



Quantum Ergodicity for Quantum Graphs without Back-Scattering

Matthew Brammall and B. Winn

Abstract. We give an estimate of the quantum variance for d -regular graphs quantised with boundary scattering matrices that prohibit back-scattering. For families of graphs that are expanders, with few short cycles, our estimate leads to quantum ergodicity for these families of graphs. Our proof is based on a uniform control of an associated random walk on the bonds of the graph. We show that recent constructions of Ramanujan graphs, and asymptotically almost surely, random d -regular graphs, satisfy the necessary conditions to conclude that quantum ergodicity holds.

1. Introduction

Quantum graphs have been suggested as an ideal model for problems in quantum chaos [33]. It is therefore surprising that there is no general theorem analogous to the Quantum Ergodicity theorem of Colin de Verdière, Šnirel'man and Zelditch [17, 46, 50] in the quantum graph setting. Quantum ergodicity has been proved for a special class of graphs derived from one-dimensional maps [5], and a general argument has been presented based on physical methods of supersymmetric field theory [24, 25]. On the other hand, it has been proved that quantum ergodicity does not hold for quantum graphs with a star-like configuration [6], and entropy bounds—which control the extent to which eigenfunctions can localise—have been derived for a few different families of quantum graphs in [32]. A recent article proves quantum ergodicity for the somewhat related problem of eigenfunctions of the discrete Laplacian on combinatorial graphs [2].

Quantum ergodicity is one of the few universal results in quantum chaos. It implies a weakened form of the semi-classical eigenfunction hypothesis [9, 48], and can be stated in the following form: let ϕ_n be an orthonormal basis of quantum wave functions with energy levels E_n , A an observable, with classical average \bar{A} . Then,

$$\lim_{E \rightarrow \infty} \frac{1}{\#\{E_n \leq E\}} \sum_{E_n \leq E} |\langle \phi_n, A\phi_n \rangle - \bar{A}|^2 = 0, \quad (1.1)$$

provided that the classical dynamics are ergodic. (For the reader who prefers to keep a concrete example in mind, one can take ϕ_n to be a sequence of normalised eigenfunctions of the Laplace–Beltrami operator on a compact Riemannian manifold of negative curvature, A can be a zeroth-order pseudo-differential operator with \bar{A} the mean value of its principal symbol, and the classical dynamics are the geodesic flow on the manifold. Quantum ergodicity in this setting was proved in [17, 46, 50]). One implication of (1.1) is that one can extract a density-one subsequence¹ of wave functions that becomes equidistributed in the large energy limit. This is equivalent to the semi-classical eigenfunction hypothesis for that subsequence [3].

Following the original proofs of quantum ergodicity, in the manifold setting, the result has been proved for a variety of situations, including Euclidian billiards [31, 51], quantised torus maps [15, 19, 38], and quantised Hamiltonian flows in \mathbb{R}^n [29].

In the present article we prove, deferring a precise statement of results to the following section, a quantum ergodicity theorem for quantum graphs quantised with the non-back-scattering boundary conditions introduced in [28], provided that the underlying graphs are expanders [30] and have not too many short cycles. These conditions are similar to those demanded for the proof of quantum ergodicity for eigenfunctions of the discrete Laplacian on combinatorial graphs in [2], although the method of proof there is quite different.

2. Notation and Statement of Results

To fix notations, we briefly describe the main definitions that we will use in this work. For further background information on quantum graphs, we refer the reader to the recent book [7].

2.1. Quantum Graphs

A quantum graph is a metric graph equipped with a differential operator acting in a space of functions defined on the bonds of the graph. We will denote by \mathfrak{V} and \mathfrak{B} , respectively, the set of vertices and bonds of the graph, with $|\mathfrak{V}| = n$ and $|\mathfrak{B}| = B$. The vector of bond lengths will be denoted $\mathbf{L} = (L_b)_{b \in \mathfrak{B}}$ where each $L_b > 0$. For us, all graphs will be undirected and simple, which means that no multiple edges are allowed, nor are loops connecting vertices to themselves. Furthermore, we will avoid considering bipartite graphs. This means that the vertex set cannot be partitioned into two sets with no connections within those sets.

Our focus will be on d -regular graphs, $d \geq 4$, which are graphs where each vertex is connected to d other vertices. This imposes the (trivial) constraint

$$2B = nd. \quad (2.1)$$

¹ Meaning that the number of terms of the subsequence in a sufficiently large interval is asymptotic to the number of terms in the interval.

Identifying each bond b of a graph \mathcal{G} with an interval $[0, L_b]$, we can define spaces $L_2(\mathcal{G})$ as the direct product of interval L^2 -spaces. We will consider metric graphs acted on by the one-dimensional (positive) Laplace operator on intervals. The associated eigenvalue problem reads

$$-\frac{d^2}{dx^2}\psi_b = k^2\psi_b, \quad b \in \mathfrak{B}, \tag{2.2}$$

and solutions are bond-wise waves, with vertices as scattering points.

A vertex scattering matrix for a vertex v of degree d is a $d \times d$ unitary matrix σ_v , where the vector of amplitudes of incoming waves $\mathbf{a}^{\text{in}} \in \mathbb{C}^d$ is related to the vector \mathbf{a}^{out} of outgoing amplitudes by

$$\mathbf{a}^{\text{out}} = \sigma_v \mathbf{a}^{\text{in}}. \tag{2.3}$$

By considering the $2B$ directed bonds, the bond-scattering matrix S is a $2B \times 2B$ matrix with b cth entry equal to 0 if bond b does not feed into a vertex v that bond c leaves, and otherwise is the corresponding element of the matrix σ_v . The matrix S so-constructed is unitary, as a consequence of the unitarity of the σ_v . The quantum evolution operator $U = U(k)$ is the $2B \times 2B$ matrix whose b cth entry is

$$U(k)_{bc} = e^{ikL_b} S_{bc}, \tag{2.4}$$

where L_b is the length² associated to the bond b . A common convention is to order the directed bonds so that bonds $b = B + 1, \dots, 2B$ are the reversals, in order, of the bonds $b = 1, \dots, B$. In that case, we can write $U(k)$ as the matrix product

$$U(k) = D(k)S, \quad \text{where } D(k) = \begin{pmatrix} e^{ik \text{diag } \mathbf{L}} & 0 \\ 0 & e^{ik \text{diag } \mathbf{L}} \end{pmatrix}, \tag{2.5}$$

however, we will sometimes adopt a different ordering for the directed bonds.

We define the spectrum of the quantum graph to be the set of non-negative values k for which the condition

$$\det(U(k) - I_{2B}) = 0 \tag{2.6}$$

is satisfied. We label the points in the spectrum as k_m , $m = 0, 1, 2, \dots$, with multiplicity, ordered so that

$$0 \leq k_0 \leq k_1 \leq k_2 \leq \dots \tag{2.7}$$

Condition (2.6) is equivalent to $U(k_m)$ having an eigenvalue 1, and we define $\Phi_m \in \mathbb{C}^{2B}$ to be the corresponding eigenvector normalised so that $\|\Phi_m\|_{\mathbb{C}^{2B}} = 1$, or if $k_m = k_{m+1} = \dots$ is a multiple root of (2.6), take the corresponding $\Phi_m, \Phi_{m+1}, \dots$ to be an arbitrary orthonormal basis of the eigenspace at 1.

If σ_v is a unitary matrix satisfying $\sigma_v^2 = I_d$, then the procedure above is equivalent to choosing a certain self-adjoint extension of the Laplace operator, in the sense that the spectrum defined above is the eigenvalue set of the self-adjoint Laplace operator, with correct multiplicity (except possibly for the eigenvalue 0, see [22, section 5]), and the components of the vectors Φ_m are

² Directed bonds have the same length as their undirected counterparts.

the amplitudes for the wave solutions to (2.2), satisfying the boundary conditions implicitly specified by the choice of extension. To give an example, the boundary conditions:

- $\psi'_{b_i}(0) = \psi'_{b_j}(0)$ for every pair of bonds b_i, b_j originating at the same vertex, and,
- $\sum \psi_b(0) = 0$, where the sum is taken over all bonds originating at a vertex,

leads, for a vertex of degree d , to the vertex scattering matrix σ_v , where

$$(\sigma_v)_{ij} = \frac{2}{d} - \delta_{ij}.$$

An alternative point of view treats vertices as scattering centres, where unitarity is a necessary condition on the matrix of transition amplitudes to ensure probability conservation. This point of view legitimises the assignment of arbitrary unitary matrices to vertices [43, 47], which allows for greater flexibility by choosing matrices with advantageous properties. In [28], a class of scattering matrices were introduced with the properties that all diagonal entries are 0, and all off-diagonal entries have equal complex amplitude. These were referred to as *equi-transmitting* matrices.

The diagonal elements of a scattering matrix give the reflection amplitude for a wave to be back-scattered into the reversal of the original bond. By setting these elements to zero, back-scattering is prohibited in the corresponding quantum graphs. In [28], it was speculated that these equi-transmitting quantum graphs would lead to new advances in the study of quantum chaos on graphs. Our proof in the present article of a quantum ergodicity theorem for quantum graphs with equi-transmitting boundary conditions can be considered as such an example.

Equi-transmitting matrices of size $d \times d$ have been proved to exist in [28] for $d = 2^n$, $d = P + 1$ where P is an odd prime number, and any d for which a skew-Hadamard matrix³ exists. Equi-transmitting matrices are further known to exist for $d = P^n + 1$ for any $n > 1$ and P any prime [23].

2.2. Notions of Graph Theory

We will appeal to a few notions of graph theory, collected here for reference. Classical graph theory is concerned with combinatorial graphs, i.e. without reference to any bond lengths. The connections are encoded by a $n \times n$ matrix C called the *connectivity matrix*, whose ij th entry is 1 if vertices i and j are connected, and 0 otherwise. If the graph is without multiple edges or loops, and d -regular, then C is symmetric, and each row contains precisely d 1's.

The (combinatorial) spectrum of a graph can be defined in a few ways, which are coincident if the graph is regular. We shall define it as the set of eigenvalues μ_1, \dots, μ_n of C . As C is symmetric, the eigenvalues are real, and we order them in decreasing order so that

$$-d \leq \mu_n \leq \dots \leq \mu_1 = d. \tag{2.8}$$

³ A skew-Hadamard matrix H is a matrix whose entries are ± 1 , with orthogonal columns, and satisfying $H + H^T = 2I_d$.

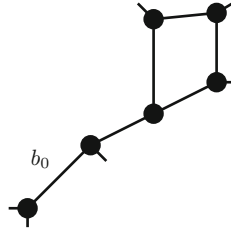


FIGURE 1. An illustrative example for the definition of $\mathcal{T}_{B,t}$. The bond b_0 belongs to $\mathcal{T}_{B,t}$ for each $t \geq 4$

The multiplicity of the eigenvalue d is the number of connected components of the graph (so that $\mu_1 = d$ —every graph has at least one component), and $\mu_n = -d$ if and only if the graph is bipartite. The eigenvalues of C excluding $\pm d$ will be called the *non-trivial* spectrum.

We will consider sequences of graphs indexed by an increasing number of vertices $n \rightarrow \infty$. Such a sequence of d -regular graphs is called a family of expanders [30] if there exists a constant $\beta > 0$ such that the non-trivial spectrum of each graph in the sequence is contained in the interval $[-d + \beta, d - \beta]$. If we can take $\beta = d - 2\sqrt{d - 1}$, the graphs are called *Ramanujan*. Ramanujan graphs are extremal in this sense, since for an increasing sequence of graphs the Alon–Boppana bound [1, 30, Theorem 5.3] implies

$$\liminf_{n \rightarrow \infty} \mu_2 \geq 2\sqrt{d - 1}. \tag{2.9}$$

We shall refer to *cycles* on a graph, which are closed paths without backtracking. We define the set $\mathcal{C}_{B,t}$ to be the set of bonds $b \in \mathfrak{B}$ of a graph that lie on a cycle of length at most t . The *girth* of a graph is the length of the shortest cycle. In particular, this means that $\mathcal{C}_{B,t} = \emptyset$ whenever t is less than the girth.

We also define the set $\mathcal{T}_{B,t}$ to be the set of directed bonds b_0 such that there exists t_1, t_2 with $t_1 + t_2 = t$ and bond b_0 is a distance at most t_1 from a cycle of length at most $2t_2$ (see Fig. 1).

The sets $\mathcal{T}_{B,t}$ and $\mathcal{C}_{B,t}$ both give a measure of the number of short cycles. The set $\mathcal{T}_{B,t}$ is more useful for our purposes, but $\mathcal{C}_{B,t}$ is easier to understand. Fortunately, their sizes are related, as the following lemma (which does not give the sharpest possible statement) makes clear.

Lemma 2.1. *Consider the sets $\mathcal{C}_{B,t}$ and $\mathcal{T}_{B,t}$ defined above, for a d -regular graph. Then,*

$$|\mathcal{T}_{B,t}| \leq \frac{(d - 1)^{t-1}}{d - 2} |\mathcal{C}_{B,2t}|, \tag{2.10}$$

where $|\cdot|$ denotes the number of elements of a set.

Proof. Clearly,

$$|\mathcal{T}_{B,t}| \leq \sum_{t_1+t_2=t} (d - 1)^{t_1} |\mathcal{C}_{B,2t_2}|. \tag{2.11}$$

Furthermore, $t_2 \geq 2$ for graphs without loops or multiple edges, so we have

$$\begin{aligned}
 |\mathcal{T}_{B,t}| &\leq \sum_{t_2=2}^t (d-1)^{t-t_2} |\mathcal{C}_{B,2t_2}| \\
 &\leq |\mathcal{C}_{B,2t}| \sum_{t_2=2}^t (d-1)^{t-t_2},
 \end{aligned}
 \tag{2.12}$$

since $|\mathcal{C}_{B,2t_2}| \leq |\mathcal{C}_{B,2t}|$. We sum the geometric series to get

$$\begin{aligned}
 |\mathcal{T}_{B,t}| &\leq \frac{|\mathcal{C}_{B,2t}|}{d-2} ((d-1)^{t-1} - 1) \\
 &\leq \frac{(d-1)^{t-1}}{d-2} |\mathcal{C}_{B,2t}|.
 \end{aligned}
 \tag{2.13}$$

□

2.3. Quantum Ergodicity for Quantum Graphs

Observables on a quantum graph will be functions that are constant on directed bonds, which can be represented by members of \mathbb{C}^{2B} . For such an observable $f \in \mathbb{C}^{2B}$, the quantisation of f , denoted $\text{Op}(f)$ is simply the diagonal $2B \times 2B$ matrix containing the entries of f :

$$\text{Op}(f) := \text{diag}\{f\}.
 \tag{2.14}$$

Let $\{\phi_j(k)\}_{j=1}^{2B}$ be an orthonormal basis of eigenvectors of the matrix $U = U(k)$. We define the quantum variance as,

$$V(f, B) := \frac{1}{2B} \lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K \sum_{j=1}^{2B} |\langle \phi_j(k), \text{Op}(f)\phi_j(k) \rangle_{\mathbb{C}^{2B}} - \frac{1}{2B} \text{Tr Op}(f)|^2 dk.
 \tag{2.15}$$

While $V(f, B)$ defined by (2.15) does not seem immediately analogous to (1.1), it was proved in [8] that averaging the second moment appearing in (2.15), involving the eigenvectors of the matrix $U(k)$, over a large window of k values is equivalent to averaging an expression similar to (1.1) involving the eigenstates $\Phi_0, \Phi_1, \Phi_2, \dots$, at least for graphs with incommensurate bond lengths. We take (2.15) as the starting point of our investigation.

It turns out (see [5] for example) that one cannot expect that $V(f, B) = 0$ for any individual fixed graph. For a fixed graph with Kirchhoff boundary conditions, a complete classification of limiting measures induced by subsequences of eigenfunctions has recently been obtained in [18]. Consequently, we consider a *family* of graphs, indexed by a sequence of increasing number of bonds $B \rightarrow \infty$. Quantum ergodicity (sometimes called asymptotic quantum ergodicity) for quantum graphs means that $V(f, B) \rightarrow 0$ as $B \rightarrow \infty$ along this sequence, [5, 25].

Our main result, stated below, estimates the size of the quantum variance for graphs quantised with equi-transmitting matrices, in terms of certain graph-theoretic properties.

Theorem 2.2. *For $d > 3$, consider a d -regular connected, non-bipartite, simple graph, on B bonds, quantised with equi-transmitting scattering matrices. Let $f \in \mathbb{C}^{2B}$ be an observable, satisfying $|f_b| < \kappa$ for all b , for some $\kappa > 0$. Let $T > 0$, and suppose that all non-trivial eigenvalues μ_i of the connectivity matrix of the graph satisfy $|\mu_i| \leq d - \beta$ for some $\beta > 0$. Then, the quantum variance (2.15) satisfies the main estimate,*

$$V(f, B) = O\left(\frac{\kappa^2}{T\beta^2}\right) + O\left(\frac{\kappa^2(d-1)^T |\mathcal{C}_{B,2T}|}{BT^2}\right), \tag{2.16}$$

where $\mathcal{C}_{B,2T}$ denotes the number of cycles as was described in Sect. 2.2 above.

It is natural to impose a uniform boundedness condition on observables for which we wish to prove that quantum ergodicity to hold. This means that the parameter κ in the statement of Theorem 2.2 is an absolute constant, independent of B . However, the estimate (2.16) makes it clear how to consider observables that grow mildly as $B \rightarrow \infty$.

Theorem 2.2 proves quantum ergodicity for families of graphs for which a parameter $T \rightarrow \infty$ can be found as $B \rightarrow \infty$, in such a way that the quantities on the right-hand side of (2.16) become negligible. We give two examples for which this is the case: the families of Ramanujan graphs constructed in [40], and random d -regular graphs [14,49]. We discuss these examples further in Sect. 6 below.

We also mention that Theorem 2.2 does not require that the bond lengths of the graph are linearly independent over \mathbb{Q} , in contrast to many other results in this field, although if the bond lengths are incommensurate, some values of constants can be improved—see the comment at the end of Sect. 6.2.

We remark that we excluded bipartite graphs from our consideration in the introduction. Our methods could be extended to include bipartite graphs, but there the notions of ergodicity would need to be generalised, since bipartite graphs can support non-uniform invariant states.

All results in the present paper would hold in the case $d = 3$, were it not for the fact that no equi-transmitting matrices of size 3×3 can exist (as a short calculation shows). Therefore, Theorem 2.2 is stated with condition $d > 3$, although in later parts of this work, d can be set equal to 3.

In most quantum ergodicity results, such as [17,46,50], the proof naturally separates into a semi-classical part, and a dynamical part. In the semi-classical part, a correspondence is established between quantum and classical time evolution, and in the dynamical part, ergodic properties of the classical dynamics are invoked to prove quantum ergodicity. We shall present our proof of Theorem 2.2 in this manner. In Sect. 3 below, we relate (see Proposition 3.3) the quantum variance to a classical random walk on the bonds of the graph. In Sect. 4 below, we analyse the equidistribution of the random walk, proving a uniform decay estimate that allows us to obtain (2.16). The final steps of the proof are carried out in Sect. 5.

3. Semi-Classical Argument

Our aim is to prove, for a suitable class of f , that $V(f, B) \rightarrow 0$ as $B \rightarrow \infty$. Without loss of generality, we can and will assume that f is chosen so that $\text{Tr Op}(f) = 0$, which will simplify the notation somewhat.

Our main estimate for the quantum variance goes back to an idea from [45] (see also [19] for a similar application of this idea). Let $T > 0$ and

$$\hat{w}_T(t) := \begin{cases} \frac{1}{T} \left(1 - \frac{|t|}{T}\right), & |t| < T, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then, for any $2B \times 2B$ unitary matrix U , and any orthonormal basis $\{\phi_j\}_{j=1}^{2B}$ of eigenvectors of U ,

$$\frac{1}{2B} \sum_{j=1}^{2B} |\langle \phi_j, \text{Op}(f)\phi_j \rangle_{\mathbb{C}^{2B}}|^2 \leq \frac{1}{2B} \sum_{t=-T}^T \hat{w}_T(t) \text{Tr}(\text{Op}(f)^* U^t \text{Op}(f) U^{-t}). \tag{3.2}$$

(To make this paper as self-contained as possible, we include a proof of (3.2) in an Appendix. See Lemma A.1). Therefore, we can estimate the quantum variance by

$$V(f, B) \leq \frac{1}{2B} \sum_{t=-T}^T \hat{w}_T(t) \left(\lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K \text{Tr}(\text{Op}(f)^* U(k)^t \text{Op}(f) U(k)^{-t}) dk \right). \tag{3.3}$$

The limit in (3.3) exists as the integrand is an almost-periodic function of k .

Let us define the matrix $\tilde{M}^{(t)}$ for $t \in \mathbb{N}_0$ as the $2B \times 2B$ matrix whose bc th entry is

$$\left(\tilde{M}^{(t)}\right)_{bc} = \lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K |U(k)^t_{bc}|^2 dk, \tag{3.4}$$

i.e. the average of the square of the bc th element of the matrix $U(k)^t$. Then, we can rewrite the quantity inside the brackets in (3.3):

Lemma 3.1. *Let $\text{Op}(f) = \text{diag}\{f\}$ where $f \in \mathbb{C}^{2B}$ and let $\tilde{M}^{(t)}$ be defined as above. Then,*

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K \text{Tr}(\text{Op}(f)^* U(k)^t \text{Op}(f) U(k)^{-t}) dk = \langle f, \tilde{M}^{(t)} f \rangle_{\mathbb{C}^{2B}}, \tag{3.5}$$

fort ≥ 0 .

Proof. We denote $F = \text{Op}(f)$. Expanding the trace, we have

$$\begin{aligned} & \text{Tr}(F^* U^t F U^{-t}) \\ &= \sum_{\substack{b_0, \dots, b_{t-1} \\ c_0, \dots, c_{t-1}}} \bar{F}_{b_0 b_0} U_{b_0 b_1} U_{b_1 b_2} \cdots U_{b_{t-1} c_0} F_{c_0 c_0} U_{c_0 c_1}^{-1} U_{c_1 c_2}^{-1} \cdots U_{c_{t-1} b_0}^{-1}, \end{aligned} \tag{3.6}$$

where the multi-sum runs over all possible choices of $2B$ bonds for each of b_0, \dots, b_{t-1} and c_0, \dots, c_{t-1} . We have also the expansions

$$U_{b_0 c_0}^t = \sum_{b_1, \dots, b_{t-1}} U_{b_0 b_1} U_{b_1 b_2} \cdots U_{b_{t-1} c_0}, \tag{3.7}$$

and

$$\begin{aligned} \overline{U}_{b_0 c_0}^t &= \sum_{c_1, \dots, c_{t-1}} \overline{U}_{b_0 c_{t-1}} \overline{U}_{c_{t-1} c_{t-2}} \cdots \overline{U}_{c_1 c_0} \\ &= \sum_{c_1, \dots, c_{t-1}} U_{c_0 c_1}^{-1} U_{c_1 c_2}^{-1} \cdots U_{c_{t-1} b_0}^{-1}, \end{aligned} \tag{3.8}$$

so that

$$\begin{aligned} &\lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K \text{Tr}(F^* U(k)^t F U(k)^{-t}) dk \\ &= \sum_{b_0, c_0} \overline{F}_{b_0 b_0} \left(\lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K |U(k)_{b_0 c_0}^t|^2 dk \right) F_{c_0 c_0} \\ &= \langle f, \tilde{M}^{(t)} f \rangle_{\mathbb{C}^{2B}}, \end{aligned} \tag{3.9}$$

□

The matrix $\tilde{M}^{(t)}$ is not easy to work with, so we introduce a second matrix M where

$$M_{bc} := |U_{bc}|^2. \tag{3.10}$$

Because U is a unitary matrix, the matrices M and $\tilde{M}^{(t)}$ are both doubly stochastic. If the graph does not contain too many cycles, then the matrix $\tilde{M}^{(t)}$ is close to M^t , in the following sense:

Proposition 3.2. *Let $T \in \mathbb{N}$ and suppose that $f \in \mathbb{C}^{2B}$ satisfies the bound $|f_b| \leq \kappa$, $b = 1, \dots, 2B$ for some $\kappa > 0$. Provided that the graph is quantised with scattering matrices that prohibit back-scattering, then*

$$|\langle f, \tilde{M}^{(t)} f \rangle_{\mathbb{C}^{2B}} - \langle f, M^t f \rangle_{\mathbb{C}^{2B}}| \leq \frac{2\kappa^2}{(d-2)(d-1)} (d-1)^t |\mathcal{C}_{B, 2T}|, \tag{3.11}$$

for all $t = 1, 2, \dots, T$.

Proof. By (2.4), we can expand $|U(k)_{b_0 c_0}^t|^2$ as

$$\begin{aligned} |U(k)_{b_0 c_0}^t|^2 &= \sum_{\substack{b_1, \dots, b_{t-1} \\ c_1, \dots, c_{t-1}}} S_{b_0 b_1} S_{b_1 b_2} \cdots S_{b_{t-1} c_0} e^{ik(L_{b_1} + L_{b_2} + \cdots + L_{b_{t-1}} + L_{c_0})} \\ &\quad \times \overline{S}_{c_1 c_0} \overline{S}_{c_2 c_1} \cdots \overline{S}_{b_0 c_{t-1}} e^{-ik(L_{c_0} + L_{c_1} + \cdots + L_{c_{t-1}})}. \end{aligned} \tag{3.12}$$

Because $S_{bc} = 0$ if directed bonds b and c are not connected, we can think of the right-hand side of equation (3.12) as a weighted sum over paths connecting b_0 to c_0 and a return path (see Fig. 2).

The crucial step in our argument is to demonstrate that, with few exceptions, the return path goes back over the same bonds *in the reverse order* as the outward path. This might not happen if, along the path, there are places

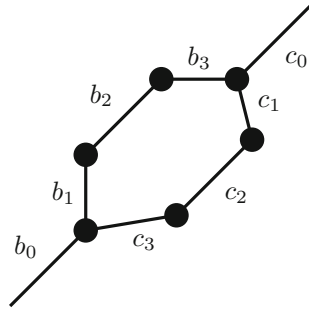


FIGURE 2. An example of a path and its return with $t = 4$. In this example, the return path is different to the outward path. If the graph does not contain too many cycles of length at most 6 then this happens only rarely

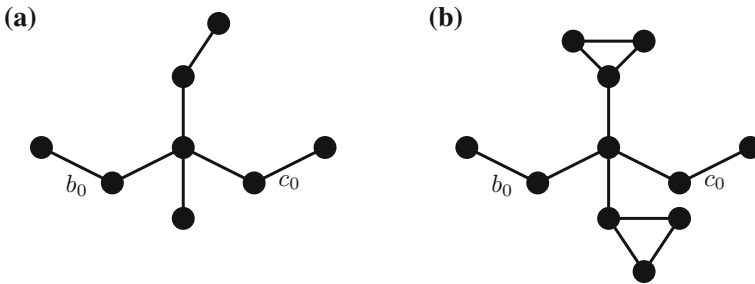


FIGURE 3. Two different possibilities for excursions along a path from b_0 to c_0 . In *a*, there is back-scattering; in *b*, no back-scattering occurs but we note that the bond b_0 belongs to the set $\mathcal{T}_{B,5}$

where at least two distinct excursions from the same vertex are made, in the sense that removing the excursions gives a shorter path from b_0 to c_0 (see Fig. 3). Since the graphs are quantised without back-scattering, the excursions may not consist of self-retracing sections, so the only possibility is if the excursions contain short cycles.

A second mechanism whereby the return path may differ from the outward path is if the path contains a short cycle of an even number of steps where the outward path takes one route, and the return path takes a different route, as could happen in the situation depicted in Fig. 2.

If the bond b_0 does not belong to the set $\mathcal{T}_{B,t}$, then there are no cycles close enough to b_0 to allow either possibility. In this case, the return path has to be the reversal of the outward path. This means that $c_1 = b_{t-1}, c_2 = b_{t-2}, \dots, c_{t-1} = b_1$ and so

$$\begin{aligned}
 |U(k)_{b_0c_0}^t|^2 &= \sum_{b_1, \dots, b_{t-1}} |S_{b_0b_1}|^2 |S_{b_1b_2}|^2 \cdots |S_{b_{t-1}c_0}|^2 \\
 &= (M^t)_{b_0c_0},
 \end{aligned}
 \tag{3.13}$$

since $|S_{bc}| = |U_{bc}|$. Since (3.13) is independent of k , the average in k in (3.4) has no effect, and we have

$$(\tilde{M}^{(t)})_{b_0c_0} = (M^t)_{b_0c_0},
 \tag{3.14}$$

for $b_0 \notin \mathcal{T}_{B,t}$. Therefore, let us define a matrix $R^{(t)}$ by

$$R^{(t)} = \tilde{M}^{(t)} - M^t,
 \tag{3.15}$$

and let us consider $(R^{(t)}f)_b$, the b th component of $R^{(t)}f$. We have proved that

$$(R^{(t)}f)_b = 0,
 \tag{3.16}$$

if $b \notin \mathcal{T}_{B,t}$. If $b \in \mathcal{T}_{B,t}$ then we can be sure that

$$|(R^{(t)}f)_b| \leq 2\kappa,
 \tag{3.17}$$

since the matrices $\tilde{M}^{(t)}$ and M (and hence M^t) are doubly stochastic. Hence,

$$\begin{aligned}
 |\langle f, \tilde{M}^{(t)}f \rangle_{\mathbb{C}^{2B}} - \langle f, M^t f \rangle_{\mathbb{C}^{2B}}| &= |\langle f, R^{(t)}f \rangle_{\mathbb{C}^{2B}}| \\
 &\leq 2\kappa^2 |\mathcal{T}_{B,t}| \\
 &\leq \frac{2\kappa^2}{d-2} (d-1)^{t-1} |\mathcal{C}_{B,2t}|,
 \end{aligned}
 \tag{3.18}$$

using Lemma 2.1,

$$\leq \frac{2\kappa^2}{d-2} (d-1)^{t-1} |\mathcal{C}_{B,2T}|,
 \tag{3.19}$$

using the fact that $|\mathcal{C}_{B,2t}| \leq |\mathcal{C}_{B,2T}|$ for $t \leq T$.

As we shall see below, quantum ergodicity essentially follows if we can prove that $\langle f, \tilde{M}^{(t)}f \rangle_{\mathbb{C}^{2B}} = o(1)$ as $t \rightarrow \infty$. However, we are able to prove this decay for the matrix M^t only, and Proposition 3.2 provides the requisite link. We remark that this procedure is reminiscent of the recent proof of quantum ergodicity for ray-splitting billiards [31]. In that work, the authors consider a probabilistic random walk on families of trajectories with the same endpoints, with transition weights given to individual trajectories, and summed over families of trajectories; the same distinction as between our matrices M^t and $\tilde{M}^{(t)}$, with quantum ergodicity likewise following from ergodicity of the latter class of random walk. In [31], the multiple trajectories arise as a result of splitting trajectories at the interface between one-or-more different billiard media; in our work, the multiple ‘‘trajectories’’ arise due to the connectivity of the graphs. □

The main result of this section is as follows.

Proposition 3.3. *Consider a d -regular graph on B bonds, quantised without back-scattering. For any $T > 0$, and any $f \in \mathbb{C}^{2B}$ satisfying $|f_b| \leq \kappa$ for each b and $\text{Op}(f) = \text{diag}\{f\}$ satisfying $\text{Tr Op}(f) = 0$,*

$$V(f, B) \leq \frac{1}{2BT} \text{Tr Op}(f)^2 + \frac{1}{B} \sum_{t=1}^T \hat{w}_T(t) (\langle f, M^t f \rangle_{\mathbb{C}^{2B}} + O(\kappa^2(d-1)^t |C_{B,2T}|)). \tag{3.20}$$

Proof. Because of cyclic invariance of trace, and the symmetry of \hat{w}_T , we can write (3.3) (extracting the $t = 0$ term) as,

$$V(f, B) \leq \frac{\text{Tr Op}(f)^2}{2BT} + \frac{1}{B} \sum_{t=1}^T \hat{w}_T(t) \left(\lim_{K \rightarrow \infty} \frac{1}{K} \int_0^K \text{Tr}(\text{Op}(f)^* U(k)^t \text{Op}(f) U(k)^{-t}) dk \right). \tag{3.21}$$

We then use Lemma 3.1 and Proposition 3.2 to get (3.20). □

To prove our quantum ergodicity result, we will need to understand the behaviour of $M^t f$, which represents a random walk on the bonds of the graph. This we do in the next section.

4. Dynamical Argument

4.1. Classical Dynamics on a Quantum Graph

The classical analogue of the quantum evolution is the Markov process on the directed bonds of the graph with the transfer matrix M , which is doubly stochastic [34].

Since M is not necessarily normal, it will be convenient to work, rather than with eigenvectors of M , with its singular vectors, defined to be the eigenvectors of the symmetric matrix $M^T M$. If we order the directed bonds in groups of d bonds departing from each vertex, the matrix $M^T M$ decomposes into block-diagonal form

$$M^T M = \begin{pmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{pmatrix},$$

where each of the n blocks is a $d \times d$ matrix

$$J = \frac{1}{(d-1)^2} \begin{pmatrix} d-1 & d-2 & \cdots & d-2 \\ d-2 & d-1 & \cdots & d-2 \\ \vdots & \vdots & \ddots & \vdots \\ d-2 & d-2 & \cdots & d-1 \end{pmatrix}.$$

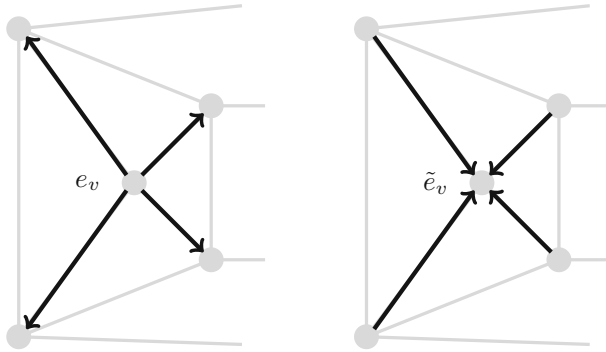


FIGURE 4. Support of the singular vectors e_j and \tilde{e}_j associated to a vertex v . See main text for the definitions

The eigenvalues of J are 1 with multiplicity one, and $(d-1)^{-2}$ with multiplicity $d-1$, and the simple eigenspace is spanned by the vector $(1, \dots, 1)^T$. Therefore, the singular values of the matrix M are: 1 with multiplicity n , and $(d-1)^{-1}$ with multiplicity $n(d-1)$. A basis for the eigenspace of $M^T M$ with eigenvalue 1 is given by the set $\{e_1, \dots, e_n\} \subseteq \mathbb{C}^{2B}$, where the j^{th} component of e_v is defined to be 1 if directed bond j points outwards from vertex v , and 0 otherwise (see Fig. 4).

We could equally well consider MM^T , which has identical spectrum to $M^T M$, and a basis of eigenvectors corresponding to the singular value 1 can be chosen to have zero components except for the *incoming* bonds of vertex v where the component is 1. We will denote these vectors by $\tilde{e}_v \in \mathbb{C}^{2B}$ for $v = 1, \dots, n$ (see Fig. 4).

Observables that are linear combinations of e_1, \dots, e_n will be called *evenly distributed around vertices* in the following subsection.

Proposition 4.1. *If the scattering matrices on the quantum graph are equi-transmitting, the action of M on the vectors e_v and \tilde{e}_v is as follows:*

$$Me_v = \tilde{e}_v, \tag{4.1}$$

$$M\tilde{e}_v = \frac{1}{d-1} \left(\sum_{w \sim v} \tilde{e}_w - e_v \right), \tag{4.2}$$

where $\sum_{w \sim v}$ is a sum over vertices w connected to v .

Proof. The action of M on a vector supported on a single directed bond b is to allocate an equal fraction to the $d-1$ directed bonds feeding into the origin of b (not including the reversal of b). For the vector Me_v , each bond directed towards v gets $d-1$ times the fraction $1/(d-1)$ of the weight 1 on each outward pointing bond in e_v . The result is a vector of weight 1 on each inward pointing vertex to v . Hence, $Me_v = \tilde{e}_v$.

In a similar way, it is clear that $M\tilde{e}_v$ has weight $1/(d-1)$ on each bond directed towards a neighbour of v , except the outward pointing bonds from v (see Fig. 5). Such a vector can be written

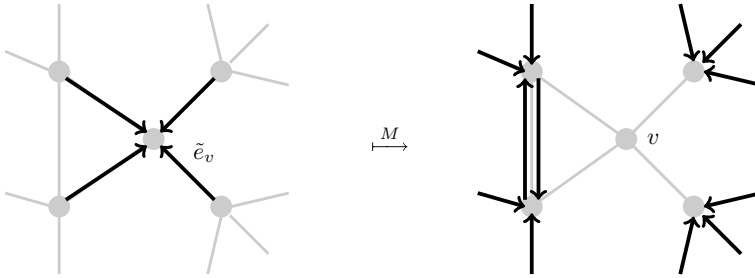


FIGURE 5. The support of $M\tilde{e}_v$. The vertex v is at the centre of the illustration

$$\frac{1}{d-1} \left(\sum_{w \sim v} \tilde{e}_w - e_v \right).$$

□

4.2. Observables Evenly Distributed Around Vertices

Let

$$G_1 := \text{span}\{e_1, \dots, e_n\} \subseteq \mathbb{C}^{2B}, \tag{4.3}$$

and let $\varphi : G_1 \rightarrow \mathbb{C}^n$ be the natural isomorphism. Let us also define $\tilde{\varphi} : G_1 \rightarrow \mathbb{C}^{2n}$ by,

$$\tilde{\varphi} : a_1 e_1 + \dots + a_n e_n \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.4}$$

We shall also work with the larger space \tilde{G}_1 , where

$$\tilde{G}_1 := \text{span}\{e_1, \dots, e_n, \tilde{e}_1, \dots, \tilde{e}_n\} \subseteq \mathbb{C}^{2B}, \tag{4.5}$$

and define $\psi : \mathbb{C}^{2n} \rightarrow \tilde{G}_1$ by

$$\psi : \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \mapsto a_1 e_1 + \dots + a_n e_n + b_1 \tilde{e}_1 + \dots + b_n \tilde{e}_n, \tag{4.6}$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{4.7}$$

We note that ψ fails to be invertible since

$$\psi \begin{pmatrix} \mathbf{e} \\ -\mathbf{e} \end{pmatrix} = 0, \quad \text{where } \mathbf{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{4.8}$$

Lemma 4.2. *If $\hat{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^{2n}$ then*

$$\|\psi(\hat{x})\|_{\mathbb{C}^{2B}} \leq \sqrt{d} (\|x\|_{\mathbb{C}^n} + \|y\|_{\mathbb{C}^n}). \tag{4.9}$$

Proof. Let us define

$$\begin{aligned} \eta &= \sum_{j=1}^n x_j e_j \\ \tilde{\eta} &= \sum_{j=1}^n y_j \tilde{e}_j, \end{aligned}$$

so that

$$\|\psi(\hat{x})\|_{\mathbb{C}^{2B}}^2 = \|\eta + \tilde{\eta}\|_{\mathbb{C}^{2B}}^2 = \|\eta\|_{\mathbb{C}^{2B}}^2 + \|\tilde{\eta}\|_{\mathbb{C}^{2B}}^2 + 2 \Re \langle \eta, \tilde{\eta} \rangle_{\mathbb{C}^{2B}}. \tag{4.10}$$

Due to orthogonality of $\{e_1, \dots, e_n\}$,

$$\|\eta\|_{\mathbb{C}^{2B}}^2 = \sum_{j=1}^n |x_j|^2 \|e_j\|_{\mathbb{C}^{2B}}^2 = d \sum_{j=1}^n |x_j|^2 = d \|x\|_{\mathbb{C}^n}^2, \tag{4.11}$$

and similarly

$$\|\tilde{\eta}\|_{\mathbb{C}^{2B}}^2 = d \|y\|_{\mathbb{C}^n}^2. \tag{4.12}$$

We also have

$$\langle \eta, \tilde{\eta} \rangle_{\mathbb{C}^{2B}} = \sum_{i,j=1}^n x_i \bar{y}_j \langle e_i, \tilde{e}_j \rangle_{\mathbb{C}^{2B}}. \tag{4.13}$$

Now, since e_j is supported on outward pointing bonds from vertex j and \tilde{e}_i is supported on inward pointing bonds to vertex i , the only way that $\langle e_i, \tilde{e}_j \rangle_{\mathbb{C}^{2B}}$ can be non-zero is if i connects to j . We have, in fact,

$$\langle e_i, \tilde{e}_j \rangle_{\mathbb{C}^{2B}} = \begin{cases} 1, & \text{if } i \sim j, \\ 0, & \text{otherwise,} \end{cases} = C_{ij}, \tag{4.14}$$

so

$$\begin{aligned} \langle \eta, \tilde{\eta} \rangle_{\mathbb{C}^{2B}} &= \sum_{i,j=1}^n x_i \bar{y}_j C_{ij} \\ &= \langle x, C y \rangle_{\mathbb{C}^n}. \end{aligned} \tag{4.15}$$

We get,

$$\begin{aligned} \|\psi(\hat{x})\|_{\mathbb{C}^{2B}}^2 &= d \|x\|_{\mathbb{C}^n}^2 + d \|y\|_{\mathbb{C}^n}^2 + 2 \Re \langle x, C y \rangle_{\mathbb{C}^n} \\ &\leq d \|x\|_{\mathbb{C}^n}^2 + d \|y\|_{\mathbb{C}^n}^2 + 2 \|x\|_{\mathbb{C}^n} \|C y\|_{\mathbb{C}^n} \\ &\leq d \|x\|_{\mathbb{C}^n}^2 + d \|y\|_{\mathbb{C}^n}^2 + 2d \|x\|_{\mathbb{C}^n} \|y\|_{\mathbb{C}^n} \\ &= d (\|x\|_{\mathbb{C}^n} + \|y\|_{\mathbb{C}^n})^2, \end{aligned} \tag{4.16}$$

proving (4.9). □

Let \hat{C} be the $2n \times 2n$ matrix,

$$\hat{C} := \begin{pmatrix} 0 & \frac{-1}{d-1}I_n \\ I_n & \frac{1}{d-1}C \end{pmatrix}. \tag{4.17}$$

Proposition 4.3. *Let $f \in G_1$ and $t = 0, 1, 2, \dots$. Then,*

$$\psi(\hat{C}^t \tilde{\varphi}(f)) = M^t f. \tag{4.18}$$

Proof. Let $\hat{x} \in \mathbb{C}^{2n}$ with $g = \psi(\hat{x})$. We first prove that

$$\psi(\hat{C}\hat{x}) = Mg. \tag{4.19}$$

Indeed, suppose that

$$g = x_1 e_1 + \dots + x_n e_n + \tilde{x}_1 \tilde{e}_1 + \dots + \tilde{x}_n \tilde{e}_n, \tag{4.20}$$

so that

$$Mg = x_1 \tilde{e}_1 + \dots + x_n \tilde{e}_n + \frac{\tilde{x}_1}{d-1} \left(\sum_{j \sim 1} \tilde{e}_j - e_1 \right) + \dots + \frac{\tilde{x}_n}{d-1} \left(\sum_{j \sim n} \tilde{e}_j - e_n \right), \tag{4.21}$$

by Proposition 4.1. However, with $\mathbf{x} = (x_1, \dots, x_n)^T$, $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ and $\hat{\mathbf{x}} = (\mathbf{x}, \tilde{\mathbf{x}})^T$, we have

$$\hat{C}\hat{\mathbf{x}} = \begin{pmatrix} -\frac{\tilde{\mathbf{x}}}{d-1} \\ \mathbf{x} + \frac{C\tilde{\mathbf{x}}}{d-1} \end{pmatrix}, \tag{4.22}$$

from which we see that $\psi(\hat{C}\hat{\mathbf{x}}) = Mg$.

To prove (4.18) in the case $t = 0$, it is immediate to observe that $\psi(\tilde{\varphi}(f)) = f$, from the definitions of ψ and $\tilde{\varphi}$. If we assume that $\psi(\hat{C}^t \tilde{\varphi}(f)) = M^t f$, then applying (4.19) with $\hat{x} = \hat{C}^t \tilde{\varphi}(f)$ proves (4.18) for the $t + 1$ case. \square

Therefore, to understand the action of M on f , we need to understand the iterates of \hat{C} .

Let \mathbf{x} be an eigenvector of C with eigenvalue μ . Then,

$$\hat{C} \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}. \tag{4.23}$$

Denoting

$$\hat{C}^t \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} =: \begin{pmatrix} y_t \mathbf{x} \\ z_t \mathbf{x} \end{pmatrix}, \tag{4.24}$$

$$= \begin{pmatrix} -\frac{z_{t-1}}{d-1} \mathbf{x} \\ (y_{t-1} + \frac{\mu z_{t-1}}{d-1}) \mathbf{x} \end{pmatrix}. \tag{4.25}$$

So

$$y_t = -\frac{z_{t-1}}{d-1}, \tag{4.26}$$

and $z_t = z_t(\mu)$ is the solution to the recurrence

$$z_t = \frac{\mu z_{t-1}}{d-1} - \frac{z_{t-2}}{d-1}, \tag{4.27}$$

with initial conditions $z_0 = 0$ and $z_1 = 1$.

Proposition 4.4. *If $|\mu| \leq d - \beta < d$, then the solutions to (4.27) satisfy*

$$|z_t(\mu)| \leq t \left(\frac{d - 1 - \beta}{d - 1} \right)^{t-1}, \tag{4.28}$$

for $t = 1, 2, \dots$

Proof. We first consider the case $|\mu| \neq 2\sqrt{d-1}$. Standard methods yield the solution to (4.27) in this case to be given by

$$z_t = \frac{\sqrt{d-1}}{2\sqrt{\omega^2-1}} \left(\left(\frac{\omega + \sqrt{\omega^2-1}}{\sqrt{d-1}} \right)^t - \left(\frac{\omega - \sqrt{\omega^2-1}}{\sqrt{d-1}} \right)^t \right), \tag{4.29}$$

where $\omega = \frac{1}{2}\mu(d-1)^{-1/2} \neq \pm 1$. Since

$$\left(\frac{\omega + \sqrt{\omega^2-1}}{\sqrt{d-1}} \right) - \left(\frac{\omega - \sqrt{\omega^2-1}}{\sqrt{d-1}} \right) = 2 \frac{\sqrt{\omega^2-1}}{\sqrt{d-1}}, \tag{4.30}$$

we may apply the inequality

$$\left| \frac{a^t - b^t}{a - b} \right| \leq t \max\{|a|, |b|\}^{t-1}, \tag{4.31}$$

to (4.29), to get

$$|z_t| \leq t \max \left\{ \left| \frac{\omega + \sqrt{\omega^2-1}}{\sqrt{d-1}} \right|, \left| \frac{\omega - \sqrt{\omega^2-1}}{\sqrt{d-1}} \right| \right\}^{t-1}. \tag{4.32}$$

For $1 < \omega \leq \frac{1}{2}(d-\beta)(d-1)^{1/2}$, we have

$$\left| \frac{\omega \pm \sqrt{\omega^2-1}}{\sqrt{d-1}} \right| \leq \frac{\omega + \sqrt{\omega^2-1}}{(d-1)^{1/2}}. \tag{4.33}$$

For such values of ω ,

$$0 \leq \omega^2 - 1 \leq \frac{(d-\beta-2)^2 - 4\beta}{4(d-1)} < \frac{(d-\beta-2)^2}{4(d-1)}, \tag{4.34}$$

so that

$$\begin{aligned} \left| \frac{\omega \pm \sqrt{\omega^2-1}}{\sqrt{d-1}} \right| &< \frac{1}{2} \frac{d-\beta}{d-1} + \frac{d-2-\beta}{2(d-1)} \\ &= \frac{d-1-\beta}{d-1}. \end{aligned} \tag{4.35}$$

A similar argument holds if $-\frac{1}{2}(d-\beta)(d-1)^{1/2} \leq \omega < -1$.

If $|\omega| < 1$ then

$$\left| \frac{\omega \pm \sqrt{\omega^2-1}}{\sqrt{d-1}} \right| = \frac{1}{\sqrt{d-1}} \leq \frac{d-1-\beta}{d-1}. \tag{4.36}$$

Finally, in the case $|\mu| = 2\sqrt{d-1}$, directly solving (4.27), we find

$$|z_t(\mu)| = \frac{t}{(d-1)^{(t-1)/2}}. \tag{4.37}$$

□

Our main result of this subsection is the following:

Proposition 4.5. *Let $f \in G_1$ with $\text{Tr Op}(f) = 0$. If all non-trivial eigenvalues μ of C satisfy the bound $|\mu| < d - \beta$ then we have*

$$\|M^t f\|_{\mathbb{C}^{2B}} \leq 2\|f\|_{\mathbb{C}^{2B}t} \left(\frac{d-1-\beta}{d-1}\right)^{t-1}, \tag{4.38}$$

for $t = 1, 2, 3, \dots$

Proof. Let x_1, \dots, x_n be an orthonormal basis of eigenvectors of C , where $x_1 = n^{-1/2}e$. Since $\text{Tr Op}(f) = 0$ we have $\langle \varphi(f), e \rangle_{\mathbb{C}^n} = 0$ and we can write

$$\varphi(f) = \alpha_2 x_2 + \dots + \alpha_n x_n, \tag{4.39}$$

where

$$\|\varphi(f)\|_{\mathbb{C}^n}^2 = |\alpha_2|^2 + \dots + |\alpha_n|^2. \tag{4.40}$$

It is also easy to see that

$$\|f\|_{\mathbb{C}^{2B}}^2 = d\|\varphi(f)\|_{\mathbb{C}^n}^2. \tag{4.41}$$

We have

$$\tilde{\varphi}(f) = \alpha_2 \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} x_n \\ 0 \end{pmatrix} \in \mathbb{C}^{2n}. \tag{4.42}$$

So, if $t = 1, 2, 3, \dots$,

$$\hat{C}^t \tilde{\varphi}(f) = \frac{\alpha_2}{d-1} \begin{pmatrix} -z_{t-1}(\mu_2)x_2 \\ (d-1)z_t(\mu_2)x_2 \end{pmatrix} + \dots + \frac{\alpha_n}{d-1} \begin{pmatrix} -z_{t-1}(\mu_n)x_n \\ (d-1)z_t(\mu_n)x_n \end{pmatrix}, \tag{4.43}$$

using (4.24).

We now use Lemma 4.2 to get

$$\begin{aligned} \|M^t f\|_{\mathbb{C}^{2B}} &= \left\| \psi \left(\hat{C}^t \tilde{\varphi}(f) \right) \right\|_{\mathbb{C}^{2B}} \\ &\leq \sqrt{d} \left(\left(\sum_{j=2}^n |\alpha_j|^2 \frac{|z_{t-1}(\mu_j)|^2}{(d-1)^2} \right)^{1/2} + \left(\sum_{j=2}^n |\alpha_j|^2 |z_t(\mu_j)|^2 \right)^{1/2} \right) \\ &\leq \sqrt{d} \left(\sum_{j=2}^n |\alpha_j|^2 \right)^{1/2} \frac{(t-1)(d-1-\beta)^{t-2} + t(d-1-\beta)^{t-1}}{(d-1)^{t-1}}, \end{aligned}$$

using Proposition 4.4,

$$\leq 2\|f\|_{\mathbb{C}^{2B}t} \left(\frac{d-1-\beta}{d-1}\right)^{t-1}. \tag{4.44}$$

□

4.3. General Mean-Zero Observables

We would like to consider a more general class of observables than those belonging to the spaces G_1 . Let $G_2 = G_1^\perp$. Thus, G_2 is the space of singular vectors of M with singular value $\frac{1}{d-1}$, and it follows that M acts on G_2 by contraction:

Lemma 4.6. *Let $g \in G_2$. Then,*

$$\|Mg\|_{\mathbb{C}^{2B}} = \frac{\|g\|_{\mathbb{C}^{2B}}}{d-1}$$

Proof. Take $g \in G_2$. Then, we have

$$\|Mg\|_{\mathbb{C}^{2B}}^2 = \langle Mg, Mg \rangle_{\mathbb{C}^{2B}} = \langle g, M^T Mg \rangle_{\mathbb{C}^{2B}} \tag{4.45}$$

$$= \frac{\|g\|_{\mathbb{C}^{2B}}^2}{(d-1)^2}. \tag{4.46}$$

□

Finally, let us put the results of Proposition 4.5 and Lemma 4.6 together.

Theorem 4.7. *Let $f \in \mathbb{C}^{2B}$ with $\text{Tr Op}(f) = 0$. Assume that all non-trivial eigenvalues μ of the connectivity matrix satisfy $|\mu| \leq d - \beta$ for some $\beta > 0$. Then, there exists a constant $K > 0$ (that does not depend on t or B) such that for $t \geq 1$,*

$$\|M^t f\|_{\mathbb{C}^{2B}} < K \|f\|_{\mathbb{C}^{2B}} t \left(\frac{d-1-\beta}{d-1} \right)^t. \tag{4.47}$$

Proof. The conditions of the theorem guarantee that

$$\|M^j g\|_{\mathbb{C}^{2B}} \leq 2 \|g\|_{\mathbb{C}^{2B}} j \left(\frac{d-1-\beta}{d-1} \right)^{j-1} \tag{4.48}$$

for any $g \in G_1$ for $j \in \mathbb{N}$, due to Proposition 4.5.

We decompose f according to $\mathbb{C}^{2B} = G_1 \oplus G_2$ as $f = f_{0,1} + f_{0,2}$, where $f_{0,1} \in G_1$ and $f_{0,2} \in G_2$, and inductively defining the sequences $\{f_{j,1}\}_{j=0}^\infty$ and $\{f_{j,2}\}_{j=0}^\infty$ by

$$Mf_{j,2} = f_{j+1,1} + f_{j+1,2}, \tag{4.49}$$

where $f_{j,i} \in G_i$ for $j \geq 0, i \in \{1, 2\}$. So upon each iteration, the component of $f_{j,2}$ that does not remain in G_2 becomes $f_{j+1,1} \in G_1$.

We have, by Lemma 4.6

$$\frac{\|f_{j,2}\|_{\mathbb{C}^{2B}}^2}{(d-1)^2} = \|Mf_{j,2}\|_{\mathbb{C}^{2B}}^2 = \|f_{j+1,1}\|_{\mathbb{C}^{2B}}^2 + \|f_{j+1,2}\|_{\mathbb{C}^{2B}}^2 \geq \|f_{j+1,i}\|_{\mathbb{C}^{2B}}^2, \tag{4.50}$$

for $i \in \{1, 2\}$, so, inductively,

$$\|f_{j,2}\|_{\mathbb{C}^{2B}} \leq \frac{\|f_{0,2}\|_{\mathbb{C}^{2B}}}{(d-1)^j} \leq \frac{\|f\|_{\mathbb{C}^{2B}}}{(d-1)^j}, \tag{4.51}$$

and

$$\|f_{j,1}\|_{\mathbb{C}^{2B}} \leq \frac{\|f_{j-1,2}\|_{\mathbb{C}^{2B}}}{d-1} \leq \frac{\|f\|_{\mathbb{C}^{2B}}}{(d-1)^j}. \tag{4.52}$$

Acting on f iteratively, we have

$$\begin{aligned}
 M^t f &= M^t f_{0,1} + M^t f_{0,2} \\
 &= M^t f_{0,1} + M^{t-1} f_{1,1} + M^{t-1} f_{1,2} \\
 &\vdots \\
 &= \sum_{j=0}^{t-1} M^{t-j} f_{j,1} + f_{t,1} + f_{t,2}.
 \end{aligned}
 \tag{4.53}$$

Thus, we have, using (4.51) and (4.52), and (4.48),

$$\begin{aligned}
 \|M^t f\|_{\mathbb{C}^{2B}} &\leq \sum_{j=0}^{t-1} \|M^{t-j} f_{j,1}\|_{\mathbb{C}^{2B}} + \|f_{t,1}\|_{\mathbb{C}^{2B}} + \|f_{t,2}\|_{\mathbb{C}^{2B}} \\
 &\leq 2 \sum_{j=0}^{t-1} (t-j) \left(\frac{d-1-\beta}{d-1}\right)^{t-j-1} \|f_{j,1}\|_{\mathbb{C}^{2B}} + \frac{2\|f\|_{\mathbb{C}^{2B}}}{(d-1)^t} \\
 &\leq 2 \sum_{j=0}^{t-1} (t-j) \left(\frac{d-1-\beta}{d-1}\right)^{t-j-1} \frac{\|f\|_{\mathbb{C}^{2B}}}{(d-1)^j} + \frac{2\|f\|_{\mathbb{C}^{2B}}}{(d-1)^t} \\
 &= \frac{2\|f\|_{\mathbb{C}^{2B}}}{(d-1)^{t-1}} \sum_{r=1}^t r(d-1-\beta)^{r-1} + \frac{2\|f\|_{\mathbb{C}^{2B}}}{(d-1)^t} \quad \text{via } r = t-j, \\
 &\leq 2t\|f\|_{\mathbb{C}^{2B}}(d-1) \frac{(d-1-\beta)^t - 1}{(d-1)^t(d-2-\beta)} + \frac{2\|f\|_{\mathbb{C}^{2B}}}{(d-1)^t} \\
 &\leq \frac{2t\|f\|_{\mathbb{C}^{2B}}(d-1)}{(d-2-\beta)} \left(\frac{d-1-\beta}{d-1}\right)^t + \frac{2\|f\|_{\mathbb{C}^{2B}}}{(d-1)^t}.
 \end{aligned}
 \tag{4.54}$$

Finally (to combine the two terms into a single), noting that

$$\frac{t(d-1)(d-1-\beta)^t}{d-2-\beta} \geq 4,
 \tag{4.55}$$

for $t \geq 1$, we get

$$\|M^t f\|_{\mathbb{C}^{2B}} \leq \frac{5(d-1)}{2(d-2-\beta)} \|f\|_{\mathbb{C}^{2B}} t \left(\frac{d-1-\beta}{d-1}\right)^t,
 \tag{4.56}$$

for $t \geq 1$.

□

5. Quantum Ergodicity for Equi-transmitting Expander Graphs

We are now able to prove Theorem 2.2. To begin with, we need a few results on certain summations.

Lemma 5.1. *Let $\theta \neq 1$ and $T > 0$. Then,*

$$\sum_{t=1}^T t\theta^t = \frac{T\theta^{T+2} + \theta - (T+1)\theta^{T+1}}{(\theta-1)^2}, \tag{5.1}$$

and consequently, if $|\theta| < 1$,

$$\sum_{t=1}^{\infty} t\theta^t = \frac{\theta}{(\theta-1)^2}. \tag{5.2}$$

Proof. The sum appearing in (5.1) is of a standard type (see formula **0.113** of [26], or [27, p. 33] for a derivation). Equation (5.2) follows by letting $T \rightarrow \infty$. \square

Lemma 5.2. *Let $T > 0$ and*

$$\hat{w}_T(t) := \begin{cases} \frac{1}{T} \left(1 - \frac{|t|}{T}\right), & |t| < T, \\ 0, & \text{otherwise,} \end{cases} \tag{5.3}$$

and $\theta \neq 1$. Then,

$$\sum_{t=1}^T \theta^t \hat{w}_T(t) = \frac{\theta}{T^2} \left(\frac{T-1 + \theta^T - T\theta}{(1-\theta)^2} \right). \tag{5.4}$$

Proof. We reverse the order of summation, to get

$$\begin{aligned} \sum_{t=1}^T \theta^t \hat{w}_T(t) &= \sum_{k=0}^{T-1} \theta^{T-k} \hat{w}_T(T-k), \quad \text{via } k = T-t, \\ &= \frac{\theta^T}{T} \sum_{k=0}^{T-1} \theta^{-k} \left(1 - \frac{T-k}{T}\right) \\ &= \frac{\theta^T}{T^2} \sum_{k=0}^{T-1} k\theta^{-k}. \end{aligned} \tag{5.5}$$

To evaluate the sum in (5.5), we use Lemma 5.1. The result is

$$\begin{aligned} \sum_{t=1}^T \theta^t \hat{w}_T(t) &= \frac{\theta^T}{T^2} \left(\frac{(T-1)\theta^{-T-1} + \theta^{-1} - T\theta^{-T}}{(\theta^{-1}-1)^2} \right) \\ &= \frac{\theta}{T^2} \left(\frac{T-1 + \theta^T - T\theta}{(1-\theta)^2} \right). \end{aligned} \tag{5.6}$$

\square

Proof Theorem 2.2. We recall equation (3.20) which provides the main estimate for $V(f, B)$:

$$\begin{aligned} V(f, B) &\leq \frac{1}{2BT} \text{Tr Op}(f)^2 \\ &\quad + \frac{1}{B} \sum_{t=1}^T \hat{w}_T(t) \left(\langle f, M^t f \rangle_{\mathcal{C}^{2B}} + O(\kappa^2(d-1)^t |\mathcal{C}_{B,2T}|) \right). \end{aligned} \tag{5.7}$$

To estimate the error term, we have the sum

$$\sum_{t=1}^T \hat{w}_T(t)(d-1)^t = \frac{d-1}{T^2} \left(\frac{T-1 + (d-1)^T - T(d-1)}{(d-2)^2} \right), \tag{5.8}$$

by Lemma 5.2. Since $d \geq 3$, we can be sure that

$$T-1 - T(d-1) < 0, \tag{5.9}$$

so

$$\sum_{t=1}^T \hat{w}_T(t)(d-1)^t < \frac{d-1}{(d-2)^2} \frac{(d-1)^T}{T^2}. \tag{5.10}$$

The first term of (5.7) is easy to bound: since $\text{Tr Op}(f)^2 \leq 2B\kappa^2$ if $|f_b| \leq \kappa$, we have

$$\frac{\text{Tr Op}(f)^2}{2BT} \leq \frac{\kappa^2}{T}. \tag{5.11}$$

The final step needed is to bound

$$\sum_{t=1}^T \hat{w}_T(t) \langle f, M^t f \rangle_{\mathbb{C}^{2B}}, \tag{5.12}$$

where, by Theorem 4.7,

$$|\langle f, M^t f \rangle_{\mathbb{C}^{2B}}| \leq \|f\|_{\mathbb{C}^{2B}} \|M^t f\|_{\mathbb{C}^{2B}} \leq K \|f\|_{\mathbb{C}^{2B}}^2 \left(\frac{d-1-\beta}{d-1} \right)^t, \tag{5.13}$$

for some constant K . Since $\hat{w}_T(t) \leq T^{-1}$ for all t , we can estimate

$$\begin{aligned} \sum_{t=1}^T \hat{w}_T(t) t \left(\frac{d-1-\beta}{d-1} \right)^t &\leq \frac{1}{T} \sum_{t=1}^{\infty} t \left(\frac{d-1-\beta}{d-1} \right)^t \\ &= \frac{1}{T} \frac{\left(\frac{d-1-\beta}{d-1} \right)}{\left(1 - \frac{d-1-\beta}{d-1} \right)^2} \\ &= \frac{1}{T} \frac{(d-1)(d-1-\beta)}{\beta^2}, \end{aligned} \tag{5.14}$$

making use of Lemma 5.1. As $\|f\|_{\mathbb{C}^{2B}}^2 \leq 2B\kappa^2$, we end up with

$$\sum_{t=1}^T \hat{w}_T(t) \langle f, M^t f \rangle_{\mathbb{C}^{2B}} = O\left(\frac{B\kappa^2}{T\beta^2} \right), \tag{5.15}$$

and combining with the other two bounds:

$$V(f, B) = O\left(\frac{\kappa^2}{T\beta^2} \right) + O\left(\frac{\kappa^2(d-1)^T |\mathcal{C}_{B,2T}|}{BT^2} \right). \tag{5.16}$$

□

Remark 5.3. *By applying the Hölder inequality, we may derive the alternative bound to (5.15):*

$$\sum_{t=1}^T \hat{w}_T(t) \langle f, M^t f \rangle_{\mathbb{C}^{2B}} = O\left(\frac{B\kappa^2}{T^{1-1/q}\beta^{1+1/p}}\right), \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad (5.17)$$

which may be useful to prove quantum ergodicity in cases of families of graphs for which $\beta \rightarrow 0$ in a slow way as $B \rightarrow \infty$.

6. Examples

In this section, we provide two examples of families of graphs for which our results prove quantum ergodicity when quantised with equi-transmitting scattering matrices.

6.1. Random Regular Graphs

There exist several models for choosing a regular graph on n vertices at random [49]. We shall consider the set $\mathcal{G}_{n,d}$ of simple, d -regular graphs on n labelled vertices. It follows from [4, 12] that the size of $\mathcal{G}_{n,d}$ obeys

$$|\mathcal{G}_{n,d}| \sim \sqrt{2}e^{(1-d^2)/4} \left(\frac{d^d n^d}{e^d (d!)^2}\right)^{n/2}, \quad \text{as } n \rightarrow \infty. \quad (6.1)$$

We make $\mathcal{G}_{n,d}$ into an ensemble of random graphs by assigning uniform probability to each element [14, 49]. These graphs are bipartite with probability $o(1)$ as $n \rightarrow \infty$, and connected (*a fortiori* d -connected) with probability $1 - o(1)$ [13].

To use Theorem 2.2 to prove that quantum ergodicity holds with probability $1 - o(1)$, we collect together some prior results showing that such random graphs are expanders, and that they do not have too many short cycles.

Random regular graphs are known to be almost Ramanujan, due to a result of Friedmann [21], which had been conjectured by Alon [1]: for any $\varepsilon > 0$, with probability $1 - o(1)$ a random d -regular graph has all non-trivial eigenvalues μ_i of its connectivity matrix bounded by $|\mu_i| \leq 2\sqrt{d-1} + \varepsilon$. This means that we can take any $\beta < d - 2\sqrt{d-1}$ in Theorem 2.2. A weaker bound valid for d even, with a simpler proof, has been given in [41], that would also serve our purpose for those values of d .

For the number of short cycles in a random d -regular graph, such questions have been considered in [39]. For d fixed, theorem 4 of [39] reads:

Theorem 6.1. *Let $k = k(n) \geq 3$ satisfy $k(d-1)^{k-1} = o(n)$. Let $S = S(n) = 20Ak(d-1)^k$ with $A = A(n) > c$ for some constant $c > 1$. The probability that the random d -regular graph on n vertices has exactly S edges which lie on cycles of length at most k is less than*

$$e^{-5(d-1)^k} \left(\frac{e}{A}\right)^{S/4k}. \quad (6.2)$$

To apply Theorem 6.1 to our situation, let $k = \frac{3}{5} \log_{d-1} n$, and $S_0 = \lfloor 42n^{3/5} \log_{d-1} n \rfloor$ and define $A(n)$ by

$$S_0 = 12A(n)n^{3/5} \log_{d-1} n. \tag{6.3}$$

Then, for n sufficiently large

$$3 < A(n) \leq 3.5, \tag{6.4}$$

and

$$k(d-1)^{k-1} = \frac{3 \log_{d-1} n}{5(d-1)} n^{3/5} = o(n), \tag{6.5}$$

so that the conditions of Theorem 6.1 are satisfied.

The probability that a random d -regular graph has *at least* S_0 edges which lie on cycles of length at most k is, according to Theorem 6.1, not more than

$$\begin{aligned} \sum_{S=S_0}^{\infty} e^{-5(d-1)^k} \left(\frac{e}{A}\right)^{S/4k} &\leq e^{-5(d-1)^k} \\ &= e^{-5n^{3/5}}, \end{aligned} \tag{6.6}$$

for n sufficiently large, since $e/A < 1$.

Let

$$T = \frac{3}{10} \log_{d-1} n = \frac{3}{10} \log_{d-1} B - \frac{3}{10} \log_{d-1} \left(\frac{d}{2}\right). \tag{6.7}$$

Then, (6.6) implies that

$$\mathbb{P} \left(|\mathcal{C}_{B,2T}| \leq 42 \left(\frac{2B}{d}\right)^{3/5} \log_{d-1} \left(\frac{2B}{d}\right) \right) \geq 1 - e^{-5(2B/d)^{3/5}}, \tag{6.8}$$

that is, with extremely high probability. We can then say that

$$\begin{aligned} &\mathbb{P} \left(\frac{(d-1)^T}{BT^2} |\mathcal{C}_{B,2T}| \rightarrow 0 \right) \\ &\geq \mathbb{P} \left(\frac{(d-1)^T}{BT^2} |\mathcal{C}_{B,2T}| \leq \frac{1400}{3} \left(\frac{2}{d}\right)^{9/10} \frac{1}{\log_{d-1}(2B/d)} B^{-1/10} \right) \\ &= \mathbb{P} \left(|\mathcal{C}_{B,2T}| \leq 42 \left(\frac{2B}{d}\right)^{3/5} \log_{d-1} \left(\frac{2B}{d}\right) \right) \\ &\geq 1 - e^{-5(2B/d)^{3/5}} \end{aligned} \tag{6.9}$$

by (6.8).

So with T given by (6.7), and with probability $1 - o(1)$, the right-hand side of (2.16) converges to 0 as $B \rightarrow \infty$, leading to quantum ergodicity for a sequence of random d -regular graphs quantised with equi-transmitting scattering matrices.

6.2. Ramanujan Graphs with Large Girth

As we have stated in Sect. 2.2, the largest theoretical value for which the parameter β can be taken in Theorem 2.2 is $\beta = d - 2\sqrt{d - 1}$, and such graphs are called Ramanujan.

Infinite families of Ramanujan graphs with $B \rightarrow \infty$ have been constructed for certain values of d only: for $d = 3$ in [16], for $d = p + 1$ where p is an odd prime in [35, 37] and, more generally, for d any prime power in [40]. An existence proof for *bipartite* Ramanujan graphs for all values of d has recently been given in [36].

We will take $d = q + 1$, where q is a prime power. It is known that equi-transmitting scattering matrices of size $d \times d$ do exist. In [40], a method of constructing non-bipartite, connected, d -regular, Ramanujan graphs on n vertices for a growing sequence of ns is given. Furthermore, it is proved that the girth $g(B)$ of such graphs satisfies

$$g(B) \geq \frac{2}{3} \log_{d-1} n = \frac{2}{3} \log_{d-1} \left(\frac{B}{2d} \right). \tag{6.10}$$

This girth estimate shows that these graphs are close to extremal, since the Moore bound [11, Ch. 23] gives a theoretical upper bound of

$$2 \log_{d-1} n(1 + o(1)), \tag{6.11}$$

for the girth of a d -regular graph on n vertices. In the case that q is a prime, the upper bound (6.10) has been shown to be an asymptotic equality [10].

Since the girth $g(B) \rightarrow \infty$ as $B \rightarrow \infty$, we can take any $T < \frac{1}{2}g(B)$ in Theorem 2.2, and the Ramanujan property shows that the first term on the right-hand side of (2.16) can be made arbitrarily small as $B \rightarrow \infty$. With this choice of T , however, it is also clear that $|C_{B,2T}| = 0$, so the second term on the right-hand side of (2.16) is absent, proving quantum ergodicity for these d -regular Ramanujan graphs quantised with equi-transmitting scattering matrices.

We end with two remarks concerning quantum ergodicity for equi-transmitting Ramanujan graphs: if the bond lengths of the quantum graph are linearly independent over \mathbb{Q} , then we can push T up to $g(B) - \varepsilon$. Second, the effective logarithmic estimate for the decay of quantum variance

$$V(f, B) = O\left(\frac{1}{\log B} \right) \tag{6.12}$$

is of the same order as can be rigorously proved in other systems [19, 44, 45, 52], but the calculations performed in [25] suggest that the true rate of decay should be algebraic (which is also consistent with what is conjectured for more general chaotic quantum systems [20, 42]). A decay rate of $1/B$ has been proved for the quantum variance for the graphs studied in [5], but this result aside, going rigorously beyond the logarithmic barrier (6.12) seems to be for quantum graphs, as with other systems, a difficult problem.

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Appendix A. Matrix Inequality

Let $T \in \mathbb{N}$. We recall that we defined

$$\hat{w}_T(t) := \begin{cases} \frac{1}{T} \left(1 - \frac{|t|}{T}\right), & |t| < T, \\ 0, & \text{otherwise.} \end{cases} \tag{A.1}$$

It is elementary to calculate the inverse Fourier transform of $\hat{w}_T(t)$, showing that

$$\int_{-T}^T \hat{w}_T(t) e^{2\pi i t x} dt = w_T(x) := \begin{cases} 2 \left(\frac{1 - \cos Tx}{T^2 x^2}\right), & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{A.2}$$

For our purposes, it suffices to note that w_T is everywhere non-negative and $w_T(0) = 1$, for all T .

The following lemma is taken from [45], p. 1463:

Lemma A.1. *Let U be an $N \times N$ unitary matrix, $\{u_j\}_{j=1}^N$ be an arbitrary orthonormal basis of U , and A be an $N \times N$ matrix. Then, if $T \in \mathbb{N}$, we have*

$$\frac{1}{N} \sum_{j=1}^N |\langle u_j, Au_j \rangle_{\mathbb{C}^N}|^2 \leq \frac{1}{N} \sum_{t=-T}^T \hat{w}_T(t) \text{Tr}(A^* U^t A U^{-t}). \tag{A.3}$$

Proof. Let us denote by θ_j the eigenphases of U , so that

$$U u_j = e^{2\pi i \theta_j} u_j. \tag{A.4}$$

By expanding the trace, we can write

$$\begin{aligned} \text{Tr}(A^* U^t A U^{-t}) &= \sum_{j=1}^N \langle u_j, A^* U^t A U^{-t} u_j \rangle_{\mathbb{C}^N} \\ &= \sum_{j=1}^N e^{-2\pi i \theta_j t} \langle Au_j, U^t Au_j \rangle_{\mathbb{C}^N}. \end{aligned} \tag{A.5}$$

By inserting the representation,

$$Au_j = \sum_{k=1}^N \langle u_k, Au_j \rangle_{\mathbb{C}^N} u_k, \tag{A.6}$$

we get

$$\text{Tr}(A^*U^tAU^{-t}) = \sum_{j,k=1}^N e^{2\pi i(\theta_k - \theta_j)t} |\langle u_k, Au_j \rangle_{\mathbb{C}^N}|^2. \tag{A.7}$$

We multiply (A.7) by $\hat{w}_T(t)$ and sum over all t , invoking the Poisson summation formula to get

$$\sum_{t=-T}^T \hat{w}_T(t) \text{Tr}(A^*U^tAU^{-t}) = \sum_{n=-\infty}^{\infty} \sum_{j,k=1}^N w_T(n + \theta_j - \theta_k) |\langle u_k, Au_j \rangle_{\mathbb{C}^N}|^2 \tag{A.8}$$

$$\geq w_T(0) \sum_{j=1}^N |\langle u_j, Au_j \rangle_{\mathbb{C}^N}|^2, \tag{A.9}$$

retaining the $j = k$ and $n = 0$ terms of the sums only. □

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Matthew Brammall
Helix Building
Kelvin Campus
West of Scotland Science Pk.
Glasgow, G20 0SP, United Kingdom
e-mail: brammall@theory.bham.ac.uk

Brian Winn
Department of Mathematical Sciences
Loughborough University
Loughborough, LE11 3TU, United Kingdom
e-mail: b.winn@loughborough.ac.uk

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