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Asymptotic Behavior of Solutions to the Drift-Diffusion Equation with Critical Dissipation

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Abstract. In this paper, the initial value problem for the drift-diffusion equation which stands for a model of a semiconductor device is studied. When the dissipative effect on the drift-diffusion equation is given by the half Laplacian, the dissipation balances to the extra force term. This case is called critical. The goal of this paper is to derive decay and asymptotic expansion of the solution to the drift-diffusion equation as time variable tends to infinity.

1. Introduction

We consider the initial value problem for the drift-diffusion equation with critical dissipation in the Euclidian space:

$$\begin{cases} \partial_t u + (-\Delta)^{1/2} u - \nabla \cdot (u \nabla \psi) = 0, & t > 0, \ x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$, $\partial_t = \partial/\partial t$, $(-\Delta)^{1/2}\varphi = \mathcal{F}^{-1}[|\xi|\mathcal{F}[\varphi]]$, $\nabla = (\partial_1, \ldots, \partial_n)$, $\partial_j = \partial/\partial x_j$ $(1 \leq j \leq n)$, $\Delta = \partial_1^2 + \cdots + \partial_n^2$ and $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a given initial data. The unknown functions u and $\psi : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ stand for the density of electrons and the potential of electromagnetic field on a semiconductor. The drift-diffusion equation is first derived from the mass-conservation law for the electrons: $\partial_t u = \nabla \cdot (\nabla u + u \nabla \psi)$ (see [21]). In this equation, the dissipation is given by the positive Laplacian $-\Delta$, which is induced by the Brownian motion. On the other hand, an electron in a semiconductor may jump from a dopant into another. In such case, it is appropriate that a dissipation on a model of semiconductor is given by an operator which is induced by the jumping process on stochastic process. The fractional Laplacian $(-\Delta)^{\theta/2}\varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]]$ is a suitable operator to illustrate the dissipation of jumping particles (cf. [3,12,19]). When $1 < \theta \leq 2$, the drift-diffusion equation with the fractional Laplacian $\partial_t u + (-\Delta)^{\theta/2} u - \nabla \cdot (u \nabla \psi) = 0$ is treated as a parabolic equation. Indeed, the L^p theory for parabolic equations provides well-posedness, global existence, decay and asymptotic expansion as $t \to \infty$ of solutions to the drift-diffusion equation in this case (we refer to [1,2,7,11,15–17,22,25]). When $\theta = 1$, the dissipation balances the nonlinear effect, which is called the critical case. Thus, the L^p theory for a parabolic equation does not work on our problem. Such a situation occurs in the studies for the quasi-geostrophic equation (cf. [4] and references therein). Now any terms on the equation contain the first-order derivatives. Then, the first equation in (1.1) is an elliptic equation of order 1. The energy method and the commutator estimate show the local and global existence, and the uniqueness of solutions of (1.1), which is already discussed in [18,24,26]. By the smoothing effect of the operator $\mathrm{e}^{-t(-\Delta)^{1/2}}$, we can confirm that this global solution satisfies that

$$u \in C^\infty((0,\infty), H^\infty(\mathbb{R}^n)). \tag{1.2}$$

Moreover, the mass-conservation law $\|u(t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$ holds if positivity of the initial data is assumed. On the other hand, Li et al. [18] proved that

$$\|u(t)\|_{L^p(\mathbb{R}^n)} \leq \|u_0\|_{L^p(\mathbb{R}^n)} \tag{1.3}$$

for any $1 \leq p \leq \infty$ and $t > 0$. Moreover, the following inequality with $1 \leq p \leq \infty$ holds for $t > 0$:

$$\|u(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-n(1-\frac{1}{p})}. \tag{1.4}$$

For $\frac{n}{n-1} < q \leq \infty$, the conservative force $\nabla\psi$ fulfills

$$\|\nabla\psi(t)\|_{L^q(\mathbb{R}^n)} \leq C(1+t)^{-n(1-\frac{1}{q})+1}. \tag{1.5}$$

The solution constructed in [18,24,26] satisfies (1.4) and (1.5), if the initial data are positive. Those inequalities are established in Propositions 2.7 and 2.8 in Sect. 2. The aim of this paper is to derive lower bounds on the decay rates of the solution. Particularly, we show asymptotic behavior of the solution as $t \to \infty$. The Duhamel formula says that the Poisson kernel

$$P(t,x) = \pi^{-\frac{n+1}{2}} \Gamma(\tfrac{n+1}{2}) \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$$

solves this linear equation $\partial_t u + (-\Delta)^{1/2} u = 0$. The Poisson kernel satisfies

$$\lambda^n P(\lambda t, \lambda x) = P(t,x)$$

for any $\lambda > 0$,

$$\|P(t)\|_{L^p(\mathbb{R}^n)} = t^{-n(1-\frac{1}{p})} \|P(1)\|_{L^p(\mathbb{R}^n)}$$

and

$$\|\nabla P(t)\|_{L^p(\mathbb{R}^n)} = t^{-n(1-\frac{1}{p})-1} \|\nabla P(1)\|_{L^p(\mathbb{R}^n)}$$

for any $1 \leq p \leq \infty$ and $t > 0$. The Duhamel formula also gives the representation of the solution of (1.1) by

$$u(t) = P(t) * u_0 + \int_0^t P(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) \mathrm{d}s, \tag{1.6}$$

where $*$ denotes the convolution for x, which is called the mild solution.

We have asymptotic behavior of the solution of (1.1) as $t \to \infty$ in the following theorem.

Theorem 1.1. *Let $n \geq 3$, $u_0 \in L^1(\mathbb{R}^n, \sqrt{1+|x|^2}\mathrm{d}x)$ and the solution u of (1.1) satisfy (1.2) and (1.4). Then*

$$\|u(t) - M_u P(t) - m_u \cdot \nabla P(t)\|_{L^p(\mathbb{R}^n)} = o(t^{-n(1-\frac{1}{p})-1})$$

as $t \to \infty$ holds for any $1 < p < \infty$, where $M_u = \int_{\mathbb{R}^n} u_0(y)\mathrm{d}y$ and $m_u = \int_{\mathbb{R}^n} (-y) u_0(y) \mathrm{d}y$.

In the two-dimensional case, we introduce

$$J(t,x) = \int_0^{t/2} \nabla P(t-s) * (P \nabla (-\Delta)^{-1} P)(s) \mathrm{d}s + \int_{t/2}^t P(t-s) * \nabla \cdot (P \nabla (-\Delta)^{-1} P)(s) \mathrm{d}s. \tag{1.7}$$

This function is well defined on $C((0,\infty); L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ and is not zero (those facts are confirmed by the similar argument as in [25, Proposition 2.9]). Furthermore, the fact that

$$\lambda^3 J(\lambda t, \lambda x) = J(t,x)$$

holds for any $\lambda > 0$ yields

$$\|J(t)\|_{L^p(\mathbb{R}^2)} = t^{-2(1-\frac{1}{p})-1} \|J(1)\|_{L^p(\mathbb{R}^2)}$$

for any $1 \leq p \leq \infty$ and $t > 0$. Namely, $J(t)$ has the same decay rate as the one of $\nabla P(t)$. But we know that there is no constant $\boldsymbol{a} \in \mathbb{R}^2$ such that $J(t) = \boldsymbol{a} \cdot \nabla P(t)$ since such a representation contradicts the relation $J(t, x_1, x_2) = J(t, |x_1|, |x_2|)$. The function $J(t)$ provides the following theorem.

Theorem 1.2. *Let $n = 2$, $u_0 \in L^1(\mathbb{R}^2, \sqrt{1+|x|^2}\mathrm{d}x)$ and the solution u of (1.1) satisfy (1.2) and (1.4). Then*

$$\|u(t) - M_u P(t) - m_u \cdot \nabla P(t) - M_u^2 J(t)\|_{L^p(\mathbb{R}^2)} = o(t^{-2(1-\frac{1}{p})-1})$$

as $t \to \infty$ holds for any $1 < p < \infty$, where $M_u = \int_{\mathbb{R}^2} u_0(y)\mathrm{d}y$ and $m_u = \int_{\mathbb{R}^2} (-y) u_0(y) \mathrm{d}y$.

We remark that the solution constructed in [18,24,26] satisfies (1.2), (1.3) and (1.4). Another important point to note here is that the asymptotic expansion in Theorems 1.1 and 1.2 is determined only by the mass and the moment of the initial data. The proofs of our main results are based on $L^p - L^q$ estimate for the mild solution. The similar argument as in [6] works in the sub-critical cases. However, their idea is not effective in the critical case. In the subcritical cases $1 < \theta \le 2$, the mild solution is written by

$$u(t) = G_\theta(t) * u_0 + \int_0^t \nabla G_\theta(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s,$$

where $G_\theta(t,x) = \mathcal{F}^{-1}[\mathrm{e}^{-t|\xi|^\theta}](x)$ is the fundamental solution of the linear equation $\partial_t u + (-\Delta)^{\theta/2}u = 0$. Unfortunately, in the critical case $\theta = 1$, this representation is not appropriate since the integration

$$\int_0^t \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s$$

may diverge to infinity. Indeed, we see that

$$\begin{aligned}&\left\| \int_0^t \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \right\|_{L^p(\mathbb{R}^n)} \\ &\quad \le C\int_0^t (t-s)^{-1}\|(u\nabla(-\Delta)^{-1}u)(s)\|_{L^p(\mathbb{R}^n)}\mathrm{d}s.\end{aligned}$$

To avoid this crux, we divide the nonlinear term into

$$\begin{aligned}&\int_0^t \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)\mathrm{d}s \\ &\quad = \int_0^{t/2} \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)\mathrm{d}s \\ &\qquad + \int_{t/2}^t \nabla(-\Delta)^{-\sigma/2}P(t-s) * (-\Delta)^{\sigma/2}(u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s\end{aligned}$$

for some $\sigma > 0$, and prepare the decay of $(-\Delta)^{\sigma/4}u$ (see Proposition 2.9 in Sect. 2). We confirm the decay of $(-\Delta)^{\sigma/2}u$ by the energy method with the aid of (1.4). In the sub-critical case $\theta > 1$, such a procedure is not required. Now, we refer the following initial value problem for the Burgers equation which is corresponding to (1.1):

$$\begin{cases} \partial_t\omega + (-\partial_x^2)^{1/2}\omega + \omega\partial_x\omega = 0,\ t>0, & x \in \mathbb{R}, \\ \omega(0,x) = \omega_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.8}$$

If smallness of ω_0 is assumed, then well-posedness, global existence in time and decay of solutions can be proved (see [8]). The nonlinear term of (1.8) decays with the same order as the one of (1.1) with $n = 2$. Namely, they fulfill $\|\omega^2\|_{L^1(\mathbb{R})} \le C(1+t)^{-1}$ and $\|u\nabla\psi\|_{L^1(\mathbb{R}^2)} \le C(1+t)^{-1}$. For the Burgers equation with the dissipative character $-\partial_x^2$, it is known that solutions tend to a self-similar solution with a logarithmic order (see [13]). For (1.8), Iwabuchi [8] derived the following estimate for any large t:

$$c(1+t)^{-3/2}\log(2+t) \le \|\omega(t) - M_\omega P(t)\|_{L^2(\mathbb{R})} \le Ct^{-3/2}\log(2+t), \tag{1.9}$$

where $M_\omega = \int_{\mathbb{R}} \omega_0(x)\mathrm{d}x$, $P = P(t,x)$ is the one-dimensional Poisson kernel, and c and C are some positive constants which satisfy $c < C$. For (1.1), the correction term $J(t)$ has been derived by the renormalization. Since the initial value problem (1.8) is represented by

$$\omega(t) = P(t) * \omega_0 - \frac{1}{2}\int_0^{t/2} \partial_x P(t-s) * (\omega^2)(s)\mathrm{d}s - \int_{t/2}^{t} P(t-s) * (\omega\partial_x\omega)(s)\mathrm{d}s, \tag{1.10}$$

applying the renormalization to (1.8), then we see

$$J_\omega(t,x) = -\frac{1}{2}\int_0^{t/2} \partial_x P(t-s) * (P^2)(s)\mathrm{d}s - \int_{t/2}^{t} P(t-s) * (P\partial_x P)(s)\mathrm{d}s$$

as the correction term. However, the integration on the first term tends to infinity. Iwabuchi gave the new correction term for the solution of (1.8) by

$$\mathcal{J}_\omega(t,x) = -\frac{1}{2}\int_0^{t/2} \partial_x P(t-s) * (P^2)(1+s)\mathrm{d}s - \int_{t/2}^{t} P(t-s) * (P\partial_x P)(1+s)\mathrm{d}s \tag{1.11}$$

and derived the asymptotic expansion as in (A.1). The correction term $\mathcal{J}_\omega(t)$ is well defined. Furthermore, we can show that

$$\|\omega(t) - M_\omega P(t) - M_\omega^2 \mathcal{J}_\omega(t)\|_{L^2(\mathbb{R})} = O(t^{-3/2}) \tag{1.12}$$

as $t \to \infty$, and

$$c\,(1+t)^{-3/2}\log(2+t) \le \|\mathcal{J}_\omega(t)\|_{L^2(\mathbb{R})} \le Ct^{-3/2}\log(2+t)$$

for any $t > 0$. The decay (1.12) is not seen in [8]. More detailed estimate will be discussed in Appendix A. In [8], (1.9) is shown by the $L^p - L^q$ type estimate for (1.10) on the corresponding Besov spaces. Our method in this paper does not need supplementary spaces. When we try to give an asymptotic expansion with higher order, we should estimate the moment of the solution. One knows that, in the Besov spaces, an estimate for the moment is complicated. We expect that our method may express an asymptotic expansion with higher order.

Notation. In this paper, we use the following notation. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we denote $x \cdot y = x_1y_1 + \cdots + x_ny_n$, $|x| = \sqrt{x\cdot x}$. We define the Fourier transform and the Fourier inverse transform by $\mathcal{F}[\varphi](\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi}\varphi(x)\mathrm{d}x$, $\mathcal{F}^{-1}[\varphi](x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\varphi(\xi)\mathrm{d}\xi$, where $i = \sqrt{-1}$. The partial derivative operators are denoted by $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(1 \le j \le n)$, $\nabla = (\partial_1, \ldots, \partial_n)$, $\Delta = \partial_1^2 + \cdots + \partial_n^2$ and $(-\Delta)^{\theta/2}\varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]]$ for $\theta > 0$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we abbreviate $\nabla^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We denote L^p and $W^{s,p}$ the Lebesgue spaces and the Sobolev spaces for $1 \le p \le \infty$ and $s > 0$ respectively. The norm of $L^p(\mathbb{R}^n)$ is represented by $\|\cdot\|_{L^p(\mathbb{R}^n)}$. For some $\sigma \ge 0$ and $a > 0$, if a function $R(t)$ satisfies $t^\sigma R(t) \to a$ as $t \to \infty$ then we write $R(t) = O(t^{-\sigma})$ as

$t \to \infty$. In the case $t^\sigma R(t) \to 0$ as $t \to \infty$, we denote $R(t) = o(t^{-\sigma})$ as $t \to \infty$. Various constants are simply denoted by C.

2. Preliminaries

In this section, we prepare several lemmas and propositions to prove our main theorems. It is well known that the fractional Laplacian is self-adjoint on $L^2(\mathbb{R}^n)$. Hence the bilinear form with the fractional Laplacian is positive. The following lemma gives the generalization of this fact.

Lemma 2.1. *Let $n \geq 2,\ p \geq 2,\ 0 \leq \sigma \leq 2$ and $\varphi \in W^{\sigma,p}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |\varphi|^{p-2}\varphi(-\Delta)^{\sigma/2}\varphi \mathrm{d}x \geq \frac{2}{p}\int_{\mathbb{R}^n} |(-\Delta)^{\sigma/4}(|\varphi|^{p/2})|^2 \mathrm{d}x$$

is fulfilled.

For the proof of Lemma 2.1, see [4,5,10]. For $k \in \mathbb{Z}_+,\ \sigma \geq 0,\ 1 \leq p \leq \infty$ and $t > 0$, the Poisson kernel fulfills

$$\|\partial_t^k(-\Delta)^{\sigma/2}P(t)\|_{L^p(\mathbb{R}^n)} = t^{-n(1-\frac{1}{p})-k-\sigma}\|\partial_t^k\nabla^\alpha P(1)\|_{L^p(\mathbb{R}^n)}.$$

Hence, Hausdorff–Young's inequality gives the following lemmas.

Lemma 2.2. *Let $n \geq 1,\ 1 \leq p \leq q \leq \infty,\ k \in \mathbb{Z}_+$ and $\sigma \geq 0$. Then, there exists a positive constant C such that*

$$\|\partial_t^k(-\Delta)^{\sigma/2}P(t) * \varphi\|_{L^q(\mathbb{R}^n)} \leq Ct^{-n(\frac{1}{p}-\frac{1}{q})-k-\sigma}\|\varphi\|_{L^p(\mathbb{R}^n)}$$

holds for any $\varphi \in L^p(\mathbb{R}^n)$ and $t > 0$.

Lemma 2.3. *Let $n \geq 1,\ k \in \mathbb{Z}_+$ and $\varphi \in L^1(\mathbb{R}^n, (1+|x|^2)^{k/2}\,\mathrm{d}x)$. Then,*

$$\left\|P(t) * \varphi - \sum_{|\alpha| \leq k}\frac{\nabla^\alpha P(t)}{\alpha!}\int_{\mathbb{R}^n}(-y)^\alpha\varphi(y)\mathrm{d}y\right\|_{L^p(\mathbb{R}^n)} = o(t^{-n(1-\frac{1}{p})-k})$$

as $t \to \infty$ holds for any $1 \leq p \leq \infty$. In addition, if $|x|^{k+1}\varphi \in L^1(\mathbb{R}^n)$, then

$$\left\|P(t) * \varphi - \sum_{|\alpha| \leq k}\frac{\nabla^\alpha P(t)}{\alpha!}\int_{\mathbb{R}^n}(-y)^\alpha\varphi(y)\mathrm{d}y\right\|_{L^p(\mathbb{R}^n)} \leq Ct^{-n(1-\frac{1}{p})-k}(1+t)^{-1}$$

is fulfilled for any $1 \leq p \leq \infty$ and $t > 0$.

Hardy–Littlewood–Sobolev's inequality yields the following inequalities.

Lemma 2.4. *Let $n \geq 2,\ 1 < \sigma < n,\ 1 < p < \frac{n}{\sigma}$ and $\frac{1}{p_*} = \frac{1}{p} - \frac{\sigma}{n}$. Then, there exist positive constants C and C' such that*

$$\|(-\Delta)^{-\sigma/2}\varphi\|_{L^{p_*}(\mathbb{R}^n)} \leq C\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$ and

$$\|\phi\|_{L^{p_*}(\mathbb{R}^n)} \leq C'\|(-\Delta)^{\sigma/2}\phi\|_{L^p(\mathbb{R}^n)}$$

for any $\phi \in W^{\sigma,p}(\mathbb{R}^n)$ are fulfilled.

For Hardy–Littlewood–Sobolev's inequality, we refer the reader to [23, 27]. Gagliardo–Nirenberg's inequality for the Riesz potentials is given by the following lemma (cf. [20] and references therein).

Lemma 2.5. *For $0<\sigma<s<n,\ 1<q,r<\infty$ and p with $\frac{1}{p}=(1-\frac{\sigma}{s})\frac{1}{q}+\frac{\sigma}{s}\frac{1}{r}$ there exists a positive constant C such that*

$$\|(-\Delta)^{\sigma/2}\varphi\|_{L^p(\mathbb{R}^n)}\le C\|\varphi\|_{L^q(\mathbb{R}^n)}^{1-\frac{\sigma}{s}}\|(-\Delta)^{s/2}\varphi\|_{L^r(\mathbb{R}^n)}^{\frac{\sigma}{s}}$$

is satisfied for any $\varphi\in L^q(\mathbb{R}^n)\cap\dot{W}^{s,r}(\mathbb{R}^n)$.

The Leibnitz rule is generalized as in the following lemma.

Lemma 2.6. *Let $1<p,q_1,q_2<\infty$ and $1<r_1,r_2\le\infty$ with $\frac{1}{p}=\frac{1}{r_1}+\frac{1}{q_1}=\frac{1}{r_2}+\frac{1}{q_2}$, and $\sigma>0$. Then,*

$$\begin{aligned}\|(-\Delta)^{\sigma/2}(fg)\|_{L^p(\mathbb{R}^n)}\le C(&\|(-\Delta)^{\sigma/2}f\|_{L^{q_1}(\mathbb{R}^n)}\|g\|_{L^{r_1}(\mathbb{R}^n)}\\&+\|f\|_{L^{r_2}(\mathbb{R}^n)}\|(-\Delta)^{\sigma/2}g\|_{L^{q_2}(\mathbb{R}^n)})\end{aligned}$$

holds for any $f\in W^{\sigma,q_1}(\mathbb{R}^n)\cap L^{r_2}(\mathbb{R}^n)$ and $g\in L^{r_1}(\mathbb{R}^n)\cap W^{\sigma,q_2}(\mathbb{R}^n)$.

For the proof, we see [9,14]. The decay (1.4) is ensured in the following proposition.

Proposition 2.7. *Let $n\ge 2,\ u_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n),\ u_0\ge 0$ and the solution u of (1.1) satisfy (1.2) and (1.3). Then there exists a positive constant C such that the inequality (1.4) holds for any $1\le p\le\infty$ and $t>0$.*

Proof. Let $p\ge 2$. We multiply (1.1) by $|u|^{p-2}u$ and obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^p(\mathbb{R}^n)}^p+p\int_{\mathbb{R}^n}|u|^{p-2}u(-\Delta)^{1/2}u\mathrm{d}x\\&\quad=p\int_{\mathbb{R}^n}|u|^{p-2}u\nabla\cdot(u\nabla(-\Delta)^{-1}u)\mathrm{d}x.\end{aligned}\tag{2.1}$$

By the integration by parts, we see that

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^{p-2}u\nabla\cdot(u\nabla(-\Delta)^{-1}u)\mathrm{d}x&=-\Big(1-\tfrac{1}{p}\Big)\int_{\mathbb{R}^n}\nabla(u^p)\cdot\nabla(-\Delta)^{-1}u\mathrm{d}x\\&=-\Big(1-\tfrac{1}{p}\Big)\int_{\mathbb{R}^n}u^pu\mathrm{d}x.\end{aligned}$$

Hence, since $u\in L^{p+1}(\mathbb{R}^3)$ and $u\ge 0$ hold, we have that

$$\int_{\mathbb{R}^n}|u|^{p-2}u\nabla\cdot(u\nabla(-\Delta)^{-1}u)\mathrm{d}x\le 0.$$

Applying this result and Lemma 2.1 into (2.1), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^p(\mathbb{R}^n)}^p+2\|(-\Delta)^{1/4}(u^{p/2})\|_{L^2(\mathbb{R}^n)}^2\le 0.$$

This inequality yields the assertion of Proposition 2.7 with $2\le p\le\infty$. For more details, we refer the reader to [15,22]. Therefore, by applying the Hölder inequality with (1.3) and (1.4) with $p=\infty$, we obtain (1.4) for $1\le p\le 2$ and complete the proof. □

Proposition 2.7 immediately leads an estimate for the potential.

Proposition 2.8. *Let $n \geq 2$ and a function $u = u(t,x)$ satisfy (1.4) for any $1 \leq p \leq \infty$ and $t > 0$. Then, $\nabla\psi = \nabla(-\Delta)^{-1}u$ satisfies (1.5) for $\frac{n}{n-1} < q \leq \infty$.*

Proof. When $\frac{n}{n-1} < q < \infty$, Lemma 2.4 and (1.4) give the desired inequality. We consider the case $q = \infty$. Since the operator $\nabla(-\Delta)^{-1}$ is written by

$$\nabla(-\Delta)^{-1}\varphi(x) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}\varphi(y)\mathrm{d}y,$$

we see

$$\begin{aligned}\left|\nabla(-\Delta)^{-1}u(t)\right| &\leq C\left(\int_{|x-y|\leq g(t)} + \int_{|x-y|\geq g(t)}\right)\frac{|u(t,y)|}{|x-y|^{n-1}}\mathrm{d}y \\ &\leq C(g(t)\|u(t)\|_{L^\infty(\mathbb{R}^n)} + g(t)^{-n+1}\|u(t)\|_{L^1(\mathbb{R}^n)})\end{aligned}$$

for any positive function $g(t)$. Hence if we choose $g(t) = 1+t$, then the above inequality and (1.4) give (1.5) with $q = \infty$. □

In the proof of the theorems, an estimate for $(-\Delta)^{1/4}u$ is required.

Proposition 2.9. *Let $n \geq 2$ and the solution u of (1.1) satisfy (1.2) and (1.4). Then, there exist positive constants C and T such that*

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(\mathbb{R}^n)} \leq Ct^{-1/2}(1+t)^{-n/2} \tag{2.2}$$

holds for any $t \geq T$.

Proof. Let $q > n+1$. We multiply (1.1) by $t^q(-\Delta)^{1/2}u$ and have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(t^q\|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R}^n)}) + 2t^q\|(-\Delta)^{1/2}u\|^2_{L^2(\mathbb{R}^n)} \\ &\quad= -t^q\int_{\mathbb{R}^n}\nabla u\cdot\nabla(-\Delta)^{-1}u(-\Delta)^{1/2}u\mathrm{d}x + t^q\int_{\mathbb{R}^n}u^2(-\Delta)^{1/2}u\mathrm{d}x \\ &\qquad+qt^{q-1}\|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.3}$$

Here, we used the relation $\int_{\mathbb{R}^n} u(-\Delta)^{1/2}u\mathrm{d}x = \|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R}^n)}$. We see by the Hölder inequality and Agmon–Douglis–Nirenberg's inequality that the first term on the right-hand side satisfies

$$\begin{aligned}&\left|t^q\int_{\mathbb{R}^3}\nabla u\cdot\nabla(-\Delta)^{-1}u(-\Delta)^{1/2}u\mathrm{d}x\right| \\ &\quad\leq Ct^q\|\nabla(-\Delta)^{-1}u(t)\|_{L^\infty(\mathbb{R}^n)}\|(-\Delta)^{1/2}u\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Hence, by employing (1.5), we have that

$$t^q\left|\int_{\mathbb{R}^3}(-\Delta)^{1/2}u\nabla u\cdot\nabla(-\Delta)^{-1}u\mathrm{d}x\right| \leq \frac{1}{3}t^q\|(-\Delta)^{1/2}u\|^2_{L^2(\mathbb{R}^n)} \tag{2.4}$$

for sufficiently large t. By the Hölder inequality, Young's inequality and the decay (1.4), we obtain

$$\begin{aligned} t^q \left| \int_{\mathbb{R}^n} u^2 (-\Delta)^{1/2} u \mathrm{d}x \right| &\leq t^q \, \|u\|_{L^4(\mathbb{R}^n)}^2 \, \|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)} \\ &\leq C t^q (1+t)^{-3n} + \frac{1}{3} t^q \|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)}^2. \quad (2.5) \end{aligned}$$

Lemma 2.5 and (1.4) provides that

$$\begin{aligned} \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R}^n)} &\leq C \|u\|_{L^2(\mathbb{R}^n)}^{1/2} \|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)}^{1/2} \\ &\leq C(1+t)^{-n/4} \|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)}^{1/2}. \end{aligned}$$

Hence, by Young's inequality, we see

$$t^{q-1} \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R}^n)}^2 \leq C t^{q-2} (1+t)^{-n} + \frac{1}{3} t^q \|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)}^2. \quad (2.6)$$

Applying (2.4), (2.5) and (2.6) into (2.3), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} (t^q \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R}^n)}^2) + t^q \|(-\Delta)^{1/2} u\|_{L^2(\mathbb{R}^n)}^2 \leq C t^{q-2} (1+t)^{-n}$$

holds for any $t \geq T$ if T is sufficiently large. Thus,

$$\begin{aligned} & t^q \|(-\Delta)^{1/4} u(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_T^t s^q \|(-\Delta)^{1/2} u(s)\|_{L^2(\mathbb{R}^n)}^2 \mathrm{d}s \\ & \quad \leq T^q \|(-\Delta)^{1/4} u(T)\|_{L^2(\mathbb{R}^n)}^2 + C \int_T^t s^{q-2} (1+s)^{-n} \mathrm{d}s \quad (2.7) \end{aligned}$$

is fulfilled for any $t \geq T$. Therefore, we obtain the desired inequality (2.2) for any $t \geq T$. □

Proposition 2.9 will prove extremely useful in the proof of our main theorems.

3. Proof of Theorems

In this section, we prove our results.

3.1. Proof of Theorem 1.1

We suppose that $t \geq 2T$, where T is appeared in Proposition 2.9. Since the solution satisfies (1.6), we see

$$\begin{aligned} & u(t) - M_u P(t) - m_u \cdot \nabla P(t) \\ & \quad = P(t) * u_0 - M_u P(t) - m_u \cdot \nabla P(t) \\ & \qquad + \int_0^{t/2} \nabla P(t-s) * (u \nabla (-\Delta)^{-1} u)(s) \mathrm{d}s \\ & \qquad + \int_{t/2}^t P(t-s) * \nabla \cdot (u \nabla (-\Delta)^{-1} u)(s) \mathrm{d}s, \quad (3.1) \end{aligned}$$

where $M_u = \int_{\mathbb{R}^n} u_0(y)\mathrm{d}y$ and $m_u = \int_{\mathbb{R}^n}(-y)u_0(y)\mathrm{d}y$. The first, second and third terms of the right-hand side of this are treated by Lemma 2.3. Since $\nabla(-\Delta)^{-1}$ is skew-adjoint on $L^2(\mathbb{R}^n)$, we see $\int_{\mathbb{R}^n}(u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}y = 0$ for any $s > 0$ and

$$\begin{aligned}&\int_0^{t/2} \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s\\&\quad= \int_0^{t/2}\int_{\mathbb{R}^n} (\nabla P(t-s,x-y) - \nabla P(t-s,x))\cdot(u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Hence, Taylor's theorem gives

$$\begin{aligned}&\int_0^{t/2} \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s\\&\quad= \sum_{|\beta|=1}\int_0^{t/2}\int_{|y|\le\sqrt{t}}\int_0^1 \nabla^\beta\nabla P(t-s,x-y+\lambda y)\\&\qquad\times(-y)^\beta(u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}\lambda\mathrm{d}y\mathrm{d}s\\&\qquad+\int_0^{t/2}\int_{|y|\ge\sqrt{t}}(\nabla P(t-s,x-y) - \nabla P(t-s,x))\\&\qquad\times(u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}y\mathrm{d}s.\end{aligned}$$

By Lemma 2.2, we derive

$$\begin{aligned}&\Big\|\int_0^{t/2}\int_{|y|\le\sqrt{t}}\int_0^1 \nabla^\beta\nabla P(t-s,x-y+\lambda y)\\&\qquad\times(-y)^\beta(u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}\lambda\mathrm{d}y\mathrm{d}s\Big\|_{L^p(\mathbb{R}^n)}\\&\quad\le Ct^{1/2}\int_0^{t/2}(t-s)^{-n(1-\frac{1}{p})-2}(1+s)^{-n+1}\mathrm{d}s \le Ct^{-n(1-\frac{1}{p})-\frac{3}{2}},\end{aligned}$$

since $n \ge 3$ and $|\beta| = 1$. Here, we used

$$\|(u\nabla(-\Delta)^{-1}u)(s)\|_{L^1(\mathbb{R}^n)} \le C(1+s)^{-n+1}, \tag{3.2}$$

which is derived by the Hölder inequality, (1.4) and (1.5). From Minkowski's inequality, Lemma 2.2 and (3.2), it is easily seen that

$$\begin{aligned}&\Big\|\int_0^{t/2}\int_{\mathbb{R}^n}(\nabla P(t-s,x-y) - \nabla P(t-s,x))\\&\qquad\times(u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}y\mathrm{d}s\Big\|_{L^p(\mathbb{R}^n)}\\&\quad\le C\int_0^{t/2}(t-s)^{-n(1-\frac{1}{p})-1}(1+s)^{-n+1}\mathrm{d}s \le Ct^{-n(1-\frac{1}{p})-1}.\end{aligned}$$

Thus, Lebesgue's monotone convergence theorem yields that

$$\Big\| \int_0^{t/2} \int_{|y|\ge\sqrt{t}} (\nabla P(t-s,x-y) - \nabla P(t-s,x)) \\ \times (u\nabla(-\Delta)^{-1}u)(s,y)\mathrm{d}y\mathrm{d}s \Big\|_{L^p(\mathbb{R}^n)} = o(t^{-n(1-\frac{1}{p})-1})$$

as $t\to\infty$. Therefore, we derive that the second term on the right-hand side of (3.1) satisfies

$$\left\| \int_0^{t/2} \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \right\|_{L^p(\mathbb{R}^n)} = o(t^{-n(1-\frac{1}{p})-1}) \tag{3.3}$$

as $t\to\infty$. The last term on the right-hand side of (3.1) is rewritten by

$$\int_{t/2}^t P(t-s) * \nabla\cdot(u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \\ = \int_{t/2}^t \nabla(-\Delta)^{-\sigma/2}P(t-s) * (-\Delta)^{\sigma/2}(u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s$$

for some $0<\sigma<\frac{1}{np}$. Lemmas 2.2 and 2.6 give that

$$\left\| \int_{t/2}^t \nabla(-\Delta)^{-\sigma/2}P(t-s) * (-\Delta)^{\sigma/2}(u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \right\|_{L^p(\mathbb{R}^n)} \\ \le C\int_{t/2}^t (t-s)^{-(1-\sigma)} \Big(\|u\|_{L^{np}(\mathbb{R}^n)} \|\nabla(-\Delta)^{-1+\frac{\sigma}{2}}u\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n)} \\ + \|(-\Delta)^{\sigma/2}u\|_{L^{np}(\mathbb{R}^n)} \|\nabla(-\Delta)^{-1}u\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n)} \Big) \mathrm{d}s.$$

By Lemma 2.4 and (1.4), we see

$$\int_{t/2}^t (t-s)^{-(1-\sigma)} \|u\|_{L^{np}(\mathbb{R}^n)} \|\nabla(-\Delta)^{-1+\frac{\sigma}{2}}u\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n)}\mathrm{d}s \\ \le C\int_{t/2}^t (t-s)^{-(1-\sigma)} s^{-n(1-\frac{1}{p})-n-\sigma+1}\mathrm{d}s \le Ct^{-n(1-\frac{1}{p})-n+1}.$$

By employing Lemma 2.5, we have

$$\|(-\Delta)^{\sigma/2}u\|_{L^{np}(\mathbb{R}^n)} \le \|u\|^{1-2\sigma}_{L^{\frac{n(1-2\sigma)p}{1-\sigma np}}(\mathbb{R}^n)} \|(-\Delta)^{1/4}u\|^{2\sigma}_{L^2(\mathbb{R}^n)}. \tag{3.4}$$

Hence, by (1.4), (1.5) and Proposition 2.9, we obtain

$$\int_{t/2}^t (t-s)^{-(1-\sigma)} \|(-\Delta)^{\sigma/2}u\|_{L^{np}(\mathbb{R}^n)} \|\nabla(-\Delta)^{-1}u\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n)}\mathrm{d}s \\ \le C\int_{t/2}^t (t-s)^{-(1-\sigma)} \|u\|^{1-2\sigma}_{L^{\frac{n(1-2\sigma)p}{1-\sigma np}}(\mathbb{R}^n)} \|(-\Delta)^{1/4}u\|^{2\sigma}_{L^2(\mathbb{R}^n)} \\ \times \|\nabla(-\Delta)^{-1}u\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n)}\mathrm{d}s \\ \le C\int_{t/2}^t (t-s)^{-(1-\sigma)} s^{-n(1-\frac{1}{p})-n-\sigma+1}\mathrm{d}s \le Ct^{-n(1-\frac{1}{p})-n+1}$$

and then

$$\left\| \int_{t/2}^{t} P(t-s) * \nabla \cdot (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \right\|_{L^p(\mathbb{R}^n)} \leq Ct^{-n(1-\frac{1}{p})-n+1} \tag{3.5}$$

for $t \geq 2T$. Applying (3.3) and (3.5) into (3.1), we complete the proof. □

We remark that (3.5) is also fulfilled in the case $n = 2$. Since Lemma 2.6 is applied in the proof, we except the case $p = 1$ from Theorem 1.1. When $p = \infty$, for the last term on the right-hand side of (3.1), we obtain

$$\begin{aligned} &\left\| \int_{t/2}^{t} \nabla(-\Delta)^{-\sigma/2} P(t-s) * (-\Delta)^{\sigma/2}(u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \right\|_{L^\infty(\mathbb{R}^n)} \\ &\quad \leq C \int_{t/2}^{t} (t-s)^{-(1-\sigma)-\frac{n}{r}} \|(-\Delta)^{\sigma/2}(u\nabla(-\Delta)^{-1}u)(s)\|_{L^r(\mathbb{R}^n)}\mathrm{d}s \end{aligned}$$

for some σ and r. Then, we can apply Lemma 2.6. However, to avoid the singularity of the integration in t, we should assume $r > n/\sigma$. On the other hand, Gagliardo–Nirenberg's inequality such as (3.4) needs the condition $r < 1/\sigma$. Since those conditions are contradictory, the case $p = \infty$ is also excepted from Theorem 1.1.

3.2. Proof of Theorem 1.2

Before proving Theorem 1.2, we prepare two propositions.

Proposition 3.1. *Let $n = 2$, $u_0 \in L^1(\mathbb{R}^2, \sqrt{1+|x|^2}\,\mathrm{d}x)$ and the solution u of (1.1) satisfy (1.2) and (1.4). Assume that $1 < p < \infty$. Then,*

$$\|u(t) - M_u P(t)\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})}(1+t)^{-1}\log(2+t) \tag{3.6}$$

holds for any $t > 0$.

Proof. Minkowski's inequality and (1.4) imply that

$$\|u(t) - M_u P(t)\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})} \tag{3.7}$$

for any $t > 0$. Since the solution is represented by (1.6), we see

$$\begin{aligned} u(t) - M_u P(t) = P(t) * u_0 - M_u P(t) \\ &+ \int_0^{t/2} \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \\ &+ \int_{t/2}^{t} P(t-s) * \nabla \cdot (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s. \end{aligned} \tag{3.8}$$

From Lemma 2.3, it is enough to estimate the third and last terms in (3.8). By applying Lemma 2.2 together with (3.2), we have

$$\begin{aligned}\left\| \int_0^{t/2} \nabla P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \right\|_{L^p(\mathbb{R}^2)} \\ \leq C \int_0^{t/2} (t-s)^{-2(1-\frac{1}{p})-1}(1+s)^{-1}\mathrm{d}s \\ \leq Ct^{-2(1-\frac{1}{p})-1}\log(1+t). \end{aligned} \tag{3.9}$$

Now, we apply (3.5) with $n = 2$ and (3.9) into (3.8), then we follow

$$\|u(t) - M_u P(t)\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})-1}\log(2+t)$$

for any $t \geq 2T$ with T which appears in Proposition 2.9. A coupling of this and (3.7) leads (3.6). □

The similar argument as above, Minkowski's inequality and (1.3) say that

$$\|u(t) - M_u P(1+t)\|_{L^p(\mathbb{R}^2)} \leq C(1+t)^{-2(1-\frac{1}{p})-1}\log(2+t)$$

holds for any $1 < p < \infty$. The assertion of the following proposition is useful in the proof of Theorem 1.2.

Proposition 3.2. *Let $n = 2, u_0 \in L^1(\mathbb{R}^2, \sqrt{1+|x|^2}\,\mathrm{d}x)$ and the solution u of (1.1) satisfy (1.2) and (1.4). Assume that $1 < p < \infty$ and $0 < \sigma < \frac{1}{4p}$. Then, there exist positive constants C and T such that*

$$\|(-\Delta)^{\sigma/2}(u(t) - M_u P(t))\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})-\sigma}(1+t)^{-1}\log(2+t)$$

holds for any $t \geq T$.

Proof. From (1.6), we have

$$(-\Delta)^{\sigma/2}\left(u(t) - M_u P(t)\right) = r_0(t) + r_1(t) + r_2(t), \tag{3.10}$$

where

$$\begin{aligned} r_0(t) &= (-\Delta)^{\sigma/2}\left(P(t) * u_0 - M_u P(t)\right), \\ r_1(t) &= \int_0^{t/2} \nabla(-\Delta)^{\sigma/2} P(t-s) * (u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s \end{aligned}$$

and

$$r_2(t) = \int_{t/2}^{t} \nabla(-\Delta)^{-\sigma/2} P(t-s) * (-\Delta)^{\sigma}(u\nabla(-\Delta)^{-1}u)(s)\mathrm{d}s.$$

We can check at once that

$$\|r_0(t)\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})-\sigma}(1+t)^{-1}.$$

The second term on the right-hand side of (3.10) satisfies

$$\begin{aligned}\|r_1(t)\|_{L^p(\mathbb{R}^2)} &\leq C\int_0^{t/2} (t-s)^{-2(1-\frac{1}{p})-1-\sigma}(1+s)^{-1}\mathrm{d}s \\ &\leq Ct^{-2(1-\frac{1}{p})-1-\sigma}\log(2+t),\end{aligned}$$

which is derived from Lemma 2.2 and (3.2). In the similar way as in the proof of Theorem 1.1, we see that

$$\|r_2(t)\|_{L^p(\mathbb{R}^2)} \le Ct^{-\sigma}(1+t)^{-2(1-\frac{1}{p})-1}$$

if t is sufficiently large. By applying those results into (3.10), we obtain the assertion of Proposition 3.2. □

Now, we prove Theorem 1.2. From (1.6) and (1.7), we see

$$\begin{aligned} &u(t) - M_u P(t) - m_u \cdot \nabla P(t) - M_u^2 J(t) \\ &\quad = P(t) * u_0 - M_u P(t) - m_u \cdot \nabla P(t) \\ &\qquad + \int_0^{t/2} \nabla P(t-s) * ((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s))\mathrm{d}s \\ &\qquad + M_u^2 \int_0^{t/2} \nabla P(t-s) * ((P\nabla(-\Delta)^{-1}P)(1+s) - (P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s \\ &\qquad + \int_{t/2}^t P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s)\big)\, \mathrm{d}s. \end{aligned} \tag{3.11}$$

Lemma 2.3 immediately gives

$$\|P(t) * u_0 - M_u P(t) - m_u \cdot \nabla P(t)\|_{L^p(\mathbb{R}^2)} = o(t^{-2(1-\frac{1}{p})-1}) \tag{3.12}$$

as $t \to \infty$. In the similar way as in the proof of Theorem 1.1, we can divide the second term on the right-hand side of (3.11) as

$$\begin{aligned} &\int_0^{t/2} \nabla P(t-s) * ((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s))\mathrm{d}s \\ &\quad = \sum_{|\beta|=1} \int_0^{t/2} \int_{|y|\le\sqrt{t}} \int_0^1 \nabla^\beta \nabla P(t-s, x-y+\lambda y) \\ &\qquad \times (-y)^\beta ((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y))\mathrm{d}y\mathrm{d}s \\ &\qquad + \int_0^{t/2} \int_{|y|>\sqrt{t}} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\ &\qquad \times ((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y))\mathrm{d}y\mathrm{d}s. \end{aligned}$$

From Lemma 2.2, the first term satisfies

$$\begin{aligned} &\bigg\| \int_0^{t/2} \int_{|y|\le\sqrt{t}} \int_0^1 \nabla^\beta \nabla P(t-s, x-y+\lambda y) \\ &\qquad \times (-y)^\beta ((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y))\mathrm{d}y\mathrm{d}s \bigg\|_{L^p(\mathbb{R}^2)} \\ &\quad \le Ct^{1/2} \int_0^{t/2} (t-s)^{-2(1-\frac{1}{p})-2} \|(u\nabla(-\Delta)^{-1}u)(s) \\ &\qquad - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s)\|_{L^1(\mathbb{R}^2)}\mathrm{d}s \\ &\quad \le Ct^{1/2} \int_0^{t/2} (t-s)^{-2(1-\frac{1}{p})-2} (1+s)^{-2} \log(2+s)\mathrm{d}s. \end{aligned}$$

In the second inequality here, we used

$$\|(u\nabla(-\Delta)^{-1}u)(s) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s)\|_{L^1(\mathbb{R}^2)} \leq C(1+s)^{-2}\log(2+s)$$

which is yielded by Lemma 2.4 and the remark under Proposition 3.1. Similarly,

$$\begin{aligned}&\Big\|\int_0^{t/2}\int_{\mathbb{R}^2}(\nabla P(t-s,x-y)-\nabla P(t-s,x))\\&\qquad\times((u\nabla(-\Delta)^{-1}u)(s,y)-M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y))\mathrm{d}y\mathrm{d}s\Big\|_{L^p(\mathbb{R}^2)}\\&\quad\leq Ct^{-2(1-\frac{1}{p})-1}\end{aligned}$$

holds. Hence,

$$\begin{aligned}&\Big\|\int_0^{t/2}\int_{|y|>\sqrt{t}}(\nabla P(t-s,x-y)-\nabla P(t-s,x))\\&\qquad\times((u\nabla(-\Delta)^{-1}u)(s,y)-M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y))\mathrm{d}y\mathrm{d}s\Big\|_{L^p(\mathbb{R}^2)}\\&\quad=o(t^{-2(1-\frac{1}{p})-1})\end{aligned}$$

as $t\to\infty$. Thus, the second term on the right-hand side of (3.11) fulfills

$$\begin{aligned}&\Big\|\int_0^{t/2}\nabla P(t-s)*((u\nabla(-\Delta)^{-1}u)(s)\\&\quad-M_u^2(P\nabla(-\Delta)^{-1}P)(1+s))\mathrm{d}s\Big\|_{L^p(\mathbb{R}^2)}\\&\quad=o(t^{-2(1-\frac{1}{p})-1})\end{aligned} \tag{3.13}$$

as $t\to\infty$. By using the relation $\int_{\mathbb{R}^2}P\nabla(-\Delta)^{-1}P\mathrm{d}y=0$ and Taylor's theorem, we obtain for the third term that

$$\begin{aligned}&\int_0^{t/2}\nabla P(t-s)*((P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\\&\quad=\int_0^{t/2}\int_{\mathbb{R}^2}(\nabla P(t-s,x-y)-\nabla P(t-s,x))\\&\qquad\times((P\nabla(-\Delta)^{-1}P)(1+s,y)-(P\nabla(-\Delta)^{-1}P)(s,y))\mathrm{d}y\mathrm{d}s\\&\quad=\sum_{|\beta|=1}\int_0^{t/2}\int_{\mathbb{R}^2}\int_0^1\nabla^\beta\nabla P(t-s,x-y+\lambda y)\\&\qquad\times(-y)^\beta((P\nabla(-\Delta)^{-1}P)(1+s,y)-(P\nabla(-\Delta)^{-1}P)(s,y))\mathrm{d}\lambda\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Therefore, by Lemma 2.2, we see

$$\begin{aligned}&\Big\|\int_0^{t/2}\nabla P(t-s)*((P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\Big\|_{L^p(\mathbb{R}^2)}\\&\quad\leq C\int_0^{t/2}(t-s)^{-2(1-\frac{1}{p})-2}\big\||y|\{(P\nabla(-\Delta)^{-1}P)(1+s)\\&\qquad-(P\nabla(-\Delta)^{-1}P)(s)\}\big\|_{L^1(\mathbb{R}^2)}\mathrm{d}s.\end{aligned}$$

It is easily seen that

$$\begin{aligned}&\||y|\{(P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s)\}\|_{L^1(\mathbb{R}^2)}\\&=\left\||y|\int_0^1\partial_t(P\nabla(-\Delta)^{-1}P)(s+\lambda)\mathrm{d}\lambda\right\|_{L^1(\mathbb{R}^2)}\le Cs^{-1}\end{aligned}$$

and

$$\begin{aligned}&\||y|\{(P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s)\}\|_{L^1(\mathbb{R}^2)}\\&\quad\le\||y|(P\nabla(-\Delta)^{-1}P)(1+s)\|_{L^1(\mathbb{R}^2)}+\||y|(P\nabla(-\Delta)^{-1}P)(s)\|_{L^1(\mathbb{R}^2)}\le C.\end{aligned}$$

A combination of them is

$$\||y|\{(P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s)\}\|_{L^1(\mathbb{R}^2)}\le C(1+s)^{-1}.$$

Consequently, the third term on the right-hand side of (3.11) satisfies

$$\begin{aligned}&\left\|\int_0^{t/2}\nabla P(t-s)*((P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\right\|_{L^p(\mathbb{R}^2)}\\&\quad\le C\int_0^{t/2}(t-s)^{-2(1-\frac{1}{p})-2}(1+s)^{-1}\mathrm{d}s\le Ct^{-2(1-\frac{1}{p})-2}\log(2+t).\end{aligned}$$

Particularly,

$$\begin{aligned}&\left\|\int_0^{t/2}\nabla P(t-s)*((P\nabla(-\Delta)^{-1}P)(1+s)-(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\right\|_{L^p(\mathbb{R}^2)}\\&\quad=o(t^{-2(1-\frac{1}{p})-1})\end{aligned}\tag{3.14}$$

as $t\to\infty$. Proposition 3.2 is used in the estimate for the last term on the right-hand side of (3.11). We choose $0<\sigma<\frac{1}{8p}$, then the last term is represented by

$$\begin{aligned}&\int_{t/2}^t P(t-s)*\nabla\cdot((u\nabla(-\Delta)^{-1}u)(s)-M_u^2(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\\&\quad=\int_{t/2}^t\nabla(-\Delta)^{-\sigma/2}P(t-s)*(-\Delta)^{\sigma/2}(u\nabla(-\Delta)^{-1}(u-M_uP))(s)\mathrm{d}s\\&\qquad+M_u\int_{t/2}^t\nabla(-\Delta)^{-\sigma/2}P(t-s)*(-\Delta)^{\sigma/2}((u-M_uP)\nabla(-\Delta)^{-1}P)(s)\mathrm{d}s.\end{aligned}$$

Therefore, by Lemmas 2.2 and 2.6, we see that

$$\left\|\int_{t/2}^t P(t-s)*\nabla\cdot((u\nabla(-\Delta)^{-1}u)(s)-M_u^2(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\right\|_{L^p(\mathbb{R}^2)}$$
$$\le C\int_{t/2}^t (t-s)^{-(1-\sigma)}$$
$$\times\Big(\|(-\Delta)^{\sigma/2}u(s)\|_{L^{2p}(\mathbb{R}^2)}\|\nabla(-\Delta)^{-1}(u(s)-M_uP(s))\|_{L^{2p}(\mathbb{R}^2)}$$
$$+\|u(s)\|_{L^{2p}(\mathbb{R}^2)}\left\|\nabla(-\Delta)^{-1+\frac{\sigma}{2}}(u(s)-M_uP(s))\right\|_{L^{2p}(\mathbb{R}^2)}$$
$$+\|(-\Delta)^{\sigma/2}(u(s)-M_uP(s))\|_{L^{2p}(\mathbb{R}^2)}\left\|\nabla(-\Delta)^{-1}P(s)\right\|_{L^{2p}(\mathbb{R}^2)}$$
$$+\|u(s)-M_uP(s)\|_{L^{2p}(\mathbb{R}^2)}\left\|\nabla(-\Delta)^{-1+\frac{\sigma}{2}}P(s)\right\|_{L^{2p}(\mathbb{R}^2)}\Big)\mathrm{d}s.$$

Now, we employ (3.4), Lemma 2.4, the decay (1.4), Propositions 3.1 and 3.2, then we obtain

$$\left\|\int_{t/2}^t P(t-s)*\nabla\cdot((u\nabla(-\Delta)^{-1}u)(s)-M_u^2(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\right\|_{L^p(\mathbb{R}^2)}$$
$$\le C\int_{t/2}^t (t-s)^{-(1-\sigma)}s^{-2(1-\frac{1}{p})-2-\sigma}\log(2+s)\mathrm{d}s.$$

Thus, we conclude that

$$\left\|\int_{t/2}^t P(t-s)*\nabla\cdot((u\nabla(-\Delta)^{-1}u)(s)-M_u^2(P\nabla(-\Delta)^{-1}P)(s))\mathrm{d}s\right\|_{L^p(\mathbb{R}^2)}$$
$$=o(t^{-2(1-\frac{1}{p})-1}) \tag{3.15}$$

as $t\to\infty$. Applying (3.12), (3.13), (3.14) and (3.15) into (3.11), we derive the desired estimate. □

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Appendix A. Asymptotic Expansion for the Burgers Equation

In this section, we discuss the asymptotic expansion of the solution of the initial value problem for the Burgers equation. It was already shown that, if ω_0 is sufficiently small in the corresponding Besov space, then the unique solution of (1.8) exists globally in time and fulfills (1.9) and

$$\begin{aligned}&\Bigg\| \omega(t) - M_\omega P(t) - M_\omega^2 \mathcal{J}_\omega(t)\\&\qquad - \left(\frac{1}{4\pi} M_\omega^2 \log(1+t) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} \omega(s,y)^2 \mathrm{d}y\mathrm{d}s + m_\omega\right) \partial_x P(t)\Bigg\|_{L^p(\mathbb{R})}\\&\quad = o(t^{-(1-\frac{1}{p})-1}) \end{aligned} \tag{A.1}$$

as $t \to \infty$ for any $1 \le p \le \infty$, where $M_\omega = \int_{\mathbb{R}} \omega_0(y)\mathrm{d}y,\ m_\omega = \int_{\mathbb{R}} (-y)\omega_0(y)\mathrm{d}y$ and $\mathcal{J}_\omega = \mathcal{J}_\omega(t,x)$ is defined by (1.11) (they are proved in [8]). We see that this asymptotic expansion contains the term which decays with logarithmic order. However, the indistinct term $\int_0^t \int_{\mathbb{R}} \omega(s,y)^2 \mathrm{d}y\mathrm{d}s$ which depends on t also appears and decay of $\mathcal{J}_\omega(t)$ is unclear. Particularly, those terms may decay with logarithmic order. Indeed, the coefficients of them fulfill $\|\omega(s)\|^2_{L^2(\mathbb{R})} \le C(1+s)^{-1}$ and $\|P(1+s)\|^2_{L^2(\mathbb{R})} = (1+s)^{-1}\|P(1)\|^2_{L^2(\mathbb{R})}$. Now, we clarify this crux. We remark that (1.12) and (A.1) are not contradictory. Indeed, we can confirm by the renormalization that logarithmic orders on $\frac{1}{4\pi} M_\omega^2 \log(1+t) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} \omega(s,y)^2 \mathrm{d}y\mathrm{d}s$ are vanishing. By L^p-L^q type estimate on the corresponding Besov space, in [8], it was already proved that the solution of (1.8) satisfies

$$\left\|\partial_x^j \left(\omega(t)^2 - M_\omega^2 P(t)^2\right)\right\|_{L^p(\mathbb{R})} \le C t^{-(1-\frac{1}{p})-1-j} (1+t)^{-1} \log(1+t) \tag{A.2}$$

for $j = 0, 1,\ 1 \le p \le \infty$ and $t > 0$. This inequality has been prepared to derive (A.1). Throughout this section, we assume (A.2). Before improving the asymptotic expansion of the solution $\omega(t)$, we introduce

$$\begin{aligned}\tilde{J}(t,x) &= -\frac{1}{2}\int_0^{t/2} \int_{\mathbb{R}} \left(\partial_x P(t-s, x-y) - \partial_x P(t,x)\right) P(s,y)^2 \mathrm{d}y\mathrm{d}s\\&\quad - \int_{t/2}^t P(t-s) * (P\partial_x P)(s)\mathrm{d}s. \end{aligned} \tag{A.3}$$

This function is well defined in $C((0,\infty), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. Indeed, since Taylor's theorem leads

$$\begin{aligned}&\int_0^{t/2} \int_{\mathbb{R}} \left(\partial_x P(t-s, x-y) - \partial_x P(t,x)\right) P(s,y)^2 \mathrm{d}y\mathrm{d}s\\&\quad = \sum_{k+l=1} \int_0^{t/2} \int_{\mathbb{R}} \int_0^1 \partial_t^k \partial_x^{1+l} P(t-s+\lambda s, x-y+\lambda y)\\&\qquad \times (-s)^k (-y)^l P(s,y)^2 \mathrm{d}\lambda\mathrm{d}y\mathrm{d}s,\end{aligned}$$

we see by Lemma 2.2 and the decay of $P(t)$ and of $\partial_x P(t)$ that

$$\|\tilde{J}(t)\|_{L^p(\mathbb{R})} \le C\int_0^{t/2} (t-s)^{-(1-\frac{1}{p})-2}\mathrm{d}s + C\int_{t/2}^t s^{-(1-\frac{1}{p})-2}\mathrm{d}s.$$

Hence, we have $\|\tilde{J}(t)\|_{L^p(\mathbb{R})} < \infty$ for any fixed $t > 0$. Moreover, since the relation $\lambda^2 \tilde{J}(\lambda t, \lambda x) = \tilde{J}(t,x)$ holds for $\lambda > 0$, this function satisfies

$$\|\tilde{J}(t)\|_{L^p(\mathbb{R})} = t^{-(1-\frac{1}{p})-1} \|\tilde{J}(1)\|_{L^p(\mathbb{R})}$$

for any $1 \le p \le \infty$ and $t > 0$. Thus, $\tilde{J}(t)$ has the same decay rate as one of $\partial_x P(t)$.

Theorem A.1. *Let the solution $\omega(t)$ of (1.8) satisfy (A.2) and the function $\tilde{J}(t)$ be defined by (A.3). Then, for any $1 \le p \le \infty$,*

$$\begin{aligned}
&\Bigg\| \omega(t) - M_\omega P(t) + \frac{1}{4\pi} M_\omega^2 \partial_x P(t) \log(2+t) - M_\omega^2 \tilde{J}(t) \\
&\quad - \left(m_\omega - \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} (\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2) \mathrm{d}y\mathrm{d}s \right. \\
&\quad \left. + \frac{1}{4\pi} M_\omega^2 \log 2 \right) \partial_x P(t) \Bigg\|_{L^p(\mathbb{R})} = o\left(t^{-(1-\frac{1}{p})-1} \right)
\end{aligned}$$

as $t \to \infty$ holds.

Proof. We first rewrite the solution. The first and the third terms on the right-hand side of (1.10) are written by

$$P(t) * \omega_0 = M_\omega P(t) + m_\omega \partial_x P(t) + \rho_0(t)$$

and

$$\int_{t/2}^t P(t-s) * (u \partial_x u)(s) \mathrm{d}s = M_\omega^2 \int_{t/2}^t P(t-s) * (P \partial_x P)(s) \mathrm{d}s - \rho_4(t)$$

respectively, where

$$\rho_0(t) = P(t) * \omega_0 - M_\omega P(t) - m_\omega \partial_x P(t)$$

and

$$\rho_4(t) = -\frac{1}{2} \int_{t/2}^t P(t-s) * \partial_x \left(\omega(s)^2 - M_\omega^2 P(s)^2 \right) \mathrm{d}s.$$

The second term is represented by

$$\begin{aligned}
&\frac{1}{2} \int_0^{t/2} \partial_x P(t-s) * (\omega^2)(s) \mathrm{d}s \\
&\quad = \frac{1}{2} \int_0^{t/2} \partial_x P(t-s) * (\omega(s)^2 - M_\omega^2 P(1+s)^2) \mathrm{d}s \\
&\qquad + \frac{1}{2} M_\omega^2 \int_0^{t/2} \partial_x P(t-s) * (P^2)(1+s) \mathrm{d}s \\
&\quad = \frac{1}{2} \partial_x P(t) \int_0^\infty \int_{\mathbb{R}} (\omega(s)^2 - M_\omega^2 P(1+s,y)^2) \mathrm{d}y\mathrm{d}s \\
&\qquad + \frac{1}{2} M_\omega^2 \partial_x P(t) \int_0^{t/2} \int_{\mathbb{R}} P(1+s,y)^2 \mathrm{d}y\mathrm{d}s \\
&\qquad + \frac{1}{2} M_\omega \int_0^{t/2} \int_{\mathbb{R}} \left(\partial_x P(t-s, x-y) - \partial_x P(t,x) \right) P(s,y)^2 \mathrm{d}y\mathrm{d}s \\
&\qquad - \rho_1(t) - \rho_2(t) - \rho_3(t)
\end{aligned}$$

where

$$\rho_1(t) = -\frac{1}{2}\int_0^{t/2}\int_{\mathbb{R}} (\partial_x P(t-s, x-y) - \partial_x P(t,x)) \times (\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2)\mathrm{d}y\mathrm{d}s,$$

$$\rho_2(t) = \frac{1}{2}\partial_x P(t,x)\int_{t/2}^{\infty}\int_{\mathbb{R}}(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2)\mathrm{d}y\mathrm{d}s$$

and

$$\rho_3(t) = -\frac{1}{2}M_\omega^2\int_0^{t/2}\int_{\mathbb{R}}(\partial_x P(t-s,x-y) - \partial_x P(t,x)) \times (P(1+s,y)^2 - P(s,y)^2)\mathrm{d}y\mathrm{d}s.$$

Hence, we obtain from (1.10) that

$$\begin{aligned}&\omega(t) - M_\omega P(t) + \frac{1}{4\pi}M_\omega^2 \partial_x P(t)\log(2+t) - M_\omega^2 \tilde{J}(t)\\ &\quad - \left(m_\omega - \frac{1}{2}\int_0^\infty\int_{\mathbb{R}}(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right)\mathrm{d}y\mathrm{d}s + \frac{1}{4\pi}M_\omega^2\log 2)\partial_x P(t)\\ &= \rho_0(t) + \rho_1(t) + \cdots + \rho_4(t).\end{aligned}$$

Here, we used the relation $\int_0^{t/2}\int_{\mathbb{R}} P(1+s,y)^2\mathrm{d}y\mathrm{d}s = \frac{1}{2\pi}(\log(2+s) - \log 2)$. Lemma 2.3 gives

$$\|\rho_0(t)\|_{L^p(\mathbb{R})} = o(t^{-(1-\frac{1}{p})-1})$$

as $t \to \infty$ for any $1 \le p \le \infty$. If we employ (A.2) with $j = 0$ on the similar way as in Sect. 3, then we conclude that

$$\|\rho_1(t)\|_{L^p(\mathbb{R})} + \|\rho_2(t)\|_{L^p(\mathbb{R})} + \|\rho_3(t)\|_{L^p(\mathbb{R})} = o(t^{-(1-\frac{1}{p})-1})$$

as $t \to \infty$. The last term is treated by (A.2) with $j = 1$. Namely, by Lemma 2.2 and (A.2), we have

$$\begin{aligned}\|\rho_4(t)\|_{L^p(\mathbb{R})} &= \frac{1}{2}\left\|\int_{t/2}^t P(t-s) * \partial_x(\omega(s)^2 - M_\omega^2 P(s)^2)\mathrm{d}s\right\|_{L^p(\mathbb{R})}\\ &\le C\int_{t/2}^t s^{-\left(1-\frac{1}{p}\right)-2}(1+s)^{-1}\log(1+s)\mathrm{d}s\end{aligned}$$

and thus

$$\|\rho_4(t)\|_{L^p(\mathbb{R})} = o(t^{-(1-\frac{1}{p})-1})$$

as $t \to \infty$. Therefore, we obtain the assertion of Theorem A.1. □

Since a coupling of Minkowski's inequality and (A.2) with $j = 0$ says that

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}}\left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right)\mathrm{d}y\mathrm{d}s\right|\\ &\quad \le C\int_0^\infty (1+s)^{-2}\log(1+s)\mathrm{d}s < \infty,\end{aligned}$$

the coefficient $\int_0^\infty \int_{\mathbb{R}} (\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2) \mathrm{d}y \mathrm{d}s$ on the asymptotic expansion in Theorem A.1 is well defined. The other terms on the asymptotic expansion are distinct and depend only on M_ω and m_ω. Furthermore, the decay of them are clear.

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