition 2.6. (*Interacting Hamiltonian*) We denote by $\mathcal{H}(\underline{A})$ the Hilbert space

$$\mathcal{H}(\underline{A}) := \text{dom}(B(\underline{A})) \oplus L^2(\Lambda) \tag{2.38}$$

with inner product

$$\langle \phi, \psi \rangle_{\mathcal{H}(\underline{A})} := \langle B(\underline{A})\phi_1, B(\underline{A})\psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \tag{2.39}$$

for $\phi = (\phi_1, \phi_2), \psi = (\psi_1, \psi_2)$. The domain of the operator $H(\underline{A})$ is given by

$$\text{dom}(H(\underline{A})) := \text{dom}(B(\underline{A})^2) \oplus \text{dom}(B(\underline{A})). \tag{2.40}$$

Notice that the specific properties of the electromagnetic potential we choose imply that

$$\langle \phi, \phi \rangle_{\mathcal{H}(\underline{A})} \geq \varepsilon \langle \phi, \phi \rangle_{L^2(\mathbb{R}^3)^2},$$

for every $\phi \in \mathcal{H}(\underline{A})$ (see Sect. 3.2 below). The scalar product (2.39) is the sesquilinear form associated with the classical field energy of the interacting Klein–Gordon Eq. (2.34). In [40, 41] (see also Sect. 3.2) it is proven that (2.34) can be represented in $L^2(\mathbb{R}^3)^2$ as a first order in time system, as in the free case.

2.1.5. Wave and Scattering Operators. Let J be the identification operator from \mathcal{H}_0 onto $\mathcal{H}(\underline{A})$ given by

$$J := \begin{pmatrix} B(\underline{A})^{-1}\chi_\Lambda B_0 & 0 \\ 0 & \chi_\Lambda \end{pmatrix}, \tag{2.41}$$

here $\chi_{(\cdot)}$ is the characteristic function of (\cdot) . The wave operators are defined as follows:

$$W_\pm(\underline{A}) := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH(\underline{A})} J e^{-itH_0}, \tag{2.42}$$

provided that the limits exist. In Sect. 3.3.1 we prove the existence of the limits, for every $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_\Phi(B)$.

The scattering operator is defined, for every $\underline{A} = (A_0, A)$ with $A \in \mathcal{A}_\Phi(B)$, by

$$S(\underline{A}) = W_+^*(\underline{A})W_-(\underline{A}). \tag{2.43}$$

2.2. Main Results

2.2.1. High-Momenta Limit of the Scattering Operator. The theorem below gives the high-momenta limit of the scattering operator, from which we reconstruct the electric potential, the magnetic field and some properties of the magnetic potentials. The derivation of this formula is the most laborious part in our paper and the core of our proofs. The proof of this theorem is deferred to Sect. 3.4.2, which is based in the results of Sect. 3.4.1. For the definition of the weighted Sobolev space $\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2$ see Sect. 3.1.

Theorem 2.7. *Set $\nu \in \mathbb{S}^2$ and $l \in \mathbb{N}$, $l \geq \zeta/2, l \geq 2$. Suppose that $\phi, \psi \in \mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2$ are supported in $\Lambda_{\kappa,\nu}$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_\Phi(B)$. Then*

$$\begin{aligned} & \langle e^{-ix \cdot \nu \nu} U S(\underline{A}) U^{-1} e^{ix \cdot \nu \nu} \phi, \psi \rangle_{L^2(\mathbb{R}^3)^2} \\ &= \left\langle \begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr (A \cdot \nu - A_0)(x+r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr (A \cdot \nu + A_0)(x+r\nu)} \end{pmatrix} \phi, \psi \right\rangle, \\ & \quad + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)} \begin{cases} O\left(\nu^{1-\zeta} + \frac{1}{\nu}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(\nu)}{\nu}\right) & \text{if } \zeta = 2. \end{cases} \end{aligned} \tag{2.44}$$

In Remark 3.17 we additionally derive high-momenta expressions for the wave operators. Notice that Theorem 2.7 does not require the functions ϕ, ψ to be compactly supported, as it is done in our previous works (see [4–7]). The function κ is introduced for two reasons: to cutoff smoothly the obstacle and to cutoff the singularities of A_0 . If A_0 has no singularities, then our result is equivalent to the results in [4–7], for the non-relativistic case: if A_0 has no singularities, and ϕ, ψ are supported in Λ_ν , we can always find some C^∞ function κ defined in \mathbb{R}^3 such that $\kappa(x) = 0$ for x in a neighborhood of K , and with $1 - \kappa$ compactly supported, such that ϕ, ψ are supported in $\Lambda_{\nu,\kappa}$.

Similar results for non-obstacle relativistic-scattering are presented in [8, 28]. There error bounds are not provided and a different class of (bounded) electromagnetic potentials is addressed.

2.2.2. Inverse Scattering Reconstruction Method. In this section we present one of our main results, namely Theorem 2.9. The proof is postponed to Sect. 3.5.

Definition 2.8. We denote by Λ_{Rec} the set of points $x \in \Lambda$ such that, for some two-dimensional plane P_x , $x + P_x \subset (\kappa^{-1}(\{1\}))^\circ$, for some C^∞ function κ defined in \mathbb{R}^3 such that $\kappa(x) = 0$ for x in a neighborhood of K , and with $1 - \kappa$ compactly supported, where for any set O we denote by O° its interior.

Theorem 2.9. *The high-momenta limit (2.44) of the scattering operator uniquely determines $B(y)$ and $A_0(y)$ for every $y \in \Lambda_{\text{Rec}}$.*

Remark 2.10. Notice that in Definition 2.8 the functions κ are not fixed, but can be conveniently selected. Furthermore, in the proof of Theorem 2.9 in Sect. 3.5.1 we give a method for the unique reconstruction of $B(y), A_0(y), y \in \Lambda_{\text{Rec}}$.

2.2.3. The Aharonov–Bohm Effect. In this section we assume that $B = 0$ and $A_0 = 0$, i.e., that the electromagnetic field vanishes in Λ . The hypothesis $A_0 = 0$ is assumed for convenience, in the spirit of the Aharonov–Bohm effect. Nevertheless some results are also valid for $A_0 \neq 0$. In the case that $A_0 \neq 0$, notable differences and similarities between the relativistic and the non-relativistic cases hold true. They are presented in Sect. 3.6.3, where the corresponding results for $A_0 \neq 0$ are proved.

Notation For any $x \in \Lambda_{\hat{v}}$ and any unit vector $\hat{v} \in \mathbb{S}^2$ we denote

$$L(x, \hat{v}) := x + \mathbb{R}\hat{v}, \tag{2.45}$$

and we give to $L(x, \hat{v})$ the orientation of \hat{v} . Suppose that $x \in \Lambda_{\hat{v}}, y \in \Lambda_{\hat{w}}, \hat{v}, \hat{w} \in \mathbb{S}^2$ satisfy $\hat{v} \cdot \hat{w} \geq 0$ and that

$$L(x, \hat{v}) \cup L(y, \hat{w}) \subset \Lambda.$$

Take $\rho > 0$ so large that

$$\begin{aligned} &\text{convex} ((x + (-\infty, -\rho]\hat{v}) \cup (y + (-\infty, -\rho]\hat{w})) \\ &\cup \text{convex} ((x + [\rho, \infty)\hat{v}) \cup (y + [\rho, \infty)\hat{w})) \subset \mathbb{R}^3 \setminus B(0; r), \end{aligned}$$

where $K \subset B(0; r)$ and the symbol $\text{convex}(\cdot)$ denotes the convex hull of the indicated set.

We denote by $\gamma(x, y, \hat{v}, \hat{w})$ the continuous, simple, oriented and closed curve with sides $x + [-\rho, \rho]\hat{v}$, oriented in the direction of \hat{v} , $y + [-\rho, \rho]\hat{w}$, oriented in the direction of $-\hat{w}$, and the oriented straight lines that join the points $x + \rho\hat{v}$ with $y + \rho\hat{w}$ and $y - \rho\hat{w}$ and $x - \rho\hat{v}$.

Results

Theorem 2.11. *Suppose that $B = 0$ and that $A_0 = 0$. Then, for any flux, Φ , and all $A \in \mathcal{A}_\Phi(0)$, the high-momenta limit of $S(\underline{A})$ in (2.44), known for $\hat{\nu}$ and $\hat{\mathbf{w}}$, determines the fluxes*

$$\int_{\gamma(x,y,\hat{\nu},\hat{\mathbf{w}})} A \tag{2.46}$$

modulo 2π , for all curves $\gamma(x, y, \hat{\nu}, \hat{\mathbf{w}})$.

Theorem 2.11 is also valid if $A_0 \neq 0$, with the restriction that (2.46) is determined only modulo π (see Sect. 3.6.3). The proof of Theorem 2.11 is done at the beginning of Sect. 3.6. This Theorem gives additionally information about the de Rham cohomology class of A , see Remarks 3.18 and 3.19.

Theorem 2.12. *Suppose that $A_0 = 0$. There is an open disjoint cover of Λ_ν , $\{\Lambda_h\}_{h \in \mathcal{I}} \cup \{\Lambda_{\text{out}}\}$, and a set of real numbers $\{F_h\}_{h \in \mathcal{I}}$ such that the following holds true: set $\phi, \psi \in \mathbf{H}^2_{(x)4l}(\mathbb{R}^3)^2$ as in Theorem 2.7, with ϕ compactly supported. For every $A \in \mathcal{A}_\Phi(0)$*

$$\begin{aligned} & \langle US(\underline{A})U^{-1} e^{i\nu \cdot x} \phi, e^{i\nu \cdot x} \psi \rangle_{L^2(\Lambda)^2} \\ &= \left\langle \left(\sum_{h \in \mathcal{I}} \begin{pmatrix} e^{iF_h} & 0 \\ 0 & e^{-iF_h} \end{pmatrix} \chi_{\Lambda_h} \phi + \chi_{\Lambda_{\text{out}}} \phi \right), \psi \right\rangle_{L^2(\Lambda)^2} \\ &+ O\left(\frac{1}{v}\right) \|\phi\|_{\mathbf{H}^2_{(x)4l}(\mathbb{R}^3)} \|\psi\|_{\mathbf{H}^2_{(x)4l}(\mathbb{R}^3)}, \end{aligned} \tag{2.47}$$

as v tends to infinity.

The sets $\{\Lambda_h\}_{h \in \mathcal{I}}$ have a geometric meaning, they are the holes of K in the direction ν . The numbers $\{F_h\}_{h \in \mathcal{I}}$ are magnetic fluxes (around handles of the obstacle) associated to the holes h , for $h \in \mathcal{I}$. The set Λ_{out} is the region without holes. This is well explained in the lines above Theorem 3.25. The proof of Theorem 2.12 is done at the end of Sect. 3.6, this theorem is actually rephrased in Theorem 3.25. The simple expression (2.47) is not anymore valid if $A_0 \neq 0$ (see Sect. 3.6.3), whereas in this case it is valid in the non-relativistic setting (see [4]).

Remark 2.13. Suppose that $A_0 = 0$ and that ϕ and ψ are compactly supported in Λ_ν , then we can always find some C^∞ function κ defined in \mathbb{R}^3 such that $\kappa(x) = 0$ for x in a neighborhood of K , and with $1 - \kappa$ compactly supported, such that ϕ and ψ are compactly supported in $\Lambda_{\kappa,\nu}$. It follows that if ϕ and ψ are compactly supported in Λ_ν and belong to $\mathbf{H}^2(\mathbb{R}^3)^2$, the conclusions of Theorem 2.12 are valid. Notice that we take $A_0 = 0$ because it is required in Theorem 2.12.

3. The Proofs

3.1. General Notation

We denote by C any finite positive constant whose value is not specified. For any $x \in \mathbb{R}^3, x \neq 0$, we denote, $\hat{x} := x/|x|$. For every $\mathbf{v} \in \mathbb{R}^3$ we designate $v := |\mathbf{v}|$. The domain of an operator or a quadratic form is denoted by $\text{dom}(X)$, where either X is an operator or a quadratic form. For every open set $O \subset \mathbb{R}^3$, we use the symbol $\langle \cdot, \cdot \rangle$ to represent the inner product in $L^2(O)$. If it is clear from the context, we use the same symbol to represent the inner product in $L^2(O)^2 := L^2(O) \oplus L^2(O)$ (in general, for every Hilbert space \mathcal{H} , we set $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$). For a general (complex) Hilbert space \mathcal{H} , its inner product is represented by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The norm of an element x in a normed vector space X is denoted by $\| \cdot \|_X$. However, if no confusion arises, we omit in general the subscript in the case that X is $L^2(O), L^2(O)^2$ or a space of operators with the corresponding operator norm.

In this paper we use a system of units in which the numerical value of the charge of the particle, the speed of light and the Plank's constant are one: $e = 1, c = 1, \hbar = 1$. The Schwartz space of rapidly decreasing C^∞ -functions in \mathbb{R}^3 is denoted by $\mathcal{S}(\mathbb{R}^3)$. For every $n \in \mathbb{N}$ and every open set O in \mathbb{R}^3 , we denote by $\mathbf{H}^n(O)$ the Sobolev space of functions with distributional derivative up to order n square integrable, and by $\mathbf{H}_0^n(O)$ the closure of $C_0^\infty(O)$ in $\mathbf{H}^n(O)$, see [1]. For every strictly positive function $\omega : O \mapsto \mathbb{R}$ we denote by $\mathbf{H}_\omega^n(O)$ the corresponding weighted Sobolev spaces. For every $\phi \in \mathbf{H}_\omega^n(O)$,

$$\|\phi\|_{\mathbf{H}_\omega^n(O)} := \sum_{\alpha_1 + \alpha_2 + \alpha_3 \leq n} \left\| \omega^{1/2} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} \phi \right\|_{L^2(O)}.$$

The open ball in \mathbb{R}^3 of radius r and center x is denoted by $B(x; r)$, and for every set O , its complement is denoted by O^c . The Fourier transform of a function $\phi \in L^2(\mathbb{R}^3)$ is denoted by

$$\widehat{\phi}(p) \equiv \mathcal{F}(\phi)(p) := \frac{1}{(2\pi)^{3/2}} \int e^{-ix \cdot p} \phi(x) dx \tag{3.1}$$

and the inverse Fourier transform by

$$\overline{\phi}(x) \equiv \mathcal{F}^{-1}(\phi)(x) := \frac{1}{(2\pi)^{3/2}} \int e^{ix \cdot p} \phi(p) dp. \tag{3.2}$$

Another useful notations that we adopt are the symbols $\langle x \rangle := (1 + x^2)^{1/2}$, $\langle \mathbf{p} \rangle := (1 + \mathbf{p}^2)^{1/2}$, for $x \in \mathbb{R}^3$ and $\mathbf{p} = -i\nabla$. In this paper we use the standard identifications between differential forms in open subsets of \mathbb{R}^3 and vector calculus (see page 348 in [4], for example).

A useful formula that we use very much in this paper is the following: for every measurable function $f : \mathbb{R}^3 \mapsto \mathbb{C}$

$$e^{-ix \cdot \nu \nu} f(\mathbf{p}) e^{ix \cdot \nu \nu} = f(\mathbf{p} + \nu \nu), \tag{3.3}$$

for every $\nu \geq 0$ and $\nu \in \mathbb{S}^2$.

3.2. The Interacting Hamiltonian

In this section we prove that $H(\underline{A})$ is a selfadjoint operator and give a proper definition of $B(\underline{A})$. Related results are presented in [40] and [41].

Let $q_{\underline{A}} : C_0^\infty(\Lambda) \times C_0^\infty(\Lambda) \rightarrow \mathbb{C}$ be the bilinear form defined by

$$q_{\underline{A}}(\phi, \psi) := \langle (\mathbf{p} - A)\phi, (\mathbf{p} - A)\psi \rangle + m^2 \langle \phi, \psi \rangle - \langle A_0^2 \phi, \psi \rangle. \tag{3.4}$$

The diamagnetic inequality (see Lemma 1.2, Chapter 9 in [37]) and (2.3) imply that $q_{\underline{A}} \geq \varepsilon$. Theorem X.17 in [32] (KLMN theorem) implies that $q_{\underline{A}}$ is closable. We denote its closure by $\bar{q}_{\underline{A}}$, it is represented by a selfadjoint operator $B(\underline{A})^2 \geq \varepsilon$. Set $B(\underline{A})$ the positive square root of $B(\underline{A})^2$. It follows that

$$\text{dom}(B(\underline{A})) = \text{dom}(\bar{q}_{\underline{A}}), \quad \bar{q}_{\underline{A}}(\phi, \psi) = (B(\underline{A})\phi, B(\underline{A})\psi), \tag{3.5}$$

for every $\phi, \psi \in \text{dom}(\bar{q}_{\underline{A}})$. Actually, (2.4) and the fact that A is bounded imply that

$$\text{dom}(\bar{q}_{\underline{A}}) = \mathbf{H}_0^1(\Lambda). \tag{3.6}$$

Proposition 3.1. *$H(\underline{A})$ is a self-adjoint operator (see Definition 2.6).*

Proof. Defining the unitary operator $U_{\underline{A}} : \mathcal{H}(\underline{A}) \mapsto L^2(\Lambda)^2$ by

$$\psi \mapsto Q \begin{pmatrix} B(\underline{A}) & 0 \\ 0 & 1 \end{pmatrix} \psi \tag{3.7}$$

we find that

$$\begin{pmatrix} 0 & i \\ -iB(\underline{A})^2 & 0 \end{pmatrix} = U_{\underline{A}}^{-1} \begin{pmatrix} B(\underline{A}) & 0 \\ 0 & -B(\underline{A}) \end{pmatrix} U_{\underline{A}}, \tag{3.8}$$

which implies that

$$\begin{pmatrix} 0 & i \\ -iB(\underline{A})^2 & 0 \end{pmatrix}$$

is self-adjoint, with domain $\text{dom}(H(\underline{A}))$. By (2.4),

$$\|A_0\phi\|^2 \leq \delta \|\mathbf{p}\phi\|^2 + C_\delta \|\phi\|^2 \leq \delta \bar{q}_{\underline{A}}(\phi, \phi) + \delta \|A_0\phi\|^2 + C_\delta \|\phi\|^2.$$

Then,

$$\|A_0\phi\|^2 \leq \tilde{\delta} \bar{q}_{\underline{A}}(\phi, \phi) + C_{\tilde{\delta}} \|\phi\|^2, \quad \tilde{\delta} := \frac{\delta}{1-\delta}, \quad C_{\tilde{\delta}} := \frac{C_\delta}{1-\delta},$$

where we use the diamagnetic inequality (Lemma 1.2, Chapter 9 in [37]). Let $\psi = (\psi_1, \psi_2) \in \text{dom}(H(\underline{A}))$, then,

$$\begin{aligned} \left\| \begin{pmatrix} 0 & 0 \\ 0 & 2A_0 \end{pmatrix} \psi \right\|_{\mathcal{H}(\underline{A})} &= 2 \|A_0\psi_2\| \\ &\leq 2\tilde{\delta}^{1/2} \left\| \begin{pmatrix} 0 & i \\ -iB(\underline{A})^2 & 0 \end{pmatrix} \psi \right\|_{\mathcal{H}(\underline{A})} + 2(C_{\tilde{\delta}})^{1/2} \|\psi\|_{\mathcal{H}(\underline{A})}, \end{aligned} \tag{3.9}$$

thus $\begin{pmatrix} 0 & 0 \\ 0 & 2A_0 \end{pmatrix}$ is small in the sense of Kato with respect to $\begin{pmatrix} 0 & i \\ -iB(\underline{A})^2 & 0 \end{pmatrix}$, with relative bound smaller than 1 (here we use that $\delta < \frac{1}{5}$). The fact that

$H(\underline{A})$ is self-adjoint follows from this last assertion and the Kato–Rellich theorem (Theorem X.12 in [32]). \square

3.3. The Wave and Scattering Operators

3.3.1. Wave Operators. In this paragraph we prove that the wave operators exist. Lemma 3.3 proves the existence of the wave operators in the case that $A \in \mathcal{A}_{\Phi}^{(\text{reg})}(B)$ (the regular case). The general case is proved in Sect. 3.3.2. In Sect. 3.3.3 we study the scattering operator. Related results are presented in [41].

We identify κ [see (2.5)] with the multiplication operator by the matrix

$$\begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} \equiv \kappa. \tag{3.10}$$

Lemma 3.2. *For every $A \in \mathcal{A}_{\Phi}(B)$, the limits (2.42) exist, if and only if, the limits $s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH(\underline{A})} \kappa e^{-itH_0}$ exist. In either case*

$$W_{\pm}(\underline{A}) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH(\underline{A})} \kappa e^{-itH_0}. \tag{3.11}$$

Proof. It is enough to prove that for every $\phi \in \mathcal{S}(\mathbb{R}^3) \oplus \mathcal{S}(\mathbb{R}^3)$

$$\lim_{t \rightarrow \pm\infty} \left\| \left[\begin{pmatrix} B(\underline{A})^{-1} \chi_{\Lambda} B_0 & 0 \\ 0 & \chi_{\Lambda} \end{pmatrix} - \kappa \right] e^{-itH_0} \phi \right\|_{\mathcal{H}(\underline{A})} = 0. \tag{3.12}$$

As B_0 has only absolutely continuous spectrum, e^{itB_0} converges weakly to zero when t tends to $\pm\infty$ (using the Riemann–Lebesgue Lemma). The Rellich–Kondrachov theorem implies that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \|f(x) e^{itB_0} \phi\| &= 0, \quad \forall \phi \in L^2(\mathbb{R}^3), \text{ if } \lim_{r \rightarrow \infty} |f(r\hat{x})| = 0, \\ &\text{uniformly on } \hat{x} \in \mathbb{S}^2. \end{aligned} \tag{3.13}$$

From this and (2.27) we deduce that (3.12) is fulfilled whenever

$$\begin{aligned} &\lim_{t \rightarrow \pm\infty} \|(\kappa B_0 - B(\underline{A})\kappa) e^{-itB_0} \psi\| \\ &= \lim_{t \rightarrow \pm\infty} \|e^{itB(\underline{A})} (\kappa B_0 - B(\underline{A})\kappa) e^{-itB_0} \psi\| = 0, \end{aligned} \tag{3.14}$$

for every $\psi \in \mathcal{S}(\mathbb{R}^3)$. We complete our proof showing the validity of (3.14).

It follows from the proof of Lemma 5.3 in [4] that the limits

$$W_{\pm}(B(\underline{A})^2; B_0^2) := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itB(\underline{A})^2} \kappa e^{-itB_0^2} \tag{3.15}$$

exist (they are actually the wave operators for the Schrödinger equation). The invariance principle (Theorem XI.23 in [33]) implies that the limits

$$W_{\pm}(B(\underline{A}); B_0) := s\text{-}\lim_{t \rightarrow \pm\infty} e^{itB(\underline{A})} \kappa e^{-itB_0} \tag{3.16}$$

exist and are equal to (3.15). Actually [33] does not consider obstacles, however, the proof also applies in this case. It follows that

$$\lim_{t \rightarrow \pm\infty} e^{itB(\underline{A})} \kappa B_0 e^{-itB_0} \psi = W_{\pm}(B(\underline{A}); B_0) B_0 \psi. \tag{3.17}$$

Now we analyze the term $e^{itB(\underline{A})}B(\underline{A})\kappa e^{-itB_0}\psi$. We prove that

$$\lim_{t \rightarrow \pm\infty} \|B(\underline{A})e^{itB(\underline{A})}\kappa e^{-itB_0}\psi\| = \|W_{\pm}(B(\underline{A}); B_0)B_0\psi\|. \tag{3.18}$$

To obtain (3.18) we compute [see (3.4) and (3.5)]

$$\begin{aligned} & \left| \|B(\underline{A})\kappa e^{-itB_0}\psi\|^2 - \|\kappa e^{-itB_0}B_0\psi\|^2 \right| \\ &= \left| \left\langle (\mathbf{p} - A)\kappa e^{-itB_0}\psi, (\mathbf{p} - A)\kappa e^{-itB_0}\psi \right\rangle \right. \\ & \quad + m^2 \left\langle \kappa e^{-itB_0}\psi, \kappa e^{-itB_0}\psi \right\rangle - \left\langle A_0^2 \kappa e^{-itB_0}\psi, \kappa e^{-itB_0}\psi \right\rangle \\ & \quad \left. - \|\kappa e^{-itB_0}B_0\psi\|^2 \right| \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty, \end{aligned} \tag{3.19}$$

where we use (3.13). Notice that we can substitute the last term $\|\kappa e^{-itB_0}B_0\psi\|^2$ by $\|e^{-itB_0}B_0\psi\|^2 = \|\mathbf{p}e^{-itB_0}\psi\|^2 + m^2\|e^{-itB_0}\psi\|^2$, using (3.13), and we can handle similarly the other terms containing κ . With the help of (3.17) and (3.18) we obtain

$$\begin{aligned} & \lim_{t \rightarrow \pm\infty} \|e^{itB(\underline{A})}(\kappa B_0 - B(\underline{A})\kappa)e^{-itB_0}\psi\|^2 \\ &= \lim_{t \rightarrow \pm\infty} \left[2\|W_{\pm}(B(\underline{A}); B_0)B_0\psi\|^2 \right. \\ & \quad - \left\langle e^{itB(\underline{A})}B(\underline{A})\kappa e^{-itB_0}\psi, W_{\pm}(B(\underline{A}); B_0)B_0\psi \right\rangle \\ & \quad \left. - \left\langle W_{\pm}(B(\underline{A}); B_0)B_0\psi, e^{itB(\underline{A})}B(\underline{A})\kappa e^{-itB_0}\psi \right\rangle \right] = 0, \end{aligned} \tag{3.20}$$

where we used that

$$\begin{aligned} & \lim_{t \rightarrow \pm\infty} \left\langle e^{itB(\underline{A})}B(\underline{A})\kappa e^{-itB_0}\psi, W_{\pm}(B(\underline{A}); B_0)B_0\psi \right\rangle \\ &= \lim_{t \rightarrow \pm\infty} \left\langle e^{itB(\underline{A})}\kappa e^{-itB_0}\psi, W_{\pm}(B(\underline{A}); B_0)B_0^2\psi \right\rangle \\ &= \|W_{\pm}(B(\underline{A}); B_0)B_0\psi\|^2. \end{aligned} \tag{3.21}$$

□

Lemma 3.3. (Existence: The Regular Case) *For every $A \in \mathcal{A}_{\mathbb{F}}^{(\text{reg})}(B)$ the limits (3.11) exist and are isometric.*

Proof. The result follows from standard techniques using Cook’s argument, (2.27) and Lemma 3.29: for $\phi = (\phi_1, \phi_2) \in \mathcal{S}(\mathbb{R}^3)^2$, with $\widehat{\phi}_1, \widehat{\phi}_2$ satisfying the properties of f in Lemma 3.29 (note that these functions are dense in $L^2(\mathbb{R}^3)^2$), we write

$$e^{itH(\underline{A})}\kappa e^{-itH_0}\phi = \kappa\phi + \int_0^t e^{isH(\underline{A})}[H(\underline{A})\kappa - \kappa H_0]e^{-isH_0}\phi. \tag{3.22}$$

Set $h(s) = \left\| [H(\underline{A})\kappa - \kappa H_0]e^{-isH_0}\phi \right\|_{\mathcal{H}(\underline{A})}$. We bound h by an integrable function using (2.27) and Corollary 3.31. This allows us to take the limit $t \rightarrow \pm\infty$ in (3.22). The isometric property follows from (2.41), (2.42) and (3.13). □

3.3.2. Existence of the Wave Operators (the General Case). In this section we prove existence of wave operators (3.11) for every magnetic potential $A \in \mathcal{A}_\Phi(B)$ using Lemma 3.3 and a change of gauge argument. We provide additionally a change of gauge formula.

Theorem 3.4. (Existence of Wave Operators and Change of Gauge Formula) *For every $\underline{A} = (A_0, A)$ with $A \in \mathcal{A}_\Phi(B)$ the limits (3.11) exist and are isometric. For every $\underline{A}^{(i)} = (A_0, A^{(i)})$, $i \in \{1, 2\}$, with $A^{(1)}, A^{(2)} \in \mathcal{A}_\Phi(B)$*

$$W_\pm(\underline{A}^{(2)}) = e^{i\lambda(x)}W_\pm(\underline{A}^{(1)})e^{-i\lambda_\infty}, \tag{3.23}$$

where $A^{(2)} - A^{(1)} = \nabla\lambda$ [see (2.10)–(2.12)].

Proof. We suppose that $A^{(1)} \in \mathcal{A}_\Phi^{\text{(reg)}}(B)$. Lemma 3.3 assures the existence of the wave operators $W_\pm(\underline{A}^{(1)})$. By proving the change of gauge formula we prove the existence of $W_\pm(\underline{A}^{(2)})$. The same proof applies for general $A^{(1)}$ and $A^{(2)}$. We prove the assertion for W_+ . The proof for W_- is analogous. A simple computation gives

$$H(\underline{A}^{(2)}) = e^{i\lambda(x)}H(\underline{A}^{(1)})e^{-i\lambda(x)}, \tag{3.24}$$

which implies that

$$W_+(\underline{A}^{(2)}) = e^{i\lambda(x)}s - \lim_{t \rightarrow \infty} e^{itH(\underline{A}^{(1)})}\kappa e^{-itH_0}e^{itH_0}\kappa' e^{-i\lambda(x)}e^{-itH_0}, \tag{3.25}$$

whenever the limit exists. Here κ' satisfies the properties of κ [see (2.5) and the text above] and (3.10). Additionally

$$\kappa'\kappa = \kappa. \tag{3.26}$$

Then we only need to prove that

$$s - \lim_{t \rightarrow \infty} e^{itH_0}\kappa' e^{-i\lambda(x)}e^{-itH_0} = e^{-i\lambda_\infty}. \tag{3.27}$$

Using (2.20)–(2.27) we obtain that

$$\begin{aligned} e^{itH_0}\kappa' e^{-i\lambda(x)}e^{-itH_0} &= U^{-1} \begin{pmatrix} e^{itB_0} & 0 \\ 0 & e^{-itB_0} \end{pmatrix} \\ &\times Q \begin{pmatrix} B_0\kappa' e^{-i\lambda(x)}B_0^{-1} & 0 \\ 0 & \kappa' e^{-i\lambda(x)} \end{pmatrix} Q^{-1} \begin{pmatrix} e^{-itB_0} & 0 \\ 0 & e^{itB_0} \end{pmatrix} U. \end{aligned} \tag{3.28}$$

By (2.13), (3.13)

$$s - \lim_{t \rightarrow \infty} [e^{-i\lambda_\infty} - e^{-i\lambda(x)}]e^{\pm itB_0} = 0. \tag{3.29}$$

Then by (3.28), (3.29),

$$\begin{aligned} s - \lim_{t \rightarrow \infty} \left[e^{itH_0}\kappa' e^{-i\lambda(x)}e^{-itH_0} - U^{-1} \begin{pmatrix} e^{itB_0} & 0 \\ 0 & e^{-itB_0} \end{pmatrix} \begin{pmatrix} e^{-i\lambda_\infty} & 0 \\ 0 & e^{-i\lambda_\infty} \end{pmatrix} \right. \\ \left. \begin{pmatrix} e^{-itB_0} & 0 \\ 0 & e^{itB_0} \end{pmatrix} U \right] &= 0. \end{aligned} \tag{3.30}$$

We recall that λ_∞ is a constant function (see the text below Remark 2.4) to prove (3.27). To deduce (3.30) we use that

$$\|B_0[e^{-i\lambda_\infty} - \kappa' e^{-i\lambda(x)}]B_0^{-1}e^{\pm itB_0}\|^2 = \|\mathbf{p}[e^{-i\lambda_\infty} - \kappa' e^{-i\lambda(x)}]B_0^{-1}e^{\pm itB_0}\|^2 + m^2\|[e^{-i\lambda_\infty} - \kappa' e^{-i\lambda(x)}]B_0^{-1}e^{\pm itB_0}\|^2 \rightarrow 0, \text{ as } t \rightarrow \pm\infty,$$

and that the terms containing derivatives of $\kappa e^{-i\lambda(x)}$ vanish as t tends to infinity, using (3.13). □

3.3.3. Scattering Operator. The following theorem proves that the scattering operator is invariant under change of gauge; it is a direct consequence of Theorem 3.4.

Theorem 3.5. *For every $\underline{A}^{(i)} = (A_0, A^{(i)})$, $i = 1, 2$, with $A^{(1)}, A^{(2)} \in \mathcal{A}_\phi(B)$,*

$$S(\underline{A}^{(2)}) = S(\underline{A}^{(1)}). \tag{3.31}$$

Remark 3.6. As in [4] we can consider change of gauge formulae when $A^{(1)}, A^{(2)}$ only have the same fluxes modulo 2π . We do not dwell on these issues here.

Remark 3.7. In this text we do not address the question of asymptotic completeness for the wave operators because it is not necessary to prove our results. Therefore, the unitarity of the scattering operator does not follow from our proofs. However, asymptotic completeness for the Klein–Gordon equation was already achieved in [41]. At this stage we point out that through asymptotic completeness one describes the asymptotic behavior of all states with absolutely continuous energy spectrum. Nevertheless, they are not all necessary to recover information from the physical system through scattering, but only some specific states. This means that solving inverse problems does not require characterizing all solutions with absolutely continuous energy but some specific ones with suitable properties. This is the reason why we do not need asymptotic completeness in our analysis.

3.4. High-Momenta Limit of the Scattering Operator

In this section we prove our main results: we give a high-momenta expression for the scattering operator, with error bounds. This formula is the content of Theorem 2.7, whose proof in the main purpose of Sect. 3.4.2. To prove our theorem we first prove the result in the special case that A is regular ($A \in \mathcal{A}_\Phi^{\text{reg}}(B)$). This is the main result of Sect. 3.4.1. The general result follows from a change of gauge formula, which is accomplished in Sect. 3.4.2. Our formula is used in Sects. 3.5 and 3.6 to reconstruct important information from the potentials and the magnetic field.

3.4.1. High-Momenta Limit for the Scattering Operator: The Regular Case.

This section is the most laborious part of our paper. Here we estimate the high-momenta limit of the scattering operator in the case that A is regular ($A \in \mathcal{A}_\Phi^{\text{reg}}(B)$). This is actually a relevant result, since the general case follows from it by a simple gauge-argument. Our main result in this section is Theorem 3.16. The whole section is devoted to prove preliminary lemmata and

theorems that we use in the proof of Theorem 3.16. To analyze the scattering operator, the wave operators are fundamental. As we can see from (3.11), the wave operators represent asymptotic limits when the time goes to plus or minus infinity. To do our proof we analyze $e^{itH(\underline{A})} \kappa e^{-itH_0}$ for finite time t , in the high-momenta regime, and then we bound the error $W_{\pm}(\underline{A}) - e^{itH(\underline{A})} \kappa e^{-itH_0}$. The high-momenta analysis of $e^{itH(\underline{A})} \kappa e^{-itH_0}$ is the most complicated part, it is done in Theorem 3.13, which is based in Lemmata 3.10 and 3.12. Lemmata 3.11 and 3.14 deal with the error $W_{\pm}(\underline{A}) - e^{itH(\underline{A})} \kappa e^{-itH_0}$. These lemmata together with Theorem 3.13 and Lemma 3.15 (which is a technical result concerning the norms we are using) are the ingredients we need to prove our main result (Theorem 3.16).

Notation

Definition 3.8. Let $\underline{A} = (A_0, A)$ with $A \in \mathcal{A}_{\Phi}(B)$. For every $v > 0$ and every $\nu \in \mathbb{S}^2$ we define [see (2.5) and the text above it]

$$Z_{\underline{A}}(t, v\nu, \kappa) = e^{it\beta(v+\nu \cdot \mathbf{p})} \begin{pmatrix} 0 & 0 \\ i2\kappa A \cdot \nu & 2\kappa A_0 \end{pmatrix} e^{-it\beta(v+\nu \cdot \mathbf{p})}. \tag{3.32}$$

Definition 3.9. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_{\Phi}(B)$. For every $\nu \in \mathbb{S}^2$ we define [see (2.5)]

$$W_{\underline{A}}(t, \nu, \kappa) = e^{it\beta\nu \cdot \mathbf{p}} \begin{pmatrix} \kappa A_0(x) & -i\kappa A \cdot \nu \\ i\kappa A \cdot \nu & \kappa A_0(x) \end{pmatrix} e^{-it\beta\nu \cdot \mathbf{p}}. \tag{3.33}$$

Notice that

$$\begin{aligned} & e^{-i \int_{-\infty}^{\infty} W_{\underline{A}}(r, v\nu, \kappa) dr} \\ &= Q^{-1} \begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr \kappa(A \cdot \nu - A_0)(x+r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr \kappa(A \cdot \nu + A_0)(x+r\nu)} \end{pmatrix} Q, \end{aligned} \tag{3.34}$$

which follows from (3.77) to (3.78).

High-Momenta Limit

Lemma 3.10. Let $\nu \in \mathbb{S}^2$ and $l \in \mathbb{N}$, $l \geq \zeta/2$. Suppose that $\phi \in \mathbf{H}_{(x)4l}^{\alpha}(\mathbb{R}^3)^2$, $\alpha \geq 2$, is supported in $\Lambda_{\kappa, \nu}$ (see (2.32)). Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_{\Phi}^{(\text{reg})}(B)$. Then for every $v \geq v_0$ (see Lemma 3.29).

$$\begin{aligned} & \int_0^t e^{isH(\underline{A})} i(H(\underline{A})\kappa - \kappa H_0) e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} \phi \, ds \\ &= \int_0^t e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} iZ_{\underline{A}}(t, v\nu, \kappa) \phi \, ds \\ &+ O\left(\int_0^{|t|} ds \left[\|\phi\|_{\mathbf{H}_{(x)4l}^{\alpha}(\mathbb{R}^3)^2} \left(\frac{1}{v(1+|s|)^{2l-1}} + \sum_{j=1}^{2l} \frac{1}{(1+|s|)^{2l-j} v^{\min(\alpha, j)}} \right) \right. \right. \\ &\left. \left. + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \frac{1}{v(1+|s|)^{\zeta-1}} \right] + \frac{|t|}{v^{\alpha}} \|\phi\|_{\mathbf{H}^{\alpha}(\mathbb{R}^3)^2} \right), \end{aligned} \tag{3.35}$$

for every $t \in \mathbb{R}$.

Proof. First we compute

$$\begin{aligned} & \int_0^t e^{isH(\underline{A})} i(H(\underline{A})\kappa - \kappa H_0) e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} \phi \\ &= \int_0^t e^{isH(\underline{A})} e^{ix \cdot v\nu} e^{-ix \cdot v\nu} i(H(\underline{A})\kappa - \kappa H_0) F_W^{-1} e^{ix \cdot v\nu} e^{-ix \cdot v\nu} \\ & \quad \times F_W e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} \phi. \end{aligned} \tag{3.36}$$

It follows from (2.22) that

$$F_W e^{\pm itH_0} F_W^{-1} = e^{\pm it\beta(\mathbf{p}^2+m^2)^{1/2}}. \tag{3.37}$$

Notice that

$$(H(\underline{A})\kappa - \kappa H_0) F_W^{-1} = \begin{pmatrix} 0 & 0 \\ \vartheta \frac{1}{(\mathbf{p}^2+m^2)^{1/2}} & 2\kappa A_0 \end{pmatrix}, \tag{3.38}$$

where

$$\vartheta = -i((\mathbf{p}^2\kappa) + 2((\mathbf{p}\kappa) \cdot \mathbf{p}) - (\mathbf{p} \cdot A)\kappa - 2A \cdot (\mathbf{p}\kappa) - 2\kappa A \cdot \mathbf{p} + \kappa A^2 - \kappa A_0^2). \tag{3.39}$$

From (3.3), (3.38) and the fact that $(H(\underline{A})\kappa - \kappa H_0) F_W^{-1}$ have only zeros in the first row follows

$$\|e^{-ix \cdot v\nu} (H(\underline{A})\kappa - \kappa H_0) F_W^{-1} e^{ix \cdot v\nu}\|_{\mathcal{L}(L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n), \mathcal{H}(\underline{A}))} \leq C, \tag{3.40}$$

where C is a constant independent of $v\nu$.

We have that

$$\begin{aligned} & e^{ix \cdot v\nu} e^{-ix \cdot v\nu} (H(\underline{A})\kappa - \kappa H_0) F_W^{-1} e^{ix \cdot v\nu} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & e^{ix \cdot v\nu} \end{pmatrix} e^{-ix \cdot v\nu} (H(\underline{A})\kappa - \kappa H_0) F_W^{-1} e^{ix \cdot v\nu}, \end{aligned} \tag{3.41}$$

and clearly

$$\left\| \begin{pmatrix} 0 & 0 \\ 0 & e^{ix \cdot v\nu} \end{pmatrix} \right\|_{\mathcal{L}(\mathcal{H}(\underline{A}), \mathcal{H}(\underline{A}))} = 1. \tag{3.42}$$

From Lemmata 3.33, 3.36 and from Eqs. (2.5), (2.9), (3.36)–(3.42), it follows that [see also (3.148) and see (2.15), (2.22) and (3.3)]

$$\begin{aligned} & \int_0^t e^{isH(\underline{A})} i(H(\underline{A})\kappa - \kappa H_0) e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} \phi \\ &= \int_0^t e^{isH(\underline{A})} e^{ix \cdot v\nu} i \begin{pmatrix} 0 & 0 \\ -i(2(\mathbf{p}\kappa) \cdot \nu - 2\kappa A \cdot \nu) & 2\kappa A_0 \end{pmatrix} e^{-is\beta(v+\nu \cdot \mathbf{p})} \phi \\ & \quad + O\left(\int_0^{|t|} ds \left[\|\phi\|_{\mathbf{H}_{(x)4l}^\alpha(\mathbb{R}^3)^2} \left(\frac{1}{v(1+|s|)^{2l-1}} + \sum_{j=1}^{2l} \frac{1}{(1+|s|)^{2l-j} v^{\min(\alpha, j)}} \right) \right. \right. \\ & \quad \left. \left. + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \frac{1}{v(1+|s|)^{\zeta-1}} \right] + \frac{|t|}{v^\alpha} \|\phi\|_{\mathbf{H}^\alpha(\mathbb{R}^3)^2} \right). \end{aligned} \tag{3.43}$$

Since ϕ is supported in $\Lambda_{\kappa,\nu}$ [see (2.32)], and using that $e^{is\nu\cdot\mathbf{p}}$ is a translation operator in $L^2(\mathbb{R}^2)$, we prove that

$$\begin{aligned} & \begin{pmatrix} 0 & 0 \\ -i(2(\mathbf{p}\kappa)\cdot\nu - 2\kappa A\cdot\nu) & 2\kappa A_0 \end{pmatrix} e^{-is\beta(v+\nu\cdot\mathbf{p})}\phi \\ &= \kappa \begin{pmatrix} 0 & 0 \\ 2i\kappa A\cdot\nu & 2\kappa A_0 \end{pmatrix} e^{-is\beta(v+\nu\cdot\mathbf{p})}\phi. \end{aligned} \tag{3.44}$$

Now we consider that, see (2.15), (2.22)

$$\begin{aligned} & e^{isH(\underline{A})}e^{ix\cdot\nu\nu}\kappa i \begin{pmatrix} 0 & 0 \\ i2\kappa A\cdot\nu & 2\kappa A_0 \end{pmatrix} \\ &= e^{isH(\underline{A})}\kappa e^{-isH_0}F_W^{-1}e^{ix\cdot\nu\nu}e^{-ix\cdot\nu\nu}F_We^{-isH_0}F_W^{-1}e^{ix\cdot\nu\nu}i \begin{pmatrix} 0 & 0 \\ i2\kappa A\cdot\nu & 2\kappa A_0 \end{pmatrix} \\ &= e^{isH(\underline{A})}\kappa e^{-isH_0}F_W^{-1}e^{ix\cdot\nu\nu}[e^{-ix\cdot\nu\nu}e^{-isB_0\beta}e^{ix\cdot\nu\nu}]i \begin{pmatrix} 0 & 0 \\ i2\kappa A\cdot\nu & 2\kappa A_0 \end{pmatrix}, \end{aligned} \tag{3.45}$$

where we use that $F_W^{-1} \begin{pmatrix} 0 & 0 \\ X & Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ X & Y \end{pmatrix}$, for every matrix of the form $\begin{pmatrix} 0 & 0 \\ X & Y \end{pmatrix}$. We additionally use that $F_W^{-1} : L^2(\mathbb{R}^3)^2 \rightarrow \mathcal{H}_0$ is unitary and that $\kappa : \mathcal{H}_0 \rightarrow \mathcal{H}(\underline{A})$ is bounded (actually κ is bounded from $\mathbf{H}^1(\mathbb{R}^3)$ with values in $\mathbf{H}^1(\Lambda)$). Eqs. (2.5), (2.9), (3.43)–(3.45), (2.23), (3.3) and Lemma 3.35 imply, see also (3.148), (3.35). \square

Lemma 3.11. *Let $\nu \in \mathbb{S}^2$ and $l \in \mathbb{N}$, $l \geq \zeta/2$. Suppose that $\phi \in \mathbf{H}_{(x)4l}^0(\mathbb{R}^3)^2 \cap \mathbf{H}^\alpha(\mathbb{R}^3)^2$, $\alpha > 0$, is supported in $\Lambda_{\kappa,\nu}$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_\Phi^{\text{reg}}(B)$. Then, for every $v \geq v_0$ (see Lemma 3.29),*

$$\begin{aligned} & \left\| \left[W_\pm(\underline{A}) - e^{itH(\underline{A})}\kappa e^{-itH_0} \right] F_W^{-1}e^{ix\cdot\nu\nu}\phi \right\|_{\mathcal{H}(\underline{A})} \\ & \leq C\|\phi\|_{\mathbf{H}_{(x)4l}^0(\mathbb{R}^3)^2} \left(\frac{1}{1+|t|} \right)^{\zeta-1} + C\|\phi\|_{\mathbf{H}^\alpha(\mathbb{R}^3)^2} \frac{1}{v^\alpha}, \end{aligned} \tag{3.46}$$

for every $t \in \mathbb{R}$, such that $\pm t \geq 0$.

Proof. We take, without loss of generality, the plus sign in $W_\pm(\underline{A})$. Set $\tau \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. By Duhamel’s formula

$$\begin{aligned} & \left[W_+(\underline{A}) - e^{itH(\underline{A})}\kappa e^{-itH_0} \right] F_W^{-1}e^{ix\cdot\nu\nu}\tau(16\mathbf{p}/v)\phi \\ &= \int_t^\infty e^{isH(\underline{A})}i(H(\underline{A})\kappa - \kappa H_0)e^{-isH_0}F_W^{-1}e^{ix\cdot\nu\nu}\tau(16\mathbf{p}/v)\phi \\ &= \int_t^\infty e^{isH(\underline{A})}e^{ix\cdot\nu\nu}e^{-ix\cdot\nu\nu}i(H(\underline{A})\kappa - \kappa H_0) \\ & \quad \times F_W^{-1}e^{ix\cdot\nu\nu}e^{-ix\cdot\nu\nu}F_We^{-isH_0}F_W^{-1}e^{ix\cdot\nu\nu}\tau(16\mathbf{p}/v)\phi. \end{aligned} \tag{3.47}$$

Using (3.37)–(3.42), it follows from Lemma 3.34 [see also (3.3), (2.5) and (2.9)] that for every $l \in \mathbb{N}$ there is a constant C_l such that

$$\begin{aligned} & \left\| e^{-ix \cdot v\nu} i(H(\underline{A})\kappa - \kappa H_0) F_W^{-1} e^{ix \cdot v\nu} e^{-ix \cdot v\nu} F_W e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} \tau(16\mathbf{p}/v)\phi \right\|_{\mathcal{H}(\underline{A})} \\ & \leq C_l \left[\|\phi\|_{\mathbf{H}_{(x),4l}^0(\mathbb{R}^3)} \left(\frac{1}{(1+|t|)^{2l}} \right) + \|\phi\|_{\mathbf{H}^0(\mathbb{R}^3)} \frac{1}{(1+|t|)^l} \right], \end{aligned} \tag{3.48}$$

for every $v > v_0$ (see Lemma 3.29), $\nu \in \mathbb{S}^2$, $t \geq 0$, and every $\phi \in \mathbf{H}_{(x),4l}^0(\mathbb{R}^3)$. Equation (3.48) together with (3.148), (3.47) and (3.42) imply the desired result. \square

Lemma 3.12. *Take $\nu \in \mathbb{S}^2$ and suppose that $\phi \in \mathbf{H}_{(x),2}^0(\mathbb{R}^3)^2$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_{\Phi}^{\text{reg}}(B)$. Then, for every $v \geq v_0$ (see Lemma 3.29) and every $t \in \mathbb{R}$*

$$\begin{aligned} & \int_0^t e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} iZ_{\underline{A}}(t, v\nu, \kappa) \phi \, ds \\ & = \int_0^t e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} iW_{\underline{A}}(t, \nu, \kappa) \phi \, ds + O\left(\frac{1}{v}\right) \|\phi\|_{\mathbf{H}_{(x),4}^0(\mathbb{R}^3)^2}, \end{aligned} \tag{3.49}$$

recall Definition 3.9.

Proof. We define

$$Z' = \begin{pmatrix} -\kappa A_0 & i\kappa A \cdot \nu \\ i\kappa A \cdot \nu & \kappa A_0 \end{pmatrix}, \quad W' = \begin{pmatrix} \kappa A_0 & -i\kappa A \cdot \nu \\ i\kappa A \cdot \nu & \kappa A_0 \end{pmatrix} \tag{3.50}$$

as Z' anti-commutes with β and W' commutes with β ,

$$Z_{\underline{A}}(s, v\nu, \kappa)\phi = (W_{\underline{A}}(s, \nu, \kappa) + e^{2is\beta\nu} e^{is\beta\nu \cdot \mathbf{p}} Z' e^{-is\beta\nu \cdot \mathbf{p}})\phi. \tag{3.51}$$

We plug (3.51) in the left-hand side of (3.49) and estimate the term corresponding to the second term in (3.51) using an integration by parts:

$$\begin{aligned} & \int_0^t e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} i e^{2is\beta\nu} e^{is\beta\nu \cdot \mathbf{p}} Z' e^{-is\beta\nu \cdot \mathbf{p}} \phi \\ & = e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} i(2i\beta\nu)^{-1} e^{2is\beta\nu} e^{is\beta\nu \cdot \mathbf{p}} Z' e^{-is\beta\nu \cdot \mathbf{p}} \Big|_0^t \\ & \quad - \int_0^t \left(\frac{\partial}{\partial s} e^{isH(\underline{A})} \kappa e^{-isH_0} \right) F_W^{-1} e^{ix \cdot v\nu} i(2i\beta\nu)^{-1} e^{2is\beta\nu} e^{is\beta\nu \cdot \mathbf{p}} Z' e^{-is\beta\nu \cdot \mathbf{p}} \phi \\ & \quad - \int_0^t e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} (2i\beta\nu)^{-1} i e^{2is\beta\nu} \left(\frac{\partial}{\partial s} e^{is\beta\nu \cdot \mathbf{p}} Z' e^{-is\beta\nu \cdot \mathbf{p}} \right) \phi \\ & = O\left(\frac{1}{v}\right) \|\phi\|_{\mathbf{H}_{(x),4}^0(\mathbb{R}^3)^2}. \end{aligned} \tag{3.52}$$

To justify the last line in Eq. (3.52), we notice that following the procedure of the proof of Corollary 3.31, using Remark 3.30, we prove that there is an integrable function $h(s)$ such that $\|Z' e^{-is\beta\nu \cdot \mathbf{p}} \phi\|_{(L^2(\mathbb{R}^n))^2} + \sum_{i=1}^3 \|(\frac{\partial}{\partial x_i} Z') e^{-is\beta\nu \cdot \mathbf{p}} \phi\|_{(L^2(\mathbb{R}^n))^2} \leq h(s) \|\phi\|_{\mathbf{H}_{(x),4}^0(\mathbb{R}^3)^2}$. We additionally notice that

$(\frac{\partial}{\partial s} e^{isH(\underline{A})} \kappa e^{-isH_0}) F_W^{-1} e^{ix \cdot v\nu} : L^2(\mathbb{R}^3)^2 \rightarrow \mathcal{H}(\underline{A})$ is bounded, independently of s and $v\nu$. Equations (3.50)–(3.52) imply (3.49). \square

Theorem 3.13. *Set $\nu \in \mathbb{S}^2$ and $l \in \mathbb{N}$, $l \geq \zeta/2$. Suppose that $\phi \in \mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2$ is supported in $\Lambda_{\kappa,\nu}$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_{\Phi}^{(\text{reg})}(B)$. Then, for every $v \geq v_0$ (see Lemma 3.29) and every $t \in \mathbb{R}$*

$$e^{itH(\underline{A})} \kappa e^{-itH_0} F_W^{-1} e^{ix \cdot v\nu} \phi = \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^t W_{\underline{A}}(r, \nu, \kappa) dr} \phi \tag{3.53}$$

$$+ O\left(\int_0^{|t|} ds \left[\|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \left(\frac{1}{v(1+|s|)^{2l-1}} + \sum_{j=1}^{2l} \frac{1}{(1+|s|)^{2l-j} v^{\min(2,j)}} \right) \right. \right. \\ \left. \left. + \|\phi\|_{\mathbf{H}^2(\mathbb{R}^3)^2} \frac{1}{v(1+|s|)^{\zeta-1}} \right] + \frac{v+|t|}{v^2} \|\phi\|_{\mathbf{H}_{(x)4}^2(\mathbb{R}^3)^2} \right). \tag{3.54}$$

Proof. By Duhamel’s formula

$$e^{itH(\underline{A})} \kappa e^{-itH_0} F_W^{-1} e^{ix \cdot v\nu} \phi - \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^t dr W_{\underline{A}}(r, \nu, \kappa)} \phi \\ = \left(e^{itH(\underline{A})} \kappa e^{-itH_0} F_W^{-1} e^{ix \cdot v\nu} e^{-i \int_0^t dr W_{\underline{A}}(r, \nu, \kappa)} - \kappa F_W^{-1} e^{ix \cdot v\nu} \right) e^{i \int_0^t dr W_{\underline{A}}(r, \nu, \kappa)} \phi \\ = \int_0^t ds \left[e^{isH(\underline{A})} i(H(\underline{A})\kappa - \kappa H_0) e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} \phi(s, t) \right. \\ \left. - e^{isH(\underline{A})} \kappa e^{-isH_0} F_W^{-1} e^{ix \cdot v\nu} iW_{\underline{A}}(s, \nu, \kappa) \phi(s, t) \right], \tag{3.55}$$

where

$$\phi(s, t) = e^{i \int_s^t dr W_{\underline{A}}(r, \nu, \kappa)} \phi. \tag{3.56}$$

The desired result follows from Lemmata 3.10 and 3.12 and Eq. (3.55), using that there is a constant C such that

$$\|\phi(s, t)\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \leq C \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2}. \tag{3.57}$$

\square

Lemma 3.14. *Let $\nu \in \mathbb{S}^2$. Suppose that $\phi \in \mathbf{H}_{(x)2\zeta}^0(\mathbb{R}^3)^2$ is supported in $\Lambda_{\kappa,\nu}$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_{\Phi}(B)$. Then*

$$\left\| \left(e^{i \int_0^{\pm\infty} W_{\underline{A}}(r, \nu, \kappa)} - e^{i \int_0^{\pm t} W_{\underline{A}}(r, \nu, \kappa)} \right) \phi \right\|_{L^2(\mathbb{R}^3)^2} \leq \frac{C}{(1+|t|)^{\zeta-1}} \|\phi\|_{\mathbf{H}_{(x)2\zeta}^0(\mathbb{R}^3)^2}, \tag{3.58}$$

for $t \geq 0$.

Proof. We take the plus sign. Let χ_t be the characteristic function of the ball in \mathbb{R}^3 of radius t and center 0. Set $\phi_t = \chi_{t/2} \phi$ and $\bar{\phi}_t = \phi - \phi_t$. It is clear that

$$\|\bar{\phi}_t\|_{L^2(\mathbb{R}^3)^2} \leq \frac{C}{(1+|t|)^{\zeta}} \|\phi\|_{\mathbf{H}_{(x)2\zeta}^0(\mathbb{R}^3)^2}, \tag{3.59}$$

for some constant C . By Duhamel’s formula and (3.59),

$$\begin{aligned} & \left\| \left(e^{i \int_0^\infty W_{\underline{A}}(r, \nu, \kappa)} - e^{i \int_0^t W_{\underline{A}}(r, \nu, \kappa)} \right) \phi \right\|_{L^2(\mathbb{R}^3)^2} \\ & \leq \left\| \int_t^\infty W_{\underline{A}}(r, \nu, \kappa) \phi_t \right\|_{L^2(\mathbb{R}^3)^2} + \left\| \int_t^\infty W_{\underline{A}}(r, \nu, \kappa) \bar{\phi}_t \right\|_{L^2(\mathbb{R}^3)^2} \\ & \leq \left\| \int_t^\infty W_{\underline{A}}(r, \nu, \kappa) \phi_t \right\|_{L^2(\mathbb{R}^3)^2} + \frac{C}{(1 + |t|)^{\zeta-1}} \|\phi\|_{\mathbf{H}^0_{(x)^\zeta}(\mathbb{R}^3)^2}. \end{aligned} \tag{3.60}$$

Next we notice, see (3.33) and (3.50), that

$$W_{\underline{A}}(r, \nu, \kappa) \phi_t = e^{ir\beta\nu \cdot \mathbf{p}} W^t e^{-ir\beta\nu \cdot \mathbf{p}} \phi_t. \tag{3.61}$$

As $e^{-ir\beta\nu \cdot \mathbf{p}} \phi_t$ is supported in the union of the balls of radius $t/2$ and centers $r\nu$ and $-r\nu$, the result follows from (2.5) and (2.6) [see also (3.60) and (3.61)]. \square

Lemma 3.15. *Set $\nu \in \mathbb{S}^2$. Suppose that $\phi, \psi \in \mathbf{H}^1(\mathbb{R}^3)^2$ are supported in $\Lambda_{\kappa, \nu}$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_\Phi^{(\text{reg})}(B)$. Then, for every $v \geq v_0$ (see Lemma 3.29)*

$$\begin{aligned} & \left| \langle \kappa F_W^{-1} e^{ix \cdot v\nu} \phi, \kappa F_W^{-1} e^{ix \cdot v\nu} \psi \rangle_{\mathcal{H}(\underline{A})} - \langle \kappa \phi, \kappa \psi \rangle_{L^2(\mathbb{R}^n)^2} \right| \\ & \leq C \frac{1}{v} \|\psi\|_{\mathbf{H}^1(\mathbb{R}^3)^2} \|\phi\|_{L^2(\mathbb{R}^3)^2}. \end{aligned} \tag{3.62}$$

Proof. Take

$$\phi = (\phi_1, \phi_2), \quad \psi = (\psi_1, \psi_2). \tag{3.63}$$

It follows that

$$\begin{aligned} & \langle \kappa F_W^{-1} e^{ix \cdot v\nu} \phi, \kappa F_W^{-1} e^{ix \cdot v\nu} \psi \rangle_{\mathcal{H}(\underline{A})} \\ & = \langle B(\underline{A}) \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \phi_1, B(\underline{A}) \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \\ & \quad + \langle \kappa \phi_2, \kappa \psi_2 \rangle_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{3.64}$$

thus, is it enough to prove that

$$\begin{aligned} & \left| \langle B(\underline{A})^2 \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \phi_1, \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \right. \\ & \quad \left. - \langle \kappa \phi_1, \kappa \psi_1 \rangle_{L^2(\mathbb{R}^3)} \right| \leq C \frac{1}{v} \|\psi\|_{\mathbf{H}^1(\mathbb{R}^3)^2} \|\phi\|_{L^2(\mathbb{R}^3)^2}. \end{aligned}$$

It is clear that

$$\begin{aligned} & \langle B(\underline{A})^2 \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \phi_1, \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \\ & = \langle e^{-ix \cdot v\nu} (i\vartheta(\mathbf{p}^2 + m^2)^{-1/2}) e^{ix \cdot v\nu} \phi_1, \kappa((\mathbf{p} + v\nu)^2 + m^2)^{-1/2} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \\ & \quad + \langle \kappa((\mathbf{p} + v\nu)^2 + m^2)^{1/2} \phi_1, \kappa((\mathbf{p} + v\nu)^2 + m^2)^{-1/2} \psi_1 \rangle_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where ϑ is defined in (3.39).

We notice that $\|e^{-ix \cdot v\nu} (i\vartheta(\mathbf{p}^2 + m^2)^{-1/2}) e^{ix \cdot v\nu}\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \leq C$ (independently of $v\nu$) and, furthermore,

$$\left\| ((\mathbf{p} + v\nu)^2 + m^2)^{-1/2} (\mathbf{p}^2 + 1)^{-1/2} (\mathbf{p}^2 + 1)^{1/2} \psi_1 \right\|_{L^2(\mathbb{R}^3)} \leq C \frac{1}{v} \|\psi_1\|_{\mathbf{H}^1(\mathbb{R}^3)}. \tag{3.65}$$

We obtain:

$$\begin{aligned}
 & \langle B(\underline{A})^2 \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \phi_1, \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \\
 &= \langle \kappa((\mathbf{p} + v\nu)^2 + m^2)^{1/2} \phi_1, \kappa((\mathbf{p} + v\nu)^2 + m^2)^{-1/2} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \\
 &+ O\left(\frac{1}{v}\right) \|\psi_1\|_{\mathbf{H}^1(\mathbb{R}^3)} \|\phi_1\|_{L^2(\mathbb{R}^3)}
 \end{aligned} \tag{3.66}$$

Set $\varsigma := 1 - \kappa^2$. The commutator $[((\mathbf{p} + v\nu)^2 + m^2)^{1/2}, \varsigma] = [((\mathbf{p} + v\nu)^2 + m^2)^{1/2}, \kappa^2] = e^{-ix \cdot v\nu} [(\mathbf{p}^2 + m^2)^{1/2}, \varsigma] e^{ix \cdot v\nu}$ is bounded if and only if $[(\mathbf{p}^2 + m^2)^{1/2}, \varsigma]$ is bounded. The latter is an integral operator in momentum space. The kernel of this operator is

$$(2\pi)^{-3} \widehat{\varsigma}(p - q) \left((p^2 + m^2)^{1/2} - (q^2 + m^2)^{1/2} \right),$$

from which it is easy to see that

$$\| [((\mathbf{p} + v\nu)^2 + m^2)^{1/2}, \kappa^2] \|_{\mathcal{L}(L^2(\mathbb{R}^3))} = \| [(\mathbf{p}^2 + m^2)^{1/2}, \kappa^2] \|_{\mathcal{L}(L^2(\mathbb{R}^3))} \leq C \tag{3.67}$$

where C does not depend on $v\nu$.

From (3.65), (3.66) and (3.67) we get

$$\begin{aligned}
 & \langle B(\underline{A})^2 \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \phi_1, \kappa(\mathbf{p}^2 + m^2)^{-1/2} e^{ix \cdot v\nu} \psi_1 \rangle_{L^2(\mathbb{R}^3)} \\
 &= \langle \kappa \phi_1, \kappa \psi_1 \rangle + O\left(\frac{1}{v}\right) \|\psi_1\|_{\mathbf{H}^1(\mathbb{R}^3)} \|\phi_1\|_{L^2(\mathbb{R}^3)}.
 \end{aligned} \tag{3.68}$$

Finally, (3.62) is obtained from (3.64) and (3.68). □

Theorem 3.16. *Set $\nu \in \mathbb{S}^2$ and $l \in \mathbb{N}$, $l \geq \zeta/2, l \geq 2$. Suppose that $\phi, \psi \in \mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2$ are supported in $\Lambda_{\kappa, \nu}$. Let $\underline{A} = (A_0, A)$, with $A \in \mathcal{A}_{\Phi}^{(\text{reg})}(B)$. Then*

$$\langle e^{-ix \cdot v\nu} F_W S(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \phi, \psi \rangle_{L^2(\mathbb{R}^3)^2} = \langle e^{-i \int_{-\infty}^{\infty} W_{\underline{A}}(r, \nu, \kappa) dr} \phi, \psi \rangle_{L^2(\mathbb{R}^3)^2} \tag{3.69}$$

$$+ \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \begin{cases} O\left(v^{1-\zeta} + \frac{1}{v}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(v)}{v}\right) & \text{if } \zeta = 2. \end{cases} \tag{3.70}$$

Proof. We assume, without loss of generality, that $v \geq v_0$ (see Lemma 3.29). Using Lemma 3.11 and Theorem 3.13 we obtain

$$\begin{aligned}
 & \langle e^{-ix \cdot v\nu} F_W S(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \phi, \psi \rangle_{L^2(\mathbb{R}^3)^2} \\
 &= \langle W_-(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \phi, W_+(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \psi \rangle_{\mathcal{H}(\underline{A})} \\
 &= \langle e^{-itH(\underline{A})} \kappa e^{itH_0} F_W^{-1} e^{ix \cdot v\nu} \phi, e^{itH(\underline{A})} \kappa e^{-itH_0} F_W^{-1} e^{ix \cdot v\nu} \psi \rangle_{\mathcal{H}(\underline{A})} \\
 &\quad + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} O\left(\left(\frac{1}{1+|t|}\right)^{\zeta-1} + \frac{1}{v^2}\right) \\
 &= \langle \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^{-t} W_{\underline{A}}(r, \nu, \kappa) dr} \phi, \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^t W_{\underline{A}}(r, \nu, \kappa) dr} \psi \rangle_{\mathcal{H}(\underline{A})} \\
 &\quad + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \left\{ O\left(\left(\frac{1}{1+|t|}\right)^{\zeta-1} + \frac{1}{v^2}\right) \right. \\
 &\quad \left. + O\left(\int_0^{|t|} ds \left[\left(\frac{1}{v(1+|s|)^{2l-1}} + \sum_{j=1}^{2l} \frac{1}{(1+|s|)^{2l-j} v^{\min(2,j)}}\right) \right. \right. \right. \\
 &\quad \quad \left. \left. \left. + \frac{1}{v(1+|s|^{\zeta-1})} \right] + \frac{v+|t|}{v^2}\right)\right\} \\
 &= \langle \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^{-t} W_{\underline{A}}(r, \nu, \kappa) dr} \phi, \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^t W_{\underline{A}}(r, \nu, \kappa) dr} \psi \rangle_{\mathcal{H}(\underline{A})} \\
 &\quad + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \begin{cases} O\left(v^{1-\zeta} + \frac{1}{v}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(v)}{v}\right) & \text{if } \zeta = 2, \end{cases} \tag{3.71}
 \end{aligned}$$

where we select $|t| = v$, after integration, and we use that $l \geq 2$. Equation (3.71), Lemma 3.15 and Lemma 3.14 (taking $|t| = v$) imply that

$$\begin{aligned}
 & \langle e^{-ix \cdot v\nu} F_W S(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \phi, \psi \rangle_{L^2(\mathbb{R}^3)^2} \\
 &= \langle e^{i \int_0^{-t} W_{\underline{A}}(r, \nu, \kappa) dr} \phi, e^{i \int_0^t W_{\underline{A}}(r, \nu, \kappa) dr} \psi \rangle_{L^2(\mathbb{R}^3)^2} \\
 &\quad + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \begin{cases} O\left(v^{1-\zeta} + \frac{1}{v}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(v)}{v}\right) & \text{if } \zeta = 2 \end{cases} \tag{3.72}
 \end{aligned}$$

$$= \langle e^{i \int_0^{-\infty} W_{\underline{A}}(r, \nu, \kappa) dr} \phi, e^{i \int_0^{\infty} W_{\underline{A}}(r, \nu, \kappa) dr} \psi \rangle_{L^2(\mathbb{R}^3)^2} \tag{3.73}$$

$$\begin{aligned}
 & \quad + \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2} \begin{cases} O\left(v^{1-\zeta} + \frac{1}{v}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(v)}{v}\right) & \text{if } \zeta = 2. \end{cases} \tag{3.74}
 \end{aligned}$$

□

Remark 3.17. In the proof of Theorem 3.16 we use high-momenta asymptotic formulas for the wave operators, which are deduced in Eq. (3.71). We actually

prove:

$$\begin{aligned}
 & \left\| W_{-}(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \phi - \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^{-\infty} W_{\underline{A}}(r, \nu, \kappa) dr} \phi \right\|_{\mathcal{H}(\underline{A})} \\
 &= \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)} \begin{cases} O\left(v^{1-\zeta} + \frac{1}{v}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(v)}{v}\right) & \text{if } \zeta = 2 \end{cases}, \\
 & \left\| W_{+}(\underline{A}) F_W^{-1} e^{ix \cdot v\nu} \psi - \kappa F_W^{-1} e^{ix \cdot v\nu} e^{i \int_0^{\infty} W_{\underline{A}}(r, \nu, \kappa) dr} \psi \right\|_{\mathcal{H}(\underline{A})} \\
 &= \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)} \begin{cases} O\left(v^{1-\zeta} + \frac{1}{v}\right) & \text{if } \zeta \neq 2 \\ O\left(\frac{\ln(v)}{v}\right) & \text{if } \zeta = 2 \end{cases}. \tag{3.75}
 \end{aligned}$$

3.4.2. High-Momenta Limit for the Scattering Operator: General Magnetic Potentials (Proof of Theorem 2.7). In this section we prove our main results: we give a high-momenta expression for the scattering operator, with error bounds. This formula is the content of Theorem 2.7. Our formula is used in Sects. 3.5 and 3.6 to reconstruct important information from the potentials and the magnetic field.

Proof of Theorem 2.7. Equation (3.31) implies that $S(\underline{A}) = S(\tilde{\underline{A}})$ for a regular magnetic potential $\tilde{A} \in \mathcal{A}_{\Phi}^{(\text{reg})}(B)$. Note that \tilde{A} always exists, by Remark 2.4, and furthermore, there is λ with $A - \tilde{A} = \nabla\lambda$. In fact λ is given by (2.11), from which we deduced in the lines below (2.13) that λ_{∞} is constant. This implies that

$$e^{-i \int_{-\infty}^{\infty} W_{\underline{A}}(r, \nu, \kappa) dr} = e^{-i \int_{-\infty}^{\infty} W_{\tilde{\underline{A}}}(r, \nu, \kappa) dr}. \tag{3.76}$$

It follows from (2.23) to (2.27) and (3.33) that

$$\begin{aligned}
 & W_{\underline{A}}(t, v\nu, \kappa) \\
 &= e^{it\beta(\nu \cdot \mathbf{p})} \left(\kappa A_0(x) - \kappa A \cdot \nu(x) \beta \right) e^{-it\beta(\nu \cdot \mathbf{p})} \\
 &= Q^{-1} \begin{pmatrix} \kappa A_0(x + \nu t) - \kappa A \cdot \nu(x + \nu t) & 0 \\ 0 & \kappa A_0(x - \nu t) + \kappa A \cdot \nu(x - \nu t) \end{pmatrix} Q, \tag{3.77}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 & e^{-i \int_{-\infty}^{\infty} W_{\underline{A}}(r, v\nu, \kappa) dr} \phi \\
 &= Q^{-1} \begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr \kappa(A \cdot \nu - A_0)(x+r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr \kappa(A \cdot \nu + A_0)(x-r\nu)} \end{pmatrix} Q \phi \\
 &= Q^{-1} \begin{pmatrix} e^{i \int_{-\infty}^{\infty} dr (A \cdot \nu - A_0)(x+r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr (A \cdot \nu + A_0)(x+r\nu)} \end{pmatrix} Q \phi. \tag{3.78}
 \end{aligned}$$

The desired result follows from Theorem 3.16 and Eqs. (3.76) and (3.78). \square

3.5. Inverse Scattering Reconstruction Method

3.5.1. Proof of Theorem 2.9. Set $y \in \Lambda_{\text{Rec}}$. Suppose that P_y is a two-dimensional plane P_y such that $y + P_y \subset \kappa^{-1}(\{1\})^\circ$, for some function $\kappa \in C^\infty(\mathbb{R}^3)$ satisfying (2.5) and the text above it (see Definition 2.8 and Remark 2.10).

Theorem 2.7 implies that the scattering operator uniquely determines

$$\left\langle \left(\begin{array}{cc} e^{i \int_{-\infty}^{\infty} dr(A \cdot \nu - A_0)(x+r\nu)} & 0 \\ 0 & e^{-i \int_{-\infty}^{\infty} dr(A \cdot \nu + A_0)(x+r\nu)} \end{array} \right) \phi, \psi \right\rangle_{L^2(\mathbb{R}^3)^2}. \quad (3.79)$$

Here we suppose that $\phi, \psi \in \mathbf{H}_{(x),4l}^2(\mathbb{R}^3)^2$ ($l \in \mathbb{N}, l \geq \zeta/2, l \geq 2$) are supported in $B(y; \delta) + P_y$, for some small enough δ such that $B(y; \delta) + P_y \subset \kappa^{-1}(\{1\})^\circ$. Selecting conveniently ϕ and ψ we obtain that the scattering matrix uniquely determines

$$e^{-2i \int_L A} = \left(e^{i \int_{-\infty}^{\infty} dr(A \cdot \nu - A_0)(x+r\nu)} \right)^{-1} e^{-i \int_{-\infty}^{\infty} dr(A \cdot \nu + A_0)(x+r\nu)} \quad (3.80)$$

$$e^{-i \int_L 2A_0} = e^{i \int_{-\infty}^{\infty} dr(A \cdot \nu - A_0)(x+r\nu)} e^{-i \int_{-\infty}^{\infty} dr(A \cdot \nu + A_0)(x+r\nu)},$$

for every line $L \subset y + P_y$. Here the integral in the first equality denotes an integral of the 1-form A and the one in the second denotes a scalar integral with respect to the Lebesgue measure in the line L . Denote by $R(A_0)(x, \nu) = \int_{-\infty}^{\infty} A_0(x + \tau\nu) d\tau$. Equation (3.80) implies that there exists an integer-valued function $n(x, \nu)$ such that $2R(x, \nu) + 2\pi n(x, \nu)$ can be recovered from the scattering operator, for every x, ν such that $x + \mathbb{R}\nu \subset y + P_y$. As R is continuous, then n can be taken to be constant. The decay properties of A_0 determine the value of n . Therefore we determine R from the scattering operator. As R describes a Radon transform, inverting this transform we can uniquely reconstruct A_0 in $y + P_y$. The full details of this procedure, as well as the reconstruction of the magnetic field, are carefully presented Theorem 6.3 in [4]. Using Theorem 6.3 in [4] we conclude that $A_0(z)$ and $B(z)$ can be uniquely reconstructed from (3.80), for every $z \in y + P_y$.

The argument of proof of the reconstruction of the magnetic field, as described in Theorem 6.3 in [4], requires the existence of unit vectors $\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_j$ — $j \in \{1, 2, 3\}$ —for every $x_0 \in \Lambda_{\text{Rec}}$ satisfying the following:

1.

$$\hat{\mathbf{u}}_j \cdot \hat{\mathbf{v}}_j = 0, \quad j \in \{1, 2, 3\},$$

2. the unit vectors

$$\hat{\mathbf{n}}_j := \hat{\mathbf{u}}_j \times \hat{\mathbf{v}}_j, \quad j = 1, 2, 3,$$

are linearly independent,

3.

$$B(x_0; \varepsilon) + p(\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_j) \subset \Lambda, \quad j = 1, 2, 3,$$

for some $\varepsilon > 0$, where $p(\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_j)$ is the two-dimensional plane generated by $\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_j$.

This is clearly possible in our case. However, we want to point out that in the proof of Theorem 6.3 in [4] we required $\hat{\mathbf{u}}_i \cdot \hat{\mathbf{v}}_j = 0$, $i, j \in \{1, 2, 3\}$. This is a typo in our previous work. The right formula is what is written in Item 1. above. The referred typo does not damage any argument in the proof of Theorem 6.3 in [4].

3.6. The Aharonov–Bohm Effect

In this section we assume that $B = 0$ and $A_0 = 0$, i.e., that the electromagnetic field vanishes in Λ . The hypothesis $A_0 = 0$ is assumed for convenience, in the spirit of the Aharonov–Bohm effect. Nevertheless some results are also valid for $A_0 \neq 0$. In Sect. 3.6.3 we explain the corresponding results. The results in this section are pretty much the same as the analogous achievements for the non-relativistic case in [4]. We follow the lines of [4], Section 7, omitting repeated proofs. However, we must present again some notation, already introduced [4], to help the reader to understand the statements of our results. In the case that $A_0 \neq 0$, notable differences between the relativistic and the non-relativistic cases hold true, see Sect. 3.6.3.

3.6.1. Theorem 2.11 and Applications.

Proof of Theorem 2.11. It follows from Theorem 2.7, (3.78) and the proof of Theorem 7.1 in [4]. □

Applications

Remark 3.18. Theorem 2.11 implies that from the high-momenta limit (2.44) for $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ we can reconstruct the fluxes

$$\int_{\alpha} A$$

for any closed curve α such that there is a surface (or chain) \mathcal{S} in Λ with $\partial\mathcal{S} = \alpha - \gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})$, because by Stokes’ theorem

$$\int_{\alpha} A = \int_{\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})} A + \int_{\mathcal{S}} B = \int_{\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})} A,$$

since $B = 0$. We recall that (see [19], page 47) $H_1(\Lambda; \mathbb{R})$ represents the first group of singular homology of Λ with coefficients in \mathbb{R} . As the coefficients are in \mathbb{R} it is, actually, a vector space. We, furthermore, recall that $H^1_{\text{de R}}(\Lambda)$ represents the de Rahm cohomology group in Λ , see [38].

Remark 3.19. As $\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})$ is a cycle, the homology class $[\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})]_{H_1(\Lambda; \mathbb{R})}$ is well defined.

We denote by

$$H_{1, \text{rec}}(\Lambda; \mathbb{R}) := \langle \{[\gamma(x, y, \hat{\mathbf{v}}, \hat{\mathbf{w}})]_{H_1(\Lambda; \mathbb{R})} : L(x, \hat{\mathbf{v}}) \cup L(x, \hat{\mathbf{w}}) \subset \Lambda\} \rangle, \tag{3.81}$$

where $\langle O \rangle$ denotes the vector space generated by O . $H_{1, \text{rec}}(\Lambda; \mathbb{R})$ is a vector subspace of $H_1(\Lambda; \mathbb{R})$. Let us denote by $H^1_{\text{de R}, \text{rec}}(\Lambda)$ the vector subspace of $H^1_{\text{de R}}(\Lambda)$ that is the dual to $H_{1, \text{rec}}(\Lambda; \mathbb{R})$, given by de Rham’s theorem (Theorem 4.17, p. 154 of [38]). Then, for all Φ and all $A \in \mathcal{A}_{\Phi}(0)$, from the

high-momenta limit (2.44) known for all \hat{v}, \hat{w} we reconstruct the projection of A into $H^1_{\text{de R, rec}}(\Lambda)$ modulo 2π , as we now show. Let

$$\left\{ [\sigma_j]_{H_1, \text{rec}(\Lambda; \mathbb{R})} \right\}_{j=1}^m,$$

be a basis of $H_1, \text{rec}(\Lambda; \mathbb{R})$, and let

$$\left\{ [A_j]_{H^1_{\text{de R, rec}}(\Lambda)} \right\}_{j=1}^m,$$

be the dual basis, i.e.,

$$\int_{\sigma_j} A_k = \delta_{j,k}, \quad j, k = 1, 2, \dots, m.$$

Let us denote by P_{rec} the projector onto $H^1_{\text{de R, rec}}(\Lambda)$. Hence, for any $A \in \mathcal{A}_\Phi(0)$

$$P_{\text{rec}} [A]_{H^1_{\text{de R}}(\Lambda)} = \sum_{j=1}^m \lambda_j [A_j]_{H^1_{\text{de R, rec}}(\Lambda)},$$

and, furthermore, as

$$\lambda_j = \int_{\sigma_j} A,$$

we reconstruct $\lambda_j, j = 1, 2, \dots, m$ (modulo 2π) from the high-momenta limit (2.44) known for all \hat{v}, \hat{w} .

3.6.2. Theorem 2.12. In this section we prove Theorem 2.12. This theorem is stated again in Theorem 3.25. We give additionally precise (explicit) definitions of the sets $\{\Lambda_h\}_{h \in \mathcal{I}}, \Lambda_{\text{out}}$ and the numbers $\{F_h\}_{h \in \mathcal{I}}$, stated in Theorem 2.12. We start by introducing some notations.

Below we give a definition of when a line $L(x, \hat{v})$ [see (2.45)] goes through holes of K . Take $r > 0$ such that $K \subset B(0; r)$. Suppose that $L(x, \hat{v}) \subset \Lambda$, and $L(x, \hat{v}) \cap B(0; r) \neq \emptyset$. We denote by $c(x, \hat{v})$ the curve consisting of the segment $L(x, \hat{v}) \cap B(0; r)$ and an arc on $\partial B(0; r)$ that connects the points $L(x, \hat{v}) \cap \partial B(0; r)$. We orient $c(x, \hat{v})$ in such a way that the segment of straight line has the orientation of \hat{v} . See Fig. 2.

Definition 3.20. (Definition 7.4 in [4]) A line $L(x, \hat{v}) \subset \Lambda$ goes through holes of K if $L(x, \hat{v}) \cap B(0; r) \neq \emptyset$ and $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} \neq 0$. Otherwise we say that $L(x, \hat{v})$ does not go through holes of K .

Definition 3.21. (Definition 7.5 in [4]) Two lines $L(x, \hat{v}), L(y, \hat{w}) \subset \Lambda$ that go through holes of K go through the same holes if $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = \pm [c(y, \hat{w})]_{H_1(\Lambda; \mathbb{R})}$. Furthermore, we say that the lines go through the same holes in the same direction if $[c(x, \hat{v})]_{H_1(\Lambda; \mathbb{R})} = [c(y, \hat{w})]_{H_1(\Lambda; \mathbb{R})}$.

Definition 3.22. For any $\hat{v} \in \mathbb{S}^2$ we denote by $\Lambda_{\hat{v}, \text{out}}$ the set of points $x \in \Lambda_{\hat{v}}$ such that $L(x, \hat{v})$ does not go through holes of K . We call this set the region without holes of $\Lambda_{\hat{v}}$. The holes of $\Lambda_{\hat{v}}$ is the set $\Lambda_{\hat{v}, \text{in}} := \Lambda_{\hat{v}} \setminus \Lambda_{\hat{v}, \text{out}}$.

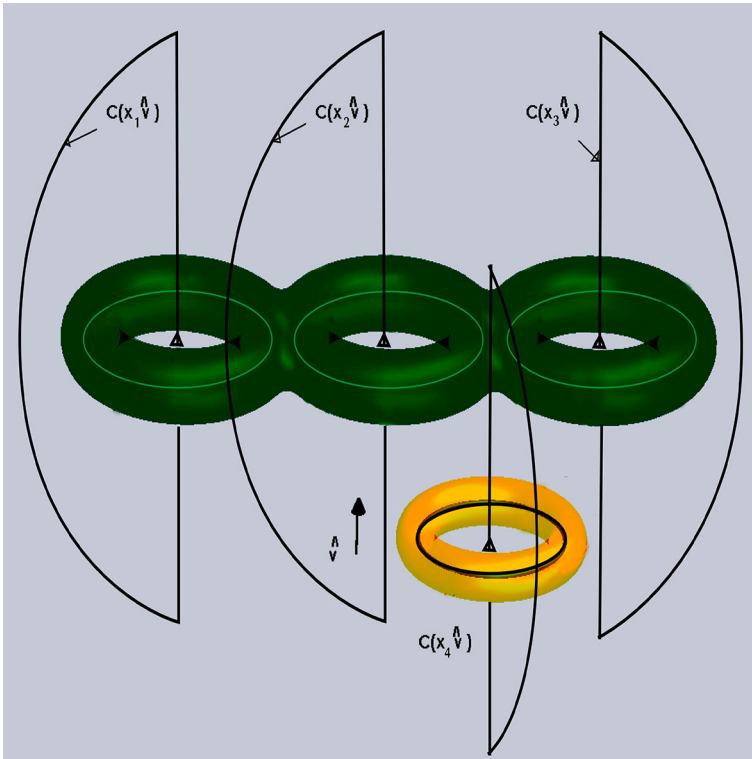


FIGURE 2. The curves $c(x, \hat{v})$

We define the following equivalence relation on $\Lambda_{\hat{v}, \text{in}}$. We say that $xR_{\hat{v}}y$ if, and only if, $L(x, \hat{v})$ and $L(y, \hat{v})$ go through the same holes and in the same direction. By $[x]$ we designate the classes of equivalence under $R_{\hat{v}}$.

We denote by $\{\Lambda_{\hat{v}, h}\}_{h \in \mathcal{I}}$ the partition of $\Lambda_{\hat{v}, \text{in}}$ given by this equivalence relation. It is defined as follows.

$$\mathcal{I} := \{[x]\}_{x \in \Lambda_{\hat{v}, \text{in}}}.$$

Given $h \in \mathcal{I}$ there is $x \in \Lambda_{\hat{v}, \text{in}}$ such that $h = [x]$. We denote,

$$\Lambda_{\hat{v}, h} := \{y \in \Lambda_{\hat{v}, \text{in}} : yR_{\hat{v}}x\}.$$

Then,

$$\Lambda_{\hat{v}, \text{in}} = \bigcup_{h \in \mathcal{I}} \Lambda_{\hat{v}, h}, \quad \Lambda_{\hat{v}, h_1} \cap \Lambda_{\hat{v}, h_2} = \emptyset, \quad h_1 \neq h_2.$$

We call $\Lambda_{\hat{v}, h}$ the holes h of K in the direction of \hat{v} . Note that

$$\{\Lambda_{\hat{v}, h}\}_{h \in \mathcal{I}} \cup \{\Lambda_{\hat{v}, \text{out}}\} \tag{3.82}$$

is an open disjoint cover of $\Lambda_{\hat{v}}$.

Definition 3.23. For any Φ , $A \in \mathcal{A}_\Phi(0)$, $\hat{\mathbf{v}} \in \mathbb{S}^2$, and $h \in \mathcal{I}$ we define,

$$F_h := \int_{c(x, \hat{\mathbf{v}})} A,$$

where x is any point in $\Lambda_{\hat{\mathbf{v}}, h}$. Note that F_h is independent the $x \in \Lambda_{\hat{\mathbf{v}}, h}$ that we choose. F_h is the flux of the magnetic field over any surface (or chain) in \mathbb{R}^3 whose boundary is $c(x, \hat{\mathbf{v}})$. We call F_h the magnetic flux on the holes h of K .

Let us take $\phi \in \mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2$ as in Theorem 2.7. We suppose, furthermore, that it is compactly supported. Then, since (3.82) is a disjoint open cover of $\Lambda_{\hat{\mathbf{v}}}$,

$$\phi = \sum_{h \in \mathcal{I}} \varphi_h + \varphi_{\text{out}}, \tag{3.83}$$

with $\varphi_h, \varphi_{\text{out}} \in \mathbf{H}_{(x)4l}^2(\mathbb{R}^3)^2$, φ_h has compact support in $\Lambda_{\hat{\mathbf{v}}, h}$, $h \in \mathcal{I}$, and φ_{out} has compact support in $\Lambda_{\hat{\mathbf{v}}, \text{out}}$. The sum is finite because ϕ has compact support. We denote, for $\mathbf{v} \in \mathbb{R}^3 \setminus \{0\}$ with $\mathbf{v}/|\mathbf{v}| = \hat{\mathbf{v}}$,

$$\phi_{\mathbf{v}} := e^{i\mathbf{v} \cdot x} \phi, \varphi_{h, \mathbf{v}} := e^{i\mathbf{v} \cdot x} \varphi_h, \varphi_{\text{out}, \mathbf{v}} := e^{i\mathbf{v} \cdot x} \varphi_{\text{out}}.$$

Remark 3.24. We remark that for every flux Φ there exists a compactly supported C^α -vector potential $\tilde{A} \in \mathcal{A}_\Phi(0)$. This holds true for the following reason: take any C^∞ gauge $A \in \mathcal{A}_\Phi(0)$ (for example the Coulomb Gauge). Take any C^∞ bounded domain \mathcal{D} containing K such that $\mathbb{R}^3 \setminus \mathcal{D}$ is simply connected. Set $x_0 \in \mathcal{D}^c$ and define

$$\lambda(x) = \int_{C_{x_0, x}} A, \tag{3.84}$$

where $C_{x_0, x}$ is any C^∞ curve connecting x_0 with x in \mathcal{D}^c . We extend λ to a C^α function $\bar{\lambda}$ defined in \mathbb{R}^3 . We take finally

$$\tilde{A}(x) = A(x) - \nabla \bar{\lambda}(x), \quad x \in \Lambda. \tag{3.85}$$

Theorem 3.25. Set $\phi, \psi \in \mathbf{H}_{(x)4l}^\alpha(\mathbb{R}^3)^2$ as in Theorem 2.7, with ϕ compactly supported. For every $A \in \mathcal{A}_\Phi(0)$

$$\begin{aligned} \langle US(\underline{A})U^{-1} \phi_{\mathbf{v}}, \psi_{\mathbf{v}} \rangle &= \left\langle \left(\sum_{h \in \mathcal{I}} \begin{pmatrix} e^{iF_h} & 0 \\ 0 & e^{-iF_h} \end{pmatrix} \varphi_{\mathbf{v}, h} + \varphi_{\text{out}, \mathbf{v}} \right), \psi_{\mathbf{v}} \right\rangle \\ &+ O\left(\frac{1}{v}\right) \|\phi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)} \|\psi\|_{\mathbf{H}_{(x)4l}^2(\mathbb{R}^3)}. \end{aligned} \tag{3.86}$$

Proof. Set $\tilde{A} \in \mathcal{A}_\Phi(0)$ be of class C^2 , compactly supported, such that $A = \tilde{A} + \nabla \lambda$, for some scalar function λ [see (2.10) and Remark 3.24]. Equation (3.31) implies that $S(\underline{A}) = S(\underline{\tilde{A}})$, with $\underline{A} = (0, \tilde{A})$. Then the desired result follows from Theorem 2.7, and the proof of Theorem 7.11 in [4]. Note that since \tilde{A} is of compact support the error term in Theorem 2.7 is of order $O\left(\frac{1}{v}\right)$. \square

Corollary 3.26. Under the conditions of Theorem 3.25, the high-momenta limit (3.86) of $S(\underline{A})$ in a single direction $\hat{\mathbf{v}}$ uniquely determines the fluxes $F_h, h \in \mathcal{I}$, modulo 2π .

Proof. The corollary follows immediately from Theorem 3.25. □

Remark 3.27. Notice that the result in Corollary 3.26 is sharp (we cannot recover more information than the fluxes modulo 2π) because the scattering operator is invariant under change of fluxes modulo 2π . This fact is achieved using gauge transformations that change fluxes modulo 2π (non-integrable factors). In order to keep our notation as simple as possible we did not use them in our analysis. However, a full detailed presentation of these is addressed in [4], for the Schrödinger equation. Our arguments in [4] can be smoothly applied to our case (see also Remark 3.6).

3.6.3. The Case $A_0 \neq 0$. Taking $A_0 \neq 0$ does not change substantially our reasoning. The results in Theorem 2.11 and Remarks 3.18 and 3.19 are deduced from the fact that the scattering operator uniquely determines

$$e^{-i \int_{-\infty}^{\infty} A \cdot \nu(x+r\nu)}, \tag{3.87}$$

which in our case holds true only when $A_0 = 0$. However, if $A_0 \neq 0$, we can recover

$$e^{-i \int_{-\infty}^{\infty} 2A \cdot \nu(x+r\nu)}, \tag{3.88}$$

see (3.80). Then if we substitute A by $2A$ in Theorem 2.11 and Remarks 3.18 and 3.19 we obtain the same results, for $A_0 \neq 0$. Some care has to be taken while considering the function κ as in (2.5), it essentially amounts to substitute K by $\kappa^{-1}(\{1\})^c$. We remark that the factor of 2 signifies a notable difference from the non-relativistic case, in which this factor is not present, see [4].

Theorem 3.25 gives a very simple formula for the high-momenta limit of the scattering operator in terms of magnetic fluxes. However, this simple formula is not anymore valid if $A_0 \neq 0$, because in this case a factors of the from $e^{\pm i \int_{-\infty}^{+\infty} A_0(x+\tau\nu)d\tau}$ must be present, see (3.79). This is also an important difference with respect to the non-relativistic case, in which the corresponding Theorem 3.25 is valid for $A_0 \neq 0$, see Theorem 7.11 in [4].

3.7. Some Technicalities: Stationary Phase Arguments

In quantum mechanics the free evolution of particles follows the classical evolution up to some error. The probability of finding the particle in a certain region enclosing the classical trajectory can be estimated. The accuracy of finding the particle close to the place where the classical particle would be depends on the wave packet spreading. These intuitive statements can be made precise by stationary phase arguments. Suppose for example that the free energy is given by the relativistic energy $B_0 = (\mathbf{p}^2 + m^2)^{1/2}$ and the initial state is a wave function $\phi \in \mathcal{S}(\mathbb{R}^3)$ whose Fourier transform is localized close to p_0 . The evolution of the particle at time t is

$$(e^{-itB_0}\phi)(x) = \frac{1}{(2\pi)^{3/2}} \int e^{i(x \cdot p - t(p^2+m^2)^{1/2})} \widehat{\phi}(p) dp. \tag{3.89}$$

Since (3.89) is an oscillatory integral, the bigger contribution is concentrated on the stationary points, i.e., the points on which the gradient in p of the

exponent vanishes:

$$\nabla_p(x \cdot p - t(p^2 + m^2)^{1/2}) = 0. \tag{3.90}$$

If the support of $\hat{\phi}$ is contained in a small ball around p_0 , this happens when

$$x \simeq t \frac{p_0}{(p_0^2 + m^2)^{1/2}}, \tag{3.91}$$

which is the description of the classical (relativistic) free trajectory with velocity $\frac{p_0}{(p_0^2 + m^2)^{1/2}}$.

This motivates the following definition that associates the (relativistic) velocity to the momentum.

Definition 3.28. (*Velocity Function*) We denote by $\mathbf{v} : \mathbb{R}^3 \mapsto \mathbb{R}^3$,

$$\mathbf{v}(p) := \frac{p}{(p^2 + m^2)^{1/2}}, \tag{3.92}$$

the function that associates to each momentum p the corresponding velocity.

If the particle is initially localized (at time $t = 0$), to a good approximation, in the ball $B(0; |t|r_0/2)$, for some $r_0 \in (0, \frac{1}{2}|x_0|)$ and $x_0 \in \mathbb{R}^3 \setminus \{0\}$, and the possible velocities are restricted (approximately) to a ball $B(x_0; r_0)$ then, at a time t , the particle is localized, to a good approximation, in the set $B(0; |t|r_0/2) + tB(x_0; r_0)$ (the initial position plus the time times the velocity). This is the content of the next Lemma, which is based on Theorem XI.14 in [33] (see also Lemma 2.1 in [39]).

Lemma 3.29. *Take $x_0 \in \mathbb{R}^3 \setminus \{0\}$, $r_0 \in (0, \frac{1}{2}|x_0|)$ and $f \in \mathcal{S}(\mathbb{R}^3)$ be such that $\mathbf{v}(\text{supp}(f)) \subset B(x_0; r_0)$. For every $l \in \mathbb{N}$ there is a constant C_l such that*

$$\left\| \chi_{(B(0; |t|r_0/2) + tB(x_0; r_0))} c \cdot e^{-itB_0} f(\mathbf{p}) \chi_{B(0; |t|r_0/2)} \right\| \leq C_l (1 + |t|)^{-l}. \tag{3.93}$$

Moreover, let $\tau \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. There exists $v_0 > 0$ and a constant C_l , for every $l \in \mathbb{N}$, such that

$$\left\| \chi_{B(\nu t; |t|/2)} c \cdot e^{-itB_0} \tau\left(\frac{16(\mathbf{p} - \nu\nu)}{v}\right) \chi_{B(0; |t|/8)} \right\| \leq C_l (1 + |t|)^{-l}, \tag{3.94}$$

for every $\nu \in \mathbb{S}^2$ and every $v \geq v_0$.

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^3)$. Using the Fourier transform we get

$$\begin{aligned} & \left[e^{-itB_0} f(\mathbf{p}) \chi_{B(0; |t|r_0/2)} \phi \right] (x) \\ &= \frac{1}{(2\pi)^3} \int dy \chi_{B(0; |t|r_0/2)} \phi(y) \int dp e^{i[(x-y) \cdot p - t(p^2 + m^2)^{1/2}]} f(p). \end{aligned} \tag{3.95}$$

We denote by

$$u_t(x, y) = \int dp e^{i[(x-y) \cdot p - t(p^2 + m^2)^{1/2}]} f(p). \tag{3.96}$$

The characteristic function in (3.93) constrains the values of x, y to satisfy

$$y \in B(0; |t|r_0/2), \quad x \notin B(0; |t|r_0/2) + tB(x_0; r_0). \tag{3.97}$$

The Corollary to Theorem XI.14 in [33] implies that for every $d \in \mathbb{N}$ there is a constant c_d such that, for x and y satisfying (3.97),

$$|u_t(x, y)| \leq c_d \frac{1}{(1 + |t| + |x - y|)^d}. \tag{3.98}$$

For $|x| \geq |t|r_0 \geq 2|y|$ we bound

$$\frac{1}{(1 + |t| + |x - y|)^d} \leq \frac{2^d}{(1 + |t| + |x|)^d}. \tag{3.99}$$

and for $|x| < |t|r_0$

$$\frac{1}{(1 + |t| + |x - y|)^d} \leq \frac{1}{(1 + |t|)^d}. \tag{3.100}$$

Equations (3.95)–(3.100) and the Cauchy–Schwartz inequality imply that there is a constant \tilde{c}_d such that

$$\left| \left[e^{-itB_0} f(\mathbf{p}) \chi_{B(0; |t|r_0/2)} \phi \right] (x) \right| \leq \tilde{c}_d |t|^{3/2} \left[\frac{1}{(1 + |t| + |x|)^d} + \frac{\chi_{B(0; |t|r_0)}}{(1 + |t|)^d} \right], \tag{3.101}$$

for x satisfying (3.97). A suitable election of d , the triangle inequality and (3.101) imply (3.93).

Equation (3.94) follows from (3.93) taking

$$f_{v,\nu} \equiv f = \tau \left(\frac{16(\mathbf{p} - v\nu)}{v} \right). \tag{3.102}$$

The necessary hypotheses for (3.93) are fulfilled for big v , taking $r_0 = 1/4$ and x_0 sufficiently close to ν to have

$$\chi_{(B(0; |t|r_0/2) + tB(x_0; r_0))^c} \cdot \chi_{B(\nu t; |t|/2)^c} = \chi_{B(\nu t; |t|/2)^c}. \tag{3.103}$$

The fact that the constants are independent of ν and v follows from (3.95) to (3.103) changing the variable of integration p in (3.95) by $z = \frac{16(p - v\nu)}{v}$, using the following estimation:

$$\begin{aligned} |u_t(x, y)| &= \left(\frac{v}{16} \right)^3 \\ &\left| \int dz \exp \left[i \left[((x - y)v/16) \cdot z - (tv/16)(z^2 + 32z \cdot \nu + 16^2 \right. \right. \right. \\ &\quad \left. \left. \left. + \left(\frac{16}{v} m \right)^2 \right)^{1/2} \right] \right] \tau(z) \right| \end{aligned} \tag{3.104}$$

(here we use that the factor $e^{i(x-y) \cdot v\nu}$ can be taken out of the integral and that it has modulus 1) and replacing $x - y$ by $\frac{v}{16}(x - y)$ and t by $\frac{v}{16}t$. Then we apply the proof of the Corollary to Theorem XI.14 in [33]. We point out that τ is independent of ν and v . Notice that we can assume that $|t| \geq 1$ because the left hand sides of (3.93) and (3.94) are bounded (we take also $v_0 \geq 1$). \square

Remark 3.30. The same conclusions of Lemma 3.29 hold true if we substitute B_0 by $\nu \cdot \mathbf{p}$ in (3.93) and (3.94). Actually, although same proof can be applied, in this case the analysis is much simpler because $e^{-it\nu \cdot \mathbf{p}}$ is a translation operator in position:

$$\left\| \chi_{(B(0;|t|r_0/2)+tB(x_0;r_0))^c} \cdot e^{-it\nu \cdot \mathbf{p}} f(\mathbf{p}) \chi_{B(0;|t|r_0/2)} \right\| \leq C_l(1+|t|)^{-l}, \quad (3.105)$$

$$\left\| \chi_{B(\nu t;|t|/2)^c} \cdot e^{-it\nu \cdot \mathbf{p}} \tau \left(\frac{16(\mathbf{p} - \nu\nu)}{v} \right) \chi_{B(0;|t|/8)} \right\| \leq C_l(1+|t|)^{-l}. \quad (3.106)$$

We only prove (3.105): let $\phi \in \mathcal{S}(\mathbb{R}^3)$. Using the Fourier transform we get

$$\begin{aligned} & \left[e^{-it\nu \cdot \mathbf{p}} f(\mathbf{p}) \chi_{B(0;|t|r_0/2)} \phi \right] (x) \\ &= \frac{1}{(2\pi)^3} \int dy \chi_{B(0;|t|r_0/2)} \phi(y) \int dp e^{i[(x-y) \cdot p - t\nu \cdot \mathbf{p}]} f(p). \end{aligned} \quad (3.107)$$

We denote by

$$u_t(x, y) = \int dp e^{i[(x-y) \cdot p - t\nu \cdot \mathbf{p}]} f(p). \quad (3.108)$$

The characteristic function in (3.105) constrains the values of x, y to satisfy

$$y \in B(0;|t|r_0/2), \quad x \notin B(0;|t|r_0/2) + tB(x_0;r_0). \quad (3.109)$$

The Corollary to Theorem XI.14 in [33] implies that for every $d \in \mathbb{N}$ there is a constant c_d such that, for x and y satisfying (3.97),

$$|u_t(x, y)| \leq c_d \frac{1}{(1+|t|+|x-y|)^d}. \quad (3.110)$$

We finish the proof following (3.99)–(3.101).

Corollary 3.31. *Suppose that $V : \mathbb{R}^3 \rightarrow \mathbb{C}$ is such that*

$$|V(x)| \leq C \frac{1}{(1+|x|)^{\alpha_V}} \quad (3.111)$$

for some $\alpha_V > 0$. Let $\phi \in \mathcal{S}(\mathbb{R}^3)$ be such that $f\hat{\phi} = \hat{\phi}$ for some f satisfying the hypothesis of Lemma 3.29.

Then there is a constant C such that

$$\|Ve^{itB_0}\phi\| \leq C \left(\frac{1}{1+|t|} \right)^{\alpha_V}, \quad t \in \mathbb{R}. \quad (3.112)$$

Proof. The result is a direct consequence of Lemma 3.29 writing $V = \left(\chi_{(B(0;|t|r_0/2)+tB(x_0;r_0))^c} + \chi_{(B(0;|t|r_0/2)+tB(x_0;r_0))^c} \right) V$ and $\phi = f(\mathbf{p})\phi = f(\mathbf{p}) \left[\chi_{B(0;|t|r_0/2)} + \chi_{B(0;|t|r_0/2)^c} \right] \phi. \quad \square$

3.7.1. Stationary Phase Arguments for High-Momenta. In this subsection we prove most of the technical results needed to derive the main achievements in this paper, which are stated and proved in Sect. 3.4. We estimate time evolution of relativistic particles, as explained at the beginning of Sect. 3.7, with the particularity that the particles we consider are very energetic. Then they behave as classical particles moving in a ballistic way, up to an error bound. More precisely, we can substitute the relativistic evolution $e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}}$ by a translation operator $e^{-it(v+\nu\cdot\mathbf{p})}$, which represents a classical free evolution with velocity ν . Here $\|\nu\| = 1$; this is in agreement with the election of our units system in which the speed of light is set to 1.

First we stress the following simple remark. We give the proof because it is used repeatedly in this paper, although it is elementary.

Remark 3.32. There is a constant C such that

$$\left\| \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu\cdot\mathbf{p})} \right] \phi \right\| \leq C \frac{|t|}{v} \|\phi\|_{\mathbf{H}^2(\mathbb{R}^3)}, \tag{3.113}$$

and

$$\left\| \frac{1}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} \phi \right\| \leq C \frac{1}{v} \|\phi\|_{\mathbf{H}^1(\mathbb{R}^3)}, \tag{3.114}$$

for every $v > 0, \nu \in \mathbb{S}^2, t \in \mathbb{R}$ and $\phi \in \mathbf{H}^2(\mathbb{R}^3)$. Furthermore, for every $\alpha, l \in \mathbb{N}$ there is a constant C_l such that

$$\left\| \frac{\mathbf{p} + v\nu}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} - \nu \right\|^l \phi \leq C_l \left(\frac{1}{v}\right)^{\min(\alpha, l)} \|\phi\|_{\mathbf{H}^\alpha(\mathbb{R}^3)}, \tag{3.115}$$

for every $v > 0, \nu \in \mathbb{S}^2$, and $\phi \in \mathbf{H}^\alpha(\mathbb{R}^3)$.

Proof. Using the fundamental theorem of calculus we find that $|e^{ia} - e^{ib}| = |1 - e^{ib-ia}| = \left| \int_0^{b-a} e^{i\tau} d\tau \right| \leq |b - a|$; for every real numbers a, b . This implies Eq. (3.113), since for $p \in \mathbb{R}^3$

$$\left| \left[e^{-it((p+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu\cdot p)} \right] \langle p \rangle^{-2} \right| \leq C \frac{|t|}{v}, \tag{3.116}$$

that is a consequence of the next calculation:

$$\begin{aligned} & \left| ((p + v\nu)^2 + m^2)^{1/2} - (v + \nu \cdot p) \right| \langle p \rangle^{-2} \\ &= \left| \frac{p^2 + m^2 - (\nu \cdot p)^2}{((p + v\nu)^2 + m^2)^{1/2} + (v + \nu \cdot p)} \right| \langle p \rangle^{-2}. \end{aligned} \tag{3.117}$$

We estimate (3.117) separately for $|p| < v/2$ and $|p| \geq v/2$. For $|p| \geq v/2$ we use the left hand side of (3.117) taking advantage of $\langle p \rangle^{-2}$. For $|p| < v/2$ we estimate the right hand side, taking into account that the denominator is bounded from below by $v/2$ and that $|p^2 + m^2 - (\nu \cdot p)^2| \langle p \rangle^{-2}$ is uniformly bounded.

Now we prove (3.115). We proceed as before taking $p \in \mathbb{R}^3$ instead of \mathbf{p} . We write $\phi = \langle \mathbf{p} \rangle^{-\alpha} \langle \mathbf{p} \rangle^\alpha \phi$. For $|p| \geq v/2$ we take advantage of $\langle p \rangle^{-\alpha}$ as before. For $|p| < v/2$ we use

$$\begin{aligned} & \left| \frac{p + v\nu}{((p + v\nu)^2 + m^2)^{1/2}} - \nu \right| \leq \left| \frac{p}{((p + v\nu)^2 + m^2)^{1/2}} \right| \\ & + \left| \frac{\left(((p + v\nu)^2 + m^2)^{1/2} - (v + \nu \cdot p) \right) \nu}{((p + v\nu)^2 + m^2)^{1/2}} \right| + \left| \frac{(\nu \cdot p)\nu}{((p + v\nu)^2 + m^2)^{1/2}} \right| \end{aligned} \tag{3.118}$$

and (3.117), without the factor $\langle p \rangle^{-2}$. Notice that in this case

$$\begin{aligned} & \left| \frac{p^2 + m^2 - (\nu \cdot p)^2}{((p + v\nu)^2 + m^2)^{1/2} + (v + \nu \cdot p)} \right| \\ & \leq |p| \left| \frac{v}{((p + v\nu)^2 + m^2)^{1/2} + (v + \nu \cdot p)} \right| + m \leq C(|p| + 1), \end{aligned}$$

for some constant C . Using similar techniques we prove Eq. (3.114). □

Lemma 3.33. *Let $\tau \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. Let $h : \mathbb{R}^3 \mapsto \mathbb{R}$ satisfy*

$$|h(x)| \leq C \left(\frac{1}{1 + |x|} \right)^\zeta, \tag{3.119}$$

for some $\zeta > 1$ and some constant C . Take $f \in C^\infty(\mathbb{R}^n)$ bounded with all derivatives bounded. For every $\alpha, l \in \mathbb{N}$ with $\alpha \geq 2$ there is a constant C_l such that

$$\begin{aligned} & \left\| h(x) f(\mathbf{p} + v\nu) \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu \cdot \mathbf{p})} \right] \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \right\| \\ & \leq C_l \left[\|\phi\|_{\mathbf{H}_{(x),4t}^\alpha(\mathbb{R}^3)} \left(\frac{1}{v(1+|t|)^{2l-1}} + \sum_{j=1}^{2l} \frac{1}{(1+|t|)^{2l-j} v^{\min(\alpha,j)}} \right) \right. \\ & \quad \left. + \|\phi\|_{\mathbf{H}^2(\mathbb{R}^3)} \frac{1}{v(1+|t|)^{\zeta-1}} \right], \end{aligned} \tag{3.120}$$

for every $v > v_0$ (see Lemma 3.29), $\nu \in \mathbb{S}^2$, $t \in \mathbb{R}$, and every $\phi \in \mathbf{H}_{(x),4t}^\alpha(\mathbb{R}^3)$.

Proof. We take $t > 0$ (without loss of generality). We use the shorthand notation

$$\psi = h(x) f(\mathbf{p} + v\nu) \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu \cdot \mathbf{p})} \right] \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \tag{3.121}$$

and write

$$\psi = \psi^{(1)} + \psi^{(2)} \tag{3.122}$$

with

$$\psi^{(1)} = \chi_{B(\nu t; |t|/2)} \psi, \quad \psi^{(2)} = \chi_{B(\nu t; |t|/2)} c \cdot \psi. \tag{3.123}$$

We estimate first $\psi^{(1)}$:

$$\begin{aligned} \|\psi^{(1)}\| &\leq \|\chi_{B(vt;|t|/2)} h(x) f(\mathbf{p} + v\nu)\| \\ &\quad \cdot \|e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu\cdot\mathbf{p})}\| \tau\left(\frac{16\mathbf{p}}{v}\right) \|\phi\| \\ &\leq C \frac{1}{v} \left(\frac{1}{1+|t|}\right)^{\zeta-1} \|\phi\|_{\mathbf{H}^2(\mathbb{R}^3)}, \end{aligned} \tag{3.124}$$

where we use Remark 3.32 and (3.119).

Now we estimate $\psi^{(2)}$. We have that

$$\begin{aligned} \|\psi^{(2)}\| &\leq C \left[\left\| \chi_{B(vt;|t|/2)} c \cdot e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} \tau\left(\frac{16\mathbf{p}}{v}\right) \frac{\chi_{B(0;|t|/8)}}{\langle x \rangle^{2l}} \right\| + \left\| \frac{\chi_{B(0;|t|/8)} c}{\langle x \rangle^{2l}} \right\| \right] \\ &\quad \cdot \left\| \left[\langle x \rangle^{2l} f(\mathbf{p} + v\nu) \left[e^{it\left[((\mathbf{p}+v\nu)^2+m^2)^{1/2} - (v+\nu\cdot\mathbf{p}) \right]} - 1 \right] \right] \phi \right\| \\ &\leq C_l \left(\frac{1}{1+|t|}\right)^{2l} \left\| \left[\langle x \rangle^{2l} f(\mathbf{p} + v\nu) \left[e^{-it\left[((\mathbf{p}+v\nu)^2+m^2)^{1/2} - (v+\nu\cdot\mathbf{p}) \right]} - 1 \right] \right] \phi \right\|, \end{aligned} \tag{3.125}$$

where we use Lemma 3.29 and (3.3). To estimate the remaining part in (3.125) we notice that the operator x is a differential operator in the Fourier transform representation (or momentum space):

$$x = i\nabla_{\mathbf{p}}. \tag{3.126}$$

Taking the commutator of $\langle x \rangle^{2l}$ with functions of \mathbf{p} produces derivatives with respect to \mathbf{p} . Taking into consideration that all derivatives of f are bounded and using (3.115) (and similar estimates) we get

$$\begin{aligned} &\left\| \left[\langle x \rangle^{2l} f(\mathbf{p} + v\nu) \left[e^{it\left[((\mathbf{p}+v\nu)^2+m^2)^{1/2} - (v+\nu\cdot\mathbf{p}) \right]} - 1 \right] \right] \phi \right\| \\ &\leq C \left\| \left[\langle x \rangle^{2l}, f(\mathbf{p} + v\nu) \left(e^{it\left[((\mathbf{p}+v\nu)^2+m^2)^{1/2} - (v+\nu\cdot\mathbf{p}) \right]} - 1 \right) \right] \phi \right\| \\ &\quad + C \left\| f(\mathbf{p} + v\nu) \left[e^{it\left[((\mathbf{p}+v\nu)^2+m^2)^{1/2} - (v+\nu\cdot\mathbf{p}) \right]} - 1 \right] \langle x \rangle^{2l} \phi \right\| \\ &\leq C \sum_{j=1}^{2l} \left(\frac{1+|t|}{v}\right)^{\min(\alpha,j)} \|\phi\|_{\mathbf{H}^{\alpha}_{(x)4l}(\mathbb{R}^3)} + C \frac{1+|t|}{v} \|\phi\|_{\mathbf{H}^{\alpha}_{(x)4l}(\mathbb{R}^3)}. \end{aligned} \tag{3.127}$$

Where we use the fact that for every $g \in \mathbf{H}^{\alpha}_{(x)4l}(\mathbb{R}^3)$ and every multi-index $\underline{l} = (l_1, l_2, l_3) \in (\mathbb{N} \cup \{0\})^3$, with $l_1 + l_2 + l_3 \leq 2l$,

$$\|\langle \mathbf{p} \rangle^{\alpha} x_1^{l_1} x_2^{l_2} x_3^{l_3} g\| \leq C \|x_1^{l_1} x_2^{l_2} x_3^{l_3} g\|_{\mathbf{H}^{\alpha}(\mathbb{R}^3)} \leq C \|g\|_{\mathbf{H}^{\alpha}_{(x)4l}(\mathbb{R}^3)}, \tag{3.128}$$

for some constant C . Equation (3.120) follows from (3.121), (3.122), (3.124), (3.125) and (3.127). □

Lemma 3.34. *Let $\tau \in C_0^\infty(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. Let $h : \mathbb{R}^3 \mapsto \mathbb{R}$ satisfy*

$$|h(x)| \leq C \left(\frac{1}{1 + |x|} \right)^\zeta, \tag{3.129}$$

for some $\zeta > 1$ and some constant C . Take $f \in C^\infty(\mathbb{R}^n)$ bounded with all derivatives bounded. For every $l \in \mathbb{N}$ there is a constant C_l such that

$$\begin{aligned} & \left\| h(x) f(\mathbf{p} + v\nu) \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} \right] \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \right\| \\ & \leq C_l \left[\|\phi\|_{\mathbf{H}_{(x)4l}^0(\mathbb{R}^3)} \left(\frac{1}{(1 + |t|)^{2l}} \right) + \|\phi\|_{\mathbf{H}^0(\mathbb{R}^3)} \frac{1}{(1 + |t|)^\zeta} \right], \end{aligned} \tag{3.130}$$

for every $v > v_0$ (see Lemma 3.29), $\nu \in \mathbb{S}^2$, $t \in \mathbb{R}$, and every $\phi \in \mathbf{H}_{(x)4l}^0(\mathbb{R}^3)$.

Proof. We take $t > 0$ (without loss of generality). We follow the procedure of the proof of Lemma 3.33. We use the shorthand notation

$$\psi = h(x)(f(\mathbf{p} + v\nu) \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} \right] \tau\left(\frac{16\mathbf{p}}{v}\right) \phi) \tag{3.131}$$

and write

$$\psi = \psi^{(1)} + \psi^{(2)} \tag{3.132}$$

with

$$\psi^{(1)} = \chi_{B(vt;|t|/2)} \psi, \quad \psi^{(2)} = \chi_{B(vt;|t|/2)^c} \psi. \tag{3.133}$$

We estimate first $\psi^{(1)}$:

$$\begin{aligned} \|\psi^{(1)}\| & \leq \|\chi_{B(vt;|t|/2)} h(x) f(\mathbf{p} + v\nu)\| \cdot \left\| e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \right\| \\ & \leq C \left(\frac{1}{1 + |t|} \right)^\zeta \|\phi\|_{\mathbf{H}^0(\mathbb{R}^3)}, \end{aligned} \tag{3.134}$$

where use (3.129).

Now we estimate $\psi^{(2)}$. We have that

$$\begin{aligned} \|\psi^{(2)}\| & \leq C \left[\left\| \chi_{B(vt;|t|/2)^c} e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} \tau\left(\frac{16\mathbf{p}}{v}\right) \frac{\chi_{B(0;|t|/8)}}{\langle x \rangle^{2l}} \right\| \right. \\ & \quad \left. + \left\| \frac{\chi_{B(0;|t|/8)^c}}{\langle x \rangle^{2l}} \right\| \right] \cdot \left\| \langle x \rangle^{2l} f(\mathbf{p} + v\nu) \phi \right\| \\ & \leq C_l \left(\frac{1}{1 + |t|} \right)^{2l} \left\| \langle x \rangle^{2l} f(\mathbf{p} + v\nu) \phi \right\|, \end{aligned} \tag{3.135}$$

where we use Lemma 3.29 and (3.3). To estimate the remaining part in (3.125) we use that $x = i\nabla_{\mathbf{p}}$. Taking the commutator of $\langle x \rangle^{2l}$ with functions of \mathbf{p} produces derivatives with respect to \mathbf{p} and as all derivatives of f are bounded

we get

$$\begin{aligned} & \left\| \left[\langle x \rangle^{2l} f(\mathbf{p} + v\nu) \phi \right] \right\| \\ & \leq \left\| \left[\langle x \rangle^{2l}, f(\mathbf{p} + v\nu) \right] \phi \right\| + \left\| f(\mathbf{p} + v\nu) \langle x \rangle^{2l} \phi \right\| \leq C \|\phi\|_{\mathbf{H}^0_{\langle x \rangle^{4l}}(\mathbb{R}^3)}. \end{aligned} \tag{3.136}$$

Equation (3.130) follows from (3.131), (3.132), (3.134), (3.135) and (3.136). \square

Lemma 3.35. *Let $\tau \in C^\infty_0(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. Let $h \in C^2(\mathbb{R}^3)$ satisfy*

$$\left| \left(\frac{\partial}{\partial x_1} \right)^{l_1} \left(\frac{\partial}{\partial x_2} \right)^{l_2} \left(\frac{\partial}{\partial x_3} \right)^{l_3} h(x) \right| \leq C \left(\frac{1}{1 + |x|} \right)^\zeta, \tag{3.137}$$

for some $\zeta > 1$, a constant C and every $l_1, l_2, l_3 \in \mathbb{N} \cup \{0\}$ with $l_1 + l_2 + l_3 \leq 2$. Take $f \in C^\infty(\mathbb{R}^n)$ bounded with all derivatives bounded. For every $l \in \mathbb{N}$, there is a constant C_l such that

$$\begin{aligned} & \left\| \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu \cdot \mathbf{p})} \right] h(x) f(\mathbf{p} + v\nu) e^{-it(v+\nu \cdot \mathbf{p})} \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \right\| \\ & \leq C \frac{1}{v} \left(\frac{1}{1 + |t|} \right)^{\zeta-1} \|\phi\|_{\mathbf{H}^2(\mathbb{R}^3)} + C_l \frac{1}{v} \left(\frac{1}{1 + |t|} \right)^{2l-1} \|\phi\|_{\mathbf{H}^2_{\langle x \rangle^{4l}}(\mathbb{R}^3)}, \end{aligned} \tag{3.138}$$

for every $v > v_0$ (see Lemma 3.29), $\nu \in \mathbb{S}^2$, $t \in \mathbb{R}$, and every $\phi \in \mathbf{H}^2_{\langle x \rangle^{4l}}(\mathbb{R}^3)$.

Proof. We take $t > 0$. Let $\theta \in C^\infty_0(\mathbb{R}^3; [0, 1])$ be equal 1 in $B(\nu; 1/2)$ and zero in $B(\nu; 2/3)^c$. Define

$$\theta_t(x) := \theta(x/t). \tag{3.139}$$

We use the shorthand

$$\begin{aligned} \psi^{(1)} &= \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu \cdot \mathbf{p})} \right] \theta_t(x) h(x) f(\mathbf{p} + v\nu) \\ & \quad \times e^{-it(v+\nu \cdot \mathbf{p})} \tau\left(\frac{16\mathbf{p}}{v}\right) \phi, \\ \psi^{(2)} &= \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu \cdot \mathbf{p})} \right] (1 - \theta_t(x)) h(x) f(\mathbf{p} + v\nu) \\ & \quad \times e^{-it(v+\nu \cdot \mathbf{p})} \tau\left(\frac{16\mathbf{p}}{v}\right) \phi. \end{aligned} \tag{3.140}$$

We analyze first $\psi^{(1)}$. We have that

$$\begin{aligned} \|\psi^{(1)}\| & \leq \left\| \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu \cdot \mathbf{p})} \right] \langle \mathbf{p} \rangle^{-2} \right\| \\ & \quad \cdot \left\| \langle \mathbf{p} \rangle^2 \theta_t(x) h(x) f(\mathbf{p} + v\nu) e^{-it(v+\nu \cdot \mathbf{p})} \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \right\| \end{aligned} \tag{3.141}$$

$$\leq C \frac{|t|}{v} \left(\frac{1}{1 + |t|} \right)^\zeta \|\phi\|_{\mathbf{H}^2(\mathbb{R}^3)}, \tag{3.142}$$

where we use (3.116) and (3.137).

Now we estimate $\psi^{(2)}$ using Remark 3.30 and (3.137):

$$\|\psi^{(2)}\| \leq \left\| \left[e^{-it((\mathbf{p}+v\nu)^2+m^2)^{1/2}} - e^{-it(v+\nu\cdot\mathbf{p})} \right] \langle \mathbf{p} \rangle^{-2} \right\| \cdot \left\| \langle \mathbf{p} \rangle^2 (1 - \theta_t(x)) h(x) f(\mathbf{p} + v\nu) e^{-it(v+\nu\cdot\mathbf{p})} \tau\left(\frac{16\mathbf{p}}{v}\right) \phi \right\| \tag{3.143}$$

$$\leq C_l \frac{|t|}{v} \left(\frac{1}{1+|t|} \right)^{2l} \|\phi\|_{\mathbf{H}^2_{(x)4l}(\mathbb{R}^3)}, \tag{3.144}$$

for every $l \in \mathbb{N}$. □

Lemma 3.36. *Let $h : \mathbb{R}^3 \mapsto \mathbb{R}$ and $b : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be such that*

$$|h(x)| \leq C \left(\frac{1}{1+|x|} \right)^\zeta, \quad \|b(x)\| \leq C \left(\frac{1}{1+|x|} \right)^\zeta, \tag{3.145}$$

for some constant C , and every $x \in \mathbb{R}^3$. It follows that there exist a constant C such that

$$\left\| h(x) \frac{1}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} e^{-it(v+\nu\cdot\mathbf{p})} \phi \right\| \leq C \left[\frac{1}{v(1+|t|)^\zeta} \|\phi\|_{\mathbf{H}^1_{(x)4l}(\mathbb{R}^3)} + \frac{1}{v^\alpha} \|\phi\|_{\mathbf{H}^\alpha(\mathbb{R}^3)} \right] \tag{3.146}$$

and

$$\left\| b(x) \cdot \left[\frac{\mathbf{p} + v\nu}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} - \nu \right] e^{-it(v+\nu\cdot\mathbf{p})} \phi \right\| \leq C \left[\frac{1}{v(1+|t|)^\zeta} \|\phi\|_{\mathbf{H}^1_{(x)4l}(\mathbb{R}^3)} + \frac{1}{v^\alpha} \|\phi\|_{\mathbf{H}^\alpha(\mathbb{R}^3)} \right], \tag{3.147}$$

for every $v > v_0$ (see Lemma 3.29), $\nu \in \mathbb{S}^2$, $t \in \mathbb{R}$, every natural number $l \geq \zeta/2$, and every $\phi \in \mathbf{H}^\alpha_{(x)4l}(\mathbb{R}^3)$, with $\alpha \in \{1, 2, \dots\}$.

Proof. We take, without loss of generality, $t > 0$. Let $\tau \in C^\infty_0(\mathbb{R}^3; [0, 1])$ be such that $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$ and it vanishes for $|x| \geq 1$. Notice that

$$\|(1 - \tau(16\mathbf{p}/v))\phi\| \leq C \frac{1}{v^\alpha} \|\phi\|_{\mathbf{H}^\alpha(\mathbb{R}^3)}. \tag{3.148}$$

We prove (3.146). The proof of (3.147) is similar, using (3.115). Define

$$\psi = h(x) \frac{1}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} e^{-it(v+\nu\cdot\mathbf{p})} \tau(16\mathbf{p}/v) \phi \tag{3.149}$$

and

$$\psi^{(1)} = \chi_{B(\nu t; |t|/2)} \psi, \quad \psi^{(2)} = \chi_{B(\nu t; |t|/2)^c} \psi. \tag{3.150}$$

We have that

$$\|\psi^{(1)}\| \leq C \|\chi_{B(\nu t; |t|/2)} h(x)\| \cdot \left\| \frac{1}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} \langle \mathbf{p} \rangle^{-1} \right\| \cdot \|\phi\|_{\mathbf{H}^1(\mathbb{R}^3)} \leq C \frac{1}{v(1+|t|)^\zeta} \|\phi\|_{\mathbf{H}^1(\mathbb{R}^3)}. \tag{3.151}$$

Now we estimate $\psi^{(2)}$ using Remark 3.30:

$$\begin{aligned} \|\psi^{(2)}\| &\leq \left[\|\chi_{B(\nu t; |t|/2)} c \cdot e^{-it(v+\nu \cdot \mathbf{p})} \tau(16\mathbf{p}/v) \chi_{B(0; |t|/8)} \langle x \rangle^{-2l} \| \right. \\ &\quad \left. + \|\chi_{B(0; |t|/8)} c \langle x \rangle^{-2l} \| \right] \cdot \left\| \langle x \rangle^{2l} \frac{1}{((\mathbf{p} + v\nu)^2 + m^2)^{1/2}} \phi \right\| \quad (3.152) \\ &\leq C \frac{1}{v(1+|t|)^{2l}} \|\phi\|_{\mathbf{H}_{\langle x \rangle^{4l}}^1(\mathbb{R}^3)}, \end{aligned}$$

where we use the procedure in (3.125)–(3.128). Equation (3.146) follows from (3.149) to (3.152). \square

References

- [1] Adams, R.A., Fournier, J.J.F.: Sobolev spaces, 2nd edn. Pure and Applied Mathematics (Amsterdam), vol. 140. Elsevier/Academic Press, Amsterdam, p. xiv + 305 (2003)
- [2] Aharonov, Y., Bohm, D.: Significance of electromagnetic potentials in the quantum theory. *Phys. Rev.* **115**, 485–491 (1959)
- [3] Arians, S.: Geometric approach to inverse scattering for the Schrödinger’s equation with magnetic and electric potentials. *J. Math. Phys.* **38**, 2761–2773 (1997)
- [4] Ballesteros, M., Weder, R.: High-velocity estimates for the scattering operator and Aharonov–Bohm effect in three dimensions. *Commun. Math. Phys.* **285**, 345–398 (2009)
- [5] Ballesteros, M., Weder, R.: The Aharonov–Bohm effect and Tonomura et al. experiments: Rigorous results. *J. Math. Phys.* **50**, 122108 (2009)
- [6] Ballesteros, M., Weder, R.: Aharonov–Bohm effect and high-velocity estimates of solutions to the Schrödinger equation. *Commun. Math. Phys.* **303**, 175–211 (2011)
- [7] Ballesteros, M., Weder, R.: High-Velocity Estimates for Schrödinger Operators in Two Dimensions: Long-Range Magnetic Potentials and Time-Dependent Inverse-Scattering. *Rev. Math. Phys.* **27**, 1550006 (2015)
- [8] Enss, V., Jung, W.: Geometrical approach to inverse scattering. In: Proceedings of the first maphysto workshop on inverse problems, 22–24 April 1999, Aarhus. *MaPhySto Miscellanea*, vol. 13. ISSN 1398-5957 (1999)
- [9] Enss, V., Weder, R.: The geometrical approach to multidimensional inverse scattering. *J. Math. Phys.* **36**, 3902–3921 (1995)
- [10] Eskin, G.: Inverse boundary value problems and the Aharonov–Bohm effect. *Inverse Probl.* **19**, 49–62 (2003)
- [11] Eskin, G.: Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov–Bohm effect. *J. Math. Phys.* **49**, 022105 (2008)
- [12] Eskin, G., Isozaki, H., O’Dell, S.: Gauge equivalence and inverse scattering for Aharonov–Bohm effect. *Commun. Partial Differ. Equ.* **35**, 2164–2194 (2010)
- [13] Eskin, G., Isozaki, H.: Gauge equivalence and inverse scattering for long-range magnetic potentials. *Russ. J. Math. Phys.* **18**(1), 54–63 (2011)

- [14] Eskin, G., Ralston, J.: The Aharonov–Bohm effect in spectral asymptotics of the magnetic Schrödinger operator. *Anal. PDE* **7**, 245–266 (2014)
- [15] Franz, W.: Elektroneninterferenzen im Magnetfeld. *Verh. D. Phys. Ges.* (3) **20**(2), 65–66 (1939)
- [16] Franz, W.: Elektroneninterferenzen im Magnetfeld. *Physikalische Berichte* **21**, 686 (1940)
- [17] Feshbach, H., Villars, F.: Elementary relativistic wave mechanics of spin 0 and spin $\frac{1}{2}$ particles. *Rev. Mod. Phys.* **30**, 24–45 (1958)
- [18] Gérard, C.: Scattering theory for Klein–Gordon equations with non-positive energy. *Ann. Henri Poincaré* **13**, 883–941 (2012)
- [19] Greenberg, M.J., Harper, J.R.: Algebraic topology, A first course. Addison-Wesley, New York (1981)
- [20] Greiner, W.: Relativistic quantum mechanics, 3rd edn. Springer, Berlin (1987)
- [21] Helffer, B.: Effet d’Aharonov–Bohm sur un état borné de l’équation de Schrödinger (French) [The Aharonov–Bohm effect on a bound state of the Schrödinger equation]. *Commun. Math. Phys.* **119**, 315–329 (1988)
- [22] Ito, H.T., Tamura, H.: Aharonov–Bohm effect in scattering by point-like magnetic fields at large separation. *Ann. Henri Poincaré* **2**, 309–359 (2001)
- [23] Ito, H.T., Tamura, H.: Aharonov–Bohm effect in scattering by a chain of point-like magnetic fields. *Asymptot. Anal.* **34**, 199–240 (2003)
- [24] Ito, H.T., Tamura, H.: Semiclassical analysis for magnetic scattering by two solenoidal fields. *J. London Math. Soc.* (2) **74**, 695–716 (2006)
- [25] Jollivet, A.: On inverse scattering for the multidimensional relativistic Newton equation at high energies. *J. Math. Phys.* **47**, 062902 (2006)
- [26] Jollivet, A.: On inverse problems for the multidimensional relativistic Newton equation at fixed energy. *Inverse Probl.* **23**, 231–242 (2007)
- [27] Jollivet, A.: On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies. *Asymptot. Anal.* **55**, 103–123 (2007)
- [28] Jung, W.: Geometrical approach to inverse scattering for the Dirac equation. *J. Math. Phys.* **38**, 39–48 (1997)
- [29] Newton, T.D., Wigner, E.P.: Localized states for elementary particles. *Rev. Mod. Phys.* **21**, 400–406 (1949)
- [30] Nicoleau, F.: A stationary approach to inverse scattering for Schrödinger operators with first order perturbation. *Commun. Partial Differ. Equ.* **22**, 527–553 (1997)
- [31] Nicoleau, F.: An inverse scattering problem with the Aharonov–Bohm effect. *J. Math. Phys.* **41**, 5223–5237 (2000)
- [32] Reed, M., Simon, B.: Methods of modern mathematical physics II Fourier analysis, Self-adjointness. Academic Press, New York (1975)
- [33] Reed, M., Simon, B.: Methods of modern mathematical physics III scattering theory. Academic Press, New York (1979)
- [34] Roux, Ph.: Scattering by a toroidal coil. *J. Phys. A* **36**, 5293–5304 (2003)
- [35] Roux, Ph., Yafaev, D.R.: On the mathematical theory of the Aharonov–Bohm effect. *J. Phys. A* **35**, 7481–7492 (2002)
- [36] Roux, Ph., Yafaev, D.R.: The scattering matrix for the Schrödinger operator with a long-range electromagnetic potential. *J. Math. Phys.* **44**, 2762–2786 (2003)

- [37] Schechter, M.: Spectra of partial differential operators, 2nd edn. North-Holland Series in Applied Mathematics and Mechanics, vol. 14. North-Holland Publishing Co., Amsterdam, p. xiv+310 (1986)
- [38] Warner, F.W.: Foundations of differentiable manifolds. Springer-Verlag, Berlin (1983)
- [39] Weder, R.: The Aharonov–Bohm effect and time-dependent inverse scattering theory. *Inverse Probl.* **18**, 1041–1056 (2002)
- [40] Weder, R.: Selfadjointness and invariance of the essential spectrum for the Klein–Gordon equation. *Helv. Phys. Acta* **50**, 105–115 (1977)
- [41] Weder, R.: Scattering theory for the Klein–Gordon equation. *J. Funct. Anal.* **27**, 100–117 (1978)
- [42] Weder, R.: High-velocity estimates, inverse scattering and topological effects. In: Edited by: Khruslov, E., Pastur, L., Shepelsky, D. (eds.) Spectral theory and differential equations: V. A. Marchenko’s 90th Anniversary Collection. American Mathematical Society Translations–Series 2. Advances in the Mathematical Sciences. vol. 233. AMS, Providence, pp. 225–251 (2014)
- [43] Yafaev, D.R.: Scattering matrix for magnetic potentials with Coulomb decay at infinity. *Integral Equ. Oper. Theory* **47**, 217–249 (2003)
- [44] Yafaev, D.R.: Scattering by magnetic fields. *St. Petersburg Math. J.* **17**, 875–895 (2006)

Miguel Ballesteros and Ricardo Weder
Departamento de Física Matemática
Instituto de Investigaciones en Matemáticas Aplicadas
y en Sistemas Universidad Nacional Autónoma de México
Apartado Postal 20-126
IIMAS-UNAM, Col. San Angel,
C.P. 01000, Mexico D.F., Mexico
e-mail: weder@unam.mx;
miguel.ballesteros@iimas.unam.mx

Communicated by Jan Dereziński.

Received: June 2, 2015.

Accepted: November 15, 2015.