Ann. Henri Poincaré 17 (2016), 1713–1736

1424-0637/16/071713-24
published online October 23, 2015
DOI 10.1007/s00023-015-0445-6

Annales Henri Poincaré

Eigenvalue Asymptotics for a Schrödinger Operator with Non-Constant Magnetic Field Along One Direction

Pablo Miranda

Abstract. We consider the discrete spectrum of the two-dimensional Hamiltonian $H = H_0 + V$, where H_0 is a Schrödinger operator with a non-constant magnetic field B that depends only on one of the spatial variables, and V is an electric potential that decays at infinity. We study the accumulation rate of the eigenvalues of H in the gaps of its essential spectrum. First, under certain general conditions on B and V, we introduce effective Hamiltonians that govern the main asymptotic term of the eigenvalue counting function. Further, we use the effective Hamiltonians to find the asymptotic behavior of the eigenvalues in the case where the potential V is a power-like decaying function and in the case where it is a compactly supported function, showing a semiclassical behavior of the eigenvalues in the first case and a non-semiclassical behavior in the second one. We also provide a criterion for the finiteness of the number of eigenvalues in the gaps of the essential spectrum of H.

1. Introduction

Let $\mathbb{R}^2 \ni (x, y) \mapsto B(x) \in \mathbb{R}_+$ be a bounded magnetic field and define the Schrödinger operator

$$H_0 := -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - b(x)\right)^2, \tag{1.1}$$

where the second component of the magnetic vector potential $\mathbb{R}^2 \ni (x, y) \mapsto (0, b(x)) \in \mathbb{R}^2$ is given by

$$b(x) = \int_0^x B(t)\,\mathrm{d}t. \tag{1.2}$$

Let $V : \mathbb{R}^2 \to [0, \infty)$ be an electric potential that decays at infinity. Set $H = H_0 + V$. It is known that the essential spectrum of H, denoted by $\sigma_{\mathrm{ess}}(H)$, satisfies

$$\sigma_{\text{ess}}(H) = \bigcup_{j\in\mathbb{N}} [\mathcal{E}_j^-, \mathcal{E}_j^+], \tag{1.3}$$

with $\mathcal{E}_j^{\pm} \in [0,\infty)$. Suppose that there exists a finite gap in the essential spectrum of H, which in our context will be equivalent to

$$\mathcal{E}_j^+ < \mathcal{E}_{j+1}^-, \tag{1.4}$$

for some $j \geq 1$ (see Sect. 2). Then, it is possible to define

$$\mathcal{N}_j(\lambda) := \text{Tr}\mathbb{1}_{(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)}(H), \quad \text{for } 0 < \lambda < \mathcal{E}_{j+1}^- - \mathcal{E}_j^+, \tag{1.5}$$

where $\mathbb{1}_\omega(\cdot)$ is the characteristic function of the set ω. The function $\mathcal{N}_j$ counts the number of eigenvalues of H on the interval $(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)$. Our purpose in this article is to describe the asymptotic behavior of $\mathcal{N}_j(\lambda)$ as λ goes to zero, for some types of non-constant magnetic fields B and electric potentials V.

The asymptotic behavior of the function $\mathcal{N}_j$ has been systematically studied in the case of a magnetic field B equal to a *constant* (see [12,18,22,27,29,30,32]). For this model exists a rather complete understanding of the behavior of $\mathcal{N}_j$, according to the decaying regime at infinity of the function V. This includes power-like, exponential and compactly supported regimes. An extension of these results was to consider the eigenvalue counting function for Schrödinger operators with *asymptotically constant* magnetic field and decaying electric potential (see [18,33,34], and for related problems see [25,28]). Other natural extensions are the Schrödinger operators with *unidirectionally constant* magnetic field presented here. This last model was first considered by A. Iwatsuka (with $V \equiv 0$) in order to give examples of magnetic Schrödinger operators with purely absolutely continuous spectrum [19]. The one particle system determined by this Hamiltonian presents some interesting transport and spectral properties which have been studied in the mathematical literature (see [8,11,15,16,20,21,35,36]), as well as in the physics literature (see e.g. [5,7,14,23,24,31]).

For the "Iwatsuka Hamiltonian", the problem of the asymptotic behavior of a counting function of the form (1.5) was already studied in [9]. In that article were considered the eigenvalues below the bottom of the essential spectrum of H, when the magnetic field $B(x)$ is a step function that changes sign at zero. The problem was also addressed in [35], for a magnetic field with similar characteristics to the one that we will study here (see (2.1)). We note that in [9] the first band function of H_0 (see Sect. 2) has a global minimum at a finite point of $\mathbb{R}$, while for the model in [35], as well as for the model in this article, the band functions have its extremal point at infinity (for us is relevant the supremum). This divergence implies that the analysis of the counting function in our cases is quite different and slightly more difficult than that in [9]. We also note that the condition required in [35] to obtain the mentioned property on the supremun of the band functions, is that the function $B(x)$ is monotone. In this article we relax somewhat this condition asking only global monotonicity to B (see (2.1)).

In [35] the behavior of $\mathcal{N}_j$ was obtained for potentials V that decay at infinity as $(x^2+y^2)^{-m/2}$ (see (2.16), (2.18) below), supposing that $0<m<1$. In Corollary 2.4 we will present a result similar to the semiclassical one given in [35], which completes the description of the first asymptotic term of $\mathcal{N}_j$ for power-like decaying potentials, that is we consider the case $m>1$. Furthermore, in Theorem 2.1 we give an effective Hamiltonian which permit us to deal with other types of decaying regimes of V. Namely, in Corollary 2.2 we give a sufficient condition that guarantees the finiteness of the number of eigenvalues of H in each gap of $\sigma_{\mathrm{ess}}(H)$. This is a geometric condition that depends on the set where B reach its supremum and the support of V.

When the condition of Corollary 2.2 does not hold, we can see that $\mathcal{N}_j$ is generically unbounded in each gap of $\sigma_{\mathrm{ess}}(H)$, as follows incidentally from Corollary 2.3 where we give asymptotic bounds for $\mathcal{N}_j$ if V is of compact support. Contrary to Corollary 2.4, the behavior of $\mathcal{N}_j$ is not semiclassical in this situation, since a semiclassical formula would imply the finiteness of the number of eigenvalues. For compact supported potentials V, a different non-semiclassical asymptotic behavior of the eigenvalue counting function was obtained in [22,30], in the constant magnetic field case. In that context the main asymptotic term is $(\ln|\ln\lambda|)^{-1}|\ln\lambda|$ which goes to infinity faster than the one presented here, $|\ln\lambda|^{1/2}$, implying that the accumulation of the eigenvalues is stronger in our case. Similar results to our was previously obtained in [3,4], for other magnetic Hamiltonians with compact supported electric potentials (see Remark after Theorem 2.1).

For non-positive potentials V we could define the functions

$$\mathcal{N}_0^-(\lambda) := \mathrm{Tr}\mathbb{1}_{(-\infty,\mathcal{E}_1^- -\lambda)}(H);\quad \mathcal{N}_j^-(\lambda) := \mathrm{Tr}\mathbb{1}_{(\mathcal{E}_j^+,\mathcal{E}_{j+1}^- -\lambda)}(H),\quad j\in\mathbb{N}.$$

Although we will present our results only for $\mathcal{N}_j$, $j\in\mathbb{N}$, they are still valid, with obvious modifications, for $\mathcal{N}_j^-$, $j\in\mathbb{Z}_+ := \{0,1,2,\ldots\}$. We omit these in order to simplify the presentation. Finally, for the case of V without definite sign we can say: If V is power-like decaying, a refinement of the analysis used here to obtain the effective Hamiltonian of Theorem 2.1, should lead to the same type of semiclassical results of Corollary 2.4 for this kind of potentials. Otherwise, if V has compact support, the situation is considerably more delicate and we do not have clear ideas of how to obtain precise results in this context. This is, to some extent, true even if the magnetic field is constant, where a description of the eigenvalue counting function has been obtained only for some particular classes of non-sign definite compact electric potentials (see [28,32]).

2. Main Results

2.1. Effective Hamiltonian

To introduce the effective Hamiltonians that govern the main asymptotic term of $\mathcal{N}_j$, we need to state more specific conditions on the magnetic field B and then recall some well-known properties of the unperturbed operator H_0.

Throughout this article we will assume the following:

(a) $B \in L^\infty(\mathbb{R})$.

(b) $B_- \le B(x) \le B_+$ a.e., for some positive constants $B_+ > B_-$.

(c) $\lim_{x\to\infty} B(x) = B_+$, and $\limsup_{x\to-\infty} B(x) < B_+$. (2.1)

Under condition (2.1) the operator defined by (1.1) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and its spectrum, denoted by $\sigma(H_0)$, is purely absolutely continuous [19,20]. Note that the potential b defined by (1.2) is an absolutely continuous strictly increasing function such that

$$B_-|x| \le |b(x)| \le B_+|x|. \tag{2.2}$$

Let $\mathcal{F}$ be the partial Fourier transform

$$(\mathcal{F}u)(x,k) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-iky} u(x,y)\,\mathrm{d}y, \quad \text{for } u \in C_0^\infty(\mathbb{R}^2).$$

Then

$$\mathcal{F} H_0 \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h(k)\,\mathrm{d}k, \tag{2.3}$$

where $h(k)$ is a self-adjoint operator acting in $L^2(\mathbb{R})$, defined by

$$h(k) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (b(x)-k)^2, \quad k \in \mathbb{R}. \tag{2.4}$$

For any $k \in \mathbb{R}$ the spectrum of the operator $h(k)$ is discrete and simple. We denote the increasing sequence of eigenvalues by $\{E_j(k)\}_{j=1}^\infty$. For any $j \in \mathbb{N}$ the *band function* $E_j(\cdot)$ is analytic as a function of $k \in \mathbb{R}$ [19,20].

Set $\mathcal{E}_j^- := \inf_{k\in\mathbb{R}} E_j(k)$; $\mathcal{E}_j^+ := \sup_{k\in\mathbb{R}} E_j(k)$, then

$$\sigma(H_0) = \bigcup_{j=1} \overline{E_j(\mathbb{R})} = \bigcup_{j=1} [\mathcal{E}_j^-, \mathcal{E}_j^+]. \tag{2.5}$$

Condition (2.1) (b) implies that $B_-(2j-1) \le E_j(k) \le B_+(2j-1)$ for all $k \in \mathbb{R}$, and (2.1) (c) implies that $\lim_{k\to\infty} E_j(k) = B_+(2j-1) = \mathcal{E}_j^+$, for all $j \in \mathbb{N}$ (see [19]).

Now we need some definitions. Put

$$\varphi_j(x) := \frac{\mathrm{H}_{j-1}(x) e^{-x^2/2}}{(\sqrt{\pi} 2^{j-1} (j-1)!)^{1/2}}, \quad x \in \mathbb{R}, \quad j \in \mathbb{N}, \tag{2.6}$$

where

$$\mathrm{H}_q(x) := (-1)^q e^{x^2} \frac{\mathrm{d}^q}{\mathrm{d}x^q} e^{-x^2}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

are the Hermite polynomials (see e.g. [1, Chapter I, Eqs. (8.5), (8.7)]). Then the real-valued function φ_j satisfies

$$-\varphi_j''(x) + x^2\varphi_j(x) = (2j-1)\varphi_j(x), \quad \|\varphi_j\|_{L^2(\mathbb{R})} = 1.$$

For $(x,\xi) \in \mathbb{R}^2$ define the function

$$\Psi_{j;x,\xi}(k) = B_+^{-1/4} e^{-ik\xi} \varphi_j\left(B_+^{1/2} x - B_+^{1/2} b^{-1}(k)\right), \quad j \in \mathbb{N}, \quad k \in \mathbb{R}. \tag{2.7}$$

The system $\{\Psi_{j;x,\xi}\}_{(x,\xi)\in\mathbb{R}^2}$ is overcomplete with respect to the measure $\frac{B_+}{2\pi}\mathrm{d}x\mathrm{d}\xi$ (see [1, Subsection 5.2.3] for the definition of an overcomplete system with respect to a given measure). Introduce the orthogonal projection

$$\mathcal{P}_{j;x,\xi} := |\Psi_{j;x,\xi}\rangle\langle\Psi_{j;x,\xi}|, \quad (x,\xi)\in\mathbb{R}^2,$$

acting in $L^2(\mathbb{R})$, and the pseudo-differential operator $\mathcal{V}_j : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as the weak integral

$$\mathcal{V}_j := \frac{B_+}{2\pi}\int_{\mathbb{R}^2} V(x,\xi)\,\mathcal{P}_{j;x,\xi}\,\mathrm{d}x\mathrm{d}\xi, \tag{2.8}$$

i.e. $\mathcal{V}_j$ is an operator with contravariant symbol V.

As already mentioned, for the potential V we will assume the following:

$$\begin{aligned} &\text{(a) } 0 \le V \in L^\infty(\mathbb{R}^2).\\ &\text{(b) } \lim_{x^2+y^2\to\infty} V(x,y) = 0. \end{aligned} \tag{2.9}$$

The diamagnetic inequality and Weyl's theorem imply that $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = \sigma(H_0)$, then (1.3) holds true. Conditions (2.9) also imply that $\mathcal{V}_j$ is a non-negative and compact operator.

Theorem 2.1. *Assume that for some $j \in \mathbb{N}$, (1.4) is true. Assume also that B satisfies (2.1), and V satisfies (2.9). Consider the band function E_j as a multiplication operator in $L^2(\mathbb{R})$. Then for each $\delta \in (0,1)$*

$$\begin{aligned} \operatorname{Tr}\mathbb{1}_{(\mathcal{E}_j^+ +\lambda,\infty)}(E_j + (1-\delta)\mathcal{V}_j) + O_\delta(1)&\\ \le \mathcal{N}_j(\lambda) \le&\\ \operatorname{Tr}\mathbb{1}_{(\mathcal{E}_j^+ +\lambda,\infty)}(E_j + (1+\delta)\mathcal{V}_j) + O_\delta(1),& \quad \lambda \downarrow 0. \end{aligned} \tag{2.10}$$

Remark. Similar results to Theorem 2.1 appear in [3] and [4]. In [3] the discrete spectrum of operators of the form $H_1 = H_{\mathrm{Hall}} + V$, is described, where

$$H_{Hall} = H_{\mathrm{Landau}} + W(x),$$

H_{Landau} being the two-dimensional Schrödinger operator with constant magnetic field, and W being a monotonic function depending only on the first variable x. In the same way, in [4] the operator $H_2 = H_{Half-Plane} + V$ is considered, where $H_{Half-Plane}$ is the Schrödinger operator with constant magnetic field defined for a half-plane, with a Dirichlet boundary condition along the edge. In both articles an eigenvalue counting function similar to (1.5) is studied. The effective Hamiltonians obtained in those articles are particular cases of the one given by Theorem 2.1, when $b^{-1}(k) = B_+^{-1}k$ in (2.7). All these three models share the particularity that the unperturbed operators $H_{\mathrm{Hall}}, H_{\mathrm{Landau}}$ and H_0 admit a direct integral decomposition with fibred operators that converge to shifted harmonic oscillators as $k \to \infty$. However, despite this similarity, the proof of Theorem 2.1 requires the use of some new ideas and presents technical difficulties that do not appear in [3] or [4].

2.2. Asymptotic Behavior of $\mathcal{N}_j(\lambda)$: Finite Number of Eigenvalues

In Corollaries 2.2, 2.3 we will see that the finiteness or the infiniteness of the number of eigenvalues of H in the gaps of $\sigma_{\mathrm{ess}}(H)$, depend on a relation between the support of V and the number

$$\mathbf{x}^+ := \inf\{x \in \mathbb{R}\,;\, B(t) = B_+ \text{ for almost all } t \text{ in } (x,\infty)\}. \tag{2.11}$$

Note that it is possible to have $\mathbf{x}^+ = \infty$.

Corollary 2.2. *Suppose that* (1.4) *is true, and that B satisfies* (2.1). *Assume also that V satisfies* (2.9) *and* $\| \int_{\mathbb{R}} V(x,y)\,\mathrm{d}y\|_{L^\infty(\mathbb{R})} < \infty$. *Then, if*

$$\mathbf{x}^+ > \sup\{x \in \mathbb{R}\,; \text{for some } \mathrm{y} \in \mathbb{R},\ (\mathrm{x},\mathrm{y}) \in \operatorname{ess\,supp} \mathrm{V}\}, \tag{2.12}$$

we have that

$$\mathcal{N}_j(\lambda) = O(1), \quad \lambda \downarrow 0. \tag{2.13}$$

2.3. Asymptotic Behavior of $\mathcal{N}_j(\lambda)$: Infinite Number of Eigenvalues for V of Compact Support

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Denote by $\mathbf{c}_-(\Omega)$ the maximal length of the vertical segments contained in $\overline{\Omega}$. Further, let $B_R((x,y)) \subset \mathbb{R}^2$ be a disk of radius $R > 0$ centered at $(x,y) \in \mathbb{R}^2$. For $a \in \mathbb{R}$, set

$$K(\Omega,a) := \{(\xi,R) \in \mathbb{R}\times\mathbb{R}_+\,;\ \text{there exists } \eta \in \mathbb{R} \text{ such that } \Omega \subset B_R((\xi + a,\eta))\},$$

and

$$\mathbf{c}_+(\Omega,a) := \inf_{(\xi,R)\in K(\Omega,a)} R\varkappa\left(\frac{\xi_+}{eR}\right),$$

where $\xi_+ := \max\{\xi,0\}$, and $\varkappa(s) := |\{t > 0\,;\, t\ln t < s\}|$, for $s \in [0,\infty)$. Here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}$.

Also define

$$\tilde{\Omega} := \{(x,y) \in \Omega; x > \mathbf{x}^+\}.$$

Corollary 2.3. *Assume that* (1.4) *holds true, and that B is a function satisfying* (2.1). *Further assume that*

$$c_- \mathbb{1}_{\Omega_-}(x,y) \le V(x,y) \le c_+ \mathbb{1}_{\Omega_+}(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{2.14}$$

where $\Omega_\pm \subset \mathbb{R}^2$ are bounded domains with Lipschitz boundaries, and $0 < c_- \le c_+ < \infty$. Then, if

$$\mathbf{x}^+ < \sup\{x \in \mathbb{R}; \text{for some } \mathrm{y} \in \mathbb{R},\ (\mathrm{x},\mathrm{y}) \in \Omega_-\},$$

the following asymptotic bounds

$$\mathcal{C}_-|\ln\lambda|^{1/2}(1+o(1)) \le \mathcal{N}_j(\lambda) \le \mathcal{C}_+|\ln\lambda|^{1/2}(1+o(1)), \quad \lambda \downarrow 0, \tag{2.15}$$

hold true with $\mathcal{C}_- := (2\pi)^{-1}\sqrt{b}\,\mathbf{c}_-(\tilde{\Omega}_-)$ *and* $\mathcal{C}_+ := e\sqrt{b}\,\mathbf{c}_+(\tilde{\Omega}_+,\mathbf{x}^+)$.

Remark. The constants $\mathcal{C}_\pm$ already appeared in [3,4], where it is shown that $\mathcal{C}_- < \mathcal{C}_+$.

2.4. Asymptotic Behavior of $\mathcal{N}_j(\lambda)$: Infinite Number of Eigenvalues for Power-like Decaying V

Now we will consider potentials V whose support is not compact. First we will assume that there exists a positive number m such that, for any pair $(\alpha, \beta) \in \mathbb{Z}_+^2$, there exists a positive constant $C_{\alpha,\beta}$, such that

$$|\partial_x^\beta \partial_\xi^\alpha V(x,\xi)| \leq C_{\alpha,\beta} \langle x, \xi\rangle^{-m-\alpha-\beta} \quad \text{for all } (x,\xi) \in \mathbb{R}^2, \tag{2.16}$$

where $\langle x, \xi\rangle = (1 + x^2 + \xi^2)^{1/2}$.

Moreover, let $s \in \mathbb{R}$ and define the volume function

$$N(\lambda, V, s) := \frac{1}{2\pi} vol\{(x,\xi) \in \mathbb{R}^2; V(x,\xi) > \lambda, x > s\}, \tag{2.17}$$

where vol denotes the Lebesgue measure in $\mathbb{R}^2$. We will assume that for some $s_0 \in \mathbb{R}$ and positive constants C and λ_0

$$N(\lambda, V, s_0) \geq C\lambda^{-2/m}, \quad 0 < \lambda < \lambda_0. \tag{2.18}$$

We say that a decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the homogeneity condition if

$$\lim_{\epsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \lambda^{2/m} \left(f(\lambda(1-\epsilon)) - f(\lambda(1+\epsilon)) \right) = 0. \tag{2.19}$$

Corollary 2.4. *Assume that (1.4) is true. Also suppose that B is a smooth function with all its derivatives bounded and for some $M > m$*

$$B_+ - B(x) = O(\langle x\rangle^{-M}), \quad x \to \infty. \tag{2.20}$$

If V satisfies (2.16) with $m > 1$, and for $s_0 \in \mathbb{R}$, $N(\lambda, V, s_0)$ satisfies (2.18) and (2.19), then we have the following asymptotic formula

$$\mathcal{N}_j(\lambda) = B_+ N(\lambda, V, s_0)(1 + o(1)), \quad \lambda \downarrow 0. \tag{2.21}$$

Remarks. (i) The smoothness condition on B is not essential. For instance, an easy modification of the arguments permits to prove Corollary 2.4 just assuming (2.1) and $\mathbf{x}^+ < \infty$.

(ii) Condition (2.16) implies that if $N(\lambda, V, s_0)$ satisfies (2.19) for some $s_0 \in \mathbb{R}$, then $N(\lambda, V, s)$ satisfies (2.19) as well, for any $s \in \mathbb{R}$. Moreover, if $N(\lambda, V, s_0)$ satisfies (2.18) then the asymptotic formula (2.21) is true for any $s \in \mathbb{R}$, since

$$\lim_{\lambda \downarrow 0} \frac{N(\lambda, V, s)}{N(\lambda, V, s_0)} = 1.$$

(iii) Results of the same type of (2.21) were obtained in [35], for non necessarily sign-definite potentials V, were the number m in (2.16) is assumed to be $0 < m < 1$, and the function B monotone.

(iv) As already mentioned in the Remark after Theorem 2.1, in [3,4] the eigenvalue counting function for magnetic Schrödinger operators similar to those considered here, was studied. In [3,4], the asymptotic behavior of these counting functions was described only for compactly supported potentials V, as in Corollary 2.2, and a slightly weaker version of Corollary 2.3 was given. Since the effective Hamiltonians obtained in [3,4] are examples of the one in Theorem 2.1, the conclusions of Corollary 2.4 are valid as well for the counting functions of the models considered in the articles [3,4].

3. Proof of the Results

3.1. Proof of Theorem 2.1

Before we begin the proof, let us set some notation and auxiliary results that we will use throughout the text. Let $r > 0$ and $T = T^*$ be a linear compact operator acting in a given Hilbert space.[1] Set

$$n_\pm(r;T) := \operatorname{Tr} \mathbb{1}_{(r,\infty)}(\pm T);$$

thus the functions $n_\pm(\cdot;T)$ are respectively the counting functions of the positive and negative eigenvalues of the operator T. If T is compact but not necessarily self-adjoint (in particular, T could act between two different Hilbert spaces), we will use also the notation

$$n_*(r;T) := n_+(r^2;T^*T), \quad r > 0;$$

thus $n_*(\cdot;T)$ is the counting function of the singular values of T. Evidently,

$$n_*(r;T) = n_*(r;T^*), \quad n_+(r;T^*T) = n_+(r;TT^*), \quad r > 0.$$

Let us recall also the well-known Weyl inequalities

$$n_+(r_1 + r_2; T_1 + T_2) \le n_+(r_1;T_1) + n_+(r_2;T_2) \tag{3.1}$$

where $r_j > 0$ and T_j, $j = 1, 2$, are linear self-adjoint compact operators (see e.g. [2, Theorem 9.2.9]), as well as the Ky Fan inequalities

$$n_*(r_1 + r_2; T_1 + T_2) \le n_*(r_1;T_1) + n_*(r_2;T_2), \quad r_1, r_2 > 0, \tag{3.2}$$

for compact but not necessarily self-adjoint T_j, $j = 1, 2$, (see e.g. [2, Theorem 9.2.9]). Further, let S_p, $p \in [1,\infty)$, be the Schatten–von Neumann class of compact operators, equipped with the norm

$$\|T\|_p := \left(-\int_0^\infty r^p \,\mathrm{d}n_*(r;T)\right)^{1/p}.$$

Then the Chebyshev-type estimate

$$n_*(r;T) \le r^{-p}\|T\|_p^p \tag{3.3}$$

holds true for any $r > 0$ and $p \in [1,\infty)$.

We start the proof by using the Birman–Schwinger principle, which give us

$$\mathcal{N}_j(\lambda) = n_-(1; V^{1/2}\left(H_0 - \mathcal{E}_j^+ - \lambda\right)^{-1} V^{1/2}) + O(1), \quad \lambda \downarrow 0. \tag{3.4}$$

To analyze the right hand side of (3.4) it is necessary to obtain further information of the operator H_0.

[1] All Hilbert spaces in this article are supposed to be separable.

First, note that from (2.3)

$$\left(H_0 - \mathcal{E}_j^+ - \lambda\right)^{-1} = \mathcal{F}^* \int_{\mathbb{R}}^{\oplus} \left(h(k) - \mathcal{E}_j^+ - \lambda\right)^{-1} \mathrm{d}k\ \mathcal{F}. \tag{3.5}$$

If $\pi_j(k)$ is the orthogonal projection of $h(k)$ corresponding to the eigenvalue $E_j(k)$, for $\lambda > 0$ and $A \in [-\infty, \infty)$ set

$$T_j(\lambda, A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \left(\mathcal{E}_j^+ - E_j(k) + \lambda\right)^{-1} \pi_j(k)\,\mathrm{d}k\,\mathcal{F}.$$

Let $l \in \mathbb{N}$, then the band function $E_l(\cdot)$ has the following property: Suppose that $l < j$, then for all $k \in \mathbb{R}$

$$E_l(k) \le B_+(2l-1) < B_+(2j-1) = \mathcal{E}_j^+.$$

Also, (2.5) and (1.4) imply that for all $l > j$

$$E_l(k) \ge \mathcal{E}_l^- \ge \mathcal{E}_{j+1}^- > \mathcal{E}_j^+,$$

for all $k \in \mathbb{R}$. Then, there exists a positive constant κ such that for all $l \ne j$ and $k \in \mathbb{R}$

$$|E_l(k) - \mathcal{E}_j^+| > \kappa. \tag{3.6}$$

Inequality (3.6) implies that, if I is the identity operator in $L^2(\mathbb{R})$, the limit

$$\lim_{\lambda \downarrow 0} \left(h(k) - \mathcal{E}_j^+ - \lambda\right)^{-1} (I - \pi_j(k)) \tag{3.7}$$

exists in the norm operator topology.

For $l = j$ we have the following result.

Lemma 3.1. *Let $j \in \mathbb{N}$, then $E_j(k) < \mathcal{E}_j^+$ for all $k \in \mathbb{R}$. Moreover, for any $A \in \mathbb{R}$ there exists $\alpha > 0$ such that $\mathcal{E}_j^+ - E_j(k) > \alpha$, for all $k < A$.*

Proof. First let us prove that for any k real, $\mathcal{E}_j^+ - E_j(k) > 0$. Let B_1 and B_2 be two functions satisfying condition (2.1), and let b_1, b_2 be the corresponding magnetic potentials as chosen in (1.2). Note that

$$b_s(x) - k = \int_{b_s^{-1}(k)}^{x} B_s(t)\,\mathrm{d}t, \quad s = 1, 2.$$

Then it is easy to see that if $B_1(x) \le B_2(x)$ a.e. in $\mathbb{R}$,

$$(b_1(x) - k)^2 \le \left(b_2(x) - b_2\left(b_1^{-1}(k)\right)\right)^2, \tag{3.8}$$

for all k, and all x in $\mathbb{R}$. For b_1, b_2, let $h(k, b_1)$, $h(k, b_2)$ be the operators defined by (2.4), and denote by $E_j(k, b_1), E_j(k, b_2)$ their associated jth eigenvalues. The inequality (3.8) implies that for all $k \in \mathbb{R}$

$$h(k, b_1) \le h\left(b_2\left(b_1^{-1}(k)\right), b_2\right), \tag{3.9}$$

and from the min–max principle we obtain that for all $k \in \mathbb{R}$, and all $j \in \mathbb{N}$

$$E_j(k, b_1) \le E_j\left(b_2\left(b_1^{-1}(k)\right), b_2\right). \tag{3.10}$$

Now, since $\limsup_{x\to-\infty} B(x) < B_+$, there exists a real number β and a non-decreasing smooth function B_β such that

$$B_\beta(x) = \begin{cases} \limsup_{x\to-\infty} B(x) & \text{if } x \leq \beta \\ B_+ & \text{if } x \geq \beta + 1, \end{cases}$$

and $B(x) \leq B_\beta(x)$ a.e. in $\mathbb{R}$. From the proof of [21, Theorem 3.2] we know that B_β non-decreasing implies that $E_j(k, b_\beta)$ is a non-decreasing function as well. Since $E_j(\cdot, b_\beta)$ is also analytic, (2.5) implies that $E_j(k, b_\beta) < \mathcal{E}_j^+$ for all $k \in \mathbb{R}$. Using (3.10) we obtain that $E_j(k) < \mathcal{E}_j^+$.

To prove the second assertion of the Lemma, just note that $E_j(\cdot, b_\beta)$ satisfies the required condition and use (3.10) again. □

Using the Weyl inequalities (3.1) together with (3.5) and (3.7), and together with Lemma 3.1, it can be easily seen that for any $r \in (0,1)$

$$\begin{aligned} &n_+(1+r; V^{1/2}T_j(\lambda, A)V^{1/2}) + O(1) \\ &\quad \leq n_-(1; V^{1/2}(H_0 - \mathcal{E}_j^+ - \lambda)^{-1}V^{1/2}) \\ &\quad \leq n_+(1-r; V^{1/2}T_j(\lambda, A)V^{1/2}) + O(1), \end{aligned} \tag{3.11}$$

as $\lambda \downarrow 0$.

Next, let $h_\infty(k)$ be the shifted harmonic oscillator

$$h_\infty(k) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(B_+x - B_+b^{-1}(k)\right)^2,$$

self-adjoint in $L^2(\mathbb{R})$, for $k \in \mathbb{R}$. The spectrum of $h_\infty(k)$ coincide with the set of Landau levels $\{B_+(2j-1) = \mathcal{E}_j^+\}_{j=1}^\infty$. Let $\pi_{j,\infty}(k)$ be the orthogonal projection of $h_\infty(k)$ corresponding to the eigenvalue $\mathcal{E}_j^+$, which can be described explicitly by

$$\pi_{j,\infty}(k) = |\Psi_{j,\infty}(\cdot, k)\rangle\langle\Psi_{j,\infty}(\cdot, k)|, \tag{3.12}$$

where $\Psi_{j,\infty}(x,k) = B_+^{1/4}\varphi_j(B_+^{1/2}x - B_+^{1/2}b^{-1}(k))$ (φ_j defined in (2.6)).

For $\lambda > 0$ and $A \in [-\infty, \infty)$, set

$$T_{j,\infty}(\lambda, A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \left(\mathcal{E}_j^+ - E_j(k) + \lambda\right)^{-1} \pi_{j,\infty}(k)\, \mathrm{d}k \mathcal{F}. \tag{3.13}$$

Our next goal is to replace $T_j(\lambda, A)$ by $T_{j,\infty}(\lambda, A)$ in inequality (3.11).

Theorem 3.2. *For any $j \in \mathbb{N}$*

$$\lim_{k\to\infty} \frac{||\pi_j(k) - \pi_{j,\infty}(k)||}{(\mathcal{E}_j^+ - E_j(k))^{1/2}} = 0.$$

The proof of this Theorem follows from the next two lemmas.

Lemma 3.3. *Define $\Lambda_k := h(k)^{-1} - h_\infty(k)^{-1}$. Then $\Lambda_k \geq 0$ and*

$$\lim_{k\to\infty} ||\Lambda_k|| = 0. \tag{3.14}$$

Proof. To see that $\Lambda_k \geq 0$, use (2.1) (b) and (3.9). To prove (3.14) we introduce the unitary operators $U_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined for any $k \in \mathbb{R}$ by

$$(U_k f)(x) = f(x + b^{-1}(k)),$$

and set

$$\tilde{h}(k) := U_k h(k) U_k^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (b(x + b^{-1}(k)) - k)^2$$

and

$$\tilde{h}_\infty := U_k h_\infty(k) U_k^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (B_+ x)^2.$$

Instead of (3.14) we will prove the equivalent statement $\lim_{k\to\infty} ||\tilde{h}(k)^{-1} - \tilde{h}_\infty^{-1}|| = 0$.

Put $d_k(x) := \tilde{h}_\infty - \tilde{h}(k) = (B_+x)^2 - (b(x+b^{-1}(k)) - k)^2$. Using two times the resolvent identity we get

$$\tilde{h}(k)^{-1} - \tilde{h}_\infty^{-1} = \tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1} + \tilde{h}(k)^{-1} d_k \tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1}. \tag{3.15}$$

We need to prove first that $\tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1}$ converges to zero in norm as $k \to \infty$.

Note that

$$\begin{aligned} |d_k(x)| &= \left|B_+x + (b(x + b^{-1}(k)) - k)\right| \left|B_+x - (b(x + b^{-1}(k)) - k)\right| \\ &= \left|\int_{b^{-1}(k)}^{b^{-1}(k)+x} B_+ + B(t)\, \mathrm{d}t\right| \left|\int_{b^{-1}(k)}^{b^{-1}(k)+x} B_+ - B(t)\, \mathrm{d}t\right| \\ &\leq 2B_+|x| \left|\int_{b^{-1}(k)}^{b^{-1}(k)+x} B_+ - B(t)\, \mathrm{d}t\right|. \end{aligned} \tag{3.16}$$

Then, since $\lim_{x\to\infty} B(x) = B_+$, and from (2.2) $b^{-1}(k) \to \infty$ for $k \to \infty$, the function $|d_k(x)|$ converges pointwise to zero when $k \to \infty$.

Denote by $D(\tilde{h}(k))$, $D(\tilde{h}_\infty)$ the domains of $\tilde{h}(k)$ and $\tilde{h}_\infty$, respectively. Using (3.8), $B_- \leq B(x) \leq B_+$ implies that

$$\left(B_- x\right)^2 \leq \left(b(x + b^{-1}(k)) - k\right)^2 \leq \left(B_+ x\right)^2, \quad \text{for all } x \in \mathbb{R}. \tag{3.17}$$

Then the domains are equal and coincide with the domain of the harmonic oscillator, i.e., $D(\tilde{h}(k)) = D(\tilde{h}_\infty) = D(-\mathrm{d}^2/\mathrm{d}x^2) \cap D(x^2)$ [10, Theorem 1].

Let $f \in L^2(\mathbb{R})$. Since $\tilde{h}_\infty^{-1} f \in D(x^2)$, for any $\epsilon > 0$ one can find $N > 0$ (independent of k) such that

$$\int\limits_{|x|>N} |(b(x + b^{-1}(k)) - k)^2 (\tilde{h}_\infty^{-1} f)(x)|^2\, \mathrm{d}x \leq \int\limits_{|x|>N} |(B_+x)^2 (\tilde{h}_\infty^{-1} f)(x)|^2\, \mathrm{d}x < \epsilon. \tag{3.18}$$

Further, $\tilde{h}_\infty^{-1} f$ is also continuous, then

$$\int_{-N}^{N} \left|d_k(x)(\tilde{h}_\infty^{-1} f)(x)\right|^2\, \mathrm{d}x \leq \sup_{x\in[-N,N]} \left|(\tilde{h}_\infty^{-1} f)(x)\right|^2 \int_{-N}^{N} |d_k(x)|^2\, \mathrm{d}x. \tag{3.19}$$

Using (3.18), (3.19) and (3.16) we can conclude that $d_k\tilde{h}_\infty^{-1}$ converges strongly to zero as $k \to \infty$. Consequently, the family $d_k\tilde{h}_\infty^{-1}$ is uniformly bounded with respect to k, and since $\tilde{h}_\infty^{-1}$ is compact we get $\|\tilde{h}_\infty^{-1}d_k\tilde{h}_\infty^{-1}\| \to 0$ for $k \to \infty$.

To finish the proof of the Lemma it only remains to show that for all $G \in D(\tilde{h}(k)) = D(\tilde{h}_\infty)$, $\|\tilde{h}(k)^{-1}d_kG\|_{L^2(\mathbb{R})} \le C\|G\|_{L^2(\mathbb{R})}$, for some constant C independent of k and G.

From [10, Theorem 1] we know that for any $g \in D(\tilde{h}(k))$

$$\frac{1}{2}||(b(x+b^{-1}(k))-k)^2g||^2_{L^2(\mathbb{R})} \le ||\tilde{h}(k)g||^2_{L^2(\mathbb{R})}. \tag{3.20}$$

Then for f in $L^2(\mathbb{R})$, if $g = \tilde{h}(k)^{-1}f$ in (3.20)

$$\frac{1}{2}||(b(x+b^{-1}(k))-k)^2\tilde{h}(k)^{-1}f||^2_{L^2(\mathbb{R})} \le ||f||^2_{L^2(\mathbb{R})}.$$

Besides, using (3.17) we get $||(B_+x)^2\tilde{h}(k)^{-1}f||^2_{L^2(\mathbb{R})} \le 2B_+^2/B_-^2||f||^2_{L^2(\mathbb{R})}$, which implies the existence of a uniform bound for $d_k\tilde{h}(k)^{-1}$, from where we can easily get the needed result for $\tilde{h}(k)^{-1}d_k$. □

Lemma 3.4. *Let Λ_k be defined as in Lemma 3.3. For all $j \in \mathbb{N}$:*

1. *There exist a constant C_j such that for all k big enough*

$$||\pi_j(k) - \pi_{j,\infty}(k)|| \le C_j||\Lambda_k\pi_{j,\infty}(k)||.$$

2. *It is satisfied the asymptotic formula*

$$\mathcal{E}_j^+ - E_j(k) = {\mathcal{E}_j^+}^2||\pi_{j,\infty}(k)\Lambda_k\pi_{j,\infty}(k)||(1+o(1)), \quad k \to \infty. \tag{3.21}$$

Proof. The proof of this Lemma uses Lemma 3.3 repeating almost word by word the proof of Propositions 3.6 and 3.7 in [4]. □

Putting together Lemmas 3.1, 3.3 and 3.4 we can proof Theorem 3.2 just by noticing that

$$\begin{aligned}\frac{||\pi_j(k)-\pi_{j,\infty}(k)||}{(E_j(k)-\mathcal{E}_j^+)^{1/2}} &\le \frac{C_j}{\mathcal{E}_j^+}\frac{\|\Lambda_k\pi_{j,\infty}(k)\|}{\|\Lambda_k^{1/2}\pi_{j,\infty}(k)\|}(1+o(1))\\ &\le \frac{C_j}{\mathcal{E}_j^+}\|\Lambda_k\|^{1/2}(1+o(1)), \quad k \to \infty.\end{aligned}$$

Proposition 3.5. *For all $A \in [-\infty,\infty)$, $r \in \mathbb{R}$, $\delta \in (0,1)$ and $j \in \mathbb{N}$*

$$\begin{aligned}&n_+\left(r(1+\delta); V^{1/2}T_{j,\infty}(\lambda,A)V^{1/2}\right) + O(1)\\ &\le n_+\left(r; V^{1/2}T_j(\lambda,A)V^{1/2}\right)\\ &\le n_+\left(r(1-\delta); V^{1/2}T_{j,\infty}(\lambda,A)V^{1/2}\right) + O(1), \quad \lambda \downarrow 0.\end{aligned} \tag{3.22}$$

Proof. First note that $n_+(r; V^{1/2}T_j(\lambda,A)V^{1/2}) = n_*(r^{1/2}; V^{1/2}T_j(\lambda,A)^{1/2})$. By Lemma 3.1, for any $\tilde{A} \in (A,\infty)$

$$\begin{aligned}&n_*\left(r; V^{1/2}\left(T_j(\lambda,A)^{1/2} - T_j(\lambda,\tilde{A})^{1/2}\right)\right) = O(1)\\ &n_*\left(r; V^{1/2}\left(T_{j,\infty}(\lambda,A)^{1/2} - T_{j,\infty}(\lambda,\tilde{A})^{1/2}\right)\right) = O(1), \quad \lambda \downarrow 0,\end{aligned}$$

since both $T_j(\lambda, A)^{1/2} - T_j(\lambda, \tilde{A})^{1/2}$ and $T_{j,\infty}(\lambda, A)^{1/2} - T_{j,\infty}(\lambda, \tilde{A})^{1/2}$ have a limit in the norm sense when $\lambda \downarrow 0$. Thanks to Theorem 3.2 it is possible to choose $\tilde{A}$ big enough such that

$$n_*(r; V^{1/2}(T_j(\lambda, \tilde{A})^{1/2} - T_{j,\infty}(\lambda, \tilde{A})^{1/2})) = 0.$$

Using the Ky-Fan inequalities (3.2) we get (3.22) (see Proposition 3.1 and Theorem 3.2 in [3] for a detailed proof of a similar result). □

Now we are ready to finish the proof of Theorem 2.1. Putting together (3.4), (3.11) and (3.22), we obtain that for any $A \in [-\infty, \infty)$ and $\delta \in (0,1)$

$$\begin{aligned} &n_+((1+\delta); V^{1/2}T_{j,\infty}(\lambda, A)V^{1/2}) + O(1) \\ &\quad \le \mathcal{N}_j(\lambda) \le n_+((1-\delta); V^{1/2}T_{j,\infty}(\lambda, A)V^{1/2}) + O(1), \quad \lambda \downarrow 0. \end{aligned} \tag{3.23}$$

For $A \in [-\infty, \infty)$ define

$$P_{j,\infty}(A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \pi_{j,\infty}(k)\, \mathrm{d}k\, \mathcal{F},$$

then, setting $A = -\infty$ we obtain that for any $r > 0$

$$\begin{aligned} &n_+\left(r; V^{1/2}T_{j,\infty}(\lambda, -\infty)V^{1/2}\right) \\ &\quad = n_+\left(r; T_{j,\infty}(\lambda, -\infty)^{1/2} V T_{j,\infty}(\lambda, -\infty)^{1/2}\right) \\ &\quad = n_+\left(r; (\mathcal{E}_j^+ - E_j(\cdot) + \lambda)^{-1/2} \mathcal{F} P_{j,\infty}(-\infty) V P_{j,\infty}(-\infty) \mathcal{F}^* \right. \\ &\qquad \left. \times (\mathcal{E}_j^+ - E_j(\cdot) + \lambda)^{-1/2}\right). \end{aligned} \tag{3.24}$$

Let $\mathcal{U} : L^2(\mathbb{R}) \to \mathcal{F}P_{j,\infty}(-\infty)\mathcal{F}^*(L^2(\mathbb{R}^2))$, defined by $(\mathcal{U}g)(x,k) = B_+^{1/4} g(k)$ $\varphi_j(B_+^{1/2}x - B_+^{1/2}b^{-1}(k))$. The operator $\mathcal{U}$ is unitary and

$$\begin{aligned} &\mathcal{U}^* \mathcal{F} P_{j,\infty}(-\infty) V P_{j,\infty}(-\infty) \mathcal{F}^* \mathcal{U} = \mathcal{V}_j, \\ &\mathcal{U}^* \left(\mathcal{E}_j^+ - E_j(\cdot) + \lambda\right)^{-1} \mathcal{U} = \left(\mathcal{E}_j^+ - E_j(\cdot) + \lambda\right)^{-1}. \end{aligned} \tag{3.25}$$

Use (3.23), (3.24) and (3.25) together with the Birman–Schwinger principle to get (2.10).

3.2. Proof of Corollary 2.2

From inequality (3.23), we see that to prove this corollary it is enough to show that for some $A \in [\infty, \infty)$ and $r \in (0,1)$

$$n_+\left(r^2; V^{1/2}T_{j,\infty}(\lambda, A)V^{1/2}\right) = n_*\left(r; T_{j,\infty}(\lambda, A)^{1/2}V^{1/2}\right) = O(1), \quad \lambda \downarrow 0. \tag{3.26}$$

The Chebyshev-type estimate (3.3), with $p = 2$, states that

$$\begin{aligned} &n_*(r; T_{j,\infty}(\lambda, A)^{1/2}V^{1/2}) \le r^{-2} \|T_{j,\infty}(\lambda, A)^{1/2}V^{1/2}\|_2^2 \\ &\quad = \frac{1}{2\pi r^2} \int_A^{\infty} \int_{\mathbb{R}^2} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \psi_{j,\infty}(x,k)^2 V(x,y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}k, \end{aligned} \tag{3.27}$$

where we have used (3.13) and (3.12). Here and in the sequel we will assume without loss of generality that $\mathbf{x}^+ = 0$. Indeed, for $\mathbf{x}^+$ finite, this follows from a translation along the x-axis and using the gauge invariance of H. If $\mathbf{x}^+$ is infinite, thanks to (3.8)-(3.9) we may replace B by a function $\widetilde{B}$ such that $\widetilde{B}(x) \geq B(x)$ and that the number $\mathbf{x}^+_{\tilde{B}} := \inf\{x \in \mathbb{R}\,;\, \tilde{B}(t) = B_+$ for almost all t in $(x, \infty)\}$ is equal to zero, and then use (3.10) in (3.30) below.

Put $X^+ := \sup\{x \in \mathbb{R}\,;$ for some $y \in \mathbb{R}, (x, y) \in \operatorname{ess\,supp} V\}$. Take $\tilde{x}$ such that $X^+ < \tilde{x} < 0 = \mathbf{x}^+$, and define the step function

$$W(x) := \begin{cases} b(\tilde{x})^2 - (B_+\tilde{x})^2 & \text{for } x < \tilde{x} \\ 0 & \text{for } x \geq \tilde{x}. \end{cases} \tag{3.28}$$

Setting $h^W(k)$ as the operator given by

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(B_+x - B_+b^{-1}(k)\right)^2 + W(x),$$

self-adjoint in $L^2(\mathbb{R})$, it is not difficult to see that for $k > 0$

$$h(k) \leq h^W(k). \tag{3.29}$$

The spectrum of $h^W(k)$ is discrete and simple. Denote by $\{E_j^W(k)\}_{j=1}^{\infty}$ the increasing sequence of eigenvalues of $h^W(k)$. Inequality (3.29) implies that

$$E_j(k) \leq E_j^W(k), \tag{3.30}$$

and then $(\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \leq (\mathcal{E}_j^+ - E_j^W(k) + \lambda)^{-1}$ for all $j \in \mathbb{N}$, $k > 0$ and $\lambda > 0$.

By Proposition 4.2 of [3], we know that there exists a positive constant C_j such that for all k big enough

$$\mathcal{E}_j^+ - E_j^W(k) \geq C_j \left(B_+b^{-1}(k)\right)^{2j-3} e^{-B_+(b^{-1}(k)-\tilde{x})^2}.$$

Then for $A > 0$ large, for any $\lambda > 0$

$$\begin{aligned} &\int_A^\infty \int_{\mathbb{R}^2} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \psi_{j,\infty}(x,k)^2 V(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}k \\ &\quad\leq \frac{1}{B_+^{2j-3}C_j} \int_A^\infty \int_{\mathbb{R}^2} k^{3-2j} e^{2k(x-\tilde{x})} e^{B_+(\tilde{x}^2-x^2)} \mathrm{H}_j \left(B_+^{1/2}x - k/B_+^{1/2}\right)^2 \\ &\quad\quad\times V(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}k, \end{aligned}$$

where we have used that $b^{-1}(k) = k/B_+$ for $k > 0$, due to $\mathbf{x}^+ = 0$. The last integral can be decomposed into a finite sum of terms of the form

$$\begin{aligned} &C_{l,n} e^{B_+\tilde{x}^2} \int_A^\infty \int_{\mathbb{R}^2} k^l x^n e^{2k(x-\tilde{x})} e^{-B_+x^2} V(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}k \\ &\quad\leq \left\| \int_{\mathbb{R}} V(x,y)\mathrm{d}y \right\|_{L^\infty(\mathbb{R})} |C_{l,n}| e^{B_+\tilde{x}^2} \int_A^\infty k^l e^{2k(X^+-\tilde{x})}\,\mathrm{d}k \int_{-\infty}^{X^+} |x|^n e^{-B_+x^2}\,\mathrm{d}x, \end{aligned}$$

for some constants $C_{l,n}$, and integers l, n. Each one of this terms is finite because of our choice of $\tilde{x}$.

3.3. Proof of Corollary 2.3

Let us first show how to obtain the upper bound in (2.15). As in the proof of Corollary 2.2, take the function W defined in (3.28), and for $A \in [-\infty, \infty)$, $\lambda > 0$ set

$$T_{j,\infty}^{W}(\lambda, A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \left(\mathcal{E}_j^+ - E_j^W(k) + \lambda\right)^{-1} \pi_{j,\infty}(k)\, \mathrm{d}k\, \mathcal{F}.$$

From (3.30), $T_{j,\infty}(\lambda, A) \leq T_{j,\infty}^{W}(\lambda, A)$, thus (3.23) implies that for all $A \in [-\infty, \infty)$ and $r \in (0,1)$

$$\mathcal{N}_j(\lambda) \leq n_+ \left(1 - r; V^{1/2} T_{j,\infty}^{W}(\lambda, A) V^{1/2}\right) + O(1), \quad \lambda \downarrow 0. \tag{3.31}$$

The asymptotic behavior of the function $n_+(1 - r; V^{1/2} T_{j,\infty}^{W}(\lambda, A) V^{1/2})$ was studied in [3] where it is shown that (Theorems 5.1 and 6.1)

$$\limsup_{\lambda \downarrow 0} \frac{n_+(1 - r; V^{1/2} T_{j,\infty}^{W}(\lambda, A) V^{1/2})}{|\ln \lambda|^{1/2}} \leq \mathcal{C}_+. \tag{3.32}$$

Putting together (3.31) and (3.32) we get the upper bound in (2.15).

For the lower bound consider the operators $h_+^N(k) := -\mathrm{d}^2/\mathrm{d}x^2 + (B_+ x - k)^2$ and $h_-^N(k) := -\mathrm{d}^2/\mathrm{d}x^2 + (B_- x - k)^2$ defined in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, both with a Neumann boundary condition at zero. From the monotonicity property with respect to the Neumann conditions, and from (3.8) we obtain that

$$h(k) \geq h_-^N(k) \oplus h_+^N(k) \tag{3.33}$$

(recall that $\mathbf{x}^+ = 0$, which implies that $b(x) = B_+ x$ for $x \geq 0$). The operators $h_\pm^N(k)$ have discrete and simple spectrum for any $k \in \mathbb{R}$. Denoting by $\{E_j^{N\pm}(k)\}_{j=1}^{\infty}$ their increasing sequences of eigenvalues, and using that

$$\lim_{k \to \infty} E_1^{N-}(k) = \infty, \quad \lim_{k \to \infty} E_j^{N+}(k) = \mathcal{E}_j^+,$$

(see e.g. [13]), we can conclude from (3.33) that for any $j \in \mathbb{N}$ there exists a constant K_j such that

$$E_j(k) \geq E_j^{N+}(k), \quad \text{for } k \geq K_j. \tag{3.34}$$

Set

$$T_{j,\infty}^{N}(\lambda, A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \left(\mathcal{E}_j^+ - E_j^{N+}(k) + \lambda\right)^{-1} \pi_{j,\infty}(k)\, \mathrm{d}k\, \mathcal{F}.$$

Inequality (3.23) along with (3.34) imply that for any $r \in (0,1)$ and $A \geq K_j$

$$\mathcal{N}_j(\lambda) \geq n_+ \left(1 + r; V^{1/2} T_{j,\infty}^{N}(\lambda, A) V^{1/2}\right) + O(1), \quad \lambda \downarrow 0. \tag{3.35}$$

Besides, it is shown in [26] that for some positive constant C_j

$$\mathcal{E}_j^+ - E_j^{N+}(k) = C_j k^{2j-1} e^{-k^2/B_+}(1 + o(1)), \quad k \to \infty.$$

Then, we can repeat the proofs of Proposition 3.7 and Corollary 3.9 in [4] in order to obtain

$$\liminf_{\lambda\downarrow 0} \frac{n_+(1+r; V^{1/2}T^N_{j,\infty}(\lambda, A)V^{1/2})}{|\ln(\lambda)|^{1/2}} \geq \mathcal{C}_-. \tag{3.36}$$

The inequalities (3.35), (3.36) imply the lower bound in (2.15).

3.4. Proof of Corollary 2.4: Upper Bound

The starting point of this proof is, as for Corollaries 2.2, 2.3, the inequalities 3.23. We will denote the operator $T_{j,\infty}(\lambda, -\infty)$ simply by $T_{j,\infty}(\lambda)$, and $P_{j,\infty}(-\infty)$ by $P_{j,\infty}$. Also from now on, without any lost of generality, we will take $s = 0$ for the function (2.17). That means, we will prove (2.21) for $N(\lambda, V, 0)$ (see Remark ii after Corollary 2.4).

Let $\varepsilon > 0$ and take a smooth function χ_ε with bounded derivatives such that $0 \leq \chi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$, $\chi_\varepsilon(x) = 0$ for $x \leq -2\varepsilon$ and $\chi_\varepsilon(x) = 1$ for $x \geq -\varepsilon$. Define

$$V_\varepsilon(x,\xi) := \chi_\varepsilon(x)V(x,\xi). \tag{3.37}$$

The Weyl's inequalities say that for any $r > 0$, $\delta \in (0,1)$, and $\lambda > 0$

$$\begin{aligned} n_+\left(r; T_{j,\infty}(\lambda)^{1/2} V T_{j,\infty}(\lambda)^{1/2}\right) &\leq n_+\left(r(1-\delta); T_{j,\infty}(\lambda)^{1/2} V_\varepsilon T_{j,\infty}(\lambda)^{1/2}\right) \\ &\quad + n_+\left(r\delta; T_{j,\infty}(\lambda)^{1/2}(V - V_\varepsilon) T_{j,\infty}(\lambda)^{1/2}\right). \end{aligned} \tag{3.38}$$

The function $V - V_\varepsilon$ is equal to zero for $x \geq -\varepsilon$. Arguing as in the proof of Corollary 2.2, we can see that for any $r > 0$

$$\begin{aligned} n_+\left(r; T_{j,\infty}(\lambda)^{1/2}(V - V_\varepsilon)T_{j,\infty}(\lambda)^{1/2}\right) &= n_*\left(\sqrt{r}; T_{j,\infty}(\lambda)^{1/2}(V - V_\varepsilon)^{1/2}\right) \\ &= O(1), \qquad \lambda \downarrow 0. \end{aligned} \tag{3.39}$$

Now, since $E_j(k) \leq \mathcal{E}_j^+$, $T_{j,\infty}(\lambda) \leq \lambda^{-1}P_{j,\infty}$, thus the min–max principle implies that for all $r > 0$ and $\lambda > 0$

$$n_+\left(r; T_{j,\infty}(\lambda)^{1/2} V_\varepsilon T_{j,\infty}(\lambda)^{1/2}\right) \leq n_+\left(r\lambda; P_{j,\infty} V_\varepsilon P_{j,\infty}\right). \tag{3.40}$$

Next, let us introduce a class of symbols suitable for our purposes. For $(x,\xi) \in \mathbb{R}^2$ consider the quadratic form in $\mathbb{R}^2$

$$g_{x,\xi}(y,\eta) = |y|^2 + \frac{|\eta|^2}{\langle x,\xi\rangle^2},$$

and for p, $q \in \mathbb{R}$, define the weight $w := \langle x\rangle^p \langle x,\xi\rangle^q$. Then, according to [17, Definition 18.4.6], consider the class of symbols $S^q_p := S(w,g)$. A symbol a is in S^q_p if for any $(\alpha,\beta) \in \mathbb{Z}^2_+$, the quantity

$$n^{p,q}_{\alpha,\beta}(a) := \sup_{(x,\xi)\in\mathbb{R}^2} \left|\langle x\rangle^{-p}\langle x,\xi\rangle^{-q+\alpha}\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \tag{3.41}$$

is finite.

For $a \in S_p^q$ we define the operator $Op^W(a)$ according to the Weyl quantization

$$(Op^W(a)u)(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} a\left(\frac{x+y}{2},\xi\right) e^{-i(x-y)\xi}u(y)\,\mathrm{d}y\,\mathrm{d}\xi,$$

for u in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Since V satisfies (2.16) it is obvious that V_ε is in S_0^{-m}. Moreover, using (2.1) (b), it is also true that the function

$$\widetilde{V}_\varepsilon(x,\xi) := V_\varepsilon(b^{-1}(x),-\xi)$$

belongs to S_0^{-m}. Due to $m>0$, the operator $Op^W(\widetilde{V}_\varepsilon)$ is compact in $L^2(\mathbb{R})$.

Using the same notation of Theorem 2.1, write $\mathcal{V}_{\varepsilon,j}$ for the pseudodifferential operator with contravariant symbol V_ε defined by (2.8).

Lemma 3.6. *For any $\varepsilon>0$ and $j\in\mathbb{N}$*

$$\mathcal{V}_{\varepsilon,j} - Op^W(\widetilde{V}_\varepsilon) = Op^W(R_1) + Op^W(R_2), \tag{3.42}$$

where the symbol $R_1 \in S_0^{-m-1}$ and $R_2 \in S_{-m}^{-m}$.

Proof. We give a sketch of the proof which is based on the proof of [36, Lemma 5.1]. Suppose that V is in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Then, from (2.8) the Weyl symbol p_V of $\mathcal{V}_{\varepsilon,j}$ is given by

$$p_V(\eta,\eta^*) = \frac{B_+}{2\pi}\int_{\mathbb{R}^3} e^{-iw\eta^*}\Psi_{j;x,\xi}\,(\eta+w/2)\,\overline{\Psi_{j;x,\xi}\,(\eta-w/2)}V_\varepsilon(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}w,$$

$\Psi_{j;x,\xi}$ being defined in (2.7). We use a first order Taylor expansion of V_ε, noticing that $\partial_1 V_\varepsilon = (\partial_1 V)\chi_\varepsilon + V(\partial_1\chi_\varepsilon)$, $\partial_2 V_\varepsilon = (\partial_2 V)\chi_\varepsilon$. Because of (2.16), $(\partial_1 V)\chi_\varepsilon, (\partial_2 V)\chi_\varepsilon \in S_0^{-m-1}$. On the other side, the partial derivative $\partial_1\chi_\varepsilon$ has compact support which implies that $V(\partial_1\chi_\varepsilon) \in S_{-p}^{-m}$ for any $p>0$, in particular for $p=m$. Now we use the same estimates given in the proof of [36, Lemma 5.1] to conclude that $\widetilde{V}_\varepsilon$ is a principal symbol for $\mathcal{V}_{\varepsilon,j}$, and that the remainder terms, coming from the Taylor expansion, satisfy the required conditions. □

For a measurable function $a:\mathbb{R}^2\to\mathbb{R}_+$ define

$$N(\lambda,a) := \frac{1}{2\pi} vol\{(x,\xi)\in\mathbb{R}^2; a(x,\xi)>\lambda\}.$$

Lemma 3.6 together with [6, Lemma 4.7] imply that there exists a positive λ_0 such that

$$\begin{aligned} n_+(\lambda; Op^W(R_1)) &= O(N(\lambda,\langle x,\xi\rangle^{-m-1})) = O(\lambda^{-\frac{2}{m+1}}),\\ n_+(\lambda; Op^W(R_2)) &= O\left(N(\lambda,\langle x,\xi\rangle^{-m}\langle x\rangle^{-m})\right) = O(\lambda^{-\frac{1}{2m}-\frac{1}{m}}), \end{aligned}$$

for $\lambda\in[0,\lambda_0]$. Then, (3.42) and the Weyl inequalities imply that for all $\delta\in(0,1)$

$$n_+(\lambda;\mathcal{V}_{\varepsilon,j}) \le n_+((1-\delta)\lambda; Op^W(\widetilde{V}_\varepsilon)) + o(\lambda^{-2/m}), \quad \lambda\downarrow 0. \tag{3.43}$$

Putting together (3.23), (3.38), (3.39), (3.40), (3.25) and (3.43) we obtain that for all $\delta \in (0,1)$

$$\mathcal{N}_j(\lambda) \leq n_+((1-\delta)\lambda; Op^W(\widetilde{V}_\varepsilon)) + o(\lambda^{-2/m}), \quad \lambda \downarrow 0. \tag{3.44}$$

Lemma 3.7. *For any $\varepsilon > 0$ the function $N(\lambda, \widetilde{V}_\varepsilon)$ satisfies the homogeneity condition* (2.19)

Proof. Note that

$$\begin{aligned}
&2\pi|N((1-\epsilon)\lambda, \widetilde{V}_\varepsilon) - N((1+\epsilon)\lambda, \widetilde{V}_\varepsilon)| \\
&\quad = \mathrm{vol}\{(x,\xi) \in \mathbb{R}^2; (1+\epsilon)\lambda \geq \chi_\varepsilon(b^{-1}(x))V(b^{-1}(x), -\xi) > (1-\epsilon)\lambda\} \\
&\quad = \mathrm{vol}\{(x,\xi) \in \mathbb{R}^2; (1+\epsilon)\lambda \geq V(b^{-1}(x), -\xi) > (1-\epsilon)\lambda,\, b^{-1}(x) \geq -\varepsilon\} \\
&\qquad + \mathrm{vol}\{(x,\xi) \in \mathbb{R}^2; (1+\epsilon)\lambda \geq \chi_\varepsilon(b^{-1}(x))V(b^{-1}(x), -\xi) > (1-\epsilon)\lambda, -2\varepsilon \\
&\qquad < b^{-1}(x) < -\varepsilon\} \\
&\quad \leq \int_{\{(x',\xi') \in \mathbb{R}^2; (1+\epsilon)\lambda \geq V(x',\xi') > (1-\epsilon)\lambda,\, x' \geq -\varepsilon\}} B(x')\, \mathrm{d}x' \mathrm{d}\xi' \\
&\qquad + \mathrm{vol}\{(x,\xi) \in \mathbb{R}^2;\, C_{0,0}\, \langle b(x)^{-1}, \xi\rangle^{-m} > (1-\epsilon)\lambda, -2\varepsilon < b^{-1}(x) < -\varepsilon\} \\
&\quad \leq B_+\left(N((1-\epsilon)\lambda, V, -\varepsilon) - N((1+\epsilon)\lambda, V, -\varepsilon)\right) + O(\lambda^{-1/m}),
\end{aligned}$$

where in the first inequality we have used the change of variables $b^{-1}(x) = x'$, $-\xi = \xi'$, that V satisfies (2.16) and that $0 \leq \chi_\varepsilon \leq 1$. Since $N(\lambda, V, -\varepsilon)$ fulfills (2.19) we obtain the required result. □

Lemma 3.8. *For any $\varepsilon > 0$, $N(\lambda, \widetilde{V}_\varepsilon)$ satisfies condition* (2.18)*. Moreover*

$$\lim_{\lambda \downarrow 0} \frac{B_+ N(\lambda, V, 0)}{N(\lambda, \widetilde{V}_\varepsilon)} = 1. \tag{3.45}$$

Proof. First let us show that

$$\lim_{\lambda \downarrow 0} \frac{N(\lambda, V_\varepsilon)}{N(\lambda, V, 0)} = 1. \tag{3.46}$$

To see this we estimate $|N(\lambda, V_\varepsilon) - N(\lambda, V, 0)|$, noticing that $\{(x,\xi) \in \mathbb{R}^2; V_\varepsilon(x,\xi) > \lambda\}$ and $\{(x,\xi) \in \mathbb{R}^2; V(x,\xi) > \lambda,\, x > 0\}$ differ in a set contained in

$$\{(x,\xi) \in \mathbb{R}^2; V_\varepsilon(x,\xi) > \lambda, -2\varepsilon < x \leq 0\}.$$

Then in view of $V_\varepsilon \in S_0^{-m}$

$$|N(\lambda, V_\varepsilon) - N(\lambda, V, 0)| = O(\lambda^{-1/m}).$$

Using that $N(\lambda, V, 0)$ satisfies property (2.18) we obtain (3.46).

Now let us prove that

$$\lim_{\lambda \downarrow 0} \frac{N(\lambda, \widetilde{V}_\varepsilon)}{B_+ N(\lambda, V_\varepsilon)} = 1. \tag{3.47}$$

Similarly to the proof of Lemma 3.7 we have

$$2\pi|B_+N(\lambda,V_\varepsilon) - N(\lambda,\widetilde{V}_\varepsilon)| = \int_{\{(x,\xi);V_\varepsilon(x,\xi)>\lambda\}} B_+ - B(x)\,\mathrm{d}x\mathrm{d}\xi$$
$$\leq C\lambda^{-1/m}\int_{-2\varepsilon}^{\lambda^{-1/m}} \langle x\rangle^{-M}\,\mathrm{d}x = o(\lambda^{-2/m}), \quad \lambda\downarrow 0. \tag{3.48}$$

Where we have used (2.20) and (2.16), and C is a positive constant independent of λ. Taking into account (3.46) along with (2.18), we obtain (3.47).

Putting together (3.46) and (3.47) we get (3.45). □

Since $N(\lambda,\widetilde{V}_\varepsilon)$ satisfies (2.18) and (2.19), it follows that it also satisfies condition (T′) of [6]. Then [6, Theorem 1.3] says that

$$\limsup_{\lambda\downarrow 0}\frac{n_+((1-\delta)\lambda; Op^W(\widetilde{V}_\varepsilon))}{N((1-\delta)\lambda,\widetilde{V}_\varepsilon)} = 1,$$

and therefore (3.44), (3.45) imply that for all $\delta\in(0,1)$

$$\limsup_{\lambda\downarrow 0}\frac{\mathcal{N}_j(\lambda)}{B_+N((1-\delta)\lambda;V,0)} \leq 1. \tag{3.49}$$

To finish the proof of the upper bound in (2.21) it only remains to note that conditions (2.18), (2.19) imply that

$$\lim_{\delta\downarrow 0}\limsup_{\lambda\downarrow 0}\frac{N((1-\delta)\lambda,V,0)}{N(\lambda,V,0)} = 1.$$

3.5. Proof of Corollary 2.4: Lower Bound

Condition (2.20) implies that there exits a smooth function $\tilde{B}$ such that $B(x)\geq\tilde{B}(x)\geq B_-$ for all $x\in\mathbb{R}$, and $B_+-\tilde{B}(x) = \tilde{C}\langle x\rangle^{-M}$, for some positive constant $\tilde{C}$ and all x sufficiently big. Using $\tilde{B}$ to define $\tilde{b}$ according to (1.2), we see that (3.10) implies

$$\left(\mathcal{E}_j^+ - E_j(k)+\lambda\right)^{-1} \geq \left(\mathcal{E}_j^+ - E_j(\tilde{b}(b^{-1}(k)),\tilde{b})+\lambda\right)^{-1}, \quad \text{for all}\, k\in\mathbb{R}, \tag{3.50}$$

where $E_j(k,\tilde{b})$ is defined as in Lemma 3.1.

Since $B_+ - \tilde{B}$ is strictly decreasing for x large, the function $\mathcal{E}_j^+ - E_j((\tilde{b}(b^{-1}(k),\tilde{b})$ is strictly decreasing for k big [21, Theorem 3.2]. We denote by ρ_j its inverse, which is defined at least in an interval of the form $(0,\gamma)$, $\gamma>0$. It is obvious that $\lim_{w\downarrow 0}\rho_j(w)=\infty$. Moreover, from Lemma 4.8 in [36], we know that (2.20) implies that $\mathcal{E}_j^+ - E_j(\tilde{b}(b^{-1}(k),\tilde{b}) = O(k^{-M})$, $k\to\infty$, and then

$$\rho_j(w) = O(w^{-1/M}), \quad w\downarrow 0. \tag{3.51}$$

For $j \in \mathbb{N}$, $\delta \in (0,1)$ and $\lambda > 0$ set $\varrho = \varrho(\lambda) := \rho_j(\delta\lambda)$. Then (3.50) implies that

$$\left(\mathcal{E}_j^+ - E_j(k) + \lambda\right)^{-1} \geq ((1+\delta)\lambda)^{-1}, \quad \text{for all } k \geq \varrho(\lambda). \tag{3.52}$$

Therefore, for all $r > 0$ and $\delta \in (0,1)$

$$\begin{aligned} n_+\left(r; V^{1/2} T_{j,\infty}(\lambda) V^{1/2}\right) &\geq n_+\left(r; V_\varepsilon^{1/2} T_{j,\infty}(\lambda) V_\varepsilon^{1/2}\right) \\ &\geq n_+\left(r; V_\varepsilon^{1/2} T_{j,\infty}(\lambda, \varrho) V_\varepsilon^{1/2}\right) \\ &\geq n_+\left(r(1+\delta)\lambda; V_\varepsilon^{1/2} P_{j,\infty}(\varrho) V_\varepsilon^{1/2}\right). \end{aligned} \tag{3.53}$$

In the first and the second inequality we have used the min–max principle, while for the third inequality we used (3.52).

Next, using the Weyl inequalities, for any $\lambda > 0$ and $\delta \in (0,1)$

$$\begin{aligned} & n_+\left(\lambda; V_\varepsilon^{1/2} P_{j,\infty}(\varrho) V_\varepsilon^{1/2}\right) \\ & \quad \geq n_+\left(\lambda(1+\delta); V_\varepsilon^{1/2} P_{j,\infty} V_\varepsilon^{1/2}\right) - n_+\left(\lambda\delta; V_\varepsilon^{1/2}(P_{j,\infty} - P_{j,\infty}(\varrho)) V_\varepsilon^{1/2}\right). \end{aligned} \tag{3.54}$$

The term $n_+(\lambda; V_\varepsilon^{1/2} P_{j,\infty} V_\varepsilon^{1/2}) = n_+(\lambda; P_{j,\infty} V_\varepsilon P_{j,\infty}) = n_+(r; \mathcal{V}_{\varepsilon,j})$ was already obtained in (3.40), and its asymptotic behavior can be estimated as in Sect. 3.4.

For the second term in (3.54) we have that (3.25) implies

$$\begin{aligned} & n_+\left(\lambda; V_\varepsilon^{1/2}(P_{j,\infty} - P_{j,\infty}(\varrho)) V_\varepsilon^{1/2}\right) \\ & \quad = n_+\left(\lambda; \int_{(-\infty,\varrho]}^{\oplus} \pi_{j,\infty}(k)\, \mathrm{d}k \mathcal{F} V_\varepsilon \mathcal{F}^* \int_{(-\infty,\varrho]}^{\oplus} \pi_{j,\infty}(k)\, \mathrm{d}k\right) \\ & \quad = n_+\left(\lambda; \mathbb{1}_{(-\infty,\varrho]} \mathcal{F} P_{j,\infty} V_\varepsilon P_{j,\infty} \mathcal{F}^* \mathbb{1}_{(-\infty,\varrho]}\right) = n_+\left(\lambda; \mathbb{1}_{(-\infty,\varrho]} \mathcal{V}_{\varepsilon,j} \mathbb{1}_{(-\infty,\varrho]}\right). \end{aligned} \tag{3.55}$$

Let $\chi_\lambda(x) := \chi_\varepsilon(-x + \rho_j(\delta\lambda)) = \chi_\varepsilon(-x + \varrho(\lambda))$, the same χ_ε of the preceding subsection. Then χ_λ is a smooth function with bounded derivatives such that $0 \leq \chi_\lambda \leq 1$, $\chi_\lambda(x) = 0$ for $x \geq \varrho(\lambda) + 2\varepsilon$ and $\chi_\lambda(x) = 1$ for $x \leq \varrho(\lambda) + \varepsilon$. It is important to note that for all positive λ, ε, and $\delta \in (0,1)$, $\chi_\lambda \in S_0^0$ and its semi-norms $n_{\alpha,\beta}^{0,0}(\chi_\lambda)$ (defined by (3.41)) are independent of δ and λ for all $(\alpha,\beta) \in \mathbb{Z}_+^2$. Indeed,

$$n_{\alpha,\beta}^{0,0}(\chi_\lambda) = \begin{cases} 0; & \alpha > 0 \\ ||\chi_\varepsilon^{(\beta)}||_{L^\infty(\mathbb{R})}; & \alpha = 0. \end{cases}$$

Write as before $Op^W(\widetilde{V}_\varepsilon)$ for the Pseudo-differential operator with Weyl symbol $\widetilde{V}_\varepsilon$. Then, since for all $\lambda > 0$, $\chi_\lambda \mathbb{1}_{(-\infty,\varrho(\lambda)]} = \mathbb{1}_{(-\infty,\varrho(\lambda)]}$, we have that

$$\begin{aligned}
&\mathbb{1}_{(-\infty,\varrho]}\mathcal{V}_{\varepsilon,j}\mathbb{1}_{(-\infty,\varrho]}\\
&= \mathbb{1}_{(-\infty,\varrho]}Op^W(\widetilde{V}_\varepsilon)\mathbb{1}_{(-\infty,\varrho]} + \mathbb{1}_{(-\infty,\varrho]}\left(\mathcal{V}_{\varepsilon,j} - Op^W(\widetilde{V}_\varepsilon)\right)\mathbb{1}_{(-\infty,\varrho]}\\
&= \mathbb{1}_{(-\infty,\varrho]}\left(Op^W(\widetilde{V}_\varepsilon)Op^W(\chi_\lambda)\right)\mathbb{1}_{(-\infty,\varrho]} + \mathbb{1}_{(-\infty,\varrho]}\left(\mathcal{V}_{\varepsilon,j} - Op^W(\widetilde{V}_\varepsilon)\right)\mathbb{1}_{(-\infty,\varrho]}.
\end{aligned} \tag{3.56}$$

The symbol $\widetilde{V}_\varepsilon \in S_0^{-m}$ and $\chi_\lambda \in S_0^0$, then it is well known that [17, Theorem 18.5.4]

$$Op^W(\widetilde{V}_\varepsilon)Op^W(\chi_\lambda) = Op^W(\widetilde{V}_\varepsilon\chi_\lambda) + Op^W(R_\lambda), \tag{3.57}$$

where $R_\lambda \in S_0^{-m-1}$, and each one of its semi-norms $n_{\alpha,\beta}^{0,-m-1}(R_\lambda)$ is polynomially bounded by a finite number of semi-norms of $\widetilde{V}_\varepsilon$ and χ_λ in S_0^{-m} and S_0^0, respectively. Since the semi-norms of χ_λ are independent of λ, [6, Lemma 4.7] implies that there exists a positive constants λ_0 such that

$$n_+(\lambda; Op^W(R_\lambda)) = O(\lambda^{-2/(m+1)}), \quad \text{for } \lambda \in (0, \lambda_0]. \tag{3.58}$$

Lemma 3.9. *For every $\varepsilon > 0$*

$$\lim_{\lambda\downarrow 0} \frac{n_+(\lambda; Op^W(\widetilde{V}_\varepsilon\chi_\lambda))}{\lambda^{-2/m}} = 0.$$

Proof. By [6, Proposition 4.1], there are positive constants C_λ and ζ such that

$$n_+\left(\lambda; Op^W\left(\widetilde{V}_\varepsilon\chi_\lambda\right)\right) \leq N\left(\lambda, \widetilde{V}_\varepsilon\chi_\lambda\right) + C_\lambda\lambda^{(-2/m)+\zeta}. \tag{3.59}$$

The constant C_λ depends polynomially on a finite number of semi-norms of the symbol $\widetilde{V}_\varepsilon\chi_\lambda$, but from composition of symbols each one of the semi-norms of $\widetilde{V}_\varepsilon\chi_\lambda$ is polynomially bounded by a finite number of semi-norms of V in S_0^{-m} and χ_λ in S_0^0. Consequently, the constant C_λ can be taken independent of λ.

The proof of (3.59) that appears in [6] is for symbols that do not depend on λ. However, it works as well in our case just introducing minor changes.

Now, since $(\widetilde{V}_\varepsilon\chi_\lambda)(x,\xi) = V(b^{-1}(x), -\xi)\chi_\varepsilon(b^{-1}(x))\chi_\lambda(x)$, where the support of $\chi_\varepsilon(b^{-1}(x))\chi_\lambda(x)$ is contained on the strip $\{(x,\xi) \in \mathbb{R}^2; b(-2\varepsilon) \leq x \leq \varrho(\lambda) + 2\varepsilon\}$, and V is in S_0^{-m}, the set $\{(x,\xi) \in \mathbb{R}^2; \widetilde{V}_\varepsilon\chi_\lambda(x,\xi) > \lambda\}$ is contained in

$$\left\{(x,\xi) \in \mathbb{R}^2; \langle b^{-1}(x), \xi\rangle^{-m} > \lambda,\, b(-2\varepsilon) \leq x \leq \varrho(\lambda) + 2\varepsilon\right\}.$$

Then,

$$N(\lambda, \widetilde{V}_\varepsilon\chi_\lambda) = O(\lambda^{-1/m}((\varrho(\lambda) + 2\varepsilon) - b(-2\varepsilon))). \tag{3.60}$$

Putting together (3.59), (3.60) and (3.51) (recalling that $M > m$), we finish the proof of the Lemma. □

Gathering (3.56), Lemma 3.6, (3.57), (3.58) and Lemma 3.9 we obtain

$$\lim_{\lambda\downarrow 0} \frac{n_+\left(\lambda; \mathbb{1}_{(-\infty,\varrho]}\mathcal{V}_{\varepsilon_j}\mathbb{1}_{(-\infty,\varrho]}\right)}{\lambda^{-2/m}} = 0, \tag{3.61}$$

thus, for all $\delta \in (0,1)$ (3.23), (3.53), (3.54), (3.55), (3.61) and Lemma 3.8 imply

$$\liminf_{\lambda\downarrow 0} \frac{\mathcal{N}_j(\lambda)}{B_+ N(\lambda(1+\delta), V, 0)} \geq \liminf_{\lambda\downarrow 0} \frac{n_+(\lambda(1+\delta); Op^W(\tilde{V}_\varepsilon))}{N(\lambda(1+\delta), \widetilde{V}_\varepsilon)}.$$

Finally, arguing as in the last part of Sect. 3.4 we can obtain the lower bound in (2.21).

Acknowledgements

This Project was partially supported by Vicerrectoría de la Investigación de la Pontificia Universidad Católica de Chile, grant *Inicio* 43/2014.

References

[1] Berezin, F., Shubin, M.: The Schrödinger Equation. Kluwer Academic Publishers Group, Dordrecht (1991)

[2] Birman, M.Sh., Solomjak, M.Z.: Spectral Theory of Selfadjoint Operators in Hilbert Space. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht (1987)

[3] Bruneau, V., Miranda, P., Raikov, G.: Discrete spectrum of quantum Hall effect Hamiltonians I. Monotone edge potentials. J. Spectr. Theory **1**, 237–272 (2011)

[4] Bruneau, V., Miranda, P., Raikov, G.: Dirichlet and Neumann eigenvalues for half-plane magnetic Hamiltonians. Rev. Math. Phys. **26**, 1450003 (2014)

[5] Calvo, M.: Exactly soluble two-dimensional electron gas in a magnetic-field barrier. Phys. Rev. B **48**, 2365–2369 (1993)

[6] Dauge, M., Robert, D.: Weyl's formula for a class of pseudodifferential operators with negative order on $L^2(R^n)$. In: Pseudodifferential Operators (Oberwolfach, 1986), Lecture Notes in Mathematics, vol. 1256, pp. 91–112. Springer, Berlin (1987)

[7] Davies, N., Patel, A., Cortijo, A., Cheianov, V., Guinea, F., Fal'ko, V.: Skipping and snake orbits of electrons: singularities and catastrophes. Phys. Rev. B **85**, 155433–155437 (2012)

[8] Dombrowski, N., Germinet, F., Raikov, G.: Quantization of edge currents along magnetic barriers and magnetic guides. Ann. Henri Poincaré **12**, 1169–1197 (2011)

[9] Dombrowski, N., Hislop, P., Soccorsi, E.: Edge currents and eigenvalue estimates for magnetic barrier Schrödinger operators. Asymptot. Anal. **89**, 331–363 (2014)

[10] Everitt, W.N., Giertz, M.: Some inequalities associated with certain ordinary differential operators. Math. Z. **126**, 308–326 (1972)

[11] Exner, P., Kovařík, H.: Magnetic strip waveguides. J. Phys. A **33**, 3297–3311 (2000)

[12] Filonov, N., Pushnitski, A.: Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains. Commun. Math. Phys. **264**, 759–772 (2006)

[13] Fournais, S., Helffer, B.: Spectral Methods in Surface Superconductivity. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston (2010)

[14] Gerhardts, R., Manolescu, A., Zwerschke, S.: Planar cyclotron motion in unidirectional superlattices defined by strong magnetic and electric fields: Traces of classical orbits in the energy spectrum. Phys. Rev. B **60**, 5536–5548 (1999)

[15] Hislop, P., Popoff, N., Raymond, N., Sunqvist, M.: Band functions in the presence of magnetic steps. arXiv:1501.02824

[16] Hislop, P.D., Soccorsi, E.: Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields. J. Math. Anal. Appl. **422**, 594–624 (2015)

[17] Hörmander, L.: The Analysis of Linear Partial Differential Operators III. Pseudodifferential Operators. Grundlehren der Mathematischen Wissenschaften, 274. Springer, Berlin (1985)

[18] Ivrii, V.: Microlocal Analysis and Precise Spectral Asymptotics. Springer Monographs in Mathematics. Springer, Berlin (1998)

[19] Iwatsuka, A.: Examples of absolutely continuous Schrödinger operators in magnetic fields. Publ. Res. Inst. Math. Sci. **21**, 385–401 (1985)

[20] Leschke, H., Warzel, S., Weichlein, A.: Energetic and dynamic properties of a quantum particle in a spatially random magnetic field with constant correlations along one direction. Ann. Henri Poincaré **7**, 335–363 (2006)

[21] Măntoiu, M., Purice, R.: Some propagation properties of the Iwatsuka model. Commun. Math. Phys. **188**, 691–708 (1997)

[22] Melgaard, M., Rozenblum, G.: Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank. Commun. Partial Differ. Equ. **28**, 697–736 (2003)

[23] Muller, J.: Effect of a nonuniform magnetic field on a two-dimensional electron gas in the ballistic regime. Phys. Rev. Lett. **68**, 385–388 (1992)

[24] Peeters, F., Reijniers, J.: Snake orbits and related magnetic edge states. J. Phys. Condens. Matter **12**, 9771–9786 (2000)

[25] Persson, M.: Eigenvalue asymptotics of the even-dimensional exterior Landau-Neumann Hamitonian. Adv. Math. Phys. **2009**, Article ID 873704 (2009)

[26] Popoff, N.: Sur le spectre de l'opérateur de Schrödinger magnétique dans un domaine diédral. PhD Thesis, Université de Rennes 1 (2012)

[27] Pushnitski, A., Rozenblum, G.: Eigenvalue clusters of the Landau Hamiltonian in the exterior of a compact domain. Doc. Math. **12**, 569–586 (2007)

[28] Pushnitski, A., Rozenblum, G.: On the spectrum of Bargmann–Toeplitz operators with symbols of a variable sign. J. Anal. Math. **114**, 317–340 (2011)

[29] Raikov, G.: Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips. Commun. Partial Differ. Equ. **15**, 407–434 (1990)

[30] Raikov, G., Warzel, S.: Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials. Rev. Math. Phys. **14**, 1051–1072 (2002)

[31] Reijniers, J., Matulis, A., Chang, K., Peeters, F.M., Vasilopoulos, P.: Confined magnetic guiding orbit states. Europhys. Lett. **59**, 749 (2002)

[32] Rozenblum, G.: On lower eigenvalue bounds for Toeplitz operators with radial symbols in Bergman spaces. J. Spectr. Theory **1**, 299–325 (2011)

[33] Rozenblum, G., Tashchiyan, G.: On the spectral properties of the perturbed Landau Hamiltonian. Commun. Partial Differ. Equ. **33**, 1048–1081 (2008)

[34] Rozenblum, G., Tashchiyan, G.: On the spectral properties of the Landau Hamiltonian perturbed by a moderately decaying magnetic field. In: Spectral and Scattering Theory for Quantum Magnetic Systems, Contemp. Math., vol. 500, pp. 169–186. Amer. Math. Soc., Providence (2009)

[35] Shirai, S.: Eigenvalue asymptotics for the Schrödinger operator with steplike magnetic field and slowly decreasing electric potential. Publ. Res. Inst. Math. Sci. **39**, 297–330 (2003)

[36] Shirai, S.: Strong-electric-field eigenvalue asymptotics for the Iwatsuka model. J. Math. Phys. **46**, 052112 (2005)

Pablo Miranda
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Vicuña Mackenna 4860, Santiago, Chile
e-mail: `pmirandar@mat.puc.cl`

Communicated by Jens Marklof.
Received: January 19, 2015.
Accepted: September 1, 2015.