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Spacelike Spherically Symmetric CMC Foliation in the Extended Schwarzschild Spacetime

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Abstract. We first summarize the characterization of smooth spacelike spherically symmetric constant mean curvature (SS-CMC) hypersurfaces in the Schwarzschild spacetime and Kruskal extension. Then use the characterization to prove special SS-CMC foliation property, and verify part of the conjecture by Malec and Ó Murchadha (Phys Rev D (3) 68:124019, 2003).

1. Introduction

Spacelike constant mean curvature (S-CMC) hypersurfaces in spacetimes are important and interesting objects in general relativity. From the geometric point of view, S-CMC hypersurfaces in spacetimes are critical points of the surface area functional with fixed enclosed volume [3]. Local maximal properties of S-CMC hypersurfaces in some conditions are proved by Brill and Flaherty [2]. These characterizations are similar to those of compact CMC hypersurfaces in Euclidean spaces.

Brill, Cavallo, and Isenberg considered spacelike spherically symmetric constant mean curvature (SS-CMC) hypersurfaces in static spacetimes, especially in Schwarzschild spacetimes [3]. From the variational principle, the CMC equation is derived. They gave descriptions of the behavior of SS-CMC hypersurfaces in the Schwarzschild spacetime through numerical integration and effective potential.

Among issues of CMC hypersurfaces, CMC foliations are important in understanding spacetimes and relativistic cosmology because York [10] suggested the concept of the CMC time functions on spacetimes. Marsden and Tipler considered the existence and uniqueness of CMC Cauchy hypersurface foliations with mean curvature as a parameter in spatially closed universes or asymptotically flat spacetimes [8]. In the paper [3], Brill–Cavallo–Isenberg conjectured that a complete CMC foliation in the extended Schwarzschild spacetime with mean curvature varied for all values can be obtained. This conjecture is answered by Eardly and Smarr on the existence in [4], and Pervez, Qadir, and Siddiqui gave a procedure which possibly produce an explicit construction with numerical evidence [9].

Malec and Ó Murchadha also considered SS-CMC hypersurfaces and CMC foliations in the Schwarzschild spacetime [6,7]. Their idea is viewing the Einstein equation as a dynamical system, then the Hamiltonian and momentum constraints give the formula of the second fundamental form of SS-CMC hypersurfaces. Through the analysis of the lapse function and the mean curvature of spherical two-surfaces, they can characterize the behavior of SS-CMC hypersurfaces. In addition, they suggested two types of SS-CMC foliation. In [6], they conjectured that the extended Schwarzschild spacetime can be foliated by a family of SS-CMC hypersurfaces with fixed mean curvature but varied another parameter. In [7], they described another SS-CMC foliation with varied mean curvature and the parameter. Both CMC foliations have phenomenon of exponentially collapsing lapse.

In this paper, we investigate the SS-CMC foliation property with fixed mean curvature and partially answer the conjecture posted by Malec and Ó Murchadha in [6]. To achieve the goal, detail study on properties of SS-CMC hypersurfaces in the Schwarzschild spacetime and Kruskal extension is necessary. Before aware of the work of Malec and Ó Murchadha in [6], we characterize all smooth SS-CMC hypersurfaces in the Kruskal extension from different points of view in [5]. Our proof of the SS-CMC foliation property highly depends on the explicit formulation obtained in [5]. For the reader's reference, we first summarize related results on the smooth SS-CMC hypersurfaces in the Schwarzschild spacetime and Kruskal extension in Sect. 2.

In Sect. 3, we concentrate on the foliation of SS-CMC family with T-axisymmetry in the Kruskal extension, which is abbreviated by TSS-CMC for convenience. We explain and reformulate the TSS-CMC foliation conjecture in Sect. 3.1, and then derive criteria for TSS-CMC family being disjoint in Sect. 3.2. These criteria are used to show that the TSS-CMC foliation holds in the Kruskal extension region II and II' as in Theorems 3.11 and 3.16. For the foliations in region I and I' with nonzero mean curvature, the estimates of the criteria are more subtle. We test some crucial cases by the numerical integrations and it looks that TSS-CMC foliation might hold in general. As the analysis for this part is technically more involving, we leave the investigation to the future.

When mean curvature is zero, the surface is called maximal hypersurface. In this case, it is much easier to prove the foliation property in region I and I' and we show that TSS-maximal hypersurface form a foliation in the whole Kruskal extension. This result was first proved in [1] by different arguments.

2. Preliminary

The Schwarzschild spacetime is a 4-dimensional time-oriented Lorentzian manifold with metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2.$$

After coordinates change, the Schwarzschild metric can be written as

$$\begin{aligned}\mathrm{d}s^2 &= \frac{16M^2\mathrm{e}^{-\frac{r}{2M}}}{r}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \\ &= \frac{16M^2\mathrm{e}^{-\frac{r}{2M}}}{r}\,\mathrm{d}U\mathrm{d}V + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2,\end{aligned} \tag{2.1}$$

where

$$\begin{cases}(r-2M)\,\mathrm{e}^{\frac{r}{2M}} = X^2 - T^2 = VU \\ \dfrac{t}{2M} = \ln\left|\dfrac{X+T}{X-T}\right| = \ln\left|\dfrac{V}{U}\right|.\end{cases} \tag{2.2}$$

It shows that $r = 2M$ is only a coordinate singularity. The Schwarzschild spacetime has a maximal analytic extension, called the Kruskal extension. It is the union of regions I, II, I', and II', where regions I and II correspond to the exterior and interior of one Schwarzschild spacetime, respectively, and regions I' and II' correspond to the exterior and interior of the other Schwarzschild spacetime. Figure 1 points out their correspondences and coordinate systems (X, T) or (U, V).

When $r = 2M$, relation (2.2) implies $U = 0$ or $V = 0$. Furthermore, the solution of (2.2) with $r = 2M$ and any finite value t is $(U, V) = (0, 0)$, so the origin of the Kruskal extension correspond to all points of $r = 2M$ and t finite value (although these points are not defined in the Schwarzschild spacetime). Similarly, if $t \to \infty$ (or $-\infty$), then (2.2) gives $U = 0$ (or $V = 0$). Hence V-axis correspond to all points $r = 2M$ and $t = \infty$, and U-axis correspond to all points $r = 2M$ and $t = -\infty$.

Sometimes we will use null coordinates (u, v) by

$$u = t - (r + 2M\ln|r - 2M|) \quad \text{and} \quad v = t + (r + 2M\ln|r - 2M|), \tag{2.3}$$

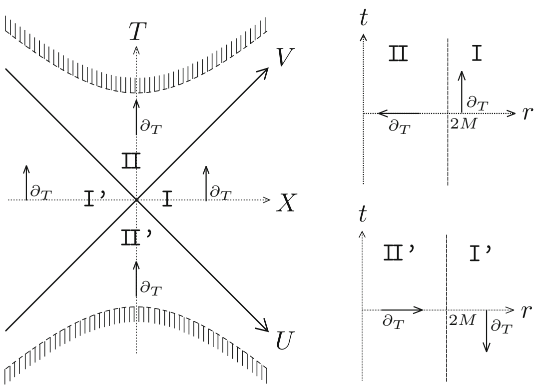

FIGURE 1. The Kruskal extension of Schwarzschild spacetimes

and relations between (U, V) and (u, v) are given by

	region I	region II	region I'	region II'
U	$\mathrm{e}^{-\frac{u}{4M}}$	$-\mathrm{e}^{-\frac{u}{4M}}$	$-\mathrm{e}^{-\frac{u}{4M}}$	$\mathrm{e}^{-\frac{u}{4M}}$
V	$\mathrm{e}^{\frac{v}{4M}}$	$\mathrm{e}^{\frac{v}{4M}}$	$-\mathrm{e}^{\frac{v}{4M}}$	$-\mathrm{e}^{\frac{v}{4M}}$.

In this article, we will take ∂_T as a future directed timelike vector field. Note that ∂_T in the two Schwarzschild spacetimes has different directions and it is indicated in Fig. 1.

In the following subsections, we will summarize the results of spacelike spherically symmetric constant mean curvature (SS-CMC for short) hypersurfaces in Schwarzschild spacetimes and Kruskal extension. These formulae and arguments are useful when dealing with CMC foliation problem. We refer to our article in ArXiv [5] for more details.

2.1. SS-CMC Solutions in Region I and I'

To understand SS-CMC hypersurfaces in the Kruskal extension, one can study and analyze SS-CMC solutions in the Schwarzschild spacetime first, and then discuss their images in the Kruskal extension.

First, we consider SS-CMC hypersurfaces in the Schwarzschild exterior, which map to the region I or I' in the Kruskal extension. Since a spacelike hypersurface in the Schwarzschild exterior can always be written as a graph of r, θ, and ϕ, particularly, an SS-CMC hypersurface in the Schwarzschild exterior is a graph of $f(r)$.[1] We use subscripts f_1 and f_3 to represent SS-CMC hypersurfaces that map to region I and I', and leave subscripts f_2 and f_4 for SS-CMC hypersurfaces that map to region II and II'.

For $f_1(r)$, the constant mean curvature equation is

$$f_1'' + \left(\left(\frac{1}{h} - (f_1')^2 h\right)\left(\frac{2h}{r} + \frac{h'}{2}\right) + \frac{h'}{h}\right) f_1' - 3H\left(\frac{1}{h} - (f_1')^2 h\right)^{\frac{3}{2}} = 0,$$

where $h(r) = 1 - \frac{2M}{r}$ and H is the constant mean curvature.[2] This is a second order ordinary differential equation, and we can solve the equation as follows:

Proposition 2.1. [5] *Suppose $\Sigma^1 = (f_1(r), r, \theta, \phi)$ is an SS-CMC hypersurface in the Schwarzschild exterior (corresponding to region* I*) with constant mean curvature H. Then*

$$f_1(r; H, c_1, \bar{c}_1) = \int_{r_1}^{r} \frac{1}{h(x)} \frac{l_1(x; H, c_1)}{\sqrt{1 + l_1^2(x; H, c_1)}}\,\mathrm{d}x + \bar{c}_1,$$

where

$$l_1(r; H, c_1) = \frac{1}{\sqrt{h(r)}}\left(Hr + \frac{c_1}{r^2}\right).$$

Here c_1 and $\bar{c}_1$ are constants, and $r_1 \in (2M, \infty)$ is fixed.

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[1] The function $f(r)$ corresponds to the height function $h(r)$ in paper [6].

[2] In paper [6], they use the terminology extrinsic curvature K, and the relation is $H = \frac{K}{3}$.

Similarly, the constant mean curvature equation of an SS-CMC hypersurface $\Sigma^3 = (f_3(r), r, \theta, \phi)$ (corresponding to region `I'`) is

$$f_3'' + \left(\left(\frac{1}{h} - (f_3')^2 h\right)\left(\frac{2h}{r} + \frac{h'}{2}\right) + \frac{h'}{h}\right) f_3' + 3H\left(\frac{1}{h} - (f_3')^2 h\right)^{\frac{3}{2}} = 0, \quad (2.4)$$

and the solution of Eq. (2.4) is the following:

Proposition 2.2. [5] *Suppose $\Sigma^3 = (f_3(r), r, \theta, \phi)$ is an SS-CMC hypersurface in the Schwarzschild exterior (corresponding to region `I'`) with constant mean curvature H. Then*

$$f_3(r; H, c_3, \bar{c}_3) = \int_{r_3}^{r} \frac{1}{h(x)} \frac{l_3(x; H, c_3)}{\sqrt{1 + l_3^2(x; H, c_3)}} \,\mathrm{d}x + \bar{c}_3,$$

where

$$l_3(r; H, c_3) = \frac{1}{\sqrt{h(r)}}\left(-Hr - \frac{c_3}{r^2}\right).$$

Here c_3 and $\bar{c}_3$ are constants, and $r_3 \in (2M, \infty)$ is fixed. We remark that for given constant mean curvature H, if $c_1 = c_3$, then $f_1'(r) = -f_3'(r)$.

We have complete discussion about the asymptotic behavior of SS-CMC hypersurfaces Σ_1 and Σ_3 in [5]. Here we just remark that for $H \neq 0$, SS-CMC hypersurfaces are asymptotic null and for $H = 0$, maximal hypersurfaces are asymptotic to the spatial infinity.

2.2. SS-CMC Solutions in Region `II` and `II'`

For SS-CMC hypersurfaces in the Schwarzschild interior, since the future timelike direction is $-\partial_r$ direction, spacelike condition gives that an SS-CMC hypersurface can be written as $(t, g(t), \theta, \phi)$ for some function $r = g(t)$.

Proposition 2.3. [6] *Each constant slice $r = r_0 \in (0, 2M)$ in the Schwarzschild interior (corresponding to region `II`) is an SS-CMC hypersurface with mean curvature*

$$H(r_0) = \frac{2r_0 - 3M}{3\sqrt{r_0^3(2M - r_0)}}.$$

These hypersurfaces are called cylindrical hypersurfaces.

For $r = g(t) \neq$ constant, we consider its inverse function, and denote $t = f_2(r)$ whenever it is defined. Since $f_2(r)$ is obtained from the inverse function, we have $f_2'(r) \neq 0$ and will allow $f_2'(r) = \infty$ or $-\infty$.

Proposition 2.4. [5] *Suppose $\Sigma^2 = (f_2(r), r, \theta, \phi)$ is an SS-CMC hypersurface in the Schwarzschild interior (corresponding to region `II`). Then*

$$f_2^*(r; H, c_2, \bar{c}_2) = \int_{r_2}^{r} \frac{1}{-h(x)} \sqrt{\frac{l_2^2(x; H, c_2)}{l_2^2(x; H, c_2) - 1}} \,\mathrm{d}x + \bar{c}_2, \quad \textit{or} \qquad (2.5)$$

$$f_2^{**}(r; H, c_2, \bar{c}_2') = \int_{r_2'}^{r} \frac{1}{h(x)} \sqrt{\frac{l_2^2(x; H, c_2)}{l_2^2(x; H, c_2) - 1}} \,\mathrm{d}x + \bar{c}_2' \qquad (2.6)$$

depending on the sign of $f_2'(r)$*, where*

$$l_2(r;H,c_2) = \frac{1}{\sqrt{-h(r)}}\left(-Hr - \frac{c_2}{r^2}\right).$$

Here $c_2, \bar{c}_2, \bar{c}_2'$ *are constants, and* r_2, r_2' *are points in the domain of* $f_2^*(r)$ *and* $f_2^{**}(r)$*, respectively.*

The function $l_2(r)$ *should satisfy* $l_2(r) > 1$*, which implies* $c_2 < 0$ *when* $H > 0$ *and* $c_2 < -8M^3H$ *when* $H \leq 0$*. We will write* $f_2(r)$ *to denote both* $f_2^*(r)$ *and* $f_2^{**}(r)$*.*

Similarly, for SS-CMC hypersurfaces in another Schwarzschild interior (corresponding to region II'), all cylindrical hypersurfaces $r = r_0 \in (0, 2M)$ are SS-CMC solutions with mean curvature $H(r_0) = \frac{3M-2r_0}{3\sqrt{r_0^3(2M-r_0)}}$. When $r \neq$ constant, we have the following results.

Proposition 2.5. [5] *Suppose* $\Sigma^4 = (f_4(r), r, \theta, \phi)$ *is an SS-CMC hypersurface in the Schwarzschild interior (corresponding to region II'). Then*

$$f_4^*(r;H,c_4,\bar{c}_4) = \int_{r_4}^{r} \frac{1}{-h(x)}\sqrt{\frac{l_4^2(x;H,c_4)}{l_4^2(x;H,c_4)-1}}\,\mathrm{d}x + \bar{c}_4, \quad \text{or} \tag{2.7}$$

$$f_4^{**}(r;H,c_4,\bar{c}_4') = \int_{r_4'}^{r} \frac{1}{h(x)}\sqrt{\frac{l_4^2(x;H,c_4)}{l_4^2(x;H,c_4)-1}}\,\mathrm{d}x + \bar{c}_4' \tag{2.8}$$

depending on the sign of $f_4'(r)$*, where*

$$l_4(r) = \frac{1}{\sqrt{-h(r)}}\left(Hr + \frac{c}{r^2}\right).$$

Here $c_4, \bar{c}_4, \bar{c}_4'$ *are constants, and* r_4, r_4' *are fixed numbers in the domain of* $f_4^*(r)$ *and* $f_4^{**}(r)$*, respectively.*

The function $l_4(r)$ *should satisfy* $l_4(r) > 1$*, which implies* $c_4 > -8M^3H$ *when* $H \geq 0$ *and* $c_4 > 0$ *when* $H < 0$*. In addition, we have* $f_4'(r) \neq 0$ *and will allow* $f_4'(r) = \pm\infty$ *at some point. We will write* $f_4(r)$ *to denote both* $f_4^*(r)$ *and* $f_4^{**}(r)$*.*

From Propositions 2.4 and 2.5, we know conditions $l_2(r) > 1$ and $l_4(r) > 1$ put restrictions on the domain of $f_2(r;H,c)$ and $f_4(r;H,c)$, respectively. Remark that c could be c_2 or c_4. Particularly, we have the equivalent conditions:

$$l_2(r) = \frac{1}{\sqrt{-h(r)}}\left(-Hr - \frac{c}{r^2}\right) > 1 \Leftrightarrow -Hr^3 - r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}} > c,$$

$$l_4(r) = \frac{1}{\sqrt{-h(r)}}\left(Hr + \frac{c}{r^2}\right) > 1 \Leftrightarrow -Hr^3 + r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}} < c.$$

Define two functions $k_H(r)$ and $\tilde{k}_H(r)$ on $(0, 2M)$ by

$$k_H(r) = -Hr^3 - r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}} \tag{2.9}$$

$$\tilde{k}_H(r) = -Hr^3 + r^{\frac{3}{2}}(2M-r)^{\frac{1}{2}}, \tag{2.10}$$

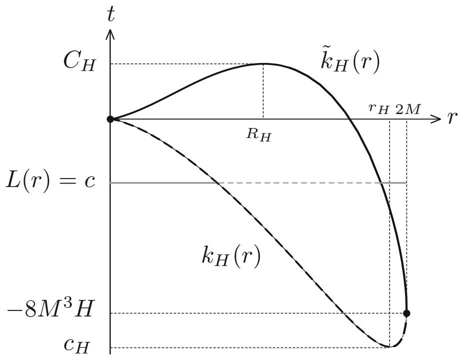

FIGURE 2. Domain of $f_2(r)$ and $f_4(r)$

then domains of $f_2(r; H, c)$ and $f_4(r; H, c)$ will be

$$D_2 = \{r \in (0, 2M) | k_H(r) > c\} \cup \{r \in (0, 2M) | k_H(r) = c \text{ and } f_2(r) \text{ is finite}\}$$

$$D_4 = \{r \in (0, 2M) | \tilde{k}_H(r) < c\} \cup \{r \in (0, 2M) | \tilde{k}_H(r) = c \text{ and } f_4(r) \text{ is finite}\}.$$

It is easy to see domains D_2 and D_4 visually. Take $H > 0$ for example and cases $H = 0$ and $H < 0$ are similar. Figure 2 illustrates the graphs of $k_H(r)$ and $\tilde{k}_H(r)$, which form a loop in the r–t plane. Given any constant c, preimage of the line $L(r) = c$ below $k_H(r)$ belongs to D_2, and preimage of the line above $\tilde{k}_H(r)$ belongs to D_4.

We still have to take care of the intersection of $L(r) = c$ and the loop, which may also belong to D_2 or D_4. Let c_H be the minimum value of $k_H(r)$ achieved at $r = r_H$ and let C_H be the maximum value of $\tilde{k}_H(r)$ achieved at $r = R_H$. After some analysis which can be found in [5], it turns out that when $c \in (c_H, C_H)$, intersections of $L(r) = c$ and $k_H(r)$ belong to D_2, and intersections of $L(r) = c$ and $\tilde{k}_H(r)$ belong to D_4. But when $c = c_H$ or $c = C_H$, their intersections do not belong to domains D_2 or D_4.

In conclusion, we have the following proposition on SS-CMC hypersurfaces:

Proposition 2.6. [5] (a) *If $c_H < c_2 < \max(0, -8M^3H)$, then $f_2(r)$ is defined on $(0, r']$ or $[r'', 2M)$ for some r' and r'', which depend on H and c_2. When we take $r_2 = r'_2 = r'(\text{or } r'')$ and $\bar{c}_2 = \bar{c}'_2$ in (2.5) and (2.6), the union $\Sigma^2 = (f_2^*(r; H, c_2, \bar{c}_2) \cup f_2^{**}(r; H, c_2, \bar{c}'_2), r, \theta, \phi)$ is a smooth SS-CMC hypersurface in the Schwarzschild interior (region* II*).*

(b) *If $\min(0, -8M^3H) < c_4 < C_H$, then $f_4(r)$ is defined on $(0, r']$ or $[r'', 2M)$ for some r' and r'', which depend on H and c_4. When we take $r_4 = r'_4 = r'(\text{or } r'')$ and $\bar{c}_4 = \bar{c}'_4$ in (2.7) and (2.8), the union $\Sigma^4 = (f_4^*(r; H, c_4, \bar{c}_4) \cup f_4^{**}(r; H, c_4, \bar{c}'_4), r, \theta, \phi)$ is a smooth SS-CMC hypersurface in the Schwarzschild interior (region* II'*).*

2.3. Smooth SS-CMC Hypersurfaces

Given an SS-CMC hypersurface in the Schwarzschild exterior or interior, as long as its domain is defined near $r = 2M$, we can discuss the behavior of SS-CMC hypersurfaces near $r = 2M$ and join two SS-CMC hypersurfaces in different region smoothly in the Kruskal extension.

Here we only list results of SS-CMC hypersurfaces needed in next section and refer to [5] for all the other cases.

(a) If $c_H < c_2 < -8M^3H$, from Proposition 2.6 (a), for every SS-CMC hypersurface Σ^2 defined near $r = 2M$, then the spacelike condition is preserved as $r \to 2M^-$. Since $f_2^* \to \infty$ as $r \to 2M^-$, the image of Σ^2 touches the interface of region `II` and `I`. We can take Σ^1 in region `I` with $c_1 = c_2$ and suitable $\bar{c}_1$ which is determined by $\bar{c}_2$ such that $\Sigma^1 \cup \Sigma^2$ is a smooth SS-CMC hypersurface in the Kruskal extension. Similarly, since $f_2^{**} \to -\infty$ as $r \to 2M^-$, the image of Σ^2 touches the interface of region `II` and `I'`. We can take Σ^3 in region `I'` with $c_3 = c_2$ and suitable $\bar{c}_3$ which is determined by $\bar{c}_2$ such that $\Sigma^1 \cup \Sigma^2 \cup \Sigma^3$ is a smooth SS-CMC hypersurface in the Kruskal extension.

(b) If $-8M^3H < c_4 < C_H$, from Proposition 2.6 (b), for every SS-CMC hypersurface Σ^4 defined near $r = 2M$, then the spacelike condition is preserved as $r \to 2M^-$. Since $f_4^{**} \to -\infty$ as $r \to 2M^-$, the image of Σ^4 touches the interface of region `II'` and `I`. We can take Σ^1 in region `I` with $c_1 = c_4$ and suitable $\bar{c}_1$ which is determined by $\bar{c}_4$ such that $\Sigma^1 \cup \Sigma^4$ is a smooth SS-CMC hypersurface in the Kruskal extension. Similarly, since $f_4^* \to \infty$ as $r \to 2M^-$, the image of Σ^4 touches the interface of region `II'` and `I'`. We can take Σ^3 in region `I'` with $c_3 = c_4$ and suitable $\bar{c}_3$ which is determined by $\bar{c}_4$ such that $\Sigma^1 \cup \Sigma^4 \cup \Sigma^3$ is a smooth SS-CMC hypersurface in the Kruskal extension.

(c) If $c_1 = -8M^3H$, from Proposition 2.1, every SS-CMC hypersurface Σ^1 is defined near $r = 2M$, and the spacelike condition is preserved as $r \to 2M^+$. Since $f_1(r)$ tends to a finite value as $r \to 2M^+$, we can extend $f_1(r)$ at $r = 2M$, and the image of Σ^1 touches the origin of the Kruskal extension. We can take Σ^3 in region `I'` with $c_3 = c_1$ and suitable $\bar{c}_3$ which is determined by $\bar{c}_1$ such that $\Sigma^1 \cup \Sigma^3$ is a smooth SS-CMC hypersurface in the Kruskal extension.

The upshot of the characterization of SS-CMC hypersurfaces is:

Theorem 2.7. [5] *For $H \in \mathbb{R}$, all smooth SS-CMC hypersurfaces and their behaviors in the Schwarzschild spacetimes or in the Kruskal extension are completely characterized, by two constants c and $\bar{c}$.*

3. CMC Foliation in the Kruskal Extension

3.1. A Conjecture of CMC Foliation

A spherically symmetric hypersurface can be represented by a curve in the Kruskal plane, and it is convenient to study the curve in null coordinates as $\Sigma = (U(s), V(s))$. In such coordinates, the spacelike condition is equivalent to the tangent vector of Σ being spacelike, that is $V'(s)U'(s) > 0$ from (2.1).

Hence the curve can be written as a graph of $V(U)$ with

$$\frac{\mathrm{d}V}{\mathrm{d}U} = \frac{V'(s)}{U'(s)} > 0.$$

That is, $V(U)$ is a monotone increasing function.

Definition 3.1. An SS-CMC hypersurface $\Sigma = (U(s), V(s))$ in the Kruskal plane is called *T-axisymmetric* if Σ is symmetric with respect to the T-axis: $U+V=0$, or equivalently, Σ satisfies the following condition:

$$\text{If } (U,V) \in \Sigma, \quad \text{then } (-V,-U) \in \Sigma. \tag{$*$}$$

In the following, we use TSS-CMC to represent T-axisymmetric SS-CMC .

In [5], we have constructed all smooth SS-CMC hypersurfaces. Every SS-CMC hypersurface is characterized by two parameters c and $\bar{c}$. Now we consider SS-CMC hypersurfaces with $c \in (c_H, C_H)$, and cylindrical hypersurfaces $r = r_H$ in region II and $r = R_H$ in region II', which correspond to $c = c_H$ and $c = C_H$, respectively. Recall that $c_H < -8M^3H \le 0 < C_H$ when $H \ge 0$ and $c_H < 0 < -8M^3H < C_H$ when $H < 0$. If we put superscripts $+$ and $-$ on functions $k_H(r)$ in (2.9) and $\tilde{k}_H(r)$ in (2.10) to represent their increasing and decreasing part, respectively, we know that each $c \in (c_H, C_H), c \ne 0$ determines two families SS-CMC hypersurfaces, which can be distinguished by $k_H^+(r), k_H^-(r), \tilde{k}_H^+(r)$ or $\tilde{k}_H^-(r)$. Denote their associated smooth SS-CMC hypersurfaces by $\Sigma^+_{H,c,\bar{c}}$, $\Sigma^-_{H,c,\bar{c}}$, $\tilde{\Sigma}^+_{H,c,\bar{c}}$ and $\tilde{\Sigma}^-_{H,c,\bar{c}}$, respectively. For $c = 0$, it determines one family of SS-CMC hypersurfaces belonging to $\tilde{\Sigma}^-_{H,c,\bar{c}}$ if $H \ge 0$, or belonging to $\Sigma^+_{H,c,\bar{c}}$ if $H < 0$. The following Proposition shows the existence of TSS-CMC hypersurfaces in each family.

Proposition 3.2. *Among the family of SS-CMC hypersurfaces $\Sigma^-_{H,c,\bar{c}}$, there exists a unique TSS-CMC hypersurface $\Sigma^-_{H,c}$. Same conclusion also holds for $\Sigma^+_{H,c,\bar{c}}$, $\tilde{\Sigma}^+_{H,c,\bar{c}}$ and $\tilde{\Sigma}^-_{H,c,\bar{c}}$. Here $c \in (c_H, C_H)$ and $H \in \mathbb{R}$.*

Proof. For $\Sigma^-_{H,c,\bar{c}}$ with $c \in (c_H, 0)$, hypersurfaces are in the Schwarzschild interior (region II). Let $r^-_{H,c}$ be the solution of $k^-_H(r) = c$. Choose $\bar{c}$ such that $f(r^-_{H,c}; H, c, \bar{c}) = 0$, where the formula of f is given in (2.5) or (2.6).[3] Denote this SS-CMC hypersurface by $\Sigma^-_{H,c}$. The hypersurface $\Sigma^-_{H,c}$ in $U+V \ge 0$ region corresponds to $f'(r) < 0$ with domain $(0, r^-_{H,c}]$ in the Schwarzschild interior, and it has nonnegative t value: $t = \int_{r^-_{H,c}}^r f'_-(x;c)\mathrm{d}x$, where $f'_- = \frac{1}{h}\sqrt{\frac{l^2}{l^2-1}}$ [see Proposition 2.4, Eq. (2.6)]. From the table below (2.3), the corresponding (U,V) coordinates are

[3] We will omit the subscripts $1, 2, 3, 4$ of f and l as long as there is no confusing.

$$U(r;c) = -\mathrm{e}^{-\frac{1}{4M}(t-r-2M\ln|r-2M|)} = -\mathrm{e}^{-\frac{1}{4M}\left(\int_{r_{H,c}^-}^r f'_-(x;c)\mathrm{d}x - r - 2M\ln|r-2M|\right)},$$
$$V(r;c) = \mathrm{e}^{\frac{1}{4M}(t+r+2M\ln|r-2M|)} = \mathrm{e}^{\frac{1}{4M}\left(\int_{r_{H,c}^-}^r f'_-(x;c)\mathrm{d}x + r + 2M\ln|r-2M|\right)}. \tag{3.1}$$

On the other hand, the hypersurface $\Sigma_{H,c}^-$ in $U+V \leq 0$ region corresponds to $f'(r) > 0$ with domain $(0, r_{H,c}^-]$ in the Schwarzschild interior, and it has nonpositive t-value: $t = \int_{r_{H,c}^-}^r f'_+(x;c)\mathrm{d}x$, where $f'_+ = \frac{1}{-h}\sqrt{\frac{l^2}{l^2-1}}$. The corresponding (U,V) coordinates are

$$U(r;c) = -\mathrm{e}^{-\frac{1}{4M}(t-r-2M\ln|r-2M|)} = -\mathrm{e}^{-\frac{1}{4M}\left(\int_{r_{H,c}^-}^r f'_+(x;c)\mathrm{d}x - r - 2M\ln|r-2M|\right)},$$
$$V(r;c) = \mathrm{e}^{\frac{1}{4M}(t+r+2M\ln|r-2M|)} = \mathrm{e}^{\frac{1}{4M}\left(\int_{r_{H,c}^-}^r f'_+(x;c)\mathrm{d}x + r + 2M\ln|r-2M|\right)}. \tag{3.2}$$

Since $f'_-(x;c) = -f'_+(x;c)$ for all $x \in (0, r_{H,c}^-]$, (3.1) and (3.2) satisfy the condition ($*$).

For $\Sigma_{H,c,\bar{c}}^+$ with $c \in (c_H, -8M^3H)$, these hypersurfaces pass through regions `I`, `II`, and `I'` [see discussion (a) before Theorem 2.7]. Let $r_{H,c}^+$ be the solution of $k_H^+(r) = c$. Choose $\bar{c}$ such that $f(r_{H,c}^+; H, c, \bar{c}) = 0$, where the formula of f is given in (2.5) or (2.6). Denote

$$\bar{f}'(r) = \frac{r^4}{(Hr^3+c_1)^2 + r^3(r-2M) - (Hr^3+c_1)\sqrt{(Hr^3+c_1)^2 + r^3(r-2M)}}, \tag{3.3}$$

then $\Sigma_{H,c}^+$ in $U+V \geq 0$ region satisfies $f'(r) = -\frac{1}{h(r)} + \bar{f}'(r)$. Therefore, we have

$$t = f(r) = \int_{r_{H,c}^+}^r \left(-\frac{1}{h(x)} + \bar{f}'(x)\right)\mathrm{d}x$$
$$= -r - 2M\ln|r-2M| + r_{H,c}^+ + 2M\ln|r_{H,c}^+ - 2M| + \int_{r_{H,c}^+}^r \bar{f}'(x)\mathrm{d}x.$$

It leads to

$$U(r;c) = \begin{cases} \mathrm{e}^{-\frac{u}{4M}} & \text{in region I} \\ -\mathrm{e}^{-\frac{u}{4M}} & \text{in region II} \end{cases} = \begin{cases} \mathrm{e}^{-\frac{1}{4M}(t-r-2M\ln|r-2M|)} & \text{in region I} \\ -\mathrm{e}^{-\frac{1}{4M}(t-r-2M\ln|r-2M|)} & \text{in region II} \end{cases}$$
$$= (r-2M)\,\mathrm{e}^{-\frac{1}{4M}\left(-2r + r_{H,c}^+ + 2M\ln|r_{H,c}^+ - 2M| + \int_{r_{H,c}^+}^r \bar{f}'(x)\,\mathrm{d}x\right)}$$
$$= (r-2M)\,\mathrm{e}^{\frac{1}{4M}\left(2r - r_{H,c}^+ - 2M\ln|r_{H,c}^+ - 2M| - \int_{r_{H,c}^+}^r \bar{f}'(x)\,\mathrm{d}x\right)}.$$
$$V(r;c) = \mathrm{e}^{\frac{v}{4M}} = \mathrm{e}^{\frac{1}{4M}(t+r+2M\ln|r-2M|)} = \mathrm{e}^{\frac{1}{4M}\left(r_{H,c}^+ + 2M\ln|r_{H,c}^+ - 2M| + \int_{r_{H,c}^+}^r \bar{f}'(x)\,\mathrm{d}x\right)}. \tag{3.4}$$

On the other hand, $\Sigma^+_{H,c}$ in $U+V \leq 0$ region satisfies $f'(r) = \frac{1}{h(r)} - \bar{f}'(r)$, and we have

$$U(r;c) = -\mathrm{e}^{-\frac{1}{4M}\left(-r^+_{H,c}-2M\ln|r^+_{H,c}-2M|-\int_{r^+_{H,c}}^r \bar{f}'(x)\,\mathrm{d}x\right)}$$
$$= -\mathrm{e}^{\frac{1}{4M}\left(r^+_{H,c}+2M\ln|r^+_{H,c}-2M|+\int_{r^+_{H,c}}^r \bar{f}'(x)\,\mathrm{d}x\right)}$$
$$V(r;c) = -(r-2M)\,\mathrm{e}^{\frac{1}{4M}\left(2r-r^+_{H,c}-2M\ln|r^+_{H,c}-2M|-\int_{r^+_{H,c}}^r \bar{f}'(x)\,\mathrm{d}x\right)}.$$

Hence $\Sigma^+_{H,c}$ satisfies condition ($*$), and $\Sigma^+_{H,c}$ is a TSS-CMC hypersurface.

When $c = -8M^3H$, $\tilde{\Sigma}^-_{H,c,\bar{c}}$ pass through region `I` and `I'` [see discussion (c) before Theorem 2.7]. Choose $\bar{c}$ such that $f(2M; H, c = -8M^3H, \bar{c}) = 0$ in the Schwarzschild exterior, and denote the hypersurface by $\tilde{\Sigma}^-_{H,-8M^3H}$. The expression of $\tilde{\Sigma}^-_{H,-8M^3H}$ is

$$\begin{cases} U(r) = \sqrt{r-2M}\,\mathrm{e}^{\frac{1}{4M}(-\int_{2M}^r f_1'(x)\,\mathrm{d}x+r)} \\ V(r) = \sqrt{r-2M}\,\mathrm{e}^{\frac{1}{4M}(\int_{2M}^r f_1'(x)\,\mathrm{d}x+r)} \end{cases} \quad \text{in region } \mathtt{I},$$

$$\begin{cases} U(r) = -\sqrt{r-2M}\,\mathrm{e}^{\frac{1}{4M}(-\int_{2M}^r f_3'(x)\,\mathrm{d}x+r)} \\ V(r) = -\sqrt{r-2M}\,\mathrm{e}^{\frac{1}{4M}(\int_{2M}^r f_3'(x)\,\mathrm{d}x+r)} \end{cases} \quad \text{in region } \mathtt{I'},$$

where

$$f_3'(r) = -f_1'(r) = -H\left(\frac{r}{r-2M}\right)^{\frac{1}{2}}\left(\frac{r(r^2+2Mr+4M^2)^2}{r^3+H^2(r-2M)(r^2+2Mr+4M^2)^2}\right)^{\frac{1}{2}}.$$

Hence $\tilde{\Sigma}^-_{H,-8M^3H}$ satisfies condition ($*$), and $\tilde{\Sigma}^-_{H,-8M^3H}$ is T-axisymmetric.

The other cases can be discussed similarly. □

With Definition 3.1 and Proposition 3.2, we can rephrase the conjecture of Malec and Ó Murchadha in [6] as TSS-CMC hypersurfaces with H fixed and c varied from c_H to C_H form a foliation. It will be explained in more details below. We will discuss the case $H > 0$ only. The case $H \leq 0$ is similar.

Consider the graphs of $\tilde{k}^+_H \cup \tilde{k}^-_H \cup k^+_H \cup k^-_H$ as a loop, which is illustrated in the left picture of Fig. 3. When we circle the loop clockwise from the origin, the first piece is $\tilde{k}^+_H$ with $c \in (0, C_H)$ and their corresponding TSS-CMC hypersurfaces are $\tilde{\Sigma}^+_{H,c}$. Let $\tilde{r}^+_{H,c}$ be the solution of $\tilde{k}^+_H(r) = c$, then the hypersurface $\tilde{\Sigma}^+_{H,c}$ intersects the r-axis at $r = \tilde{r}^+_{H,c}$ in the Schwarzschild interior (region `II'`) and from (2.2), $\tilde{\Sigma}^+_{H,c}$ intersects T-axis at $T = -\sqrt{2M - \tilde{r}^+_{H,c}}\,\mathrm{e}^{\frac{\tilde{r}^+_{H,c}}{4M}}$ in the Kruskal plane. The increasing property of $\tilde{k}^+_H$ implies that $\tilde{r}^+_{H,c}$ increases as c increases, so the T-intercept increases as well.

These TSS-CMC hypersurfaces $\tilde{\Sigma}^+_{H,c}$ from $c = 0$ to $c = C_H$ are conjectured to form a foliation between two hyperbolas $r = 0$ and $r = R_H$, where R_H is the solution to $\tilde{k}^+_H(r) = C_H$. When $c = C_H$, we take the cylindrical hypersurface $r = R_H$ to be the TSS-CMC one, and call it $\tilde{\Sigma}^+_{H,C_H}$. Denote $\tilde{\Sigma}^+_{H,0<c\leq C_H} = \{\tilde{\Sigma}^+_{H,c} | 0 < c \leq C_H\}$.

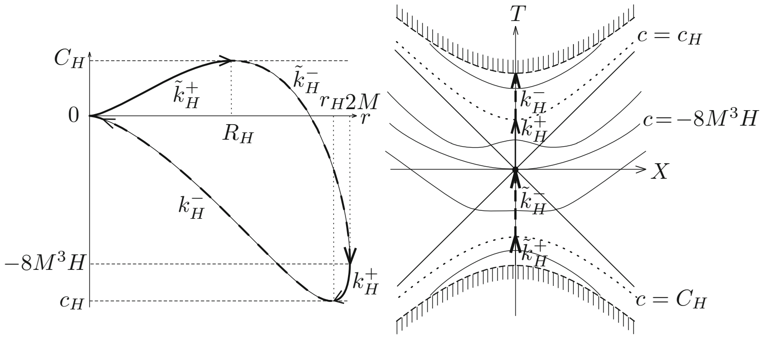

FIGURE 3. Graphs of $k_H(r) = k_H^- \cup k_H^+$ and $\tilde{k}_H(r) = \tilde{k}_H^+ \cup \tilde{k}_H^-$, and a possible TSS-CMC foliation for $H > 0$

The second piece is $\tilde{k}_H^-$ with c decreasing from C_H to $-8M^3H$. Let $\tilde{r}_{H,c}^-$ be the solution of $\tilde{k}_H^-(r) = c$, then the corresponding TSS-CMC hypersurface $\tilde{\Sigma}_{H,c}^-$ intersects the r-axis at $r = \tilde{r}_{H,c}^-$ in the Schwarzschild interior (region II'), and $\tilde{\Sigma}_{H,c}^-$ intersects T-axis at $T = -\sqrt{2M - \tilde{r}_{H,c}^-}\, e^{\frac{\tilde{r}_{H,c}^-}{4M}}$ in the Kruskal plane. The T-intercept increases when c decreases. When $c = -8M^3H$, the TSS-CMC hypersurface $\tilde{\Sigma}_{H,-8M^3H}^-$ passes through the origin in the Kruskal plane. Hence $\tilde{\Sigma}_{H,c}^-$ for c ranging from C_H to $-8M^3H$ lie between the hyperbola $r = R_H$ and $\tilde{\Sigma}_{H,-8M^3H}^-$, and they are conjectured to form a foliation. Denote $\tilde{\Sigma}_{H,C_H>c\geq -8M^3H}^- = \{\tilde{\Sigma}_{H,c}^- | C_H > c \geq -8M^3H\}$.

The third piece is k_H^+ with c decreasing from $-8M^3H$ to c_H. If $r_{H,c}^+$ is the solution of $k_H^+(r) = c$, the TSS-CMC hypersurface $\Sigma_{H,c}^+$ intersects T-axis at $T = \sqrt{2M - r_{H,c}^+}\, e^{\frac{r_{H,c}^+}{4M}}$, and the T-intercept increases when c decreases. The critical value c_H determines a TSS-CMC hypersurface $r = r_H$, which is a cylindrical hypersurface, and we call it Σ_{H,c_H}^+. Denote $\Sigma_{H,-8M^3H>c>c_H}^+ = \{\Sigma_{H,c}^+ | -8M^3H > c > c_H\}$. It is conjectured that TSS-CMC hypersurfaces in $\Sigma_{H,-8M^3H>c>c_H}^+$ form a foliation between $\tilde{\Sigma}_{H,-8M^3H}^-$ and the hyperbola $r = r_H$.

Finally, consider k_H^- with c increasing from c_H to 0. Denote $\Sigma_{H,c_H\leq c<0}^- = \{\Sigma_{H,c}^- | c_H \leq c < 0\}$. These TSS-CMC hypersurfaces $\Sigma_{H,c}^-$ lie between two hyperbolas $r = r_H$ and $r = 0$ in region II with the T-intercepts $T = \sqrt{2M - r_{H,c}^-}\, e^{\frac{r_{H,c}^-}{4M}}$, where $r_{H,c}^-$ is the solution of $k_H^-(r) = c$, and the T-intercept increases when c increases. So $\Sigma_{H,c_H\leq c<0}^-$ are conjectured to form a foliation between two hyperbolas $r = r_H$ and $r = 0$.

As we move along the loop from $\tilde{k}_H^+$, $\tilde{k}_H^-$, k_H^+ to k_H^-, the change of their T-intercepts is indicated in the right picture of Fig. 3. In these notions, Malec and Ó Murchadha's conjecture can be summarized as:

Conjecture 3.3 (Malec, Edward and Ó Murchadha, Niall [6]). *Given any constant mean curvature H, the Kruskal extension can be foliated by the TSS-CMC hypersurfaces* $\{\Sigma_H\} = \tilde{\Sigma}^+_{H,0<c\le C_H} \cup \tilde{\Sigma}^-_{H,C_H>c>-8M^3H} \cup \Sigma^+_{H,-8M^3H\ge c>c_H} \cup \Sigma^-_{H,c_H\le c<0}$.

3.2. Criteria for a Global CMC Foliation

From the construction in Sect. 3.1, each TSS-CMC hypersurface in $\{\Sigma_H\}$ has different T-intercept. The continuity property implies that $\{\Sigma_H\}$ forms a local foliation near the T-axis. That is, for any two TSS-CMC hypersurfaces in $\{\Sigma_H\}$, there exists an open set O in the Kruskal extension such that they are disjoint in O near the T-axis. Conjecture 3.3 claims that $\{\Sigma_H\}$ is a global TSS-CMC foliation. In other words, the open set can be taken as the Kruskal extension and $\{\Sigma_H\}$ covers the whole Kruskal extension.

To answer the conjecture, we will first derive criteria that hypersurfaces in $\{\Sigma_H\}$ are disjoint. The T-axisymmetry implies that it suffices to consider the hypersurfaces in $U+V>0$ region. We restrict to this case from now on. First, consider the family of TSS-CMC hypersurfaces $\Sigma^-_{H,c_H<c<0}$. The V-coordinate of $\Sigma^-_{H,c}$ is given in (3.1):

$$V(r;c) = \mathrm{e}^{\frac{1}{4M}(t+r+2M\ln|r-2M|)} = \mathrm{e}^{\frac{1}{4M}\left(\int_{r^-_{H,c}}^r f'_-(x;c)\,\mathrm{d}x + r + 2M\ln|r-2M|\right)},$$

where

$$\int_{r^-_{H,c}}^r f'_-(x;c)\,\mathrm{d}x = \int_{r^-_{H,c}}^r \frac{x}{x-2M}\frac{-Hx^3-c}{\sqrt{(Hx^3+c)^2+x^3(x-2M)}}\,\mathrm{d}x. \tag{3.5}$$

We remark that when $c\in(c_H,0)$, the integral can be extended to a finite value at $r=0$. So $V(0;c)$ is defined, and we can use $V(0;c)$ to derive a criterion to detect whether $\Sigma^-_{H,c}, c\in(c_H,0)$ are disjoint.

Proposition 3.4. *If* $\frac{\mathrm{d}V(0;c)}{\mathrm{d}c}\le 0$ *for all TSS-CMC hypersurfaces in* $\Sigma^-_{H,c_H<c<0}$, *then the hypersurfaces are disjoint. If* $\frac{\mathrm{d}V(0;c)}{\mathrm{d}c}>0$ *at* $c=c_0$, *then* Σ^-_{H,c_0} *will intersect some other hypersurface in* $\Sigma^-_{H,c_H<c<0}$.

Proof. For $0>c_1>c_2>c_H$, and for $r\in(0,2M)$ where $V(r;c_1)$ and $V(r;c_2)$ are defined, we have

$$\frac{V(r;c_1)}{V(r;c_2)} = \frac{V(0;c_1)}{V(0;c_2)}\,\mathrm{e}^{\frac{1}{4M}\int_0^r (f'_-(x;c_1)-f'_-(x;c_2))\mathrm{d}x}.$$

The condition $\frac{\mathrm{d}V(0;c)}{\mathrm{d}c}\le 0$ implies $\frac{V(0;c_1)}{V(0;c_2)}\le 1$. Furthermore, from Proposition 2.4, we have

$$\frac{\mathrm{d}f'_-}{\mathrm{d}c} = \frac{1}{h(r)}\frac{-1}{(l^2-1)^{\frac{3}{2}}}\frac{\mathrm{d}l}{\mathrm{d}c} < 0,$$

so $f'_-(r;c_1)-f'_-(r;c_2)<0$, which implies $\mathrm{e}^{\frac{1}{4M}\int_0^r (f'_-(x;c_1)-f'_-(x;c_2))\mathrm{d}x} < 1$. Therefore, $V(r;c_1)<V(r;c_2)$ for all r is defined, and hence $\Sigma^-_{H,c_H<c<0}$ are disjoint because an intersection point must have the same r by (2.2).

If $\frac{\mathrm{d}V(0;c_0)}{\mathrm{d}c} > 0$, there exists $c_1 > c_0$ and $\varepsilon > 0$ such that $V(0;c_1) - V(0;c_0) = \varepsilon > 0$, which implies $\frac{V(0;c_1)}{V(0;c_0)} > 1$. Consider the function $F(r) = \frac{V(r;c_1)}{V(r;c_0)}$ defined on $r \in [0, r^-_{H,c_1}]$, then

$$F(r^-_{H,c_1}) = \mathrm{e}^{-\frac{1}{4M}\int_{r_{H,c_0}}^{r_{H,c_1}} f'_-(x,c_0)\,\mathrm{d}x} < 1$$

because of $r^-_{H,c_1} < r^-_{H,c_0}$ and $f'_-(x,c_0) < 0$. By the intermediate value theorem, there exists $r_0 \in (0, r^-_{H,c_1})$ such that $F(r_0) = 1$, so $(U(r_0;c_0), V(r_0;c_0))$ is an intersection point. □

Similar arguments give a criterion for $\tilde{\Sigma}^+_{H,0<c<C_H}$.

Proposition 3.5. *If* $\frac{\mathrm{d}U(0;c)}{\mathrm{d}c} \geq 0$ *for all* TSS-CMC *hypersurfaces in* $\tilde{\Sigma}^+_{H,0<c<C_H}$*, then the hypersurfaces are disjoint. If* $\frac{\mathrm{d}U(0;c)}{\mathrm{d}c} < 0$ *at* $c = c_0$*, then* $\tilde{\Sigma}^+_{H,c_0}$ *will intersect some other hypersurface in* $\tilde{\Sigma}^+_{H,0<c<C_H}$*.*

Next, we consider $\Sigma^+_{H,-8M^3H\geq c>c_H}$. The V-coordinate of $\Sigma^+_{H,c}$ in $U+V > 0$ region is given in (3.4):

$$V(r;c) = \mathrm{e}^{\frac{1}{4M}(t+r+2M\ln|r-2M|)} = \mathrm{e}^{\frac{1}{4M}\left(r^+_{H,c}+2M\ln|r^+_{H,c}-2M|+\int_{r^+_{H,c}}^{r}\bar{f}'(x)\mathrm{d}x\right)}.$$

Proposition 3.6. *Given* $r' > r_H$*, if* $\frac{\mathrm{d}V(r';c)}{\mathrm{d}c} \leq 0$ *for all* TSS-CMC *hypersurfaces in the family* $\Sigma^+_{H,-8M^3H\geq c>c_H}$*, then the hypersurfaces are disjoint for all* $r \in [r_H, r']$ *defined. If* $\frac{\mathrm{d}V(r';c)}{\mathrm{d}c} > 0$ *at* $c = c_0$*, then* Σ^+_{H,c_0} *intersects some other hypersurface in* $\Sigma^+_{H,-8M^3H\geq c>c_H}$*.*

Proof. For $-8M^3H \geq c_1 > c_2 > c_H$ and $r < r'$, we have

$$\frac{V(r;c_1)}{V(r;c_2)} = \frac{V(r';c_1)}{V(r';c_2)}\,\mathrm{e}^{\frac{1}{4M}\int_r^{r'}(\bar{f}'(x;c_2)-\bar{f}'(x;c_1))\mathrm{d}x}.$$

The condition $\frac{\mathrm{d}V(r';c)}{\mathrm{d}c} \leq 0$ implies $\frac{V(r';c_1)}{V(r';c_2)} \leq 1$. One can check $\frac{\mathrm{d}\bar{f}'}{\mathrm{d}c} > 0$ and it gives $\bar{f}'(r;c_1) > \bar{f}'(r;c_2)$, so $\mathrm{e}^{\frac{1}{4M}\int_r^{r'}(\bar{f}'(x;c_2)-\bar{f}'(x;c_1))\mathrm{d}x} < 1$. Therefore, $\frac{V(r;c_1)}{V(r;c_2)} < 1$ for all r defined. Since $UV = (r-2M)\mathrm{e}^{\frac{r}{2M}}$, it shows that $\Sigma^+_{H,-8M^3H\geq c>c_H}$ are disjoint.

If $\frac{\mathrm{d}V(r';c_0)}{\mathrm{d}c} > 0$, there exists c_1 with $c_1 < c_0$ and $\varepsilon > 0$ such that $V(r';c_0) - V(r';c_1) = \varepsilon > 0$, which implies $\frac{V(r';c_0)}{V(r';c_1)} > 1$. Let $F(r) = \frac{V(r;c_0)}{V(r;c_1)}$, which is defined on $r \in [r^+_{H,c_0}, \infty)$. Since

$$F(r^+_{H,c_0}) = \frac{V(r^+_{H,c_0};c_0)}{V(r^+_{H,c_0};c_1)} < \frac{V(r^+_{H,c_0};c_0)}{V(r^+_{H,c_1};c_1)} < 1,$$

by the intermediate value theorem, there is $r_0 \in (r^+_{H,c_0}, r')$ such that $F(r_0) = 1$, and $(U(r_0;c_0), V(r_0;c_0))$ is the intersection point. □

We also have a criterion for $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$.

Proposition 3.7. *Given $r' > R_H$, if $\frac{\mathrm{d}U(r';c)}{\mathrm{d}c} \geq 0$ for all TSS-CMC hypersurfaces in the family $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$, then the hypersurfaces are disjoint for $r \in [R_H, r']$ defined. If $\frac{\mathrm{d}U(r';c)}{\mathrm{d}c} < 0$ at $c = c_0$, then $\tilde{\Sigma}^-_{H,c_0}$ intersects some other hypersurface in $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$.*

For TSS-CMC hypersurfaces in families $\Sigma^-_{H,c_H<c<0}$ and $\Sigma^+_{H,-8M^3H\geq c>c_H}$, V-coordinates are positive in $U+V>0$ region, so the criteria for disjoint hypersurfaces can be replaced by $\frac{\mathrm{d}\ln V}{\mathrm{d}c} \leq 0$. Similarly, for TSS-CMC hypersurfaces in $\tilde{\Sigma}^+_{H,0<c<C_H}$ and $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$, U-coordinates are positive in $U+V>0$ region, so the criteria for disjoint hypersurfaces can be replaced by $\frac{\mathrm{d}\ln U}{\mathrm{d}c} \geq 0$.

Finally, we remark that two families $\Sigma^+_{H,-8M^3H\geq c>c_H}$ and $\Sigma^-_{H,c_H<c<0}$ must be disjoint because the cylindrical hypersurface $r = r_H$ is a barrier between them. Similarly, $r = R_H$ is a barrier between $\tilde{\Sigma}^+_{H,0<c<C_H}$ and $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$.

3.3. CMC Foliation for $\boldsymbol{\Sigma^-_{H,c_H\leq c<0}}$ in Region II and $\boldsymbol{\tilde{\Sigma}^+_{H,0<c\leq C_H}}$ in Region II'

In the following, we use $R = R(H,c)$ to denote $r^-_{H,c}$ ($r^+_{H,c}, \tilde{r}^+_{H,c}$, or $\tilde{r}^-_{H,c}$) in convenience when it does not cause confusion.

Proposition 3.8. *For all $H \in \mathbb{R}$, each TSS-CMC hypersurface in $\Sigma^-_{H,c_H<c<0}$ satisfies $\frac{\mathrm{d}\ln V(0;c)}{\mathrm{d}c} \leq 0$ so that they are disjoint.*

Proof. From the calculation in the Appendix A, we have the formula (A.8):

$$\frac{\mathrm{d}\ln V(0;c(R))}{\mathrm{d}c} = -\frac{1}{4MJ(R)\sqrt{-h(R)}}\left(\int_0^R \frac{H\cdot F(x,R)+G(x,R)}{(R-x)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x - 1\right),$$

where $F(x,R)$ and $G(x,R)$ are as (A.2) and (A.4):

$$\begin{aligned} F(x,R) &= x^2(-3x^2(x+R-2M)+(2x-3M)(x^2+Rx+R^2)) \\ &= x^2((3M-x)(x^2-R^2)+xR(R-3M-x)) \quad \text{and} \\ G(x,R) &= x^2\sqrt{-h(R)}(x(R-3M)+R(x-3M)). \end{aligned}$$

Because both $F(x,R)$ and $G(x,R)$ are negative functions on $x\in[0,R]$, and $J(R) = -3HR^{\frac{3}{2}}(2M-R)+(2R-3M) < 0$, we get $\frac{\mathrm{d}\ln V(0;c)}{\mathrm{d}c} \leq 0$ if $H \geq 0$.

If $H < 0$, we can still show that $H\cdot F(x,R)+G(x,R) \leq 0$ for all $x \in [0,R]$, which also implies $\frac{\mathrm{d}\ln V(0;c)}{\mathrm{d}c} \leq 0$. The proof goes as below. Define $a = \frac{2M}{r_H}, b = \frac{2M}{R}$, and $z = \frac{x}{R}$. We have relations $h(R) = 1-\frac{2M}{R} = 1-b$, and from (C.1)

$$HR = \frac{a}{b}\frac{1}{\sqrt{a-1}}\left(\frac{4-3a}{6}\right).$$

Hence $R < r_H$, $H < 0$, and $x \in [0, R]$ imply $b > a > \frac{4}{3}$ and $0 \le z \le 1$. Furthermore, we have

$$\begin{aligned}
&H \cdot F(x,R) \\
&\quad = \frac{a}{b}\frac{R^4z^2}{\sqrt{a-1}}\left(\frac{4-3a}{6}\right)\left(-3z^2(z+1-b)+\left(2z-\frac{3}{2}b\right)(z^2+z+1)\right) \\
&\quad = \frac{a}{b}\frac{R^4z^2}{\sqrt{a-1}}\left(\frac{3a-4}{6}\right)\left(z(z-1)(z+2)+\frac{3}{2}b\left(-\left(z-\frac{1}{2}\right)^2+\frac{5}{4}\right)\right).
\end{aligned}$$

The term $z(z-1)(z+2)$ is negative on $z \in [0,1]$, so

$$H \cdot F(x,R) \le \frac{a}{b}\frac{R^4z^2}{\sqrt{a-1}}\left(\frac{3a-4}{6}\right)(3b) = \frac{R^4z^2a}{\sqrt{a-1}}\left(\frac{3a-4}{2}\right).$$

For $G(x,R)$, because $b > \frac{4}{3}$, we have

$$\begin{aligned}
G(x,R) &= R^4z^2\sqrt{b-1}\left(z\left(1-\frac{3}{2}b\right)+\left(z-\frac{3}{2}b\right)\right) \\
&= R^4z^2\sqrt{b-1}\left(z\left(2-\frac{3}{2}b\right)-\frac{3}{2}b\right) \\
&\le -\frac{3R^4z^2b\sqrt{b-1}}{2} \le -\frac{3R^4z^2a\sqrt{a-1}}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
H \cdot F(x,R) + G(x,R) &\le \frac{R^4z^2}{\sqrt{a-1}}\left(\frac{a(3a-4)}{2} - \frac{3a(a-1)}{2}\right) \\
&= -\frac{a}{2}\frac{R^4z^2}{\sqrt{a-1}} < 0.
\end{aligned}$$

□

To show $\Sigma^-_{H,c_H<c<0}$ forms a foliation between $r = 0$ and $r = r_H$ in the Kruskal extension II, we still need to prove that every point can be covered by some TSS-CMC in $\Sigma^-_{H,c_H<c<0}$. From (3.5), $f(0; H, c) = \lim_{r\to 0} f(r; H, c)$ is defined. We will estimate $f(0; H, c)$ when $c \to 0$ or $c \to c_H$.

Proposition 3.9. *For TSS-CMC hypersurfaces in* $\Sigma^-_{H,c_H<c<0}$*, we have*

$$\lim_{c\to c_H} f(0; H, c) = \infty, \quad \text{and} \quad \lim_{c\to 0} f(0; H, c) = 0.$$

Proof. First, we know

$$f(0; H, c) \ge \int_{\frac{r_H}{2}}^{R} \frac{r}{r-2M}\frac{Hr^3 + c(R)}{\sqrt{(Hr^3 + c(R))^2 + r^3(r-2M)}}\,\mathrm{d}r \quad \text{for all } R > \frac{r_H}{2},$$

where $c(R) = -HR^3 - R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$. Consider

$$\begin{aligned}
&(Hr^3 + c(R))^2 + r^3(r-2M) \\
&\quad = (R-r)P_1(r, R, \deg = 5) \\
&\quad = (R-r)^2P_2(r, R, \deg = 4) + (R-r)(2R^2)(3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}} + 3M - 2R),
\end{aligned}$$

where P_1 and P_2 are polynomials with respect to r. Because $(Hr^3 + c(R))^2 + r^3(r-2M) \geq 0$, we know $P_1(r, R, \deg = 5) > 0$ for all $r \in [0, R]$. We also have $3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}} + 3M - 2R > 0$ for $R < r_H$. Let

$$Q_1(R) = \max_{r\in[\frac{r_H}{2},R]} P_2(r, R, \deg = 4),$$

$$Q_2(R) = 2R^2(3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}} + 3M - 2R) > 0,$$

$$m(R) = \min_{r\in[\frac{r_H}{2},R]} \frac{r(Hr^3 + c(R))}{r-2M}, \quad \text{and} \quad m = \min_{R\in[\frac{r_H}{2},r_H]} m(R) > 0.$$

We have

$$f(0; H, c) \geq m \int_{\frac{r_H}{2}}^{R} \frac{1}{\sqrt{(R-r)^2 Q_1(R) + (R-r)Q_2(R)}}\, \mathrm{d}r.$$

The quantity $Q_2 \to 0$ and Q_1 bounded as $R \to r_H$ $(c \to c_H)$ imply $f(0; H, c) \to \infty$ as $c \to c_H$. This can also be seen by direct computation that the integration gives

$$\frac{m}{\sqrt{Q_1}} \ln \left| \frac{2Q_1}{Q_2}\left(R - \frac{r_H}{2} + \frac{Q_2}{2Q_1}\right) + \sqrt{\left(\frac{2Q_1}{Q_2}\left(R - \frac{r_H}{2} + \frac{Q_2}{2Q_1}\right)\right)^2 - 1}\right|.$$

When c tends to 0, we split $f(0; H, c)$ into two parts

$$\begin{aligned} f(0; H, c) &= \int_0^{\frac{R}{2}} \frac{1}{-h(r)} \sqrt{\frac{l^2(r;c)}{l^2(r;c)-1}}\, \mathrm{d}r + \int_{\frac{R}{2}}^{R} \frac{1}{-h(r)} \sqrt{\frac{l^2(r;c)}{l^2(r;c)-1}}\, \mathrm{d}r \\ &= (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

The square root term of the first part has a maximum value at $r = r_* = r_*(R)$, so

$$(\mathrm{I}) \leq \sqrt{\frac{l^2(r_*;c)}{l^2(r_*;c)-1}} \int_0^{\frac{R}{2}} \frac{1}{-h(r)}\, \mathrm{d}r = \sqrt{\frac{l^2(r_*;c)}{l^2(r_*;c)-1}} \left(-\frac{R}{2} + 2M \ln\left|\frac{2M}{2M - \frac{R}{2}}\right|\right).$$

For the second part,

$$\begin{aligned} (\mathrm{II}) &= \int_{\frac{R}{2}}^{R} \frac{1}{-h(r)} \frac{-Hr^3 - c}{\sqrt{(Hr^3+c)^2 + r^3(r-2M)}}\, \mathrm{d}r \\ &\leq Q_3(R) \int_{\frac{R}{2}}^{R} \frac{1}{\sqrt{R-r}}\, \mathrm{d}r = Q_3(R)\sqrt{2R}, \end{aligned}$$

where $Q_3(R) = \max_{r\in[\frac{R}{2},R]} \frac{1}{-h(r)} \frac{-Hr^3 - c(R)}{\sqrt{P_1(r,R,\deg=5)}}$. Hence we have

$$0 \leq f(0; H, c) \leq \sqrt{\frac{l^2(r_*;c)}{l^2(r_*;c)-1}} \left(-\frac{R}{2} + 2M \ln\left|\frac{2M}{2M - \frac{R}{2}}\right|\right) + Q_3(R)\sqrt{2R}.$$

As $c \to 0$, we have $R \to 0$ and $l^2(r_*; c)$ bounded away from 1 as well as $Q_3(R)$ being bounded. So right hand side of the above inequality tends to zero when $R \to 0$, and it gives $f(0; H, c) \to 0$ as $c \to 0$. □

Since $\lim_{c\to c_H} f(0,H,c) = \infty$ and $\lim_{c\to 0} f(0;H,c) = 0$, for any level set $t = t_0 > 0$ there is c_0 such that $t_0 = f(0;H,c_0)$ and $f(0;H,c) > t_0$ for all $c \in (c_H, c_0)$. For given $H \in \mathbb{R}$ and $c \in (c_H, 0)$, because $f(0;H,c) > t_0 > 0$ and $f(R;H,c) = 0$, there exists $r = r(c,t_0) \in (0,R)$ such that $f(r;H,c) = t_0$.

Proposition 3.10. *The* TSS-CMC *family* $\Sigma^-_{H,c_H<c<0}$ *pointwise converges to the cylindrical hypersurface* $r = r_H$ *as* $c \to c_H^+$.

Proof. For SS-CMC hypersurfaces

$$\left(f(r;H,c) = \int_r^R \frac{x}{x-2M}\frac{Hx^3+c}{\sqrt{(Hx^3+c)^2+x^3(x-2M)}}\,\mathrm{d}x, r, \theta, \phi\right),$$

$f(r;H,c)$ is a continuous function with respect to the parameter c. To prove the proposition, it suffices to show that

$$\lim_{c\to c_H} r(c,t_0) = r_H.$$

Fix $t_0 > 0$, for any $c \in (c_H, c_0)$ there exists $r_0 = r_0(R)$ such that

$$t_0 = \int_{r_0(R)}^R \frac{r}{r-2M}\frac{Hr^3+c}{\sqrt{(Hr^3+c)^2+r^3(r-2M)}}\,\mathrm{d}r.$$

Note that $c = -HR^3 - R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$, where c and R can determine each other uniquely in the family $\Sigma^-_{H,c_H<c<0}$ ($\tilde{\Sigma}^+_{H,0<c<C_H}$, $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$, and $\Sigma^+_{H,-8M^3H\geq c>c_H}$). Hence we can use R as parameter instead. Letting c tend to c_H, if $r_0(R) \not\to r_H$, then right hand side will be unbounded, and it contradicts to the finite value of left hand side. Hence we have $r_0(R) \to r_H$. □

Combining all the results above gives the following theorem.

Theorem 3.11. *For all* $H \in \mathbb{R}$, *the* TSS-CMC *family* $\Sigma^-_{H,c_H<c<0}$ *forms a foliation between two cylindrical hypersurfaces* $r = 0$ *and* $r = r_H$ *in the Kruskal extension* II.

The same arguments lead to the CMC foliation for $\tilde{\Sigma}^+_{H,0<c\leq C_H}$:

Theorem 3.12. *For all* $H \in \mathbb{R}$, *the* TSS-CMC *family* $\tilde{\Sigma}^+_{H,0<c\leq C_H}$ *forms a foliation between two cylindrical hypersurfaces* $r = 0$ *and* $r = R_H$ *in the Kruskal extension* II'.

3.4. CMC Foliation for $\Sigma^+_{H,-8M^3H>c>c_H}$ and $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$

Proposition 3.13. *There exists a constant* $C > 0$ *such that for any given* $H \geq -C$, $\frac{\mathrm{d}\ln V(2M;c)}{\mathrm{d}c} \leq 0$, *which means hypersurfaces in* $\Sigma^+_{H,-8M^3H>c>c_H}$ *are disjoint in region* II *when* $H \geq -C$.

Proof. We refer to the Appendix B for the calculation of $\frac{\mathrm{d}\ln V(2M;c(R))}{\mathrm{d}c}$, which gives

$$\frac{\mathrm{d}\ln V(2M;c(R))}{\mathrm{d}c} = \frac{1}{4MJ(R)\sqrt{-h(R)}}\left(\int_R^{2M}\frac{H\cdot F(x,R)+G(x,R)}{(x-R)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x - 1\right)$$

in (B.2). Note that $c(R) = -HR^3 - R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$, $J(R) > 0$ is given in (A.6) and $F(x,R), G(x,R)$ are in (A.3), (A.4), respectively. Function $G(x,R)$ is negative on $[R,2M]$ because $R < 2M$ and $x < 2M$.

Next we show that $F(x,R)$ is negative on $[R,2M]$ when $H \geq 0$. By the change of variables $b = \frac{2M}{R}$ and $z = \frac{x}{R}$, $F(x,R)$ can be expressed as

$$R^5 z^2 \left(-z^3 + \left(\frac{3b}{2} - 1\right) z^2 + \left(2 - \frac{3b}{2}\right) z - \frac{3b}{2}\right).$$

Let $\bar{F}(z,b) = -z^3 + \left(\frac{3b}{2} - 1\right) z^2 + \left(2 - \frac{3b}{2}\right) z - \frac{3b}{2}$. Note that $H \geq 0$ implies $b < \frac{4}{3}$. Since the coefficient of the highest order term of $\bar{F}$ is negative and $\bar{F}(\frac{3b-4}{2}, b) = \bar{F}(0,b) = \bar{F}(1,b) = -\frac{3b}{2} < 0$, we have $\bar{F}(z,b) < 0$ for $z \in [1,b]$ with $b < \frac{4}{3}$, as $\frac{3b-4}{2} < 0 < 1$ in this case. So $F(x,R)$ is negative on $[R,2M]$ when $H \geq 0$ and thus $\frac{\mathrm{d} \ln V(2M;c(R))}{\mathrm{d}c} < 0$ when $H \geq 0$. By the continuity, we get $\frac{\mathrm{d} \ln V(2M;c(R))}{\mathrm{d}c} \leq 0$ for $H \geq -C$. □

Proposition 3.14. *For TSS-CMC hypersurfaces in* $\Sigma^+_{H,-8M^3H>c>c_H}$ *in region* II, *we have*

$$\lim_{c \to c_H} V(2M;c) = \infty, \quad \textit{and} \quad \lim_{c \to -8M^3H} V(2M;c) = 0.$$

Proof. Recall that from (3.4),

$$V(r;c) = \mathrm{e}^{\frac{1}{4M}\left(r^+_{H,c} + 2M \ln |r^+_{H,c} - 2M| + \int_{r^+_{H,c}}^{r} \bar{f}'(x)\,\mathrm{d}x\right)},$$

where

$$\bar{f}'(r) = \frac{r^4}{(Hr^3+c)^2 + r^3(r-2M) - (Hr^3+c)\sqrt{(Hr^3+c)^2 + r^3(r-2M)}}.$$

is given in (3.3). Notice that

$$\begin{aligned}(Hr^3 + c(R))^2 &+ r^3(r-2M) \\ &= (r-R)^2 P_3(r,R,\deg = 4) + (r-R)(2R^2)(-3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}} - 3M + 2R)\end{aligned} \tag{3.6}$$

with $-3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}} - 3M + 2R > 0$ for $R > r_H$. Let

$$\begin{aligned}
Q_4(R) &= \max_{r \in [R,2M]} P_3(r,R,\deg = 4), \\
Q_5(R) &= 2R^2(-3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}} - 3M + 2R) > 0, \\
m(R) &= \min_{r \in [R,2M]} \frac{r^4}{\sqrt{(Hr^3+c)^2 + r^3(r-2M)} - (Hr^3+c)}, \quad \text{and} \\
m &= \min_{R \in [r_H, 2M]} m(R) > 0.
\end{aligned}$$

We have

$$\int_R^{2M} \bar{f}'(r, c(R))\,\mathrm{d}r \geq m \int_R^{2M} \frac{1}{\sqrt{(r-R)^2 Q_4(R) + (r-R) Q_5(R)}}\,\mathrm{d}r.$$

The quantity $Q_5 \to 0$ and Q_4 bounded as $R \to r_H$ $(c \to c_H)$ imply $V(2M; c) \to \infty$ as $c \to c_H$.

Now we look at the case $c \to 0$, that is $R \to 2M$. To study the limit $\lim_{R\to 2M} \int_R^{2M} \bar{f}'(r, c(R))\,\mathrm{d}r$, we need to estimate the denominator of $\bar{f}'(r)$. From (3.6) and

$$\lim_{R\to 2M}\left(\min_{r\in[R,2M]}(r-R)P_3(r,R,\deg{=}4)+(2R^2)(-3HR^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}-3M+2R)\right)$$
$$= 8M^3,$$

$$\lim_{R\to 2M}\min_{r\in[R,2M]}\frac{-(Hr^3+c)}{(2M-R)^{\frac{1}{2}}} = \lim_{R\to 2M}\min_{r\in[R,2M]}\frac{-Hr^3+HR^3+R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}}{(2M-R)^{\frac{1}{2}}}$$
$$= (2M)^{\frac{3}{2}},$$

for R close to $2M$, we have the following estimate

$$\int_R^{2M}\bar{f}'(r,c(R))\mathrm{d}r \le \int_R^{2M}\frac{(2M)^4}{M^3(r-R)+M^3(2M-R)^{\frac{1}{2}}(r-R)^{\frac{1}{2}}}\,\mathrm{d}r$$
$$\le 16M\int_R^{2M}\frac{1}{(r-R)+(2M-R)^{\frac{1}{2}}(r-R)^{\frac{1}{2}}}\,\mathrm{d}r = 32M\ln 2.$$

Hence

$$\lim_{c\to 0}V(2M;c) = \lim_{R\to 2M}\sqrt{2M-R}\;\mathrm{e}^{\frac{1}{4M}\left(R+\int_R^{2M}\bar{f}'(x)\,\mathrm{d}x\right)} = 0.$$

□

Proposition 3.15. *The* TSS-CMC *family $\Sigma_{H,-8M^3H>c>c_H}$ in region* II *pointwise converges to the cylindrical hypersurface $r = r_H$ as $c \to c_H$.*

Proof. Given $t_0 > 0$ and $c \in (c_H, -8M^3H)$, there uniquely exists $r_0 = r_0(R) \in [R, 2M)$ such that

$$t_0 = \int_R^{r_0(R)} f'(x,c(R))\,\mathrm{d}x = \int_R^{r_0(R)}\frac{x}{x-2M}\frac{Hx^3+c}{\sqrt{(Hx^3+c)^2+x^3(x-2M)}}\,\mathrm{d}x.$$

Letting c tend to c_H, if $r_0(R) \not\to r_H$ then right hand side will be unbounded, and it contradicts to the finite value of left hand side. Hence we have $r_0(R) \to r_H$. □

The case $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$ can be treated similarly, and we can conclude that

Theorem 3.16. *There exists a constant $C > 0$ such that for any given $H \ge -C$, the* TSS-CMC *family $\Sigma^+_{H,-8M^3H\ge c>c_H}$ forms a foliation in region* II*, and for any given $H \le C$, the* TSS-CMC *family $\tilde{\Sigma}^-_{H,C_H>c>-8M^3H}$ forms a foliation in region* II'.

3.5. Maximal Hypersurfaces Foliation in the Kruskal Extension

In this subsection, we will show that T-axisymmetric spacelike spherically symmetric maximal hypersurfaces form a foliation in the whole Kruskal extension. From Theorems 3.11 and 3.16, we know that $\{\Sigma_{H=0}\}$ forms a foliation in region `II` and `II'`. Using the above method, we can also prove that $\{\Sigma_{H=0}\}$ forms a foliation in region `I` and `I'`. Thus it forms a foliation in the whole Kruskal extension. This reproves the result of Beig and Ó Murchadha in [1].

When $H = 0$, from (2.9) and (2.10), we have $k_H(r) = -\tilde{k}_H(r)$. Since C_H is the maximum value of $\tilde{k}_H(r)$ and c_H is the minimum value of $k_H(r)$, we get $C_H = -c_H$. So two hypersurfaces $\Sigma^+_{H=0,c}$ and $\tilde{\Sigma}^-_{H=0,-c}, c \in (c_H, 0)$ are symmetric about the X-axis in the Kruskal extension. To prove the maximal hypersurfaces foliation, it suffices to show the case of $\Sigma^+_{H=0,0\geq c>c_H}$.

Remark that Proposition 3.6 implies that in $\Sigma^+_{H=0,0>c>c_H}$, if the limit $\lim_{r\to\infty} \frac{\mathrm{d}\ln V(r,c)}{\mathrm{d}c} \leq 0$, then hypersurfaces are disjoint. Referring to the computation of $\frac{\mathrm{d}\ln V(r,c)}{\mathrm{d}c}$ in Appendices B and C, we put $H = 0$ in (B.1). In this case, $a = \frac{4}{3}$, and from (C.2) and (C.3) we get

$$4M(2R-3M)\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c} = \int_1^{\frac{r}{R}} \frac{z^2\left(\left(2-\frac{3}{2}b\right)z-\frac{3}{2}b\right)}{(z-1)^{\frac{1}{2}}\left(z^3+z^2+z+1-b(z^2+z+1)\right)^{\frac{3}{2}}}\,\mathrm{d}z - \left.\frac{1}{\sqrt{(b-1)+z^3(z-b)}}\right|_{z=\frac{r}{R}}. \tag{3.7}$$

Consider the limit of (3.7) as r tends to infinity and let $y = z-1$:

$$\int_1^{\infty} \frac{z^2\left(\left(2-\frac{3}{2}b\right)z-\frac{3}{2}b\right)}{(z-1)^{\frac{1}{2}}\left((z^3+z^2+z+1)-b(z^2+z+1)\right)^{\frac{3}{2}}}\,\mathrm{d}z = \int_0^{\infty} \frac{\left(2-\frac{3}{2}b\right)y^3+(6-6b)y^2+\left(6-\frac{15}{2}b\right)y+(2-3b)}{y^{\frac{1}{2}}\left(y^3+(4-b)y^2+(6-3b)y+(4-3b)\right)^{\frac{3}{2}}}\,\mathrm{d}y. \tag{3.8}$$

In this case, $1 \leq b \leq \frac{4}{3}$, and the denominator can be bounded by

$$y(y+1)^2 \leq y^3+(4-b)y^2+(6-3b)y+(4-3b) \leq (y+1)^3.$$

Hence (3.8) has the following estimate:

$$\begin{aligned}
&\int_0^{\infty} \frac{\left(2-\frac{3}{2}b\right)y^3+(6-6b)y^2+\left(6-\frac{15}{2}b\right)y+(2-3b)}{y^{\frac{1}{2}}\left(y^3+(4-b)y^2+(6-3b)y+(4-3b)\right)^{\frac{3}{2}}}\,\mathrm{d}y \\
&\leq \int_0^{\infty}\left(\frac{\left(2-\frac{3}{2}b\right)y^3}{y^{\frac{1}{2}}(y(y+1)^2)^{\frac{3}{2}}}+\frac{(6-6b)y^2}{y^{\frac{1}{2}}((y+1)^3)^{\frac{3}{2}}}+\frac{\left(6-\frac{15}{2}b\right)y}{y^{\frac{1}{2}}((y+1)^3)^{\frac{3}{2}}}+\frac{2-3b}{y^{\frac{1}{2}}((y+1)^3)^{\frac{3}{2}}}\right)\mathrm{d}y \\
&= \frac{1}{2}\left(2-\frac{3}{2}b\right)+\frac{4}{35}(6-6b)+\frac{16}{105}\left(6-\frac{15}{2}b\right)+\frac{32}{35}(2-3b) \\
&= \frac{31}{7}-\frac{149}{28}b \leq -\frac{25}{28} < 0.
\end{aligned}$$

So we have $\lim_{r\to\infty}\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c} < 0$, which means all maximal hypersurfaces are disjoint.

To show these maximal hypersurfaces cover the whole Kruskal extension, it suffices to show that for all fixed $r > 2M$, $\lim_{c\to c_H} V(r;c) = \infty$ and $\lim_{c\to 0} \frac{U(r;c)}{V(r;c)} = 1$. Since $V(r;c) \geq V(2M;c)$ and $V(2M;c) \to \infty$ as $c \to c_H$ by Proposition 3.14, we get $\lim_{c\to c_H} V(r;c) = \infty$.

From (3.3), (3.4), and $c = -R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$, we have

$$\begin{aligned}\frac{U(r;c(R))}{V(r;c(R))} &= \mathrm{e}^{\frac{1}{2M}(r+2M\ln|r-2M|-R-2M\ln|R-2M|-\int_R^r \bar{f}'(x)\,\mathrm{d}x)}\\ &= \mathrm{e}^{\frac{1}{2M}(\int_R^r(1+\frac{2M}{x-2M}-\bar{f}'(x,c(R)))\,\mathrm{d}x)}\\ &= \mathrm{e}^{\frac{R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}}{2M}\left(\int_R^r \frac{x}{(x-2M)\sqrt{x-R}\sqrt{x^3-(2M-R)(x^2+Rx+R^2)}}\,\mathrm{d}x\right)}.\end{aligned}$$

The positive function $\frac{x}{\sqrt{x^3-(2M-R)(x^2+Rx+R^2)}}$ is bounded by $\frac{4}{\sqrt{3M}}$ as long as $R \geq \frac{7M}{4}$ and $x > R$. We also have

$$\int \frac{1}{(x-2M)\sqrt{x-R}}\mathrm{d}x = \frac{1}{\sqrt{2M-R}}\ln\left|\frac{\sqrt{x-R}-\sqrt{2M-R}}{\sqrt{x-R}+\sqrt{2M-R}}\right| + C.$$

Hence

$$\begin{aligned}1 \leq \lim_{c\to 0}\frac{U(r;c(R))}{V(r;c(R))} &\leq \lim_{R\to 2M} \mathrm{e}^{\frac{R^{\frac{3}{2}}}{2M}\ln\left|\frac{\sqrt{x-R}-\sqrt{2M-R}}{\sqrt{x-R}+\sqrt{2M-R}}\right|\Big|_{x=R}^{x=r}}\\ &= \lim_{R\to 2M}\left|\frac{\sqrt{r-R}-\sqrt{2M-R}}{\sqrt{r-R}+\sqrt{2M-R}}\right|^{\frac{R^{\frac{3}{2}}}{2M}} = 1.\end{aligned}$$

In conclusion, we get the foliation theorem.

Theorem 3.17. *If $H = 0$, the foliation Conjecture 3.3 is true.*

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Appendices

Appendix A. Formula of $\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c}$ in $\Sigma^-_{H,c_H<c<0}$

The aim of the Appendices A and B is to derive the formula of $\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c}$. First of all, we discuss TSS-CMC hypersurfaces in $\Sigma^-_{H,c_H<c<0}$. For $r \in [0,R)$, by the chain rule, we have

$$\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c} = \frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}c} = \frac{1}{4M}\left(\frac{\mathrm{d}}{\mathrm{d}R}\int_R^r f'(x;c(R))\,\mathrm{d}x\right)\frac{\mathrm{d}R}{\mathrm{d}c},$$

where

$$f'(x,c(R)) = \frac{1}{h(x)}\sqrt{\frac{l^2(x,c(R))}{l^2(x,c(R))-1}} \text{ and } l(x,c(R)) = \frac{1}{\sqrt{-h(R)}}\left(-Hx-\frac{c(R)}{x^2}\right)$$

from (2.6) and $c(R) = -HR^3 - R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$. Some rearrangements give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}\int_R^r f'(x;c(R))\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}R}\int_r^R \frac{A(x,R)}{h(x)\sqrt{A^2(x,R)+B(x)}}\,\mathrm{d}x\\ &= \frac{\mathrm{d}}{\mathrm{d}R}\int_r^R \frac{A(x,R)}{h(x)\sqrt{(R-x)P(x,R)}}\,\mathrm{d}x,\end{aligned}$$

where $A(x,R) = Hx^3 - HR^3 - R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$, $B(x) = x^3(x-2M)$, and $P(x,R) \neq 0$. Since $\int_R^r f'(x;c(R))\mathrm{d}x$ is an improper integral, we have to be careful. For $\varepsilon > 0$, define

$$\phi_\varepsilon(R) = \int_r^{R-\varepsilon} \frac{A(x,R)}{h(x)\sqrt{A^2(x,R)+B(x)}}\,\mathrm{d}x = \int_r^{R-\varepsilon} \frac{A(x,R)}{h(x)\sqrt{(R-x)P(x,R)}}\,\mathrm{d}x.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}&\frac{\mathrm{d}\phi_\varepsilon(R)}{\mathrm{d}R}\\ &= \frac{A(R-\varepsilon,R)}{h(R-\varepsilon)\sqrt{A^2(R-\varepsilon,R)+B(R-\varepsilon)}} + \int_r^{R-\varepsilon}\frac{1}{h(x)}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{A(x,R)}{\sqrt{A^2(x,R)+B(x)}}\right)\mathrm{d}x\\ &= \frac{1}{h(R-\varepsilon)}\int_r^{R-\varepsilon}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{A(x,R)}{\sqrt{A^2(x,R)+B(x)}}\right) + \frac{h(R-\varepsilon)}{h(x)}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{A(x,R)}{\sqrt{A^2(x,R)+B(x)}}\right)\right]\mathrm{d}x\\ &\quad + \frac{1}{h(R-\varepsilon)}\frac{A(r,R)}{\sqrt{A^2(r,R)+B(r)}}.\end{aligned} \tag{A.1}$$

A direct computation shows that the terms with order $(R-x)^{-\frac{3}{2}}$ in the integrand all have ε in their coefficients. Therefore $\frac{\mathrm{d}\phi_\varepsilon(R)}{\mathrm{d}R}$ converges uniformly and it implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}R}\int_R^r f'(x;c(R))\,\mathrm{d}x\\ &\quad = \frac{1}{h(R)}\int_r^R \frac{H\cdot F(x,R)+G(x,R)}{(R-x)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x + \frac{1}{h(R)}\frac{A(r,R)}{\sqrt{A^2(r,R)+B(r)}},\end{aligned}$$

where

$$\begin{aligned}F(x,R) &= x^2(-3x^2(x+R-2M)+(2x-3M)(x^2+Rx+R^2))\\ &= x^2((3M-x)(x^2-R^2)+xR(R-3M-x)) &\text{(A.2)}\\ &= x^2(-x^3+(3M-R)x^2+(2R^2-3MR)x-3MR^2), &\text{(A.3)}\\ G(x,R) &= x^2\sqrt{-h(R)}(x(R-3M)+R(x-3M)). &\text{(A.4)}\end{aligned}$$

The function $c(R) = -HR^3 - R^{\frac{3}{2}}(2M-R)^{\frac{1}{2}}$ implies

$$\frac{\mathrm{d}R}{\mathrm{d}c} = \frac{\sqrt{-h(R)}}{-3HR^{\frac{3}{2}}(2M-R)+(2R-3M)} \tag{A.5}$$

and we denote

$$J(R) = -3HR^{\frac{3}{2}}(2M-R)+(2R-3M). \tag{A.6}$$

In conclusion, we have

$$\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c} = -\frac{1}{4MJ(R)\sqrt{-h(R)}} \times \left(\int_r^R \frac{H\cdot F(x,R)+G(x,R)}{(R-x)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x + \frac{A(r,R)}{\sqrt{A^2(r,R)+B(r)}}\right). \tag{A.7}$$

Taking $r=0$ in (A.7) and $\frac{A(0,R)}{\sqrt{A^2(0,R)+B(0)}} = -1$ give

$$\frac{\mathrm{d}\ln V(0;c(R))}{\mathrm{d}c} = -\frac{1}{4MJ(R)\sqrt{-h(R)}}\left(\int_0^R \frac{H\cdot F(x,R)+G(x,R)}{(R-x)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x - 1\right) \tag{A.8}$$

The criteria in Proposition 3.4 implies that if $\frac{\mathrm{d}\ln V(0;c)}{\mathrm{d}c} = \lim_{r\to 0}\frac{\mathrm{d}\ln V(r;c)}{\mathrm{d}c} \le 0$, then hypersurfaces in $\Sigma^-_{H,c_H<c<0}$ are disjoint. Remark that $\frac{\mathrm{d}R}{\mathrm{d}c} < 0$ for $\Sigma^-_{H,c_H<c<0}$, which implies $J(R)<0$.

Appendix B. Formula of $\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c}$ in $\Sigma^+_{H,-8M^3H>c>c_H}$

Next, we consider TSS-CMC hypersurfaces in $\Sigma^+_{H,-8M^3H>c>c_H}$. For $r\in(R,\infty)$, by the chain rule, we have $\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c} = \frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}c}$ and formula (3.4) implies

$$4M\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}R} = \lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}R}\left(\int_{R+\varepsilon}^r \bar{f}'(x;R)\,\mathrm{d}x + R + 2M\ln|R-2M|\right).$$

Since $\bar{f}'$ is smooth on $r\in[R+\varepsilon,r]$, we can use the fundamental theorem of calculus to get

$$\begin{aligned}
&4M\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}R}\\
&= \lim_{\varepsilon\to 0}\left(-\bar{f}'(R+\varepsilon;R) + \int_{R+\varepsilon}^r \frac{\mathrm{d}}{\mathrm{d}R}\bar{f}'(x;R)\,\mathrm{d}x + \frac{1}{h(R+\varepsilon)}\right)\\
&= \lim_{\varepsilon\to 0}\left(-f'(R+\varepsilon;R) + \int_{R+\varepsilon}^r \frac{\mathrm{d}}{\mathrm{d}R}f'(x;R)\,\mathrm{d}x\right)\\
&= \lim_{\varepsilon\to 0}\left[-\frac{A(R+\varepsilon,R)}{h(R+\varepsilon)\sqrt{A^2(R+\varepsilon,R)+B(R+\varepsilon)}} + \int_{R+\varepsilon}^r \frac{1}{h(x)}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{A(x,R)}{\sqrt{A^2(x,R)+B(x)}}\right)\mathrm{d}x\right]
\end{aligned}$$

Similar to the formula (A.1) and its argument, we get

$$4M\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}R} = -\frac{1}{h(R)}\int_R^r \frac{H\cdot F(x,R)+G(x,R)}{(R-x)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x - \frac{1}{h(R)}\frac{A(r,R)}{\sqrt{A^2(r,R)+B(r)}},$$

where $F(x,R), G(x,R)$ are as (A.2) and (A.4), respectively. From (A.5), we get

$$\frac{\mathrm{d}\ln V(r;c(R))}{\mathrm{d}c} = \frac{1}{4MJ(R)\sqrt{-h(R)}}\left(\int_R^r \frac{H\cdot F(x,R)+G(x,R)}{(x-R)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x + \frac{A(r,R)}{\sqrt{A^2(r,R)+B(r)}}\right), \tag{B.1}$$

where $J(R) = -3HR^{\frac{3}{2}}(2M-R)+(2R-3M)$. In particular, taking $r = 2M$, formula (B.1) becomes

$$\frac{\mathrm{d}\ln V(2M;c(R))}{\mathrm{d}c} = \frac{1}{4MJ(R)\sqrt{-h(R)}}\left(\int_R^{2M} \frac{H\cdot F(x,R)+G(x,R)}{(x-R)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x - 1\right). \tag{B.2}$$

Remark that Proposition 3.6 implies that in $\Sigma^+_{H,-8M^3H>c>c_H}$, if the limit $\lim_{r\to\infty}\frac{\mathrm{d}\ln V(r,c)}{\mathrm{d}c} \le 0$, then hypersurfaces are disjoint. Furthermore, we have $\frac{\mathrm{d}R}{\mathrm{d}c} > 0$, which implies $J(R) > 0$.

Appendix C. Change of Variables

We shall use the following change of variables for better control. Define $a = \frac{2M}{r_H}$, $b = \frac{2M}{R}$, and $z = \frac{x}{R}$. Then $h(R) = 1-b$ and

$$HR = \frac{2r_H - 3M}{3\sqrt{r_H^3(2M-r_H)}}R = \frac{R}{r_H}\frac{1}{\sqrt{\frac{2M}{r_H}-1}}\left(\frac{4-\frac{6M}{r_H}}{6}\right) = \frac{a}{b}\frac{1}{\sqrt{a-1}}\left(\frac{4-3a}{6}\right). \tag{C.1}$$

Furthermore, we have

$$H\cdot F(x,R) = \frac{a}{b}\frac{R^4z^2}{\sqrt{a-1}}\left(\frac{4-3a}{6}\right)\left(-3z^2(z+1-b)+\left(2z-\frac{3}{2}b\right)(z^2+z+1)\right),$$

$$G(x,R) = R^4z^2\sqrt{b-1}\left(z\left(1-\frac{3}{2}b\right)+\left(z-\frac{3}{2}b\right)\right),$$

and

$$P(x,R) = R^3\left(\left(\frac{a(4-3a)}{6b(a-1)^{\frac{1}{2}}}\right)^2(z-1)(z^2+z+1)^2 - \frac{a(4-3a)}{3b(a-1)^{\frac{1}{2}}}(z^2+z+1)(b-1)^{\frac{1}{2}} + (z^3+z^2+z+1)-b(z^2+z+1).\right)$$

Hence

$$\int_R^r \frac{H\cdot F(x,R)+G(x,R)}{(x-R)^{\frac{1}{2}}(P(x,R))^{\frac{3}{2}}}\,\mathrm{d}x = \int_1^{\frac{r}{R}} \frac{\tilde{F}(z,a,b)+\tilde{G}(z,b)}{(z-1)^{\frac{1}{2}}(\tilde{P}(z,a,b))^{\frac{3}{2}}}\,\mathrm{d}z, \tag{C.2}$$

where

$$\tilde{F}(z,a,b) = \frac{a(4-3a)}{6b(a-1)^{\frac{1}{2}}} z^2 \left(-3z^2(z+1-b) + \left(2z-\frac{3}{2}b\right)(z^2+z+1)\right),$$

$$\tilde{G}(z,b) = z^2(b-1)^{\frac{1}{2}} \left(\left(2-\frac{3}{2}b\right)z - \frac{3}{2}b\right),$$

and

$$\begin{aligned}\tilde{P}(z,a,b) &= \left(\frac{a(4-3a)}{6b(a-1)^{\frac{1}{2}}}\right)^2 (z-1)(z^2+z+1)^2 - \frac{a(4-3a)}{3b(a-1)^{\frac{1}{2}}}(z^2+z+1)(b-1)^{\frac{1}{2}}\\ &\quad + (z^3+z^2+z+1) - b(z^2+z+1).\end{aligned}$$

We also have

$$\begin{aligned}\frac{A(r,R)}{\sqrt{A^2(r,R)+B(r)}} &= \left.\frac{A(x,R)}{\sqrt{A^2(x,R)+B(x)}}\right|_{x=r}\\ &= \left.\frac{\frac{a(4-3a)}{6b(a-1)^{\frac{1}{2}}}(z^3-1)-(b-1)^{\frac{1}{2}}}{\sqrt{\left(\frac{a(4-3a)}{6b(a-1)^{\frac{1}{2}}}(z^3-1)-(b-1)^{\frac{1}{2}}\right)^2 + z^3(z-b)}}\right|_{z=\frac{r}{R}}.\end{aligned} \tag{C.3}$$

These change of variables are helpful to get better estimates on the criteria (A.8) and (B.1).

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