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Inverse Scattering at High Energies for Classical Relativistic Particles in a Long-Range Electromagnetic Field

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Abstract. We define scattering data for the relativistic Newton equation in a static external electromagnetic field $(-\nabla V, B) \in C^1(\mathbb{R}^n, \mathbb{R}^n) \times C^1(\mathbb{R}^n, A_n(\mathbb{R})), n \geq 2$, that decays at infinity like $r^{-\alpha-1}$ for some $\alpha \in (0, 1]$, where $A_n(\mathbb{R})$ is the space of $n \times n$ antisymmetric matrices. We prove, in particular, that the short-range part of $(\nabla V, B)$ can be reconstructed from the high-energy asymptotics of the scattering data provided that the long-range tail of $(\nabla V, B)$ is known. We consider also inverse scattering in other asymptotic regimes. This work generalizes [Jollivet (Asympt Anal 55:103–123, 2007)] where a short-range electromagnetic field was considered.

1. Introduction

Consider the multidimensional relativistic Newton equation in a static external electromagnetic field:

$$\dot{p}(t) = F(x(t), \dot{x}(t)) := -\nabla V(x(t)) + \frac{1}{c}B(x(t))\dot{x}(t),$$
$$p(t) = \frac{\dot{x}(t)}{\sqrt{1 - \frac{|\dot{x}(t)|^2}{c^2}}}, \quad x(t) \in \mathbb{R}^n,\ t \in \mathbb{R},\ n \geq 2, \tag{1.1}$$

where $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, and where $V \in C^2(\mathbb{R}^n, \mathbb{R}), B(x)$ is the $n \times n$ real antisymmetric matrix with elements $B_{i,k}, 1 \leq i, k \leq n$, and where B satisfies the closure condition

$$\frac{\partial}{\partial x_i}B_{k,m}(x) + \frac{\partial}{\partial x_m}B_{i,k}(x) + \frac{\partial}{\partial x_k}B_{m,i}(x) = 0, \tag{1.2}$$

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for $x \in \mathbb{R}^n$ and for $i, k, m = 1 \ldots n$. The constant c is positive, and for $\sigma \in (0, +\infty)$ we will denote by $\mathcal{B}(0, \sigma)$ [resp. $\overline{\mathcal{B}(0, \sigma)}$] the open (resp. closed) Euclidean ball of center 0 and radius σ.

When $n = 3$ the Eq. (1.1) is the equation of motion of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by (V, B) (see [6] and, for example, [23, Section 17]). In this equation, $x, \dot{x}, p$ denote the position, the velocity and the impulse of the particle, respectively, and t is the time, and c is the speed of light.

We also assume throughout this paper that F satisfies the following conditions

$$F = F^l + F^s, \tag{1.3}$$

where $F^l(x, v) := -\nabla V^l(x) + \frac{1}{c} B^l(x) v, F^s(x, v) = -\nabla V^s(x) + \frac{1}{c} B^s(x) v$ and $(V^l, V^s) \in (C^2(\mathbb{R}^n, \mathbb{R}))^2, (B^l, B^s) \in (C^1(\mathbb{R}^n, A_n(\mathbb{R})))^2$, and where

$$\begin{aligned} |\partial_x^{j_1} V^l(x)| &\leq \beta^l_{|j_1|}(1 + |x|)^{-(\alpha + |j_1|)}, \\ |\partial_x^{j_2} B^l_{i,k}(x)| &\leq \beta^l_{|j_2|+1}(1 + |x|)^{-(\alpha + |j_2| + 1)}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} |\partial_x^{j_1} V^s(x)| &\leq \beta^s_{|j_1|+1}(1 + |x|)^{-(\alpha + 1 + |j_1|)}, \\ |\partial_x^{j_2} B^s_{i,k}(x)| &\leq \beta^s_{|j_2|+2}(1 + |x|)^{-(\alpha + |j_2| + 2)}, \end{aligned} \tag{1.5}$$

for $x \in \mathbb{R}^n, |j_1| \leq 2$ and $|j_2| \leq 1$ and for some $\alpha \in (0, 1]$ (here j is the multi-index $j = (j^1, \ldots, j^n) \in (\mathbb{N} \cup \{0\})^n, |j| = \sum_{m=1}^n j^m$, and β^l_m and $\beta^s_{m'}$ are positive real constants for $m = 0, 1, 2$ and for $m' = 1, 2, 3$). Note that the assumption $0 < \alpha \leq 1$ includes the decay rate at infinity of the magnetic field of a magnetic monopole in $\mathbb{R}^3$ [5], and that it also includes the decay rate at infinity of a Coulombian potential. Indeed for a Coulombian potential $V^l(x) = \frac{1}{|x|}$ or for a field $(B^l_{2,3}(x), -B^l_{1,3}(x), B^l_{1,2}(x)) = \frac{x}{|x|^3}$ of a magnetic monopole in $\mathbb{R}^3 \backslash \{0\}$, estimates (1.4) are satisfied uniformly for $|x| > \varepsilon$ and $\alpha = 1$ for any $\varepsilon > 0$. Although our electromagnetic fields are assumed to be smooth on the entire space, our study may provide interesting results even in presence of singularities.

For Eq. (1.1) the energy

$$E = c^2 \sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t)) \tag{1.6}$$

is an integral of motion.

For $\sigma \in [0, +\infty)$ set

$$\mu(\sigma) = \sqrt{\frac{2\sigma}{\frac{\sigma}{c^2} + \sqrt{\frac{\sigma^2}{c^4} + 4}}} \tag{1.7}$$

and $\mu^l = \mu\big(2^6 \alpha^{-1} n^2 \max(\beta^l_1, \beta^l_2)\big)$. Then under conditions (1.4) the following is valid (see Sect. 4 for a proof): for any $v \in \mathcal{B}(0, c), |v| \geq \mu^l$, there exists a unique solution $z_\pm(v, .)$ of the equation

$$\dot{p}(t) = F^l(z(t), \dot{z}(t)),$$
$$p(t) = \frac{\dot{z}(t)}{\sqrt{1 - \frac{|\dot{z}(t)|^2}{c^2}}},\ t \in \mathbb{R}, \tag{1.8}$$

so that

$$\dot{z}_\pm(v,t) - v = o(1), \quad \text{as } t \to \pm\infty,\ z_\pm(v,0) = 0, \tag{1.9}$$

and

$$\sup_{\mathbb{R}} |\dot{z}_\pm(v,.) - v| \leq \frac{2^{\frac{7}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha|v|}. \tag{1.10}$$

When $F^l \equiv 0$ then we have $z_\pm(v,t) = tv$ for $(t,v) \in \mathbb{R} \times \mathcal{B}(0,c), v \neq 0$.

Then under conditions (1.4) and (1.5), the following is valid: for any $(v_-, x_-) \in \mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l) \times \mathbb{R}^n$, the Eq. (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = z_-(v_-, t) + x_- + y_-(t), \tag{1.11}$$

where $|\dot{y}_-(t)| + |y_-(t)| \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in \mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l) \times \mathbb{R}^n$,

$$x(t) = z_+(v_+, t) + x_+ + y_+(t), \tag{1.12}$$

for a unique $(v_+, x_+) \in \mathcal{B}(0,c) \times \mathbb{R}^n$, where $|v_+| = |v_-| \geq \mu^l$ by conservation of the energy (1.6), and where $v_+ =: a(v_-, x_-), x_+ =: b(v_-, x_-)$, and $|\dot{y}_+(t)| + |y_+(t)| \to 0$, as $t \to +\infty$. A solution x of (1.1) that satisfies (1.11) and (1.12) for some $(v_-, x_-), v_- \neq 0$, is called a scattering solution.

We call the map $S : (\mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l)) \times \mathbb{R}^n \to (\mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l)) \times \mathbb{R}^n$ given by the formulas

$$v_+ = a(v_-, x_-), \quad x_+ = b(v_-, x_-), \tag{1.13}$$

the scattering map for the Eq. (1.1). We call $a(v_-, x_-), b(v_-, x_-)$ the scattering data for the Eq. (1.1), and we define

$$a_{sc}(v_-, x_-) = a(v_-, x_-) - v_-, \quad b_{sc}(v_-, x_-) = b(v_-, x_-) - x_-. \tag{1.14}$$

We denote by $\mathcal{D}(S)$ the set of definition of S. The scattering map S is continuous, and $\mathrm{Mes}(((\mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l)) \times \mathbb{R}^n)\backslash\mathcal{D}(S)) = 0$ for the Lebesgue measure on $\mathcal{B}(0,c) \times \mathbb{R}^n$. In addition, the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where $\mathcal{M} = \{(v_-, x_-) \in \mathcal{B}(0,c) \times \mathbb{R}^n \mid v_- \neq 0, v_- \cdot x_- = 0\}$. Indeed we have

$$a(v_-, x_- + t_0 v_-) = a(v_-, x_-),\ b(v_-, x_- + t_0 v_-) = b(v_-, x_-) + t_0 a(v_-, x_-), \tag{1.15}$$

for any $t_0 \in \mathbb{R}$ and for $(v_-, x_-) \in \mathcal{D}(S)$.

We refer the reader to [2–4,13,14,22,29,31,32] and references therein for the forward classical scattering theory. Our definition of the scattering map is derived from constructions given in [4,13].

One can imagine the following experimental setting that allows to measure the scattering data without knowing the electromagnetic field (V, B)

inside a (a priori bounded) region of interest. First choose an electromagnetic field (V^l, B^l) that generates the same long-range effects as (V, B) does. Then compute the solutions $z_\pm(v, .)$ of Eq. (1.8). Then for a fixed $(v_-, x_-) \in (\mathcal{B}(0, c)\backslash\mathcal{B}(0, \mu^l)) \times \mathbb{R}^n$ send a particle far away from the region of interest with a trajectory asymptotic to $x_- + z_-(v_-, .)$ at large and negative times. When the particle escapes any bounded region of the space at finite time, then detect the particle and find $S(v_-, x_-) = (v_+, x_+)$ so that the trajectory of the particle is asymptotic to $x_+ + z_+(v_+, .)$ at large and positive times far away from the bounded region of interest.

In this paper, we consider the following inverse scattering problem for Eq. (1.1):

$$\text{Given } S \text{ and given the long-range tail } F^l \text{ of the force } F, \text{ find } F^s. \tag{1.16}$$

The main results of the present work consist in estimates and asymptotics for the scattering data (a_{sc}, b_{sc}) and scattering solutions for the Eq. (1.1) and in application of these asymptotics and estimates to the inverse scattering problem (1.16) at high energies. Our main results include, in particular, Theorem 1.1 given below that provides the high-energy asymptotics of the scattering data and the Born approximation of the scattering data at fixed energy.

Consider

$$T\mathbb{S}^{n-1} := \{(\theta, x) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \mid \theta \cdot x = 0\},$$

and for any $m \in \mathbb{N}$ consider the X-ray transform P defined by

$$Pf(\theta, x) := \int_{-\infty}^{+\infty} f(t\theta + x)\mathrm{d}t$$

for any function $f \in C(\mathbb{R}^n, \mathbb{R}^m)$ so that $|f(x)| = O(|x|^{-\tilde{\beta}})$ as $|x| \to +\infty$ for some $\tilde{\beta} > 1$. For $(\sigma, \tilde{\beta}, r, \tilde{\alpha}) \in (0, +\infty)^2 \times (0, \min(1, 2^{-\frac{3}{2}}c)) \times (0, 1]$, let $\rho_0 = \rho_0(\sigma, r, \tilde{\beta}, \tilde{\alpha})$ be defined as the root of the equation

$$1 = \frac{24n^2\tilde{\beta}\sqrt{1 - \frac{\rho_0^2}{c^2}}\left(1 + \sigma + \frac{1}{c}\right)\left(1 + \frac{1}{\frac{\rho_0}{2^{\frac{3}{2}}} - r}\right)}{\tilde{\alpha}\left(\frac{\rho_0}{2^{\frac{3}{2}}} - r\right) r(1 - r)^{\tilde{\alpha}+2}}, \ \rho_0 \in (2^{\frac{3}{2}}r, c). \tag{1.17}$$

Set

$$\begin{aligned} W(v, x) := &\int_{-\infty}^{0} \Big(g\Big(g^{-1}(v) + \int_{-\infty}^{\sigma} F^l(z_-(v, \tau) + x, \dot{z}_-(v, \tau))\mathrm{d}\tau\Big) \\ &- g\left(g^{-1}(v) + \int_{-\infty}^{\sigma} F^l(z_-(v, \tau), \dot{z}_-(v, \tau))\mathrm{d}\tau\right)\Big)\,\mathrm{d}\sigma \\ &+ \int_0^{+\infty} \Big(g\Big(g^{-1}(a(v, x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v, x), \tau) + x, \dot{z}_+(a(v, x), \tau))\mathrm{d}\tau\Big) \\ &- g\left(g^{-1}(a(v, x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v, x), \tau), \dot{z}_+(a(v, x), \tau))\mathrm{d}\tau\right)\Big)\,\mathrm{d}\sigma, \end{aligned} \tag{1.18}$$

for $(v,x) \in \mathcal{D}(S)$, where

$$g(x) := \frac{x}{\sqrt{1+\frac{|x|^2}{c^2}}} \text{ for } x \in \mathbb{R}^n, \quad \text{and} \quad g^{-1}(x') := \frac{x'}{\sqrt{1-\frac{|x'|^2}{c^2}}} \text{ for } x' \in \mathcal{B}(0,c). \tag{1.19}$$

Then we have the following results.

Theorem 1.1. *Let* $(\theta,x) \in T\mathbb{S}^{n-1}$. *Under conditions* (1.4) *and* (1.5) *the following limits are valid*

$$\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}} a_{sc}(\rho\theta,x) = \int_{-\infty}^{+\infty} F(\tau\theta+x,c\theta)\mathrm{d}\tau, \tag{1.20}$$

$$\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}(b_{sc}(\rho\theta,x)-W(\rho\theta,x)) = \int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau\theta+x,c\theta)\mathrm{d}\tau\mathrm{d}\sigma$$
$$-\int_0^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau\theta+x,c\theta)\mathrm{d}\tau\mathrm{d}\sigma + PV^s(\theta,x)\theta. \tag{1.21}$$

In addition,

$$\left|\frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}} a_{sc}(\rho\theta,x) - \int_{-\infty}^{+\infty} F(\tau\theta+x,\rho\theta)\mathrm{d}\tau\right|$$
$$\le \beta^2\sqrt{1-\frac{\rho^2}{c^2}}\frac{700n^4\rho\Big(|x|+1\Big)\big(\frac{1}{c}+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\big)}{\alpha^2(\frac{\rho}{2\sqrt{2}}-r)^2(1-r)^{2\alpha+2}}, \tag{1.22}$$

$$\left|\frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\Big(b_{sc}(\rho\theta,x)-W(\rho\theta,x)\Big) - \int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau\theta+x,\rho\theta)\mathrm{d}\tau\mathrm{d}\sigma\right.$$
$$\left.+\int_0^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau\theta+x,\rho\theta)\mathrm{d}\tau\mathrm{d}\sigma - PV^s(\theta,x)\frac{\rho^2\theta}{c^2}\right|$$
$$\le \beta^2\sqrt{1-\frac{\rho^2}{c^2}}\frac{700n^4\rho^2\Big(|x|+1)(\frac{1}{c}+1)\Big(1+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\Big)}{\alpha^2(\alpha+1)(\frac{\rho}{2\sqrt{2}}-r)^3(1-r)^{2\alpha+1}}, \tag{1.23}$$

for $r \in (0,\min(1,2^{-\frac{3}{2}}c))$ *and for* $\rho \in (\rho_0(|x|,r,\beta,\alpha),c)$, *where* $\beta = \max(\beta_1^l,\beta_2^l,\beta_2^s,\beta_3^s)$.

The vector W defined by (1.18) is known from the scattering data and from F^l. Then from (1.20) and [17, Proposition 1.1] and inversion of the X-ray transform (see [10,24,26,28]) it follows that F can be reconstructed from a_{sc}. From (1.21) one can prove the following statements (see [17, Proposition 1.2] and subsequent comments therein): the potential V^s is uniquely determined up to its radial part by the high-energy asymptotics of b_{sc} and W; the magnetic field B^s can be reconstructed from the high-energy asymptotics of b_{sc} and W when $n \geq 3$, and up to its radial part when $n = 2$.

The estimates (1.22) and (1.23) also give the asymptotics of a_{sc}, b_{sc}, when the parameters α, n, ρ, θ and x are fixed and β decreases to 0. In that regime the leading term of $a_{sc}(\rho\theta, x)$ and $b_{sc}(\rho\theta, x) - W(\rho\theta, x)$ for $(\theta, x) \in T\mathbb{S}^{n-1}$ and for $\rho \in (\rho_0(|x|, r, \beta, \alpha), c)$ are given by

$$\frac{\sqrt{1-\frac{\rho^2}{c^2}}}{\rho}\int_{-\infty}^{+\infty} F(\tau\theta + x, \rho\theta)\mathrm{d}\tau, \tag{1.24}$$

$$\frac{\sqrt{1-\frac{\rho^2}{c^2}}}{\rho^2}\left(\int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau\theta + x, \rho\theta)\mathrm{d}\tau\mathrm{d}\sigma \right.$$
$$\left. - \int_{0}^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau\theta + x, \rho\theta)\mathrm{d}\tau\mathrm{d}\sigma + PV^s(\theta, x)\frac{\rho^2\theta}{c^2}\right), \tag{1.25}$$

respectively. Therefore, Theorem 1.1 gives the Born approximation for the scattering data at fixed energy when the electromagnetic field is sufficiently weak, and one can prove the following statements (see [17, Remark 1.1]): the force F can be reconstructed from the Born approximation (1.24) of a_{sc} at fixed energy; V^s can be reconstructed from W and the Born approximation (1.25) of b_{sc} at fixed energy; B^s can be reconstructed from (1.25) when $n \geq 3$, and up to its radial part when $n = 2$.

Theorem 1.1 is a generalization of [17, formulas (1.7a), (1.7b), (1.8a) and (1.8b)] where inverse scattering for the relativistic multidimensional Newton equation was studied in the short-range case ($F^l \equiv 0$). The formulas [17, (1.7b) and (1.8b)] also provide the approximation of the scattering data $(a_{sc}(v_-, x_-), b_{sc}(v_-, x_-))$ for the short-range case ($F^l \equiv 0$) when the parameters α, n, v_- and β are fixed and $|x_-| \to +\infty$. Such an asymptotic regime is not covered by Theorem 1.1. Therefore, we shall modify in Sect. 3 the definition of the scattering map to study these modified scattering data in the following three asymptotic regimes: at high energies, Born approximation at fixed energy, and when the parameters α, n, v_- and β are fixed and $|x_-| \to +\infty$.

Inverse scattering at high energies for the nonrelativistic multidimensional Newton equation in a short-range potential V was first studied by Novikov [26]. Then inverse scattering at high energies was studied by Jollivet [17] for equation (1.1) in a short-range electromagnetic field and by Jollivet [18] for the nonrelativistic Newton equation in a long-range potential V. We develop the approach of Novikov [26] Jollivet [17,18] to obtain our results.

For inverse scattering at fixed energy for the multidimensional Newton equation, see for example [19] and references therein.

For inverse scattering at high energies in quantum mechanics, see [1,7–9,11,12,15,16,21,25,27,30] and references therein.

Our paper is organized as follows. In Sect. 2, we transform the differential equation (1.1) with initial conditions (1.11) in an integral equation which takes the form $y_- = A(y_-)$. Then we study the nonlinear integral operator A on a suitable space (Lemma 2.1) and we give estimates for the deflection $y_-(t)$ in (1.11) and for the scattering data $a_{sc}(v_-, x_-), b_{sc}(v_-, x_-)$ (Theorem 2.3). Then we prove Theorem 1.1. Note that we work with small angle scattering compared to the dynamics generated by F^l with respect to the "free" solutions $z_-(v_-, t)$: In particular, the angle between the vectors $\dot{x}(t) = \dot{z}_-(v_-, t) + \dot{y}_-(t)$ and $\dot{z}_-(v_-, t)$ goes to zero when the parameters $\beta, \alpha, n, v_-/|v_-|, x_-$ are fixed and $|v_-|$ increases. In Sect. 3, we change the definition of the scattering map so that one can obtain for the modified scattering data $(\tilde{a}_{sc}(v_-, x_-), \tilde{b}_{sc}(v_-, x_-))$ their approximation at high energies, or their Born approximation at fixed energy, or their approximation when the parameters α, n, v_- and β are fixed and $|x_-| \to +\infty$ (Theorem 3.2, Corollary 3.3). Sections 4, 5 and 6 are devoted to proofs of Lemmas 2.1, 2.2 and Theorem 2.3.

2. Scattering Solutions

2.1. Integral Equation

For the rest of the text $\beta_2 = \max(\beta_2^l, \beta_2^s)$, and $H(f(\tau), \dot{f}(\tau))$ is shortened to $H(f)(\tau)$ for any $(f, \tau) \in C^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}$, where H stands for F, F^s or F^l.

Let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$ and $|v_-| \geq \mu^l$. Then the function y_- in (1.11) satisfies the integral equation $y_- = A(y_-)$ where

$$A(f)(t) = \int_{-\infty}^{t} \dot{A}(f)(\sigma)\mathrm{d}\sigma, \tag{2.1}$$

$$\begin{aligned}\dot{A}(f)(t) = g\Big(g^{-1}(v_-) + \int_{-\infty}^{t} F(z_-(v_-, .) + x_- + f)(\tau)\mathrm{d}\tau\Big)\\ - g\Big(g^{-1}(v_-) + \int_{-\infty}^{t} F^l(z_-(v_-, .))(\tau)\mathrm{d}\tau\Big),\end{aligned} \tag{2.2}$$

for $t \in \mathbb{R}$ and for $f \in C^1(\mathbb{R}, \mathbb{R}^n), \sup_{t\in(-\infty,0]}(|f(t)| + |t||\dot{f}(t)|) < \infty$. We have $A(f) \in C^2(\mathbb{R}, \mathbb{R}^n)$ for $f \in C^1(\mathbb{R}, \mathbb{R}^n)$ so that $\sup_{t\in(-\infty,0]}(|f(t)|+|t||\dot{f}(t)|) < \infty$ [see (4.2), (4.6), (4.7)].

For $r \in (0, 1)$ and for $|v_-| \geq \mu^l, |v_-| \geq 2^{\frac{3}{2}} r$, we introduce the following complete metric space M_{r,v_-} endowed with the following norm $\|.\|$

$$M_{r,v_-} = \left\{ f \in C^1(\mathbb{R}, \mathbb{R}^n) \mid \sup_{\mathbb{R}} |\dot{z}_-(v_-, .) + \dot{f}| \leq c,\ \|f\| \leq r \right\}, \tag{2.3}$$

$$\begin{aligned}\|f\| = \max\Bigg(\sup_{t\in(-\infty,0)} \max\left(1, \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)\right) |\dot{f}(t)|,\\ \sup_{(0,+\infty)} |\dot{f}|, \sup_{(-\infty,0)} |f| \Bigg).\end{aligned} \tag{2.4}$$

The space M_{r,v_-} is a convex subset of $C^1(\mathbb{R}, \mathbb{R}^n)$. Then we have the following estimate and contraction estimate for the map A restricted to M_{r,v_-}.

Lemma 2.1. *Let* $(v_-,x_-)\in\big(\mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l)\big)\times\mathbb{R}^n, v_-\cdot x_-=0,$ *and let* $r\in\big(0,\min(\frac{|v_-|}{2^{\frac{3}{2}}},1)\big)$*. When*

$$\frac{2^{\frac{5}{2}}n\max(\beta_1^l,\beta_2)\sqrt{1-\frac{|v_-|^2}{c^2}}}{\alpha\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2\Big(1-r\Big)^{\alpha+1}}\leq 1, \tag{2.5}$$

then the following estimates are valid

$$\|A(f)\|\leq\lambda_1(n,\alpha,\beta_1^l,\beta_2,|x_-|,|v_-|,r)$$
$$:=\frac{4n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\Big(\frac{r\beta_1^l}{c}+2\beta_2\Big(n^{\frac{1}{2}}(|x_-|+r)+1\Big)\Big)\Big(1+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\Big)}{\alpha\left(\frac{|v_-|}{2\sqrt{2}}-r\right)(1-r)^{\alpha+1}}, \tag{2.6}$$

and

$$\|A(f_1)-A(f_2)\|\leq\lambda_2(n,\alpha,\beta_1^l,\beta_2,\beta_3^s,|v_-|,r)\|f_1-f_2\|, \tag{2.7}$$

$$\lambda_2(n,\alpha,\beta_1^l,\beta_2,\beta_3^s,|v_-|,r)$$
$$:=\frac{4n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\Big(\frac{\beta_1^l+\beta_2}{c}+2n^{\frac{1}{2}}(\beta_2+\beta_3^s)\Big)\Big(1+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\Big)}{\alpha\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)(1-r)^{\alpha+2}},$$

for $(f,f_1,f_2)\in M_{r,v_-}^3$.

A proof of Lemma 2.1 is given in Sect. 5.
We also need the following result.

Lemma 2.2. *Let* $(v_-,x_-)\in(\mathcal{B}(0,c)\backslash\mathcal{B}(0,\mu^l))\times\mathbb{R}^n, v_-\cdot x_-=0,$ *and let* $r\in\big(0,\min(\frac{|v_-|}{2^{\frac{3}{2}}},1)\big)$*. When* $y_-\in M_{r,v_-}$ *is a fixed point for the map* A *then* $x:=z_-(v_-,.)+x_-+y_-$ *is a scattering solution for Eq. (1.1) and*

$$x(t)=z_+(a(v_-,x_-),t)+b(v_-,x_-)+y_+(t), \tag{2.8}$$

for $t\geq 0$*, where*

$$a(v_-,x_-):=g\Big(g^{-1}(v_-)+\int_{-\infty}^{+\infty}F(x)(\tau)\mathrm{d}\tau\Big), \tag{2.9}$$

$$b(v_-,x_-):=x_-+A(y_-)(0)-y_+(0)=x_-+y_-(0)-y_+(0), \tag{2.10}$$

$$\begin{aligned}y_+(t)\ =\ -\int_t^{+\infty}\Big(g\Big(g^{-1}(a(v_-,x_-))-\int_\sigma^{+\infty}F(x)(\tau)\mathrm{d}\tau\Big)\\ -g\Big(g^{-1}(a(v_-,x_-))-\int_\sigma^{+\infty}F^l(z_+(a(v_-,x_-),.))(\tau)\mathrm{d}\tau\Big)\Big)\mathrm{d}\sigma,\end{aligned} \tag{2.11}$$

for $t\geq 0$.

Lemma 2.2 is proved in Sect. 4.

2.2. Estimates on the Scattering Solutions

In this section, our main results consist in estimates and asymptotics for the scattering data (a_{sc}, b_{sc}) and scattering solutions for the Eq. (1.1).

Theorem 2.3. *Under the assumptions of Lemma 2.2 and when*

$$\frac{24n^2\sqrt{1-\frac{|v_-|^2}{c^2}}\max(\beta_1^l,\beta_2)\Big(1+\frac{1}{c}\Big)\Big(1+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\Big)}{\alpha\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)(1-r)^{\alpha+1}} \leq 1, \tag{2.12}$$

then the following estimates are valid:

$$|\dot{y}_-(t)| \leq \frac{8n^2\beta_2\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}(|x_-|+2)}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1-r-\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)t\Big)^{\alpha+1}}, \tag{2.13}$$

for $t \leq 0$,

$$\sup_{(0,+\infty)}|\dot{y}_-| \leq \frac{16n^2\beta_2\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}(|x_-|+2)}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)(1-r)^{\alpha+1}}. \tag{2.14}$$

In addition,

$$|a_{sc}(v_-,x_-)| \leq \frac{8n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}}{\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)\left(1+\frac{|x_-|}{\sqrt{2}}-r\right)^{\alpha}}\left(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)(1+\frac{|x_-|}{\sqrt{2}}-r)}\right), \tag{2.15}$$

$$|b_{sc}(v_-,x_-)| \leq \frac{2n^2\beta_2(9|x_-|+15)\sqrt{1-\frac{|v_-|^2}{c^2}}}{\alpha(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2\Big(1-r\Big)^{\alpha}}, \tag{2.16}$$

$$|\dot{y}_+(t)| \leq \frac{2n^2\beta_2(5|x_-|+7)\sqrt{1-\frac{|v_-|^2}{c^2}}}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1-r+t(\frac{|v_-|}{2^{\frac{3}{2}}}-r)\Big)^{\alpha+1}}, \tag{2.17}$$

for $t \geq 0$, *and*

$$\begin{aligned}&|a_{sc}(v_-,x_-)-\sqrt{1-\frac{|v_-|^2}{c^2}}\int_{-\infty}^{+\infty}F(\tau v_-+x_-,v_-)\mathrm{d}\tau|\\ &\quad\leq\frac{700n^4\beta^2\Big(1-\frac{|v_-|^2}{c^2}\Big)(|x_-|+1)\Big(\frac{1}{c}+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\Big)}{\alpha^2\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2(1-r)^{2\alpha+2}},\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\Big|b_{sc}(v_-,x_-)-W(v_-,x_-)-\sqrt{1-\frac{|v_-|^2}{c^2}}\Big(\int_{-\infty}^{0}\int_{-\infty}^{\sigma}F^s(\tau v_-+x_-,v_-)\mathrm{d}\tau\mathrm{d}\sigma\\ &\quad-\int_0^{+\infty}\int_\sigma^{+\infty}F^s(\tau v_-+x_-,v_-)\mathrm{d}\tau\mathrm{d}\sigma+\int_{-\infty}^{+\infty}V^s(\sigma v_-+x_-)\mathrm{d}\sigma\frac{v_-}{c^2}\Big)\Big|\end{aligned}$$

$$\leq \frac{700n^4\beta^2\Big(1-\frac{|v_-|^2}{c^2}\Big)\Big(|x_-|+1\Big)(\frac{1}{c}+1)\Big(1+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\Big)}{\alpha^2(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^3(1-r)^{2\alpha+1}}, \tag{2.19}$$

where $\beta = \max(\beta_1^l, \beta_2, \beta_3^s)$.

Theorem 2.3 is proved in Sect. 6.

Proof of Theorem 1.1. Let $(\theta, x) \in T\mathbb{S}^{n-1}$ and let $(r, \rho) \in (0, \min(1, 2^{-\frac{3}{2}}c)) \times (0, +\infty), \rho > \rho_0(|x|, r, \beta, \alpha)$, where ρ_0 is defined in (1.17). Set $(v_-, x_-) = (\rho\theta, x)$. Then note that

$$\max\Big(\lambda_0, \frac{\lambda_1}{r}, \lambda_2, \lambda_3\Big) \leq \frac{24n^2\sqrt{1-\frac{|v_-|^2}{c^2}}\beta(1+|x_-|+\frac{1}{c})\Big(1+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\Big)}{\alpha\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)r(1-r)^{\alpha+2}} < 1, \tag{2.20}$$

where λ_1 and λ_2 are defined in (2.6) and (2.7), respectively, and where $\lambda_0 := \frac{2^6n^2\max(\beta_1^l,\beta_2)\sqrt{1-\frac{|v_-|^2}{c^2}}}{\alpha|v_-|^2}$ and λ_3 is the left-hand side of (2.12) (we also used (1.17)). From estimate (2.20) and Lemma 2.1 we obtain: $|v_-| \geq \mu^l$ and A is a contraction in M_{r,v_-}, and Theorem 2.3 holds for the unique fixed point $y_- \in M_{r,v_-}$ of A. The estimate (2.18) and (2.19) hold and they provide the estimates (1.22) and (1.23), which proves Theorem 1.1. □

2.3. Motivations for Changing the Definition of the Scattering Map

For a solution x at a nonzero energy for Eq. (1.1) we say that it is a scattering solution when there exists $\varepsilon > 0$ so that $1+|x(t)| \geq \varepsilon(1+|t|)$ for $t \in \mathbb{R}$ (see [4]). In the "Introduction" and in the previous subsections we choose to parametrize the scattering solutions of equation (1.1) by the solutions $z_\pm(v, .)$ of the Eq. (1.8) [see the asymptotic behaviors (1.11) and (1.12)], and then to formulate the inverse scattering problem (1.16) using this parametrization. We obtain the estimates (1.22) and (1.23) that provide the high-energy asymptotics and the Born approximation at fixed energy of the scattering data. However, these estimates do not provide the asymptotics of the scattering data (a_{sc}, b_{sc}) when the parameters α, n, ρ and β are fixed and $|x| \to +\infty$. Motivated by this disadvantage, we introduce a new family of "free" solutions $z_\pm(v, x, .)$ of Eq. (1.8) that will be used for parametrizing some unbounded solutions of the Newton equation (1.1) at nonzero energy and for measuring their deviation. Those free solutions $z_\pm(v, x, .)$ have the properties that $\lim_{t\to\pm\infty} \dot z_\pm(v, x, t) = v$ and $z_\pm(v, x, 0) = x$. In addition, for $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n, v_- \cdot x_- = 0$, so that the energy $E = c^2(1-|v_-|^2/c^2)^{-1/2}$ is sufficiently "high", then there exists a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ of Eq. (1.1) such that

$$x(t) = z_-(v_-, x_-, t) + y_-(t), \tag{2.21}$$

where $|\dot y_-(t)| + |y_-(t)| \to 0$, as $t \to -\infty$. Such a solution also satisfies:

$$x(t) = z_+(\tilde a, \tilde b, t) + y_+(t), \tag{2.22}$$

where $|\dot{y}_+(t)| + |y_+(t)| \to 0$, as $t \to +\infty$ for a unique $(\tilde{a}, \tilde{b}) \in \mathcal{B}(0, c) \times \mathbb{R}^n$. The map $\tilde{S}$ defined by $\tilde{S}(v_-, x_-) = (\tilde{a}, \tilde{b})$ at sufficiently high energies is our modified scattering map.

In the next section, we provide more details on its definition, and we study its asymptotics in the three regimes: at high energies, its Born approximation at fixed energy, and at fixed energy when the "impact parameter" $|x_-| \to +\infty$.

3. A Modified Scattering Map

First let us introduce a new family of "free" solutions $z_\pm(v, x, .)$ of Eq. (1.8) that will be used for parametrizing some unbounded solutions of the Newton equation (1.1) at nonzero energy and for measuring their deviation. We set

$$\mu^l_\sigma := \mu\Big(2^8\alpha^{-1}n^2 \max(\beta^l_1, \beta^l_2)\Big(\frac{1}{2} + \frac{\sigma}{\sqrt{2}}\Big)^{-\alpha}\Big), \tag{3.1}$$

for $\sigma \ge 0$, where the function μ is defined by (1.7).

Let $(v, x) \in \mathcal{B}(0, c) \times \mathbb{R}^n$ so that $v \cdot x = 0$ and $|v| \ge \mu^l_{|x|}$. Then the following is valid (see [20]): for any $(w, q) \in \mathcal{B}(v, \frac{|v|}{2^{\frac{5}{2}}}) \times \overline{\mathcal{B}(0, \frac{1}{2})}$, $|w| = |v|$, there exists a unique solution $z_\pm(w, x + q, .)$ of the Eq. (1.8) so that

$$\dot{z}_\pm(w, x + q, t) - w = o(1) \text{ as } t \to \pm\infty,\ z_\pm(w, x + q, 0) = x + q, \tag{3.2}$$

and

$$\sup_{\mathbb{R}} |\dot{z}_\pm(w, x + q, .) - w| \le \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta^l_1 \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha|v|\Big(1 + \frac{|x|}{\sqrt{2}} - |q|\Big)^{\alpha}}. \tag{3.3}$$

Under conditions (1.4) and (1.5), the following is valid: for any $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$ so that $|v_-| \ge \mu^l_{|x_-|}$, then the Eq. (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = z_-(v_-, x_-, t) + y_-(t), \tag{3.4}$$

where $\dot{y}_-(t) \to 0,\ y_-(t) \to 0,$ as $t \to -\infty$.

In addition, the function y_- in (3.4) satisfies the integral equation $y_- = \mathcal{A}(y_-)$ where

$$\mathcal{A}(f)(t) = \int_{-\infty}^t \dot{\mathcal{A}}(f)(\sigma)\mathrm{d}\sigma, \tag{3.5}$$

$$\dot{\mathcal{A}}(f)(t) = g\Big(g^{-1}(v_-) + \int_{-\infty}^t F(z_-(v_-, x_-, .) + f)(\tau)\mathrm{d}\tau\Big) - g\Big(g^{-1}(v_-) + \int_{-\infty}^t F^l(z_-(v_-, x_-, .))(\tau)\mathrm{d}\tau\Big), \tag{3.6}$$

for $t \in \mathbb{R}$ and for $f \in C^1(\mathbb{R}, \mathbb{R}^n)$, $\sup_{t\in(-\infty,0]}(|f(t)| + |t||\dot{f}(t)|) < \infty$. We remind that $H(f(\tau), \dot{f}(\tau))$ is shortened to $H(f)(\tau)$ for any $(f, \tau) \in C^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}$ above and in the rest of the text, where H stands for F, F^s or F^l.

For $r \in \left(0, \min\left(1, \frac{|v_-|}{2^{\frac{3}{2}}}\right)\right)$, we introduce the following metric space M_{r,v_-,x_-} endowed with the following norm $\|.\|_*$

$$M_{r,v_-,x_-} = \left\{ f \in C^1(\mathbb{R}, \mathbb{R}^n) \mid \sup_{\mathbb{R}} |\dot{z}_-(v_-, x_-, .) + \dot{f}| \leq c, \ \|f\|_* \leq r \right\}, \quad (3.7)$$

$$\|f\|_* = \max\left(\sup_{t \in (-\infty,0)} \max\left(1, \left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)|t|\right)\right) |\dot{f}(t)|, \right.$$
$$\left. \sup_{(0,+\infty)} |\dot{f}|, \sup_{(-\infty,0)} |f| \right). \quad (3.8)$$

The space M_{r,v_-,x_-} is a convex subset of $C^1(\mathbb{R}, \mathbb{R}^n)$. We study the map $\mathcal{A}$ defined by (3.5) and (3.6) on the metric space M_{r,v_-,x_-}. Set

$$\tilde{k}(v_-, x_-, f) := g\Big(g^{-1}(v_-) + \int_{-\infty}^{+\infty} F(z_-(v_-, x_-, .) + f)(t)\mathrm{d}t\Big), \quad (3.9)$$

for $f \in M_{r,v_-,x_-}$. For the rest of the section we also set $\beta_2 = \max(\beta_2^l, \beta_2^s)$.

The following Lemma 3.1 is the analog of Lemma 2.1.

Lemma 3.1. *Let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n, v_- \cdot x_- = 0, |v_-| \geq \mu^l_{|x_-|}$, and let $r \in \left(0, \min(1, \frac{|v_-|}{2^{\frac{3}{2}}})\right)$. When*

$$\frac{2^{\frac{3}{2}} n \max(\beta_1^l, \beta_2) \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(1 + \frac{1}{1 + \frac{|x_-|}{\sqrt{2}} - r}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^\alpha} \leq 1, \quad (3.10)$$

then the following estimates are valid

$$\|\mathcal{A}(f)\|_* \leq \tilde{\lambda}_1(n, \alpha, \beta_1^l, \beta_2, |x_-|, |v_-|, r), \quad (3.11)$$

$$\tilde{\lambda}_1 := \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{\beta_1^l r}{c} + 4\beta_2(n^{\frac{1}{2}} r + 1)\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \times \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r} + \frac{1}{1 - r + \frac{|x_-|}{\sqrt{2}}}\right),$$

and

$$\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_* \leq \tilde{\lambda}_2(n, \alpha, \beta_1^l, \beta_2, \beta_3^s, |x_-|, |v_-|, r)\|f_1 - f_2\|_*, \quad (3.12)$$

$$\tilde{\lambda}_2 := \frac{4n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{\beta_1^l}{c} + 2\beta_2 n^{\frac{1}{2}} + \frac{\frac{\beta_2}{c} + 2\beta_3^s n^{\frac{1}{2}}}{1 - r + \frac{|x_-|}{\sqrt{2}}}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \times \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r} + \frac{1}{1 - r + \frac{|x_-|}{\sqrt{2}}}\right),$$

for $(f, f_1, f_2) \in M^3_{r,v_-,x_-}$. *In addition, we have*

$$|\tilde{k}(v_-, x_-, f) - v_-| \leq \frac{8n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}}{\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\left(1+\frac{|x_-|}{\sqrt{2}}-r\right)^{\alpha}}\left(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)\left(1+\frac{|x_-|}{\sqrt{2}}-r\right)}\right), \tag{3.13}$$

for $f \in M_{r,v_-,x_-}$.

Let $r \in \left(0, \min(\frac{1}{2}, 2^{-\frac{3}{2}}c)\right)$ and let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n, v_- \cdot x_- = 0$ so that $|v_-| > \tilde{\rho}_0(|x_-|, r, \beta, \alpha)$, where $\beta = \max(\beta_1^l, \beta_2, \beta_3^s)$ and $\tilde{\rho}_0$ is defined as the root of the following equation

$$1 = \frac{72n^2(c^{-1}+1)\beta\sqrt{1-\frac{\tilde{\rho}_0^2}{c^2}}\left(1+\frac{1}{\frac{\tilde{\rho}_0}{2^{\frac{3}{2}}}-r}\right)}{\alpha r\left(\frac{\tilde{\rho}_0}{2^{\frac{3}{2}}}-r\right)\left(\frac{1}{2}+\frac{|x_-|}{\sqrt{2}}\right)^{\alpha}}, \quad \tilde{\rho}_0 \in (2^{\frac{3}{2}}r, c). \tag{3.14}$$

From (3.14) and Lemma 3.1 we obtain: $|v_-| \geq \mu^l_{|x_-|}$ [see (3.1)], $\mathcal{A}$ is a contraction in M_{r,v_-,x_-}, and $\mathcal{A}$ has a unique fixed point $y_- \in M_{r,v_-,x_-}$ of $\mathcal{A}$. The function $x := z_-(v_-, x_-, .) + y_-$ is a scattering solution of (1.1) (in the sense given in Sect. 2.3). We set

$$\tilde{a}(v_-, x_-) := \tilde{k}(v_-, x_-, y_-). \tag{3.15}$$

By conservation of energy $|\tilde{a}(v_-, x_-)| = \lim_{t\to\infty} |\dot{x}(t)| = |v_-|$. From (3.13) and (3.14) ($|v_-| > \tilde{\rho}_0(|x_-|, r, \beta, \alpha)$) it follows that $|\tilde{a}(v_-, x_-) - v_-| \leq 2^{-\frac{5}{2}}|v_-|$, and we can consider the free solution $z_+(\tilde{a}(v_-, x_-), x_- + q, .)$ for any $q \in \overline{\mathcal{B}(0, \frac{1}{2})}$. Furthermore, the following decomposition holds

$$x(t) = z_+(\tilde{a}(v_-, x_-), x_- + q, t) + \mathcal{G}_{v_-,x_-}(q) - q + h(v_-, x_-, q, t), \tag{3.16}$$

where

$$\mathcal{G}_{v_-,x_-}(q) := y_-(0) - h(v_-, x_-, q, 0), \tag{3.17}$$

and

$$\begin{aligned} h(v_-, x_-, q, t) := -\int_t^{+\infty} &\Big(g\Big(g^{-1}(\tilde{a}(v_-, x_-)) - \int_\sigma^{+\infty} F(x)(\tau))\mathrm{d}\tau\Big) \\ &- g\Big(g^{-1}(\tilde{a}(v_-, x_-)) - \int_\sigma^{+\infty} F^l(z_+(\tilde{a}(v_-, x_-), x_- + q, .))(\tau)\mathrm{d}\tau\Big)\Big)\mathrm{d}\sigma, \end{aligned} \tag{3.18}$$

for $t \geq 0$ and for $q \in \overline{\mathcal{B}(0, \frac{1}{2})}$.

Then under the assumption $|v_-| > \tilde{\rho}_0(|x_-|, r, \beta, \alpha)$ it can be proved that $\mathcal{G}_{v_-,x_-}$ is a $\frac{1}{7}$-contraction map from $\overline{\mathcal{B}(0, \frac{1}{2})}$ into $\overline{\mathcal{B}(0, \frac{1}{2})}$, and we denote by $\tilde{b}_{sc}(v_-, x_-)$ its unique fixed point in $\overline{\mathcal{B}(0, \frac{1}{2})}$. We set $\tilde{b}(v_-, x_-) :=$

$x_- + \tilde{b}_{sc}(v_-, x_-)$ and $\tilde{a}_{sc}(v_-, x_-) := \tilde{a}(v_-, x_-) - v_-$. The decomposition (3.16) becomes

$$z_-(v_-, x_-, t) + y_-(t) = z_+(\tilde{a}(v_-, x_-), \tilde{b}(v_-, x_-), t) + y_+(t), \tag{3.19}$$
$$y_+(t) = h(v_-, x_-, \tilde{b}_{sc}(v_-, x_-), t), \tag{3.20}$$

for $t \geq 0$. The map $(\tilde{a}_{sc}, \tilde{b}_{sc})$ defined on the set

$$\tilde{\mathcal{M}} = \Big\{(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n \mid v_- \cdot x_- = 0,\ |v_-| > \tilde{\rho}_0(|x_-|, r, \beta, \alpha) \text{ for some } r \in \Big(0, \min\Big(\frac{1}{2}, \frac{c}{2^{\frac{3}{2}}}\Big)\Big)\Big\}$$

is our modified scattering map. The inverse scattering problem for equation (1.1) can now be formulated as follows

$$\text{Given } (\tilde{a}_{sc}, \tilde{b}_{sc}) \text{ and } F^l,\ \text{ find } F^s. \tag{3.21}$$

Then set

$$\tilde{W}(v_-, x_-) = g\Big(g^{-1}(v_-) + \int_{-\infty}^{+\infty} F^l(z_-(v_-, x_-, .))(\tau)\mathrm{d}\tau\Big) - v_-. \tag{3.22}$$

Note that $\tilde{W}$ is known from F^l. We obtain the following analog of Theorem 2.3.

Theorem 3.2. *Let $r \in \Big(0, \min(\frac{1}{2}, 2^{-\frac{3}{2}}c)\Big)$ and let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n, v_- \cdot x_- = 0$ so that $|v_-| \geq \tilde{\rho}_0(|x_-|, r, \beta, \alpha)$ where $\beta = \max(\beta_1^l, \beta_2, \beta_3^s)$ and $\tilde{\rho}_0$ is defined by (3.14). Then we have:*

$$|\dot{y}_-(t)| \leq \frac{16n^2\Big(1 - \frac{|v_-|^2}{c^2}\Big)^{\frac{1}{2}}\beta_2}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 - r + \frac{|x_-|}{\sqrt{2}} - \Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)t\Big)^{\alpha+1}}, \tag{3.23}$$

for $t \leq 0$;

$$\sup_{(0,+\infty)} |\dot{y}_-| \leq \frac{32n^2\Big(1 - \frac{|v_-|^2}{c^2}\Big)^{\frac{1}{2}}\beta_2}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 - r + \frac{|x_-|}{\sqrt{2}}\Big)^{\alpha+1}}; \tag{3.24}$$

and

$$|\tilde{a}_{sc}(v_-, x_-)| \leq \frac{24n^{\frac{3}{2}}\Big(1 - \frac{|v_-|^2}{c^2}\Big)^{\frac{1}{2}}\max(\beta_1^l, \beta_2)}{\alpha\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\Big)^{\alpha}}, \tag{3.25}$$

$$|\tilde{b}_{sc}(v_-, x_-)| \leq \frac{28n^2\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\beta_2}{\alpha(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}}-r\right)^2\left(\frac{1}{2}+\frac{|x_-|}{\sqrt{2}}\right)^{\alpha}}, \tag{3.26}$$

$$|\dot{y}_+(t)| \leq \frac{16n^2\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\max(\beta_1^l, \beta_2)}{(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\left(\frac{1}{2}+\frac{|x_-|}{\sqrt{2}}+t\frac{|v_-|}{2\sqrt{2}}\right)^{\alpha+1}}, \tag{3.27}$$

for $t \geq 0$. *In addition,*

$$\begin{aligned}&\left|\tilde{a}_{sc}(v_-, x_-) - \tilde{W}(v_-, x_-) - \sqrt{1-\frac{|v_-|^2}{c^2}}\int_{-\infty}^{+\infty} F^s(\tau v_- + x_-, v_-)\mathrm{d}\tau\right| \\ &\quad\leq \frac{700n^4\beta^2\left(1-\frac{|v_-|^2}{c^2}\right)\left(\frac{1}{c}+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\right)}{\alpha(\alpha+1)\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)^2\left(\frac{1}{2}+\frac{|x_-|}{\sqrt{2}}\right)^{2\alpha+1}},\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\Bigg|\tilde{b}_{sc}(v_-, x_-) - \sqrt{1-\frac{|v_-|^2}{c^2}}\Big(\int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau v_- + x_-, v_-)\mathrm{d}\tau\mathrm{d}\sigma \\ &\quad - \int_0^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau v_- + x_-, v_-)\mathrm{d}\tau\mathrm{d}\sigma + \int_{-\infty}^{+\infty} V^s(\tau v_- + x_-)\mathrm{d}\tau\frac{v_-}{c^2}\Big)\Bigg| \\ &\quad\leq \frac{1000n^4\beta^2\left(1-\frac{|v_-|^2}{c^2}\right)(\frac{1}{c}+1)\left(1+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\right)}{\alpha^2(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}}-r\right)^3\left(\frac{1}{2}+\frac{|x_-|}{\sqrt{2}}\right)^{2\alpha}}.\end{aligned} \tag{3.29}$$

Estimates (3.28) and (3.29) provide the high-energy asymptotics of the modified scattering data. The analog of formulas (1.20) and (1.21) are the first two formulas of the following Corollary.

Corollary 3.3. *Let* $(\theta, x) \in T\mathbb{S}^{n-1}$. *We have*

$$\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{a}_{sc}(\rho\theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta)\mathrm{d}\tau, \tag{3.30}$$

$$\begin{aligned}\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{b}_{sc}(\rho\theta, x) &= \int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau\theta + x, c\theta)\mathrm{d}\tau\mathrm{d}\sigma \\ &\quad - \int_0^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau\theta + x, c\theta)\mathrm{d}\tau\mathrm{d}\sigma + PV^s(\theta, x)\theta.\end{aligned} \tag{3.31}$$

In addition, when $x \neq 0$ *we have*

$$\rho\Big(\tilde{a}_{sc}(\rho\theta, \eta x) - \tilde{W}(\rho\theta, \eta x)\Big) = \sqrt{1-\frac{\rho^2}{c^2}}\int_{-\infty}^{+\infty} F^s(\tau\theta + \eta x, \rho\theta)\mathrm{d}\tau + O(\eta^{-2\alpha-1}), \tag{3.32}$$

$$\rho^2\tilde{b}_{sc}(\rho\theta,\eta x) = \sqrt{1-\frac{\rho^2}{c^2}}\Big(\int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau\theta+\eta x,\rho\theta)\mathrm{d}\tau\mathrm{d}\sigma - \int_0^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau\theta+\eta x,\rho\theta)\mathrm{d}\tau\mathrm{d}\sigma + \int_{-\infty}^{+\infty} V^s(\tau\theta+\eta x)\mathrm{d}\tau\frac{\rho^2\theta}{c^2}\Big) + O(\eta^{-2\alpha}), \tag{3.33}$$

for $\rho > \tilde{\rho}_0(\eta|x|, r, \beta, \alpha)$, *as* $\eta \to +\infty$.

Then F can be reconstructed from the high-energy asymptotics of $\tilde{a}_{sc}$, and from (3.31) one can prove the following statements: The potential V^s is uniquely determined up to its radial part by $\tilde{b}_{sc}$; the magnetic field B^s can be reconstructed from $\tilde{b}_{sc}$ when $n \geq 3$, and up to its radial part when $n = 2$.

Formulas (3.32) and (3.33) provide the first leading term in the asymptotics of the modified scattering data $(\tilde{a}_{sc}(\rho\theta, x), \tilde{b}_{sc}(\rho\theta, x))$ when the parameters α, n, s, θ and β are fixed and $|x|$ increases to $+\infty$.

The estimates (3.28) and (3.29) also gives the Born approximation for the modified scattering data at fixed energy when the electromagnetic field is sufficiently weak. The Born approximation at fixed energy of $\tilde{a}_{sc}(\rho\theta, x)$ and $\tilde{b}_{sc}(\rho\theta, x)$ for $(\theta, x) \in T\mathbb{S}^{n-1}$ and for $\tilde{\rho} \in (\tilde{\rho}_0(|x|, r, \beta, \alpha), c)$ are given by (1.24) and (1.25), respectively, and we have: The force F can be reconstructed from the Born approximation (1.24) of $\tilde{a}_{sc}$ at fixed energy; V^s can be reconstructed from the Born approximation (1.25) of $\tilde{b}_{sc}$ at fixed energy; B^s can be reconstructed from (1.25) when $n \geq 3$, and up to its radial part when $n = 2$.

We refer the reader to the preprint [20] for a proof of Lemma 3.1 and Theorem 3.2 and Corollary 3.3.

4. Preliminary Estimates and Proof of Lemma 2.2

For the rest of the text we use the following properties of the function $g : \mathbb{R}^n \to \mathcal{B}(0, c)$ defined by (1.19):

$$|\nabla g_i(x)|^2 \leq \frac{1}{1+\frac{|x|^2}{c^2}}, \tag{4.1}$$

$$|g(x)-g(y)| \leq \sqrt{n}|x-y| \sup_{\varepsilon\in(0,1)} \frac{1}{\sqrt{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}}, \tag{4.2}$$

$$|\nabla g_i(x)-\nabla g_i(y)| \leq \frac{3\sqrt{n}}{c}|x-y| \sup_{\varepsilon\in(0,1)} \frac{1}{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}, \tag{4.3}$$

$$\nabla g_i(x) = \frac{1}{(1+\frac{|x|^2}{c^2})^{\frac{1}{2}}}e_i - \frac{x_i x}{c^2(1+\frac{|x|^2}{c^2})^{\frac{3}{2}}}, \tag{4.4}$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, and for $i = 1 \ldots n$ where $g = (g_1, \ldots, g_n)$, and where the ith component of the vector e_i is equal to 1 and all others components of e_i are equal to zero.

We also used the following properties of the forces (F^l, F^s):

$$|F^l(x,v)| \leq 2n\beta_1^l(1+|x|)^{-\alpha-1}, \tag{4.5}$$

$$|F^s(x,v)| \leq 2n\beta_2^s(1+|x|)^{-\alpha-2}, \tag{4.6}$$

$$\begin{aligned}|F^l(x,v)-F^l(x',v')| \leq{}& \frac{n\beta_1^l}{c}|v-v'|\sup_{\varepsilon\in(0,1)}(1+|(1-\varepsilon)x+\varepsilon x'|)^{-\alpha-1}\\ &+2n^{\frac{3}{2}}\beta_2^l|x-x'|\sup_{\varepsilon\in(0,1)}(1+|(1-\varepsilon)x+\varepsilon x'|)^{-\alpha-2},\end{aligned} \tag{4.7}$$

$$\begin{aligned}|F^s(x,v)-F^s(x',v')| \leq{}& \frac{n\beta_2^s}{c}|v-v'|\sup_{\varepsilon\in(0,1)}(1+|(1-\varepsilon)x+\varepsilon x'|)^{-\alpha-2}\\ &+2n^{\frac{3}{2}}\beta_3^s|x-x'|\sup_{\varepsilon\in(0,1)}(1+|(1-\varepsilon)x+\varepsilon x'|)^{-\alpha-3},\end{aligned} \tag{4.8}$$

for $(x,x',v,v')\in(\mathbb{R}^n)^4, \max(|v|,|v'|)\leq c$.

Proof. We prove the existence and uniqueness of the solution z_+ (similarly one can prove the existence and uniqueness of z_-). Set $C = 2^{\frac{7}{2}}n^{\frac{3}{2}}\beta_1^l\alpha^{-1}|v|^{-1} \times \sqrt{1-\frac{|v|^2}{c^2}}$. Let $\mathcal{V}$ be the complete metric space defined by

$$\mathcal{V} := \left\{ f\in C^1(\mathbb{R},\mathbb{R}^n) \mid f(0)=0 \text{ and } \sup_{\mathbb{R}}|\dot f|\leq C,\ \sup_{\mathbb{R}}|v+\dot f|\leq c\right\},$$

endowed with the norm $\|f\|_{\mathcal{V}} := \sup_{\mathbb{R}}|\dot f|$. Note that $\mathcal{V}$ is a convex subset of $C^1(\mathbb{R},\mathbb{R}^n)$. We consider the integral equations

$$G(f)(t) := \int_0^t \dot G(f)(s)\mathrm{d}s, \tag{4.9}$$

$$\dot G(f)(t) := g\Big(g^{-1}(v) - \int_t^{+\infty} F^l(.v+f)(\tau)\mathrm{d}\tau\Big) - v, \tag{4.10}$$

for $f\in\mathcal{V}$ and for $t\in\mathbb{R}$. Then we have [see also (1.19)]

$$G(f)(0)=0, \quad |v+\dot G(f)(t)|<c \quad \text{for } t\in\mathbb{R}. \tag{4.11}$$

We use the following estimate (4.12)

$$|\tau v + f(\tau)| \geq \Big(|v|-C\Big)|\tau| \geq \frac{|v|}{\sqrt{2}}|\tau|, \tag{4.12}$$

for $\tau\in\mathbb{R}$ and $f\in\mathcal{V}$ (we used the estimate $|f(t)|\leq C|t|$ for $t\in\mathbb{R}$ and for $f\in\mathcal{V}$, and we used the assumption $|v|\geq\mu^l$). Using (4.5) and (4.12) we obtain that

$$\int_t^{+\infty}|F^l(.v+f)(\tau)|\mathrm{d}\tau \leq 2n\beta_1^l\int_t^{+\infty}\Big(1+\frac{|v|}{\sqrt{2}}|\tau|\Big)^{-\alpha-1}\mathrm{d}\tau \leq \frac{2^{\frac{5}{2}}n\beta_1^l}{\alpha|v|}, \tag{4.13}$$

for $t\in\mathbb{R}$ and for $f\in\mathcal{V}$. We used the integral value $\int_{-\infty}^{+\infty}(a+b|\tau|)^{-\alpha-1}\mathrm{d}\tau=\frac{2}{b\alpha a^{\alpha}}$ for $a>0$ and $b>0$. Then we obtain

$$\begin{aligned}&\Big|g^{-1}(v)-\int_t^{+\infty}(\varepsilon F^l(.v+f_1)(\tau)+\eta F^l(.v+f_2)(\tau))\mathrm{d}\tau\Big|\\&\quad\geq|g^{-1}(v)|-\frac{2^{\frac{5}{2}}n\beta_1^l}{\alpha|v|}\geq\frac{|v|}{2\sqrt{1-\frac{|v|^2}{c^2}}},\end{aligned}\tag{4.14}$$

for $(f_1,f_2)\in\mathcal{V}^2$, and for $(\varepsilon,\eta,t)\in(0,1)^2\times\mathbb{R},\varepsilon+\eta\leq 1$ (we used again the assumption $|v|\geq\mu^l$). Therefore combining (4.10), (4.13), (4.14) and (4.2) we obtain

$$\begin{aligned}|\dot G(f)(t)|&\leq\frac{2^{\frac{5}{2}}n^{\frac{3}{2}}\beta_1^l}{\alpha|v|}\sup_{\varepsilon\in(0,1)}\frac{1}{\sqrt{1+\frac{\left|g^{-1}(v)-\varepsilon\int_t^{+\infty}F^l(.v+f)(\tau)\mathrm{d}\tau\right|^2}{c^2}}}\\&\leq\frac{2^{\frac{7}{2}}n^{\frac{3}{2}}\beta_1^l\sqrt{1-\frac{|v|^2}{c^2}}}{\alpha|v|},\end{aligned}\tag{4.15}$$

for $t\in\mathbb{R}$ and $f\in\mathcal{V}$.

Now let $(f_1,f_2)\in\mathcal{V}^2$. Then using (4.2), (4.10) and (4.14) we have

$$|\dot G(f_1)(t)-\dot G(f_2)(t)|\leq 2n^{\frac{1}{2}}\sqrt{1-\frac{|v|^2}{c^2}}\int_t^{+\infty}\Big|F^l(.v+f_1)-F^l(.v+f_2)\Big|(\tau)\mathrm{d}\tau\tag{4.16}$$

for $t\in\mathbb{R}$. Using (4.7) and (4.12) we have

$$\begin{aligned}&|F^l(.v+f_1)-F^l(.v+f_2)|(\tau)\leq\frac{n\beta_1^l}{c}\Big(1+\frac{|v|}{2^{\frac{1}{2}}}|\tau|\Big)^{-\alpha-1}\sup_{\mathbb{R}}|\dot f_1-\dot f_2|\\&\quad+2n^{\frac{3}{2}}\beta_2^l\Big(1+\frac{|v|}{2^{\frac{1}{2}}}|\tau|\Big)^{-\alpha-2}|\tau|\sup_{s\in\mathbb{R}\setminus\{0\}}\frac{|(f_1-f_2)(s)|}{|s|}\\&\leq\Big(\frac{n\beta_1^l}{c}+\frac{2^{\frac{3}{2}}n^{\frac{3}{2}}}{|v|}\beta_2^l\Big)\Big(1+\frac{|v|}{2^{\frac{1}{2}}}|\tau|\Big)^{-\alpha-1}\|f_1-f_2\|_{\mathcal{V}}\end{aligned}\tag{4.17}$$

for $\tau\in\mathbb{R}$. Therefore, we obtain

$$|\dot G(f_1)(t)-\dot G(f_2)(t)|\leq\frac{n^{\frac{3}{2}}2^{\frac{5}{2}}\sqrt{1-\frac{|v|^2}{c^2}}\Big(\frac{\beta_1^l}{c}+\frac{2^{\frac{3}{2}}n^{\frac{1}{2}}}{|v|}\beta_2^l\Big)}{\alpha|v|}\|f_1-f_2\|_{\mathcal{V}},\tag{4.18}$$

for $t\in\mathbb{R}$.

From (4.11), (4.15), (4.18) and $|v|\geq\mu^l$ it follows that the operator G is a $\frac{1}{2}$-contraction map from $\mathcal{V}$ to $\mathcal{V}$. Set $z_+(v,t)=tv+f_v(t)$ for $t\in\mathbb{R}$, where f_v denotes the unique fixed point of G in $\mathcal{V}$. Then $z_+(v,.)$ satisfies (1.8), (1.9), (1.10). □

Before proving Lemma 2.2 we recall the following standard result.

Lemma 4.1. *Let $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ satisfy equation (1.1) and let $z \in C^2(\mathbb{R}, \mathbb{R}^n)$ satisfy equation (1.8). Assume that there exists $v \in \mathcal{B}(0, c), v \neq 0$, so that $\dot{x}(t) \to v$ and $\dot{z}(t) \to v$ as $t \to +\infty$. Then*

$$\sup_{(0,+\infty)} |x - z| < \infty \quad \text{and} \quad \sup_{t\in(0,+\infty)} (1 + t)|\dot{x} - \dot{z}|(t) < \infty. \tag{4.19}$$

Proof of Lemma 2.2. We need the following preliminary estimate (4.21). From the formula $g(\tau) = g(0) + \int_0^\tau \dot{g}(s)ds$ for $g \in C^1(\mathbb{R}, \mathbb{R}^n)$ it follows that,

$$|(f_1 - f_2)(\tau)| \leq \sup_{(-\infty,0)} |f_1 - f_2| + |\tau| \sup_{(0,+\infty)} |\dot{f}_1 - \dot{f}_2|, \tag{4.20}$$

for $(\tau, f_1, f_2) \in \mathbb{R} \times M^2_{r,v_-}$. Hence

$$\begin{aligned} |z_-(v_-, \tau) + x + f(\tau)| &\geq |x + \tau v_-| - |z_-(v_-, \tau) - \tau v_-| - |f(\tau)| \\ &\geq \frac{|x|}{\sqrt{2}} - r + |\tau|\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big), \end{aligned} \tag{4.21}$$

for $(f, \tau) \in M_{r,v_-} \times \mathbb{R}$ and for any $x \in \mathbb{R}^n$ so that $x \cdot v_- = 0$. We used the inequality $|x + \tau v_-| \geq \frac{|x|}{\sqrt{2}} + |\tau|\frac{|v_-|}{\sqrt{2}}$ $(x \cdot v_- = 0)$ and (4.20) (for $(f_1, f_2) = (f, 0)$ and $\|f\| \leq r$), and we used (1.10) and the condition $|v_-| \geq \mu^l$.

Hence the integral $\int_{-\infty}^{+\infty} F(z_-(v_-, .) + x_- + f)(\tau)\mathrm{d}\tau$ is absolutely convergent for any $f \in M_{r,v_-}$. And when $y_- \in M_{r,v_-}$ is a fixed point for A then $z_-(v_-, .) + x_- + y_-$ satisfies Eq. (1.1) [see (2.1), (2.2)] and $\dot{z}_-(v_-, t) + \dot{y}_-(t) = g\Big(g^{-1}(v_-) + \int_{-\infty}^t F(z_-(v_-, .) + x_- + y_-)(\tau)\mathrm{d}\tau\Big) \to a(v_-, x_-)$ as $t \to +\infty$, where $a(v_-, x_-)$ is defined in (2.9). Then from Lemma 4.1 it follows that $\sup_{(0,+\infty)} |z_-(v_-, .) + x_- + y_- - z_+(a(v_-, x_-), .)| < +\infty$ and $\sup_{t\in(0,+\infty)}(1 + t)|\dot{z}_-(v_-, t) + \dot{y}_-(t) - \dot{z}_+(a(v_-, x_-), t)| < \infty$. Using these latter estimates and $y_- \in M_{r,v_-}$, and using (4.6), (4.7) and (4.21) we obtain that the integral on the right hand side of (2.11) is absolutely convergent. Then the decomposition (2.8) follows from the equality $A(y_-) = y_-$ and (2.1) and (2.2) and straightforward computations. □

5. Proof of Lemma 2.1

Proof of Lemma 2.1. We shorten $z_-(v_-, .)$ to z_- in this paragraph. We first prove the estimates (5.9), (5.10) and (5.12) given below. Let $f \in M_{r,v_-}$. Using (4.6) and (4.21) we have

$$|F^s(z_- + \varepsilon x_- + f)(\tau)| \leq \frac{2n\beta_2}{\Big(1 + \frac{\varepsilon|x_-|}{\sqrt{2}} - r + |\tau|\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big)^{\alpha+2}}, \tag{5.1}$$

for $\tau \in \mathbb{R}$ and for $\varepsilon \in [0, 1]$. Integrating both sides of (5.1) over $(-\infty, t)$, we obtain

$$\int_{-\infty}^t \Big|F^s(z_- + \varepsilon x_- + f)(\tau)\Big|\mathrm{d}\tau \leq \frac{4n\beta_2}{(\alpha + 1)\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 + \frac{\varepsilon|x_-|}{\sqrt{2}} - r\Big)^{\alpha+1}}, \tag{5.2}$$

for $t \in \mathbb{R}$ and for $\varepsilon \in [0,1]$. Similarly using (4.5) instead of (4.6) we have

$$\int_{-\infty}^{t} \Big|F^l(z_- + \varepsilon x_- + f)(\tau)\Big| \mathrm{d}\tau \le \frac{4n\beta_1^l}{\alpha\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 + \frac{\varepsilon|x_-|}{\sqrt{2}} - r\Big)^{\alpha}}, \tag{5.3}$$

for $t \in \mathbb{R}$ and for $\varepsilon \in [0,1]$. We combine (5.2) and (5.3), and we use (2.5), and we have

$$\begin{aligned} &\Big|g^{-1}(v_-) + \int_{-\infty}^{t} \Big(\eta_1 F(z_- + x_- + f_1)(\tau) + \eta_2(F^l + \mu F^s)(z_- + \varepsilon x_- + f_2)(\tau)\Big)\mathrm{d}\tau\Big| \\ &\quad \ge |g^{-1}(v_-)| - \frac{8n\max(\beta_1^l, \beta_2)}{\alpha\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 - r\Big)^{\alpha+1}} \ge \frac{1}{2}|g^{-1}(v_-)|, \end{aligned} \tag{5.4}$$

for $(f_1, f_2) \in M^2_{r,v_-}$ and for $(t, \eta_1, \eta_2, \mu, \varepsilon) \in \mathbb{R} \times [0,1]^4$ so that $\eta_1 + \eta_2 \le 1$. Then from (2.2), (5.4) and (4.2) it follows that

$$|\dot{A}(f)(t)| \le 2n^{\frac{1}{2}}\sqrt{1 - \frac{|v_-|^2}{c^2}}\Big|\int_{-\infty}^{t}\Big(F(z_- + x_- + f)(\tau) - F^l(z_-)(\tau)\Big)\mathrm{d}\tau\Big|, \tag{5.5}$$

for $t \in \mathbb{R}$. Using (5.1) we have

$$\begin{aligned} &\int_{-\infty}^{t} \Big|F^s(z_- + x_- + f)(\tau)\Big|\mathrm{d}\tau \\ &\quad \le \frac{2n\beta_2}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 + \frac{|x_-|}{\sqrt{2}} - r - t\Big(\frac{|v_-|}{2^{\frac{3}{2}}} - r\Big)\Big)^{\alpha+1}}, \end{aligned} \tag{5.6}$$

for $t \le 0$. Using (4.7) and (4.21) (with $''x'' = (1 - \varepsilon + \varepsilon\mu)x_-, (\varepsilon, \mu) \in [0,1]^2$) we obtain

$$\begin{aligned} &\Big|F^l(z_- + x_- + f_1)(\tau) - F^l(z_- + \mu x_- + f_2)(\tau)\Big| \\ &\quad \le \frac{\frac{n\beta_1^l}{c}|\dot{f}_1 - \dot{f}_2|(\tau)}{\Big(1 - r + \Big(\frac{|v_-|}{2^{\frac{3}{2}}} - r\Big)|\tau|\Big)^{\alpha+1}} + \frac{2n^{\frac{3}{2}}\beta_2\Big((1-\mu)|x_-| + |f_1 - f_2|(\tau)\Big)}{\Big(1 - r + \Big(\frac{|v_-|}{2^{\frac{3}{2}}} - r\Big)|\tau|\Big)^{\alpha+2}} \end{aligned} \tag{5.7}$$

for $(f_1, f_2) \in M^2_{r,v_-}$ and for $(\tau, \mu) \in \mathbb{R} \times [0,1]$. We integrate (5.7) over $(-\infty, t)$, and we use the estimates $|f(\tau)| \le r$ and $|\dot{f}(\tau)| \le r(1 - r + (\frac{|v_-|}{2\sqrt{2}} - r)|\tau|)^{-1}$ for $\tau \le 0$, and we obtain

$$\begin{aligned} &\int_{-\infty}^{t} \Big|F^l(z_- + x_- + f)(\tau) - F^l(z_-)(\tau)\Big|\mathrm{d}\tau \\ &\quad \le \Big(\frac{n\beta_1^l r}{c} + 2n^{\frac{3}{2}}\beta_2(|x_-| + r)\Big)\int_{-\infty}^{t}\Big(1 - r + \Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)|\tau|\Big)^{-\alpha-2}\mathrm{d}\tau \\ &\quad \le \frac{\frac{nr\beta_1^l}{c} + 2n^{\frac{3}{2}}\beta_2(|x_-| + r)}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)\Big(1 - r - \Big(\frac{|v_-|}{2\sqrt{2}} - r\Big)t\Big)^{\alpha+1}}, \end{aligned} \tag{5.8}$$

for $t \le 0$.

Combining (5.5), (5.8), (5.6) we obtain

$$|\dot{A}(f)(t)| \leq \frac{2n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\left(\frac{r\beta_1^l}{c}+2\beta_2(n^{\frac{1}{2}}(|x_-|+r)+1)\right)}{(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\left(1-r-\left(\frac{|v_-|}{2\sqrt{2}}-r\right)t\right)^{\alpha+1}}, \tag{5.9}$$

$$|A(f)(t)| \leq \frac{2n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\left(\frac{r\beta_1^l}{c}+2\beta_2(n^{\frac{1}{2}}(|x_-|+r)+1)\right)}{\alpha(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}}-r\right)^2\left(1-r-\left(\frac{|v_-|}{2\sqrt{2}}-r\right)t\right)^{\alpha}}, \tag{5.10}$$

for $t \leq 0$.

Let $t \geq 0$ and $|v_-| > 2\sqrt{2}r, r < 1$. Integrating (5.7) over $(0,t)$ and using the estimate $\sup_{(0,+\infty)}|\dot{f}| \leq r$ and (4.20) [for $(f_1,f_2)=(f,0)$ and $\|f\| \leq r$], we obtain

$$\begin{aligned}
&\int_0^t \left|F^l(z_-+x_-+f)(\tau)-F^l(z_-)(\tau)\right|\mathrm{d}\tau \\
&\leq \frac{nr\beta_1^l}{c}\int_0^t\left(1-r+\left(\frac{|v_-|}{2\sqrt{2}}-r\right)|\tau|\right)^{-\alpha-1}\mathrm{d}\tau \\
&\quad + 2n^{\frac{3}{2}}\beta_2\int_0^t\left(1-r+\left(\frac{|v_-|}{2\sqrt{2}}-r\right)|\tau|\right)^{-\alpha-2}(|x_-|+r+r|\tau|)\mathrm{d}\tau \\
&\leq \frac{n}{\left(\frac{|v_-|}{2\sqrt{2}}-r\right)(1-r)^{\alpha}}\left(\frac{\frac{r\beta_1^l}{c}}{\alpha}+\frac{2n^{\frac{1}{2}}\beta_2(|x_-|+r)}{(\alpha+1)(1-r)}+\frac{2n^{\frac{1}{2}}\beta_2 r}{\alpha\left(\frac{|v_-|}{2\sqrt{2}}-r\right)}\right).
\end{aligned} \tag{5.11}$$

Hence combining (5.5), (5.2), (5.11) and (5.8) (for "$t=0$"), we obtain

$$|\dot{A}(f)(t)| \leq \frac{2n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}}-r\right)(1-r)^{\alpha}}\left(\frac{\frac{r\beta_1^l}{c}+4\beta_2(n^{\frac{1}{2}}(|x_-|+r)+1)}{(\alpha+1)(1-r)}+\frac{r\beta_1^l}{c\alpha}+\frac{2n^{\frac{1}{2}}\beta_2 r}{\alpha\left(\frac{|v_-|}{2\sqrt{2}}-r\right)}\right). \tag{5.12}$$

Then estimate (2.6) follows from (5.9), (5.10) and (5.12).

Now we prove the estimates (5.18), (5.19) and (5.21) given below. Estimate (2.7) follows from those latter estimates. Let $|v_-| > 2\sqrt{2}r, r < 1$, and let $(f_1,f_2) \in M^2_{r,v_-}$. From (4.8) and (4.21) it follows that

$$\begin{aligned}
&\left|F^s(z_-+x_-+f_1)(\tau)-F^s(z_-+x_-+f_2)(\tau)\right| \\
&\leq \frac{\frac{n\beta_2}{c}|\dot{f}_1-\dot{f}_2|(\tau)}{\left(1-r+\frac{|x_-|}{\sqrt{2}}+\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)|\tau|\right)^{\alpha+2}}+\frac{2n^{\frac{3}{2}}\beta_3^s|f_1-f_2|(\tau)}{\left(1-r+\frac{|x_-|}{\sqrt{2}}+\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)|\tau|\right)^{\alpha+3}}
\end{aligned} \tag{5.13}$$

for $\tau \in \mathbb{R}$. Note that

$$\dot{A}(f_1)(t) - \dot{A}(f_2)(t) = g\Big(g^{-1}(v_-) + \int_{-\infty}^{t} F(z_- + x_- + f_1)(\tau)\mathrm{d}\tau\Big) - g\Big(g^{-1}(v_-) + \int_{-\infty}^{t} F(z_- + x_- + f_2)(\tau)\mathrm{d}\tau\Big). \quad (5.14)$$

Hence using (4.2) we obtain

$$|\dot{A}(f_1)(t) - \dot{A}(f_2)(t)| \leq 2n^{\frac{1}{2}}\sqrt{1 - \frac{|v_-|^2}{c^2}} J(t), \quad (5.15)$$

$$J(t) := \int_{-\infty}^{t} |F(z_- + x_- + f_2)(\tau) - F(z_- + x_- + f_1)(\tau)|\mathrm{d}\tau, \quad (5.16)$$

for $t \in \mathbb{R}$. Let $t \leq 0$. We integrate (5.7) and (5.13) over $(-\infty, t)$, and we use the estimates $|(f_1 - f_2)(\tau)| \leq \|f_1 - f_2\|$ and $|(\dot{f}_1 - \dot{f}_2)(\tau)| \leq (1 - r + (\frac{|v_-|}{2\sqrt{2}} - r)|\tau|)^{-1}\|f_1 - f_2\|$ for $\tau \leq 0$, and we have

$$J(t) \leq \|f_1 - f_2\| \times \int_{-\infty}^{t} \Big(\frac{\left(\frac{n\beta_1^l}{c} + 2n^{\frac{3}{2}}\beta_2\right)}{\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|\tau|\right)^{\alpha+2}} + \frac{\left(\frac{n\beta_2}{c} + 2n^{\frac{3}{2}}\beta_3^s\right)}{\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|\tau|\right)^{\alpha+3}}\Big)\mathrm{d}\tau. \quad (5.17)$$

We also use (5.15) and we obtain

$$|\dot{A}(f_1)(t) - \dot{A}(f_2)(t)| \leq \frac{2n^{\frac{3}{2}}\left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\|f_1 - f_2\|}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right)\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)^{\alpha+1}} \times \Big(\frac{\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2}{(\alpha+1)} + \frac{\frac{\beta_2}{c} + 2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2)\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)}\Big), \quad (5.18)$$

$$|A(f_1)(t) - A(f_2)(t)| \leq \frac{2n^{\frac{3}{2}}\left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\|f_1 - f_2\|}{(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)^{\alpha}} \times \Big(\frac{\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2}{\alpha} + \frac{\frac{\beta_2}{c} + 2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2)\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)}\Big). \quad (5.19)$$

Let $t \geq 0$ and $|v_-| > 2\sqrt{2}r, r < 1$, and let $(f_1, f_2) \in M_{r,v_-}^2$. We integrate (5.7) and (5.13) over $(0, t)$, and we use the estimates (4.20) and $|\dot{f}_1(\tau) - \dot{f}_2(\tau)| \leq \|f_1 - f_2\|$ for $\tau \in \mathbb{R}$, and we have

$$
\begin{aligned}
J(t)-J(0)&=\int_0^t |F(x_-+z_-+f_2)(\tau)-F(x_-+z_-+f_1)(\tau)|\mathrm{d}\tau\\
&\le \int_0^t \frac{\frac{n\beta_1^l}{c}\|f_1-f_2\|\mathrm{d}\tau}{\left(1-r+\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\tau\right)^{\alpha+1}}+\int_0^t \frac{2n^{\frac{3}{2}}\beta_2(1+\tau)\|f_1-f_2\|\mathrm{d}\tau}{\left(1-r+\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\tau\right)^{\alpha+2}}\\
&\quad+\int_0^t \frac{\frac{n\beta_2}{c}\|f_1-f_2\|\mathrm{d}\tau}{\left(1-r+\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\tau\right)^{\alpha+2}}+\int_0^t \frac{2n^{\frac{3}{2}}\beta_3^s(1+\tau)\|f_1-f_2\|\mathrm{d}\tau}{\left(1-r+\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\tau\right)^{\alpha+3}}\\
&\le \frac{n\|f_2-f_1\|}{(1-r)^\alpha\left(\frac{|v_-|}{2\sqrt{2}}-r\right)}\Bigg(\frac{\beta_1^l}{c\alpha}+\frac{\beta_2}{c(\alpha+1)(1-r)}+\frac{2n^{\frac{1}{2}}}{(1-r)}\Big(\frac{\beta_2}{\alpha+1}\\
&\quad+\frac{\beta_3^s}{(\alpha+2)(1-r)}\Big)+\frac{2n^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}}-r\right)}\Big(\frac{\beta_2}{\alpha}+\frac{\beta_3^s}{(\alpha+1)(1-r)}\Big)\Bigg).
\end{aligned}
\tag{5.20}
$$

Using (5.15), (5.17) (with "$t=0$") and (5.20) we obtain

$$
\begin{aligned}
|\dot A(f_1)(t)-\dot A(f_2)(t)|&\le \frac{2n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\|f_1-f_2\|}{\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\left(1-r\right)^{\alpha}}\\
&\times\Bigg[\frac{\beta_1^l}{c(\alpha+1)\left(1-r\right)}+\frac{\beta_2}{c(\alpha+2)\left(1-r\right)^2}\\
&+\Big(\frac{\beta_1^l}{c\alpha}+\frac{\beta_2}{c(\alpha+1)(1-r)}\Big)+\frac{4n^{\frac{1}{2}}}{(1-r)}\Big(\frac{\beta_2}{\alpha+1}+\frac{\beta_3^s}{(\alpha+2)(1-r)}\Big)\\
&+\frac{2n^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}}-r\right)}\Big(\frac{\beta_2}{\alpha}+\frac{\beta_3^s}{(\alpha+1)(1-r)}\Big)\Bigg].
\end{aligned}
\tag{5.21}
$$

□

6. Proof of Theorem 2.3

Estimate (2.15) follows from (4.2), (5.4), (5.2) and (5.3) (with "t" replaced by $+\infty$). We prove (2.13). From the identity $A(y_-)=y_-$ and from computations similar to (5.9) it follows that

$$
\begin{aligned}
\sup_{(-\infty,t)}|\dot y_-|&\le \frac{2n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\beta_1^l\sup_{(-\infty,t)}|\dot y_-|}{c\alpha\left(\frac{|v_-|}{2\sqrt{2}}-r\right)(1-r)^\alpha}\\
&\quad+\frac{4n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}\beta_2(n^{\frac{1}{2}}(|x_-|+r)+1)}{(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}}-r\right)\left(1-r-\left(\frac{|v_-|}{2\sqrt{2}}-r\right)t\right)^{\alpha+1}},
\end{aligned}
\tag{6.1}
$$

for $t \le 0$. Then we use condition (2.12) and we obtain

$$\frac{1}{2}\sup_{(-\infty,t)}|\dot{y}_-| \le \frac{4n^{\frac{3}{2}}\beta_2\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}(n^{\frac{1}{2}}(|x_-|+r)+1)}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1-r-\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)t\Big)^{\alpha+1}}, \tag{6.2}$$

for $t \le 0$, which proves (2.13). We prove (2.14). From computations similar to (5.12) it follows that

$$\begin{aligned}\sup_{\mathbb{R}}|\dot{y}_-| &= \sup_{\mathbb{R}}|\dot{A}(y_-)|\\ &\le \frac{2n^{\frac{3}{2}}\left(1-\frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}}{(\frac{|v_-|}{2\sqrt{2}}-r)(1-r)^{\alpha}}\Big(\frac{4\beta_2(n^{\frac{1}{2}}(|x_-|+r)+1)}{(\alpha+1)\Big(1-r\Big)}\\ &\quad+\sup_{\mathbb{R}}|\dot{y}_-|\Big(\frac{2\beta_1^l}{c\alpha}+\frac{2n^{\frac{1}{2}}\beta_2}{\alpha(\frac{|v_-|}{2\sqrt{2}}-r)}\Big)\Big).\end{aligned} \tag{6.3}$$

Then we use condition (2.12) and we obtain (2.14).

For the rest of this Section we shorten $z_-(v_-,.), a_{sc}(v_-,x_-), a(v_-,x_-)$ and $b_{sc}(v_-,x_-)$ to z_-, a_{sc}, a and b_{sc}.

We prove (2.18). First note that

$$a_{sc} - \sqrt{1-\frac{|v_-|^2}{c^2}}\int_{-\infty}^{+\infty}F(.v_-+x_-)(\tau)\mathrm{d}\tau = \Delta_{1,1}+\Delta_{1,2}, \tag{6.4}$$

where

$$\Delta_{1,1} := \lim_{t\to+\infty}\Big(\dot{A}(y_-)-\dot{A}(0)\Big)(t), \tag{6.5}$$

$$\begin{aligned}\Delta_{1,2} :=& \Big(\Big\langle\nabla g_j(g^{-1}(v_-)),\int_{-\infty}^{+\infty}\Big(F(z_-+x_-)(\tau)-F(.v_-+x_-)(\tau)\Big)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\\ &+\Big(\int_0^1\Big\langle\nabla g_j\Big(g^{-1}(v_-)+\varepsilon\int_{-\infty}^{+\infty}F(z_-+x_-)(\tau)\mathrm{d}\tau\Big)-\nabla g_j(g^{-1}(v_-)),\\ &\int_{-\infty}^{+\infty}F(z_-+x_-)(\tau)\mathrm{d}\tau\Big\rangle\mathrm{d}\varepsilon\Big)_{j=1\dots n},\end{aligned} \tag{6.6}$$

and where $\langle .,.\rangle$ denotes the scalar product in $\mathbb{R}^n$. For the decomposition (6.4) we used (4.4) and the identity $\langle v_-, \int_{-\infty}^{+\infty}F(.v_-+x_-)(\tau)\mathrm{d}\tau\rangle = 0$ to obtain

$$\begin{aligned}&\Big(\Big\langle\nabla g_j(g^{-1}(v_-)),\int_{-\infty}^{+\infty}F(.v_-+x_-)(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\\ &=\sqrt{1-\frac{|v_-|^2}{c^2}}\int_{-\infty}^{+\infty}F(.v_-+x_-)(\tau)\mathrm{d}\tau.\end{aligned} \tag{6.7}$$

We prove the estimates (6.10) and (6.15) given below that provide a bound for $\Delta_{1,i}, i = 1, 2$. Then adding those bounds and using the decomposition (6.4), we obtain (2.18). First we use (4.2) and we have

$$|\Delta_{1,1}| \leq 2n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\int_{-\infty}^{+\infty}|F(z_-+x_-+y_-)(\tau)-F(z_-+x_-)(\tau)|\mathrm{d}\tau. \tag{6.8}$$

Hence from (4.7), (4.8) and from the estimate (4.20) for $(f_1, f_2) = (y_-, 0)$, it follows that

$$\begin{aligned}|\Delta_{1,1}| \leq 2n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}&\left[\frac{\frac{2n\sup_{\mathbb{R}}|\dot{y}_-|}{c}\left(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)\left(1-r+\frac{|x_-|}{\sqrt{2}}\right)}\right)}{\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)\left(1-r+\frac{|x_-|}{\sqrt{2}}\right)^{\alpha}}\right.\\
&+\frac{4n^{\frac{3}{2}}\sup_{(-\infty,0)}|y_-|}{\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)\left(1-r+\frac{|x_-|}{\sqrt{2}}\right)^{\alpha+1}}\left(\frac{\beta_2^l}{\alpha+1}+\frac{\beta_3^s}{(\alpha+2)\left(1-r+\frac{|x_-|}{\sqrt{2}}\right)}\right)\\
&\left.+\frac{2n^{\frac{3}{2}}\frac{\sup_{(0,+\infty)}|\dot{y}_-|}{(\frac{|v_-|}{2^{\frac{3}{2}}}-r)}}{\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)\left(1-r+\frac{|x_-|}{\sqrt{2}}\right)^{\alpha}}\left(\frac{\beta_2^l}{\alpha}+\frac{\beta_3^s}{(\alpha+1)\left(1-r+\frac{|x_-|}{\sqrt{2}}\right)}\right)\right].\end{aligned} \tag{6.9}$$

Then we use (2.13) and (2.14) for bounds on $\sup_{(0,+\infty)}|\dot{y}_-|, \sup_{\mathbb{R}}|\dot{y}_-|$ and $\sup_{(-\infty,0)}|y_-|$, and we obtain

$$|\Delta_{1,1}| \leq \frac{256n^4\beta^2\left(1-\frac{|v_-|^2}{c^2}\right)(|x_-|+2)\left(\frac{1}{c}+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\right)}{\alpha(\alpha+1)\left(\frac{|v_-|}{2^{\frac{3}{2}}}-r\right)^2(1-r)^{2\alpha+2}}. \tag{6.10}$$

We use (4.1), (4.3) and (5.4), and we obtain

$$\begin{aligned}|\Delta_{1,2}| \leq n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}&\int_{-\infty}^{+\infty}\Big|F(z_-+x_-)(\tau)-F(.v_-+x_-)(\tau)\Big|\mathrm{d}\tau\\
&+\frac{12n}{c}\left(1-\frac{|v_-|^2}{c^2}\right)\Big|\int_{-\infty}^{+\infty}F(z_-+x_-)(\tau)\mathrm{d}\tau\Big|^2.\end{aligned} \tag{6.11}$$

Using (5.2) and (5.3) (with "(f, r)" = $(0, 0)$) we obtain

$$\Big|\int_{-\infty}^{+\infty}F(z_-+x_-)(\tau)\mathrm{d}\tau\Big| \leq \frac{2^{\frac{7}{2}}n}{|v_-|\left(1+\frac{|x_-|}{\sqrt{2}}\right)^{\alpha}}\left(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)\left(1+\frac{|x_-|}{\sqrt{2}}\right)}\right). \tag{6.12}$$

Then we use (4.7) and (1.10), and we have

$$\int_{-\infty}^{+\infty}\Big|F^l(z_-+x_-)(\tau)-F^l(.v_-+x_-)(\tau)\Big|\mathrm{d}\tau$$
$$\leq\int_{-\infty}^{+\infty}\frac{\frac{n\beta_1^l}{c}\sup_{\mathbb{R}}|\dot z_--v_-|\mathrm{d}\tau}{\Big(1+\frac{|x_-|}{\sqrt2}+|\tau|\frac{|v_-|}{2^{\frac32}}\Big)^{\alpha+1}}+\int_{-\infty}^{+\infty}\frac{2n^{\frac32}\beta_2|\tau|\sup_{\mathbb{R}}|\dot z_--v_-|\mathrm{d}\tau}{\Big(1+\frac{|x_-|}{\sqrt2}+|\tau|\frac{|v_-|}{2^{\frac32}}\Big)^{\alpha+2}}$$
$$\leq\frac{2^6n^{\frac52}\beta_1^l\sqrt{1-\frac{|v_-|^2}{c^2}}}{\alpha^2|v_-|^2\Big(1+\frac{|x_-|}{\sqrt2}\Big)^{\alpha}}\Big(\frac{\beta_1^l}{c}+\frac{2^{\frac52}n^{\frac12}\beta_2}{|v_-|}\Big). \tag{6.13}$$

Similarly we use (4.8) and (1.10), and we obtain

$$\int_{-\infty}^{+\infty}\Big|F^s(z_-+x_-)-F^s(.v_-+x_-)\Big|(\tau)\mathrm{d}\tau$$
$$\leq\frac{2^6n^{\frac52}\beta_1^l\Big(1-\frac{|v_-|^2}{c^2}\Big)^{\frac12}\Big(\frac{\beta_2}{c}+\frac{2^{\frac52}n^{\frac12}\beta_3^s}{|v_-|}\Big)}{\alpha(\alpha+1)|v_-|^2\Big(1+\frac{|x_-|}{\sqrt2}\Big)^{\alpha+1}}. \tag{6.14}$$

Combining (6.11), (6.12), (6.13) and (6.14), we obtain

$$|\Delta_{1,2}|\leq\frac{2^9\cdot3n^3\Big(1-\frac{|v_-|^2}{c^2}\Big)}{c|v_-|^2\Big(1+\frac{|x_-|}{\sqrt2}\Big)^{2\alpha}}\Big(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)\Big(1+\frac{|x_-|}{\sqrt2}\Big)}\Big)^2$$
$$+\frac{2^6n^3\beta_1^l\Big(1-\frac{|v_-|^2}{c^2}\Big)}{\alpha|v_-|^2\Big(1+\frac{|x_-|}{\sqrt2}\Big)^{\alpha}}\Big(\frac1c\Big(\frac{\beta_1^l}{\alpha}+\frac{\beta_2}{(\alpha+1)\Big(1+\frac{|x_-|}{\sqrt2}\Big)}\Big)$$
$$+\frac{2^{\frac52}n^{\frac12}}{|v_-|}\Big(\frac{\beta_2}{\alpha}+\frac{\beta_3^s}{(\alpha+1)\Big(1+\frac{|x_-|}{\sqrt2}\Big)}\Big)\Big). \tag{6.15}$$

Now we prove (2.16) and (2.17). We rewrite y_+ as follows

$$y_+=h_0+h_1, \tag{6.16}$$

where

$$h_0(t):=\int_t^{+\infty}\Big(\Big\langle\int_0^1\nabla g_j\Big(g^{-1}(a)-\varepsilon\int_\sigma^{+\infty}F(z_-+x_-+y_-)(\tau)\mathrm{d}\tau$$
$$-(1-\varepsilon)\int_\sigma^{+\infty}F^l(z_+(a,.))(\tau)\mathrm{d}\tau\Big)\mathrm{d}\varepsilon,$$
$$\int_\sigma^{+\infty}F^s(z_-+x_-+y_-)(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\ldots n}\mathrm{d}\sigma, \tag{6.17}$$

$$h_1(t) := \int_t^{+\infty} \Big(\Big\langle \int_0^1 \nabla g_j \Big(g^{-1}(a) - \varepsilon \int_\sigma^{+\infty} F(z_- + x_- + y_-)(\tau) \mathrm{d}\tau - (1-\varepsilon) \int_\sigma^{+\infty} F^l(z_+(a,.))(\tau)\mathrm{d}\tau \Big) \mathrm{d}\varepsilon, \int_\sigma^{+\infty} \Big(F^l(z_- + x_- + y_-) - F^l(z_+(a,.)) \Big)(\tau)\mathrm{d}\tau \Big\rangle \Big)_{j=1\ldots n} \mathrm{d}\sigma, \quad (6.18)$$

for $t \geq 0$. We estimate $\dot{h}_0$. We also need the following estimate (6.20). For $\varepsilon,\ \varepsilon' \in (0,1)$ and $\tau \geq 0$ we have

$$\begin{aligned} &|(1-\varepsilon)(z_-(\tau) + y_-(\tau)) + \varepsilon z_+(a,\tau) + \varepsilon' x_-| \\ &\quad \geq |\varepsilon' x_- + \tau v_-| - (1-\varepsilon)|z_-(\tau) - \tau v_-| - r\tau - r - \varepsilon |z_+(a,\tau) - \tau a| - \varepsilon |a_{sc}|\tau \\ &\quad \geq \varepsilon' \frac{|x_-|}{\sqrt{2}} - r + \tau \left(\frac{|v_-|}{\sqrt{2}} - r - \frac{2^{\frac{7}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha |v_-|} \right) - \varepsilon |a_{sc}|\tau. \end{aligned} \quad (6.19)$$

We used (1.10) for "v" $= v_-$ and for "v" $= a$, and we used the identity $|v_-| = |a|$ that holds by conservation of energy. Then we use (2.12), (2.15) and (6.19), and we have for $(\varepsilon, \varepsilon') \in (0,1)^2$ and $\tau \geq 0$

$$|(1-\varepsilon)(z_-(\tau) + y_-(\tau)) + \varepsilon z_+(a,\tau) + \varepsilon' x_-| \geq \varepsilon' \frac{|x_-|}{\sqrt{2}} - r + \tau \Big(\frac{|v_-|}{2\sqrt{2}} - r \Big). \quad (6.20)$$

Then the following estimate that is similar to (5.4) holds

$$\begin{aligned} &|g^{-1}(a) - \eta_1 \int_{-\infty}^{+\infty} F(z_- + x_- + y_-)(\tau)\mathrm{d}\tau - \int_\sigma^{+\infty} \Big(\eta_2 F(z_- + x_- + y_-) \\ &\quad + \eta_3 F^l(z_+(a,.)) + \eta_4 F^l(z_+(a,.) + x_-) \Big)(\tau)\mathrm{d}\tau| \geq \frac{1}{2} |g^{-1}(v_-)|, \end{aligned} \quad (6.21)$$

for $(\sigma, \eta_1, \eta_2, \eta_3, \eta_4) \in [0, +\infty) \times [0,1]^4, \eta_1 + \eta_2 + \eta_3 + \eta_4 \leq 1$.

From (6.21), (4.1), (4.6) and (6.17) it follows that

$$\begin{aligned} |\dot{h}_0(t)| &\leq 2n^{\frac{1}{2}} \Big(1 - \frac{|v_-|^2}{c^2} \Big)^{\frac{1}{2}} \int_t^{+\infty} |F^s(z_- + x_- + y_-)|(\tau)\mathrm{d}\tau \\ &\leq \frac{4\beta_2 n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}}}{(\alpha+1) \Big(\frac{|v_-|}{2^{\frac{3}{2}}} - r \Big) \Big(1 + \frac{|x_-|}{\sqrt{2}} - r + \Big(\frac{|v_-|}{2^{\frac{3}{2}}} - r \Big) t \Big)^{\alpha+1}}, \end{aligned} \quad (6.22)$$

for $t\geq 0$. Now set

$$\delta_r := \max\Big(\sup_{(0,+\infty)}|b_{sc}+y_+|, \sup_{s\in(0,+\infty)}\Big(1-r+\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)s\Big)|\dot{y}_+(s)|\Big). \quad (6.23)$$

We remind that δ_r is finite by Lemma 4.1 (for "(x,z)" $=(z_-+x_-+y_-, z_+(a,.))$). Then we use (4.1), (4.7), (6.21) and (6.20), and we obtain

$$|\dot{h}_1(t)| \leq 2n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\int_t^{+\infty}\frac{2n^{\frac{3}{2}}\beta_2|x_-|+(\frac{n\beta_1^l}{c}+2n^{\frac{3}{2}}\beta_2)\delta_r}{\Big(1-r+\tau\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)^{\alpha+2}}d\tau$$
$$\leq \frac{2n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\Big(2n^{\frac{1}{2}}\beta_2|x_-|+(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2)\delta_r\Big)}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1-r+t\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big)^{\alpha+1}}, \quad (6.24)$$

for $t\geq 0$. Hence combining (6.16), (6.22) and (6.24) we have

$$|\dot{y}_+(t)| \leq \frac{2n^{\frac{3}{2}}\Big(2\beta_2+2n^{\frac{1}{2}}\beta_2|x_-|+(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2)\delta_r\Big)\sqrt{1-\frac{|v_-|^2}{c^2}}}{(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)\Big(1-r+t\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)\Big)^{\alpha+1}}, \quad (6.25)$$

for $t\geq 0$. In addition, from (2.10) and (2.13) it follows that

$$|b_{sc}+y_+(0)| = |y_-(0)| \leq \frac{8n^2\beta_2\Big(1-\frac{|v_-|^2}{c^2}\Big)^{\frac{1}{2}}(|x_-|+2)}{\alpha(\alpha+1)\Big(\frac{|v_-|}{2\sqrt{2}}-r\Big)^2\Big(1-r\Big)^{\alpha}}. \quad (6.26)$$

From (6.23), (6.25) and (6.26) it follows that

$$\delta_r \leq \frac{2n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\Big(\beta_2(n^{\frac{1}{2}}(6|x_-|+8)+2)+(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2)\delta_r\Big)\max\Big(1,\frac{1}{\alpha\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)}\Big)}{(\alpha+1)\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)(1-r)^{\alpha}}. \quad (6.27)$$

Then we use condition (2.12) and we obtain

$$\frac{\delta_r}{2} \leq \frac{4n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\Big(\beta_2(n^{\frac{1}{2}}(3|x_-|+4)+1)\Big)}{(\alpha+1)\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)(1-r)^{\alpha}}\max\Big(1,\frac{1}{\alpha\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)}\Big). \quad (6.28)$$

Estimate (6.28) and condition (2.12) also provide the following estimate

$$\Big(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2\Big)\delta_r \leq \beta_2(n^{\frac{1}{2}}(3|x_-|+4)+1). \quad (6.29)$$

Estimate (2.17) follows from (6.25) and (6.29). Then we integrate over $(0,+\infty)$ both sides of (2.17) and we obtain a bound on $|y_+(0)|$. Then we add this bound with the bound on $|y_-(0)|$ from (2.13), and we use (2.10) and we obtain (2.16).

It remains to prove (2.19). From (2.10), (6.16), (6.17) and (6.18) at $t=0$ it follows by straightforward computations that

$$\sum_{j=1}^{8}\Delta_{2,j}=b_{sc}(v_-,x_-)-W(v_-,x_-)-\Big(\Big\langle\nabla g_j(g^{-1}(v_-)),\\ \int_{-\infty}^{0}\int_{-\infty}^{\sigma}F^s(.v_-+x_-)(\tau)\mathrm{d}\tau\mathrm{d}\sigma-\int_0^{+\infty}\int_{\sigma}^{+\infty}F^s(.v_-+x_-)(\tau)\mathrm{d}\tau\mathrm{d}\sigma\Big\rangle\Big)_{j=1\dots n}, \tag{6.30}$$

where

$$\Delta_{2,1}:=A(y_-)(0)-A(0)(0), \tag{6.31}$$

$$\Delta_{2,2}:=\int_{-\infty}^{0}\Big(\Big\langle\nabla g_j\Big(g^{-1}(v_-)+\varepsilon\int_{-\infty}^{\sigma}F(z_-+x_-)(\tau)\mathrm{d}\tau\\ +(1-\varepsilon)\int_{-\infty}^{\sigma}F^l(z_-+x_-)(\tau)\mathrm{d}\tau\Big)-\nabla g_j\Big(g^{-1}(v_-)\Big),\\ \int_{-\infty}^{\sigma}F^s(z_-+x_-)(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\mathrm{d}\sigma, \tag{6.32}$$

$$\Delta_{2,3}:=\int_{-\infty}^{0}\Big(\Big\langle\nabla g_j(g^{-1}(v_-)),\\ \int_{-\infty}^{\sigma}(F^s(z_-+x_-)-F^s(.v_-+x_-))(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\mathrm{d}\sigma, \tag{6.33}$$

$$\Delta_{2,4}:=-\int_0^{+\infty}\Big(\Big\langle\int_0^1\Big(\nabla g_j\Big(g^{-1}(a)-\varepsilon\int_{\sigma}^{+\infty}F(z_-+x_-+y_-)(\tau)\mathrm{d}\tau\\ -(1-\varepsilon)\int_{\sigma}^{+\infty}F^l(z_+(a,.))(\tau)\mathrm{d}\tau\Big)-\nabla g_j(g^{-1}(v_-))\Big)\mathrm{d}\varepsilon,\\ \int_{\sigma}^{+\infty}F^s(z_-+x_-+y_-)(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\mathrm{d}\sigma, \tag{6.34}$$

$$\Delta_{2,5}:=-\int_0^{+\infty}\Big(\Big\langle\nabla g_j(g^{-1}(v_-)),\int_{\sigma}^{+\infty}(F^s(z_-+x_-+y_-)\\ -F^s(z_-+x_-))(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\mathrm{d}\sigma, \tag{6.35}$$

$$\Delta_{2,6}:=-\int_0^{+\infty}\Big(\Big\langle\nabla g_j(g^{-1}(v_-)),\int_{\sigma}^{+\infty}(F^s(z_-+x_-)\\ -F^s(.v_-+x_-))(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\dots n}\mathrm{d}\sigma, \tag{6.36}$$

$$\Delta_{2,7} := -\int_0^{+\infty}\Big(\Big\langle\int_0^1 \nabla g_j\Big(g^{-1}(a)-\varepsilon\int_\sigma^{+\infty}F(z_-+x_-+y_-)(\tau)\mathrm{d}\tau \\ -(1-\varepsilon)\int_\sigma^{+\infty}F^l(z_+(a,.))(\tau)\mathrm{d}\tau\Big)\mathrm{d}\varepsilon, \\ \int_\sigma^{+\infty}\Big(F^l(z_-+x_-+y_-)-F^l(z_+(a,.)+x_-)\Big)(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\ldots n}\mathrm{d}\sigma, \tag{6.37}$$

$$\Delta_{2,8} := -\int_0^{+\infty}\Big(\Big\langle\int_0^1\Big(\nabla g_j\Big(g^{-1}(a)-\varepsilon\int_\sigma^{+\infty}F(z_-+x_-+y_-)(\tau)\mathrm{d}\tau \\ -(1-\varepsilon)\int_\sigma^{+\infty}F^l(z_+(a,.))(\tau)\mathrm{d}\tau\Big) \\ -\nabla g_j\Big(g^{-1}(a)-\varepsilon\int_\sigma^{+\infty}F^l(z_+(a,.)+x_-)(\tau)\mathrm{d}\tau \\ -(1-\varepsilon)\int_\sigma^{+\infty}F^l(z_+(a,.))(\tau)\mathrm{d}\tau\Big)\Big)\mathrm{d}\varepsilon, \\ \int_\sigma^{+\infty}\Big(F^l(z_+(a,.)+x_-)-F^l(z_+(a,.))\Big)(\tau)\mathrm{d}\tau\Big\rangle\Big)_{j=1\ldots n}\mathrm{d}\sigma. \tag{6.38}$$

From the derivation of (5.19) and from (2.13) it follows that

$$|\Delta_{2,1}| \leq \frac{2n^{\frac{3}{2}}\Big(1-\frac{|v_-|^2}{c^2}\Big)^{\frac{1}{2}}\Big(\frac{\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2}{\alpha}+\frac{\frac{\beta_2}{c}+2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2)(1-r)}\Big)}{(\alpha+1)(\frac{|v_-|}{2\sqrt{2}}-r)^2\Big(1-r\Big)^{\alpha}} \\ \times\max\Big(\sup_{(-\infty,0)}|y_-|,\sup_{s\in(-\infty,0)}\Big(1-r+\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)|s|\Big)|\dot{y}_-(s)|\Big) \\ \leq \frac{48n^4\beta^2\Big(1-\frac{|v_-|^2}{c^2}\Big)(\frac{1}{c}+1)(|x_-|+2)\Big(1+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r}\Big)}{\alpha^2(\alpha+1)^2\Big(\frac{|v_-|}{2^{\frac{3}{2}}}-r\Big)^3(1-r)^{2\alpha+1}}. \tag{6.39}$$

Using (4.3), (5.4) and then (4.5), (4.6) and (4.21) (with "(f,r)" $=(0,0)$) we have

$$|\Delta_{2,2}| \leq \frac{12n}{c}\Big(1-\frac{|v_-|^2}{c^2}\Big)\int_{-\infty}^0\max\Big(\int_{-\infty}^\sigma|F(z_-(v_-,.)+x_-)(\tau)|\mathrm{d}\tau, \\ \int_{-\infty}^\sigma|F^l(z_-(v_-,.)+x_-)(\tau)|\mathrm{d}\tau\Big)\int_{-\infty}^\sigma|F^s(z_-(v_-,.)+x_-)(\tau)|\mathrm{d}\tau\mathrm{d}\sigma \\ \leq\int_{-\infty}^0\Big(\frac{2^{\frac{5}{2}}\beta_1^l n}{\alpha|v_-|\Big(1+\frac{|x_-|}{\sqrt{2}}\Big)^{\alpha}}+\frac{2^{\frac{5}{2}}\beta_2 n}{(\alpha+1)|v_-|\Big(1+\frac{|x_-|}{\sqrt{2}}\Big)^{\alpha+1}}\Big)$$

$$\times \frac{\frac{12n^2}{c}(1-\frac{|v_-|^2}{c^2})2^{\frac{5}{2}}\beta_2 \mathrm{d}\sigma}{(\alpha+1)|v_-|\Big(1+\frac{|x_-|}{\sqrt{2}}+\frac{|v_-|}{2\sqrt{2}}|\sigma|\Big)^{\alpha+1}}$$
$$\leq \frac{96n^3\Big(1-\frac{|v_-|^2}{c^2}\Big)\max(\beta_1^l,\beta_2)^2}{c\alpha^2(\alpha+1)\Big(\frac{|v_-|}{2^{\frac{3}{2}}}\Big)^3\Big(1+\frac{|x_-|}{\sqrt{2}}\Big)^{2\alpha}}. \tag{6.40}$$

We use (4.1), (4.8), (4.21) (with "(f,r)" $=(0,0)$) and (1.10), and we obtain

$$\max(|\Delta_{2,3}|,|\Delta_{2,6}|) \leq \frac{2^{\frac{7}{2}}n^2\beta_1^l(1-\frac{|v_-|^2}{c^2})}{\alpha|v_-|}\int_0^{+\infty}\int_\sigma^{+\infty}\Big[\frac{n\beta_2}{c}(1+\frac{|x_-|}{\sqrt{2}}$$
$$+\frac{|v_-|}{2\sqrt{2}}\tau)^{-\alpha-2}+2n^{\frac{3}{2}}\beta_3^s\tau(1+\frac{|x_-|}{\sqrt{2}}+\frac{|v_-|}{2\sqrt{2}}\tau)^{-\alpha-3}\Big]\mathrm{d}\tau\mathrm{d}\sigma$$
$$\leq \frac{4n^3\max(\beta_1^l,\beta_2,\beta_3^s)^2(1-\frac{|v_-|^2}{c^2})}{\alpha^2(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}})^3(1+\frac{|x_-|}{\sqrt{2}})^\alpha}\Big(\frac{1}{c}+\frac{2n^{\frac{1}{2}}}{\frac{|v_-|}{2^{\frac{3}{2}}}}\Big). \tag{6.41}$$

We use (4.3) and (6.21), and then we use (4.5), (4.6) and (6.20), and we have

$$|\Delta_{2,4}| \leq \frac{12n}{c}\Big(1-\frac{|v_-|^2}{c^2}\Big)\int_0^{+\infty}\Big(\int_{-\infty}^{+\infty}|F(z_-+x_-+y_-)(\tau)|\mathrm{d}\tau$$
$$+\int_\sigma^{+\infty}|F^l(z_+(a,.))(\tau)|\mathrm{d}\tau\Big)\int_\sigma^{+\infty}|F^s(z_-+x_-+y_-)(s)|ds\mathrm{d}\sigma$$
$$\leq \frac{12n}{c}\Big(1-\frac{|v_-|^2}{c^2}\Big)\int_0^{+\infty}\Big(\int_0^{+\infty}\frac{4n\beta_2\mathrm{d}\tau}{(1+\frac{|x_-|}{\sqrt{2}}-r+(\frac{|v_-|}{2\sqrt{2}}-r)\tau)^{\alpha+2}}$$
$$+\int_0^{+\infty}\frac{6n\beta_1^l\mathrm{d}\tau}{(1-r+(\frac{|v_-|}{2\sqrt{2}}-r)\tau)^{\alpha+1}}\Big)$$
$$\times\int_\sigma^{+\infty}\frac{2n\beta_2 ds\mathrm{d}\sigma}{(1+\frac{|x_-|}{\sqrt{2}}-r+(\frac{|v_-|}{2\sqrt{2}}-r)s)^{\alpha+2}}$$
$$\leq \frac{48n^3(1-\frac{|v_-|^2}{c^2})\max(\beta_1^l,\beta_2)^2}{c\alpha^2(\alpha+1)(\frac{|v_-|}{2\sqrt{2}}-r)^3(1-r)^{2\alpha}}\Big(3+\frac{2}{1+\frac{|x_-|}{\sqrt{2}}-r}\Big). \tag{6.42}$$

We use (4.1), (5.13), (2.13) (for a bound on $\sup_{(-\infty,0)}|y_-|$) and (2.14), and we obtain

$$|\Delta_{2,5}| \leq n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\int_0^{+\infty}\int_\sigma^{+\infty}|F^s(z_-+x_-+y_-)(\tau)$$
$$-F^s(z_-+x_-)(\tau)|\mathrm{d}\tau\mathrm{d}\sigma$$
$$\leq n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\int_0^{+\infty}\int_\sigma^{+\infty}\Big(\frac{\frac{n\beta_2\sup_{(0,+\infty)}|\dot{y}_-|}{c}}{(1+\frac{|x_-|}{\sqrt{2}}-r+(\frac{|v_-|}{2\sqrt{2}}-r)\tau)^{\alpha+2}}$$
$$+\frac{2n^{\frac{3}{2}}\beta_3^s(\sup_{(-\infty,0)}|y_-|+\tau\sup_{(0,+\infty)}|\dot{y}_-|)}{(1+\frac{|x_-|}{\sqrt{2}}-r+(\frac{|v_-|}{2\sqrt{2}}-r)\tau)^{\alpha+3}}\Big)\mathrm{d}\tau\mathrm{d}\sigma$$

$$\leq \frac{n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\sup_{(0,+\infty)}|\dot{y}_-|\left(\frac{\beta_2}{c}+\frac{2\sqrt{n}\beta_3^s}{\frac{|v_-|}{2\sqrt{2}}-r}\right)}{\alpha(\alpha+1)(\frac{|v_-|}{2\sqrt{2}}-r)^2(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha}}$$
$$+\frac{2n^2\sqrt{1-\frac{|v_-|^2}{c^2}}\beta_3^s\sup_{(-\infty,0)}|y_-|}{(\alpha+1)(\alpha+2)(\frac{|v_-|}{2\sqrt{2}}-r)^2(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha+1}}$$
$$\leq \frac{40n^4(1-\frac{|v_-|^2}{c^2})\max(\beta_1^l,\beta_2,\beta_3^s)^2\Big(|x_-|+2\Big)\left(\frac{1}{c}+\frac{1}{\frac{|v_-|}{2\sqrt{2}}-r}\right)}{\alpha(\alpha+1)^2(\frac{|v_-|}{2\sqrt{2}}-r)^3(1-r)^{2\alpha+1}}. \quad (6.43)$$

From (6.21), (4.1), (4.7), (6.20) and (6.28) it follows that

$$|\Delta_{2,7}| \leq 2n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\int_0^{+\infty}\int_\sigma^{+\infty}\frac{(\frac{n\beta_1^l}{c}+2n^{\frac{3}{2}}\beta_2)\delta_r \mathrm{d}\tau\mathrm{d}\sigma}{(1-r+(\frac{|v_-|}{2^{\frac{3}{2}}}-r)\tau)^{\alpha+2}}$$
$$\leq \frac{32n^4(1-\frac{|v_-|^2}{c^2})\beta^2(3|x_-|+5)(1+\frac{1}{c})(1+\frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}}-r})}{\alpha^2(\alpha+1)^2(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^3(1-r)^{2\alpha}}. \quad (6.44)$$

Combining (6.20), (4.3), (6.21), (4.7), (4.5) and (4.6) we obtain

$$|\Delta_{2,8}| \leq \frac{12n(1-\frac{|v_-|^2}{c^2})}{c}\int_0^{+\infty}\int_\sigma^{+\infty}\Big(|F(z_-+x_-+y_-)(\tau)|$$
$$+|F^l(z_+(a,.)+x_-)(\tau)|\Big)\mathrm{d}\tau\int_\sigma^{+\infty}\frac{2n^{\frac{3}{2}}\beta_2|x_-|}{(1+\frac{|v_-|}{2^{\frac{3}{2}}}\tau)^{\alpha+2}}\mathrm{d}\tau\mathrm{d}\sigma$$
$$\leq \frac{48n^{\frac{7}{2}}\beta_2|x_-|(1-\frac{|v_-|^2}{c^2})\left(\frac{2\beta_1^l}{\alpha}+\frac{\beta_2^s}{(\alpha+1)(1+\frac{|x_-|}{\sqrt{2}}-r)}\right)}{c\alpha(\alpha+1)\left(\frac{|v_-|}{2^{\frac{3}{2}}}\right)^2(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha}}. \quad (6.45)$$

Then we add the bounds on the right-hand sides of (6.39)–(6.45), and we use (6.30), and we obtain (2.19). □

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References

[1] Arians, S.: Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials. J. Math. Phys. **38**(6), 2761–2773 (1997)

[2] Cook J.M.: Banach algebras and asymptotic mechanics. 1967 Cargèse Lectures in Theoretical Physics: Application of Mathematics to Problems in Theoretical Physics (Cargèse, 1965), pp. 209–245. Gordon and Breach Science Publications, New York (1967)

[3] Dereziński, J.: Large time behavior of classical N-body systems. Commun. Math. Phys. **148**, 503–520 (1992)

[4] Derezinski, J., Gérard, C.: Scattering theory of classical and quantum N-particle systems. Springer, Berlin, Heidelberg (1997)

[5] Dirac, P.A.M.: Quantised singularities in the electromagnetic field. Proc. R. Soc. Lond. A **133**, 60–72 (1931)

[6] Einstein, A.: Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen. Jahrbuch der Radioaktivität Und Elektronik **4**, 411–462 (1907)

[7] Enss, V., Weder, R.: The geometrical approach to multidimensional inverse scattering. J. Math. Phys. **36**(8), 3902–3921 (1995)

[8] Eskin, G., Ralston, J.: Inverse scattering problems for Schrödinger operators with magnetic and electric potentials. Inverse problems in wave propagation (Minneapolis, MN, 1995), IMA Volumes in Mathematics and its Applications, 90, pp. 147–166. Springer, New York (1997)

[9] Faddeev, L.D.: Uniqueness of solution of the inverse scattering problem. Vestnik. Leningrad. Univ. **11**(7), 126–130 (1956)

[10] Gel'fand, I.M., Gindikin, S.G., Graev, M.I.: Integral geometry in affine and projective spaces. Itogi Nauki i Tekhniki, Sovr. Prob. Mat. **16**, 53–226 (1980) (Russian).

[11] Hachem, G.: The Faddeev formula in the inverse scattering for Dirac operators. Helv. Phys. Acta **72**(5&6), 301–315 (1999)

[12] Henkin, G.M., Novikov, R.G.: A multidimensional inverse problem in quantum and acoustic scattering. Inverse Probl. **4**, 103–121 (1988)

[13] Herbst, I.W.: Classical scattering with long range forces. Commun. Math. Phys. **35**,193–214 (1974)

[14] Hunziker, W.: The S-matrix in classical mechanics. Commun. Math. Phys. **8**, 282–299 (1968)

[15] Isozaki, H.: Inverse scattering theory for Dirac operators. Ann. Inst. H. Poincaré Phys. Théor. **66**(2), 237–270 (1997)

[16] Ito, H.T.: High-energy behavior of the scattering amplitude for a Dirac operator. Publ. RIMS Kyoto Univ. **31**, 1107–1133 (1995)

[17] Jollivet, A.: On inverse scattering in electromagnetic field in classical relativistic mechanics at high energies. Asympt. Anal. **55**(1&2), 103–123 (2007)

[18] Jollivet, A.: Inverse scattering at high energies for the multidimensional Newton equation in a long range potential. To appear in Asympt. Anal., preprint (2013). arXiv:1306.3638

[19] Jollivet, A.: On inverse scattering at fixed energy for the multidimensional Newton equation in a non-compactly supported field. J. Inverse Ill-Posed Probl. **21**(6), 713–734 (2013)

[20] Jollivet, A.: On inverse scattering at high energies for the multidimensional relativistic Newton equation in a long range electromagnetic field. Preprint (2013) arXiv:1401.0182

[21] Jung, W.: Geometric approach to inverse scattering for Dirac equation. J. Math. Phys. **38**(1), 39–48 (1997)

[22] Loss, M., Thaller, B.: Scattering of particles by long-range magnetic fields. Ann. Phys. **176**(1), 159–180 (1987)

[23] Landau, L.D., Lifschitz, E.M.: The classical theory of fields. Pergamon Press, New York (1971)

[24] Natterer, F.: The Mathematics of Computerized Tomography. B. G. Teubner, Stuttgart, Wiley, Chichester and New York (1986)

[25] Nicoleau, F.: A stationary approach to inverse scattering for Schrödinger operators with first order perturbation. Commun. Partial Differ. Equ. **22**(3&4), 527–553 (1997)

[26] Novikov, R.G.: Small angle scattering and X-ray transform in classical mechanics. Ark. Mat. **37**, 141–169 (1999)

[27] Novikov, R.G.: The $\bar{\partial}$-approach to monochromatic inverse scattering in three dimensions. J. Geom. Anal **18**(2), 612–631 (2008)

[28] Radon, J.: Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. Ber. Verh. SÄchs. Akad. Wiss. Leipzig, Math.-Nat. K **1**(69), 262–277 (1917)

[29] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. III. Scattering Theory, p. xv+463. Academic Press (Harcourt Brace Jovanovich, Publishers), New York, London (1979)

[30] Shiota, T.: An inverse problem for the wave equation with first order perturbation. Amer. J. Math. **107**(1), 241–251 (1985)

[31] Simon, B.: Wave operators for classical particle scattering. Commun. Math. Phys. **23**, 37–48 (1971)

[32] Yajima, K.: Classical scattering for relativistic particles. J. Fac. Sci., Univ. Tokyo Sect. I A **29**, 599–611 (1982)

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