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An Alexandrov–Fenchel-Type Inequality in Hyperbolic Space with an Application to a Penrose Inequality

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Abstract. We prove a sharp Alexandrov–Fenchel-type inequality for star-shaped, strictly mean convex hypersurfaces in hyperbolic n-space, $n \geq 3$. The argument uses two new monotone quantities along the inverse mean curvature flow. As an application we establish, in any dimension, an optimal Penrose inequality for asymptotically hyperbolic graphs carrying a minimal horizon, with the equality occurring if and only if the graph is an anti-de Sitter–Schwarzschild solution. This sharpens previous results by Dahl–Gicquaud–Sakovich and settles, for this class of initial data sets, the conjectured Penrose inequality for time-symmetric space–times with negative cosmological constant. We also explain how our methods can be easily adapted to derive an optimal Penrose inequality for asymptotically locally hyperbolic graphs in any dimension $n \geq 3$. When the horizon has the topology of a compact surface of genus at least one, this provides an affirmative answer, for this class of initial data sets, to a question posed by Gibbons, Chruściel and Simon on the validity of a Penrose-type inequality for exotic black holes.

1. Introduction

If $\Sigma \subset \mathbb{R}^n$ is a convex hypersurface, then the Alexandrov–Fenchel inequalities say that

$$\int_\Sigma \sigma_k(\kappa)\mathrm{d}\Sigma \geq C_{n,k}\left(\int_\Sigma \sigma_{k-1}(\kappa)\mathrm{d}\Sigma\right)^{\frac{n-k-1}{n-k}}, \tag{1.1}$$

where $\sigma_k(\kappa)$, $1 \leq k \leq n-1$, is the kth elementary symmetric function of the principal curvature vector $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$ of Σ and $C_{n,k} > 0$ is a universal constant. Moreover, the equality holds in (1.1) if and only if Σ is a round

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sphere. Classically, (1.1) follows from the general theory of mixed volumes so that convexity is used in an essential way; see [32]. Recently, however, Guan and Li [21] used a suitable normalization of a certain inverse curvature flow to extend the validity of (1.1), with the corresponding rigidity statement, for any Σ which is star-shaped and k-convex [which means that $\sigma_i(\kappa) \geq 0$ for $i = 1, \dots, k$].

An interesting question is to establish versions of these inequalities for appropriate classes of hypersurfaces in more general ambient manifolds, preferably with a corresponding rigidity statement for the case of equality. Here we focus on the case $k = 1$ of (1.1), namely

$$c_n \int_\Sigma H \mathrm{d}\Sigma \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{1.2}$$

where A is the area, $H = \sigma_1(\kappa)$ is the mean curvature,

$$c_n = \frac{1}{2(n-1)\omega_{n-1}},$$

and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. We take a first step toward solving this problem by establishing a natural analogue of (1.2) for star-shaped, strictly mean convex hypersurfaces in hyperbolic n-space, $n \geq 3$; see Theorem 1.1. The proof is partly inspired by [21] and uses two new monotone quantities for the inverse mean curvature flow in hyperbolic space. The precise asymptotics for this flow, which is a key ingredient in our analysis, has been recently established by Gerhard [18,19]; see [14] for previous work on this subject. Also, a Heintze–Karcher-type inequality due to Brendle [4] plays a key role in our proof. We also make use of a special case of a sharp geometric inequality by Brendle et al. [5]. We note that Gallego and Solanes [15] proved related isoperimetric inequalities using integral-geometric methods, but their results do not seem to be sharp.

The inequality (1.2) has recently become relevant in the context of the Penrose inequality for asymptotically flat graphs carrying a minimal horizon [11,27] and for asymptotically hyperbolic graphs carrying a constant mean curvature horizon [13]. As an application of Theorem 1.1 we establish an optimal Penrose inequality for asymptotically hyperbolic graphs carrying a minimal horizon, including the rigidity statement according to which the equality holds only if (M, g) is the graph realization of an anti-de Sitter–Schwarzschild solution; see Theorem 1.2. This Penrose inequality improves recent results by Dahl et al. [10] and settles, for this class of initial data sets, the conjectured Penrose inequality for time-symmetric space-times with negative cosmological constant [6,29]. We remark that the proof of the rigidity statement follows from the arguments in a recent paper by Huang and Wu [24], as adapted to the asymptotically hyperbolic case in [10, Section 5]; see also [12].

To explain our results, let us consider the hyperbolic n-space $\mathbb{H}^n$ with coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ and endowed with the metric

$$g_1 = dr^2 + \sinh^2 r\, h, \tag{1.3}$$

where r is the geodesic distance to a chosen origin corresponding to $r = 0$ and h is the round metric on $\mathbb{S}^{n-1}$. We say that a closed, embedded hypersurface $\Sigma \subset \mathbb{H}^n$ is *star-shaped* if it can be written as a radial graph over a geodesic sphere centered at the origin. Also, it is *strictly mean convex* if its mean curvature H is positive everywhere. We also consider $\rho_1 \colon \mathbb{H}^n \to \mathbb{R}$,

$$\rho_1(r) = \cosh r. \tag{1.4}$$

With this notation at hand we can state the hyperbolic Alexandrov–Fenchel-type inequality.

Theorem 1.1. *If $\Sigma \subset \mathbb{H}^n$ is a star-shaped and strictly mean convex hypersurface, then*

$$c_n \int_\Sigma \rho_1 H \mathrm{d}\Sigma \geq \frac{1}{2}\left(\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right), \tag{1.5}$$

where A is the area of Σ. Moreover, the equality holds if and only if Σ is a geodesic sphere centered at the origin.

We now explain the relevance of this result for a certain Penrose inequality. Recall that a Riemannian manifold (M^n, g) is said to be *asymptotically hyperbolic* (AH) if there exists a compact subset $K \subset M$ and a diffeomorphism $\Psi \colon M - K \to \mathbb{H}^n - K_0$, where $K_0 \subset \mathbb{H}^n$ is compact, such that

$$\|\Psi_* g - g_1\|_{g_1} = O(e^{-\sigma r}), \quad \|\nabla_{g_1} \Psi_* g\|_{g_1} = O(e^{-\sigma r}), \tag{1.6}$$

as $r \to +\infty$, for some $\sigma > n/2$. We also assume that the difference between scalar curvatures, namely

$$\mathfrak{R}_g = R_g + n(n-1),$$

is such that $\rho_1 \mathfrak{R}_g$ is integrable. For any chart at infinity as in (1.6) it is possible to associate a mass-like invariant $\mathfrak{m}_\Psi$ which lies in $\mathbb{L}^{n+1}$, the Lorentzian space endowed with the metric

$$(z, w) = z_0 w_0 - z_1 w_1 - \cdots - z_n w_n; \tag{1.7}$$

see Sect. 2 for more details on this construction. It turns out that the causal character of $\mathfrak{m}_\Psi$ is invariant under coordinate changes at infinity. Moreover, the numerical invariant $\mathfrak{m}_{(M,g)}$ defined by

$$\mathfrak{m}^2_{(M,g)} = |(\mathfrak{m}_\Psi, \mathfrak{m}_\Psi)| \tag{1.8}$$

does *not* depend on the chart Ψ and is termed the *mass* of (M, g). It is natural to choose $\mathfrak{m}_{(M,g)} > 0$ if $\mathfrak{m}_\Psi$ is time-like and future directed.

The positive mass conjecture in this context asserts that if $\mathfrak{R}_g \geq 0$, then $\mathfrak{m}_\Psi$ is time-like and future-directed or vanishes, the latter occurring only if (M, g) is isometric to $(\mathbb{H}^n, g_1)$. Equivalently, $\mathfrak{m}_{(M,g)} \geq 0$ with equality holding only for hyperbolic space. This has been proved for the spin case by Chruściel and Herzlich [7], generalizing a previous contribution by Wang [34]; see also [2] for a similar result in low dimensions with the spin condition removed. Moreover, if M carries a (possibly disconnected) compact, outermost minimal boundary Γ (a *horizon*) of area A, then the corresponding Penrose conjecture says that

$$\mathfrak{m}_{(M,g)} \geq \frac{1}{2}\left(\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right), \tag{1.9}$$

with equality holding only if (M, g) is the (exterior) anti-de Sitter–Schwarzschild solution. We refer to Sect. 2 and the surveys [6,29] for background on this conjecture.

Progress in establishing (1.9) has been restricted so far to the case of graphs, as we now pass to explain. Recall that the metric

$$\overline{g}_1 = \rho_1^2 d\tau^2 + g_1, \quad \tau \in \mathbb{R}, \tag{1.10}$$

realizes $\mathbb{H}^n \times \mathbb{R}$ as the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$. Using this model we, then, say that a complete immersed hypersurface $M \subset \mathbb{H}^{n+1}$ is *AH* if there exists a compact subset $K \subset M$ such that $M - K$ can be written as a vertical graph associated with a smooth function $u\colon \mathbb{H}^n - K_0 \to \mathbb{R}$, where $K_0 \subset \mathbb{H}^n$ is compact, so that (1.6) holds for the chart Ψ given by $\Psi(x, u(x)) = x$, $x \in M - K_0$; see Definition 2.1 below. As explained in [13], if additionally M carries a minimal horizon Γ, then we may assume that $\mathfrak{m}_\Psi$ is time-like and future oriented so that after composing Ψ with an isometry we have

$$\mathfrak{m}_{(M,g)} = \mathfrak{m}_\Psi(\rho_1), \tag{1.11}$$

where here we use that $\rho_1 = z_0|_{\mathbb{H}^n}$ if we view $\mathbb{H}^n \subset \mathbb{L}^{n+1}$ as the standard hyperboloid. Charts with this property are called *balanced*. Now let M be *balanced* in the sense that nonparametric coordinates at infinity are balanced as above. Moreover, assume that Γ lies on a totally geodesic hypersurface $P \subset \mathbb{H}^{n+1}$ defined by $\tau = \tau_0$, $\tau_0 \in \mathbb{R}$ and that M meets P orthogonally along Γ, so that Γ is minimal (hence, a horizon indeed). Under these conditions and starting from (1.11) it is shown in [13] that

$$\mathfrak{m}_{(M,g)} = c_n \int_M \Theta \mathfrak{R}_g dM + c_n \int_\Gamma \rho_1 H d\Gamma, \tag{1.12}$$

where $\Theta = \langle N, \partial/\partial t\rangle$, with N being the unit normal to M pointing upward at infinity and H being the mean curvature of $\Gamma \subset P$ with respect to its inward pointing unit normal; compare to the more general formula in (2.24).

We remark that if M is a graph, then (1.12) has been previously proved in [10]. If this is the case, so that $\Theta > 0$, and if we assume further that $\mathfrak{R}_g \geq 0$, then we obtain from (1.12) that

$$\mathfrak{m}_{(M,g)} \geq c_n \int_\Gamma \rho H d\Gamma. \tag{1.13}$$

In [10] this estimate is used to obtain several sub-optimal versions of (1.9). For instance, assuming that $\Gamma \subset P = \mathbb{H}^n$ is h-convex (in the sense that all principal curvatures are at least 1) and encloses the origin of P, the authors show that

$$\mathfrak{m}_{(M,g)} \geq \frac{1}{2}\left(\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \sinh r_{\text{in}} \frac{A}{\omega_{n-1}}\right),$$

where r_{in} is the radius of the largest geodesic ball centered at the origin and contained in the region enclosed by Γ. Notice that this only yields the conjectured inequality (1.9) if Γ is a geodesic sphere centered at the origin. In view of Theorem 1.1, however, if we take $\Sigma = \Gamma$ we immediately obtain the first statement in the following theorem; a more general result, covering the locally hyperbolic case, can be found in Theorem 2.2 below. Recall that a hypersurface is said to be *mean convex* if its mean curvature is non-negative everywhere.

Theorem 1.2. *Let $(M, g) \subset \mathbb{H}^{n+1}$ be a balanced AH graph carrying a minimal horizon Γ as above. If we assume further that $\Gamma \subset P = \mathbb{H}^n$ is star-shaped (with respect to the origin) and mean convex, then (1.9) holds if $R_g \geq -n(n-1)$. Moreover, the equality occurs if and only if (M, g) is the graph realization of an (exterior) anti-de Sitter–Schwarzschild solution [which is obtained by taking $\epsilon = 1$ in (2.23) below].*

As remarked above, the rigidity statement requires a separate argument and is based on results in a recent preprint by Huang and Wu [24].

This paper is organized as follows: In Sect. 2 we recall the definitions and main properties of mass-type invariants for asymptotically locally hyperbolic (ALH) manifolds. The proofs of Theorems 1.1 and 1.2, which use the material on the inverse mean curvature flow discussed in Sect. 3 and in Appendix A, are presented in Sect. 4. We observe that the Penrose inequality (1.9) admits a natural generalization to the case in which the geometry at infinity is asymptotically *locally* hyperbolic. In Appendix B we indicate how the arguments leading to the proofs of Theorems 1.1 and 1.2 can be adapted to prove a sharp Penrose-type inequality for ALH graphs, which is described in Theorem 2.2 below.

2. Mass-Type Invariants and Penrose Inequalities for Asymptotically Locally Hyperbolic Manifolds

In this section we review the main properties of mass-type invariants for asymptotically hyperbolic manifolds; see [7,8,23,30] for more details. The class of invariants presented here, which appear in natural generalizations of the classical Penrose inequality, is just a special case of a much more general construction due to Chruściel, Herzlich and Nagy [7,8,23]. Their definition applies in particular to the class of ALH manifolds we consider here.

We start by describing the corresponding *locally hyperbolic* (LH) reference metrics. Fix $\epsilon = 0, \pm 1$ and let (N^{n-1}, h) be a closed space form of sectional curvature ϵ. In the product manifold $P_\epsilon = I_\epsilon \times N$, where $I_{-1} = (1, +\infty)$ and $I_0 = I_1 = (0, +\infty)$, define the metric

$$g_\epsilon = \frac{d\tilde{r}^2}{\rho_\epsilon(\tilde{r})^2} + \tilde{r}^2 h, \quad \tilde{r} \in I_\epsilon, \tag{2.14}$$

where

$$\rho_\epsilon(\tilde{r}) = \sqrt{\tilde{r}^2 + \epsilon}. \tag{2.15}$$

It is easy to check that (P_ϵ, g_ϵ) is locally hyperbolic in the sense that its sectional curvature is constant and equal to -1. We also note that the manifold $Q_\epsilon = \mathbb{R} \times P_\epsilon$, endowed with the metric

$$\overline{g}_\epsilon = \rho_\epsilon^2 \mathrm{d}\tau^2 + g_\epsilon, \tag{2.16}$$

is locally hyperbolic as well. Moreover, if $\tau_0 \in \mathbb{R}$, then the horizontal slice $P_\epsilon^{\tau_0} \subset Q_\epsilon$ given by $\tau = \tau_0$ is totally geodesic. This follows from the fact that the vertical vector field ∂_τ is Killing. In particular, each $P_\epsilon^{\tau_0}$ can be naturally identified to P_ϵ. Notice that if we take $\epsilon = 1$ and (N, h) to be a round sphere, then we recover hyperbolic space as in the Introduction; compare (2.14)–(2.15) with (1.3)–(1.4) after setting $\tilde{r} = \sinh r$.

Let (M^n, g) be a complete n-dimensional Riemannian manifold, possibly carrying a compact inner boundary Γ. For simplicity we assume that M has a unique end, say E. We say that (M, g) is *ALH* if there exists a chart Ψ taking E to the end of P_ϵ corresponding to $\tilde{r} = +\infty$ so that, as $\tilde{r} \to +\infty$,

$$\|\Psi_* g - g_\epsilon\|_{g_\epsilon} + \|d\Psi_* g\|_{g_\epsilon} = O\left(\tilde{r}^{-\sigma}\right), \tag{2.17}$$

for some $\sigma > n/2$. We also assume that $\mathfrak{R}_g = R_g + n(n-1)$ is such that $\rho_\epsilon \mathfrak{R}_g$ is integrable. If $\epsilon = 1$ and $N = \mathbb{S}^{n-1}$ we say that (M, g) is *AH*.

It turns out that each model space P_ϵ is *static* in the sense that the space

$$\mathcal{N}_{g_\epsilon} = \{f \in C^\infty(P_\epsilon);\ (\Delta_{g_\epsilon} f) g_\epsilon - \nabla^2_{g_\epsilon} f + f \mathrm{Ric}_{g_\epsilon} = 0\} \tag{2.18}$$

is non-trivial, as we can easily check that $\rho_\epsilon \in \mathcal{N}_{g_\epsilon}$. If Ψ is a chart at infinity as above, we define the corresponding mass functional $\mathfrak{m}_\Psi : \mathcal{N}_{g_\epsilon} \to \mathbb{R}$ by

$$\mathfrak{m}_\Psi(\varphi) = \lim_{r \to +\infty} c_n \int_{N_r} \left(\varphi \left(\mathrm{div}_{g_\epsilon} e - \mathrm{dtr}_{g_\epsilon} e\right) - i_{\nabla_{g_\epsilon}\varphi} e + (\mathrm{tr}_{g_\epsilon} e) \mathrm{d}\varphi\right)(\nu_r) \mathrm{d}N_r, \tag{2.19}$$

where $e = \Psi_* g - g_\epsilon$, ν_r is the unit normal to $N_r = \{r\} \times N$ and

$$c_n = \frac{1}{2(n-1)\vartheta_{n-1}}, \quad \vartheta_{n-1} = \mathrm{area}(N, h).$$

If Φ is another chart at infinity one verifies that

$$\mathfrak{m}_\Phi(\varphi) = \mathfrak{m}_\Psi(\varphi \circ \mathcal{I}^{-1}), \tag{2.20}$$

where $\mathcal{I} \in \mathrm{Isom}(P_\epsilon)$ satisfies

$$\|\Phi \circ \Psi^{-1} - \mathcal{I}\|_{g_\epsilon} = O(\tilde{r}^{-\sigma}).$$

Thus, to get a numerical invariant out of this scheme we need a detailed description of the structure of the action of $\mathrm{Isom}(P_\epsilon)$ on $\mathcal{N}_{g_\epsilon}$ appearing on the right-hand side of (2.20).

We first consider the case $\epsilon = 1$ and $N_1 = \mathbb{S}^{n-1}$ so that $P_1 = \mathbb{H}^n$. Here, $\mathcal{N}_{g_1}$ is generated by $\{z_i|_{\mathbb{H}^n}\}_{i=0}^n$, where we view $\mathbb{H}^n \subset \mathbb{L}^{n+1}$, the Lorentz space endowed with the metric (1.7). Since the action of $\mathrm{Isom}(\mathbb{H}^n)$ on $\mathcal{N}_{g_1} = \mathbb{L}^{n+1}$ preserves (1.7), with $\rho_1 = z_0$ being time-like and future oriented, it follows that the real number $\mathfrak{m}_{(M,g)}$ defined up to a sign by (1.8) does *not* depend on the chart Ψ and is termed the *mass* of (M, g). We note that the causal character of $\mathfrak{m}_\Psi$ is also invariant under coordinate changes at infinity, so it is natural to choose $\mathfrak{m}_{(M,g)} > 0$ if $\mathfrak{m}_\Psi$ is time-like and future directed. As already observed in Sect. 1, if M carries a minimal horizon Γ, then we may assume, after composition with an isometry, that (1.11) holds.

In contrast to the hyperbolic case described in the previous paragraph, it is known that for $\epsilon \leq 0$ the space $\mathcal{N}_{g_\epsilon}$ is one-dimensional, generated by ρ_ϵ. Thus, in all the cases considered here a mass-type numerical invariant $\mathfrak{m}_{(M,g)}$ is obtained by evaluating the right-hand side of (2.19) on $\varphi = \rho_\epsilon$.

An ALH manifold (M, g) as above can be thought of as the initial data set of a time-symmetric solution of the Einstein field equations with negative cosmological constant. The invariant $\mathfrak{m}_{(M,g)}$ is then interpreted as the total mass of the solution. Physical reasoning predicts that $\mathfrak{m}_{(M,g)}$ should have the appropriate sign under the relevant dominant energy condition, namely $\mathfrak{R}_g \geq 0$ [equivalently, $R_g \geq -n(n-1)$]. When M carries a compact minimal horizon Γ one expects the invariant to satisfy a Penrose-type inequality in the sense that it should be bounded from below by a suitable expression involving the area $|\Gamma|$ of Γ. In order to figure out the correct form of this inequality, we consider the so-called *Kottler black hole metrics*, which are deformations of the LH metrics g_ϵ above.

Let us introduce a real parameter $m > 0$ and consider the metric

$$g_{\epsilon,m} = \frac{\mathrm{d}\tilde{r}^2}{\rho_{\epsilon,m}(\tilde{r})^2} + \tilde{r}^2 h, \tag{2.21}$$

where

$$\rho_{\epsilon,m}(\tilde{r}) = \sqrt{\tilde{r}^2 + \epsilon - \frac{2m}{\tilde{r}^{n-2}}}.$$

For each m as above, it is easy to see that the function

$$\tilde{r} \mapsto f_{\epsilon,m}(\tilde{r}) = \tilde{r}^n + \epsilon\tilde{r}^{n-2} - 2m$$

is strictly positive for $\tilde{r} > \tilde{r}_{\epsilon,m}$, where $\tilde{r}_{\epsilon,m}$ is the unique positive zero of $f_{\epsilon,m}$. Thus, the metric $g_{\epsilon,m}$ is well defined on the product $P_{\epsilon,m} = I_{\epsilon,m} \times N$, where $I_{\epsilon,m} = \{\tilde{r}; \tilde{r} > \tilde{r}_{\epsilon,m}\}$. Moreover, it extends smoothly to the slice $\tilde{r} = \tilde{r}_{\epsilon,m}$, the so-called *horizon*, denoted $\mathcal{H}_{\epsilon,m}$. This terminology can be justified as follows: The metric $g_{\epsilon,m}$ is *static* in the sense that $\rho_{\epsilon,m} \in \mathcal{N}_{g_{\epsilon,m}}$. It is well-known that this is equivalent to the assertion that the Lorentzian metric

$$\tilde{g}_{\epsilon,m} = -\rho_{m,\epsilon}^2 \mathrm{d}\tau^2 + g_{\epsilon,m},$$

defined on $Q_{\epsilon,m} = \mathbb{R} \times P_{\epsilon,m}$, is a solution to the vacuum field equations with negative cosmological constant:

$$\mathrm{Ric}_{\tilde{g}_{\epsilon,m}} = -n\tilde{g}_{\epsilon,m}.$$

Moreover, the null hypersurface $\tilde{r} = \tilde{r}_{\epsilon,m}$ defines the event horizon surrounding the central singularity $\tilde{r} = 0$. This justifies the horizon terminology and explains why $g_{\epsilon,m}$ is termed a black hole metric.

A computation shows that if $(\theta_1, \ldots, \theta_{n-1})$ are coordinates in $N_{\tilde{r}}$, then the sectional curvatures of $g_{\epsilon,m}$ are

$$K(\partial_{\tilde{r}}, \partial_{\theta_i}) = -1 - (n-2)\frac{m}{\tilde{r}^n}$$

and

$$K(\partial_{\theta_i}, \partial_{\theta_j}) = -1 + \frac{2m}{\tilde{r}^{n-2}}.$$

This not only shows that $g_{\epsilon,m}$ satisfies the appropriate dominant energy condition, since its scalar curvature is $R_{g_{\epsilon,m}} = -n(n-1)$, but also suggests that $g_{\epsilon,m}$ is ALH in the sense described above. In fact, a straightforward computation gives

$$\|g_{\epsilon,m} - g_\epsilon\|_{g_\epsilon} + \|\mathrm{d}g_{\epsilon,m}\|_{g_\epsilon} = O\left(\tilde{r}^{-n}\right),$$

as expected. Using (2.19) with $\varphi = \rho_\epsilon$ we finally conclude that $\mathfrak{m}_{(P_{\epsilon,m}, g_{\epsilon,m})} = m$, which shows that m should be interpreted as the total mass of $g_{\epsilon,m}$.

One immediately finds that the area $|\mathcal{H}_{\epsilon,m}|$ of the horizon $\mathcal{H}_{\epsilon,m}$ of $(P_{\epsilon,m}, g_{\epsilon,m})$ relates to its mass m by means of

$$m = \frac{1}{2}\left(\left(\frac{|\mathcal{H}_{\epsilon,m}|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \epsilon\left(\frac{|\mathcal{H}_{\epsilon,m}|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right).$$

Thus, in analogy with the standard Penrose inequality (1.9), it is natural to conjecture that if (M, g) is an n-dimensional ALH manifold (with respect to the reference metric g_ϵ) carrying an outermost minimal horizon Γ and satisfying $R_g \geq -n(n-1)$ everywhere, then there holds

$$\mathfrak{m}_{(M,g)} \geq \frac{1}{2}\left(\left(\frac{|\Gamma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \epsilon\left(\frac{|\Gamma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right), \tag{2.22}$$

with the equality occurring if and only if g is isometric to the corresponding black hole metric.

Remark 2.1. For $\epsilon \leq 0$ and in the physical dimension $n = 3$, Eq. (2.22) first appears in [9] as a conjectured inequality whose veracity would follow in case the use of the so-called Geroch's monotonicity of the Hawking mass under the inverse mean curvature flow, as envisaged by Gibbons [20], could be justified. Contrary to this rather optimistic initial expectation, Neves [31] has shown that, at least in the AH case, the convergence properties of the flow at infinity are insufficient to implement Geroch's scheme. Similar remarks should also apply in the general ALH context, even though Lee and Neves [28] have recently established that Geroch's strategy works in the so-called 'non-positive mass range'; see Remark 2.3. Despite these negative results, Theorem 2.2 below confirms that our methods apply to handle the special case of graphs in *any* dimension $n \geq 3$.

To motivate our setting we observe that each $(P_{\epsilon,m}, g_{\epsilon,m})$ can be isometrically immersed as a radially symmetric vertical graph inside $(Q_\epsilon, \overline{g}_\epsilon)$: the defining function $u_{\epsilon,m} \colon I_{\epsilon,m} \to \mathbb{R}$ satisfies $u_{\epsilon,m}(\tilde{r}_{\epsilon,m}) = 0$ and

$$\rho_\epsilon(\tilde{r})^2 \left(\frac{\mathrm{d}u_{\epsilon,m}}{\mathrm{d}\tilde{r}^2}\right)^2 = \frac{1}{\rho_{\epsilon,m}(\tilde{r})^2} - \frac{1}{\rho_\epsilon(\tilde{r})^2} \tag{2.23}$$

It is clear from this that the horizon $\mathcal{H}_{\epsilon,m}$ lies on the totally geodesic horizontal slice P_ϵ^0, with the intersection $M \cap P_\epsilon^0$ being orthogonal along $\mathcal{H}_{\epsilon,m}$. This motivates us to consider a more general class of hypersurfaces in $(Q_\epsilon, \overline{g}_\epsilon)$.

Definition 2.1. We say that a complete hypersurface $(M, g) \subset (Q_\epsilon, \overline{g}_\epsilon)$, possibly carrying a compact inner boundary Γ, is *ALH* if there exists a compact set $K \subset M$ so that $M - K$ can be written as a graph over the end E_0 of the horizontal slice $P_\epsilon^0 \subset Q_\epsilon$, with the graph being associated to a smooth function u such that the asymptotic condition (2.17) holds for the nonparametric chart $\Psi_u(x, u(x)) = x$, $x \in E_0$. Moreover, we assume that $\rho_\epsilon \mathfrak{R}_g$ is integrable. As usual, if $\epsilon = 1$ and $N = \mathbb{S}^{n-1}$, then we say that (M, g) is *AH*.

Under these conditions, the mass $\mathfrak{m}_{(M,g)}$ can be computed by taking $\Psi = \Psi_u$ in (2.19); as in the "Introduction" we assume that Ψ_u is balanced if $\epsilon = 1$. More precisely, if the inner boundary Γ lies on some totally geodesic, horizontal slice $P_\epsilon^{\tau_0}$, which we of course identify with P_ϵ, and moreover that the intersection $M \cap P_\epsilon^{\tau_0}$ is orthogonal along Γ, so that $\Gamma \subset M$ is minimal and hence a horizon indeed, then the computations in [13] actually give the following integral formula for the mass:

$$\mathfrak{m}_{(M,g)} = c_n \int_M \Theta \left(R_g + n(n-1)\right) \mathrm{d}M + c_n \int_\Gamma \rho_\epsilon H \mathrm{d}\Gamma, \tag{2.24}$$

where $\Theta = \langle \partial/\partial t, N \rangle$, N is the unit normal to M, which we choose so as to point upward at infinity, and H is the mean curvature of $\Gamma \subset P_\epsilon$ with respect to its *inward* pointing unit normal, which means that the normal points in the direction *opposite* to the end of P_ϵ given by $\tilde{r} = +\infty$. In particular, if $R_g \geq -n(n-1)$ and M is a graph ($\Theta > 0$), then

$$\mathfrak{m}_{(M,g)} \geq c_n \int_\Gamma \rho_\epsilon H \mathrm{d}\Gamma. \tag{2.25}$$

We are now in a position to state the following Alexandrov–Fenchel-type inequality, which extends Theorem 1.1 to the case $\epsilon \leq 0$.

Theorem 2.1. *Let $\Sigma \subset P_\epsilon = P$ be a compact embedded hypersurface which is* star-shaped *in the sense that it can be written as a radial graph over a slice $N_{\tilde{r}} = \{\tilde{r}\} \times N$ and* strictly mean convex *in the sense that its mean curvature satisfies $H > 0$. Then there holds*

$$c_n \int_\Sigma \rho_\epsilon H \mathrm{d}\Sigma \geq \frac{1}{2} \left(\left(\frac{|\Sigma|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} + \epsilon \left(\frac{|\Sigma|}{\vartheta_{n-1}} \right)^{\frac{n-2}{n-1}} \right), \tag{2.26}$$

with the equality occurring if and only if Σ is a slice.

By making $\Sigma = \Gamma$ and combining (2.25) and (2.26) we immediately obtain the following sharp Penrose-type inequality, which extends Theorem 1.2 to the case $\epsilon \leq 0$.

Theorem 2.2. *If $M \subset Q_{0,\epsilon}$ is an ALH graph as above, so that its horizon $\Gamma \subset P_\epsilon^{\tau_0}$ is star-shaped and mean convex in the sense that $H \geq 0$, then*

$$\mathfrak{m}_{(M,g)} \geq \frac{1}{2}\left(\left(\frac{|\Gamma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \epsilon\left(\frac{|\Gamma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right), \tag{2.27}$$

with the equality holding if and only if (M, g) is (congruent to) the graph realization (2.23) of the corresponding black hole solution.

Remark 2.2. Under the conditions of Theorem 2.2, the mass $\mathfrak{m}_{(M,g)}$ is always positive due to (2.25). In particular, if $\epsilon = -1$ the lower bounds (2.26) and (2.27) only are effective if we further assume that $|\Gamma| > \vartheta_{n-1}$.

As already observed, the following corollary provides a positive answer to a question posed by Gibbons [20] and Chruściel and Simon [9] for the class of initial data sets we consider.

Corollary 2.1. *If the horizon Γ is a surface of genus $\gamma \geq 1$, then*

$$\mathfrak{m}_{(M,g)} \geq \left(\frac{4\pi}{\vartheta_2}\right)^{3/2}\sqrt{\frac{|\Gamma|}{16\pi}}\left(1 - \gamma + \frac{|\Gamma|}{4\pi}\right), \tag{2.28}$$

where for $\gamma = 1$ we assume the normalization $\vartheta_2 = 4\pi$. Moreover, the equality holds if and only if (M, g) is (congruent to) the graph realization of the corresponding black hole solution.

Proof. If $\gamma \geq 2$ this follows by taking $n = 3$ and $\epsilon = -1$ in the theorem and observing that Gauss–Bonnet gives $\vartheta_2 = 4\pi(\gamma - 1)$. If $\gamma = 1$ we take $\epsilon = 0$ and use the normalization. □

The proofs of Theorems 2.1 and 2.2 are straightforward adaptations of the proofs of Theorems 1.1 and 1.2. The necessary modifications are briefly described in Appendix B. We note that further results in this direction have been obtained in [16].

Remark 2.3. As discussed in [9], when $\epsilon = -1$ the Kottler metrics (2.21) also describe static black hole solutions when the parameter m becomes negative up to a certain critical value, namely

$$m_{\text{crit}} = -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

In this regard we mention that Lee and Neves [28] used the Huisken–Ilmanen's formulation of the inverse mean curvature flow to establish a Penrose-type inequality for conformally compact ALH 3-manifolds in this mass range. More precisely, they prove (2.28) with the mass replaced by the supremum of the mass aspect function, which is assumed to be non-positive along the boundary at infinity. In particular, their manifolds always have *non-positive* mass while our graphs necessarily satisfy $\mathfrak{m}_{(M,g)} > 0$; see Remark 2.2. Thus, their result and Corollary 2.1 are in a sense complementary to each other.

3. Geometric Flows for Hypersurfaces

As mentioned above, the proof of Theorem 1.1 uses the inverse mean curvature flow recently studied by Gerhardt [18,19]; see also [14]. As a preparation for the argument, let us start by considering a closed, isometrically immersed hypersurface $\Sigma \subset \mathbb{H}^n$ with unit normal ξ. As usual, we denote by g_1 both the standard metric on $\mathbb{H}^n$ and its restriction to Σ. Also, b is the second fundamental form of Σ. For simplicity we set $D = \nabla_{g_1}$. Thus, if X and Y are vector fields tangent to Σ,

$$b(X,Y) = g_1(aX,Y) = \langle aX,Y\rangle,$$

where

$$aX = -D_X\xi,$$

is the shape operator. As before, we denote by $\kappa = (\kappa_1,\ldots,\kappa_{n-1})$ the principal curvature vector of Σ, so that

$$H = \sigma_1(\kappa) = \mathrm{tr}_{g_1} b \tag{3.29}$$

is the *mean curvature*. It follows from the Cauchy–Schwarz inequality that

$$(n-1)|a|^2 \geq H^2, \tag{3.30}$$

with the equality occurring at a given point if and only if Σ is umbilical there. We also consider the *extrinsic scalar curvature* of the immersion, namely

$$K = \sigma_2(\kappa) = \sum_{i<j} \kappa_i\kappa_j = \frac{1}{2}\left(H^2 - |a|^2\right). \tag{3.31}$$

Notice that these invariants are related by the Newton–MacLaurin inequality:

$$2K \leq \frac{n-2}{n-1}H^2, \tag{3.32}$$

with the equality holding at a given point only if Σ is umbilical there [22]. We also recall the *support function*

$$p = \langle D\rho, \xi\rangle, \tag{3.33}$$

where we set $\rho = \rho_1$ for simplicity. This relates to ρ and H by means of the following Minkowski identity:

$$\Delta\rho = (n-1)\rho + Hp, \tag{3.34}$$

where $\Delta = \mathrm{div} \circ \nabla$ is the Laplacian of $g_1|_\Sigma$. This is a consequence of the fact that the vector field $D\rho$ is *conformal*, that is,

$$D_X D\rho = \rho X, \tag{3.35}$$

for any vector field X on $\mathbb{H}^n$. Another useful consequence of (3.35) is the formula

$$\mathrm{div}\,(G\nabla\rho) = (n-2)\rho H + 2pK, \tag{3.36}$$

where

$$G = HI - a \tag{3.37}$$

is the *Newton tensor* of a; see [1] for further details.

We now consider an one-parameter family $X(t,\cdot)\colon \Sigma^{n-1}\to\mathbb{H}^n$, $t\in[0,\epsilon)$, of closed, isometrically immersed hypersurfaces evolving according to

$$\frac{\partial X}{\partial t}=F\xi, \tag{3.38}$$

where ξ is the unit normal to $\Sigma_t=X(t,\cdot)$ and F is a general speed function. To save notation we also denote the evolving hypersurface simply by Σ whenever no confusion arises. The following evolution equations are well-known [35]:

Proposition 3.1. *Under the flow* (3.38) *we have*:

1. *The unit normal evolves as*

$$\frac{\partial \xi}{\partial t}=-\nabla F. \tag{3.39}$$

2. *The area element* $\mathrm{d}\Sigma$ *evolves as*

$$\frac{\partial}{\partial t}\mathrm{d}\Sigma=-FH\mathrm{d}\Sigma. \tag{3.40}$$

In particular, if A *is the area of* Σ*, then*

$$\frac{\mathrm{d}A}{\mathrm{d}t}=-\int_\Sigma FH\mathrm{d}\Sigma. \tag{3.41}$$

3. *The mean curvature evolves as*

$$\frac{\partial H}{\partial t}=\Delta F+(|a|^2-(n-1))F. \tag{3.42}$$

If Σ is star-shaped and mean convex, then our conventions imply that ξ is the *inward* pointing unit normal vector. Thus, in the model (1.3), Σ can be graphically represented by means of a map of the type

$$\theta\in\mathbb{S}^{n-1}\mapsto(u(\theta),\theta)\in\mathbb{H}^n, \tag{3.43}$$

for some smooth function u. In particular, if $\theta=(\theta_1,\ldots,\theta_{n-1})$ is a local coordinate system on $\mathbb{S}^{n-1}$ and $E_i=\partial/\partial\theta_i$, then the tangent space to the graph is spanned by

$$Z_i=u_i\frac{\partial}{\partial r}+E_i,\quad u_i=E_i(u),\quad i=1,\ldots,n-1, \tag{3.44}$$

so we can take

$$\xi=\frac{1}{W}\left(\frac{u^i}{\dot\rho(u)^2}E_i-\frac{\partial}{\partial r}\right),\quad W=\sqrt{1+|\nabla v|_h^2}, \tag{3.45}$$

where

$$v=\varphi(u),\quad \dot\varphi=1/\dot\rho, \tag{3.46}$$

with

$$\dot\rho(u)=\sinh u;$$

see [18] or [14]. Also,

$$p=-\frac{\sinh u}{W}. \tag{3.47}$$

Notice that $p\leq 0$.

From now on we assume that $\Sigma = \Sigma_t = X(t, \cdot)$ is a one-parameter family of star-shaped, strictly mean convex hypersurfaces evolving according to (3.38). This assumption will be justified later on for the flows we shall consider; see Proposition 3.6 and Remark 3.1.

Proposition 3.2. *Under the above conditions, the function ρ evolves along the flow* (3.38) *according to*

$$\frac{\partial \rho}{\partial t} = pF. \tag{3.48}$$

Proof. As noted above, we can graphically represent Σ by (3.43), where u is time dependent so that (3.38) implies

$$\frac{\partial u}{\partial t} = -\frac{F}{W}.$$

Since $u = r$ along Σ we have

$$\frac{\partial \rho}{\partial t} = \sinh u \frac{\partial u}{\partial t} = -\frac{\sinh u\, F}{W},$$

and the result follows from (3.47). □

The following proposition computes the variation of the curvature integral:

$$\mathcal{I}(\Sigma) = \int_\Sigma \rho H \mathrm{d}\Sigma \tag{3.49}$$

on the left-hand side of (1.5).

Proposition 3.3. *Along the flow* (3.38) *we have*

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = 2\int_\Sigma pHF\mathrm{d}\Sigma - 2\int_\Sigma \rho KF \mathrm{d}\Sigma. \tag{3.50}$$

Proof. Using Propositions 3.1 and 3.2,

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} &= \int_\Sigma \frac{\partial \rho}{\partial t} H \mathrm{d}\Sigma + \int_\Sigma \rho \frac{\partial H}{\partial t} \mathrm{d}\Sigma + \int_\Sigma \rho H \frac{\partial}{\partial t} \mathrm{d}\Sigma \\
&= \int_\Sigma pHF\mathrm{d}\Sigma + \int_\Sigma \rho\left(\Delta F + \left(|a|^2 - (n-1)\right) F\right)\mathrm{d}\Sigma \\
&\quad - \int_\Sigma \rho H^2 F \mathrm{d}\Sigma \\
&= \int_M pHF\mathrm{d}\Sigma + \int_\Sigma F\Delta\rho \mathrm{d}\Sigma + \int_\Sigma \rho\left(|a|^2 - (n-1)\right) F\mathrm{d}\Sigma \\
&\quad - \int_\Sigma \rho H^2 f \mathrm{d}\Sigma,
\end{aligned}$$

and the result follows, after some cancelations, from (3.31) and (3.34). □

Proposition 3.4. *Along the flow* (3.38), *the support function evolves according to*

$$\frac{\partial p}{\partial t} = F\rho - \langle \nabla \rho, \nabla F\rangle. \tag{3.51}$$

As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma p\mathrm{d}\Sigma = n\int_\Sigma F\rho\mathrm{d}\Sigma. \tag{3.52}$$

Proof. Using (3.33), (3.35) and (3.39) we compute

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\langle D\rho,\xi\rangle \\ &= F\langle D_\xi D\rho,\xi\rangle + \langle D\rho, D_{\partial/\partial t}\xi\rangle \\ &= F\rho - \langle\nabla\rho,\nabla F\rangle,\end{aligned}$$

which proves (3.51). Now, using this and (3.40),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma p\mathrm{d}\Sigma &= \int_\Sigma \frac{\partial p}{\partial t}\mathrm{d}\Sigma + \int_\Sigma p\frac{\partial}{\partial t}\mathrm{d}\Sigma \\ &= \int_\Sigma F\rho\mathrm{d}\Sigma - \int_\Sigma\langle\nabla\rho,\nabla F\rangle\mathrm{d}\Sigma - \int_\Sigma pFH\mathrm{d}\Sigma \\ &= \int_\Sigma F\rho\mathrm{d}\Sigma + \int_\Sigma F\Delta\rho\mathrm{d}\Sigma - \int_\Sigma pFH\mathrm{d}\Sigma,\end{aligned}$$

so that (3.52) follows from (3.34). □

The following proposition, proved in [4], plays a central role in our argument.

Proposition 3.5. *If $\Sigma\subset\mathbb{H}^n$ is star-shaped and strictly mean convex, then*

$$(n-1)\int_\Sigma\frac{\rho}{H}\mathrm{d}\Sigma \geq -\int_\Sigma p\mathrm{d}\Sigma. \tag{3.53}$$

Moreover, the equality holds if and only if Σ is totally umbilical.

Proof. This is a rather special case of the Heintze–Karcher-type inequality proved in [4], so we merely sketch the elegant argument there. The idea is to let Σ flow under

$$\frac{\partial X}{\partial t} = \rho\xi, \tag{3.54}$$

so we take $F=\rho$ in (3.38). Using (3.42), (3.48), (3.34) and (3.30) we see that, as long as the flow exists,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\rho}{H} &= \frac{1}{H}\frac{\partial\rho}{\partial t} - \frac{\rho}{H^2}\frac{\partial H}{\partial t} \\ &= \frac{p\rho}{H} - \frac{\rho}{H^2}\left(\Delta\rho + \left(|a|^2-(n-1)\right)\rho\right) \\ &= -\frac{\rho^2}{H^2}|a|^2 \\ &\leq -\frac{\rho^2}{n-1},\end{aligned}$$

so that by (3.40),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\frac{\rho}{H}\mathrm{d}\Sigma \leq -\frac{n}{n-1}\int_\Sigma\rho^2\mathrm{d}\Sigma.$$

Combining this with (3.52) we finally get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((n-1)\int_\Sigma \frac{\rho}{H}\mathrm{d}\Sigma + \int_\Sigma p\mathrm{d}\Sigma\right) \leq 0,$$

that is, the quantity within parenthesis is monotone non-increasing along the flow (3.54). The next step is to investigate the asymptotic behavior of solutions of (3.54). This might appear problematic at first sight but the key observation is that (3.54) is equivalent to the standard flow by *inward* parallel hypersurfaces ($F = 1$) in the conformal metric

$$\tilde{g}_1 = \rho^{-2} g_1 = \frac{\mathrm{d}r^2}{\cosh^2 r} + \tanh^2 r h.$$

Thus, any solution becomes extinct in a certain finite time $t_* > 0$ so that

$$\lim_{t\to t_*}(n-1)\int_\Sigma \frac{\rho}{H}\mathrm{d}\Sigma + \int_\Sigma p\mathrm{d}\Sigma = 0,$$

as desired. In fact, an additional complication arises from the fact that the flow might develop singularities before the extinction time due to the appearance of cut points but, as explained in [4], a regularization procedure can be implemented to take care of this. □

Remark 3.1. It follows from the computation above that

$$\frac{\partial}{\partial t}\frac{H}{\rho} \geq \frac{H^2}{n-1},$$

which implies that strict mean convexity is preserved under (3.54).

From now on we specialize to the flow

$$\frac{\partial X}{\partial t} = -\frac{\xi}{H}, \tag{3.55}$$

so that $F = -1/H$. This is the famous *inverse mean curvature flow*, which has been extensively studied in a variety of contexts [17,25,31,33]. Here we will make use of recent results by Gerhardt [18,19] for evolving hypersurfaces in hyperbolic space, which we collect below.

Proposition 3.6. *If the initial hypersurface is star-shaped and strictly mean convex, then the corresponding solution is defined for all $t > 0$ and expands the evolving hypersurfaces toward infinity while maintaining star-shapedness and strictly mean convexity. Moreover, the hypersurfaces become strictly convex exponentially fast and also more and more umbilical in the sense that*

$$|b_i^j - \delta_i^j| \leq C e^{-\frac{t}{n-1}}, \quad t > 0, \tag{3.56}$$

that is, the principal curvatures are uniformly bounded and converge exponentially fast to 1. Moreover, there exists $f\colon \mathbb{S}^{n-1} \to \mathbb{R}$ smooth so that, as $t \to +\infty$, the graphing function u satisfies

$$\lim_{t\to+\infty}\left\| u - \frac{t}{n-1} - f(\theta)\right\|_{C^\infty(\mathbb{S}^{n-1})} = 0. \tag{3.57}$$

In particular,

$$\rho(u) = \cosh u = O(e^{\frac{t}{n-1}}), \quad \dot{\rho}(u) = \sinh u = O(e^{\frac{t}{n-1}}), \tag{3.58}$$

and

$$|\nabla u|_h + |\nabla^2 u|_h = O(1). \tag{3.59}$$

Remark 3.2. It is claimed in [18] that the function f above is actually a constant, which means that the flow would deform the induced metric on the hypersurface to a round one after a suitable scaling. This is, however, not correct, as the concrete example in [26] shows. The correct asymptotics (3.57) appears in [19].

4. The Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 involves the consideration of two new monotone quantities along the solution of (3.55) with Σ as the initial hypersurface. Thus, for any closed $\Sigma \subset \mathbb{H}^n$ we set

$$\mathcal{J}(\Sigma) = -\int_\Sigma p \mathrm{d}\Sigma, \tag{4.60}$$

$$\mathcal{K}(\Sigma) = \omega_{n-1}\mathcal{A}(\Sigma)^{\frac{n}{n-1}}, \tag{4.61}$$

where $\mathcal{A}(\Sigma) = A/\omega_{n-1}$, and

$$\mathcal{L}(\Sigma) = \frac{\mathcal{I}(\Sigma) - (n-1)\mathcal{K}(\Sigma)}{\mathcal{A}(\Sigma)^{\frac{n-2}{n-1}}}. \tag{4.62}$$

To save notation, sometimes we write $\mathcal{I}(t) = \mathcal{I}(\Sigma_t)$, etc. As we shall see below, the new monotone quantities are $\mathcal{L}$ and $\mathcal{A}^{-\frac{n}{n-1}}(\mathcal{J} - \mathcal{K})$.

Proposition 4.1. *On a geodesic sphere we have*

$$\mathcal{L} \geq (n-1)\omega_{n-1}. \tag{4.63}$$

Moreover, the equality holds if and only if the geodesic sphere is centered at the origin.

Proof. If a geodesic sphere has radius r, then its area is $A = \omega_{n-1}\sinh^{n-1} r$ and its mean curvature is $H = (n-1)\coth r$. Furthermore, if it is centered at the origin, then its support function is $p = -\sinh r$ by (3.47). The equality in (4.63) then follows by a direct computation. On the other hand, if $\Sigma \subset \mathbb{H}^n$ is *any* geodesic sphere of radius r, then

$$K = \frac{(n-1)(n-2)}{2}\coth^2 r,$$

so that (3.36) yields

$$\begin{aligned}\int_\Sigma \rho H \mathrm{d}\Sigma &= -\frac{2}{n-2}\int_\Sigma pK\mathrm{d}\Sigma \\ &= -(n-1)\coth^2 r\int_\Sigma p\mathrm{d}\Sigma.\end{aligned}$$

Furthermore, if B is the geodesic ball bounded by Σ, (3.33), (3.35) and the divergence theorem imply

$$\int_\Sigma p \mathrm{d}\Sigma = -\int_B \Delta_{\mathbb{H}^n} \rho \mathrm{d}\mathbb{H}^n = -n \int_B \rho \mathrm{d}\mathbb{H}^n,$$

so that

$$\mathcal{I}(\Sigma) = \int_\Sigma \rho H \mathrm{d}\Sigma = n(n-1)\coth^2 r \int_B \rho \mathrm{d}\mathbb{H}^n.$$

It is clear from (4.62) and this way of writing $\mathcal{I}(\Sigma)$ as a volume integral involving ρ that the strict inequality in (4.63) holds if Σ is not centered at the origin. □

Remark 4.1. Inequality (4.63) above just means that the inequality in Theorem 1.1 holds for any geodesic sphere, with the equality occurring if and only if it is centered at the origin.

Proposition 4.2. *If the initial hypersurface Σ in (3.55) is star-shaped and strictly mean convex, then*

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = \mathcal{A} \tag{4.64}$$

and

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \frac{n}{n-1}\mathcal{K}. \tag{4.65}$$

Also,

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t} \geq \frac{n}{n-1}\mathcal{J}, \tag{4.66}$$

with the equality occurring if and only if Σ is totally umbilical.

Proof. The relation (4.65) follows from (4.64), which is a consequence of (3.41) with $F = -1/H$. Also, Eq. (4.66) follows immediately from (3.52) and (3.53). □

The above result is crucial in establishing the existence of monotone quantities for the flow (3.55).

Proposition 4.3. *If Σ is star-shaped and strictly mean convex, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathcal{J}-\mathcal{K}}{\mathcal{A}^{\frac{n}{n-1}}} \geq 0, \tag{4.67}$$

along any solution of (3.55). Also, in any interval where $\mathcal{J} \leq \mathcal{K}$, there holds

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} \leq 0. \tag{4.68}$$

Moreover, if the equality holds in any of these inequalities for some t, then Σ_t is totally umbilical.

Proof. By (4.65) and (4.66) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{J}-\mathcal{K}\right)\geq\frac{n}{n-1}\left(\mathcal{J}-\mathcal{K}\right),$$

which by (4.64) clearly yields (4.67). Moreover, by (3.50) with $F=-1/H$,

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t}=2\int_{\Sigma}\frac{\rho K}{H}\mathrm{d}\Sigma+2\mathcal{J},$$

so that by (3.32),

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t}\leq\frac{n-2}{n-1}\mathcal{I}+2\mathcal{J}.$$

From (4.65), after a rearrangement of terms, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{I}-(n-1)\mathcal{K}\right)\leq\frac{n-2}{n-1}\left(\mathcal{I}-(n-1)\mathcal{K}\right)+2\left(\mathcal{J}-\mathcal{K}\right),$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{I}-(n-1)\mathcal{K}\right)\leq\frac{n-2}{n-1}\left(\mathcal{I}-(n-1)\mathcal{K}\right),$$

whenever $\mathcal{J}\leq\mathcal{K}$. In the presence of (4.64), this immediately gives (4.68). Finally, if the equality holds in either (4.67) or in (4.68), then it holds in (3.53) as well. □

We start the proof of Theorem 1.1 by noticing that in [5, Theorem 1.1] the authors establish a sharp geometric inequality for strictly mean convex, star-shaped hypersurfaces in the anti-deSitter–Schwarzschild space. By sending the mass parameter to zero, it follows from their work that if we set

$$\mathcal{M}=\frac{\mathcal{I}-(n-1)\mathcal{J}}{\mathcal{A}^{\frac{n-2}{n-1}}},\tag{4.69}$$

then there holds

$$\mathcal{M}(\Sigma)\geq(n-1)\omega_{n-1},\tag{4.70}$$

for any $\Sigma\subset\mathbb{H}^n$ strictly mean convex and star-shaped, with the equality holding if and only if Σ is a geodesic sphere centered at the origin. Notice that this implies (1.5) whenever $\mathcal{J}(\Sigma)\geq\mathcal{K}(\Sigma)$, so we may assume that $\mathcal{J}(\Sigma)<\mathcal{K}(\Sigma)$.

We now let Σ flow under (3.55). In case $\mathcal{J}(\Sigma_t)>\mathcal{K}(\Sigma_t)$ for some $t>0$, let t_0 be the first value of the time parameter so that $\mathcal{J}(\Sigma_{t_0})=\mathcal{K}(\Sigma_{t_0})$. Notice that t_0 exists because by (4.67) the quantity $\mathcal{A}^{-\frac{n}{n-1}}(\mathcal{J}-\mathcal{K})$ is monotone nondecreasing along *any* solution of (3.55). Since $\mathcal{J}(\Sigma_t)\leq\mathcal{K}(\Sigma_t)$ for $t\leq t_0$, it then follows from Proposition 4.3 that $\mathcal{L}(\Sigma)\geq\mathcal{L}(\Sigma_{t_0})=\mathcal{M}(\Sigma_{t_0})\geq(n-1)\omega_{n-1}$, where we used (4.70) in the last step. Thus, our main inequality (1.5) is also established in this case, so it remains to consider the case in which $\mathcal{J}(\Sigma_t)<\mathcal{K}(\Sigma_t)$ for any $t>0$. However, if this is the case, then it follows again by Proposition 4.3 that $\mathcal{L}$ is monotone nonincreasing for all $t>0$. But by Proposition A.1 we have

$$\liminf_{t\to+\infty}\mathcal{L}(t)\geq(n-1)\omega_{n-1},$$

so that

$$\mathcal{L}(0) \geq (n-1)\omega_{n-1},$$

which is just a rewriting of (1.5). Finally, we note that whenever the equality holds, then it also holds in (3.53), which implies that Σ is a geodesic sphere necessarily centered at the origin by Remark 4.1. This completes the proof of Theorem 1.1.

As remarked in the Introduction, the Penrose inequality in Theorem 1.2 follows immediately from (1.13) and the Alexandrov–Fenchel inequality (1.5) in Theorem 1.1. On the other hand, the rigidity statement follows from the arguments in [10, Section 5]. This completes the proof of Theorem 1.2.

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Appendix A. The Asymptotic Behavior of $\mathcal{L}$

In this appendix we present a proof of the following proposition, which provides the expected limiting estimate for the quantity $\mathcal{L}$ along solutions of the inverse mean curvature flow. This asymptotic behavior is used in the proof of Theorem 1.1.

Proposition A.1. *If Σ_t is a solution of (3.55) with Σ_0 strictly mean convex and star-shaped, then*

$$\liminf_{t\to+\infty} \mathcal{L}(\Sigma_t) \geq (n-1)\omega_{n-1}. \tag{A.71}$$

We write the evolving hypersurfaces as graphs of a function $u = u(t,\theta)$, $\theta \in \mathbb{S}^{n-1}$. Recall that $\rho(u) = \cosh u$ so that $\dot\rho(u) = \sinh u$ and

$$\rho^2 = \dot\rho^2 + 1. \tag{A.72}$$

Also, if $v = \varphi(u)$, $\dot\varphi = 1/\dot\rho$, as in (3.46), then it follows from (3.58) and (3.59) that

$$|\nabla v|_h + |\nabla^2 v|_h = O\left(e^{-\frac{t}{n-1}}\right) \tag{A.73}$$

and

$$|\rho(u) - \dot\rho(u)| = O\left(e^{-\frac{t}{n-1}}\right). \tag{A.74}$$

Moreover, by (3.45),

$$W^{-1} = 1 - \frac{1}{2}|\nabla v|_h^2 + O\left(e^{-\frac{4t}{n-1}}\right). \tag{A.75}$$

The induced metric is

$$g_{1ij} = \dot{\rho}^2 (h_{ij} + v_i v_j) \tag{A.76}$$

so that

$$\sqrt{\det g_1} = \dot{\rho}^{n-1} \sqrt{\det h} \left(1 + \frac{1}{2} |\nabla v|_h^2 + O\left(e^{-\frac{4t}{n-1}} \right) \right), \tag{A.77}$$

so we get the expansions

$$\mathcal{A}(\Sigma_t) = \fint \dot{\rho}^{n-1} + O\left(e^{\frac{(n-3)t}{n-1}} \right), \tag{A.78}$$

and

$$\mathcal{A}(\Sigma_t)^{\frac{n-2}{n-1}} = \left(\fint \dot{\rho}^{n-1} \right)^{\frac{n-2}{n-1}} + O\left(e^{\frac{(n-4)t}{n-1}} \right), \tag{A.79}$$

where

$$\fint = \frac{1}{\omega_{n-1}} \int$$

and the integration is over $\mathbb{S}^{n-1}$.

Recall that our intention is to estimate from below the function

$$\mathcal{L}(\Sigma_t) = \frac{\int_{\Sigma_t} \rho H \mathrm{d}\Sigma_t - (n-1)\omega_{n-1} \left(\mathcal{A}(\Sigma_t) \right)^{\frac{n}{n-1}}}{\mathcal{A}(\Sigma_t)^{\frac{n-2}{n-1}}}.$$

In terms of v, the second fundamental form of the evolving hypersurface is

$$b_{ij} = \frac{\dot{\rho}}{W} \left(\rho (h_{ij} + v_i v_j) - (\nabla^2 v)_{ij} \right).$$

Notice also that by (A.76) the inverse metric is

$$g_1^{ij} = \dot{\rho}^{-2} \left(h^{ij} - \frac{v^i v^j}{W^2} \right),$$

where $v^i = h^{ij} v_j$, so that the shape operator is

$$a_j^i = g_1^{ik} b_{kj} = \frac{\rho}{W \dot{\rho}} \delta_j^i - \frac{1}{W \dot{\rho}} \tilde{h}^{ik} (\nabla^2 v)_{kj},$$

where

$$\tilde{h}^{ij} = h^{ij} - \frac{v^i v^j}{W^2},$$

and from this we see that

$$\rho H = (n-1) W^{-1} \frac{\rho^2}{\dot{\rho}} - W^{-1} \frac{\rho}{\dot{\rho}} \Delta v + O\left(e^{-\frac{3t}{n-1}} \right). \tag{A.80}$$

Thus, if we combine this with (A.77) and (A.75) we obtain

$$\begin{aligned}\int_{\Sigma_t} \rho H \mathrm{d}\Sigma_t &= (n-1)\int \rho^2 \dot{\rho}^{n-2} - \int \dot{\rho}^{n-1}\Delta v + O\left(e^{\frac{(n-3)t}{n-1}}\right) \\ &= (n-1)\int \rho^2\dot{\rho}^{n-2} + (n-1)\int \dot{\rho}^{n-2}\langle\nabla\dot{\rho},\nabla v\rangle_h + O\left(e^{\frac{(n-3)t}{n-1}}\right) \\ &= (n-1)\int \rho^2\dot{\rho}^{n-2} + (n-1)\int \rho^2\dot{\rho}^{n-2}|\nabla v|_h^2 + O\left(e^{\frac{(n-3)t}{n-1}}\right).\end{aligned}$$

On the other hand, by Hölder inequality and (A.77) we find that

$$\begin{aligned}\omega_{n-1}\mathcal{A}(\Sigma_t)^{\frac{n}{n-1}} &\leq \int (\sqrt{\det g_1})^{\frac{n}{n-1}} \\ &= \int \dot{\rho}^n + \frac{n}{2(n-1)}\int \dot{\rho}^n|\nabla v|_h^2 + O\left(e^{\frac{n-4}{n-1}t}\right).\end{aligned}$$

Thus, by (A.72) we obtain

$$\begin{aligned}\int_{\Sigma_t}\rho H\mathrm{d}\Sigma_t - (n-1)\omega_{n-1}\mathcal{A}(\Sigma_t)^{\frac{n}{n-1}} &\geq (n-1)\int \dot{\rho}^{n-2} + \frac{n-2}{2}\int \dot{\rho}^n|\nabla v|_h^2 \\ &\quad + O\left(e^{\frac{n-3}{n-1}t}\right) \\ &= (n-1)\int \dot{\rho}^{n-2} \\ &\quad + \frac{n-2}{2}\int \dot{\rho}^{n-4}|\nabla\dot{\rho}|_h^2 + O\left(e^{\frac{n-3}{n-1}t}\right),\end{aligned}$$

so if we take (A.79) into account we see that proving (A.71) amounts to checking that

$$(n-1)\int \dot{\rho}^{n-2} + \frac{n-2}{2}\int \dot{\rho}^{n-4}|\nabla\dot{\rho}|_h^2 \geq (n-1)\omega_{n-1}^{\frac{1}{n-1}}\left(\int \dot{\rho}^{n-1}\right)^{\frac{n-2}{n-1}}. \tag{A.81}$$

But, as observed in [5], this is an immediate consequence of a sharp Sobolev type inequality by Beckner [3]. This completes the proof of Proposition A.1.

Appendix B. The Asymptotically Locally Hyperbolic Case

In this appendix we briefly describe how the argument leading to Theorem 1.2 can be easily adapted to recover its generalization given by Theorem 2.2.

Clearly, the key step is to prove the Alexandrov–Fenchel-type inequality (2.26) in Theorem 2.1. We start by observing that Propositions 3.2–3.4 remain true with ρ replaced by ρ_ϵ, since their proofs only use that the ambient manifold is locally hyperbolic and carries the conformal field $\nabla_{g_\epsilon}\rho_\epsilon$; see (3.35). Also, since this ambient manifold satisfies the structural conditions in the main result in [4], the analogue of Proposition 3.5 also holds true. Taken together, these facts imply that the analogue of Proposition 4.3 still holds true, so that the proof of (2.26) boils down to checking that

$$\liminf_{t\to+\infty}\mathcal{L}(\Sigma_t) \geq (n-1)\vartheta_{n-1}\epsilon, \tag{B.82}$$

where Σ_t is the solution to the inverse mean curvature flow having Σ as initial hypersurface; compare with (A.71). We note that the left-hand side of (B.82) makes sense because the analogue of Proposition 3.6 remains true, which follows straightforwardly from the methods in [18,19]. Taking into account that (A.72) should be replaced by $\rho^2 = \dot{\rho}^2 + \epsilon$, we see that (B.82) reduces to proving that

$$(n-1)\epsilon \int_N \dot{\rho}^{n-2} + \frac{n-2}{2} \int_N \dot{\rho}^{n-4} |\nabla \dot{\rho}|_h^2 \geq (n-1)\epsilon \omega_{n-1}^{\frac{1}{n-1}} \left(\int_N \dot{\rho}^{n-1} \right)^{\frac{n-2}{n-1}};$$

compared to (A.81). Since the validity of this inequality is immediate for $\epsilon = 0, -1$, and the rigidity statement follows from a simple adaptation of the arguments in [10, Section 5], the proof of Theorem 2.2 is completed.

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