



Poisson Eigenvalue Statistics for Random Schrödinger Operators on Regular Graphs

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Abstract. For random operators it is conjectured that spectral properties of an infinite-volume operator are related to the distribution of spectral gaps of finite-volume approximations. In particular, localization and pure point spectrum in infinite volume is expected to correspond to Poisson eigenvalue statistics. Motivated by results on the Anderson model on the infinite tree we consider random Schrödinger operators on finite regular graphs. We study local spectral statistics: we analyze the number of eigenvalues in intervals with length comparable to the inverse of the number of vertices of the graph, in the limit where this number tends to infinity. We show that the random point process generated by the rescaled eigenvalues converges in certain spectral regimes of localization to a Poisson process. The corresponding result on the lattice was proved by Minami. However, due to the geometric structure of regular graphs the known methods turn out to be difficult to adapt. Therefore, we develop a new approach based on direct comparison of eigenvectors.

1. Introduction

For random operators with extensive disorder, it is generally conjectured that there is a connection between spectral properties of a random operator in an infinite-volume setup and the distribution of spectral gaps of finite-volume approximations; see for example [10, 23, 48] and references therein. In particular, it is expected that within regimes of localization and pure point spectrum of the infinite volume operator, the local spectral statistics of finite volume approximations is close to Poisson statistics.

An intuitive argument that goes back to Dyson [22] can be based on perturbation theory: the eigenvalues of a finite-volume operator repel each other and this level repulsion is comparable to the overlap of the corresponding

eigenvectors. For eigenvalues in spectral regimes of localization, the eigenvectors are typically localized in different regions and have negligible overlap. Hence, in these spectral regimes level repulsion is expected to have vanishing effect and eigenvalues should be fairly independent. This is the guiding idea that eigenvalue statistics has to be studied by analyzing eigenvectors.

An important example for random operators is the Anderson model of random Schrödinger operators [17, 46, 49]. On the lattice \mathbb{Z}^d , $d \geq 1$, a random Schrödinger operator is given by the lattice Laplacian plus a random potential. Anderson localization on the lattice, in particular the existence of regimes of pure point spectrum, was proved first in [28] using a multi-scale analysis and later in [1, 4] using bounds on fractional moments of the Green function.

On the lattice, the aforementioned relation of localization and spectral statistics was also established rigorously: by restricting the random Schrödinger operator on the infinite lattice to a cube C_r of side length $r \in \mathbb{N}$, one obtains a random symmetric $n_r \times n_r$ matrix, where n_r denotes the number of lattice points in C_r . So there are n_r eigenvalues E_j and (assuming that the random potential has bounded support) these eigenvalues accumulate in a uniformly bounded interval. To study the local spectral statistics, one considers the random eigenvalue point process on the scale of the mean eigenvalue spacing, that is on the scale $1/n_r$. Thus, one studies the point process generated by the shifted, rescaled points $n_r(E_j - E)$, $j = 1, \dots, n_r$, in the limit $r \rightarrow \infty$. If E lies within a spectral regime of localization, this random point process converges in distribution to a Poisson process. This was proved in one dimension [43], where the entire spectrum is pure point, and in higher dimensions [42] where the regime of localization is characterized by exponential decay of fractional moments of the Green function. This result, in particular the relevant regime of localization, was extended in [30, 31, 52] and the implications on the distribution of eigenvectors were studied in [13, 35, 44]. These results are proved on the lattice \mathbb{Z}^d . Poisson eigenvalue statistics was also derived on the single-ended Canopy graph [7], but so far no rigorous results have been found for regular graphs.

Here, we prove Poisson eigenvalue statistics in certain spectral regimes of localization for a large class of graphs including regular graphs. In a regular graph of degree $K + 1$, $K \in \mathbb{N}$, each vertex is connected by an edge to $K + 1$ other vertices. As explained below it is difficult to adapt the existing methods to show Poisson statistics to regular graphs, so we have to develop a new approach.

The motivation to consider regular graphs stems from the fact that random regular graphs are appropriate finite-volume approximations of infinite regular trees. A random regular graph of degree $K + 1$ with n vertices—a graph chosen from the ensemble of all such graphs with uniform probability—typically coincides locally with a regular infinite tree of degree $K + 1$. (We refer to [15] for details about random graph models.) This property was used to prove that the spectral measure of the graph Laplacian on a random regular graph converges to the spectral measure on the infinite tree as n tends to infinity [40, 41].

Consequently, the integrated density of states on a random regular graph approximates the integrated density of states on the infinite tree. Denoting by $N_n(t)$ the number of eigenvalues below $t \in \mathbb{R}$ of the adjacency matrix of a random regular graph with n vertices, one has $\lim_{n \rightarrow \infty} N_n(t)/n = 0$ for $t \leq -2\sqrt{K}$ and

$$\lim_{n \rightarrow \infty} \frac{1}{n} N_n(t) = \frac{K+1}{2\pi} \int_{-2\sqrt{K}}^t \frac{\sqrt{4K - \lambda^2}}{(K+1)^2 - \lambda^2} d\lambda$$

for $t > -2\sqrt{K}$, both almost surely [40, 41]. The measure on the right-hand side is known as the Kesten–McKay law and gives the spectral measure on the infinite $K+1$ -regular tree. This result is remarkable since other finite-volume approximations—for example the integrated density of states on a finite regular tree—do not converge to the integrated density of states on the infinity tree [7].

This convergence of spectral measures was generalized to show that the Laplacian and random Schrödinger operators on random regular graphs approximate the corresponding operators on the infinite tree in various ways [11, 16, 21, 29, 47, 51].

In turn, the Anderson model on the infinite tree is one of the most studied models of random operators starting with the seminal work of Anderson [12]. It is generally conjectured that the Anderson model shows a phase transition from localization to delocalization. However, the existence of delocalized eigenvectors and regimes of absolutely continuous spectrum has been established rigorously only on trees [5, 8, 27, 37]. It is a frequently discussed question if this phase transition can also be seen in the local spectral statistics on regular graphs that approximate trees. It is conjectured that random Schrödinger operators on random regular graphs show a transition from localization, where the eigenvalue point process converges to a Poisson process, to delocalization, where eigenvalue statistics is governed by level repulsion that is familiar from random matrix theory [7, 20, 25, 33, 53]. There are physical arguments and numerical results [14, 24, 39, 45], but so far there have been no rigorous proofs for neither part of the conjecture.

To explain the difficulties that arise in trying to extend the known methods to prove Poisson statistics to regular graphs, fix a vertex x in a regular graph of degree $K+1$. For $r \in \mathbb{N}$ consider the neighborhood $B_r(x)$ of all vertices that are at distance at most r from x . Together with this neighborhood, consider also its inner boundary $\partial B_r(x)$ of vertices that are at distance r from x . Then both the number of points in $B_r(x)$ and the number of points in the boundary $\partial B_r(x)$ are of order K^r . In particular, the ratio of boundary to volume does not go to zero as r grows. On the lattice \mathbb{Z}^d the same ratio decays with rate $1/r$ and this decay is a crucial ingredient in the existing proofs of Poisson eigenvalue statistics.

For example in [42], eigenvalue statistics is studied by analyzing the Green function in the cube C_r . This cube is decomposed into smaller cubes and the Green function is decoupled at the boundaries of the smaller cubes. This decoupling leads to independence and eventually to Poisson statistics. However, this strategy relies on the fact that the error due to decoupling along the

boundaries has negligible effect. Hence, this strategy cannot be adapted to regular graphs because of the non-vanishing contribution of boundary terms.

In this article we study regular graphs and more general graphs with $n \in \mathbb{N}$ vertices and uniformly bounded degree in the limit $n \rightarrow \infty$. This includes all regular graphs with fixed degree and in particular random regular graphs. We show that the rescaled eigenvalue process converges—in a certain regime of localization that is specified below—in distribution to a Poisson process. To circumvent the difficulties mentioned above, we use a new approach. We do not work primarily with the Green function, but we analyze eigenvectors directly. In particular, we do not use an a priori decomposition, but we adapt the decomposition to the location of the eigenvectors. This is realized by comparing eigenvectors of random Schrödinger operators on the graph with eigenvectors of local restrictions of the operators.

In the next section, we first introduce the relevant notation about graphs, random Schrödinger operators, and eigenvalue processes. In (5) we state and discuss the localization assumption given in terms of exponential decay of fractional moments of the Green function. Then in Theorem 1 and Corollary 2, we formulate the main results. In Sect. 3 we give the strategy of proof and explain the structure of the remainder of the article.

2. Main Result

We consider simple undirected connected graphs \mathcal{G}_n with $n \in \mathbb{N}$ vertices. For two vertices $x, y \in \mathcal{G}_n$, we write $d(x, y)$ for the distance between x and y , that is for the length of the shortest path in \mathcal{G}_n connecting x and y . For a vertex $x \in \mathcal{G}_n$ and $r > 0$, let

$$B_r(x) = \{y \in \mathcal{G}_n : d(x, y) \leq r\}$$

denote the r -neighborhood of x . For any subset $\mathcal{B} \subset \mathcal{G}_n$ we write $\partial\mathcal{B} = \{x \in \mathcal{B} : \text{dist}(x, \mathcal{G}_n \setminus \mathcal{B}) = 1\}$ for its inner boundary and $|\mathcal{B}|$ for the number of vertices in \mathcal{B} . More generally, for a set A , we write $|A|$ for the number of elements in A , while for an interval $I \subset \mathbb{R}$ we write $|I|$ for its length.

The (maximal) degree of a graph is the maximal number of edges emerging from a vertex. We assume that the degree of the graphs \mathcal{G}_n is uniformly bounded by $K + 1$ with $K \geq 2$. Then for any $n \in \mathbb{N}$, $x \in \mathcal{G}_n$, and $r > 0$, we get

$$|B_r(x)| \leq 1 + (K + 1) \sum_{m=0}^{r-1} K^m \leq 3K^r \tag{1}$$

and

$$|\partial B_r(x)| \leq (K + 1)K^{r-1} \leq \frac{3}{2}K^r. \tag{2}$$

This is satisfied, in particular, for $(K + 1)$ -regular graphs where each vertex is connected by an edge to $K + 1$ other vertices. However, our results are valid for all simple undirected connected graphs that satisfy the uniform bounds (1) and (2).

We study the distribution of eigenvalues of random Schrödinger operators

$$H_n(\omega) = A_n + \alpha V_n(\omega), \tag{3}$$

with domain $\ell^2(\mathcal{G}_n)$ in the limit $n \rightarrow \infty$. Here, A_n denotes the adjacency matrix of \mathcal{G}_n ,

$$(A_n\phi)(x) = - \sum_{y \in \mathcal{G}_n: d(y,x)=1} \phi(y), \quad x \in \mathcal{G}_n, \quad \phi \in \ell^2(\mathcal{G}_n)$$

that corresponds to the graph Laplacian with the diagonal terms removed. The random potential $V_n(\omega)$ acts as a multiplication operator,

$$(V_n(\omega)\phi)(x) = \omega_x\phi(x), \quad x \in \mathcal{G}_n, \quad \phi \in \ell^2(\mathcal{G}_n),$$

where $(\omega_x)_{x \in \mathcal{G}_n}$ is a collection of independent identically distributed real random variables. We assume that the single-site distribution $\rho(dt) = \text{Prob}(\omega_x \in dt)$, $x \in \mathcal{G}_n$, is absolutely continuous with bounded density such that

$$\|\rho\|_\infty = \sup_{t \in \mathbb{R}} |\rho(t)| < \infty.$$

We also assume that the support of ρ is bounded such that $\text{supp } \rho = [-\rho_0, \rho_0]$ with $\rho_0 < \infty$. With \mathbb{P} and \mathbb{E} , we denote probability and expectation with respect to the distribution of $\omega = (\omega_x)_{x \in \mathcal{G}_n}$. Finally, $\alpha > 0$ is a parameter controlling the strength of the disorder.

We denote by $(E_j(\omega))_{j=1}^n = \sigma(H_n(\omega))$ the eigenvalues of the operator $H_n(\omega)$ and by $(\phi_j(\omega))_{j=1}^n$ the corresponding orthonormal eigenvectors. For eigenvalues with multiplicities, we count them according to the multiplicity and we choose the eigenvectors as an orthonormal basis of the corresponding eigenspace. (We can assume, however, that almost surely all eigenvalues are simple.) For an interval $J \subset \mathbb{R}$, let $N(J)$ denote the number of eigenvalues in J .

The spectrum of $H_n(\omega)$ is almost surely contained in the interval $[-K - 1 - \alpha\rho_0, K + 1 + \alpha\rho_0]$. Thus, the eigenvalues accumulate in a bounded interval. To study local spectral statistics, we are interested in the random eigenvalue point process on the scale of the mean eigenvalue spacing, that is on the scale $1/n$. In particular, we consider random variables $N(I_n)$, where I_n are suitable intervals with length of order $1/n$. On the infinite $(K + 1)$ -regular tree, the spectrum of the corresponding random Schrödinger operator is almost surely contained in the interval $[-2\sqrt{K} - \alpha\rho_0, 2\sqrt{K} + \alpha\rho_0]$ (see for example [26, 36, 38, 46]). On random regular graphs, it follows that the support of the spectral measure of $H_n(\omega)$ converges to this interval as $n \rightarrow \infty$ [29, 40].

Thus for fixed $E \in [-2\sqrt{K} - \alpha\rho_0, 2\sqrt{K} + \alpha\rho_0]$, we consider the random point process $\nu_n^{(E)}$ generated by the rescaled eigenvalues $\{n(E_j - E)\}_{j=1}^n$:

$$\nu_n^{(E)} = \sum_{j=1}^n \delta_{n(E_j - E)}.$$

Here, $\delta_{n(E_j - E)}$ denotes the Dirac measure: for a Borel set $A \subset \mathbb{R}$, $\delta_{n(E_j - E)}(A) = 1$ if $n(E_j - E) \in A$ and $\delta_{n(E_j - E)}(A) = 0$ otherwise. For any bounded interval $I \subset \mathbb{R}$, we denote by

$$I_n = E + I/n = \{t \in \mathbb{R} : n(t - E) \in I\}$$

the rescaled interval centered at E such that

$$\nu_n^{(E)}(I) = N(I_n). \tag{4}$$

Our goal is to show that $\nu_n^{(E)}$ converges—for values of E in suitable spectral regimes of localization—to a Poisson process.

We prove this convergence to a Poisson process in spectral regimes of localization, where fractional moments of the Green function decay exponentially. A localization assumption in terms of decay of the Green function is also used for example in [7, 42, 52]. To state the precise assumption, we denote by

$$G_{n,\alpha}(x, y; z) = \langle \delta_x, (H_n(\omega) - z)^{-1} \delta_y \rangle, \quad x, y \in \mathcal{G}_n, \quad z \in \mathbb{C}_+,$$

the matrix elements of the Green function of $H_n(\omega)$. Here, $\delta_x \in \ell^2(\mathcal{G}_n)$ is such that $\delta_x(u) = 1$ for $x = u$ and $\delta_x(u) = 0$ otherwise.

We assume that there is an open interval $I_0 \subset [-2\sqrt{K} - \alpha\rho_0, 2\sqrt{K} + \alpha\rho_0]$ and that there are constants $n_0 \in \mathbb{N}$ and $s \in (0, 1)$ such that for all $n \geq n_0$ and all $z = E + i\zeta$ with $E \in I_0$ and $\zeta > 0$, the estimate

$$\mathbb{E} [|G_{n,\alpha}(x, y; z)|^s] \leq C_s \exp(-\mu_s d(x, y)) \tag{5}$$

holds for all $x, y \in \mathcal{G}_n$ with μ_s large enough and with a uniform constant C_s . We emphasize that the right-hand side is independent of $n, E \in I_0$, and $\zeta > 0$.

Exponential decay of this type can be proved with established methods developed to derive localization via fractional moments of the Green function [1–4, 6] (see also [9, Ch. 7] for a condensed presentation). In particular, bounds of this form hold for large disorder (that means large α) or extreme energies (that means E close to the spectral edges). On the infinite tree, even the Green function of the adjacency matrix (corresponding to $\alpha = 0$) decays exponentially with rate $(\ln K)/2$. Localization and pure point spectrum for random Schrödinger operators is proved in regimes where (5) holds with $\mu_s > \ln K$, so that the bound is summable over the tree [1].

However, on trees as well as on finite graphs with bounded degree there are regimes such that (5) holds with arbitrarily large decay rate. With the methods mentioned above, one can derive, for example, that for given μ_0 , there is α_0 such that for $\alpha \geq \alpha_0$ the bound (5) holds with $\mu_s = \mu_0$ for all $E \in [-2\sqrt{K} - \alpha\rho_0, 2\sqrt{K} + \alpha\rho_0]$ and $\zeta > 0$. In this case, we could choose $I_0 = (-2\sqrt{K} - \alpha\rho_0, 2\sqrt{K} + \alpha\rho_0)$.

Theorem 1. *Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be a sequence of graphs such that \mathcal{G}_n has n vertices and such that the bounds (1) and (2) are satisfied for all $n \in \mathbb{N}$. Assume that $I_0 \subset \mathbb{R}$ is an open interval, such that the fractional moment bound (5) holds with $\mu_s \geq 43 \ln K$.*

Then for all $E \in I_0$ and all bounded intervals $I \subset \mathbb{R}$, the limit

$$\lim_{n \rightarrow \infty} \left| \mathbb{E} \left[\exp \left(-t \nu_n^{(E)}(I) \right) \right] - \exp \left(-\mathbb{E} \left[\nu_n^{(E)}(I) \right] (1 - e^{-t}) \right) \right| = 0$$

holds uniformly in $t \geq 0$.

Let \mathcal{P}_λ denote a random variable, Poisson distributed with parameter $\lambda \geq 0$. Then,

$$\mathbb{E} [\exp (-t\mathcal{P}_\lambda)] = \exp (-\lambda (1 - e^{-t})) .$$

Hence, Theorem 1 shows that the Laplace transform of the random variable $\nu_n^{(E)}(I)$ and thus the distribution of $\nu_n^{(E)}(I)$ are close to Poisson distribution with parameter $\mathbb{E}[\nu_n^{(E)}(I)]$. From general results of convergence of point processes (see for example [34, Ch. 14]), we deduce that $\nu_n^{(E)}$ converges to a Poisson point process:

Let us write $\bar{\nu}_n^{(E)}$ for the measure $\mathbb{E}[\nu_n^{(E)}]$. By (4), the Wegner estimate (14) implies

$$\bar{\nu}_n^{(E)}(I) \leq \|\rho\|_\infty n|I_n| = \|\rho\|_\infty |I|$$

for any interval $I \subset \mathbb{R}$. Thus, $\bar{\nu}_n$ is absolutely continuous with respect to Lebesgue measure with uniformly bounded density. In particular,

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} f(\tau) \bar{\nu}_n^{(E)}(d\tau) \leq \|\rho\|_\infty \int_{\mathbb{R}} f(\tau) d\tau < \infty$$

for every continuous function $f : \mathbb{R} \mapsto [0, \infty)$ with compact support. From properties of the vague topology, it follows that there is a subsequence $(n_k)_{k \in \mathbb{N}}$ and a Borel measure $\bar{\nu}^{(E)}$ such that $\bar{\nu}_{n_k}^{(E)}$ converges vaguely to $\bar{\nu}^{(E)}$ [34, Thm. A2.3]. In particular, $\bar{\nu}^{(E)}$ is again absolutely continuous with respect to Lebesgue measure and

$$\bar{\nu}_{n_k}^{(E)}(B) \rightarrow \bar{\nu}^{(E)}(B) \tag{6}$$

holds for any bounded Borel set $B \subset \mathbb{R}$ [34, Thm. A2.3]. We combine this with Theorem 1 and obtain

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[\exp \left(-t \nu_{n_k}^{(E)}(I) \right) \right] = \exp \left(-\bar{\nu}^{(E)}(I) (1 - e^{-t}) \right)$$

for any bounded interval $I \subset \mathbb{R}$. In particular, $\nu_{n_k}^{(E)}(I)$ converges in distribution to $\mathcal{P}_{\bar{\nu}^{(E)}(I)}$. From [34, Thm. 14.16], we conclude that the process $\nu_{n_k}^{(E)}$ converges in distribution to a Poisson process with intensity measure $\bar{\nu}^{(E)}$. We summarize these findings in the following statement.

Corollary 2. *Under the conditions of Theorem 1, there is a subsequence $(\mathcal{G}_{n_k})_{k \in \mathbb{N}}$ and a Borel measure $\bar{\nu}^{(E)}$ such that the eigenvalue process $\nu_{n_k}^{(E)}$ converges in distribution to a Poisson process with intensity $\bar{\nu}^{(E)}$.*

Let us conclude this section with two remarks about the results.

First, we note that the assumption about the rate of decay, $\mu_s \geq 43 \ln K$, is stronger than necessary. It is conjectured that the eigenvalue process converges to a Poisson process at least in regimes where the limiting infinite volume operator has pure point spectrum. On the tree, pure point spectrum exists in regimes where (5) holds with $\mu_s > \ln K$ [1]. So, our results do not cover the optimal range of values. However, they establish the existence of spectral

regimes where the local distribution of eigenvalues is given by Poisson statistics. So in these regimes, they prove the conjectured relation of localization and eigenvalue statistics.

If one is only interested in a statement of the form of Theorem 1 without information about the intensity measure, the assumption about the decay could be relaxed to $\mu_s > 18 \ln K$ and maybe even further. However, it seems to be impossible to reach the optimal condition $\mu_s > \ln K$ with the methods discussed here.

The second remark concerns the intensity measure of the limiting Poisson process. On the lattice \mathbb{Z}^d , one can choose the subsequence $(n_k)_{k \in \mathbb{N}}$ with $n_k = k^d$ and consider cubes of side length k . Then, (6) holds for Lebesgue almost every E with

$$\bar{\nu}^{(E)}(B) = D(E) |B|;$$

see for example [42]. Here, $D(E)$ denotes the density of states at E of the random Schrödinger operator on the lattice. Thus, the intensity measure is given by Lebesgue measure times the density of states. A similar result holds for the Canopy graph [7]. In general, the intensity measure depends on the choice of graphs. But if we consider random regular graphs, then it is reasonable to conjecture that the limiting intensity measure in Corollary 2 is given by Lebesgue measure times the density of states of the random Schrödinger operator on the infinite tree.

3. Strategy of Proof

Our strategy is based on the fact that the bound (5) on fractional moments of the Green function implies exponential localization of eigenvectors. Indeed, similar to that in [1, 6], we get the following result about exponential localization that we prove in the appendix.

Proposition 3. *For a graph \mathcal{G}_n satisfying (2), let $I \subset \mathbb{R}$ be such that the fractional moment bound (5) holds for $E \in I$ with exponent $\mu_s > 2 \ln K$ and let*

$$\ln K < \mu < \mu_s - \ln K.$$

Then there is a constant $C_{\text{loc}} > 0$ and a random variable $X_n(\omega)$ with

$$\mathbb{E} \left[X_n^{2(a-1)/a} \right] \leq C_{\text{loc}} n |I|, \quad 1 < a < 2 - \frac{\ln K}{\mu}, \tag{7}$$

such that the following holds: for each eigenvalue $E_j \in I$, there is a vertex $x_j \in \mathcal{G}_n$ such that the corresponding $\ell^2(\mathcal{G}_n)$ -normalized eigenvector ϕ_j satisfies

$$|\phi_j(x)| \leq X_n \exp(-\mu d(x, x_j)) \tag{8}$$

for all $x \in \mathcal{G}_n$.

Note that the bound in (7) includes a factor n . To get a bound independent of n , we will apply Proposition 3 to intervals I_n with length of order $1/n$.

The proof of Theorem 1 is based on the comparison of the eigenvalue process $\nu_n^{(E)}$ with an auxiliary process $\eta_n^{(E)}$. In Sect. 6.1 we construct this process such that

$$\eta_n^{(E)}(I) = \sum_{x \in \mathcal{G}_n} b_x^{(E)}(I),$$

where $(b_x^{(E)}(I))_{x \in \mathcal{G}_n}$ are random Bernoulli variables. We define these variables in terms of local operators $H_n^{(x)}(\omega)$. For each $x \in \mathcal{G}_n$ the operator $H_n^{(x)}(\omega)$ is the restriction of $H_n(\omega)$ to the neighborhood $B_{R_n}(x)$, where $R_n \in \mathbb{N}$ is chosen in (12). The construction of η_n also depends on a small parameter τ_n chosen in (11) that controls how well the local operators $H_n^{(x)}$ approximate H_n .

The definition of b_x in terms of these local operators allows to show that b_x and b_y are independent if x and y are sufficiently far away from each other; see Lemma 11. Based on a local Minami estimate, we also show that $b_x b_y$ is typically zero if x and y are close to each other; see Lemma 12. In Sect. 6.2, we use these facts to prove that η_n is close to a Poisson process:

Proposition 4. *Let \mathcal{G}_n satisfy (1) and (2) and let $I \subset \mathbb{R}$ and $E \in \mathbb{R}$. For each $R_n \in \mathbb{N}$ and $0 < \tau_n \leq 1/6\sqrt{K}n$, the bound*

$$\begin{aligned} & \left| \mathbb{E} \left[\exp \left(-t\eta_n^{(E)}(I) \right) \right] - \exp \left(-\mathbb{E} \left[\eta_n^{(E)}(I) \right] (1 - e^{-t}) \right) \right| \\ & \leq 81 \|\rho\|_\infty^2 (|I| + 1)^2 K^{8R_n} n^{-1} \end{aligned}$$

holds for all $t \geq 0$.

To complete the proof of Theorem 1, we have to compare $\nu_n^{(E)}(I)$ and $\eta_n^{(E)}(I)$. The exponential decay of eigenvectors allows to show the following estimate that we prove in Sect. 6.3.

Proposition 5. *Let \mathcal{G}_n satisfy (1) and (2) and let $I_0 \subset \mathbb{R}$ be such that the fractional moment bound (5) holds with $\mu_s > 2 \ln K$. Let $R_n \in \mathbb{N}$ and $\tau_n \leq 1/6\sqrt{K}n$ and set*

$$C_n = \tau_n^2 e^{(\mu_s - 2 \ln K)R_n}. \tag{9}$$

Then, there is a constant $C > 0$ such that for each $E \in I_0$ and each interval $I \subset \mathbb{R}$, the estimates

$$\begin{aligned} & \mathbb{E} \left[\left| \eta_n^{(E)}(I) - \nu_n^{(E)}(I) \right| \right] \\ & \leq C\sqrt{K} (1 + \|\rho\|_\infty^2 + |I|) \left(\tau_n n + \tau_n K^{2R_n} + C_n^{-2(a-1)/a} \right) n \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left[\left| e^{-t\eta_n^{(E)}(I)} - e^{-t\nu_n^{(E)}(I)} \right| \right] \\ & \leq C\sqrt{K} (1 + \|\rho\|_\infty^2 + |I|) \left(\tau_n n + \tau_n K^{2R_n} + C_n^{-2(a-1)/a} \right) \end{aligned}$$

hold for all $t \geq 0$ and all $n \in \mathbb{N}$ with $I_n \subset I_0$. Here, $1 < a < 2 - \ln K / (\mu_s - \ln K)$ denotes the parameter from Proposition 3.

Based on these results, we can prove Theorem 1. We have to choose the parameters $R_n \in \mathbb{N}$ and $\tau_n > 0$ in such a way that the error terms in Proposition 4 and Proposition 5 become small.

Proof of Theorem 1. We estimate

$$\begin{aligned} \left| \mathbb{E} \left[e^{-t\nu_n^{(E)}(I)} \right] - e^{-\mathbb{E}[\nu_n^{(E)}(I)](1-e^{-t})} \right| &\leq \mathbb{E} \left[\left| e^{-t\nu_n^{(E)}(I)} - e^{-t\eta_n^{(E)}(I)} \right| \right] \\ &\quad + \left| \mathbb{E} \left[e^{-t\eta_n^{(E)}(I)} \right] - e^{-\mathbb{E}[\eta_n^{(E)}(I)](1-e^{-t})} \right| \\ &\quad + 1 - e^{-\mathbb{E}[\eta_n^{(E)}(I) - \nu_n^{(E)}(I)](1-e^{-t})}. \end{aligned}$$

We remark that $\mu, K, \|\rho\|_\infty$, and $|I|$ are bounded. We also note that for $E \in I_0$, we have $I_n \subset I_0$ for n large enough. Under the assumption of Theorem 1, we can thus apply Proposition 4 and Proposition 5 to estimate

$$\begin{aligned} &\left| \mathbb{E} \left[e^{-t\nu_n^{(E)}(I)} \right] - e^{-\mathbb{E}[\nu_n^{(E)}(I)](1-e^{-t})} \right| \\ &\leq C \left(\frac{K^{8R_n}}{n} + \left(\tau_n n + \tau_n K^{2R_n} + C_n^{-2(a-1)/a} \right) n \right) \end{aligned} \tag{10}$$

with C_n given in (9) and with a constant $C > 0$ independent of n and $t \geq 0$.

We optimize the error terms

$$\frac{K^{8R_n}}{n}, \quad \tau_n n^2, \quad \tau_n^{-4(a-1)/a} e^{-2(\mu_s - 2 \ln K)R_n(a-1)/a} n.$$

This leads to

$$\tau_n = \frac{K^{8R_n}}{n^3} \tag{11}$$

and

$$R_n = \left\lceil \frac{(7a - 6) \ln n}{(a - 1)\mu_s + (18a - 14) \ln K} \right\rceil. \tag{12}$$

With this choice of parameters all error terms in (10) are bounded by a constant times

$$\exp \left(- \frac{(a - 1)\mu_s - (38a - 34) \ln K}{(a - 1)\mu_s + (18a - 14) \ln K} \ln n \right).$$

Thus, to ensure that the bound tends to zero we have to show that

$$\left(\frac{\mu_s}{\ln K} - 34 \right) \frac{a - 1}{a} > 4. \tag{13}$$

The assumption that $\mu_s \geq 43 \ln K$ implies that we can choose $\mu_s > 2(\sqrt{101} + 11) \ln K$ and thus $a > (4\sqrt{101} + 41)/(2\sqrt{101} + 21)$ in Proposition 3. This implies (13) and $\tau_n \leq 1/6\sqrt{K}n$ for n large enough and the proof of Theorem 1 is complete. \square

In the next section, we collect some general estimates that will be used in subsequent proofs. In Sect. 5, we use the methods from [18] to derive an estimate about Poisson approximation of weakly dependent Bernoulli variables. In Sect. 6, we construct the auxiliary process η_n and prove Proposition 4 and Proposition 5.

4. Some General Estimates

In this section we record some general estimates that will be used in the proofs. We consider an arbitrary finite simple undirected connected graph \mathcal{G} with degree $K + 1$ and a random Schrödinger operator $H(\omega)$ in $\ell^2(\mathcal{G})$ of the form (3). As before, we denote by $(E_j)_{j=1}^{|\mathcal{G}|} = \sigma(H)$ and $(\phi_j)_{j=1}^{|\mathcal{G}|}$ the eigenvalues and corresponding $\ell^2(\mathcal{G})$ -normalized eigenvectors of $H(\omega)$.

First, we state the Wegner estimate about the mean number of eigenvalues in an interval [54]. For any $J \subset \mathbb{R}$,

$$\mathbb{E} [N(J)] \leq \|\rho\|_\infty |\mathcal{G}| |J| \tag{14}$$

and it follows that

$$\mathbb{P} [N(J) \geq 1] \leq \|\rho\|_\infty |\mathcal{G}| |J|. \tag{15}$$

This fundamental bound was generalized by Minami [42] to higher numbers of eigenvalues; see also [13, 19, 32, 50]: for $k \in \mathbb{N}$,

$$\mathbb{P} [N(J) \geq k] \leq \frac{1}{k!} \|\rho\|_\infty^k |\mathcal{G}|^k |J|^k. \tag{16}$$

Next, we collect several consequences of exponential localization of eigenvectors.

Lemma 6. *Let $\psi \in \ell^2(\mathcal{G})$ be an approximate eigenvector of H . That means we assume that there is a neighborhood $\mathcal{B} \subset \mathcal{G}$ and constants $\lambda \in \mathbb{R}$ and $\tau > 0$, such that $H\psi(x) = \lambda\psi(x)$ for $x \in \mathcal{B}$, $\psi(x) = 0$ for $x \notin \mathcal{B}$, $\|\psi\| = 1$, and*

$$\sum_{x \in \partial\mathcal{B}} |\psi(x)|^2 \leq \tau^2.$$

Then there exists an eigenvalue $E_j \in \sigma(H)$ such that $|\lambda - E_j| \leq \sqrt{K}\tau$.

Moreover, if $N((\lambda - \epsilon, \lambda + \epsilon)) \leq 1$ for some $\epsilon > 0$, then the eigenvector ϕ_j corresponding to E_j satisfies

$$|\langle \psi, \phi_j \rangle|^2 \geq 1 - K\tau^2\epsilon^{-2}. \tag{17}$$

Proof. The eigenvectors of $H(\omega)$ form a basis of $\ell^2(\mathcal{G}_n)$; hence,

$$\|(H - \lambda)\psi\|^2 = \sum_{j=1}^{|\mathcal{G}|} |\langle \psi, \phi_j \rangle|^2 (\lambda - E_j)^2 \geq \min_j (\lambda - E_j)^2.$$

Moreover, by assumption,

$$\|(H - \lambda)\psi\|^2 = \sum_{x \notin \mathcal{B}} \left| \sum_{y: d(y,x)=1} \psi(y) \right|^2 \leq K \sum_{y \in \partial\mathcal{B}} |\psi(y)|^2 \leq K\tau^2.$$

Combining these bounds yields the first claim. To prove the second claim, we use these relations again and estimate

$$\sum_{i: |E_i - \lambda| \geq \epsilon} |\langle \psi, \phi_i \rangle|^2 \leq \frac{1}{\epsilon^2} \sum_{i: |E_i - \lambda| \geq \epsilon} |\langle \psi, \phi_i \rangle|^2 (\lambda - E_i)^2 \leq K\tau^2\epsilon^{-2}.$$

Since ψ is normalized, this implies

$$\sum_{i:|E_i-\lambda|<\epsilon} |\langle \psi, \phi_i \rangle|^2 \geq 1 - K\tau^2\epsilon^{-2}$$

and the claim follows from the assumption $N((\lambda - \epsilon, \lambda + \epsilon)) \leq 1$. □

Let us now show that the assumption about the number of eigenvalues is justified with high probability for appropriate intervals.

Lemma 7. *Let $I \subset \mathbb{R}$ be a bounded interval and for $\epsilon \leq |I|$ let $\Delta(\epsilon)$ denote the event that there are two eigenvalues $E_j, E_k \in I \cap \sigma(H)$ with $j \neq k$ satisfying $|E_j - E_k| < \epsilon$. Then,*

$$\mathbb{P}[\Delta(\epsilon)] \leq 2\|\rho\|_\infty^2 |I| \epsilon |\mathcal{G}|^2.$$

Proof. The proof is based on the Minami estimate (16). To apply this result we cover the interval I by intervals of length 2ϵ . Choose $a \in \mathbb{R}$ such that $I = [a, a + |I|]$. Let $M(\epsilon) = \lfloor |I|/\epsilon \rfloor$ and for $m = 1, \dots, M(\epsilon)$, we choose intervals $J_m = [a + (m - 1)\epsilon, a + (m + 1)\epsilon]$.

In the event $\Delta(\epsilon)$, there exists $m \in \{1, \dots, M(\epsilon)\}$ such that E_j and E_k both lie in J_m . Hence, by (16) with $k = 2$, we get

$$\mathbb{P}[\Delta(\epsilon)] \leq \mathbb{P}[\exists m \in \{1, \dots, M(\epsilon)\} : N(J_m) \geq 2] \leq M(\epsilon) \frac{1}{2} \|\rho\|_\infty^2 |\mathcal{G}|^2 (2\epsilon)^2$$

and the result follows from the bound $M(\epsilon) \leq |I|/\epsilon$. □

Finally, we need the following simple consequences of the bound (17).

Lemma 8. *Assume that there are two normalized vectors $\psi_1, \psi_2 \in \ell^2(\mathcal{G})$ and an eigenvector ϕ_j of H such that the estimates $|\langle \psi_1, \phi_j \rangle|^2 \geq 1 - \delta^2$ and $|\langle \psi_2, \phi_j \rangle|^2 \geq 1 - \delta^2$ hold for some $\delta > 0$. Then, we have*

$$|\langle \psi_1, \psi_2 \rangle| \geq 1 - 2\delta^2.$$

Proof. The eigenvectors of H form a basis of $\ell^2(\mathcal{G})$; hence,

$$\langle \psi_1, \psi_2 \rangle = \sum_{k=1}^{|\mathcal{G}|} \langle \psi_1, \phi_k \rangle \langle \phi_k, \psi_2 \rangle = \langle \psi_1, \phi_j \rangle \langle \phi_j, \psi_2 \rangle + \sum_{k \neq j} \langle \psi_1, \phi_k \rangle \langle \phi_k, \psi_2 \rangle.$$

By assumption, $|\langle \psi_1, \phi_j \rangle| |\langle \phi_j, \psi_2 \rangle| \geq 1 - \delta^2$ and $\sum_{k \neq j} |\langle \psi_1, \phi_k \rangle|^2 \leq \delta^2$ and the same bound holds for ψ_2 . It follows that

$$|\langle \psi_1, \psi_2 \rangle| \geq 1 - \delta^2 - \left(\sum_{k \neq j} |\langle \psi_1, \phi_k \rangle|^2 \right)^{1/2} \left(\sum_{k \neq j} |\langle \psi_2, \phi_k \rangle|^2 \right)^{1/2} \geq 1 - 2\delta^2$$

and the proof is complete. □

Lemma 9. *Assume that a normalized vector $\psi \in \ell^2(\mathcal{G})$ satisfies $|\langle \psi, \phi_j \rangle|^2 \geq 1 - \delta^2$ for an eigenvector ϕ_j of H and $\delta > 0$. Then, for each $x \in \mathcal{G}$,*

$$|\psi(x)| \leq |\phi_j(x)| + \delta.$$

Proof. Again, we use that the eigenvectors of H form a basis and estimate

$$\begin{aligned} |\psi(x)| &\leq |\langle \phi_j, \psi \rangle| |\phi_j(x)| + \sum_{k \neq j} |\langle \phi_k, \psi \rangle| |\phi_k(x)| \\ &\leq |\phi_j(x)| + \left(\sum_{k \neq j} |\langle \phi_k, \psi \rangle|^2 \right)^{1/2} \left(\sum_{k=1}^{|\mathcal{G}|} |\phi_k(x)|^2 \right)^{1/2}. \end{aligned}$$

Thus, the claim follows from the identity $\sum_{k=1}^{|\mathcal{G}|} |\phi_k(x)|^2 = 1$. □

5. Poisson Approximation

Here, we prove the following result of Poisson approximation of a weakly dependent random point process on a graph \mathcal{G} . This is the basis for the proof of Proposition 4. We adapt the Chen–Stein method from [18].

Lemma 10. *Let $(b_x)_{x \in \mathcal{G}}$ be a collection of random Bernoulli variables. Assume that there is $\varrho \in \mathbb{N}$ such that b_x is independent of $(b_y)_{y \notin B_\varrho(x)}$ for all $x \in \mathcal{G}$. We write $\Lambda = \sum_{x \in \mathcal{G}} b_x$ and $\bar{\lambda} = \sum_{x \in \mathcal{G}} \mathbb{E}[b_x]$. Then the bound*

$$\left| \mathbb{E} [e^{-t\Lambda}] - e^{-\bar{\lambda}(1-e^{-t})} \right| \leq \sum_{x \in \mathcal{G}} \sum_{y \in B_\varrho(x) \setminus \{x\}} \mathbb{E} [b_x b_y] + \sum_{x \in \mathcal{G}} \sum_{y \in B_\varrho(x)} \mathbb{E} [b_x] \mathbb{E} [b_y]$$

holds for all $t \geq 0$.

Proof. We fix $t \geq 0$ and for $x \in \mathcal{G}$, we write $p_x = \mathbb{E} [b_x]$. We define a function $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ depending on t and $\bar{\lambda}$. We set $f(0) = 0$ and for $m \in \mathbb{N}$

$$\begin{aligned} f(m) &= - \int_{e^{-t}}^1 e^{-\bar{\lambda}(y-e^{-t})} y^{m-1} dy \\ &= -(m-1)! \bar{\lambda}^{-m} \sum_{k=0}^{m-1} \frac{\bar{\lambda}^k}{k!} \left(e^{-tk} - e^{-\bar{\lambda}(1-e^{-t})} \right) \end{aligned}$$

such that, for $m \in \mathbb{N}_0$,

$$e^{-tm} - e^{-\bar{\lambda}(1-e^{-t})} = mf(m) - \bar{\lambda}f(m+1). \tag{18}$$

Below, we will show that

$$\begin{aligned} &|\mathbb{E} [\Lambda f(\Lambda) - \bar{\lambda}f(\Lambda + 1)]| \\ &\leq \sup_{m \in \mathbb{N}_0} |\partial f(m)| \sum_{x \in \mathcal{G}} \left(\sum_{y \in B_\varrho(x) \setminus \{x\}} \mathbb{E} [b_x b_y] + \sum_{y \in B_\varrho(x)} p_x p_y \right) \end{aligned} \tag{19}$$

with $\partial f(m) = f(m+1) - f(m)$. An elementary estimate shows that $|\partial f(m)| \leq 1$ for all $m \in \mathbb{N}_0$ and all $t \geq 0$ and $\bar{\lambda} \geq 0$. Thus, the claim of the proposition follows from (18) and (19).

To prove (19), we write

$$\Lambda^{(x)} = \sum_{y \in \mathcal{G} \setminus \{x\}} b_y \quad \text{and} \quad \Gamma^{(x)} = \sum_{y \in \mathcal{G} \setminus B_\varrho(x)} b_y.$$

Then for all $x \in \mathcal{G}$, we have $b_x f(\Lambda) = b_x f(\Lambda^{(x)} + 1)$. Hence, we can write

$$\begin{aligned} \Lambda f(\Lambda) - \bar{\lambda} f(\Lambda + 1) &= \sum_{x \in \mathcal{G}} b_x \left(f(\Lambda^{(x)} + 1) - f(\Gamma^{(x)} + 1) \right) \\ &\quad + \sum_{x \in \mathcal{G}} (b_x - p_x) f(\Gamma^{(x)} + 1) \\ &\quad + \sum_{x \in \mathcal{G}} p_x \left(f(\Gamma^{(x)} + 1) - f(\Lambda + 1) \right). \end{aligned}$$

By assumption, b_x and $\Gamma^{(x)}$ are independent, so the expectation of the second summand is zero. Thus, we obtain

$$\begin{aligned} \mathbb{E} [\Lambda f(\Lambda) - \bar{\lambda} f(\Lambda + 1)] &= \sum_{x \in \mathcal{G}} \mathbb{E} \left[b_x \left(f(\Lambda^{(x)} + 1) - f(\Gamma^{(x)} + 1) \right) \right] \\ &\quad + \sum_{x \in \mathcal{G}} p_x \mathbb{E} \left[f(\Gamma^{(x)} + 1) - f(\Lambda + 1) \right]. \end{aligned} \tag{20}$$

Let us now fix $x \in \mathcal{G}$ and let $x = y_1, y_2, y_3, \dots, y_{|B_\varrho(x)|}$ denote the vertices in $B_\varrho(x)$. We write $Y_0^{(x)} = \Gamma^{(x)}$ and $Y_j^{(x)} = \Gamma^{(x)} + \sum_{k=1}^j b_{y_k}$ for $j = 1, \dots, |B_\varrho(x)|$, such that $Y_{|B_\varrho(x)|}^{(x)} = \Lambda$. Then, for $j = 1, \dots, |B_\varrho(x)|$, we have

$$f(Y_{j-1}^{(x)} + 1) - f(Y_j^{(x)} + 1) = b_{y_j} \left(f(Y_{j-1}^{(x)} + 1) - f(Y_{j-1}^{(x)} + 2) \right)$$

and it follows that

$$\begin{aligned} f(\Gamma^{(x)} + 1) - f(\Lambda + 1) &= \sum_{j=1}^{|B_\varrho(x)|} \left(f(Y_{j-1}^{(x)} + 1) - f(Y_j^{(x)} + 1) \right) \\ &= \sum_{j=1}^{|B_\varrho(x)|} b_{y_j} \left(f(Y_{j-1}^{(x)} + 1) - f(Y_{j-1}^{(x)} + 2) \right) \\ &= - \sum_{j=1}^{|B_\varrho(x)|} b_{y_j} \partial f(Y_{j-1}^{(x)} + 1). \end{aligned} \tag{21}$$

Similarly, we write $Z_1^{(x)} = \Gamma^{(x)}$ and $Z_j^{(x)} = \Gamma^{(x)} + \sum_{k=2}^j b_{y_k}$, for $j = 2, \dots, |B_\varrho(x)|$, such that $Z_{|B_\varrho(x)|}^{(x)} = \Lambda^{(x)}$. Then, for $j = 2, \dots, |B_\varrho(x)|$, we have

$$f(Z_j^{(x)} + 1) - f(Z_{j-1}^{(x)} + 1) = b_{y_j} \left(f(Z_{j-1}^{(x)} + 2) - f(Z_{j-1}^{(x)} + 1) \right)$$

and it follows that

$$\begin{aligned}
 f(\Lambda^{(x)} + 1) - f(\Gamma^{(x)} + 1) &= \sum_{j=2}^{|B_\varrho(x)|} \left(f(Z_j^{(x)} + 1) - f(Z_{j-1}^{(x)} + 1) \right) \\
 &= \sum_{j=2}^{|B_\varrho(x)|} b_{y_j} \left(f(Z_{j-1}^{(x)} + 2) - f(Z_{j-1}^{(x)} + 1) \right) \\
 &= \sum_{j=2}^{|B_\varrho(x)|} b_{y_j} \partial f(Z_{j-1}^{(x)} + 1).
 \end{aligned} \tag{22}$$

Inserting (21) and (22) into (20), we find

$$\begin{aligned}
 \mathbb{E} [\Lambda f(\Lambda) - \bar{\lambda} f(\Lambda + 1)] &= \sum_{x \in \mathcal{G}} \sum_{j=2}^{|B_\varrho(x)|} \mathbb{E} \left[b_x b_{y_j} \partial f(Z_{j-1}^{(x)} + 1) \right] \\
 &\quad - \sum_{x \in \mathcal{G}} \sum_{j=1}^{|B_\varrho(x)|} p_x \mathbb{E} \left[b_{y_j} \partial f(Y_{j-1}^{(x)} + 1) \right]
 \end{aligned}$$

and (19) follows. This completes the proof. □

6. The Auxiliary Process

In this section, we construct an auxiliary process $\eta_n^{(E)}$ and we show that it is close to the eigenvalue process and to a Poisson process. In particular, we prove Proposition 4 and Proposition 5. We fix a bounded interval $I \subset \mathbb{R}$ and we set $I_n = E + I/n$. The construction of η_n depends on the parameters $R_n \in \mathbb{N}$ and $0 < \tau_n \leq 1/6\sqrt{K}n$ specified in (12) and (11). To shorten the notation, we write $B_n(x) = B_{R_n}(x)$.

6.1. Construction of the Process

For each $x \in \mathcal{G}_n$, we define an auxiliary operator $H_n^{(x)}(\omega)$ to be the restriction of $H_n(\omega)$ to $\ell^2(B_n(x))$ with Neumann boundary conditions:

$$\left(H_n^{(x)}(\omega)\phi \right) (y) = - \sum_{\substack{u \in B_n(x) \\ d(u,y)=1}} \phi(u) + \alpha \omega_y \phi(y)$$

for $y \in B_m(x)$ and $(H_n^{(x)}(\omega)\phi)(y) = 0$ for $y \notin B_m(x)$. We emphasize that the operator $H_n^{(x)}(\omega)$ and therefore also its eigenvectors and eigenvalues are independent of the values of the potential $V_n(\omega)$ outside of the neighborhood $B_n(x)$ and thus depend only on $(\omega_y)_{y \in B_n(x)}$. To shorten the notation, we do not always write the dependence on ω .

Next, we construct a random subset $\mathcal{F}_n \subset \mathcal{G}_n$ as follows: a vertex $x \in \mathcal{G}_n$ belongs to \mathcal{F}_n if and only if

- (i) There exists an eigenvalue $\xi^{(x)} \in \sigma \left(H_n^{(x)} \right) \cap I_n$ and

(ii) the corresponding $\ell^2(B_n(x))$ -normalized eigenvector $\psi^{(x)}$ satisfies

$$\sum_{y \in \partial B_n(x)} |\psi^{(x)}(y)|^2 \leq \tau_n^2.$$

For $x \in \mathcal{F}_n$, we extend the function $\psi^{(x)}$ by zero to \mathcal{G}_n and for $x \in \mathcal{G}_n \setminus \mathcal{F}_n$ we set $\xi^{(x)} = 0$ and $\psi^{(x)} \equiv 0$ on \mathcal{G}_n .

For $x \in \mathcal{F}_n$, the function $\psi^{(x)}$ is an approximate eigenvector in the sense of Lemma 6. Indeed, $\|\psi^{(x)}\| = 1$ and $\psi^{(x)}(y) = 0$ for $y \notin B_n(x)$. Moreover, since $H_n^{(x)}$ and H_n coincide locally in $B_n(x)$ this also implies that $H_n \psi^{(x)}(y) = \xi^{(x)} \psi^{(x)}(y)$ for $y \in B_n(x)$. Thus from Lemma 6, we get the following error estimate: for any $x \in \mathcal{F}_n$ there exists an eigenvalue E_j of H_n such that the bound

$$|E_j - \xi^{(x)}| \leq \sqrt{K} \tau_n \tag{23}$$

holds.

Remark. Loosely speaking, if a vertex x lies in \mathcal{F}_n then an eigenvector of H_n should be located close to x . Indeed, the bound (17) from Lemma 6 shows that typically the eigenvector ϕ_j of H_n corresponding to E_j has large overlap with $\psi^{(x)}$. This suggests that \mathcal{F}_n is decomposed into clusters, neighborhoods around the vertices $x_j, j = 1, \dots, \nu_n(I)$, the localization centers of the eigenvectors of H_n . So to construct a process close to ν_n , we have to count the number of clusters and we have to thin out the set \mathcal{F}_n .

To construct a further subset \mathcal{E}_n of \mathcal{F}_n , we define, for $x \in \mathcal{F}_n$, a cluster

$$\mathcal{C}(x) = \left\{ y \in \mathcal{F}_n \cap B_{2R_n}(x) : \left| \xi^{(x)} - \xi^{(y)} \right| \leq 2\sqrt{K} \tau_n \right\}.$$

For $x \in \mathcal{G}_n$, we set $x \in \mathcal{E}_n$ if and only if $x \in \mathcal{F}_n$ and

$$\left| \psi^{(x)}(x) \right| \geq \left| \psi^{(y)}(y) \right|$$

for all $y \in \mathcal{C}(x)$. We write $\eta_n^{(E)}(I)$ for the number of vertices in \mathcal{E}_n . Finally, for each $x \in \mathcal{G}_n$ we define a random Bernoulli variable $b_x = 1$ if $x \in \mathcal{E}_n$ and $b_x = 0$; otherwise so that

$$\eta_n^{(E)}(I) = |\mathcal{E}_n| = \sum_{x \in \mathcal{G}_n} b_x. \tag{24}$$

We remark that the random sets \mathcal{F}_n and \mathcal{E}_n and the random variables $(b_x)_{x \in \mathcal{G}_n}$ depend on the interval I and on $E \in I_0$. However, we fix I and E throughout this section, so we often omit writing the dependence on I and E .

6.2. Proof of Propostion 4: Independence and Poisson Approximation

To show that η_n is close to a Poisson process, we use Lemma 10. To apply this result to the auxiliary process, we need the following lemmas.

Lemma 11. *Assume that $x_1, x_2 \in \mathcal{G}_n$ satisfy $d(x_1, x_2) > 6R_n$. Then, b_{x_1} and b_{x_2} are independent.*

Proof. For any vertex $y \in \mathcal{G}_n$, the event $\{y \in \mathcal{F}_n\}$ depends only on the eigenvalues and eigenvectors of the operator $H_n^{(y)}$ and is thus measurable with respect to $(\omega_u)_{u \in B_n(y)}$. Accordingly, also the random variables $\xi^{(y)}$ and $|\psi^{(y)}(y)|$ are measurable with respect to $(\omega_u)_{u \in B_n(y)}$. In turn, any event $\{x \in \mathcal{E}_n\}$ depends only on the random variables $\xi^{(y)}$ and $|\psi^{(y)}(y)|$ with $y \in B_{2R_n}(x)$. Hence, the event $\{x \in \mathcal{E}_n\}$ is measurable with respect to

$$\bigcup_{y \in B_{2R_n}(x)} (\omega_u)_{u \in B_n(y)} = (\omega_u)_{u \in B_{3R_n}(x)}.$$

We see that the Bernoulli variable b_x depends only on the random potential in the neighborhood $B_{3R_n}(x)$. This implies the result. \square

Lemma 12. *For two distinct vertices $x, y \in \mathcal{G}_n$, we have*

$$\mathbb{E} [b_x b_y] \leq 18 \|\rho\|_\infty^2 \left(|I| + 6\sqrt{K}n\tau_n \right)^2 K^{2R_n} n^{-2}.$$

Proof. We consider the subset $B_n(x, y) = B_n(x) \cup B_n(y)$ and we define the operator $H_n^{(x,y)}$ to be the restriction of H_n to $\ell^2(B_n(x, y))$ with Neumann boundary conditions.

Let $\tilde{I}_n \subset \mathbb{R}$ denote the interval constructed from I_n by enlarging it by $3\sqrt{K}\tau_n$ at both ends. Assume that $b_x = b_y = 1$. Then, we will show that the operator $H_n^{(x,y)}$ has almost surely more than one eigenvalue in \tilde{I}_n .

We argue by contradiction and assume that there is only one eigenvalue in \tilde{I}_n . The assumption $b_x = b_y = 1$ implies that both x and y lie in \mathcal{F}_n . By definition of \mathcal{F}_n , there are eigenvalues $\xi^{(x)} \in \sigma(H_n^{(x)}) \cap I_n$ and $\xi^{(y)} \in \sigma(H_n^{(y)}) \cap I_n$ and the corresponding eigenvectors satisfy

$$\sum_{u \in \partial B_n(x)} |\psi^{(x)}(u)|^2 \leq \tau_n^2 \quad \text{and} \quad \sum_{u \in \partial B_n(y)} |\psi^{(y)}(u)|^2 \leq \tau_n^2.$$

We extend $\psi^{(x)}$ and $\psi^{(y)}$ by zero to $B_n(x, y)$ and use them as approximate eigenvectors for the operator $H_n^{(x,y)}$. Applying Lemma 6 with $\mathcal{G} = B_n(x, y)$, $H = H_n^{(x,y)}$, and $\lambda = \xi^{(x)}$ and $\lambda = \xi^{(y)}$, respectively, yield eigenvalues $\mu^{(x)}$ and $\mu^{(y)}$ of $H_n^{(x,y)}$ satisfying $|\mu^{(x)} - \xi^{(x)}| \leq \sqrt{K}\tau_n$ and $|\mu^{(y)} - \xi^{(y)}| \leq \sqrt{K}\tau_n$. Since both $\xi^{(x)}$ and $\xi^{(y)}$ lie in I_n , we see that both $\mu^{(x)}$ and $\mu^{(y)}$ lie in \tilde{I}_n . By assumption, it follows that $\mu^{(x)} = \mu^{(y)}$ and that

$$\left| \xi^{(x)} - \xi^{(y)} \right| \leq 2\sqrt{K}\tau_n. \tag{25}$$

Moreover, the assumption that \tilde{I}_n contains only one eigenvalue of $H_n^{(x,y)}$ also implies that there is only one eigenvalue in $(\xi^{(x)} - 2\sqrt{K}\tau_n, \xi^{(x)} + 2\sqrt{K}\tau_n)$ and that there is only one eigenvalue in $(\xi^{(y)} - 2\sqrt{K}\tau_n, \xi^{(y)} + 2\sqrt{K}\tau_n)$. Thus, Lemma 6 also gives existence of an eigenvector Φ of $H_n^{(x,y)}$ corresponding to $\mu^{(x)} = \mu^{(y)}$ such that

$$\left| \langle \Phi, \psi^{(x)} \rangle \right| \geq 1 - K\tau_n^2(2\sqrt{K}\tau_n)^{-2} = \frac{3}{4} \quad \text{and} \quad \left| \langle \Phi, \psi^{(y)} \rangle \right| \geq \frac{3}{4}.$$

Thus, Lemma 8 yields $|\langle \psi^{(y)}, \psi^{(x)} \rangle| > 0$ and in particular

$$d(x, y) \leq 2R_n. \tag{26}$$

The bounds (25) and (26) show that $y \in \mathcal{C}(x)$ and $b_x = b_y = 1$ implies $|\psi^{(x)}(x)| = |\psi^{(y)}(y)|$, which is false almost surely for $x \neq y$. We have reached a contradiction.

We have shown that $b_x b_y = 1$ implies that the operator $H_n^{(x,y)}$ has almost surely more than one eigenvalue in the interval \tilde{I}_n . Hence, the Minami estimate (16) gives

$$\mathbb{E}[b_x b_y] = \mathbb{P}[b_x b_y = 1] \leq \mathbb{P}\left[|\sigma(H_n^{(x,y)}) \cap \tilde{I}_n| \geq 2\right] \leq \frac{1}{2} \|\rho\|_\infty^2 |\tilde{I}_n|^2 |B_n(x, y)|^2.$$

Inserting $|\tilde{I}_n| = (|I| + 6\sqrt{K}\tau_n n)/n$ and $|B_n(x, y)| \leq 2|B_n(x)| \leq 6K^{R_n}$ completes the proof. \square

Proof of Proposition 4. In view of Lemma 11, we can apply Lemma 10 with $\mathcal{G} = \mathcal{G}_n$ and $\varrho = 6R_n$. It remains to estimate $\mathbb{E}[b_x]$ and $\mathbb{E}[b_x b_y]$ for $x \neq y$.

From Lemma 12, (1), and $\tau_n \leq 1/6\sqrt{K}n$ we obtain, for any $x \in \mathcal{G}_n$,

$$\begin{aligned} \sum_{y \in B_{6R_n}(x) \setminus \{x\}} \mathbb{E}[b_x b_y] &\leq 18 \|\rho\|_\infty^2 \left(|I| + 6\sqrt{K}n\tau_n\right)^2 K^{2R_n} n^{-2} |B_{6R_n}(x)| \\ &\leq 54 \|\rho\|_\infty^2 (|I| + 1)^2 K^{8R_n} n^{-2}. \end{aligned}$$

It follows that

$$\sum_{x \in \mathcal{G}_n} \sum_{y \in B_{6R_n}(x) \setminus \{x\}} \mathbb{E}[b_x b_y] \leq 54 \|\rho\|_\infty^2 (|I| + 1)^2 K^{8R_n} n^{-1}. \tag{27}$$

To estimate $\mathbb{E}[b_x]$, we note that $b_x = 1$ implies, in particular, $x \in \mathcal{F}_n$. So applying the Wegner estimate (15) we obtain, for all $x \in \mathcal{G}_n$,

$$\begin{aligned} \mathbb{E}[b_x] = \mathbb{P}[b_x = 1] &\leq \mathbb{P}\left[|\sigma(H_n^{(x)}) \cap I_n| \geq 1\right] \\ &\leq \|\rho\|_\infty |I_n| |B_n(x)| \leq 3 \|\rho\|_\infty |I| K^{R_n} n^{-1}. \end{aligned}$$

Hence, it follows that

$$\sum_{x \in \mathcal{G}_n} \sum_{y \in B_{6R_n}} \mathbb{E}[b_x] \mathbb{E}[b_y] \leq 27 \|\rho\|_\infty^2 |I|^2 K^{8R_n} n^{-1}. \tag{28}$$

Thus, the claim follows from Lemma 10 and (27) and (28). \square

6.3. Proof of Proposition 5: Error Estimates

In this subsection, we show that $\eta_n^{(E)}$ is close to the eigenvalue process $\nu_n^{(E)}$. We consider an open interval $I_0 \subset \mathbb{R}$ such that the fractional moment bound (5) holds with exponent $\mu_s > 2 \ln K$. We fix $E \in I_0$ and assume that $n \in \mathbb{N}$ is large enough such that $I_n \subset I_0$.

We choose $a, b \in \mathbb{R}$ such that $I = (a, b)$ and $I_n = (E + a/n, E + b/n)$. Let now $E_j, j = 1, \dots, \nu_n^{(E)}(I)$, denote the eigenvalues of H_n in I_n . As before, for an eigenvalue $E_j \in \sigma(H_n)$ we denote by ϕ_j the corresponding eigenvector.

Let us write $\epsilon_n = 6\sqrt{K}\tau_n$ for short and consider the set I_n^e , a small set around the end points of the interval I_n :

$$I_n^e = (E + a/n - \epsilon_n, E + a/n + \epsilon_n) \cup (E + b/n - \epsilon_n, E + b/n + \epsilon_n).$$

By $N(I_n^e)$, we denote the number of eigenvalues of H_n in I_n^e . The Wegner estimate (15) implies

$$\mathbb{P}[N(I_n^e) \geq 1] \leq \|\rho\|_\infty |I_n^e| n \leq 4\|\rho\|_\infty \epsilon_n n. \tag{29}$$

To apply Lemma 6, we also need the following estimate. By $\Delta(\epsilon_n)$, we denote the event that there are two eigenvalues $E_j, E_k \in \sigma(H_n) \cap I_n$ with $j \neq k$ satisfying $|E_j - E_k| \leq 2\epsilon_n$. By Lemma, 7 we have

$$\mathbb{P}[\Delta(\epsilon_n)] \leq 4\|\rho\|_\infty^2 |I_n| \epsilon_n n^2 = 4\|\rho\|_\infty^2 |I| \epsilon_n n. \tag{30}$$

Finally, we remark that the event $\{|\psi^{(x)}(x)| = |\psi^{(y)}(y)|\}$ has probability zero for any two distinct vertices $x, y \in \mathcal{F}_n$. Thus, the event

$$O_n = \bigcup_{\substack{(x,y) \in \mathcal{F}_n \times \mathcal{F}_n \\ x \neq y}} \left\{ |\psi^{(x)}(x)| = |\psi^{(y)}(y)| \right\}$$

has probability zero. By $\Delta(\epsilon_n)^c$ and O_n^c , we denote the complement of the corresponding event.

Lemma 13. *On the event $\Omega_n = \{N(I_n^e) = 0\} \cap \Delta(\epsilon_n)^c \cap O_n^c$, we have $\eta_n^{(E)}(I) \leq \nu_n^{(E)}(I)$.*

Proof. Recall the definition of η_n as the number of vertices in \mathcal{E}_n ; see (24). We write $\mathcal{E}_n = \{e_i\}_{i=1}^{\eta_n}$. By definition, each vertex e_i lies in \mathcal{F}_n . Thus, there is $\xi^{(e_i)} \in \sigma(H_n^{(e_i)}) \cap I_n$ and $E_{j_i} \in \sigma(H_n)$ such that (23) holds for E_{j_i} and $\xi^{(e_i)}$. On the event Ω_n , we have $N(I_n^e) = 0$ and it follows that $E_{j_i} \in I_n$. Hence, with each vertex $e_i, i = 1, \dots, \eta_n$, we can associate an eigenvalue $E_{j_i} \in \sigma(H_n) \cap I_n$.

Now, assume that one eigenvalue $E_j \in \sigma(H_n) \cap I_n$ is associated with two vertices $e_i, e_k \in \mathcal{E}_n$ such that $j_i = j_k$ and $E_{j_i} = E_{j_k}$. Then, by (23), we have

$$\left| \xi^{(e_i)} - \xi^{(e_k)} \right| \leq \left| \xi^{(e_i)} - E_{j_i} \right| + \left| E_{j_k} - \xi^{(e_k)} \right| \leq 2\sqrt{K}\tau_n. \tag{31}$$

Recall from Sect. 6.1 that the existence of E_{j_i} and E_{j_k} is implied by Lemma 6 with $H = H_n$ and $\lambda = \xi^{(e_i)}$ and $\lambda = \xi^{(e_k)}$, respectively. On Ω_n , the event $\Delta(\epsilon_n)$ does not happen so that the eigenvalues of H_n in I_n have distance at least $2\epsilon_n$ from each other. We also have $N(I_n^e) = 0$ and thus all conditions of Lemma 6 are satisfied. Hence, the eigenvectors $\psi^{(e_i)}$ and $\psi^{(e_k)}$ corresponding to $\xi^{(e_i)}$ and $\xi^{(e_k)}$, respectively, satisfy

$$\left| \left\langle \psi^{(e_i)}, \phi_j \right\rangle \right|^2 \geq 1 - K\tau_n^2 \epsilon_n^{-2} \quad \text{and} \quad \left| \left\langle \psi^{(e_k)}, \phi_j \right\rangle \right|^2 \geq 1 - K\tau_n^2 \epsilon_n^{-2},$$

where ϕ_j denotes the eigenvector of H_n corresponding to $E_{j_i} = E_{j_k}$. Thus, Lemma 8 and the identity $\epsilon_n = 6\sqrt{K}\tau_n$ imply

$$\left| \left\langle \psi^{(e_i)}, \psi^{(e_k)} \right\rangle \right| \geq 1 - 2K\tau_n^2 \epsilon_n^{-2} > 0$$

and

$$d(e_i, e_k) \leq 2R_n. \tag{32}$$

The relations (31) and (32) show that the vertex e_i belongs to the cluster $\mathcal{C}(e_k)$. Both vertices e_i and e_k lie in \mathcal{E}_n , so it follows that $|\psi^{(e_i)}(e_i)| = |\psi^{(e_k)}(e_k)|$ and, thus, on O_n^c , $e_i = e_k$. Hence, for each vertex e_i , $i = 1, \dots, \eta_n$, there is a distinct eigenvalue $E_{j_i} \in \sigma(H_n) \cap I_n$ and the proof is complete. \square

To prove the lower bound $\eta_n^{(E)}(I) \geq \nu_n^{(E)}(I)$, we introduce the following local events. Similar as above, we denote by $\Delta^{(x)}(\epsilon_n)$ the event that there are two eigenvalues

$$\xi_j^{(x)}, \xi_k^{(x)} \in \sigma(H_n^{(x)}) \cap I_n$$

with $j \neq k$ satisfying $|\xi_j^{(x)} - \xi_k^{(x)}| \leq 2\epsilon_n$. Then Lemma 7 yields, for all $x \in \mathcal{G}_n$,

$$\mathbb{P} \left[\Delta^{(x)}(\epsilon_n) \right] \leq 4\|\rho\|_\infty^2 |I_n| \epsilon_n |B_n(x)|^2 \leq 36\|\rho\|_\infty^2 |I| K^{2R_n} \epsilon_n n^{-1}. \tag{33}$$

We also need the following estimate, a variant of Lemma 6. In the remainder of this section, we choose $\ln K < \mu < \mu_s - \ln K$ as in Proposition 3 and write $\mu_K = \mu - (\ln K)/2$ for short. Moreover, for $\phi \in \ell^2(\mathcal{G}_n)$ and $\hat{x} \in \mathcal{G}_n$, we write $\phi|_{B_n(\hat{x})}$ for the restricted vector that is set to be zero outside of $B_n(\hat{x})$.

Lemma 14. *Let ϕ be a $\ell^2(\mathcal{G}_n)$ -normalized eigenvector of H_n with corresponding eigenvalue E . Assume that there is a constant*

$$0 < C_n \leq \sqrt{\frac{e^{2\mu_K} - 1}{3}} e^{\mu_K R_n} \tag{34}$$

and a vertex $\hat{x} \in \mathcal{G}_n$, such that, for all $x \in \mathcal{G}_n$,

$$|\phi(x)| \leq C_n e^{-\mu d(x, \hat{x})}. \tag{35}$$

Then there exists an eigenvalue $\xi \in \sigma(H_n^{(\hat{x})})$ satisfying

$$|E - \xi| \leq \sqrt{3K} C_n e^{-\mu_K (R_n + 1)}$$

and the restricted vector $\phi|_{B_n(\hat{x})}$ satisfies

$$\|\phi|_{B_n(\hat{x})}\|^2 \geq 1 - \frac{3}{2} C_n^2 \frac{e^{-2\mu_K R_n}}{e^{2\mu_K} - 1}. \tag{36}$$

Assume, in addition, that the event $\Delta^{(\hat{x})}(\epsilon_n)$ does not happen. Then the $\ell^2(B_n(\hat{x}))$ -normalized eigenvector ψ of $H_n^{(\hat{x})}$ corresponding to ξ satisfies

$$\left| \langle \psi, \tilde{\phi} \rangle \right|^2 \geq 1 - 3K C_n^2 e^{-2\mu_K (R_n + 1)} \epsilon_n^{-2},$$

where $\tilde{\phi}$ denotes the normalized vector $\tilde{\phi} = \|\phi|_{B_n(\hat{x})}\|^{-1} \phi|_{B_n(\hat{x})}$.

Proof. We have to show that $\tilde{\phi}$ is an approximate eigenvector for the operator $H_n^{(\hat{x})}$ with domain $\ell^2(B_n(\hat{x}))$. The operators H_n and $H_n^{(\hat{x})}$ coincide in the interior of $B_n(\hat{x})$, such that

$$\left\| \left(H_n^{(\hat{x})} - E \right) \phi \right\|_{\ell^2(B_n(\hat{x}))}^2 = \sum_{x \in \partial B_n(\hat{x})} \left| \sum_{y \notin B_n(\hat{x}): d(y,x)=1} \phi(y) \right|^2.$$

Hence, from the Schwarz inequality and from (35) and (2), we obtain

$$\left\| \left(H_n^{(\hat{x})} - E \right) \phi \right\|_{\ell^2(B_n(\hat{x}))}^2 \leq K \sum_{\partial B_{R_n+1}(\hat{x})} |\phi(y)|^2 \leq \frac{3}{2} K C_n^2 e^{-2\mu_K(R_n+1)}. \tag{37}$$

To estimate the effect of the normalization of $\tilde{\phi}$, we use (35) and (2) again to get

$$\begin{aligned} \sum_{y \notin B_n(\hat{x})} |\phi(y)|^2 &= \sum_{m > R_n} \sum_{d(y,\hat{x})=m} |\phi(y)|^2 \\ &\leq \frac{3}{2} \sum_{m > R_n} K^m C_n^2 e^{-2\mu_K m} = \frac{3}{2} C_n^2 \frac{e^{-2\mu_K R_n}}{e^{2\mu_K} - 1}. \end{aligned}$$

Since ϕ is normalized, this implies (36). Inserting this into (37) and applying (34) to simplify yields

$$\begin{aligned} &\left\| \left(H_n^{(\hat{x})} - E \right) \tilde{\phi} \right\|_{\ell^2(B_n(\hat{x}))}^2 \\ &\leq \frac{3K C_n^2 e^{-2\mu_K(R_n+1)}}{2 - 3C_n^2(e^{2\mu_K} - 1)^{-1} e^{-2\mu_K R_n}} \leq 3K C_n^2 e^{-2\mu_K(R_n+1)}. \end{aligned}$$

The remainder of the proof follows the same arguments as the proof of Lemma 6. □

Now, we proceed to give the proof of the lower bound.

Lemma 15. *Let the constant $C_n > 0$ satisfy*

$$C_n \leq \tau_n^2 e^{(\mu - \ln K)R_n}. \tag{38}$$

Define the event

$$\Omega'_n = \{N(I_n^e) = 0\} \cap \Delta(\epsilon_n)^c \cap \{X_n \leq C_n\} \cap \bigcap_{x \in \mathcal{G}_n} \Delta^{(x)}(\epsilon_n)^c \cap O_n^c$$

with X_n from Proposition 3. Then, for n large enough, on Ω'_n we have

$$\eta_n^{(E)}(I) = \nu_n^{(E)}(I).$$

Proof. The upper bound follows directly from Lemma 13 and the fact that $\Omega'_n \subset \Omega_n$. Let us proceed to prove the lower bound $\eta_n \geq \nu_n$.

Here, we use the assumption that within I_0 the fractional moment bound (5) is satisfied: for n large enough, we have $I_n \subset I_0$ and Proposition 3 yields that, for each eigenvalue $E_j \in \sigma(H_n) \cap I_n$, the corresponding eigenvector ϕ_j

satisfies (8). In addition, on Ω'_n , we have $X_n(\omega) \leq C_n$. Thus, there exists a vertex $x_j \in \mathcal{G}_n$ such that, for all $x \in \mathcal{G}_n$,

$$|\phi_j(x)| \leq C_n e^{-\mu d(x, x_j)}. \tag{39}$$

First, we need to show that $x_j \in \mathcal{F}_n$, so we have to verify conditions (i) and (ii) from Sect. 6.1.

Note that (38) combined with $\tau_n \leq 1/6\sqrt{K}n$ implies (34). Hence, Lemma 14 applied to ϕ_j yields the existence of an eigenvalue $\xi^{(x_j)} \in \sigma(H_n^{(x_j)})$ satisfying $|E_j - \xi^{(x_j)}| \leq \sqrt{3K}C_n e^{-\mu\kappa(R_n+1)}$. From (38) and the fact that $\tau_n \leq 1/6\sqrt{K}n$, we conclude

$$|E_j - \xi^{(x_j)}| \leq \sqrt{K}\tau_n.$$

On Ω'_n , we have $N(I_n^e) = 0$ and it follows that $\xi^{(x_j)} \in I_n$ and that x_j satisfies (i).

To verify condition (ii), let $\psi^{(x_j)}$ be the eigenvector corresponding to $\xi^{(x_j)}$ and let $\tilde{\phi}_j$ denote the normalized restricted eigenvector corresponding to E_j :

$$\tilde{\phi}_j = \left\| \phi_j|_{B_n(x_j)} \right\|^{-1} \phi_j|_{B_n(x_j)}.$$

The event $\Delta^{(x_j)}(\epsilon_n)$ does not occur on Ω'_n ; thus Lemma 14 implies

$$\left| \langle \tilde{\phi}_j, \psi^{(x_j)} \rangle \right|^2 \geq 1 - 3KC_n^2 e^{-2\mu\kappa(R_n+1)} \epsilon_n^{-2}.$$

Hence, by Lemma 9 and (2), we find

$$\sum_{y \in \partial B_n(x_j)} \left| \psi^{(x_j)}(y) \right|^2 \leq 2 \left(\sum_{y \in \partial B_n(x_j)} \left| \tilde{\phi}_j(y) \right|^2 + \frac{9}{2} K^{R_n+1} C_n^2 e^{-2\mu\kappa(R_n+1)} \epsilon_n^{-2} \right).$$

By (2), (36), (34), and (39), the first summand is bounded by $3C_n^2 e^{-2\mu\kappa R_n}$. Inserting the identities $\epsilon_n = 6\sqrt{K}\tau_n$ and $\mu_K = \mu - (\ln K)/2$ and using the fact that $\tau_n \leq 1/6\sqrt{K}n$, we obtain

$$\begin{aligned} \sum_{y \in \partial B_n(x_j)} \left| \psi^{(x_j)}(y) \right|^2 &\leq \frac{1}{4} C_n^2 K^{2R_n+1} e^{-2\mu(R_n+1)} \tau_n^{-2} \left(\frac{24e^{2\mu} \tau_n^2}{K^{R_n+1}} + 1 \right) \\ &\leq C_n^2 e^{-2(\mu - \ln K)R_n} \tau_n^{-2}. \end{aligned}$$

By (38), we see that the right-hand side is bounded by τ_n^2 so that x_j satisfies condition (ii). Hence, $x_j \in \mathcal{F}_n$.

To associate with E_j a vertex from \mathcal{E}_n , write $\mathcal{E}_n = \{e_i\}_{i=1}^{\eta_n}$ and consider

$$e_{i_j} = \arg \max_{y \in \mathcal{C}(x_j)} \left| \psi^{(y)}(y) \right|.$$

We will now show that

$$e_{i_j} \in \mathcal{E}_n. \tag{40}$$

Let $u \in \mathcal{C}(e_{i_j})$. Then, since $e_{i_j} \in \mathcal{C}(x_j)$,

$$\left| \xi^{(u)} - \xi^{(x_j)} \right| \leq \left| \xi^{(u)} - \xi^{(e_{i_j})} \right| + \left| \xi^{(e_{i_j})} - \xi^{(x_j)} \right| \leq 4\sqrt{K}\tau_n.$$

By (23), there is an eigenvalue $E^{(u)} \in \sigma(H_n)$ such that $|E^{(u)} - \xi^{(u)}| \leq \sqrt{K}\tau_n$. Combining this with $6\sqrt{K}\tau_n = \epsilon_n$, we get

$$|E^{(u)} - E_j| \leq |E^{(u)} - \xi^{(u)}| + |\xi^{(u)} - \xi^{(x_j)}| + |\xi^{(x_j)} - E_j| \leq 6\sqrt{K}\tau_n = \epsilon_n.$$

On Ω'_n the event $\Delta(\epsilon_n)$ does not happen, so we conclude that $E^{(u)} = E_j$. In particular, we obtain

$$|\xi^{(u)} - \xi^{(x_j)}| \leq |\xi^{(u)} - E^{(u)}| + |E_j - \xi^{(x_j)}| \leq 2\sqrt{K}\tau_n.$$

Moreover, we can apply Lemma 6 using the same argument that leads to (32) to show $d(u, x_j) \leq 2R_n$ and we find $u \in \mathcal{C}(x_j)$. This shows that $\mathcal{C}(e_{i_j}) \subset \mathcal{C}(x_j)$, so that

$$e_{i_j} = \arg \max_{y \in \mathcal{C}(e_{i_j})} |\psi^{(y)}(y)|.$$

On O_n^c , this proves (40). So with each eigenvalue $E_j \in \sigma(H_n) \cap I_n$, $j = 1, \dots, \nu_n$, we can associate a vertex $e_{i_j} \in \mathcal{E}_n$.

Assume now that $e_{i_j} = e_{j_k}$ for $j, k \in \{1, \dots, \nu_n\}$. In particular, we get $\xi^{(e_{i_j})} = \xi^{(e_{i_k})}$ and, since $e_{i_j} \in \mathcal{C}(x_j)$ and $e_{i_k} \in \mathcal{C}(x_k)$,

$$\begin{aligned} |E_j - E_k| &\leq |E_j - \xi^{(x_j)}| + |\xi^{(x_j)} - \xi^{(e_{i_j})}| + |\xi^{(x_k)} - \xi^{(e_{i_k})}| + |\xi^{(x_k)} - E_k| \\ &\leq 6\sqrt{K}\tau_n = \epsilon_n. \end{aligned}$$

On Ω'_n , it follows that $E_j = E_k$. So with each eigenvalue E_j , $j = 1, \dots, \nu_n$, we can associate a distinct vertex e_{i_j} . This proves the lower bound $\eta_n \geq \nu_n$ and completes the proof. \square

With Lemma 15 at hand, we can complete the proof of Proposition 5.

Proof of Proposition 5. We note the obvious bounds $\nu_n^{(E)}(I) \leq n$ and $\eta_n^{(E)}(I) \leq n$. Thus, Lemma 15 shows

$$\mathbb{E} \left[\left| \nu_n^{(E)}(I) - \eta_n^{(E)}(I) \right| \right] \leq n\mathbb{P} [\Omega_n^c]$$

and

$$\mathbb{E} \left[\left| e^{-t\nu_n^{(E)}(I)} - e^{-t\eta_n^{(E)}(I)} \right| \right] \leq \mathbb{P} [\Omega_n^c]$$

for all $t \geq 0$. Combining (29), (30), (7), and (33) yields

$$\mathbb{P} [\Omega_n^c] \leq 4\|\rho\|_\infty \epsilon_n n + 4\|\rho\|_\infty^2 |I| \epsilon_n n + C_{\text{loc}} n |I_n| C_n^{-2(1-a)/a} + 36\|\rho\|_\infty^2 |I| K^{2R_n} \epsilon_n$$

and the claim follows from the identities $|I_n| = |I|/n$ and $\epsilon_n = 6\sqrt{K}\tau_n$. \square

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Appendix A. Exponential Localization of Eigenvectors

In this section, we show that exponentially decaying bounds on fractional moments of the Green function imply exponential localization of eigenvectors as stated in Proposition 3.

Proof of Proposition 3. This result follows from an estimate of the eigenfunction correlator in the spirit of [1, 2, 6]. Let $P_{\{\cdot\}}$ denote the spectral projection corresponding to $H_n(\omega)$ and for $x, y \in \mathcal{G}_n$ define the eigenfunction correlator

$$Q(x, y; I) = \sum_{E \in \sigma(H_n) \cap I} |\langle \delta_x, P_{\{E\}} \delta_y \rangle| = \sum_{E \in \sigma(H_n) \cap I} |\phi_E(x)| |\phi_E(y)|,$$

where ϕ_E denote the eigenvector corresponding to $E \in \sigma(H_n(\omega))$. It was noticed in [1] that the eigenfunction correlator can be bounded in terms of fractional moments of the Green function. Using the Schwarz inequality, we estimate

$$\begin{aligned} \sum_{E \in \sigma(H_n) \cap I} |\phi_E(x)| |\phi_E(y)| &\leq \left(\sum_{E \in \sigma(H_n) \cap I} |\phi_E(x)|^{2-s} |\phi_E(y)|^s \right)^{1/2} \\ &\quad \times \left(\sum_{E \in \sigma(H_n) \cap I} |\phi_E(x)|^s |\phi_E(y)|^{2-s} \right)^{1/2}. \end{aligned}$$

Let σ_x denote the spectral measure of $H_n(\omega)$ associated with vertex $x \in \mathcal{G}_n$. Then it follows that

$$\begin{aligned} Q(x, y; I) &\leq \left(\int_I |\langle \delta_x, P_{\{E\}} \delta_x \rangle|^{-s} |\langle \delta_x, P_{\{E\}} \delta_y \rangle|^s \sigma_x(dE) \right)^{1/2} \\ &\quad \times \left(\int_I |\langle \delta_y, P_{\{E\}} \delta_y \rangle|^{-s} |\langle \delta_y, P_{\{E\}} \delta_x \rangle|^s \sigma_y(dE) \right)^{1/2}. \end{aligned}$$

Now, one can apply the results of [1, Sect. 3] to estimate

$$\mathbb{E}[Q(x, y; I)] \leq C \int_I \mathbb{E}[|G_{n,\alpha}(x, y; E + i0)|^s] dE;$$

see also [9, Thm. 6.8] for a streamlined presentation. Hence, the fractional moment bound (5) implies

$$\mathbb{E}[Q(x, y; I)] \leq C|I|e^{-\mu_s d(x,y)} \tag{41}$$

with a uniform constant $C > 0$.

Let us now choose μ such that $\ln K < \mu < \mu_s - \ln K$ and estimate

$$Q(x, y; I) = Q(x, y; I)e^{\mu d(x,y)} e^{-\mu d(x,y)} \leq A(x, \omega)e^{-\mu d(x,y)}, \tag{42}$$

where we have introduced the random variable

$$A(x, \omega) = \sum_{v \in \mathcal{G}_n} e^{\mu d(x,v)} Q(x, v; I).$$

By (41) and (2), we have, for each fixed $x \in \mathcal{G}_n$,

$$\mathbb{E}[A(x)] \leq C|I| \sum_{v \in \mathcal{G}_n} e^{-(\mu_s - \mu)d(x,v)} \leq \frac{3}{2} \frac{C|I|}{1 - Ke^{-(\mu_s - \mu)}}.$$

Now we set, for $1 < a < 2 - \ln K/\mu$,

$$X_n(\omega) = \left(\frac{3}{2} \frac{1}{1 - Ke^{-(2-a)\mu}} \right)^{1/(2a-2)} \left(\sum_{x \in \mathcal{G}_n} A(x, \omega) \right)^{a/(2a-2)}$$

and we obtain

$$\mathbb{E} \left[X_n^{2(a-1)/a} \right] \leq \frac{3}{2} \frac{C|I|n}{1 - Ke^{-(\mu_s - \mu)}} \left(\frac{3}{2} \frac{1}{1 - Ke^{-(2-a)\mu}} \right)^{1/a}.$$

This yields (7) with

$$C_{\text{loc}} = \frac{3}{2} \frac{C}{1 - Ke^{-(\mu_s - \mu)}} \left(\frac{3}{2} \frac{1}{1 - Ke^{-(2-a)\mu}} \right)^{1/a}.$$

To show that (8) follows, we choose, for each eigenvector ϕ_j with eigenvalue $E_j \in I$, the random vertex $x_j(\omega) = \arg \max |\phi_j(x)|$. Then the definition of $Q(x, y; I)$ combined with estimate (42) implies, for all $x \in \mathcal{G}_n$,

$$|\phi_j(x)| \leq \frac{1}{|\phi_j(x_j)|} Q(x, x_j; I) \leq \frac{A(x_j, \omega)}{|\phi_j(x_j)|} e^{-\mu d(x, x_j)}. \tag{43}$$

To estimate $|\phi_j(x_j)|$ from below, we note that the normalization of ϕ_j implies, for any $1 < a < 2 - \ln K/\mu$,

$$1 = \sum_{x \in \mathcal{G}_n} |\phi_j(x)|^2 = \sum_{x \in \mathcal{G}_n} |\phi_j(x)|^a |\phi_j(x)|^{2-a} \leq |\phi_j(x_j)|^a \sum_{x \in \mathcal{G}_n} |\phi_j(x)|^{2-a}.$$

Inserting (43) and then using (2) yield

$$\begin{aligned} 1 &\leq |\phi_j(x_j)|^{2a-2} A(x_j, \omega)^{2-a} \sum_{x \in \mathcal{G}_n} e^{-(2-a)\mu d(x, x_j)} \\ &\leq |\phi_j(x_j)|^{2a-2} A(x_j, \omega)^{2-a} \frac{3}{2} \frac{1}{1 - Ke^{-(2-a)\mu}}. \end{aligned}$$

Solving for $|\phi_j(x_j)|$, we obtain

$$|\phi_j(x_j)| \geq \left(\frac{2}{3} \left(1 - Ke^{-(2-a)\mu} \right) \right)^{1/(2a-2)} \frac{1}{A(x_j, \omega)^{(2-a)/(2a-2)}}.$$

Finally, we insert this estimate into (43) and we see that

$$\begin{aligned} |\phi_j(x)| &\leq \left(\frac{3}{2} \frac{1}{1 - Ke^{-(2-a)\mu}} \right)^{1/(2a-2)} A(x_j, \omega)^{a/(2a-2)} e^{-\mu d(x, x_j)} \\ &\leq X_n(\omega) e^{-\mu d(x, x_j)}. \end{aligned}$$

This completes the proof. □

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