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Gravitational Collapse and the Vlasov–Poisson System

Gerhard Rein and Lukas Taegert

Abstract. A self-gravitating homogeneous ball of a fluid with pressure zero where the fluid particles are initially at rest collapses to a point in finite time. We prove that this gravitational collapse can be approximated arbitrarily closely by suitable solutions of the Vlasov–Poisson system which are known to exist globally in time.

1. Introduction

Perhaps the simplest example of a matter distribution which collapses under the influence of its own, self-consistent gravitational field is a homogeneous ball of an ideal, compressible fluid with the equation of state that the pressure is identically zero—this is usually referred to as dust—and with the particles initially being at rest. In suitable units, the radius $r(t)$ of this ball is determined by the initial value problem

$$\ddot{r} = -\frac{1}{r^2}, \quad r(0) = 1, \quad \dot{r}(0) = 0.$$

The mass density is given by

$$\rho(t, x) = \frac{3}{4\pi} \frac{1}{r^3(t)} \mathbf{1}_{B_{r(t)}(0)}(x), \quad t \geq 0, \quad x \in \mathbb{R}^3.$$

Here, $\mathbf{1}_S$ denotes the indicator function of the set S and $B_r(z)$ is the ball of radius $r > 0$ centered at $z \in \mathbb{R}^3$. There exists a time $T > 0$ such that $r(t) > 0$ exists on $[0, T[$ with $\lim_{t \to T} r(t) = 0$, i.e. the dust ball collapses to a point in finite time. It is easy to check that the above mass density solves the pressure-less Euler–Poisson system

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1}$$

$$\partial_t u + (u \cdot \partial_x) u = -\partial_x U, \tag{1.2}$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x| \to \infty} U(t, x) = 0, \tag{1.3}$$

with velocity field

$$u(t,x) = \frac{\dot r(t)}{r(t)} x;$$

$U = U(t,x)$ denotes the induced gravitational potential. The fact that such a matter distribution collapses is not surprising since there is no mechanism which opposes gravity, the pressure being put to zero by the choice of the equation of state. This situation changes significantly if dust is replaced by a collisionless gas as matter model in which case smooth, compactly supported initial data launch solutions which exist globally in time and do not undergo a gravitational collapse in the above sense, cf. [3,5]. As in dust, the particles in a collisionless gas interact only by gravity, but the particle ensemble is now given in terms of a density function $f = f(t,x,v) \geq 0$ on phase space where $t \in \mathbb{R},\ x, v \in \mathbb{R}^3$ stand for time, position, and velocity, and f obeys the Vlasov–Poisson system which consists of the Vlasov equation

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{1.4}$$

coupled to the Poisson equation (1.3) via the definition

$$\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v \tag{1.5}$$

of the spatial mass density in terms of the phase space density f; unless indicated otherwise, integrals always extend over $\mathbb{R}^3$. In astrophysics, the Vlasov–Poisson system (1.3), (1.4), (1.5) is used as a model for galaxies or globular clusters, cf. [1]. The relation between its solutions and those of the pressure-less Euler–Poisson system (1.1), (1.2), (1.3) was investigated in [2]. If (ρ, u) is a solution of the former system, then formally $f(t,x,v) = \rho(t,x)\,\delta(v - u(t,x))$ satisfies the latter system where δ denotes the Dirac distribution. However, when we speak of the Vlasov–Poisson system in the present paper, we only consider genuine functions on phase space (which will actually be smooth) as solutions, and we ask the question whether the collapsing solution of the pressure-less Euler–Poisson system presented at the beginning of this introduction can be approximated by suitable smooth solutions of the Vlasov–Poisson system. This is indeed possible, cf. Theorem 2.2 below, and yields the existence of nearly collapsing solutions of the Vlasov–Poisson system.

To make the latter more precise, we recall the definitions of the kinetic and the potential energy associated with a solution:

$$E_{\mathrm{kin}}(t) = \frac{1}{2} \iint |v|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x,$$
$$E_{\mathrm{pot}}(t) = -\frac{1}{8\pi} \int |\partial_x U(t,x)|^2\,\mathrm{d}x = \frac{1}{2} \iint \frac{\rho(t,x)\,\rho(t,y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x.$$

The total energy $E_{\mathrm{kin}}(t) + E_{\mathrm{pot}}(t)$ is conserved. We also recall that a solution of the Vlasov–Poisson system is *spherically symmetric* if $f(t,x,v) = f(t,Ax,Av)$ for any rotation $A \in \mathrm{SO}(3)$. For a spherically symmetric solution and by abuse of notation, $\rho(t,x) = \rho(t,r)$ and

$$\partial_x U(t,x) = \frac{m(t,r)}{r^2}\frac{x}{r}, \quad |x| = r,$$

where

$$m(t,r) = 4\pi \int_0^r s^2 \rho(t,s)\,\mathrm{d}s$$

is the mass contained in the ball of radius $r > 0$ centered at the origin. It is well known that spherically symmetric initial data launch spherically symmetric solutions of the Vlasov–Poisson system, cf. [6].

Theorem 1.1. *For any constants $C_1, C_2 > 0$ there exists a smooth, spherically symmetric solution f of the Vlasov–Poisson system such that initially*

$$\|\rho(0)\|_\infty, \quad E_{\mathrm{kin}}(0), \quad -E_{\mathrm{pot}}(0), \quad \sup_{r>0} \frac{m(0,r)}{r} \le C_1,$$

but for some time $t > 0$,

$$\|\rho(t)\|_\infty, \quad E_{\mathrm{kin}}(t), \quad -E_{\mathrm{pot}}(t), \quad \sup_{r>0} \frac{m(t,r)}{r} > C_2.$$

By choosing C_1 small and C_2 large, we see that the matter distribution is initially dilute and the particles are nearly at rest, but at some later time t the matter is very concentrated with large total kinetic and potential energy. In this sense, Vlasov–Poisson solutions can be very close to a gravitational collapse even though—as opposed to the case of the pressure-less Euler–Poisson system—the quantities in the above theorem remain bounded on any bounded time interval. Besides the wish to understand better the relation of the two systems under consideration, there is a more specific motivation for the current investigation which also explains why the term m/r is considered in the theorem above. This motivation originates in general relativity.

In 1939, Oppenheimer and Snyder [4] showed how a black hole can develop from regular data. Much as in our introductory example, they considered a spherically symmetric, asymptotically flat spacetime with a dilute homogeneous ball of dust as matter model and showed that a trapped surface and hence a black hole form in the evolution. If r denotes the area radius, then the condition $2m/r > 1$ indicates that the sphere of radius r is trapped and the spacetime contains a black hole. Here, m is the appropriate general relativistic analogue of the mass function introduced above, the so-called quasilocal ADM mass, and $m(t, r = \infty)$ is a conserved quantity, the ADM mass. The Oppenheimer–Snyder example suffers from the fact that there is no pressure in the matter model, and it remains unclear if a similar calculation is possible with a matter model which is not pressure-less. The analysis in the present note is intended as a blue-print for the analogous analysis in the general relativistic setting which will lead to Oppenheimer–Snyder type solutions which collapse to a black hole, but with the physically more realistic Vlasov equation as a matter model, cf. [8].

Both the Oppenheimer–Snyder solution and its Newtonian analogue can be obtained by taking a spatially homogeneous solution with a big crunch singularity in the future, cutting a suitable, spatially finite piece from it and extending it by vacuum. If the cutting is done along the trajectory of a dust particle, a consistent, asymptotically flat solution of the desired form is obtained. In the present analysis, we follow the same recipe. We first introduce a class of spatially homogeneous, cosmological solutions of the Vlasov–Poisson system with a singularity in the future; such solutions, which do of course not satisfy the boundary condition in (1.3), and their perturbations were considered in [7]. From such a spatially homogeneous solution, we cut a ball centered at the origin at time $t = 0$ and extend it smoothly by vacuum. This provides the initial data for the Vlasov–Poisson solution. The latter will for some time have a spatially homogeneous region at the center, the size of which we can control. By choosing the original homogeneous solution sufficiently close to a dust solution in a suitable sense, the time for which the homogeneous core persists can be pushed as close to the collapse time of the homogeneous solution as we wish. This will prove Theorem 2.2 from which Theorem 1.1 will follow.

2. Solutions with a Homogeneous Core

We first recall the construction of spatially homogeneous solutions to the Vlasov–Poisson system. To do so, we fix a continuously differentiable function $H\colon \mathbb{R} \to [0,\infty[$ with support $\operatorname{supp} H \subset [0,1]$ and

$$\int H(|v|^2)\,\mathrm{d}v = \frac{3}{4\pi}.$$

For $\epsilon \in]0,1]$ let

$$H_\epsilon := \frac{1}{\epsilon^3} H\left(\frac{\cdot}{\epsilon^2}\right) \tag{2.1}$$

so that $\operatorname{supp} H_\epsilon \subset [0,\epsilon^2]$ and

$$\int H_\epsilon(|v|^2)\,\mathrm{d}v = \frac{3}{4\pi}.$$

Let $a\colon [0,T[\to]0,\infty[$ be the maximal solution of

$$\ddot{a} = -\frac{1}{a^2},\ \ a(0) = \mathring{a}, \quad \dot{a}(0) = 0, \tag{2.2}$$

where $\mathring{a} > 0$ is prescribed. A straight forward computation shows that

$$h_\epsilon\colon [0,T[\times\mathbb{R}^3\times\mathbb{R}^3 \to [0,\infty[, \quad h_\epsilon(t,x,v) := H_\epsilon(|a(t)v - \dot{a}(t)x|^2) \tag{2.3}$$

is a spherically symmetric solution of the Vlasov–Poisson system—where the boundary condition at spatial infinity is dropped—with

$$\rho_h(t,r) = \frac{3}{4\pi a^3(t)}, \quad m_h(t,r) = \frac{r^3}{a^3(t)},$$

and

$$\partial_x U_h(t,x) = \frac{x}{a^3(t)}, \tag{2.4}$$

i.e. the macroscopic quantities related to this spatially homogeneous solution h_ϵ do actually not depend on ϵ. It is well known that (2.2) cannot be solved explicitly. The following information on the behavior of a will be useful.

Lemma 2.1. *Let* $a\colon [0,T[\to]0,\infty[$ *be the maximal solution of* (2.2). *Then,* $T = \frac{\pi}{2\sqrt{2}}\mathring{a}^{3/2}$, *$a$ is strictly decreasing on* $[0,T[$ *with* $\lim_{t\to T} a(t) = 0$, *and for all* $t \in]0,T[$,

$$\dot{a}(t) = -\sqrt{2}\sqrt{\frac{1}{a(t)} - \frac{1}{\mathring{a}}} < 0, \tag{2.5}$$

$$\frac{a(t)}{\mathring{a}}\sqrt{\frac{\mathring{a}}{a(t)} - 1} + \arctan\sqrt{\frac{\mathring{a}}{a(t)} - 1} = \sqrt{2}\mathring{a}^{-3/2}t. \tag{2.6}$$

Proof. As long as the solution exists, $\ddot{a} < 0$ and hence $\dot{a}(t) < \dot{a}(0) = 0$ for $t > 0$. We multiply the differential equation in (2.2) by $\dot{a}$ and integrate to find that

$$\frac{1}{2}\dot{a}^2(t) = \frac{1}{a(t)} - \frac{1}{\mathring{a}} \tag{2.7}$$

on $[0,T[$ which yields (2.5). Using the substitution

$$b = \sqrt{\frac{1}{a(t)} - \frac{1}{\mathring{a}}},$$

this equation can be integrated once more to yield (2.6). Since $\lim_{t\to T} a(t) = 0$, the formula for T is obtained by taking the corresponding limit in (2.6), and the proof is complete. □

Remark. Let $r(t) = a(t)/\mathring{a},\ t \in [0,T[$. Then

$$\rho(t,x) := \frac{3}{4\pi a^3(t)}\mathbf{1}_{B_{r(t)}(0)}(x), \quad u(t,x) := \frac{\dot{a}(t)}{a(t)}x$$

defines a solution of the pressure-less Euler–Poisson system which coincides with our introductory collapse example if $\mathring{a} = 1$.

This solution can be viewed as follows. We start with the spatially homogeneous solution with density ρ_h and the given velocity field and cut from it a spherically symmetric piece the boundary of which is given by the curve $r = r(t)$. It should be noted that this curve is precisely the trajectory of the particle which starts at radius $r = 1$ with zero initial velocity.

An analogous boundary curve for a corresponding cut in the Vlasov case does not exist since there is at each point in space a distribution of particles with different velocities. We, therefore, proceed as follows. We choose a family of cutoff functions $\phi_\epsilon \in C^\infty([0,\infty[)$, $\epsilon \in]0,1]$, such that

$$0 \le \phi_\epsilon \le 1, \quad \phi_\epsilon(r) = 1 \quad \text{for} \quad r \le 1, \quad \phi_\epsilon(r) = 0 \quad \text{for} \quad r > 1 + \epsilon.$$

The initial data

$$\mathring{f}_\epsilon(x,v) := h_\epsilon(0,x,v)\phi_\epsilon(|x|) \tag{2.8}$$

launch a smooth, global, spherically symmetric solution f_ϵ of the Vlasov–Poisson system; notice that initially this solution coincides with the homogeneous one on $B_1(0) \times \mathbb{R}^3$. We aim to show that for ϵ small a homogeneous core at the center persists arbitrarily closely up to the collapse time T of the homogeneous solution from which fact Theorem 1.1 will follow. To define the boundary of the homogeneous core, we place a point mass which is slightly larger than the total mass of the initial data at the origin and consider the trajectory of a particle which moves radially inward in the corresponding potential and starts at the cutoff radius 1 with an initial radial velocity which in modulus is larger than the initial radial velocities of the Vlasov particles. To make this precise, we define a strict upper bound for the total mass of f_ϵ by

$$M_\epsilon := \iint \mathring{f}_\epsilon(x,v)\,\mathrm{d}v\,\mathrm{d}x + \epsilon \tag{2.9}$$

and let $r_\epsilon\colon [0,T_\epsilon[\to]0,\infty[$ be the maximal solution of the initial value problem

$$\ddot{r} = -\frac{M_\epsilon}{r^2}, \quad r(0) = 1, \quad \dot{r}(0) = -\epsilon. \tag{2.10}$$

We can now state our main result.

Theorem 2.2. *Let f_ϵ and r_ϵ be defined as above for $\epsilon \in]0,1]$. Then, the following holds.*

(a) $T_\epsilon < T$ *for* $\epsilon \in]0,1]$ *with* $T_\epsilon \to T$ *for* $\epsilon \to 0$.
(b) $r_\epsilon(t) \leq a(t)/\mathring{a}$ *for* $t \in [0,T_\epsilon[$ *and* $\epsilon \in]0,1]$, *and* $r_\epsilon(t) \to a(t)/\mathring{a}$ *for* $\epsilon \to 0$, *uniformly on any time interval* $[0,T'] \subset [0,T[$.
(c) $f_\epsilon(t,x,v) = h_\epsilon(t,x,v)$ *for* $\epsilon \in]0,1]$, $t \in [0,T_\epsilon[$, $|x| \leq r_\epsilon(t)$, *and* $v \in \mathbb{R}^3$.

The theorem will be proven in a number of steps in the next section. We first indicate how it implies Theorem 1.1.

Proof of Theorem 1.1. For all $\epsilon \in]0,1]$, the following estimates hold at time $t = 0$. First of all, (2.8) and the properties of H_ϵ and ϕ_ϵ imply that

$$0 \leq \rho(0,x) \leq \frac{3}{4\pi\mathring{a}^3}\mathbf{1}_{B_2(0)}(x), \quad x \in \mathbb{R}^3.$$

This in turn implies that

$$m(0,r) \leq \frac{r^3}{\mathring{a}^3} \quad \text{for} \quad r \leq 2, \quad m(0,r) \leq \frac{8}{\mathring{a}^3} \quad \text{for} \quad r > 2.$$

In particular,

$$\sup_{r>0} \frac{m(0,r)}{r} \leq \frac{4}{\mathring{a}^3}.$$

Moreover,

$$E_{\mathrm{kin}}(0) \leq \frac{1}{2}\frac{32\pi}{3}\int |v|^2 H_\epsilon(\mathring{a}^2|v|^2)\,\mathrm{d}v = \frac{16\pi}{3}4\pi\int_0^1 s^4 H(s^2)\,\mathrm{d}s\frac{1}{\mathring{a}^5}.$$

Finally,

$$-E_{\mathrm{pot}}(0) \leq \frac{1}{2}\left(\frac{3}{4\pi\mathring{a}^3}\right)^2 \int_{|x|\leq 2}\int_{|y|\leq 2} \frac{1}{|x-y|}\mathrm{d}y\,\mathrm{d}x.$$

These estimates show that by choosing $\mathring{a}$ sufficiently large the estimates at $t = 0$ in Theorem 1.1 hold.

Consider now some $\epsilon \in]0, 1]$ and $0 < t < T_\epsilon$. Theorem 2.2(c) implies that

$$\rho(t, x) = \frac{3}{4\pi a^3(t)}, \quad |x| \le r_\epsilon(t).$$

This in turn implies that

$$m(t, r) = \frac{r^3}{a^3(t)}, \quad r \le r_\epsilon(t)$$

so that in particular

$$\sup_{r>0} \frac{m(t, r)}{r} \ge \frac{r_\epsilon^2(t)}{a^3(t)}.$$

Finally,

$$\begin{aligned} -E_{\mathrm{pot}}(t) &\ge \frac{1}{2}\left(\frac{3}{4\pi a^3(t)}\right)^2 \int_{|x|\le r_\epsilon(t)} \int_{|y|\le r_\epsilon(t)} \frac{1}{|x-y|} \mathrm{d}y\, \mathrm{d}x \\ &= \frac{1}{2}\left(\frac{3}{4\pi}\right)^2 \int_{|x|\le 1} \int_{|y|\le 1} \frac{1}{|x-y|}\, \mathrm{d}y\, \mathrm{d}x \frac{r_\epsilon^5(t)}{a^6(t)}. \end{aligned}$$

Using parts (a) and (b) of Theorem 2.2 together with the fact that $\lim_{t\to T} a(t) = 0$, all these quantities can be made large in the sense of Theorem 1.1 by making ϵ small and choosing t close to T; when doing this $\mathring{a}$ and hence the estimates at $t = 0$ remain unchanged. The fact that the kinetic energy behaves in the same way follows from conservation of energy, and the proof is complete. □

3. Proof of Theorem 2.2

We first observe that the parameter $\mathring{a}$ was used only to make sure that the initial estimates in Theorem 1.1 hold. Since it plays no role in the proof of Theorem 2.2, we can for the rest of this paper simplify our notation by choosing $\mathring{a} = 1$.

3.1. Proof of Parts (a) and (b) of Theorem 2.2

The initial data for the functions a and r_ϵ imply that $r_\epsilon < a$ on some interval $]0, t^*[\subset [0, T[\cap [0, T_\epsilon[$ where we choose $t^* > 0$ maximal. The definition (2.9) together with the properties of ϕ_ϵ implies that

$$1 < M_\epsilon < (1 + \epsilon)^3 + \epsilon.$$

We use the lower bound to conclude from the differential equations for a and r_ϵ that on $]0, t^*[$ the estimate $\ddot{r}_\epsilon < \ddot{a}$ holds, and since $\dot{r}_\epsilon(0) = -\epsilon < \dot{a}(0)$ also $\dot{r}_\epsilon < \dot{a}$. This implies that $t^* = \min(T, T_\epsilon)$. Since the maximal existence time T, respectively, T_ϵ is determined by the fact that the function a, respectively, r_ϵ becomes zero there and since the difference $a - r_\epsilon$ is positive and strictly increasing as long as both functions exist we can conclude that $T_\epsilon < T$ and $r_\epsilon < a$ on $]0, T_\epsilon[$.

As in the proof of Lemma 2.1, we see that r_ϵ is a strictly decreasing function with $\dot r_\epsilon < -\epsilon$, and

$$(\dot r_\epsilon(t))^2 - \epsilon^2 = 2M_\epsilon \left(\frac{1}{r_\epsilon(t)} - 1\right).$$

Hence

$$\dot r_\epsilon(t) = -\sqrt{2M_\epsilon}\sqrt{\frac{1}{r_\epsilon(t)} - C_\epsilon} \quad \text{with} \quad C_\epsilon := 1 - \frac{\epsilon^2}{2M_\epsilon}.$$

We note that $\lim_{\epsilon\to 0} M_\epsilon = 1$ and $0 < C_\epsilon < 1$ with $\lim_{\epsilon\to 0} C_\epsilon = 1$. Let us define a function $F\colon [0,1] \to \mathbb{R}$ by

$$F(r) := r\sqrt{\frac{1}{r} - 1} + \arctan\sqrt{\frac{1}{r} - 1} \quad \text{for} \quad r \in]0,1], \quad F(0) := \frac{\pi}{2}.$$

This function is continuous and differentiable on $]0,1[$ with $F'(r) = -1/\sqrt{\frac{1}{r} - 1} < 0$. Hence $F\colon [0,1] \to [0,\pi/2]$ is strictly decreasing and onto, with a continuous inverse. Using F, the above differential equation for r_ϵ can be integrated and yields the relation

$$F(C_\epsilon r_\epsilon(t)) - F(C_\epsilon) = \sqrt{2M_\epsilon C_\epsilon^3}\, t, \quad t \in [0, T_\epsilon[.$$

Since $\lim_{t\to T_\epsilon} r_\epsilon(t) = 0$ and $\lim_{r\to 0} F(r) = \pi/2$, it follows that

$$T_\epsilon = \frac{1}{\sqrt{2M_\epsilon C_\epsilon^3}}\left(\frac{\pi}{2} - F(C_\epsilon)\right) \to \frac{\pi}{2\sqrt{2}} = T$$

for $\epsilon \to 0$ as claimed. Now fix some $t \in [0, T[$. Then for ϵ sufficiently small, $t \in [0, T_\epsilon[$, and

$$r_\epsilon(t) = \frac{1}{C_\epsilon} F^{-1}\left(F(C_\epsilon) + \sqrt{2M_\epsilon C_\epsilon^3}\, t\right) \to F^{-1}\left(\sqrt{2}\, t\right) = a(t)$$

as $\epsilon \to 0$; here we used the limit behavior of M_ϵ and C_ϵ and the identity (2.6). This proves the desired limit for $r_\epsilon(t)$, and since $0 \le a(s) - r_\epsilon(s) \le a(t) - r_\epsilon(t)$ on $[0,t]$ the limit is uniform on compact subintervals of $[0,T[$. Parts (a) and (b) of Theorem 2.2 are proven. □

3.2. The Behavior of Characteristics

To prove part (c) of Theorem 2.2, we use the fact that a smooth function f solves the Vlasov equation (1.4) if and only if it is constant along its characteristics, i.e. along the solutions of the characteristic system

$$\dot x = v, \quad \dot v = -\partial_x U(s,x). \tag{3.1}$$

In particular, if $[0,\infty[\ni s \mapsto (X(s,t,x,v), V(s,t,x,v))$ denotes the solution of the characteristic system with initial data $(X(t,t,x,v), V(t,t,x,v)) = (x,v)$ with $t \ge 0$ and $x, v \in \mathbb{R}^3$ prescribed, then a solution of the Vlasov–Poisson system f is related to its initial data $\mathring{f}$ by the relation

$$f(t,x,v) = \mathring{f}(X(0,t,x,v), V(0,t,x,v)), \quad t \in \mathbb{R}, \quad x, v \in \mathbb{R}^3.$$

To show that the solution f_ϵ launched by $\mathring{f}_\epsilon$ has a homogeneous core bounded by the curve $r = r_\epsilon$, we need to control characteristics which cross this curve.

This analysis will be facilitated by the spherical symmetry of the solutions in question. To exploit this symmetry, we define for $x, v \in \mathbb{R}^3$ corresponding spherical variables by

$$r = |x|, \quad w = \frac{x \cdot v}{r}, \quad L = |x \times v|^2. \tag{3.2}$$

If $(x(s), v(s))$ solves (3.1), then in spherical variables,

$$\dot{r} = w, \quad \dot{w} = \frac{L}{r^3} - \frac{m(s,r)}{r^2}, \quad \dot{L} = 0;$$

spherical symmetry of the gravitational field implies that angular momentum and also its square L are constant along particle trajectories.

We first establish some control on characteristics under the assumption that the gravitational field is generated by a mass distribution of total mass bounded by M_ϵ.

Lemma 3.1. *Let $\partial_x U = \partial_x U(t,x) = m(t,r)x/r^3$ be a spherically symmetric and continuously differentiable gravitational field defined on $[0,T[\times\mathbb{R}^3$ with the property that $0 \le m(t,r) < M_\epsilon$ for $t \in [0,T[$, $r \ge 0$, and some $\epsilon \in]0,1]$. Let $[0,T[\ni s \mapsto (x(s), v(s))$ be a solution of the corresponding characteristic system (3.1) with spherical representation $(r(s), w(s), L)$ as described in (3.2).*

(a) *If $r(0) \ge 1$ and $|v(0)| < \epsilon$, then $r(s) > r_\epsilon(s)$ for all $s \in]0, T_\epsilon[$.*
(b) *If $r(0) \le 1$ and $|r(t)| \ge r_\epsilon(t)$ for some $t \in]0, T_\epsilon[$, then $r(s) > r_\epsilon(s)$ for all $s \in]t, T_\epsilon[$.*

Proof. We first consider part (a). Since by assumption $r(0) \ge 1 = r_\epsilon(0)$ and $\dot{r}(0) > -\epsilon = \dot{r}_\epsilon(0)$, there exists $t \in]0, T_\epsilon]$ such that $r(s) > r_\epsilon(s)$ for $s \in]0, t[$, and we choose t maximal. On the interval $]0, t[$,

$$\ddot{r} = \frac{L}{r^3} - \frac{m(s,r)}{r^2} > -\frac{M_\epsilon}{r^2} > -\frac{M_\epsilon}{r_\epsilon^2} = \ddot{r}_\epsilon. \tag{3.3}$$

If $t < T_\epsilon$, we use the assumptions at $s = 0$ and integrate (3.3) twice to find that $r(t) > r_\epsilon(t)$ which contradicts the maximality of t. Hence $t = T_\epsilon$, and part (a) is proven.

As to part (b), we first show that there exists a time $t' \in]0, t]$ such that

$$r(t') \ge r_\epsilon(t') \quad \text{and} \quad w(t') \ge \dot{r}_\epsilon(t'). \tag{3.4}$$

This can be seen as follows. If $w(t) \ge \dot{r}_\epsilon(t)$ we choose $t' = t$. If $w(t) < \dot{r}_\epsilon(t)$, there exists $t^* \in [0, t[$ such that $r(s) > r_\epsilon(s)$ for $s \in]t^*, t[$, and we choose t^* minimal. Assuming that $w(s) < \dot{r}_\epsilon(s)$ on $]t^*, t[$ it would follow that $r(t^*) > r_\epsilon(t^*)$. If $t^* > 0$, this contradicts the minimality of t^*, and if $t^* = 0$ it contradicts the assumption $r(0) \le 1 = r_\epsilon(0)$ in part (b). Hence there must exist a time t' such that (3.4) holds. For any time $s \in [t', T_\epsilon[$ such that $r(s) \ge r_\epsilon(s)$ it follows that $\ddot{r}(s) > \ddot{r}_\epsilon(s)$, cf. (3.3). In particular, this holds for $s = t'$. Hence there exists $t^* \in]t', T_\epsilon]$ such that $r(s) > r_\epsilon(s)$ for $s \in]t', t^*[$, and we choose t^* maximal. Assuming $t^* < T_\epsilon$ we integrate the inequality $\ddot{r}(s) > \ddot{r}_\epsilon(s)$ twice starting at t' and using the properties (3.4) to conclude that $r(t^*) > r_\epsilon(t^*)$ in contradiction to the maximality of t^*. Hence $t^* = T_\epsilon$, and the proof of part (b) is complete. $\square$

Next, we consider the characteristics of the homogeneous solution h_ϵ.

Lemma 3.2. *Let $\epsilon \in]0,1]$ and let $[0,T[\ni s \mapsto (x(s),v(s))$ be a characteristic curve of the homogeneous solution h_ϵ, i.e. in (3.1) the field $\partial_x U$ is given by (2.4), with spherical representation $(r(s),w(s),L)$ as described in (3.2). If*

$$t \in [0,T_\epsilon[\quad \textit{with} \quad h_\epsilon(t,x(t),v(t)) > 0 \quad \textit{and} \quad r(t) \le r_\epsilon(t),$$

then $r(s) < r_\epsilon(s)$ for all $s \in [0,t[$.

Proof. Since we want to use the spherical representation of the given characteristic, we first assume that $x(s) \neq 0$ for $s \in]0,t[$. To exploit the fact that the characteristics of the homogeneous solution h_ϵ remain close to the trajectories of the corresponding dust particles, it is convenient to rewrite the characteristic equations in coordinates which are co-moving with the particles of the corresponding homogeneous dust solution, i.e.

$$\begin{aligned}\tilde r(s) &:= \frac{r(s)}{a(s)},\\ \tilde w(s) &:= a^2(s)\dot{\tilde r}(s) = a(s)w(s) - \dot a(s) r(s).\end{aligned}$$

The separating curve is transformed accordingly, i.e.

$$\begin{aligned}\tilde r_\epsilon(s) &:= \frac{r_\epsilon(s)}{a(s)},\\ \tilde w_\epsilon(s) &:= a^2(s)\dot{\tilde r}_\epsilon(s) = a(s)\dot r_\epsilon(s) - \dot a(s) r_\epsilon(s).\end{aligned}$$

Using the already established part (b) of Theorem 2.2 and recalling that we took $\mathring a = 1$, we find that on the interval $]0,T_\epsilon[$,

$$\dot{\tilde w}_\epsilon = a\,\ddot r_\epsilon - \ddot a\, r_\epsilon = \frac{r_\epsilon}{a^2} - a\frac{M_\epsilon}{r_\epsilon^2} \le \frac{1}{a} - \frac{M_\epsilon}{a} = \frac{1}{a}(1-M_\epsilon) < 0.$$

Thus, $\tilde w_\epsilon(s) < \tilde w_\epsilon(0) = -\epsilon$, and hence

$$\dot{\tilde r}_\epsilon(s) < -\frac{\epsilon}{a^2(s)}, \quad s \in]0,T_\epsilon[. \tag{3.5}$$

To compare this to the given, mass-carrying characteristic of the homogeneous solution, we observe that by the definition (2.1) of H_ϵ,

$$\begin{aligned}\epsilon^2 &> |a(s)v(s) - \dot a(s)x(s)|^2\\ &= a^2(s)|v(s)|^2 - 2a(s)\dot a(s)(x\cdot v)(s) + \dot a^2(s)|x(s)|^2\\ &= a^2(s)\frac{L}{r^2(s)} + (a(s)w(s) - \dot a(s)r(s))^2\\ &= \frac{L}{\tilde r^2(s)} + \tilde w^2(s).\end{aligned}$$

Thus, for $s \in]0,t[$, it follows that $\tilde w(s) > -\epsilon$ which by definition of $\tilde w$ and (3.5) implies that

$$\dot{\tilde r}(s) > -\frac{\epsilon}{a^2(s)} > \dot{\tilde r}_\epsilon(s).$$

Since $\tilde{r}(t) \le \tilde{r}_\epsilon(t)$, it follows that $\tilde{r}(s) < \tilde{r}_\epsilon(s)$ and hence $r(s) < r_\epsilon(s)$ on $[0,t[$ as desired.

So far we assumed that $x(s) \neq 0$ on the interval $]0,t[$. For times $s \in [0,t[$ where $x(s) = 0$ the assertion of the lemma holds since $r_\epsilon(s) > 0$. If the function $x(s)$ has zeros but is not identically zero, we let $[s_1, s_2] \subset [0,t]$ be an interval with non-empty interior and such that $x(s) \neq 0$ on $]s_1, s_2[$; we choose the interval maximal with this property. Then, $x(s_2) = 0$ or $s_2 = t$, and in either case $r(s_2) \le r_\epsilon(s_2)$ so that the above argument on the interval $[0,t]$ now applies to $[s_1, s_2]$ and implies that $r(s) < r_\epsilon(s)$ on $[s_1, s_2[$. Since the interval $[0,t[$ is the union of such subintervals and a set of points where $x(s) = 0$, the proof is complete. □

3.3. Proof of Part (c) of Theorem 2.2

Using the above information on characteristics, we can prove the remaining assertion of Theorem 2.2. Since in these arguments the parameter ϵ remains fixed, we write $f = f_\epsilon$ for the solution of the Vlasov–Poisson system launched by the initial data $\mathring{f} = \mathring{f}_\epsilon$ specified in (2.8) and h for the homogeneous solution. We recall that $\iint \mathring{f} < M_\epsilon$,

$$\mathring{f}(x,v) = h(0,x,v), \quad (x,v) \in B_1(0) \times \mathbb{R}^3,$$

and

$$\mathring{f}(x,v) = 0 \quad \text{if} \quad |v| \ge \epsilon;$$

the latter follows from the definition (2.3) of the homogeneous solution h and (2.8). We define

$$I := \left\{(t,x,v) \in [0,T_\epsilon[\times \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| < r_\epsilon(t)\right\},$$

and we have to show that $f|_I = h|_I$. To prove this, we first establish the following assertion for these functions on the boundary of I:

$$\begin{aligned} &\forall t \in]0,T_\epsilon[,\ x,v \in \mathbb{R}^3 \quad \text{with} \quad |x| = r_\epsilon(t) \quad \text{and} \quad w = \frac{x \cdot v}{r} \le \dot{r}_\epsilon(t): \\ &f(t,x,v) = 0 = h(t,x,v). \end{aligned} \tag{3.6}$$

To prove the assertion for f, we consider the characteristic curve $(x(s), v(s)) = (X,V)(s,t,x,v)$ of f which at time $s = t$ passes through a boundary point as specified in (3.6) with $w < \dot{r}_\epsilon(t)$. Then, there exists some $\delta > 0$ such that $|x(s)| < r_\epsilon(s)$ for $t < s < t + \delta$ and $|x(s)| > r_\epsilon(s)$ for $t - \delta < s < t$. Hence Lemma 3.1 (b) implies that $|x(0)| > 1$, and Lemma 3.1 (a) implies that $|v(0)| \ge \epsilon$ and hence $f(t,x,v) = \mathring{f}(x(0), v(0)) = 0$; by continuity, the assertion also holds if $w = \dot{r}_\epsilon(t)$. To prove (3.6) for h, we argue in the same way using a characteristic curve of h, and Lemma 3.2 implies that $h(t,x,v) = 0$.

The idea now is that there can be at most one solution of the Vlasov–Poisson system on I which has given data at $t = 0$ and satisfies the boundary condition (3.6). If we take the difference of the Vlasov equations for f and h, we find that

$$\partial_t(f-h) + v \cdot \partial_x(f-h) - \partial_x U_f \cdot \partial_v(f-h) = \partial_x(U_f - U_h) \cdot \partial_v h \tag{3.7}$$

which holds for all $t \in [0, T_\epsilon[$ and $x, v \in \mathbb{R}^3$; U_f and U_h denote the potentials induced by f, respectively, h. We consider a characteristic curve $(x(s), v(s)) = (X, V)(s, t, x, v)$ of f with $|x| < r_\epsilon(t)$ and define

$$s^* := \sup\{s \in [0, t] \mid |x(\tau)| < r_\epsilon(\tau),\ s \leq \tau \leq t\}$$

so that $|x(s)| < r_\epsilon(s)$ on $]s^*, t]$. Integrating (3.7) along the characteristic curve implies that

$$\begin{aligned}(f-h)(t,x,v) &= (f-h)(s^*, x(s^*), v(s^*)) \\ &\quad + \int_t^{s^*} \left(\partial_x(U_f - U_g) \cdot \partial_v h\right)(s, x(s), v(s))\,\mathrm{d}s \\ &= \int_t^{s^*} \left(\partial_x(U_f - U_g) \cdot \partial_v h\right)(s, x(s), v(s))\,\mathrm{d}s; \end{aligned} \tag{3.8}$$

notice that either $s^* = 0$ in which case the first term on the right hand side vanishes because both functions have the same initial data for $|x| \leq 1$, or $s^* > 0$ in which case $|x(s^*)| = r_\epsilon(s^*)$ and $w(s^*) \leq \dot{r}_\epsilon(s^*)$, and the first term on the right hand side vanishes due to (3.6). For $t \in [0, T_\epsilon[$, we define

$$D(t) := \sup\{|f - h|(t, x, v) \mid |x| \leq r_\epsilon(t),\ v \in \mathbb{R}^3\}.$$

Spherical symmetry and standard estimates imply that

$$\sup\{|\partial_x U_f - \partial_x U_h|(t, x) \mid |x| \leq r_\epsilon(t)\} \leq C D(t);$$

notice that the velocity supports of both f and h are bounded, uniformly on $[0, T_\epsilon[$, and hence $|\rho_f(t, x) - \rho_h(t, x)| \leq C D(t)$ for $|x| \leq r_\epsilon(t)$. Hence, (3.8) implies that

$$D(t) \leq C \int_0^t D(s)\,\mathrm{d}s$$

so that $D(t) = 0$ for $t \in [0, T_\epsilon[$, and the proof of Theorem 2.2 is complete.

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Gerhard Rein and Lukas Taegert
Fakultät für Mathematik, Physik und Informatik
Universität Bayreuth
95440 Bayreuth
Germany
e-mail: gerhard.rein@uni-bayreuth.de;
lukas.taegert@uni-bayreuth.de

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