On the Focusing Effect of Gravitational Waves

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Abstract. We show that a suitable choice of the initial data from the past null infinity of Christodoulou's work on the formation of black holes (The formation of black holes in general relativity. Monographs in Mathematics, European Mathematical Society, Zürich, 2009) will lead to the formation of a near Schwarzschild region up to the past null infinity.

1. Introduction

Black holes are a central concept in general relativity and are expected to form in generic gravitational collapsing according to the weak cosmic censorship conjecture. A related concept is the trapped surface, which is an embedded space-like two-surface with negative mean curvature with respect to both future null normals. Penrose proved that under some suitable physical conditions, the presence of a closed trapped surface implies that the spacetime is not future geodesically complete. Therefore, the formation of trapped surfaces is an important step to understand the mechanism of gravitational collapsing. In [1], Christodoulou discovered a remarkable mechanism responsible for the formation of trapped surfaces, which is the first result concerning the *dynamical* formation of trapped surfaces without symmetry in vacuum case. Based on the techniques he and Klainerman developed in [2] proving the global stability of the Minkowski spacetime, he managed to understand the cancellations among different components of the connection and curvature and how various geometric quantities propagate. He also proved a version of the above result for data prescribed at the past null infinity. From the point of view of partial differential equations, it established the first result on the long time dynamics in general relativity for general initial data which are not

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necessarily close to the Minkowski spacetime. In [3], Klainerman and Rodnianski extended Christodoulou's result. They gave a significant simplification of Christodoulou's result based on studying a larger class of initial data introduced by a parabolic scaling. They also required less regularity. The price for a simplification is that the propagation estimates are weaker than those of Christodoulou's. However, we can actually start with the existence result in [3] and then assume the same restriction on the data as that of Christodoulou's. This procedure allows one to recover Christodoulou's stronger results.

It is natural to study more precise behavior of the spacetime constructed in [1] in order to understand the mechanism in gravitational collapsing. The focusing of gravitational waves, which is described by the short pulse ansatz introduced by Christodoulou, is responsible for the formation of trapped surfaces considered in [1]. In this paper, we study what will happen in the spacetime after the focusing of such gravitational waves. Precisely, this region corresponds to the advanced time $u \geq \delta$ (which will be defined in the context). It is proved that if the total incoming energy of the gravitational waves from the past null infinity in each direction is the same, then a region that is close to a Schwarzschild spacetime will form. The mass parameter of this Schwarzschild spacetime happens to be the total incoming energy (up to a constant multiple). The following picture shows the basic result of this paper. We will construct the *Near Schwarzschild* region. The precise statement will be given in the context. A similar statement is given in [5].

Finally, we remark that the formation of the near Schwarzschild region may help to understand the mechanism of the formation of trapped surfaces and other related problems, since we have a good understanding of the Schwarzschild spacetime. For instance, a finite version of the result is used in [4] to construct asymptotically flat Cauchy initial data. We also remark that our result does not touch the future null infinity.

2. Preliminaries

2.1. Preliminaries on Geometry

Before stating the main results, we introduce the geometric setup and main result in [1]. We use $\mathcal{D}$ to denote the underlying spacetime and use g to denote the background 3+1 dimensional Lorentzian metric. We also use ∇ to denote the Levi-Civita connection of the metric g .

Recall that we have two optical functions $\underline{u}$ and u defined on $\mathcal{D}$ such that

$$g(\nabla \underline{u}, \nabla \underline{u}) = g(\nabla u, \nabla u) = 0.$$

The spacetime $\mathcal{D}$ is spanned by the level sets of $\underline{u}$ and u . The functions u and $\underline{u}$ increase towards the future. We use C_u to denote the outgoing null hypersurfaces which are level surfaces of u and use $\underline{C}_{\underline{u}}$ to denote the incoming null hypersurfaces which are level surfaces of $\underline{u}$. We also use $S_{\underline{u},u} = \underline{C}_{\underline{u}} \cap C_u$ to denote the space-like two sphere. The notation $\underline{C}_{\underline{u}}^{[u',u'']}$ refers to the part of the incoming cone $\underline{C}_{\underline{u}}$ where $u' \leq u \leq u''$ and the notation $C_u^{[\underline{u}',\underline{u}'']}$ refers to the part of the outgoing cone C_u where $\underline{u}' \leq \underline{u} \leq \underline{u}''$.

For the initial null hypersurface C_{u_0} where $u_0 < -1$, we require that $C_{u_0}^{[u_0,0]}$ is a flat light cone in Minkowski spacetime and S_{0,u_0} is the standard sphere with radius $|u_0|$. Thus, by solving Einstein vacuum equations, we know the future domain of dependence of $C_{u_0}^{[u_0,0]}$ is flat. It is equivalent to saying that the past of $\underline{C}_0$ can be isometrically embedded into Minkowski spacetime. In particular, the incoming null hypersurface $\underline{C}_0$ coincides with an incoming light cone of the Minkowski spacetime.

We are ready to define various geometric quantities. The positive function Ω is defined by the formula $\Omega^{-2} = -2g(\nabla \underline{u}, \nabla u)$. We then define the normalized null pair (e_3, e_4) by $e_3 = -2\Omega \nabla \underline{u}$ and $e_4 = -2\Omega \nabla u$. We also need two more null vector fields $\underline{L} = \Omega e_3$ and $L = \Omega e_4$. We remark that the flows generated by $\underline{L}$ and L preserve the double null foliation. On a given two sphere $S_{\underline{u},u}$ we choose a local orthonormal frame (e_1, e_2) . We call (e_1, e_2, e_3, e_4) a *null frame*. As a convention, throughout the paper, we use capital Latin letters $A, B, C, \dots$ to denote an index from 1 to 2, e.g. e_A denotes either e_1 or e_2 .

Let ϕ be a tangential tensorfield on $\mathcal{D}$. By definition, ϕ being tangential means that ϕ is *a priori* a tensorfield defined on the spacetime $\mathcal{D}$ and all the possible contractions of ϕ with either e_3 or e_4 are zeros. We use $D\phi$ and $\underline{D}\phi$ to denote the projection to $S_{\underline{u},u}$ of usual Lie derivatives $\mathcal{L}_L\phi$ and $\mathcal{L}_{\underline{L}}\phi$. The spacetime metric g induces a Riemannian metric $\mathcal{g}$ on $S_{\underline{u},u}$. We use d and ∇ to denote the exterior differential and covariant derivative (with respect to $\mathcal{g}$) on $S_{\underline{u},u}$.

Let $(\theta^A)_{A=1,2}$ be a local coordinate system on the two sphere S_{0,u_0} . We can extend θ^A 's to the whole $\mathcal{D}$ by first setting $L(\theta^A) = 0$ on C_{u_0} , and then setting $\underline{L}(\theta^A) = 0$ on $\mathcal{D}$. Therefore, we obtain a coordinate system $(\underline{u}, u, \theta^A)$ on $\mathcal{D}$. In such coordinates, the Lorentzian metric g takes the following form

$$g = -2\Omega^2(d\underline{u} \otimes du + du \otimes d\underline{u}) + \mathcal{g}_{AB}(d\theta^A - b^A d\underline{u}) \otimes (d\theta^B - b^B d\underline{u}). \quad (2.1)$$

The null vectors $\underline{L}$ and L can be computed as $\underline{L} = \partial_u$ and $L = \partial_u + b^A \partial_{\theta_A}$. By construction, we have $b^A(\underline{u}, u_0, \theta) = 0$. In addition, we also require $\Omega(\underline{u}, u_0, \theta) = \Omega(0, u, \theta) = 1$.

We recall the definitions of null connection coefficients. Roughly speaking, the following quantities are Christoffel symbols of ∇ according to the null frame (e_1, e_2, e_3, e_4) :

$$\begin{aligned}\chi_{AB} &= g(\nabla_A e_4, e_B), \eta_A = -\frac{1}{2}g(\nabla_3 e_A, e_4), \omega = \frac{1}{2}\Omega g(\nabla_4 e_3, e_4), \\ \underline{\chi}_{AB} &= g(\nabla_A e_3, e_B), \underline{\eta}_A = -\frac{1}{2}g(\nabla_4 e_A, e_3), \underline{\omega} = \frac{1}{2}\Omega g(\nabla_3 e_4, e_3).\end{aligned}$$

They are all tangential tensorfields. We define $\chi' = \Omega^{-1}\chi$, $\underline{\chi}' = \Omega^{-1}\underline{\chi}$ and $\zeta = \frac{1}{2}(\eta - \underline{\eta})$. The trace of χ and $\underline{\chi}$ will play an important role in Einstein field equations and they are defined by $\text{tr}\chi = \not{g}^{AB}\chi_{AB}$ and $\text{tr}\underline{\chi} = \not{g}^{AB}\underline{\chi}_{AB}$. We remark that the trace is taken with respect to the metric $\not{g}$ and the indices are raised by $\not{g}$. By definition, we can check directly the following identities $\not{d}\log\Omega = \frac{1}{2}(\eta + \underline{\eta})$, $D\log\Omega = \omega$, $\underline{D}\log\Omega = \underline{\omega}$ and $\underline{D}b = 4\Omega^2\zeta^\sharp$. Here, $\zeta^\sharp$ is the vector field dual to the 1-form ζ . In the sequel, we will suppress the sign $\sharp$ and use the metric $\not{g}$ to identify ζ and $\zeta^\sharp$.

We can also decompose the curvature tensor into null curvature components:

$$\begin{aligned}\alpha_{AB} &= R(e_A, e_4, e_B, e_4), \quad \beta_A = \frac{1}{2}R(e_A, e_4, e_3, e_4), \quad \rho = \frac{1}{4}R(e_3, e_4, e_3, e_4), \\ \underline{\alpha}_{AB} &= R(e_A, e_3, e_B, e_3), \quad \underline{\beta}_A = \frac{1}{2}R(e_A, e_4, e_3, e_4), \quad \sigma = \frac{1}{4}R(e_3, e_4, e_A, e_B)\not{\epsilon}^{AB},\end{aligned}$$

where $\not{\epsilon}$ is the volume form on $S_{\underline{u}, u}$.

In order to express the Einstein vacuum equations with respect to a null frame, we have to introduce some operators. For a symmetric tangential 2-tensorfield θ , we use $\hat{\theta}$ and $\text{tr}\theta$ to denote the trace-free part and trace of θ (with respect to $\not{g}$). If θ is trace free, $\hat{D}\theta$ and $\underline{\hat{D}}\theta$ refer to the trace-free part of $D\theta$ and $\underline{D}\theta$. Let ξ be a tangential 1-form. We define some products and operators for later use. For the products, we define $(\theta_1, \theta_2) = \not{g}^{AC}\not{g}^{BD}(\theta_1)_{AB}(\theta_2)_{CD}$ and $(\xi_1, \xi_2) = \not{g}^{AB}(\xi_1)_A(\xi_2)_B$. This also leads to the following norms $|\theta|^2 = (\theta, \theta)$ and $|\xi|^2 = (\xi, \xi)$. We then define the contractions $(\theta \cdot \xi)_A = \theta_A{}^B \xi_B$, $(\theta_1 \cdot \theta_2)_{AB} = (\theta_1)_A{}^C (\theta_2)_{CB}$, $\theta_1 \wedge \theta_2 = \not{\epsilon}^{AC}\not{\epsilon}^{BD}(\theta_1)_{AB}(\theta_2)_{CD}$ and $\xi_1 \hat{\otimes} \xi_2 = \xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1 - (\xi_1, \xi_2)\not{g}$. The Hodge dual for ξ is defined by ${}^*\xi = \not{\epsilon}_A{}^C \xi_C$. For the operators, we define $\text{div}\xi = \nabla^A \xi_A$, $\text{curl}\xi = \not{\epsilon}^{AB}\nabla_A \xi_B$ and $(\text{div}\theta)_A = \nabla^B \theta_{AB}$. We finally define a traceless operator $(\nabla \hat{\otimes} \xi)_{AB} = (\nabla \xi)_{AB} + (\nabla \xi)_{BA} - \text{div}\xi \not{g}_{AB}$.

We use a null frame (e_1, e_2, e_3, e_4) to decompose the Einstein vacuum equations $\text{Ric}(g) = 0$ into components. This leads to the following null structure equations (where K is the Gauss curvature of $S_{\underline{u}, u}$):

$$\hat{D}\chi' = -\alpha, \quad D\text{tr}\chi' = -\frac{1}{2}\Omega^2(\text{tr}\chi')^2 - \Omega^2|\chi'|^2, \quad (2.2)$$

$$\underline{\hat{D}}\underline{\chi}' = -\underline{\alpha}, \quad \underline{D}\text{tr}\underline{\chi}' = -\frac{1}{2}\Omega^2(\text{tr}\underline{\chi}')^2 - \Omega^2|\underline{\chi}'|^2, \quad (2.3)$$

$$D\eta = \Omega(\chi \cdot \underline{\eta} - \beta), \quad \underline{D}\underline{\eta} = \Omega(\underline{\chi} \cdot \eta + \underline{\beta}), \quad (2.4)$$

$$D\underline{\omega} = \Omega^2(2(\eta, \underline{\eta}) - |\eta|^2 - \rho), \quad (2.5)$$

$$\underline{D}\omega = \Omega^2(2(\eta, \underline{\eta}) - |\underline{\eta}|^2 - \rho), \quad (2.6)$$

$$K = -\frac{1}{4}\text{tr}\chi\text{tr}\underline{\chi} + \frac{1}{2}(\widehat{\chi}, \widehat{\underline{\chi}}) - \rho, \quad (2.7)$$

$$\text{div}\widehat{\chi}' = \frac{1}{2}\not{d}\text{tr}\chi' - \widehat{\chi}' \cdot \eta + \frac{1}{2}\text{tr}\chi'\eta - \Omega^{-1}\beta, \quad (2.8)$$

$$\text{div}\widehat{\underline{\chi}}' = \frac{1}{2}\not{d}\text{tr}\underline{\chi}' - \widehat{\underline{\chi}}' \cdot \underline{\eta} + \frac{1}{2}\text{tr}\underline{\chi}'\underline{\eta} - \Omega^{-1}\underline{\beta}, \quad (2.9)$$

$$\text{curl}\eta = \sigma - \frac{1}{2}\widehat{\chi} \wedge \widehat{\underline{\chi}}, \quad \text{curl}\underline{\eta} = -\sigma + \frac{1}{2}\widehat{\chi} \wedge \widehat{\underline{\chi}}, \quad (2.10)$$

$$\widehat{D}(\Omega\widehat{\underline{\chi}}) = \Omega^2 \left(\nabla \widehat{\otimes} \underline{\eta} + \underline{\eta} \widehat{\otimes} \underline{\eta} + \frac{1}{2}\text{tr}\chi\widehat{\underline{\chi}} - \frac{1}{2}\text{tr}\chi\widehat{\chi} \right), \quad (2.11)$$

$$D(\Omega\text{tr}\underline{\chi}) = \Omega^2 \left(2\text{div}\underline{\eta} + 2|\underline{\eta}|^2 - (\widehat{\chi}, \widehat{\underline{\chi}}) - \frac{1}{2}\text{tr}\chi\text{tr}\underline{\chi} + 2\rho \right), \quad (2.12)$$

$$\widehat{\underline{D}}(\Omega\widehat{\chi}) = \Omega^2 \left(\nabla \widehat{\otimes} \eta + \eta \widehat{\otimes} \eta + \frac{1}{2}\text{tr}\chi\widehat{\chi} - \frac{1}{2}\text{tr}\chi\widehat{\underline{\chi}} \right), \quad (2.13)$$

$$\underline{D}(\Omega\text{tr}\chi) = \Omega^2 \left(2\text{div}\eta + 2|\eta|^2 - (\widehat{\chi}, \widehat{\underline{\chi}}) - \frac{1}{2}\text{tr}\chi\text{tr}\underline{\chi} + 2\rho \right). \quad (2.14)$$

We also use the null frame to decompose second Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ into components. This leads the following null Bianchi equations,

$$\widehat{\underline{D}}\alpha - \frac{1}{2}\Omega\text{tr}\underline{\chi}\alpha + 2\underline{\omega}\alpha + \Omega\{-\nabla \widehat{\otimes} \beta - (4\underline{\eta} + \zeta)\widehat{\otimes} \beta + 3\underline{\chi}\rho + 3^*\widehat{\chi}\sigma\} = 0, \quad (2.15)$$

$$\widehat{D}\underline{\alpha} - \frac{1}{2}\Omega\text{tr}\chi\underline{\alpha} + 2\omega\underline{\alpha} + \Omega\{\nabla \widehat{\otimes} \underline{\beta} + (4\underline{\eta} - \zeta)\widehat{\otimes} \underline{\beta} + 3\underline{\chi}\rho - 3^*\widehat{\chi}\sigma\} = 0, \quad (2.16)$$

$$D\beta + \frac{3}{2}\Omega\text{tr}\chi\beta - \Omega\widehat{\chi} \cdot \beta - \omega\beta - \Omega\{\text{div}\alpha + (\underline{\eta} + 2\zeta) \cdot \alpha\} = 0, \quad (2.17)$$

$$\underline{D}\underline{\beta} + \frac{3}{2}\Omega\text{tr}\underline{\chi}\underline{\beta} - \Omega\widehat{\underline{\chi}} \cdot \underline{\beta} - \underline{\omega}\underline{\beta} + \Omega\{\text{div}\underline{\alpha} + (\eta - 2\zeta) \cdot \underline{\alpha}\} = 0, \quad (2.18)$$

$$\underline{D}\beta + \frac{1}{2}\Omega\text{tr}\underline{\chi}\beta - \Omega\widehat{\underline{\chi}} \cdot \beta + \underline{\omega}\beta - \Omega\{\not{d}\rho + ^*\not{d}\sigma + 3\underline{\eta}\rho + 3^*\underline{\eta}\sigma + 2\widehat{\chi} \cdot \underline{\beta}\} = 0, \quad (2.19)$$

$$D\underline{\beta} + \frac{1}{2}\Omega\text{tr}\chi\underline{\beta} - \Omega\widehat{\chi} \cdot \underline{\beta} + \omega\underline{\beta} + \Omega\{\not{d}\rho - ^*\not{d}\sigma + 3\underline{\eta}\rho - 3^*\underline{\eta}\sigma - 2\widehat{\underline{\chi}} \cdot \beta\} = 0, \quad (2.20)$$

$$D\rho + \frac{3}{2}\Omega\mathrm{tr}\chi\rho - \Omega\left\{\mathrm{div}\beta + (2\underline{\eta} + \zeta, \beta) - \frac{1}{2}(\widehat{\chi}, \alpha)\right\} = 0, \quad (2.21)$$

$$\underline{D}\rho + \frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\rho + \Omega\left\{\mathrm{div}\underline{\beta} + (2\underline{\eta} - \zeta, \underline{\beta}) + \frac{1}{2}(\widehat{\chi}, \underline{\alpha})\right\} = 0, \quad (2.22)$$

$$D\sigma + \frac{3}{2}\Omega\mathrm{tr}\chi\sigma + \Omega\left\{\mathrm{curl}\beta + (2\underline{\eta} + \zeta, {}^*\beta) - \frac{1}{2}\widehat{\chi} \wedge \alpha\right\} = 0, \quad (2.23)$$

$$\underline{D}\sigma + \frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\sigma + \Omega\left\{\mathrm{curl}\underline{\beta} + (2\underline{\eta} - \zeta, {}^*\underline{\beta}) + \frac{1}{2}\widehat{\chi} \wedge \underline{\alpha}\right\} = 0. \quad (2.24)$$

2.2. Earlier Results

The initial data consist of the conformal geometry on C_{u_0} . Let $\Phi_{\underline{u}}$ be the one parameter group generated by L . We can rewrite the induced metric $\mathcal{g}|_{S_{\underline{u}, u_0}}$ uniquely as $\mathcal{g}|_{S_{\underline{u}, u_0}} = (\phi|_{S_{\underline{u}, u_0}})^2 \widehat{\mathcal{g}}|_{S_{\underline{u}, u_0}}$, where $\phi|_{S_{\underline{u}, u_0}}$ is a positive function, such that the metric $\Phi_{\underline{u}}^* \widehat{\mathcal{g}}|_{S_{\underline{u}, u_0}}$ on S_{0, u_0} has the same volume form as $\mathcal{g}|_{S_{0, u_0}}$. In this language, we only need to specify $\Phi_{\underline{u}}^* \widehat{\mathcal{g}}|_{S_{\underline{u}, u_0}}$ freely on S_{0, u_0} since it yields the conformal geometry on C_{u_0} .

Let $\{(U_1, (\theta_1^A)), (U_2, (\theta_2^A))\}_{A=1,2}$ be the two stereographic charts on S_{0, u_0} . Thus, the round metric $\mathcal{g}|_{S_{0, u_0}}$ is expressed as

$$(\mathcal{g}|_{S_{0, u_0}})_{AB}(\theta) = \frac{|u_0|^2}{(1 + \frac{1}{4}|\theta|^2)^2} \delta_{AB}$$

with $\theta = \theta_1$ or θ_2 and $|\theta|^2 = |\theta^1|^2 + |\theta^2|^2$. Since we require that $\widehat{\mathcal{g}}(\underline{u}) = \Phi_{\underline{u}}^* \widehat{\mathcal{g}}|_{S_{\underline{u}, u_0}}$ on S_{0, u_0} has the same volume form as $\mathcal{g}|_{S_{0, u_0}}$, that is,

$$\det(\widehat{\mathcal{g}}(\underline{u})_{AB})(\theta) = \det(\mathcal{g}(0)_{AB})(\theta) = \frac{|u_0|^4}{(1 + \frac{1}{4}|\theta|^2)^4},$$

$\widehat{\mathcal{g}}(\underline{u})$ is given by

$$\widehat{\mathcal{g}}(\underline{u})_{AB}(\theta) = \frac{|u_0|^2}{(1 + \frac{1}{4}|\theta|^2)^2} m_{AB}(\underline{u}, \theta) = \frac{|u_0|^2}{(1 + \frac{1}{4}|\theta|^2)^2} \exp \psi_{AB}(\underline{u}, \theta), \quad (2.25)$$

where m_{AB} takes value in the set of positive definite symmetric matrices with determinant one and ψ_{AB} takes value in the set of symmetric trace-free matrices. After this reduction, to prescribe initial data, we only need to specify a function

$$\psi : [0, \delta + 1] \times S_{0, u_0} \longrightarrow \widehat{S}_2, \quad (\underline{u}, \theta) \mapsto \psi_{AB}(\underline{u}, \theta),$$

where $\widehat{S}_2$ denotes the set of 2×2 symmetric trace-free matrices, and δ is a small parameter. Here ψ is a function which represents in two charts, respectively, and satisfies compatible condition.

In Christodoulou's work [1], he only provided data on $C_{u_0}^{[0,\delta]}$, i.e. $\underline{u} \in [0, \delta]$. First he chose a smooth compactly supported $\widehat{S}_2$ -valued function $\psi_0 \in C_c^\infty((0, 1) \times S_{0,u_0})$. Then he called the following specific data

$$\psi(\underline{u}, \theta) = \frac{\delta^{\frac{1}{2}}}{|u_0|} \psi_0\left(\frac{\underline{u}}{\delta}, \theta\right), \quad (2.26)$$

the *short pulse ansatz* and he called ψ_0 the *seed data*.

In [4], we require the following

$$\int_0^1 \left| \frac{\partial \psi_0}{\partial s}(s, \theta) \right|^2 ds = 16m_0 \quad (2.27)$$

and then extend ψ by zero for $\underline{u} \geq \delta$. The condition above means that the total incoming gravitational energy is the same for all directions, as mentioned in the Introduction. Combining the above two parts, we obtain a smooth characteristic initial data on $C_{u_0}^{[0,\delta+1]}$ which is compactly supported in $C_{u_0}^{(0,\delta)}$.

Christodoulou proved the existence of the solution of vacuum Einstein equations in the region $0 \leq \underline{u} \leq \delta$, $u_0 \leq u \leq -1 - \underline{u}$ for δ sufficiently small. We summarize part of the results concerning the geometry on the final incoming null cone $\underline{C}_\delta$ under the condition (2.27) which can be found in Theorem 16.2 in [1] and Lemma 3 in Section 4 of [4].

Proposition 1. *Fix some large integer k and $N \geq 15$. Then if δ is sufficiently small depending on the C^{k+N} norm of the seed data ψ_0 , the quantities*

$$\begin{aligned} & \|(|u|\nabla)^i (|u|\widehat{\chi})\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i \left(|u|^3 \left(\rho + \frac{2m_0}{|u|^3} \right) \right)\|_{L^2(S_{\delta,u})}, \\ & \|(|u|\nabla)^i (|u|^3 \sigma)\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i (|u|^2 \beta)\|_{L^2(S_{\delta,u})}, \\ & \|(|u|\nabla)^i (|u|\alpha)\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i \left(|u|^2 \left(\text{tr}\chi - \left(\frac{2}{|u|} - \frac{4m_0}{|u|^2} \right) \right) \right)\|_{L^2(S_{\delta,u})}, \\ & \|(|u|\nabla)^i \left(|u|^2 \left(\omega - m_0 \left(\frac{1}{|u|^2} - \frac{1}{|u_0|^2} \right) \right) \right)\|_{L^2(S_{\delta,u})} \end{aligned}$$

for all $i \leq k$ are bounded by $\delta^{1/2}|u|$ times a constant depending on the C^{k+N} norm of the seed data. Also the following quantities

$$\begin{aligned} & \|(|u|\nabla)^i (|u|^2 \widehat{\underline{\chi}})\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i \left(|u|^2 \left(\text{tr}\underline{\chi} + \frac{2}{|u|} \right) \right)\|_{L^2(S_{\delta,u})}, \\ & \|(|u|\nabla)^i (|u|^{9/2} \underline{\alpha})\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i (|u|^4 \underline{\beta})\|_{L^2(S_{\delta,u})}, \\ & \|(|u|\nabla)^i (|u|^2 \underline{\eta})\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i (|u|^2 \underline{\eta})\|_{L^2(S_{\delta,u})}, \|(|u|\nabla)^i (|u|^3 \underline{\omega})\|_{L^2(S_{\delta,u})} \end{aligned}$$

for all $i \leq k$ are bounded by $\delta^{1/2}|u|$ times a constant depending on the C^{k+N} norm of the seed data. We also have similar bounds in $L^4(S_{\delta,u})$ and L^∞ using the Sobolev inequalities.

The complete detail of the proof of the above proposition can be found in [4], in which the statement given is not exactly the same as above for technical reason. For the sake of completeness, we draw an outline of the proof of the

above proposition starting from assuming the estimates in [1] hold. Remember that the sharp energy estimates are only obtained for two derivatives of the curvature components in [1]; therefore, the first step of proving the above proposition is to do sharp higher order energy estimates. We shall prove that the short pulse hierarchy is still valid in the higher order cases. More precisely, we introduce the following norms:

$$\begin{aligned}\mathcal{R}_i(u) &= \delta^{-\frac{1}{2}}|u|^{-1}(\|\delta^{\frac{3}{2}}|u|(|u|\nabla)^i\alpha\| + \|\delta^{\frac{1}{2}}|u|^2(|u|\nabla)^i\beta\| \\ &\quad + \| |u|^3(|u|\nabla)^i(\rho, \sigma)\| + \|\delta^{-1}|u|^4(|u|\nabla)^i\underline{\beta}\|), \\ \Gamma_{i+1}(u) &= \delta^{-\frac{1}{2}}|u|^{-1}\|\delta^{\frac{1}{2}}|u|(|u|\nabla)^{i+1}\widehat{\chi}\| + \left\| |u|^2(|u|\nabla)^{i+1}\left(\mathrm{tr}\chi - \frac{2}{|u|}\right) \right\| \\ &\quad + \|\delta^{-\frac{1}{2}}|u|^2(|u|\nabla)^{i+1}\widehat{\underline{\chi}}\| + \left\| \delta^{-1}|u|^3(|u|\nabla)^{i+1}\left(\mathrm{tr}\underline{\chi} + \frac{2}{|u|} - \frac{2u}{|u|^2}\right) \right\| \\ &\quad + \| |u|^2(|u|\nabla)^{i+1}(\eta, \underline{\eta})\| + \|\delta^{-1}|u|^3(|u|\nabla)^{i+1}\underline{\omega}\| \\ &\quad + \|\delta^{\frac{1}{2}}|u|(|u|\nabla)^{i+1}\omega\|,\end{aligned}$$

where all the norms $\|\cdot\|$ are taken with respect to $\|\cdot\|_{L^2(C_u^{[0,\delta]})}$; on the incoming cone $\underline{C}_u$, we also introduce

$$\underline{\mathcal{R}}_i[\underline{\alpha}](u) = \| |u|^{-\frac{3}{2}}\delta^{-\frac{3}{2}}|u|^{\frac{9}{2}}(|u|\nabla)^i\underline{\alpha}\|_{L^2(\underline{C}_u)}.$$

Christodoulou proved that $\mathcal{R}_i(u)$, $\Gamma_{i+1}(u)$ and $\underline{\mathcal{R}}_i[\underline{\alpha}](u)$ are bounded by a constant depending on C^8 norms of the seed data for $i \leq 2$. Then we should first prove that the quantities are bounded by a constant depending on C^{i+N} norms of the seed data for large i if δ is chosen sufficiently small. The proof is based on standard energy method. One should compare the above norms $\mathcal{R}_i(u)$, $\Gamma_{i+1}(u)$ and $\underline{\mathcal{R}}_i[\underline{\alpha}](u)$ to the norms listed in Proposition 1, and then find that the second part of Proposition 1 follows immediately by Sobolev type inequalities.

It is also easy to see that the first part of Proposition 1 actually improves the so-called *smallness* of various geometric quantities (i.e., the connection coefficients and curvature components) compared to those derived by simply higher order energy estimates, i.e., $\mathcal{R}_i(u)$, $\Gamma_{i+1}(u)$ and $\underline{\mathcal{R}}_i[\underline{\alpha}](u)$. That is to say, some geometric quantities which are originally bounded by δ to some nonpositive power, can be bounded by δ to some positive power under the additional condition (2.27). This can be done as follows. We first show that the first part of the proposition holds on S_{δ, u_0} , based on the vanishing of $\widehat{\chi}$ on S_{δ, u_0} and the Eqs. (2.2), (2.7), (2.8), (2.10) (see Lemma 1 in the context). Then we integrate the Eqs. (2.6), (2.13)–(2.15), (2.19), (2.22), (2.24) in sequence along the incoming null cone $\underline{C}_\delta$. An important observation is when we integrate the equations, although we only use the original bounds of various geometric quantities without improving the *smallness*, it turns out that the right-hand side of the equations will not prevent the improvement of the *smallness* of the left-hand side. This fact is not essentially related to the additional condition (2.27), and should be considered as a special structure of the vacuum Einstein

equations. Therefore, there are no obstructions for the *smallness* on the sphere S_{δ, u_0} propagating deep into the spacetime.

We remark that the estimates stated in the above proposition are not optimal if we restrict our attention to the null cone $\underline{C}_\delta$. We only state the estimates which are maintained in the region we are going to construct.

3. Main Result

This section is devoted to the statements of the main results which we are going to prove. We define another spacetime metric g_{m_0} on the future development of $C_{u_0}^{[\delta, \delta+1]} \cup \underline{C}_\delta^{[u_0, -1-\delta]}$. The explicit formula of g_{m_0} is given in the Appendix but we give a description here using the language of characteristic initial data problem. To distinguish various geometric quantities relative to the new metric from the old one, we shall use a lower index m_0 for its connection coefficients and curvature components. The metric g_{m_0} is determined by its initial value on $C_{u_0}^{[\delta, \delta+1]} \cup \underline{C}_\delta^{[u_0, -1-\delta]}$, which consists of the following quantities: the induced metric $\mathcal{g}_{m_0}(\delta, u_0)$ on S_{δ, u_0} which is the round metric of the sphere with radius $|u_0|$, the torsion $\zeta_{m_0}(\delta, u_0) \equiv 0$ on S_{δ, u_0} , two null expansions $\text{tr}\chi_{m_0}(\delta, u_0) \equiv \frac{2}{|u_0|} - \frac{4m_0}{|u_0|^2}$ and $\text{tr}\underline{\chi}_{m_0}(\delta, u_0) \equiv -\frac{2}{|u_0|}$ on S_{δ, u_0} , the lapse function $\Omega_{m_0} \equiv 1$ on $C_{u_0} \cup \underline{C}_\delta$ and two shears $\hat{\chi}_{m_0} \equiv 0$ on C_{u_0} and $\hat{\underline{\chi}}_{m_0} \equiv 0$ on $\underline{C}_\delta$. By the spherical symmetry of these data, it is easy to see g_{m_0} is locally isometric to the Schwarzschild metric with mass m_0 .

From now on we use the notations $\tilde{\Gamma}$ and $\tilde{R}$ to denote the differences $\Gamma - \Gamma_{m_0}$ and $R - R_{m_0}$ for the connection coefficients and curvature components that are non-vanishing in g_{m_0} . We will also use this notation to denote the differences between various quantities relative to g and g_{m_0} . We will prove the following main theorem in this paper.

Theorem 1. *Suppose that the Schwarzschild metric g_{m_0} exists in the region $\delta \leq \underline{u} \leq \delta + 1$, $u_0 \leq u \leq -1 - \underline{u}$. Let $\varepsilon > 0$ and $k > 0$ a large integer, then if δ is sufficiently small depending on the C^{k+N} norm of the seed data, the solution g also exists in the same region, and satisfies*

$$\begin{aligned} \mathcal{E}_i^u(u) &\triangleq |u|^{-1} \|(|u|\nabla)^i (|u|\alpha, |u|^2\beta, |u|^3\tilde{\rho}, |u|^3\sigma, |u|^4\underline{\beta})\|_{L^2(C_{\underline{u}}^u)} \leq \varepsilon, \\ \mathcal{F}_i^u(u) &\triangleq \| |u'|^{-1} (|u'|\nabla)^i (|u'|\beta, |u'|^2\tilde{\rho}, |u'|^2\tilde{\sigma}, |u'|^3\underline{\beta}, |u'|^4\underline{\alpha}) \|_{L^2(\underline{C}_{\underline{u}}^u)} \leq \varepsilon \end{aligned}$$

for all $i \leq k$. Also

$$\begin{aligned} \mathcal{O}_i^2(\underline{u}, u) &\triangleq |u|^{-1} \|(|u|\nabla)^i (|u|\hat{\chi}, |u|^2\widetilde{\text{tr}\chi}, |u|^2\widetilde{\text{tr}\underline{\chi}}, \\ &\quad |u|^2\eta, |u|^2\underline{\eta}, |u|^{3/2}\tilde{\omega}, |u|^3\widetilde{\underline{\omega}})\|_{L^2(S_{\underline{u}, u})} \leq \varepsilon \end{aligned}$$

for all $i \leq k$.

The proof of the above theorem is the main part of this paper. The crucial point is, by Proposition 1, various geometric quantities on the incoming null cone $\underline{C}_\delta$ are close to those on an incoming null cone in the Schwarzschild spacetime g_{m_0} which we defined above. We then prove in the following Proposition 2

that the geometric quantities on $C_{u_0}^{[\delta, \delta+1]}$ are close to those on an outgoing null cone in g_{m_0} . We should remark that the closeness is measured with a weight $|u|$ which allows us to push the initial data to past null infinity. For each u_0 , the existence of the solution g in the region $\delta \leq \underline{u} \leq \delta + 1$, $u_0 \leq u \leq u_1$ follows by standard perturbation argument, by which the bounds obtained depend on u_0 (one can also refer to [4]). Therefore we need to modify the bootstrap argument in proving existence a little bit to derive estimates with precise u_0 dependence.

We then formulate here the initial data problem from the past null infinity as in [1]. The solution of the vacuum Einstein equations will be constructed up to $u \rightarrow -\infty$ and $0 \leq \underline{u} \leq \delta + 1$. The past of $\underline{C}_0$ is required to be the Minkowski spacetime. In particular, $\underline{C}_0$ coincides with an incoming null cone in the Minkowski spacetime from the past null infinity. Then, a smooth function

$$\psi_\infty : [0, \delta + 1] \times S^2 \rightarrow \widehat{S}_2, \quad (\underline{u}, \theta) \mapsto \psi_\infty(\underline{u}, \theta)$$

is said to be the initial data of such problem if

$$|u|\psi(\underline{u}, u, \theta) = \psi_\infty(\underline{u}, \theta) + O\left(\frac{1}{|u|}\right)$$

as $u \rightarrow -\infty$ where ψ is defined as in (2.25). Equivalently, the initial data can also be represented by a 2-covariant tensor value function $\widehat{\chi}_\infty$ defined on $[0, \delta + 1] \times S^2$ which is trace free relative to the standard metric on S^2 if the shear $\widehat{\chi}$ satisfies

$$|u|^{-1}\widehat{\chi}(\underline{u}, u, \theta) = \widehat{\chi}_\infty(\underline{u}, \theta) + O\left(\frac{1}{|u|}\right), \quad \text{as } u \rightarrow -\infty.$$

These two kinds of formulation are connected by the relation

$$\widehat{\chi}_\infty(\underline{u}, \theta) = \frac{1}{2(1 + \frac{1}{4}|\theta|^2)^2} \frac{\partial \psi}{\partial \underline{u}}(\underline{u}, \theta).$$

In our case, $\psi_\infty(\underline{u}, \theta) = \delta^{1/2}\psi_0(\frac{u}{\delta}, \theta)$ for $0 \leq \underline{u} \leq \delta$, and $\psi_\infty(\underline{u}, \theta) \equiv 0$ for $\delta \leq \underline{u} \leq \delta + 1$, where ψ_0 is the seed data. Therefore, $\widehat{\chi}_\infty(\underline{u}, \theta) = \frac{1}{2\delta^{1/2}(1 + \frac{1}{4}|\theta|^2)^2} \frac{\partial \psi_0}{\partial s}(\frac{u}{\delta}, \theta)$. An analog to the condition (2.27) is

$$16m_0 = \int_0^1 \left| \frac{\partial \psi_0}{\partial s}(s, \theta) \right|^2 ds = 4 \int_0^\delta \left| \widehat{\chi}_\infty \Big|_{\mathring{g}}^2 d\underline{u}. \quad (3.1)$$

The function $|u|^2 e(\underline{u}, u, \theta) = \frac{1}{2}|u|^2 |\widehat{\chi}|_{\mathring{g}}^2 \rightarrow e_\infty(\underline{u}, \theta) \triangleq \frac{1}{2} |\widehat{\chi}_\infty|_{\mathring{g}}^2$. Physically, e_∞ is the incoming radiative energy per unit solid angle up to a constant multiple. Then the above condition means the total energy in the advanced time interval $[0, \delta]$ is the same in each direction. We remark that in finite case, assuming the condition (2.27), the corresponding more physical condition $\int_0^\delta |u|^2 |\widehat{\chi}|^2 d\underline{u} = 4m_0$ is valid only up to an error of size $\delta^{1/2}$ (see [4]). For the above initial data, we are going to prove the following.

Theorem 2. *Suppose that g_{m_0} is the Schwarzschild metric defined in the region $\delta \leq \underline{u} \leq \delta + 1$, $-\infty < u \leq -1 - \underline{u}$. For the initial data problem from the past*

null infinity described above, and a given small $\varepsilon > 0$, large integer $k > 0$, then if δ is sufficiently small depending on the C^{k+N} norm of the seed data, there exists a smooth solution of vacuum Einstein equations g in the region $0 \leq \underline{u} \leq \delta + 1$, $-\infty < u < -1 - \underline{u}$, and is ε -close to g_{m_0} in C^{k-4} norm (in appropriate sense) in the region $\delta \leq \underline{u} \leq \delta + 1$, $-\infty < u < -1 - \underline{u}$.

This theorem is a direct consequence of Theorem 1. We will choose a sequence $u_{0,n} \rightarrow -\infty$ and construct a sequence of solution g_n . Since the estimates we will derive do not depend on u_0 , we can then perform a limiting argument using Arzelà–Ascoli theorem. Similar argument is also performed in [1].

4. Proof of the Main Result

4.1. Estimates on the Initial Null Cone

Firstly, we consider the geometry on the initial null cone $C_{u_0}^{[\delta, \delta+1]}$. We write $\mathcal{g}(\underline{u}) = \phi^2 \widehat{\mathcal{g}}(\underline{u})$. Then $\widehat{\mathcal{g}}(\underline{u})$ for $0 \leq \underline{u} \leq \delta$ is given by the short pulse ansatz and for $\delta \leq \underline{u} \leq \delta + 1$, $\widehat{\mathcal{g}}(\underline{u})$ is constant, which is equivalent to the zero extension of $\widehat{\chi}$ to $C_{u_0}^{[\delta, \delta+1]}$. It is not hard to conclude that, using the null structure equation for $D\text{tr}\chi'$, we have

$$\frac{\partial^2 \phi}{\partial \underline{u}^2} + e\phi = 0$$

where $e = \frac{1}{2}|\widehat{\chi}|_{\widehat{\mathcal{g}}}^2$. Note that $e = 0$ for $\delta \leq \underline{u} \leq \delta + 1$, so $\phi(\underline{u}) = \frac{\phi(\delta)\text{tr}\chi(\delta)}{2}(\underline{u} - \delta) + \phi(\delta)$ for $\delta \leq \underline{u} \leq \delta + 1$. On the other hand, let ϕ_{m_0} be defined as $\Phi_{\underline{u}-\delta}^*(\mathcal{g}_{m_0}|_{S_{\underline{u}, u_0}}) = \phi_{m_0}\mathcal{g}_{m_0}|_{S_{\delta, u_0}}$ where $\mathcal{g}_{m_0}|_{S_{\delta, u_0}}$ is the round metric of radius $|u_0|$ by definition. Then $\phi_{m_0}(\underline{u}) = (\frac{1}{|\underline{u}|} - \frac{2m_0}{|\underline{u}|^2})(\underline{u} - \delta) + 1$. By computing $\phi - \phi_{m_0}$ for $\delta \leq \underline{u} \leq \delta + 1$ and $|u_0|^{-2}\widehat{\mathcal{g}} - \mathcal{g}$ as in Chapter 2 in [1], we have $|\partial_\theta^k(\mathcal{g}_{AB} - (\mathcal{g}_{m_0})_{AB})| \lesssim_k \delta^{1/2}|u_0|$. Here ∂_θ means the ordinary partial derivatives in both stereographic charts, the norm is not taken with respect to $\mathcal{g}$ but simply the sum of square of all components and the notation $C \lesssim_k D$ means $C \leq \mathcal{A}_k D$ where $\mathcal{A}_k$ depends only on the C^{k+N} norm of the seed data. In this subsection, k is an arbitrary positive integer, but not the integer given in Theorem 1. The above facts illustrate the relation $\nabla \sim |u|^{-1}$ which is also valid in [1].

The geometry on $C_{u_0}^{[\delta, \delta+1]}$ is estimated based on Chapter 2 of [1] and Lemma 2 in [4] concerning the geometry of S_{δ, u_0} , which can be summarized in the lemma below.

Lemma 1. *On the sphere S_{δ, u_0} , we have $\alpha \equiv 0$, $\omega \equiv 0$ and for all k*

$$\begin{aligned} & |u_0|^k \left(|u_0|^2 |\nabla^k (\text{tr}\chi - (\frac{2}{|u_0|} - \frac{4m_0}{|u_0|^2}))| + |u_0|^2 |\nabla^k \beta| \right. \\ & \left. + |u_0|^3 |\nabla^k \rho + \frac{2m_0}{|u_0|^3})|, |u_0|^3 |\nabla^k \sigma| \right) \lesssim_k \delta^{\frac{1}{2}}. \end{aligned}$$

Also, we have

$$\begin{aligned} & |u_0|^k \left(|u_0|^2 |\nabla^k \underline{\chi}| + |u_0|^2 |\nabla^k (\text{tr} \underline{\chi} + \frac{2}{|u_0|})| + |u_0|^2 |\nabla^k (\eta, \underline{\eta})| \right. \\ & \quad \left. + |u_0|^3 |\nabla^k \underline{\omega}| + |u_0|^4 |\nabla^k \underline{\beta}| + |u_0|^5 |\nabla^k \underline{\alpha}| \right) \lesssim_k \delta^{\frac{1}{2}}. \end{aligned}$$

We remark that the second part is simply the result in Chapter 2 of [1] and the first part is an improvement based on the condition (2.27). Notice that the proof of Proposition 1 is actually based on the above lemma.

We then consider the geometry on $C_{u_0}^{[\delta, \delta+1]}$. We will prove the following, which states that the geometric quantities on the outgoing null cone $C_{u_0}^{[\delta, \delta+1]}$ are close to those on an outgoing null cone in g_{m_0} .

Proposition 2. $\mathcal{E}_k^{\delta+1}(u_0) + \mathcal{O}_k^2(\underline{u}, u_0) \lesssim_k \delta^{1/2}$ for all k .

Proof. We first compute $\widetilde{\text{tr}} \chi$ which can be directly solved as $\widetilde{\text{tr}} \chi(\underline{u}) = \frac{2}{\underline{u}+2/\text{tr} \chi(\delta)} - \frac{2}{\underline{u}+2/\text{tr} \chi_{m_0}(\delta)}$. Then

$$|(|u_0| \nabla)^k \widetilde{\text{tr}} \chi| \lesssim_k \delta^{1/2} |u_0|^{-2}.$$

Next we consider the component $\eta = -\underline{\eta}$ by the structure equation for $D\eta$, which can be written on C_{u_0} as $D\eta + \text{tr} \chi \eta = -\frac{1}{2} \text{d}\text{tr} \chi$. By commuting ∇^k with this equation and the Gronwall's inequality we have $|(|u_0| \nabla)^k \eta| \lesssim_k \delta^{1/2} |u_0|^{-2}$.

We have to discuss first the Gauss curvature K of $S_{\underline{u}, u_0}$. We use the estimate $|\partial_\theta^k (\not{g}_{AB} - (\not{g}_{m_0})_{AB})| \lesssim_k \delta^{1/2} |u_0|$ to obtain $|(|u_0| \nabla)^k \tilde{K}| \lesssim_k \delta^{1/2} |u_0|^{-3}$. Then we eliminate ρ in the equation for $D\text{tr} \chi$ and the Gauss equation to derive the propagation equation for $\widetilde{\text{tr}} \chi$: $D\widetilde{\text{tr}} \chi + \text{tr} \chi \widetilde{\text{tr}} \chi = -2\tilde{K} - \text{d}\not{\chi} \eta + |\eta|^2$ and the propagation equation for $\widehat{\chi}$: $D\widehat{\chi} - \frac{1}{2} \text{tr} \chi \widehat{\chi} = -\nabla \widehat{\otimes} \eta + \eta \widehat{\otimes} \eta$. We can obtain in the same manner as η that $|(|u_0| \nabla)^k \widetilde{\text{tr}} \chi| \lesssim_k \delta^{1/2} |u_0|^{-2}$ and $|(|u_0| \nabla)^k \widehat{\chi}| \lesssim_k \delta^{1/2} |u_0|^{-2}$.

We rewrite the Gauss equation as $\tilde{\rho} = -\frac{1}{4}(\text{tr} \chi \text{tr} \chi - \text{tr} \chi_{m_0} \text{tr} \chi_{m_0}) - \tilde{K}$ and the Ricci equation (the equation for $\text{curl} \eta$) to derive that $|(|u_0| \nabla)^k \tilde{\rho}| + |(|u_0| \nabla)^k \sigma| \lesssim_k \delta^{1/2} |u_0|^{-3}$. We use the equation for $\text{d}\not{\chi} \widehat{\chi}'$ to derive that $|(|u_0| \nabla)^k \beta| \lesssim_k \delta^{1/2} |u_0|^{-2}$ (Note here that we should have the bound $\delta^{1/2} |u_0|^{-3}$ but this bound is not consistent deep into the spacetime). Finally, we estimate $\underline{\omega}$, $\underline{\beta}$ and $\underline{\alpha}$ using the structure equations for $D\underline{\omega}$, $D\underline{\beta}$ and $\widehat{D}\underline{\alpha}$ to derive that $|(|u_0| \nabla)^k \underline{\omega}| \lesssim_k \delta^{1/2} |u_0|^{-3}$, $|(|u_0| \nabla)^k \underline{\beta}| \lesssim_k \delta^{1/2} |u_0|^{-4}$, $|(|u_0| \nabla)^k \underline{\alpha}| \lesssim_k \delta^{1/2} |u_0|^{-5}$.

We have derived the estimates for $\widetilde{\text{tr}} \chi$, η , $\tilde{K}$, $\widetilde{\text{tr}} \chi$, $\widehat{\chi}$, $\underline{\omega}$ and the curvature components $\tilde{\rho}$, σ , β , $\underline{\beta}$, $\underline{\alpha}$ above. Also remember we require $\widehat{\chi} \equiv 0$ and so that $\alpha \equiv 0$ on $C_{u_0}^{[\delta, \delta+1]}$, and $\Omega \equiv 1$ and so that $\omega \equiv 0$ on $C_{u_0}^{[\delta, \delta+1]}$. Therefore, we complete the proof of this proposition. $\square$

After controlling the geometry on $C_{u_0}^{[\delta, \delta+1]}$, we are in the position to derive estimates for the solution g in the region $\delta \leq \underline{u} \leq \delta + 1$, $u_0 \leq u \leq u_1$.

4.2. Proof of Theorem 1

We are going to prove Theorem 1 in this subsection. We first assume that in the region M_c which is enclosed by $\underline{C}_\delta$, C_{u_0} , $\underline{C}_{\delta+1}$ and the space-like hypersurface $\underline{u} + u = c$, the solution g exists and satisfies:

Bootstrap Assumption

$$|u|^{-1} \|(|u|\nabla^i)(|u|^3\tilde{\rho})\|_{L^2(C_{\underline{u}}^\#)} + \| |u'|^{-1} (|u'|\nabla)^i (|u'|^2\tilde{\rho}) \|_{L^2(\underline{C}_{\underline{u}}^u)} \leq \Delta_0 \quad (4.1)$$

for all $i \leq k$. For connection coefficients, we assume that for all $i \leq k$,

$$\begin{aligned} \mathcal{O}_i^2(\underline{u}, u) &= \| |u|^{-1} \| (|u|\nabla)^i (|u|\widehat{\chi}, |u|^2\widetilde{\text{tr}\chi}, |u|^2\widehat{\underline{\chi}}, |u|^2\widetilde{\text{tr}\underline{\chi}}, \\ &\quad |u|^2\eta, |u|^2\eta, |u|^{3/2}\widetilde{\omega}, |u|^3\widetilde{\omega}) \|_{L^2(S_{\underline{u}, u})} \leq \Delta_0. \end{aligned} \quad (4.2)$$

We also assume the following

$$\mathcal{O}^\infty(\underline{u}, u) = \| |u|\widehat{\chi}, |u|^2\widetilde{\text{tr}\chi}, |u|^2\widehat{\underline{\chi}}, |u|^2\widetilde{\text{tr}\underline{\chi}}, |u|^2\eta, |u|^2\eta, |u|^{3/2}\widetilde{\omega}, |u|^3\widetilde{\omega} \|_{L^\infty} \leq \Delta_0. \quad (4.3)$$

Under the assumption (4.3), we have the following Sobolev inequalities

$$\|\xi\|_{L^4(S)} \lesssim |u|^{\frac{1}{2}} \|\nabla \xi\|_{L^2(S)} + |u|^{-\frac{1}{2}} \|\xi\|_{L^2(S)},$$

$$\|\xi\|_{L^\infty(S)} \lesssim |u|^{\frac{1}{2}} \|\nabla \xi\|_{L^4(S)} + |u|^{-\frac{1}{2}} \|\xi\|_{L^4(S)},$$

here ξ is a tangential tensorfield, and the notation $A \lesssim B$ means $A \leq C(m_0, \Delta_0)B$.

We are going to prove the estimates in Theorem 1 hold in M_c first. It is sufficient to prove the estimates based on the above bootstrap assumption for sufficiently small ε . We only derive estimates for up to the second order derivatives of the curvature components for simplicity and to emphasize the fact that two derivatives of curvature are enough to guarantee the existence. Higher order cases can be considered in a similar way. Firstly, we derive estimates for the connection coefficients in terms of the curvature components.

4.2.1. Zero Order Estimates. Consider $\widehat{\chi}$ first. Rewrite the structure equation for $\widehat{D}\widehat{\chi}'$ as

$$D|\widehat{\chi}'|^2 + 2\Omega\text{tr}\chi|\widehat{\chi}'|^2 \leq 2|\widehat{\chi}'||\alpha|.$$

Then by the Sobolev inequalities, and the Gronwall's inequality (which is valid under the assumption (4.3), see Lemma 4.4 and 4.6 in [1]), we have

$$\begin{aligned} \| |u|\widehat{\chi}' \|_{L^\infty(S_{\underline{u}, u})} &\lesssim \| |u|\widehat{\chi}' \|_{L^\infty(S_{\delta, u})} + \int_\delta^u \| |u|\alpha \|_{L^\infty(S_{\underline{u}', u})} d\underline{u}' \\ &\lesssim |u|^{-1} \sum_{i=0}^2 (\| (|u|\nabla)^i (|u|\widehat{\chi}') \|_{L^2(S_{\delta, u})} + \| (|u|\nabla)^i (|u|\alpha) \|_{L^2(C_{\underline{u}}^\#)}) \\ &\lesssim \mathcal{O}_{\leq 2}^2(\delta, u) + \mathcal{E}_{\leq 2}^u(u). \end{aligned} \quad (4.4)$$

We will omit the lower index ≤ 2 in this section.

For $\widehat{\chi}'$, the equation for $\widehat{\underline{D}}\widehat{\underline{\chi}}$ can be written as

$$D|\widehat{\underline{\chi}}'|^2 + 2\Omega\text{tr}\widehat{\underline{\chi}}|\widehat{\underline{\chi}}'|^2 \leq 2|\widehat{\underline{\chi}}'|\underline{\alpha}|.$$

We use the Gronwall's inequality again but should care about the integration of $\Omega\text{tr}\widehat{\underline{\chi}}$ along a generator of $\underline{C}_\delta$. We write

$$\begin{aligned} & \left| \int_{u'}^u \Omega\text{tr}\widehat{\underline{\chi}} du'' + \int_{u'}^u \frac{2}{|u''|} du'' \right| \\ &= \left| \int_{u'}^u \widetilde{\Omega\text{tr}\widehat{\underline{\chi}}} du'' + \int_{u'}^u \left((\Omega\text{tr}\widehat{\underline{\chi}})_{m_0} + \frac{2}{|u''|} \right) du'' \right| \lesssim |u|^{-2}. \end{aligned}$$

By Lemma 4 in the Appendix about $(\Omega\text{tr}\widehat{\underline{\chi}})_{m_0}$, we conclude that

$$\exp\left(-\int_{u_0}^u \Omega\text{tr}\widehat{\underline{\chi}} du''\right) \lesssim \frac{|u_0|^2}{|u|^2}, \exp\left(\int_u^{u'} \Omega\text{tr}\widehat{\underline{\chi}} du''\right) \lesssim \frac{|u'|^2}{|u|^2}. \quad (4.5)$$

Then the Gronwall's inequality and Sobolev inequality give

$$\begin{aligned} \| |u|^2 \widehat{\underline{\chi}}' \|_{L^\infty(S_{\underline{u}, u})} &\lesssim \| |u_0|^2 \widehat{\underline{\chi}}' \|_{L^\infty(S_{\underline{u}, u_0})} + \int_{u_0}^u \| |u'|^2 \underline{\alpha} \|_{L^\infty(S_{\underline{u}, u'})} du' \\ &\lesssim \mathcal{O}(\underline{u}, u_0) + \mathcal{F}^u(\underline{u}). \end{aligned}$$

The power 2 of the weight $|u|^2$, $|u_0|^2$ and $|u'|^2$ appears as a consequence of the coefficient of the term $\Omega\text{tr}\widehat{\underline{\chi}}$. For a tensor equation $\widehat{\underline{D}}\phi$, the term $\lambda\Omega\text{tr}\widehat{\underline{\chi}}|\phi|^2$ appears naturally when deriving the equation for $\widehat{\underline{D}}|\phi|^2$. The power also depends on the norm (L^2 , L^4 or L^∞) we take. One can refer to Lemma 4.5 and 4.7 in [1].

We go to η and $\underline{\eta}$ next. We consider the equations for $D\eta$ and $\widehat{\underline{D}}\underline{\eta}$ as a coupled system and we derive

$$\| \eta \|_{L^\infty(S_{\underline{u}, u})} \lesssim \| \eta \|_{L^\infty(S_{\delta, u})} + \frac{1}{|u|} \int_\delta^u \| \underline{\eta} \|_{L^\infty(S_{\underline{u}', u})} d\underline{u}' + \int_\delta^u \| \beta \|_{L^\infty(S_{\underline{u}', u})} d\underline{u}', \quad (4.6)$$

$$\begin{aligned} \| |u| \underline{\eta} \|_{L^\infty(S_{\underline{u}, u})} &\lesssim \| |u_0| \underline{\eta} \|_{L^\infty(S_{\underline{u}, u_0})} \\ &+ \int_{u_0}^u \| \eta \|_{L^\infty(S_{\underline{u}, u'})} du' + \int_{u_0}^u \| |u'| \underline{\beta} \|_{L^\infty(S_{\underline{u}, u'})} du'. \end{aligned} \quad (4.7)$$

We multiply $|u|^2$ with the first inequality and substitute the second inequality to the first, then take $\sup_{\delta \leq \underline{u}' \leq \underline{u}}$ on both sides, we have

$$\begin{aligned} \sup_{\delta \leq \underline{u}' \leq \underline{u}} \| |u|^2 \eta \|_{L^\infty(S_{\underline{u}, u})} &\lesssim \| |u|^2 \eta \|_{L^\infty(S_{\delta, u})} + \sup_{\delta \leq \underline{u}' \leq \underline{u}} \| |u_0| \underline{\eta} \|_{L^\infty(S_{\underline{u}', u_0})} \\ &+ \int_{u_0}^u \sup_{\delta \leq \underline{u}' \leq \underline{u}} \| \eta \|_{L^\infty(S_{\underline{u}', u'})} du' \\ &+ \int_{u_0}^u \sup_{\delta \leq \underline{u}' \leq \underline{u}} \| |u'| \underline{\beta} \|_{L^\infty(S_{\underline{u}', u'})} du' \\ &+ \int_\delta^u \| |u|^2 \beta \|_{L^\infty(S_{\underline{u}', u})} d\underline{u}'. \end{aligned}$$

Denote the right-hand side again by f except the third term. Then we have

$$\sup_{\delta \leq \underline{u}' \leq \underline{u}} \|\eta\|_{L^\infty(S_{\underline{u}', u})} \lesssim |u|^{-2} \int_{u_0}^u \sup_{\delta \leq \underline{u}' \leq \underline{u}} \|\eta\|_{L^\infty(S_{\underline{u}', u'})} d\underline{u}' + |u|^{-2} f$$

where

$$\|f\|_{L^\infty(S_{\underline{u}, u})} \lesssim \mathcal{O}(\delta, u) + \sup_{\delta \leq \underline{u}' \leq \underline{u}} \mathcal{O}(\underline{u}', u_0) + \sup_{\delta \leq \underline{u}' \leq \underline{u}} \mathcal{F}^u(\underline{u}') + \mathcal{E}^u(u).$$

Thanks to the weight $|u|^{-2}$ we can apply the Gronwall's inequality and we can compute

$$\begin{aligned} & \sup_{\delta \leq \underline{u}' \leq \underline{u}} \| |u|^2 \eta \|_{L^\infty(S_{\underline{u}', u})} \\ & \lesssim \mathcal{P}(\underline{u}, u) \triangleq \sup_{\delta \leq \underline{u}' \leq \underline{u}} (\mathcal{O}(\underline{u}', u_0) + \mathcal{F}^u(\underline{u}')) + \sup_{u_0 \leq u' \leq u} (\mathcal{O}(\delta, u') + \mathcal{E}^u(u')). \end{aligned} \quad (4.8)$$

Substituting the above back to (4.7) immediately gives

$$\| |u|^2 \eta \|_{L^\infty(\underline{u}', u)} \lesssim \mathcal{P}(\underline{u}, u). \quad (4.9)$$

We consider $\widetilde{\omega}$ using the renormalized form of the equation for $D\omega$:

$$D\widetilde{\omega} = \Omega^2(2(\eta, \underline{\eta}) - |\eta|^2) - \widetilde{\Omega}^2 \rho + \Omega_{m_0}^2 \widetilde{\rho}.$$

By the equation $D\widetilde{\log \Omega} = \widetilde{\omega}$, we estimate

$$\begin{aligned} \|\widetilde{\Omega}\|_{L^\infty(S_{\underline{u}, u})} & \lesssim \|\widetilde{\log \Omega}\|_{L^\infty(S_{\underline{u}, u})} \lesssim \int_{u_0}^u \|\widetilde{\omega}\|_{L^\infty(S_{\underline{u}, u'})} d\underline{u}' \\ & \lesssim |u|^{-2} \sup_{u_0 \leq u' \leq u} \| |u'|^3 \widetilde{\omega} \|_{L^\infty(S_{\underline{u}, u'})}. \end{aligned}$$

Then the equation for $D\widetilde{\omega}$ gives

$$\begin{aligned} \| |u|^3 \widetilde{\omega} \|_{L^\infty(S_{\underline{u}, u})} & \lesssim \| |u|^3 \widetilde{\omega} \|_{L^\infty(S_{\delta, u})} + |u|^{-1} \mathcal{P}(\underline{u}, u) \\ & \quad + \int_{\delta}^{\underline{u}} \|\widetilde{\Omega}\|_{L^\infty(S_{\underline{u}', u})} d\underline{u}' + \int_{\delta}^{\underline{u}} \| |u|^3 \rho \|_{L^\infty(S_{\underline{u}', u})} d\underline{u}' \\ & \lesssim \| |u|^3 \widetilde{\omega} \|_{L^\infty(S_{\delta, u})} \\ & \quad + \int_{\delta}^{\underline{u}} |u|^{-2} \sup_{u_0 \leq u' \leq u} \| |u'|^3 \widetilde{\omega} \|_{L^\infty(S_{\underline{u}', u'})} d\underline{u}' + \mathcal{P}(\underline{u}, u). \end{aligned}$$

By the Gronwall's inequality, we finally obtain

$$\sup_{u_0 \leq u' \leq u} \| |u'|^3 \widetilde{\omega} \|_{L^\infty(S_{\underline{u}, u'})} \lesssim \mathcal{P}(\underline{u}, u).$$

As a byproduct, we also have $\| |u|^2 \widetilde{\Omega} \|_{L^\infty(S_{\underline{u}, u})} \lesssim \mathcal{P}(\underline{u}, u)$.

We also consider the renormalized equation for $D\widetilde{\omega}$ as follows.

$$D\widetilde{\omega} = \Omega^2(2(\eta, \underline{\eta}) - |\eta|^2) - \widetilde{\Omega}^2 \rho + \Omega_{m_0}^2 \widetilde{\rho}.$$

Therefore, we have

$$\|\widetilde{\omega}\|_{L^\infty(\underline{u}, u)} \lesssim |u|^{-3} \mathcal{P}(\underline{u}, u) + |u|^{-3/2} \mathcal{F}^u(\underline{u}) \lesssim |u|^{-3/2} \mathcal{P}(\underline{u}, u).$$

Finally, we consider $\text{tr}\chi'$ and $\text{tr}\underline{\chi}'$. We only derive estimate for $\text{tr}\underline{\chi}$ in detail, and the estimate for $\text{tr}\chi$ follows in a similar way. The normalized form of the equation $\underline{D}\text{tr}\underline{\chi}'$ writes in the following form

$$\underline{D}\widetilde{\text{tr}\underline{\chi}'} = -\frac{1}{2}((\Omega\text{tr}\underline{\chi} + (\Omega\text{tr}\chi)_{m_0})\widetilde{\text{tr}\underline{\chi}'} + \widetilde{\Omega^2\text{tr}\underline{\chi}'}(\text{tr}\chi')_{m_0}) - \Omega^2|\underline{\chi}'|^2.$$

We then apply the Gronwall's inequality. Note that we also need the estimate (4.5), and the estimates for $\widetilde{\Omega}$ and $\widetilde{\chi}'$ derived above,

$$\begin{aligned} \| |u|^2 \widetilde{\text{tr}\underline{\chi}'} \|_{L^\infty(S_{\underline{u}, u})} &\lesssim \| |u_0|^2 \widetilde{\text{tr}\underline{\chi}'} \|_{L^\infty(S_{\delta, u_0})} \\ &\quad + \int_{u_0}^u |u'|^2 (|u'|^{-2} \|\widetilde{\Omega}\|_{L^\infty(S_{\underline{u}, u'})} + |u'|^{-2} \|\widetilde{\chi}'\|_{L^\infty(S_{\underline{u}, u'})}) du'. \end{aligned}$$

Combining the above estimates, we conclude

$$\| |u|^2 \widetilde{\text{tr}\underline{\chi}'} \|_{L^\infty(S_{\underline{u}, u})} \lesssim \mathcal{O}(\underline{u}, u_0) + \mathcal{F}^u(\underline{u}) + \sup_{u_0 \leq u' \leq u} (\mathcal{O}(\delta, u') + \mathcal{E}^u(u')).$$

4.2.2. Estimates for the Derivatives. For the first order derivative of $\widehat{\chi}'$ and $\text{tr}\chi'$, we commute ∇ with the equation $D\widehat{\chi}' = \Omega^2|\widehat{\chi}'|^2\sharp - \alpha$ (this equation is equivalent to $\widehat{D}\widehat{\chi}' = -\alpha$ because $\widehat{\chi}'$ is trace free) and the structure equation for $D\text{tr}\chi'$, we have

$$\begin{aligned} D\nabla\widehat{\chi}' &= \nabla(\Omega\chi) \cdot \widehat{\chi}' + \nabla(\Omega^2)|\widehat{\chi}'|^2\sharp + \Omega^2(\widehat{\chi}', \nabla\widehat{\chi}')\sharp - \nabla\alpha, \\ D\sharp\text{tr}\chi' &= -\Omega\text{tr}\chi\sharp\text{tr}\chi' - \text{tr}\chi\text{tr}\chi'\sharp\Omega - \sharp(\Omega^2|\widehat{\chi}'|^2). \end{aligned}$$

The first term on the right-hand side is simply the commutator $[D, \nabla]\widehat{\chi}'$ ¹. We further rewrite $\nabla(\Omega\chi) = \sharp(\Omega^2) \cdot \chi \cdot \widehat{\chi}' + \Omega^2\nabla\widehat{\chi}' \cdot \widehat{\chi}' + \Omega^2\sharp\text{tr}\chi' \cdot \widehat{\chi}'$ and $\sharp(\Omega^2|\widehat{\chi}'|^2) = \sharp(\Omega^2)|\widehat{\chi}'|^2 + 2\Omega^2(\widehat{\chi}', \nabla\widehat{\chi}')$. The unknowns of the above coupled system are $\nabla\widehat{\chi}'$ and $\sharp\text{tr}\chi'$, the other terms are viewed as known (Notice that $\sharp\Omega = (\eta + \underline{\eta})/2$). We can write

$$\begin{aligned} D\nabla\widehat{\chi}' &= i \cdot \nabla\widehat{\chi}' + h \cdot \sharp\text{tr}\chi' + s, \\ D\sharp\text{tr}\chi' &= g \cdot \nabla\widehat{\chi}' + f \cdot \sharp\text{tr}\chi' + r. \end{aligned}$$

By the Gronwall's inequality, We have

$$\begin{aligned} \|\nabla\widehat{\chi}\|_{L^4(S_{\underline{u}, u})} &\lesssim \int_{\delta}^u \|h \cdot \sharp\text{tr}\chi'\|_{L^4(S_{\underline{u}', u})} d\underline{u}' + \int_{\delta}^u \|s\|_{L^4(S_{\underline{u}', u})} d\underline{u}', \\ \|\sharp\text{tr}\chi'\|_{L^4(S_{\underline{u}, u})} &\lesssim \int_{\delta}^u \|g \cdot \nabla\widehat{\chi}'\|_{L^4(S_{\underline{u}', u})} d\underline{u}' + \int_{\delta}^u \|r\|_{L^4(S_{\underline{u}', u})} d\underline{u}'. \end{aligned}$$

Multiply the first by $|u|^2$ and the second by $|u|^3$, then sum them up. To estimate the terms in r and s except $\nabla\alpha$, we simply bound the factor $\widehat{\chi}'$ by (4.4) or bound $\sharp\Omega$ by (4.8), (4.9), and bound the other factors by the bootstrap assumption. We apply the Gronwall's inequality again, and conclude that

$$|u|^{-1/2} (\|(|u|\nabla)(|u|\widehat{\chi}')\|_{L^4(S_{\underline{u}, u})} + \|(|u|\nabla)(|u|^2\widetilde{\text{tr}\underline{\chi}'}\|_{L^4(S_{\underline{u}, u})}) \lesssim \mathcal{P}(\underline{u}, u). \quad (4.10)$$

¹ See Lemma 4.1 in [1] for the precise formulas.

For the derivatives of η and $\underline{\eta}$, we again commute ∇ 's to the equations for $D\eta$ and $\underline{D}\eta$. The equations look like the following.

$$\begin{aligned} D\nabla\eta &= \nabla(\Omega\chi) \cdot \eta + \nabla(\Omega\chi) \cdot \underline{\eta} + \Omega\chi \cdot \nabla\eta - \mathfrak{d}\Omega \cdot \beta - \Omega\nabla\beta, \\ \underline{D}\nabla\underline{\eta} &= \nabla(\Omega\chi) \cdot \underline{\eta} + \nabla(\Omega\chi) \cdot \eta + \Omega\chi \cdot \nabla\underline{\eta} + \mathfrak{d}\Omega \cdot \beta + \Omega\nabla\beta \end{aligned}$$

Let a and $\underline{a}$ denote the right-hand sides of the above equations respectively except the terms $\Omega\chi \cdot \nabla\eta$ and $\Omega\chi \cdot \nabla\underline{\eta}$. Then the $L^4(S_{\underline{u},u})$ norms of a and $\underline{a}$ except the terms involving β and $\underline{\beta}$ can be estimated using (4.10) and the bootstrap assumption. Therefore we perform the same argument as in deriving (4.8) and (4.9), i.e., the zero order estimates for η and $\underline{\eta}$, to deal with the coupled system for $\nabla\eta$ and $\nabla\underline{\eta}$. Finally,

$$|u|^{-1/2}(\|(|u|\nabla)(|u|^2\eta)\|_{L^4(S_{\underline{u},u})} + \|(|u|\nabla)(|u|^2\underline{\eta})\|_{L^4(S_{\underline{u},u})}) \lesssim \mathcal{P}(\underline{u}, u). \quad (4.11)$$

For the derivatives of $\underline{\omega}$ and ω , we again commute ∇ 's to the equations for $D\underline{\omega}$ and $\underline{D}\omega$, and deduce that

$$\begin{aligned} D\mathfrak{d}\underline{\omega} &= \mathfrak{d}(\Omega^2)(2(\eta, \underline{\eta}) - |\eta|^2 - \rho) + \Omega^2\nabla(2(\eta, \underline{\eta}) - |\eta|^2 - \rho), \\ \underline{D}\mathfrak{d}\omega &= \mathfrak{d}(\Omega^2)(2(\underline{\eta}, \eta) - |\underline{\eta}|^2 - \rho) + \Omega^2\nabla(2(\underline{\eta}, \eta) - |\underline{\eta}|^2 - \rho). \end{aligned}$$

The above two equations are not coupled. The right-hand sides except the terms involving ρ and $\mathfrak{d}(\Omega^2)$ can be estimated easily by (4.8), (4.9), (4.11) and the bootstrap assumption. The quantity $\mathfrak{d}(\Omega^2)$ is estimated by

$$\begin{aligned} |u|^{-1/2}\|u|\mathfrak{d}(\Omega^2)\|_{L^4(S_{\underline{u},u})} &\lesssim |u|^{-1/2}\|u|\mathfrak{d}(\log \Omega)\|_{L^4(S_{\underline{u},u})} \\ &\lesssim |u|^{-2} \sup_{u_0 \leq u' \leq u} \left(|u|^{-1/2}\|u'|^3|u'|\mathfrak{d}\omega\|_{L^4(S_{\underline{u},u'})} \right) \end{aligned}$$

using the equation $\underline{D}\mathfrak{d}\log \Omega = \mathfrak{d}\underline{\omega}$. Therefore, as in the zero order estimates of $\underline{\omega}$ and ω , we have

$$|u|^{-1/2}(\|(|u|\nabla)(|u|^3\underline{\omega})\|_{L^4(S_{\underline{u},u})} + \|(|u|\nabla)(|u|^3\widetilde{\omega})\|_{L^4(S_{\underline{u},u})}) \lesssim \mathcal{P}(\underline{u}, u).$$

As a byproduct, we have

$$|u|^{-1/2}\|u|\mathfrak{d}(\Omega^2)\|_{L^4(S_{\underline{u},u})} \lesssim \mathcal{P}(\underline{u}, u).$$

Then we commute ∇ with the equation $\underline{D}\widehat{\chi}' = \Omega^2|\widehat{\chi}'|^2\mathfrak{g} - \underline{\alpha}$ (this equation is equivalent to $\widehat{D}\widehat{\chi}' = -\underline{\alpha}$ because $\widehat{\chi}'$ is trace free) and the equation for $\underline{D}\text{tr}\chi'$, we have

$$\begin{aligned} \underline{D}\nabla\widehat{\chi}' &= \nabla(\Omega\chi) \cdot \widehat{\chi}' + \nabla(\Omega^2)|\widehat{\chi}'|^2\mathfrak{g} + \Omega^2(\widehat{\chi}', \nabla\widehat{\chi}')\mathfrak{g} - \nabla\underline{\alpha}, \\ \underline{D}\mathfrak{d}\text{tr}\chi' &= -\Omega\text{tr}\chi\mathfrak{d}\text{tr}\chi' - \text{tr}\chi\text{tr}\chi'\mathfrak{d}\Omega - \mathfrak{d}(\Omega^2|\widehat{\chi}'|^2). \end{aligned}$$

Similar argument gives the following

$$|u|^{-1/2}(\|(|u|\nabla)(|u|^2\widehat{\chi}')\|_{L^4(S_{\underline{u},u})} + \|(|u|\nabla)(|u|^2\widetilde{\text{tr}\chi'})\|_{L^4(S_{\underline{u},u})}) \lesssim \mathcal{P}(\underline{u}, u). \quad (4.12)$$

Notice that we need (4.5) here, as deriving estimates for $\widehat{\chi}'$ itself.

We can then commute one more ∇ to above all equations, and estimate the second order derivatives of the connection coefficients in a similar way. Then all the estimates can be summarized as follows.

Lemma 2. $\mathcal{O}_{\leq 2}^2(\underline{u}, u) \lesssim \mathcal{P}(\underline{u}, u)$ where $\mathcal{P}(\underline{u}, u)$ is defined in (4.8).

4.2.3. Estimates for the Curvature. We are in the position to derive estimates for the curvature components. Instead of ρ itself, we need to use the so-called renormalized Bianchi equations for $\tilde{\rho}$ below. Together with the original Bianchi equations for $\widehat{D}\underline{\alpha}$, $\widehat{D}\alpha$, $D\beta$, $\underline{D}\beta$, $D\sigma$, $\underline{D}\sigma$, all ten equations form a complete system.

$$D\tilde{\rho} + \frac{3}{2}\Omega\text{tr}\chi\tilde{\rho} + \frac{3}{2}\widetilde{\Omega\text{tr}\chi\rho_{m_0}} - \Omega \left\{ \text{div} \beta + (2\underline{\eta} + \zeta, \beta) - \frac{1}{2}(\widehat{\chi}, \alpha) \right\} = 0, \quad (4.13)$$

$$\underline{D}\tilde{\rho} + \frac{3}{2}\Omega\text{tr}\chi\tilde{\rho} + \frac{3}{2}\widetilde{\Omega\text{tr}\chi\rho_{m_0}} + \Omega \left\{ \text{div} \underline{\beta} + (2\underline{\eta} - \zeta, \underline{\beta}) - \frac{1}{2}(\widehat{\chi}, \underline{\alpha}) \right\} = 0, \quad (4.14)$$

$$\underline{D}\beta + \frac{1}{2}\Omega\text{tr}\chi\beta - \Omega\widehat{\chi} \cdot \beta + \underline{\omega}\beta - \Omega \left\{ \text{div} \tilde{\rho} + {}^*\text{div} \sigma + 3\underline{\eta}\rho + 3{}^*\eta\sigma + 2\widehat{\chi} \cdot \beta \right\} = 0, \quad (4.15)$$

$$D\underline{\beta} + \frac{1}{2}\Omega\text{tr}\chi\underline{\beta} - \Omega\widehat{\chi} \cdot \underline{\beta} + \omega\underline{\beta} + \Omega \left\{ \text{div} \tilde{\rho} - {}^*\text{div} \sigma + 3\underline{\eta}\rho - 3{}^*\eta\sigma - 2\widehat{\chi} \cdot \underline{\beta} \right\} = 0. \quad (4.16)$$

We commute ∇^i with the Bianchi equations and write down the energy inequalities with $|u|$ weighted. For the Bianchi equations for $\widehat{D}\nabla^i\alpha$ and $D\nabla^i\beta$, we write

$$\begin{aligned} \widehat{D}\nabla^i\alpha - \frac{1}{2}\Omega\text{tr}\chi\nabla^i\alpha - \Omega\nabla\widehat{\otimes}\nabla^i\beta &= E_3^i(\alpha), \\ D\nabla^i\beta - \Omega\widehat{\chi} \cdot \nabla^i\beta - \Omega\text{div} \nabla^i\alpha &= E_4^i(\beta). \end{aligned}$$

where $E_3^i(\alpha)$ and $E_4^i(\beta)$ are terms which can be derived explicitly using the Bianchi equations by direct computation, and the operators $\nabla\widehat{\otimes}$ and div apply to the last indices. Remember $\underline{D}\not{g}^{-1} = -2\Omega\underline{\chi} = -2\Omega\widehat{\chi} - \Omega\text{tr}\chi\not{g}$ and $\underline{D}d\mu_{\not{g}} = \Omega\text{tr}\chi d\mu_{\not{g}}$, we compute carefully

$$\begin{aligned} &\underline{D}(|\nabla^i\alpha|^2 d\mu_{\not{g}}) \\ &= \underline{D}(\not{g}^{A_1 B_1} \dots \not{g}^{A_i B_i} \not{g}^{C_1 D_1} \not{g}^{C_2 D_2} \nabla_{A_1} \dots \nabla_{A_i} \alpha_{C_1 C_2} \nabla_{B_2} \dots \nabla_{B_i} \alpha_{D_1 D_2} d\mu_{\not{g}}) \\ &= (-\Omega\widehat{\chi} \cdot \nabla^i\alpha \cdot \nabla^i\alpha - (i+1)\Omega\text{tr}\chi|\nabla^i\alpha|^2 + 2(\nabla^i\alpha, \widehat{D}\nabla^i\alpha))d\mu_{\not{g}}. \end{aligned}$$

The first term on the right-hand side is simply a schematic notation but the coefficient of the second term is important. This is related to the decay rates of the components. We then compute

$$\underline{D}(|u|^{2i}|\nabla^i\alpha|^2 d\mu_{\not{g}}) + D(2|u|^{2i}|\nabla^i\beta|^2 d\mu_{\not{g}})$$

$$\begin{aligned}
&= |u|^{2i}(-i(\Omega \text{tr} \underline{\chi} + \frac{2}{|u|})|\nabla^i \alpha|^2 + 4\text{div}(\Omega \nabla^i \alpha \cdot \nabla^i \beta) \\
&\quad - \Omega \widehat{\chi} \cdot \nabla^i \alpha \cdot \nabla^i \alpha - 4\Omega \nabla^i \alpha (\text{d} \log \Omega, \nabla^i \beta)) \text{d}\mu_{\mathcal{G}} \\
&\quad + ((\nabla^i \alpha, 2E_3^i(\alpha)) + 2(\nabla^i \beta, 2E_4^i(\beta))) \text{d}\mu_{\mathcal{G}}.
\end{aligned}$$

Denote the right-hand side of the above identity except the div term by $|u|^{2i} \tau_0^{(i)} \text{d}\mu_{\mathcal{G}}$. Then we integrate the above identity on $M_{\underline{u}, u} \cap M_c$ where $M_{\underline{u}, u}$ is the region $\delta \leq \underline{u}' \leq \underline{u}$, $u_0 \leq u' \leq u$. We use C_u and $\underline{C}_{\underline{u}}$ to denote their parts in M_c . Therefore we have

$$\begin{aligned}
&\int_{C_u} |u|^{2i} |\nabla^i \alpha|^2 + \int_{\underline{C}_{\underline{u}}} 2|u|^{2i} |\nabla^i \beta|^2 \\
&\lesssim \int_{C_{u_0}} |u_0|^{2i} |\nabla^i \alpha|^2 + \int_{\underline{C}_{\delta}} 2|u|^{2i} |\nabla^i \beta| + \iint_{M_{\underline{u}, u} \cap M_c} |u|^{2i} |\tau_0|. \quad (4.17)
\end{aligned}$$

We do similar computation for the other Bianchi equations. We write

$$\begin{aligned}
D\nabla^i \beta + \frac{1}{2}\Omega \text{tr} \underline{\chi} \beta - \Omega \widehat{\chi} \cdot \nabla^i \beta - \Omega(\text{d}\nabla^i \tilde{\rho} + {}^* \text{d}\nabla^i \sigma) &= E_3^i(\beta), \\
D\nabla^i \tilde{\rho} + \frac{1}{2}\Omega \text{tr} \chi \nabla^i \tilde{\rho} - \Omega \text{div} \nabla^i \beta &= E_4^i(\tilde{\rho}), \\
D\nabla^i \sigma + \frac{1}{2}\Omega \text{tr} \chi \nabla^i \sigma + \Omega \text{curl} \nabla^i \beta &= E_4^i(\sigma).
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{C_u} |u|^{2i+2} |\nabla^i \beta|^2 + \int_{\underline{C}_{\underline{u}}} |u|^{2i+2} (|\nabla^i \tilde{\rho}|^2 + |\nabla^i \sigma|^2) \\
&\lesssim \int_{C_{u_0}} |u_0|^{2i+2} |\nabla^i \beta|^2 + \int_{\underline{C}_{\delta}} |u|^{2i+2} (|\nabla^i \tilde{\rho}|^2 + |\nabla^i \sigma|^2) \\
&\quad + \iint_{M_{\underline{u}, u} \cap M_c} |u|^{2i+2} |\tau_1^{(i)}| \quad (4.18)
\end{aligned}$$

where

$$\begin{aligned}
\tau_1^{(i)} &= -(i+1)\left(\frac{2}{|u|} + \Omega \text{tr} \underline{\chi}\right) |\nabla^i \beta|^2 + 2\Omega(\text{d} \log \Omega, \nabla^i \tilde{\rho} \nabla^i \beta + \nabla^i \sigma {}^* \nabla^i \beta) \\
&\quad - \Omega \widehat{\chi} \cdot \nabla^i \beta \cdot \nabla^i \beta + (\nabla^i \beta, 2E_3^i(\beta)) + (\nabla^i \tilde{\rho}, 2E_4^i(\tilde{\rho})) + (\nabla^i \sigma, 2E_4^i(\sigma)).
\end{aligned}$$

Similarly, we write

$$\begin{aligned}
D\nabla^i \tilde{\rho} + \frac{3}{2}\Omega \text{tr} \underline{\chi} \nabla^i \tilde{\rho} + \Omega \text{div} \nabla^i \underline{\beta} &= E_3^i(\tilde{\rho}), \\
D\nabla^i \sigma + \frac{3}{2}\Omega \text{tr} \chi \nabla^i \sigma + \Omega \text{curl} \nabla^i \underline{\beta} &= E_3^i(\sigma), \\
D\nabla^i \underline{\beta} - \Omega \widehat{\chi} \cdot \nabla^i \underline{\beta} + \Omega(\text{d}\nabla^i \tilde{\rho} - {}^* \text{d}\nabla^i \sigma) &= E_4^i(\underline{\beta}),
\end{aligned}$$

and

$$D\nabla^i \underline{\beta} + \frac{3}{2}\Omega \text{tr} \underline{\chi} \underline{\beta} - \Omega \widehat{\chi} \cdot \nabla^i \underline{\beta} + \Omega \text{div} \nabla^i \underline{\alpha} = E_3^i(\underline{\beta}),$$

$$\widehat{D}\nabla^i \underline{\alpha} - \frac{1}{2}\Omega \text{tr} \chi \nabla^i \underline{\alpha} + \Omega \nabla \widehat{\otimes} \nabla^i \underline{\beta} = E_4^i(\underline{\alpha}).$$

Therefore, we have

$$\begin{aligned} & \int_{C_u} |u|^{2i+4} (|\nabla^i \widetilde{\rho}|^2 + |\nabla^i \sigma|^2) + \int_{\underline{C}_u} |u|^{2i+4} |\nabla^i \underline{\beta}| \\ & \lesssim \int_{C_{u_0}} |u_0|^{2i+4} (|\nabla^i \widetilde{\rho}|^2 + |\nabla^i \sigma|^2) + \int_{\underline{C}_\delta} |u|^{2i+4} |\nabla^i \underline{\beta}| \\ & \quad + \iint_{M_{\underline{u}, u} \cap M_c} |u|^{2i+4} |\tau_2^{(i)}| \end{aligned} \quad (4.19)$$

where

$$\begin{aligned} \tau_2^{(i)} = & - (i+2)(\Omega \text{tr} \chi + \frac{2}{|u|})(|\nabla^i \widetilde{\rho}|^2 + |\nabla^i \sigma|^2) \\ & + 2\Omega(\not{d} \log \Omega, \nabla^i \widetilde{\rho} \nabla^i \underline{\beta} - \nabla^i \sigma * \nabla^i \underline{\beta}) \\ & - \Omega \widehat{\chi} \cdot \nabla^i \widetilde{\rho} \cdot \nabla^i \widetilde{\rho} + (\nabla^i \widetilde{\rho}, 2E_3^k(\widetilde{\rho})) + (\nabla^i \sigma, 2E_3^k(\sigma)) + (\nabla^i \underline{\beta}, 2E_4^k(\underline{\beta})). \end{aligned}$$

and

$$\begin{aligned} & \int_{C_u} 2|u|^{2i+6} |\nabla^i \underline{\beta}|^2 + \int_{\underline{C}_u} |u|^{2i+6} |\nabla^i \underline{\alpha}|^2 \\ & \lesssim \int_{C_{u_0}} 2|u_0|^{2i+6} |\nabla^i \underline{\beta}|^2 + \int_{\underline{C}_\delta} |u|^{2i+6} |\nabla^i \underline{\alpha}|^2 + \iint_{M_{\underline{u}, u} \cap M_c} |u|^{2i+6} |\tau_3^{(i)}|. \end{aligned} \quad (4.20)$$

where

$$\begin{aligned} \tau_3^{(i)} = & - 2(i+3)(\Omega \text{tr} \chi + \frac{2}{|u|}) |\nabla^i \underline{\beta}|^2 + \Omega \widehat{\chi} \cdot \nabla^i \underline{\beta} \cdot \nabla^i \underline{\beta} \\ & + 4\Omega \nabla^i \underline{\alpha} (\not{d} \log \Omega, \nabla^i \underline{\beta}) + 2(\nabla^i \underline{\beta}, 2E_4^k(\underline{\beta})) + (\nabla^i \underline{\alpha}, 2E_3^k(\underline{\alpha})). \end{aligned}$$

We will estimate $|\tau_0^{(i)}|$, $|\tau_1^{(i)}|$, $|\tau_2^{(i)}|$, $|\tau_3^{(i)}|$, for $i \leq 2$ in terms of $\mathcal{E}^u(u)$ and $\mathcal{F}^u(\underline{u})$. Denote $(\tau_j^{(i)})_I$ be the first three terms of $\tau_j^{(i)}$. For example, $(\tau_0^{(i)})_I = -i(\Omega \text{tr} \chi + \frac{2}{|u|}) |\nabla^i \alpha|^2 - \Omega \widehat{\chi} \cdot \nabla^i \alpha \cdot \nabla^i \alpha - 4\Omega \nabla^i \alpha (\not{d} \log \Omega, \nabla^i \beta)$. The terms $\Omega \text{tr} \chi + \frac{2}{|u|}$, $\Omega \widehat{\chi}$ and $\not{d} \log \Omega$ are bounded by the bootstrap assumption. By the Sobolev inequalities and Hölder inequality, we have

$$\begin{aligned} & \sum_{j=0}^2 \sum_{i=0}^2 \iint_{M_{\underline{u}, u} \cap M_c} |u|^{2i+2j} |(\tau_j^{(i)})_I| \lesssim \int_{u_0}^u \frac{1}{|u'|^2} \mathcal{E}^u(u')^2 du' \\ & \sum_{i=0}^2 \iint_{M_{\underline{u}, u} \cap M_c} |u|^{2i+6} |(\tau_3^{(i)})_I| \lesssim \int_{u_0}^u \frac{1}{|u'|^2} \mathcal{E}^u(u')^2 du' + \int_\delta^{\underline{u}} \frac{1}{|\underline{u}|} \mathcal{F}^u(\underline{u}')^2 d\underline{u}'. \end{aligned}$$

The remaining terms of $\tau_j^{(i)}$, which we will call $(\tau_j^{(i)})_{II}$, consist of the terms of three types: the commutator of ∇ and D or $\underline{D}$; the commutators of

∇ and the Hodge operators; ∇ applies to the terms in the form $\tilde{\Gamma} \cdot R$ or $\Gamma \cdot \tilde{R}$. For example, we write

$$\begin{aligned} E_3^i(\alpha) &= [\underline{D}, \nabla^i] \alpha - (\Omega \nabla \hat{\otimes} \nabla^i \beta - \nabla^i (\Omega \nabla \hat{\otimes} \beta)) \\ &\quad + \frac{1}{2} \sum_{j=0}^{i-1} \nabla^j \nabla (\Omega \text{tr} \underline{\chi}) \cdot \nabla^{i-1-j} \alpha + \nabla^i E_3^0(\alpha). \end{aligned}$$

The first part has the form $\sum_{j=0}^{i-1} \nabla^j \nabla (\Omega \underline{\chi}) \cdot \nabla^{i-1-j} \alpha$. The factors of the derivatives of $\Omega \underline{\chi}$ are simply bounded by the bootstrap assumption. The second part has the form $\sum_{j=0}^{i-1} \nabla^j K \cdot \nabla^{i-1-j} \beta$ where K is the Gauss curvature of $S_{\underline{u}, u}$. By expressing K using the Gauss equation $K = -\frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} + \frac{1}{2} (\hat{\chi}, \hat{\chi}) - \rho$, we can bound K by the bootstrap assumption for the connection coefficients and for ρ . Terms of the third part (the last two terms) have the form $\sum_{j=0}^i \nabla^j (\Omega \Gamma) \cdot \nabla^{i-j} R$ but the terms like $\Omega \text{tr} \underline{\chi} \nabla^i R$ do not appear. We shall pay more attention to the terms involving ρ itself, say $\nabla^i (\Omega \hat{\chi}) \rho$. We bound ρ directly by the bootstrap assumption and bound $\nabla^i (\Omega \hat{\chi})$ in terms of the curvature components derived in last subsection. We shall also pay attention to the terms involving $\underline{\omega}$ itself. By Lemma 4 in the Appendix about $\underline{\omega}_{m_0}$ and the bootstrap assumption, we conclude $\underline{\omega}$ decays like $|u|^{-3}$ and does not make trouble like $\Omega \text{tr} \underline{\chi}$. So we use the bootstrap assumption, Sobolev inequalities and Hölder inequality, and obtain

$$\begin{aligned} &\iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i} |(\nabla^i \alpha, 2E_3^i(\alpha))| \\ &\lesssim \int_{u_0}^u |u'|^{-2} \mathcal{E}^{\underline{u}}(u') du' + \iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i} |\nabla^i (\Omega \hat{\chi}) \rho| |\nabla^i \alpha|. \end{aligned}$$

We also estimate in a similar way that

$$\iint_{M_{\underline{u}, u} \cap M_c} |2(\nabla^i \beta, 2E_4^i(\beta))| \lesssim \int_{u_0}^u |u'|^{-2} \mathcal{E}^{\underline{u}}(u')^2 du'.$$

For the other $(\tau_j^{(i)})_{II}$, we estimate

$$\begin{aligned} \iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i+2} |(\tau_1^{(i)})_{II}| &\lesssim \int_{u_0}^u |u'|^{-2} \mathcal{E}^{\underline{u}}(u')^2 du' \\ &\quad + \iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i+2} (|\nabla^i \widetilde{\Omega \text{tr} \chi} \rho_{m_0}| + |\nabla^i (\Omega \eta) \rho|) |\nabla^i \tilde{\rho}|. \end{aligned}$$

To estimate $(\tau_2^{(i)})_{II}$, note that $\underline{\alpha}$ appears in the equation for $\underline{D} \tilde{\rho}$ so we need $\mathcal{F}^u(\underline{u})$ involved in the estimate. By careful computation, we have

$$\begin{aligned} &\iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i+4} |(\tau_2^{(i)})_{II}| \\ &\lesssim \int_{u_0}^u |u'|^{-2} \mathcal{E}^{\underline{u}}(u')^2 du' + \int_{\delta}^{\underline{u}} |u|^{-1} \mathcal{F}^u(\underline{u})^2 d\underline{u} \end{aligned}$$

$$+ \iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i+4} (|\nabla^i \widetilde{\Omega \text{tr} \chi} \rho_{m_0}| |\nabla^i \widetilde{\rho}| + |\nabla^i (\Omega \underline{\eta}) \rho| |\nabla^i \underline{\beta}|).$$

We also estimate

$$\begin{aligned} \iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i+6} |(\tau_3^{(i)})_{II}| &\lesssim \int_{u_0}^u |u'|^{-2} \mathcal{E}^u(u')^2 du' + \int_{\delta}^{\underline{u}} |u|^{-1} \mathcal{F}^u(\underline{u}')^2 d\underline{u}' \\ &\quad + \iint_{M_{\underline{u}, u} \cap M_c} |u'|^{2i+6} |\nabla^i (\Omega \widehat{\chi}) \rho| |\nabla^i \underline{\alpha}|. \end{aligned}$$

The ρ terms above can be estimated by

$$\begin{aligned} &|u|^{-1/2} \left(\sup_{u_0 \leq u' \leq u} \mathcal{O}(\underline{u}, u')^2 + \sup_{\delta \leq \underline{u}' \leq \underline{u}} \mathcal{O}(\underline{u}', u)^2 \right) \\ &+ \int_{u_0}^u |u'|^{-3/2} \sup_{u_0 \leq u'' \leq u'} \mathcal{E}^u(u'')^2 du' + \int_{\delta}^{\underline{u}} |u|^{-1/2} \sup_{\delta \leq \underline{u}'' \leq \underline{u}'} \mathcal{F}^u(\underline{u}')^2 d\underline{u}'. \end{aligned}$$

Define

$$\mathcal{E}(u) = \sup_{u_0 \leq u' \leq u} \mathcal{E}(u')^2, \quad \mathcal{F}(\underline{u}) = \sup_{\delta \leq \underline{u}' \leq \underline{u}} \mathcal{F}(\underline{u}')^2$$

Then we sum up all the energy inequalities (4.17)–(4.20) and substitute the above estimates, we have

$$\begin{aligned} \mathcal{E}(u) + \mathcal{F}(\underline{u}) &\lesssim \mathcal{E}(u_0) + \mathcal{F}(\delta) \\ &\quad + |u|^{-1/2} \left(\sup_{u_0 \leq u' \leq u_1} \mathcal{O}(\delta, u')^2 + \sup_{\delta \leq \underline{u}' \leq \delta+1} \mathcal{O}(\underline{u}', u_0)^2 \right) \\ &\quad + \int_{u_0}^u |u'|^{-3/2} \mathcal{E}(u') du' + \int_{\delta}^{\underline{u}} |u|^{-1/2} \mathcal{F}(\underline{u}') d\underline{u}'. \end{aligned}$$

By the Gronwall's inequality, thanks to the factor $|u'|^{-3/2}$, we have

$$\begin{aligned} &\mathcal{E}(u) + \mathcal{F}(\underline{u}) \\ &\lesssim \mathcal{E}(u_0) + \mathcal{F}(\delta) + |u|^{-1/2} \left(\sup_{u_0 \leq u' \leq u_1} \mathcal{O}(\delta, u')^2 + \sup_{\delta \leq \underline{u}' \leq \delta+1} \mathcal{O}(\underline{u}', u_0)^2 \right). \end{aligned}$$

Remark 1. We should remark here that a similar inequality to the above one holds for higher order energy. Explicitly, assuming up to the i th order derivatives of the connection coefficients and the curvature components are appropriately bounded, i.e., $\mathcal{E}_{\leq i}(u)$, $\mathcal{F}_{\leq i}(\underline{u})$, $\mathcal{O}_{\leq i}^2(\underline{u}, u)$ are bounded, an $(i+1)$ th order version of the above inequality holds. This statement can be proved, by commuting more ∇ to the null structure equations and null Bianchi equations, and performing the argument in this section, especially Sects. 4.2.2 and 4.2.3.

By Proposition 1 and 2, we can choose δ sufficiently small such that the right-hand side is not larger than a given small $\varepsilon > 0$ which is smaller than $\frac{1}{2}\Delta_0$, and recover the bootstrap assumption (4.1) for $i \leq 2$. Then we perform the argument of estimating the connection coefficients in terms of the curvature components above and also recover the bootstrap assumption (4.2)

for the connection for $i \leq 2$ if we choose $\varepsilon > 0$ sufficiently small. The bootstrap assumption (4.3) is recovered by the Sobolev inequalities. Therefore we have proved that the estimates in Theorem 1 hold for $i \leq 2$.

By Remark 1, provided that we can choose δ sufficiently small such that

$$\mathcal{E}_i(u_0) + \mathcal{F}_i(\delta) + |u|^{-1} \left(\sup_{u_0 \leq u' \leq u_1} \mathcal{O}_i(\delta, u')^2 + \sup_{\delta \leq \underline{u}' \leq \delta+1} \mathcal{O}_i(\underline{u}', u_0)^2 \right)$$

is not larger than a given small $\varepsilon > 0$, which is certainly the case for $i \leq k$ because of Proposition 1 and 2, we can prove the estimates in Theorem 1 hold for $i \leq k$. Therefore we have proved the following a priori estimates.

Proposition 3. *Given Proposition 1 and 2, the estimates in Theorem 1 hold in M_c .*

The existence part of Theorem 1 follows by Proposition 3, the standard local existence result for Cauchy problem and for characteristic initial data problem (see [6]), using a continuity argument. This part is the same as constructing the spacetime in [1], provided a priori estimates. We omit the detail here. Finally, we complete the proof of Theorem 1.

5. Solution up to the Past Infinity

We are going to prove Theorem 2 in this section. We will construct a solution in the region $\mathcal{D}_\infty : 0 \leq \underline{u} \leq \delta + 1, -\infty < u \leq -1 - \underline{u}$. We choose a sequence $u_{0,n} \rightarrow -\infty$ and construct a solution g_n in the region $\mathcal{D}_n : 0 \leq \underline{u} \leq \delta + 1, u_{0,n} < u < -1 - \delta$ with the same seed data. We consider the components of g_n in the coordinates $(\underline{u}, u, \theta^A)$, say $\Omega_n, b_n^A, (\mathcal{g}_n)_{AB}$ (see (2.1)). Notice that the Schwarzschild metric g_{m_0} also depends on n by our construction. We omit the index n in the following. The norm $|\cdot|$ is taken by the metric δ^{AB} in fixed coordinate systems $(\underline{u}, u, \theta^A)$.

Lemma 3. *Let $\mu = (\mu_3, \mu_4, \mu_A)$ be a multi-index and ∂^μ be*

$$\frac{\partial^{\mu_A}}{\partial(\theta^A)^{\mu_A}} \frac{\partial^{\mu_4}}{\partial \underline{u}^{\mu_4}} \frac{\partial^{\mu_3}}{\partial u^{\mu_3}}.$$

Then for $|\mu| \leq k-4$, in the region $\delta \leq \underline{u} \leq \delta+1, u_{0,n} < u < -1 - \underline{u}$, we have

$$|\partial^\mu (|u|^{-2} \widetilde{\mathcal{G}})| + |u|^{1/2} |\partial^\mu \widetilde{\Omega}| + |u| |\partial^\mu b^A| \lesssim \frac{\varepsilon}{|u|^{1+\mu_3}}.$$

Proof. We have already estimated $||u|^2 \widetilde{\Omega}| \lesssim \varepsilon$. Now we have the following equation

$$\frac{\partial \widetilde{\mathcal{G}}_{AB}}{\partial u} = 2\Omega \widehat{\chi}_{AB} + \widetilde{\Omega \text{tr} \chi} (\mathcal{g}_{m_0})_{AB} + \Omega \text{tr} \chi \widetilde{\mathcal{G}}_{AB}.$$

Then the right-hand side is bounded by

$$\begin{aligned} |\Omega \widehat{\chi}_{AB}| &\lesssim |\Omega \widehat{\chi}|_{\mathcal{G}} |\partial_{\theta^A}|_{\mathcal{G}} |\partial_{\theta^B}|_{\mathcal{G}} \\ &= |\Omega \widehat{\chi}|_{\mathcal{G}} ((\mathcal{g}_{m_0})_{AA} (\mathcal{g}_{m_0})_{BB} + (\mathcal{g}_{m_0})_{AA} \widetilde{\mathcal{G}}_{AA} + (\mathcal{g}_{m_0})_{BB} \widetilde{\mathcal{G}}_{BB} + \widetilde{\mathcal{G}}_{AA} \widetilde{\mathcal{G}}_{BB}) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial u} \partial^\mu \Gamma_{AB}^C &= \partial^\mu \frac{\partial \Gamma_{AB}^C}{\partial u} \\ &= \frac{1}{2} \partial^\mu \left(g^{CD} (\nabla_A (\Omega_{\underline{\chi}})_{BD} + \nabla_B (\Omega_{\underline{\chi}})_{AD} - \nabla_D (\Omega_{\underline{\chi}})_{AB}) \right). \end{aligned}$$

Therefore, we only need to consider $\mathbf{D}^\mu \nabla \widehat{\chi}$ and $\mathbf{D}^\mu \nabla (\Omega \text{tr} \chi)$ and then integrate the equation along u direction. Finally, we write

$$\frac{\partial \left(\partial^\mu \widetilde{\not{g}}_{AB} \right)}{\partial \underline{u}} = \partial^\mu \left(\frac{\partial \widetilde{\not{g}}_{AB}}{\partial \underline{u}} \right) = \partial^\mu \left(2\widetilde{\Omega} \chi - b^C \frac{\partial \not{g}_{AB}}{\partial \theta^C} \right).$$

Therefore we only need to consider $\mathbf{D}^\mu \widehat{\chi}$ and $\mathbf{D}^\mu \Omega \text{tr} \chi$.

Finally we go to the derivatives of b^A . We first write

$$\frac{\partial (\partial^\mu (b^A))}{\partial u} = \partial^\mu (4\Omega^2 \zeta^A).$$

By $\zeta = \frac{\eta - \underline{\eta}}{2}$, we only need to consider $\mathbf{D}^\mu (\eta, \underline{\eta})$. Let $\bar{\partial}$ be ∂_{θ^A} or $\partial_{\underline{u}}$. We only need to consider the case $\mu_3 = 0$. Besides $\bar{\partial}$, we take one more $\partial_{\underline{u}}$, then

$$\partial_{\underline{u}} \bar{\partial} \partial^\mu (b^A) = \bar{\partial} \partial^\mu (4\Omega^2 \zeta^A).$$

Therefore we only need to consider $\nabla \mathbf{D}^\mu (\eta, \underline{\eta})$ or $D \mathbf{D}^\mu (\eta, \underline{\eta})$ and then integrate the equations along u direction.

By Theorem 1 proved in the previous section, the derivatives involving only ∇ of the connection and curvature are already bounded. The D and $\underline{D}$ derivatives are expressed in terms of ∇ using the structure equations and Bianchi equations, except the derivatives without $\underline{D}$ of $\widetilde{\omega}$ and α and derivatives without D of $\widetilde{\omega}$ and $\underline{\alpha}$. This is because we have no equations for $D\omega$, $\underline{D}\omega$, $D\alpha$ and $\underline{D}\alpha$. To derive estimates for these terms, we need to take one more D or $\underline{D}$ to derive the ordinary differential equations along D or $\underline{D}$ direction, as we have done for $\bar{\partial} \partial^\mu b^A$ for $\mu_3 = 0$ above. Finally, since we have the $L^2(S_{\underline{u}, u})$ bounds for the k th order derivatives of the connection coefficients, we have the L^∞ bounds for the $(k-2)$ th order derivatives of the connection coefficients. The connection coefficients are essentially the first order derivatives of the metric components, and we have lost one order derivatives when integrating the ODEs. So the lemma holds for $|\mu| \leq k-4$. $\square$

Remark 2. We should remark that, if we pursue optimal regularity, this is not the correct way to derive the bounds for purely D derivatives for α , and purely $\underline{D}$ derivatives for $\underline{\alpha}$. We should also start from the bounds for the k th order derivatives of the curvature components (not the connection coefficients) to gain one order derivatives back. But in our case, for simplicity, we do not care about how many derivatives are used.

Let $m = k - 4$. The above lemma together with the result in [1] about the asymptotic behavior of the solution in the region $0 \leq \underline{u} \leq \delta$, $u_{0,n} < u < -1 - \underline{u}$, says that we have uniform $C^m(\mathcal{D}_n)$ bounds for every n . Therefore we can take subsequences of $(\not{g}_n)_{AB}$, Ω_n and $(b_n)^A$, which converge to some functions $(\not{g}_\infty)_{AB}$, Ω_∞ and $(b_\infty)^A$ in $C^m(\mathcal{D}_\infty)$, in the sense that they converge in $C^m(\mathcal{D}_n)$ for every n . The metric written in the form

$$\begin{aligned} g_\infty = & -2\Omega_\infty^2 (d\underline{u} \otimes d\underline{u} + d\underline{u} \otimes d\underline{u}) \\ & + (\not{g}_\infty)_{AB} (d\theta^A - (b_\infty)^A d\underline{u}) \otimes (d\theta^B - (b_\infty)^B d\underline{u}) \end{aligned}$$

is then a C^m solution to the vacuum Einstein equations in $\mathcal{D}_\infty$ with initial data ψ_∞ from the past null infinity. By passing to the limit, the estimates in the above lemma for g in the region $\delta \leq \underline{u} \leq \delta + 1$ are satisfied by g_∞ , i.e., g_∞ is ε -close to g_{m_0} in the sense

$$|\partial^\mu(|u|^{-2}((\not{g}_\infty - \not{g}_{m_0})_{AB}))| + |u|^{1/2}|\partial^\mu(\Omega_\infty - \Omega_{m_0})| + |u||\partial^\mu(b_\infty)^A| \lesssim \frac{\varepsilon}{|u|^{1+\mu_3}}$$

for $|\mu| \leq m$, in the region $\delta \leq \underline{u} \leq \delta + 1$, $-\infty < u < -1 - \underline{u}$.

By Remark 1 and the argument in the above lemma, we actually conclude that, fixing $l > m$, the components of g_n have uniform bounds in $C^l(\mathcal{D}_n)$. Therefore we can take a subsequence of g_n which converges to a limit $g_\infty^{(l)}$ in the sense that $g_n \rightarrow g_\infty^{(l)}$ in $C^l(\mathcal{D}_n)$ for every n . Then $g_n \rightarrow g_\infty^{(l)}$ also in $C^m(\mathcal{D}_n)$ and therefore $g_\infty = g_\infty^{(l)}$. So g_∞ is in the class C^l . Since l is arbitrary, we finally conclude that g_∞ is smooth.

Appendix: A Double Null Foliation in Schwarzschild spacetime

It is well known that the Schwarzschild spacetime expressed in the Boyer-Lindquist coordinate is

$$-\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

And if we define r^* such that $dr^* = (1 - \frac{2m}{r})^{-1}dr$, and let $u' = t - r^*$, $v' = t + r^*$, we have the null coordinate expression

$$-\left(1 - \frac{2m}{r}\right) du' dv' + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

However, the (v', u') coordinates degenerate at $r = 2m$. So we need to reparametrize the null coordinates, that is, find two functions $\underline{u} = \underline{u}(v')$, $u = u(u')$. Adapting to the solution g_{m_0} defined in the context, we define $\underline{u}, u$ such that:

$$\begin{aligned} dt - \left(1 - \frac{2m}{r}\right)^{-1} dr &= 2 \left(1 - \frac{2m}{|u|}\right)^{-1} du \\ dt + \left(1 - \frac{2m}{r}\right)^{-1} dr &= 2 \left(1 - \frac{2m}{|u_0|}\right) \left(1 - \frac{2m}{(1 - 2m/|u_0|)\underline{u} + |u_0|}\right)^{-1} d\underline{u} \end{aligned}$$

with the condition that, $\underline{u} = 0$, $u = u_0$ when $t = 0$, $r = |u_0|$ where $u_0 < -2m$ is a fixed constant. Then the metric takes the form:

$$\begin{aligned} &-4 \left(1 - \frac{2m}{r}\right) \left(1 - \frac{2m}{|u_0|}\right) \left(1 - \frac{2m}{(1 - 2m/|u_0|)\underline{u} + |u_0|}\right)^{-1} \left(1 - \frac{2m}{|u|}\right)^{-1} d\underline{u} du \\ &+ r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

It is easy to see that the coefficient of $d\underline{u} du$ equals to constant -4 for $\underline{u} = 0$ or $u = u_0$ and not hard to check that on the sphere $\underline{u} = 0$, $u = u_0$,

$$\text{tr}\chi \Big|_{\underline{u}=0, u=u_0} = \frac{2}{r} \frac{\partial r}{\partial \underline{u}} \Big|_{\underline{u}=0, u=u_0} = \frac{2}{|u_0|} - \frac{4m}{|u_0|^2},$$

$$\text{tr}\underline{\chi}\Big|_{\underline{u}=0, u=u_0} = \frac{2}{r} \frac{\partial r}{\partial u}\Big|_{\underline{u}=0, u=u_0} = -\frac{2}{|u_0|}$$

where $\text{tr}\chi$ and $\text{tr}\underline{\chi}$ are defined in the $(\underline{u}, u)$ coordinates. Therefore, the coordinates $(\underline{u}, u)$ defined above give exactly the double null foliation we have used to define g_{m0} in the context, thanks to the uniqueness of the solution. The null hypersurface $\underline{u} = 0$ here corresponds to the null cone $\underline{C}_\delta$. Then we need the following lemma in this paper.

Lemma 4. *For $0 \leq \underline{u} \leq 1$, we have*

$$|(\Omega \text{tr}\underline{\chi})_m + \frac{2}{|u|}| + |u||\underline{\omega}_m| \leq C(m) \frac{1}{|u|^2}$$

Proof. We omit the lower index m in this proof. Observe that

$$\Omega \text{tr}\underline{\chi} = \frac{2}{r} \frac{\partial r}{\partial u} = -\frac{2}{r} \left(1 - \frac{2m}{r}\right) \left(1 - \frac{2m}{|u|}\right)^{-1}.$$

Then we hope to expand r by u near $u = -\infty$. It is not hard to see $r \rightarrow \infty$ when $u \rightarrow -\infty$, then $\lim_{u \rightarrow -\infty} \frac{r}{u} = \lim_{u \rightarrow -\infty} \frac{\partial r}{\partial u} = -1$. So there is a bounded continuous function a such that $\frac{1}{r} = \frac{1}{|u|} + a(\underline{u}, u) \frac{1}{|u|^2}$. So we may write

$$\begin{aligned} & -\frac{2}{r} \left(1 - \frac{2m}{r}\right) \left(1 - \frac{2m}{|u|}\right)^{-1} \\ &= -\left(\frac{2}{|u|} + \frac{a}{|u|^2}\right) \left(1 - \frac{2m}{|u|} - \frac{2ma}{|u|^2}\right) \left(1 + \frac{2m}{|u|} + O_m\left(\frac{1}{|u|^2}\right)\right) \\ &= -\frac{2}{|u|} + a'(\underline{u}, u, m) \frac{1}{|u|^2}, \end{aligned}$$

where a' is also a bounded continuous function. Therefore, we have proven the inequality for $\Omega \text{tr}\underline{\chi}$. For $\underline{\omega}$, we write $\underline{\omega} = \frac{\partial \log \Omega}{\partial u} = \frac{1}{2} \Omega^{-2} \frac{\partial \Omega^2}{\partial u}$ where $\frac{1}{2} \Omega^2$ can be read in the exact form of the metric:

$$\frac{1}{2} \Omega^2 = \left(1 - \frac{2m}{r}\right) \left(1 - \frac{2m}{|u_0|}\right) \left(1 - \frac{2m}{(1 - 2m/|u_0|)\underline{u} + |u_0|}\right)^{-1} \left(1 - \frac{2m}{|u|}\right)^{-1}.$$

Expanding $\frac{1}{2} \Omega^2$ near $u = -\infty$ in a similar way gives the inequality for $\underline{\omega}$ and we have proven the lemma. $\square$

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