



Resonances and Partial Delocalization on the Complete Graph

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Abstract. Random operators may acquire extended states formed from a multitude of mutually resonating local quasi-modes. This mechanics is explored here in the context of the random Schrödinger operator on the complete graph. The operator exhibits local quasi-modes mixed through a single channel. While most of its spectrum consists of localized eigenfunctions, under appropriate conditions it includes also bands of states which are delocalized in the ℓ^1 -though not in ℓ^2 -sense, where the eigenvalues have the statistics of Šeba spectra. The analysis proceeds through some general observations on the scaling limits of random functions in the Herglotz–Pick class. The results are in agreement with a heuristic condition for the emergence of resonant delocalization, which is stated in terms of the tunneling amplitude among quasi-modes.

1. Random Schrödinger Operator on the Complete Graph

1.1. The Operator and Its Phase Diagram

The random Schrödinger operator on the complete graph \mathcal{K}_M is given by the $M \times M$ matrix:

$$H_M := -|\varphi_0\rangle\langle\varphi_0| + \kappa_M V, \quad (1.1)$$

with

$$\langle\varphi_0| = (1, 1, \dots, 1)/\sqrt{M} \quad \text{and} \quad \kappa_M := \frac{\lambda}{\sqrt{2 \ln M}}. \quad (1.2)$$

The rank-one operator $-|\varphi_0\rangle\langle\varphi_0|$, which plays the role of the kinetic term, is a multiple of the adjacency matrix on the complete graph of M vertices ($\times(-1)/M$). The second term is a multiple of a random potential V , which is a diagonal matrix with independent entries $(V(1), \dots, V(M))$ with the common probability density

$$\varrho(v) := \exp(-v^2/2) / \sqrt{2\pi}. \quad (1.3)$$

The gaussian distribution is chosen here mainly for concreteness sake; most of the analysis can be adapted to other distributions with continuous densities, with suitable adjustments in the scale of the coupling constant $\lambda > 0$.

At $\lambda = 0$, the spectrum of H_M consists of two levels. The ground state is non-degenerate, at energy $E = -1$ and given by the “extended state” φ_0 , and the other energy level is $(M - 1)$ -fold degenerate, at energy $E = 0$. The degeneracy is split as soon as $\lambda \neq 0$. For $M \gg 1$, as λ is increased the hitherto degenerate levels spread at rates proportional to λ , being asymptotically dense in the interval $[-\lambda, \lambda]$.

This model is studied here as a case study of resonant delocalization. The δ function states which are localized at sites of unusually high values of the potential V (whose maximum is typically close to $\sqrt{2 \ln M}$) form approximate eigenfunctions, or “quasi -modes”. The kinetic term allows tunneling between such states, and under the right conditions the operator’s eigenfunctions take the form of hybridized mixtures of localized states. Of particular interest is the consequent emergence of a narrow spectral band at which the eigenstates are semi-delocalized. Following is an outline of the results established in this work.

The energy of the extended state which starts as φ_0 changes with λ at a much slower rate ($o(1)$). At $\lambda = 1 + o(1)$ a first order transition occurs at the spectral edge: the extended state is passed through an ‘avoided crossing’ by a localized state which is supported mainly on the minimum of the potential (Theorem 2.1). As λ is further increased the operator continues to have an extended state at an energy close to (-1) , which is repeatedly passed by localized states as λ is increased over the interval $(1, \sqrt{2})$. The extended state and the localized state passing it hybridize for only a “brief instant” on the λ scale. The signature of that is that for $\lambda < \sqrt{2}$, in energies away from $E = 0$ where the quasi-modes are initially bunched up, at any a-priori chosen value of λ the operator has only strongly localized states except for one which is a slightly perturbed version of $|\varphi_0\rangle$ (Theorems 6.3 and 6.4).

Another transition happens when λ passes the value $\sqrt{2}$. Beyond that more massive hybridization occurs and a small band of semi-extended states emerges at energies $E = -1 + o(1)$ (Theorem 6.1). Similar semi-delocalization is found at energies close to $E = 0$ for all $\lambda > 0$.

1.2. The Spectral Scaling Limit

In discussing the random operator’s spectrum in the limit $M \rightarrow \infty$ we consider its blown-up picture under a magnification in which it appears as a random point process with mean spacing of order 1. In the regime of resonant delocalization the limit is given by what we call the Šeba process, after the prior appearance of similar spectra in the Šeba billiard. It appears also in other contexts, which are mentioned in Sect. 3 where the process is defined.

In Sect. 4, we present some tools which yield a classification of the different possibilities for the scaling limits of spectra of similar characteristic equations, and for establishing convergence. These may be of independent interest, involving some general results about the possible limits for the set of roots of

a characteristic equation which is expressed in terms of random functions in the Herglotz–Pick class.

In Sect. 7, we compare the results on the emergence of narrow bands or semi-delocalization with a heuristic criterion which is based on a condition for hybridization among two resonating quasi-modes. For this purpose, the standard notion of quasi-modes is enhanced here with a definition of tunneling amplitude which is natural for systems with random potential. The heuristic criterion is found to be of relevance also in the present context, even though the hybridization studied here is of somewhat more extensive nature. Also discussed there are the different notions of delocalization which are of relevance for operators with long range hopping.

1.3. Relation to Past Works

Our motivation to study resonant delocalization as a mechanism for the formation of bands of extended states was in part motivated by recent results on random Schrödinger operators on tree graphs [1, 2]. The mechanism plays there a role even in regimes of very low density of states, and it is of interest to understand its role in other systems with rapid growth of volume reached by n steps. (Tree graphs are also of interest from other perspectives; e.g., the location of the mobility edge is affected by multifractality effects on which more can be found in [1, 8, 14, 21].)

The random Schrödinger operator on the complete graph is particularly amenable to analysis, and, by a number of different methods, it has already attracted attention in various contexts: Anderson localization [9, 22], quantum chaos [4–6, 10–12, 20, 23], and adiabatic quantum computation [18]. Our discussion overlaps in part with [9], which focused more on the localization phase, and [22] which highlighted a SUSY calculation by which partial localization results were obtained. However, these works have not addressed the delocalization phenomena on which we focus. In particular, the transition in which many localized modes resonating through a single channel turn into a band of spatially delocalized states, seems to not have been discussed. The description of this phenomenon, and the general tools which are presented here, form the main points of this work.

2. The Macro and Micro Perspectives

2.1. The Spectral Range on the Macroscopic Scale

While our main focus here is on the nature of eigenfunctions, it is natural to first determine the range of the spectrum $\sigma(H_M)$ and the spectral density. This can be obtained simply through the following two observations:

1. Since $H_M = -|\varphi_0\rangle\langle\varphi_0| + \kappa_M V$ differs from $\kappa_M V$ by just a rank-one perturbation, the eigenvalues of the two interlace. Hence the spectral density of states of H_M is that of its potential part, and given by $\rho(E/\kappa_M)$ (as was already discovered through a somewhat more involved SUSY calculation [22]).

- Due to well understood large deviations, the values which the random potential V assumes over the M point set \mathcal{K}_M typically span the interval $[-\sqrt{2 \ln M}, \sqrt{2 \ln M}]$ up to fluctuations of order $1/\sqrt{\ln M}$. In the normalization selected in (1.1), for fixed λ , the two terms of H_M are (typically) of comparable norms:

$$\|T\| = 1, \quad \|\kappa_M V\| = \lambda + \mathcal{O}\left(\frac{1}{\sqrt{\ln M}}\right) \tag{2.1}$$

where the second equality holds in a probability sense. (This normalization may remind one of a familiar feature of the random energy model, cf. (A.1) for a precise statement.)

The emerging picture is summarized in the following statement, in which we employ the notion of the Hausdorff distance between two subsets of a metric space, here $I, J \subset \mathbb{R}$:

$$d_H(I, J) := \max \left\{ \sup_{x \in I} \inf_{y \in J} |x - y|, \sup_{y \in J} \inf_{x \in I} |x - y| \right\}. \tag{2.2}$$

Theorem 2.1 (Spectrum and ground state). *For the sequence of operators H_M , with $M \rightarrow \infty$ at fixed $\lambda > 0$:*

- For large M the spectrum $\sigma(H_M)$ of H_M is typically close, in the Hausdorff distance d_H , to the non-random set

$$S(\lambda) = \{-1\} \cup [-\lambda, \lambda] \tag{2.3}$$

in the sense that for any $\varepsilon > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P}(d_H(\sigma(H_M), S(\lambda)) > \varepsilon) = 0. \tag{2.4}$$

- The ground state energy and the corresponding ground state function ψ_0 satisfy with asymptotically full probability:

$$\begin{aligned} \text{for } \lambda \in (0, 1) : \quad & \min \sigma(H_M) = -1 - \kappa_M^2 + \mathcal{O}(\kappa_M^4), \\ & \frac{\|\psi_0\|_2}{\|\psi_0\|_\infty} = \Theta(\sqrt{M}), \\ \text{for } \lambda \in (1, \infty) : \quad & \min \sigma(H_M) = \min \sigma(\kappa_M V) + \frac{\mathcal{O}(1)}{M}, \\ & 1 \leq \frac{\|\psi_0\|_2}{\|\psi_0\|_\infty} \leq 1 + \frac{\mathcal{O}(1)}{\kappa_M \sqrt{M}}. \end{aligned} \tag{2.5}$$

In (2.5) we employ the following adaptation of the Bachmann–Landau notation: $f_M = \mathcal{O}(g_M)$ means that for all sufficiently large M , except for events of asymptotically vanishing probability $|f_M| \leq Cg_M$, and $f_M = \Theta(g_M)$ means that with similar exception $cg_M \leq f_M \leq Cg_M$ for some $C, c \in (0, \infty)$, with constants which are independent of the realization of the randomness. The proof of Theorem 2.1 is presented here in Appendix A.

2.2. The Characteristic Equation

Further insight can be obtained from the characteristic equations which determines the spectrum. A rank-one perturbation argument yields (as in [9]):

Proposition 2.2. *For any potential V which is non-degenerate (i.e. $V(x) \neq V(y)$ except for $x = y$) the spectrum of H_M consists of the collection of energies E for which*

$$F_M(E) := \frac{1}{M} \sum_{x=1}^M \frac{1}{\kappa_M V(x) - E} = 1, \tag{2.6}$$

and the corresponding eigenfunctions are given by:

$$\psi_E(x) = \frac{\text{Const.}}{\kappa_M V(x) - E}. \tag{2.7}$$

(To which it may be added that in the case discussed here degenerate potentials occur only with probability 0.)

Proof. The resolvent expansion $\frac{1}{H_M - z} = \sum_{n=0}^{\infty} \frac{1}{\kappa_M V - z} \left[|\varphi_0\rangle \langle \varphi_0| \frac{1}{\kappa_M V - z} \right]^n$ allows to deduce, by standard arguments, that for any $z \in \mathbb{C} \setminus \mathbb{R}$:

$$\frac{1}{H_M - z} = \frac{1}{\kappa_M V - z} + [1 - F_M(z)]^{-1} \frac{1}{\kappa_M V - z} |\varphi_0\rangle \langle \varphi_0| \frac{1}{\kappa_M V - z} \tag{2.8}$$

and, in particular, $\langle \varphi_0, (H_M - z)^{-1} \varphi_0 \rangle = (F_M(z)^{-1} - 1)^{-1}$. The spectrum and eigenfunctions of H_M are then read from the poles and residues of its resolvent. \square

Localization properties of the corresponding eigenfunctions (2.7) are expressed here through the ratios $\|\psi_E\|_p / \|\psi_E\|_{\infty}$, where

$$\|\psi_E\|_p^p = \sum_{x=1}^M |\psi_E(x)|^p, \quad \|\psi_E\|_{\infty} = \max_x |\psi_E(x)|. \tag{2.9}$$

and of particular relevance here will be the cases $p = 2$ and $p = 1$. The relation with the usual participation ratio is discussed in Sect. 7.2.

Our main objective is to identify conditions under which the eigenfunctions delocalize, and present a mechanism by which multiple eigenfunction hybridization occurs due to resonances among many local quasi-modes of small energy gaps.

2.3. The Microscopic Scale

As stated in (2.6), the eigenvalues of H_M form a level set of the random function $F_M(E)$. The mean gap between the values of $\kappa_M V$ in the vicinity of an energy $\mathcal{E} \in \mathbb{R}$, and thus also between the intertwining eigenvalues of H_M , is given by

$$\Delta_M(\mathcal{E}) := \frac{\kappa_M}{M \varrho(\mathcal{E}/\kappa_M)} = \sqrt{2\pi} \kappa_M / M^{1-(\mathcal{E}/\lambda)^2}. \tag{2.10}$$

Upon suitable amplification of the energy scale the spectrum acquires the form of a random point process whose mean spacing is of order one. Their locations are given in terms of the rescaled energy parameter

$$u := \frac{E - \mathcal{E}_M}{\Delta_M(\mathcal{E}_M)}, \tag{2.11}$$

where we also allow for the center of the scaling window (\mathcal{E}_M) to slightly vary with M . The rescaled eigenvalues $u_{M,n}$ are simple and intertwine with the collection of similarly rescaled values of the random potential $\kappa_M V$ in the vicinity of \mathcal{E}_M :

$$\omega_{M,n} := \frac{\kappa_M V(x_n) - \mathcal{E}_M}{\Delta_M(\mathcal{E}_M)}. \tag{2.12}$$

These points are labeled here in increasing order relative to the reference energy \mathcal{E}_M , so that:

$$\dots \leq \omega_{M,-1} \leq \omega_{M,0} \leq 0 < \omega_{M,1} < \dots, \quad \text{and } \omega_{M,n-1} < u_{M,n} \leq \omega_{M,n}. \tag{2.13}$$

In discussing the corresponding eigenfunctions, we chose the constant in (2.7) as $\text{Const.} = \Delta_M(\mathcal{E}_M) =: \Delta_M$, so that:

$$\psi_n(x_m) = \frac{1}{\omega_{M,m} - u_{M,n}}, \tag{2.14}$$

with (x_n) ordered by the values of $V(x)$, as above.

For a microscopic perspective on the characteristic equation (2.6), we rewrite F_M in terms of the rescaled energy parameter (2.11) as $F_M(E) = \frac{1}{M\Delta_M} \sum_n (\omega_{M,n} - u)^{-1}$. Splitting the sum into two parts, (2.6) can be presented as:

$$\boxed{S_{M,\omega}(u, L) = M\Delta_M - T_{M,\omega}(u, L)} \tag{2.15}$$

with

$$S_{M,\omega}(u, L) := \sum_n \frac{\mathbb{1}[|\omega_{M,n}| \leq L]}{\omega_{M,n} - u}, \quad T_{M,\omega}(u, L) := \sum_n \frac{\mathbb{1}[|\omega_{M,n}| > L]}{\omega_{M,n} - u}. \tag{2.16}$$

The cutoff parameter $L = L_M$ will be taken to increase with M at a rate such that

$$1 \ll L_M \ll M^{1-(\mathcal{E}_M/\lambda)^2} / \sqrt{\ln M}. \tag{2.17}$$

The lower bound on L in (2.10) (i.e. the requirement that $L \rightarrow \infty$) ensures that the restricted sum in $S_{M,\omega}(u, L)$ extends over all the terms in the “scaling window”, which is described by the limiting point process. The upper bound aims at keeping the sum in $S_{M,\omega}$ symmetrically balanced with respect to $u = 0$, in distributional sense. The term $T_{M,\omega}(u, L)$ captures the contribution of the singularities which fall beyond the range described by the scaling limit.

Within the above scaling window the functions $S_{M,\omega}(u, L)$ and $T_{M,\omega}(u, L)$ exhibit quite different dependence on the energy parameter u . In the next section we shall describe some relevant results on the limiting behavior of each of these terms. This would yield a short list of possible characteristics of the limiting behavior of the eigenvalue within the scaling window, and of the corresponding eigenstates.

3. The Šeba Process

A characteristic equation similar to (2.15) is known to occur also in other contexts, including Šeba graphs [20, 23], singular perturbations of certain chaotic billiards [4, 5, 10–12] and in random matrix theory [19]. In a number of examples, the singularities of $S_{M,\omega}$ converge to a Poisson point process, as in our case, however, the term on the right is replaced by a constant $\alpha \in \mathbb{R}$. We shall refer to the collection of the solutions of the corresponding equation as the α -Šeba process, after Ref. [23]. This point process would form one of the limiting situations encountered in our context. Let us turn to its definition.

3.1. Definition

Under the mapping which is described by (2.12) the collection of rescaled values of the random potential $(\omega_{M,n})$ converges in distribution to a Poisson process of intensity 1 (i.e. mean density dx). We refer to the latter as the *standard Poisson process*. Its configurations are countably infinite discrete random subsets $\omega \subset \mathbb{R}$.¹

For any given configuration $\omega \subset \mathbb{R}$, we shall refer to the following function as its Borel–Stieltjes transform

$$S_\omega(\zeta) := \lim_{n \rightarrow \infty} \sum_{v \in \omega \cap [-n, n]} \frac{1}{v - \zeta}, \tag{3.1}$$

assuming that the limit exists for all $\zeta \in \mathbb{C}^+$.

Proposition 3.1 (Theorem 4.1 in [3]). *For almost every realization ω of the standard Poisson process:*

1. *The limit (3.1) exists almost surely, simultaneously for all $\zeta \in \mathbb{C}^+$, and*

$$\lim_{\eta \rightarrow \infty} S_\omega(i\eta) = i\pi. \tag{3.2}$$

2. *Along the real line the random function $S_\omega(x)$ has only simple poles. Between any consecutive pair of such a gap Δv , the function increases monotonously from $-\infty$ to $+\infty$, with slope $S'_\omega(x) \geq 1/|\Delta v|^2$.*

¹ The scaling limit appears differently at the spectral edges, where the rescaled collection of potential values converges to a Poisson process with intensity $e^{\mp u} du$ (cf. [13]). We will, however, not need to discuss this process; the results presented here for the spectral edges can be obtained through less detailed information.

3. The thus defined Stieltjes–Poisson random function $S_\omega(\zeta)$ is a shift-covariant functional of ω , in the sense that

$$S_{\mathcal{T}_b\omega}(\zeta) = S_\omega(\zeta + b) \tag{3.3}$$

for all $b \in \mathbb{R}$ and $\zeta \in \mathbb{C}^+$, with $\mathcal{T}_b\omega$ the point configuration shifted to the left by b .

Definition 3.2. Let $S_\omega(\zeta)$ be the Borel–Stieltjes transform of a standard Poisson process ω whose points are labeled in increasing order relative to $0 \in \mathbb{R}$, and for $\alpha \in \mathbb{R}$ let $(u_n(\alpha, \omega) =: u_n)$ be the set of solutions of the equation

$$S_\omega(u) = -\alpha \tag{3.4}$$

ordered as in (2.13).

1. We refer to the intertwined point process $(\{\omega_n, u_n\})$, as the Šeba process at parameter $\alpha \in \mathbb{R}$.
2. For any given ω and $\alpha \in \mathbb{R}$, we refer to the points in (u_n) as the Šeba-eigenvalues, and for each $n \in \mathbb{Z}$ regard as the corresponding Šeba-eigenfunction the function $\Psi_n : \omega \mapsto \mathbb{C}$ defined by:

$$\Psi_n(v) := \Psi_n^{(\omega, \alpha)}(v) := \frac{1}{v - u_n}, \quad v \in \omega. \tag{3.5}$$

The terminology is motivated by the comparison of (2.6) and (2.7) with (3.4) and (3.5). The Šeba-eigenfunctions’ norms

$$\|\Psi_n\|_p := \left(\sum_{v \in \omega} |\Psi_n(v)|^p \right)^{1/p} \tag{3.6}$$

satisfy:

$$\|\Psi_n\|_\infty = \frac{1}{\text{dist}(u_n(\alpha, \omega), \omega)}, \quad \|\Psi_n\|_2^2 := S'_\omega(u_n). \tag{3.7}$$

For $p < \infty$, $\|\Psi_n\|_p$ may not yet capture all the relevant information about the eigenfunctions of a finite system whose spectrum within the scaling window the Šeba process may approximate, since the finite systems’ wavefunctions have also weight in regions which asymptotically will be off scale (see Sect. 6). Let us nevertheless note that due to the fact that $\lim_{n \rightarrow \infty} \omega_n/n = 1$ (by the ergodic theorem applied to the Poisson process), one has:

Lemma 3.3. For any Šeba eigenvalue process at $\alpha \in \mathbb{R}$, with probability one all the Šeba-eigenfunctions are almost surely ℓ^1 -delocalized,

$$\|\Psi_u\|_1 / \|\Psi_u\|_\infty = \infty, \tag{3.8}$$

yet also localized in the ℓ^2 -sense, satisfying:

$$\|\Psi_u\|_2 / \|\Psi_u\|_\infty \in (1, \infty) \tag{3.9}$$

and $\|\Psi_u\|_\infty \in (0, \infty)$ for all $u \in S_\omega^{-1}(\{\alpha\})$.

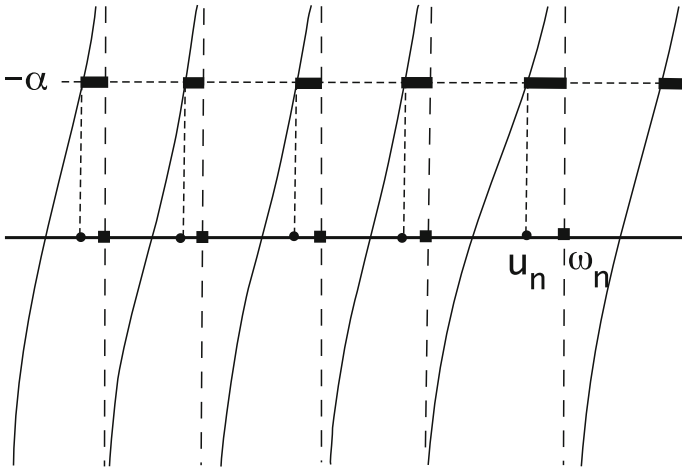


FIGURE 1. The Šeba process: a schematic depiction of the solutions of (3.4) which are discussed in Lemma 3.4. The function satisfies $S_\omega(u) \geq -\alpha$ throughout the intervals $[u_n, \omega_n]$, and $S_\omega(u) \leq -\alpha$ throughout $(\omega_{n-1}, u_n]$. This is used in the proof that for $|\alpha| \gg 1$ the solutions typically lie very close to points of ω , on a side determined by $\text{sgn}(\alpha)$

A different situation is found in the limiting case $|\alpha| \rightarrow \infty$: the Šeba eigenvalues coalesce then with the poles of S_ω and the Šeba eigenfunctions (once normalized in ℓ^p -sense at some $p > 1$) get to be totally localized, each at a single point of ω . This is quantified in the following estimate, which in Sect. 6 will be used in the discussion of the scaling limits in situations where one finds eigenfunction localization.

Lemma 3.4. *The Šeba process at level α satisfies for any $W, t > 0$:*

$$\mathbb{P} \left(\max_{n: |u_n| \leq W} \text{dist}(u_n(\alpha, \omega), \omega \cup \{-W, W\}) \geq t \frac{2W}{\max\{|\alpha|, 1\}} \right) \leq \frac{1}{t}, \quad (3.10)$$

and for $t > |\alpha|$:

$$\mathbb{P} \left(\min_{n: |u_n| \leq W} \text{dist}(u_n(\alpha, \omega), \omega) \leq \frac{1}{t} \right) \leq \frac{2W}{t - |\alpha|}. \quad (3.11)$$

Proof. For any $q > 0$, if $\text{dist}(u_n(\alpha, \omega), \omega) \geq q$ for some $u_n(\alpha, \omega) \in [-W, W]$, then by the monotonicity of the function $S_\omega(u)$ between its singularities (Fig. 1) it satisfies $|S_\omega(u)| \geq |\alpha|$ throughout either the interval $[u_n, \omega_n]$ (in case $\alpha < 0$) or throughout the interval $(\omega_{n-1}, u_n]$ (in case $\alpha > 0$). This implies that, regardless of the sign of α ,

$$\int_{-W}^W \mathbf{1}[\text{sgn}(-\alpha)S_\omega(u) \geq |\alpha|] du \geq q. \quad (3.12)$$

To bound the probability that (3.12) occurs, we recall that for each $u \in \mathbb{R}$ the value of $S_\omega(u)$ has the probability distribution of a Cauchy variable with barycentre $i\pi$ [3]. Thus:

$$\begin{aligned} \mathbb{E} \left(\int_{-W}^W \mathbb{1}[\text{sgn}(-\alpha)S_\omega(u) \geq |\alpha|] du \right) &= \int_{-W}^W \mathbb{P}(\text{sgn}(\alpha)S_\omega(u) \geq |\alpha|) du \\ &\leq \frac{2W}{|\alpha|}. \end{aligned} \tag{3.13}$$

Through the Chebyshev inequality this allows to conclude that for any $q > 0$:

$$\mathbb{P} \left(\int_{-W}^W \mathbb{1}[\text{sgn}(-\alpha)S_\omega(u) \geq |\alpha|] du \geq q \right) \leq \frac{2W}{|\alpha|q}. \tag{3.14}$$

The choice $q = t \frac{2W}{\max\{|\alpha|, 1\}}$ yields the first bound claimed in (3.10).

In the proof of the second bound it is convenient to employ the function:

$$S_\omega^*(x) := \lim_{\varepsilon \rightarrow 0} \sum_{v \in \omega} \frac{\mathbb{1}[|v - x| \geq \varepsilon]}{v - x} \tag{3.15}$$

in which we omit the contribution of the site of ω which is closest to x .

Let now $t > |\alpha|$. We note that if for some $u_n(\alpha, \omega) \in [-W, W]$

$$\text{dist}(u_n(\alpha, \omega), \omega) \leq 1/t \tag{3.16}$$

then depending on the sign, $\sigma = \pm$, of the shortest path from that point to ω , we find that for one of the sites $\omega_n \in [-W, W] + \sigma t^{-1}$ the function S_ω^* satisfies,

$$\sigma S_\omega^*(\omega_n - \sigma t^{-1}) \geq t - |\alpha|. \tag{3.17}$$

For a Poisson process the probability that there is a Poisson point within an ε neighborhood of a given point $x \in \mathbb{R}$, and the contribution to $S_\omega(x)$ from all other sites of ω , are independent quantities. Thus, by a calculation similar to (3.13), for either of the two values (± 1) of σ :

$$\lim_{M \rightarrow \infty} \mathbb{E} \left(\sum_{v \in \omega \cap ([-W, W] + \sigma t^{-1})} \mathbb{1}[\sigma S_\omega^*(v - \sigma t^{-1}) > t - |\alpha|] \right) \leq \frac{2W}{t - |\alpha|}. \tag{3.18}$$

Since the probability that (3.16) holds for some $u_n(\alpha, \omega) \in [-W, W]$ is dominated by the mean of the number of such points we conclude that for any $t > |\alpha|$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\min_{n: |u_n| < W} \text{dist}(u_n, \omega) < t^{-1} \right) \leq \frac{2W}{t - |\alpha|}. \tag{3.19}$$

which completes the proof. □

4. Scaling Limits of Herglotz–Pick Functions

To proceed with (2.15) let us present three basic results on the possible scaling limits of the random functions $S_{M,\omega}(u, L)$ and $T_{M,\omega}(u, L)$. Both belong to the Herglotz–Pick class (HP), of functions with analytic extension to the upper half plane which assume there only values in \mathbb{C}^+ (More on this class of functions can be read in, e.g., [15, 16].) By the Herglotz representation theorem any HP function admits a unique representation in the form:

$$F(z) = a_F z + b_F + \int \left(\frac{1}{x - z} - \frac{x}{1 + x^2} \right) \mu_F(dx) \tag{4.1}$$

with $a_F \geq 0$, $b_F \in \mathbb{R}$ and μ_F a Borel measure satisfying: $\int (1 + x^2)^{-1} \mu_F(dx) < \infty$ In the cases discussed here the ‘spectral measure’ μ_F is of pure-point type, i.e. μ_F consists of discrete point masses.

Associated with any open interval (a, b) is the subclass $P(a, b)$ of functions which are analytic in (a, b) , or equivalently for which $\mu_F((a, b)) = 0$.

4.1. The Oscillatory Part

A natural topology for the set of HP functions is that of uniform convergence on compact subsets of \mathbb{C}^+ . The topology is metrizable, with a metric in which this class of functions forms a complete separable metric space. Basic properties of the corresponding notion of convergence, and its extensions to random functions in this class, are discussed in [3]. Of particular relevance is the following implication of [3, Theorems 3.1 and 6.1]. We shall apply it to the function $S_{M,\omega}$, taking $\widehat{\omega}_M$ to represent the symmetrically truncated process

$$\widehat{\omega}_M = \omega_M \mathbb{1}[|\omega_{M,n}| < L_M], \tag{4.2}$$

with L_M satisfying (2.17) (see Lemma 5.1).

Theorem 4.1. *Let $\widehat{\omega}_M$ be a sequence of random point processes. Suppose that as $M \rightarrow \infty$:*

- (i) $\widehat{\omega}_M$ converge in distribution to the standard Poisson process,
- (ii) for all $\varepsilon > 0$:

$$\lim_{W \rightarrow \infty} \limsup_{M \rightarrow \infty} \mathbb{P} \left(\left| \sum_{v \in \widehat{\omega}_M} \frac{\mathbb{1}[|v| > W]}{v} \right| \geq \varepsilon \right) = 0, \tag{4.3}$$

and

$$\lim_{W \rightarrow \infty} \limsup_{M \rightarrow \infty} \mathbb{P} \left(\sum_{v \in \widehat{\omega}_M} \frac{\mathbb{1}[|v| > W]}{v^2} \geq \varepsilon \right) = 0. \tag{4.4}$$

Then

$$\widehat{S}_{M,\widehat{\omega}}(z) := \sum_{v \in \widehat{\omega}_M} \frac{1}{v - z}. \tag{4.5}$$

converge in distribution to the Stieltjes–Poisson random function S_ω .

The convergence is in the sense of probability distribution of random functions in the HP class, which are elements of a space whose topology is based on pointwise convergence on \mathbb{C}^+ (as in [3]).

Proof of Theorem 4.1. Theorem 6.1 in [3] states that the assertion is implied by assumption i. and the requirement that for all $\varepsilon > 0$:

$$\lim_{\eta \rightarrow \infty} \limsup_{M \rightarrow \infty} \mathbb{P} \left(\left| \widehat{S}_M(i\eta) - i\pi \right| \geq \varepsilon \right) = 0, \tag{4.6}$$

where $i\pi$ is the corresponding distributional limit of $S(i\eta)$. To show that this condition is implied by assumption (ii), let us consider separately the contribution to the limit of the real and imaginary parts of $\widehat{S}_M(i\eta) - i\pi$. Condition (i) and (4.4) entail the distributional convergence of $\text{Im } \widehat{S}_M(i\eta) \xrightarrow{\mathcal{D}} \text{Im } S(i\eta)$ as $M \rightarrow \infty$ for every $\eta > 0$ and hence, using (3.2), one may deduce the distributional convergence of $\text{Im} [\widehat{S}_M(i\eta) - i\pi]$ to 0 in the limit seen in (4.6). To bound the real part, we let $W = \eta^2$, and split:

$$\begin{aligned} \text{Re } \widehat{S}_M(i\eta) &= \sum_n \frac{\widehat{\omega}_{M,n}}{\widehat{\omega}_{M,n}^2 + \eta^2} \mathbb{1}[|\widehat{\omega}_{M,n}| \leq \eta^2] + \sum_n \frac{\mathbb{1}[|\widehat{\omega}_{M,n}| > \eta^2]}{\widehat{\omega}_{M,n}} \\ &\quad - \sum_n \frac{\eta^2}{\widehat{\omega}_{M,n}(\widehat{\omega}_{M,n}^2 + \eta^2)} \mathbb{1}[|\widehat{\omega}_{M,n}| > \eta^2]. \end{aligned} \tag{4.7}$$

The last term on the right side is bounded by $\eta^{-1} \text{Im } \widehat{S}_M(i\eta)$ and hence, by the just established statement it converges in probability in the double limit $M \rightarrow \infty$ and $\eta \rightarrow \infty$ to zero. The second term converges to 0 in probability by assumption (ii). By assumption (i) for each fixed $0 < \eta < \infty$ the first term converges in distribution as $M \rightarrow \infty$ to the corresponding sum for the Poisson process ω_P : $\text{Re } \sum_{v \in \widehat{\omega}_P \cap [-\eta^2, \eta^2]} (v - i\eta)^{-1}$ and hence as $\eta \rightarrow \infty$ to zero by (3.2). This concludes the proof of (4.6). \square

4.2. Linearity at the Tail

The second preparatory statement addresses possible limits of the tail contributions to Herlotz–Pick functions due to spectral components whose support moves away to infinity. The following implies that in their restrictions to any fixed window $[-W, W]$ the functions are asymptotically linear, on the relevant scale. It would be applied to the tail function $[T_{M,\omega}(u, L_M) - M\Delta_M]$ which is of interest here.

Theorem 4.2. *Let $F(z)$ be a function in the class $P(-L, L)$. Then for any $W < L/10$ and $u, u_0, u_1 \in [-W, W]$:*

$$\left| \frac{F(u) - F(u_0)}{u - u_0} - \frac{F(u_1) - F(u_0)}{u_1 - u_0} \right| \leq \frac{6W}{L} \frac{F(u_1) - F(u_0)}{u_1 - u_0} \tag{4.8}$$

Proof. The spectral representation (4.1) yields

$$F'(z) = a_F + \int_{|x|>L} \frac{1}{(x - z)^2} \mu_F(dx) \ (\geq 0), \tag{4.9}$$

from which it follows that for any $u \in [-W, W]$:

$$\frac{d}{du} \ln F'(u) = \frac{F''(u)}{F'(u)} \leq \frac{2}{L - W} \tag{4.10}$$

since the bound is valid for the ratio of each pair of corresponding spectral components in the integrals yielding F'' and F' , the latter being an integral of positive terms. Hence, for any $x, y \in [-W, W]$:

$$\frac{F'(x)}{F'(y)} \leq \exp\left(\frac{4W}{L - W}\right) \tag{4.11}$$

which can be restated in the form:

$$|F'(x) - F'(y)| \leq F'(y) \left[\exp\left(\frac{4W}{L - W}\right) - 1 \right]. \tag{4.12}$$

Thus, through an application of the mean-value theorem:

$$\begin{aligned} & \left| \frac{F(u) - F(u_0)}{u - u_0} - \frac{F(u_1) - F(u_0)}{u_1 - u_0} \right| \\ & \leq \frac{F(u_1) - F(u_0)}{u_1 - u_0} \left[\exp\left(\frac{4W}{L - W}\right) - 1 \right]. \end{aligned} \tag{4.13}$$

Using: $[e^{4/(1-x)} - 1] \leq 6x$ for $x \leq 1/10$, we get (4.8). □

The asymptotic linearity implies that one has only the following finite list of possibilities for the scaling limits of random functions in the class $P(-L, L)$ (up to a restriction to subsequences, to guarantee consistency).

Definition 4.3. A sequence of random monotone increasing functions $K_M(u)$ is said to have a *generalized linear limit* if the corresponding condition in the following list holds, for all $0 < W < \infty$ and $\varepsilon > 0$:

1. *regular linear limit*
for some $a, b \in \mathbb{R}, a > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{u \in [-W, W]} |K_M(u) - (au + b)| \leq \varepsilon \right) = 1 \tag{4.14}$$

- 2 $_{\pm}$. *singular (+∞) or singular (−∞) limit*
for any $b \in \mathbb{R}$, and σ given by the corresponding choice of ± 1 :

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\min_{u \in [-W, W]} \text{sgn}(\sigma) K_M(u) > b \right) = 1 \tag{4.15}$$

3. *singular limit with transition*
there is a “transition point”, $\tau \in \mathbb{R}$, such that for any $b \in \mathbb{R}$ and $\varepsilon > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} (\pm K_M(\tau \pm \varepsilon) > b) = 1. \tag{4.16}$$

for each value of the \pm sign.

In essence, the above is the list of possible scaling limits of linear functions, $\alpha(u) = au + b$ with $a \geq 0$, allowing the parameters to assume diverging values. The distinction here is between the cases:

1. both a and b assume a finite limit,

2. one of the parameters dominates over the other (either $|a| \ll |b|$, or $|a| \ll |b|$), with $\pm = \text{sign}(a/b)$,
3. the two diverge but the ratio a/b has a finite limit.

One may note that for functions to which Theorem 4.2 applies, just point-wise convergence, in the sense that

$$\mathcal{D} - \lim_{M \rightarrow \infty} K_{M,\omega}(u) \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \tag{4.17}$$

for a dense collection of $u \in \mathbb{R}$, implies the uniformity which is required in Definition 4.3, cases (1) and (3). (The argument can be seen in the proof of Lemma 5.5 below.)

4.3. Convergence to the Šeba Process

Theorems 4.1 and 4.2 carry the following implication for the limiting behavior of the solutions of an equation of the form (2.15).

Theorem 4.4. *Let $\widehat{\omega}_M$ be a sequence of point processes satisfying the assumptions (i) and (ii) of Theorem 4.1 and $\widehat{S}_{M,\widehat{\omega}}(u)$ the sequence of the related functions*

$$\widehat{S}_{M,\widehat{\omega}}(u) = \sum_{v \in \widehat{\omega}_M} \frac{1}{v - u}, \tag{4.18}$$

and let $K_M(u)$ be a sequence of random Herglotz–Pick functions with generalized linear limiting behavior. Then the intertwined point process consisting of the support of $\widehat{\omega}_M$ and the solution set of equation

$$\widehat{S}_{M,\widehat{\omega}}(u) + K_M(u) = 0 \tag{4.19}$$

has the following limit:

1. If K_M has a regular linear limit given by a constant function (i.e. type 1 limit in the sense of Theorem 4.2 with $a = 0, |b| < \infty$), then the intertwined point processes converge to the Šeba process at the corresponding level ($\alpha = b$).
2. If K_M has a singular $(+\infty)$ or singular $(-\infty)$ limit (type 2_{\pm}), then the solution set coincides asymptotically with $\widehat{\omega}$.
3. If the limit is singular with transition (type 3) then the solution set converges to the limit of $\widehat{\omega} \cup \{\tau\}$, where the added point is the transition point of K .

It may be added that the outcome is not affected by possible correlations between the two random functions $\widehat{S}_{M,\widehat{\omega}}$ and K_M .

Proof. 1. Let $W > 0$ be a fixed, but arbitrarily large number. According to Theorem 4.1, for any fixed $u \in \mathbb{R}$, $\widehat{S}_{M,\widehat{\omega}}(u)$ converges in distribution to the Stieltjes–Poisson random function $S_{\omega}(x)$. By Skorokhod’s representation theorem [7, Section 1.6], $\widehat{S}_{M,\widehat{\omega}}(u)$ and $S_{\omega}(u)$ can be realized on the same probability space (i.e. the two probability measures “coupled”), so that $\widehat{S}_{M,\widehat{\omega}}(u) \rightarrow S_{\omega}(u)$ with probability one. The coupling can be implemented simultaneously for

all $u \in [-W, W] \cap \mathbb{Q}$ (which forms a countable collection of sites), i.e. with probability one:

$$\forall u \in [-W, W] \cap \mathbb{Q} : \quad \widehat{S}_{M,\widehat{\omega}}(u) \rightarrow S_\omega(u). \tag{4.20}$$

Let us show that with probability one

$$\forall u \in [-W, W] \setminus \omega : \quad \widehat{S}_{M,\widehat{\omega}}(u) \rightarrow S_\omega(u). \tag{4.21}$$

By the assumed convergence of $\widehat{\omega}_M$ to the Poisson process, and an additional application of the Skorokhod representation theorem, we may additionally assume that with probability one $\widehat{\omega}_M \rightarrow \omega$. Any point $u \in [-W, W] \setminus \omega$ has a neighborhood I_u which is disjoint from ω . Therefore any smaller neighborhood $I'_u \subset I_u$ is disjoint from all $\widehat{\omega}_M$ for all $M \geq M_u$ sufficiently large. The functions $\widehat{S}_{M,\widehat{\omega}}$ are monotone increasing on I'_u (for $M \geq M_u$), and converge to a monotone increasing function S_ω on a dense subset of I'_u by (4.20). Since $\widehat{S}_{M,\widehat{\omega}}$ is continuous on I'_u , this implies that $\widehat{S}_{M,\widehat{\omega}}$ converge to S_ω uniformly on I'_u , and in particular $\widehat{S}_{M,\widehat{\omega}}(u) \rightarrow S_\omega(u)$, proving (4.21).

Let $\{u_{M,n}\}$ be the finite collection of solution of the equation (4.19) within $[-W, W]$. According to (4.21), for any $\epsilon > 0$ there exists $M_\epsilon > 0$ such that for all n and all $M \geq M_\epsilon$:

$$|K_M(u_{M,n}) - b| < \epsilon. \tag{4.22}$$

Between any pair of consecutive poles v_M, v'_M of $\widehat{S}_{M,\widehat{\omega}}$ the function $\widehat{S}_{M,\widehat{\omega}}(u) + K_M(u)$ is continuous and monotone increasing from $(-\infty)$ to $(+\infty)$. Therefore Eq. (4.19) has a unique solution $u_{M,n}$ there. By item 2 of Proposition 3.1 (which bounds the derivative of $\widehat{S}_{M,\widehat{\omega}}$ from below), for all $M \geq M_\epsilon$:

$$|u_n - u_{M,n}| \leq \epsilon |v_M - v'_M|^2. \tag{4.23}$$

In combination with assumption (i), this implies the stated convergence to the Šeba process.

2. For $|b| \gg 1$, the above argument implies that the solutions of (4.19) are at distances of order $O(|b|^{-1})$ from the sites of ω (the sign of b determining the side from which the approach happens). Direct adaptation of the argument to the limit $|b| \rightarrow \infty$ implies the statement in case 2.

3. In case 3, the functions $K_M(u_{M,n})$ exhibit a sharp transition whose location converges to τ from $(-\infty)$ to $(+\infty)$ (as asymptotic values). Thus, by an extension of the above argument, the solution set converges to $\omega \cup \tau$. \square

5. Relevant Estimates

5.1. The Oscillatory Term $S_{M,\omega}(u)$

The following estimate shows that under the condition (2.17), the assumptions of Theorem 4.1 apply to the function $S_{M,\omega}(u, L_M)$ which is defined in (2.16). Consequently, in the scaling limit this component of the characteristic equation converges to the Stieltjes–Poisson random function.

Lemma 5.1. *Let \mathcal{E}_M be a sequence of energies in $(-\lambda, \lambda)$, L_M a sequence of cutoff values satisfying*

$$\lim_{M \rightarrow \infty} s_M = 0, \quad s_M := L_M \frac{\sqrt{\ln M}}{M^{1-(\mathcal{E}_M/\lambda)^2}} \tag{5.1}$$

and $\omega_{M,n}$ the process of rescaled potential values defined relative to \mathcal{E}_M through the scaling relation (2.12). Then the quantities

$$Y_{M,W} = \sum_n \frac{\mathbf{1}[W < |\omega_{M,n}| < L_M]}{\omega_{M,n}} \tag{5.2}$$

are of mean and variance satisfying, for each $W < \infty$:

$$\lim_{M \rightarrow \infty} \mathbb{E}[Y_{M,W}] = 0, \quad \lim_{M \rightarrow \infty} \text{Var}(Y_{M,W}) \leq \frac{C}{W} \tag{5.3}$$

with a uniform $C < \infty$.

Proof. Taking the expectation value and expressing it in terms of the macroscopic variables, one gets

$$\begin{aligned} \mathbb{E}[Y_{M,W}] &= \frac{M\Delta_M}{\kappa_M} \int_{W\Delta_M/\kappa_M}^{L_M\Delta_M/\kappa_M} \frac{\varrho\left(\frac{\mathcal{E}_M}{\kappa_M} + v\right) - \varrho\left(\frac{\mathcal{E}_M}{\kappa_M} - v\right)}{v} \, dv \\ &\leq C \frac{M\Delta_M}{\kappa_M} \max_{|x| \leq L_M\Delta_M} \varrho\left(\frac{\mathcal{E}_M + x}{\kappa_M}\right) \frac{|\mathcal{E}_M| + |x|}{\kappa_M} \frac{L_M\Delta_M}{\kappa_M} \\ &\leq C \frac{\Delta_M}{\kappa_M^2} L_M (1 + s_M)^2 = C s_M (1 + s_M)^2, \end{aligned} \tag{5.4}$$

where $C < \infty$ is a constant and use was made of (2.10) and the observation that the above shift by x does not cause a significant change in $\log \varrho$.

To estimate the variance, we note that despite our labeling convention the sum in (5.2) can also be regarded as over M independently and identically distributed variables ω_n with values in \mathbb{R} . The variables' independence easily implies for the sum's variance:

$$\begin{aligned} \text{Var}(Y_{M,W}) &\leq \mathbb{E} \left[\sum_n \frac{\mathbf{1}[W < |\omega_{M,n}| < L_M]}{|\omega_{M,n}|^2} \right] \\ &\leq \frac{M\Delta_M^2}{\kappa_M^2} \int_{W\Delta_M/\kappa_M}^{L_M\Delta_M/\kappa_M} \frac{\varrho\left(\frac{\mathcal{E}_M}{\kappa_M} + v\right) + \varrho\left(\frac{\mathcal{E}_M}{\kappa_M} - v\right)}{v^2} \, dv \\ &\leq C \frac{M\Delta_M^2}{\kappa_M^2} \max_{|x| \leq L_M\Delta_M} \varrho\left(\frac{\mathcal{E}_M + x}{\kappa_M}\right) \frac{\kappa_M}{W\Delta_M} \\ &\leq \frac{C}{W} (1 + s_M), \end{aligned} \tag{5.5}$$

where again $C < \infty$ is a constant. □

5.2. The Tail Term

The other component of the characteristic equation (2.15) is the function

$$R_{M,\omega}(u) := T_{M,\omega}(u, L_M) - M\Delta_M. \tag{5.6}$$

Let us first consider its mean value.

Lemma 5.2. *Let \mathcal{E}_M be a convergent sequence of energies, with $\lim_{M \rightarrow \infty} \mathcal{E}_M \in (-\lambda, \lambda)$, and $L_M \rightarrow \infty$ a sequence of cutoff values satisfying (5.1). Then for any $u \in \mathbb{R}$:*

$$\begin{aligned} & \lim_{M \rightarrow \infty} (\mathbb{E}[T_M(u, L_M)] - M\Delta_M \widehat{\varrho}_M(\mathcal{E}_M + u\Delta_M)) \\ &= - \lim_{M \rightarrow \infty} \mathbb{E}[S_M(u, L_M)] = 0 \end{aligned} \tag{5.7}$$

with $\widehat{\varrho}_M$ the function

$$\widehat{\varrho}_M(\mathcal{E}) := \int \frac{\varrho(v) dv}{\kappa_M v - \mathcal{E}}, \tag{5.8}$$

the dash in \int indicating a principal-value integral (i.e. up to scaling, the Hilbert transform of ϱ).

Proof. The first equality follows directly from the definition of the different terms. Taking advantage of that, we turn to estimate the expectation value $\mathbb{E}(S_M(u, L_M))$. It can be written as a sum of two terms. One of them is

$$\begin{aligned} & M\Delta_M \varrho\left(\frac{\mathcal{E}_M + u\Delta_M}{\kappa_M}\right) \int \frac{\mathbb{1}[|\kappa_M v - \mathcal{E}_M| \leq \Delta_M L_M]}{\kappa_M v - \mathcal{E}_M - u\Delta_M} dv \\ &= \frac{\varrho\left(\frac{\mathcal{E}_M + u\Delta_M}{\kappa_M}\right)}{\varrho\left(\frac{\mathcal{E}_M}{\kappa_M}\right)} \ln \frac{L_M - u}{L_M} \rightarrow 0. \end{aligned} \tag{5.9}$$

This tends to zero since $L_M \rightarrow \infty$, and the effect of the shift in the argument of ϱ is controllable as in (5.4).

The second term in modulus equals

$$\begin{aligned} & M\Delta_M \left| \int \mathbb{1}[|\kappa_M v - \mathcal{E}_M| \leq \Delta_M L] \frac{\varrho(v) - \varrho((\mathcal{E}_M + u\Delta_M)/\kappa_M)}{\kappa_M v - \mathcal{E}_M - u\Delta_M} dv \right| \\ & \leq C \frac{M\Delta_M}{\kappa_M} \max_{|x| \leq L_M \Delta_M} \varrho\left(\frac{\mathcal{E}_M + x}{\kappa_M}\right) \frac{|\mathcal{E}_M| + |x|}{\kappa_M} \frac{L_M \Delta_M}{\kappa_M} \leq C \frac{L_M \sqrt{\ln M}}{M^{1-(\mathcal{E}_M/\lambda)^2}}. \end{aligned} \tag{5.10}$$

The last inequality is the same as in (5.4). The right side vanishes in the limit $M \rightarrow \infty$ by (5.1). □

We shall be interested in locating the energies at which $\mathbb{E}[R_{M,\omega}(u)]$ is of order 1 (which is where significant wave function hybridization may occur). In terms of the mean, this corresponds to:

$$M\Delta_M |\widehat{\varrho}_M(\mathcal{E}) - 1| \lesssim 1. \tag{5.11}$$

Away from $\mathcal{E} = 0$ (i.e. where $M\Delta_M \rightarrow \infty$) this condition is satisfied only for \mathcal{E} in the vicinity of the solutions of

$$\widehat{\varrho}_M(\mathcal{E}) - 1 = 0. \tag{5.12}$$

Looking closer, one finds that:

(i) for each $\mathcal{E} \neq 0$:

$$\lim_{M \rightarrow \infty} \widehat{\varrho}_M(\mathcal{E}) = -1/\mathcal{E} \tag{5.13}$$

- (ii) the principal-value cutoff rounds off the singularity at $\mathcal{E} = 0$, so that for finite M the functions $\widehat{\varrho}_M(\mathcal{E})$ are continuous, changing sign at $\mathcal{E} = 0$,
- (iii) for $\mathcal{E} \notin \{-1, 0\}$, the factor $M\Delta_M$ causes R_M to diverge.

More precisely:

Lemma 5.3. *For large enough M , Eq. (5.12) has exactly two solutions, one in the vicinity of $\mathcal{E} = -1$, which will be denoted $\widehat{\mathcal{E}}_M^{(-1)}$, and the other in the vicinity of $\mathcal{E} = 0$, denoted here as $\widehat{\mathcal{E}}_M^{(0)}$. For $M \rightarrow \infty$ these behave as:*

$$\begin{aligned} \widehat{\mathcal{E}}_M^{(-1)} &= -1 - \kappa_M^2 + \mathcal{O}(\kappa_M^4) \\ \widehat{\mathcal{E}}_M^{(0)} &= -\frac{\kappa_M^2}{2\sqrt{\pi}} + \mathcal{O}(\kappa_M^4), \end{aligned} \tag{5.14}$$

Proof. The Hilbert transform of the Gaussian probability density $\varrho(v)$ is an odd continuous function which satisfies the asymptotics:

$$\int \frac{\varrho(v) dv}{v - \xi} = \begin{cases} -2\sqrt{\pi} [\xi - \frac{2}{3}\xi^3 + \mathcal{O}(|\xi|^5)] & \text{for small } \xi, \\ -\xi^{-1} - \xi^{-3} + o(\xi^{-3}) & \text{for large } \xi. \end{cases} \tag{5.15}$$

Since $\kappa_M \rightarrow 0$ the claim immediately follows therefrom. □

To reach beyond the mean, and in particular apply Theorem 4.4 to the function $R_{M,\omega}(u)$, we need information on the function’s fluctuations. The following estimates will be of help.

Lemma 5.4. *For any $\lambda > 0$, $\mathcal{E}_M \in (-\lambda, \lambda)$, $1 < L_M < M^{1-(\mathcal{E}_M/\lambda)^2}/\sqrt{\ln M}$, and $|u| < L_M/2$:*

$$\begin{aligned} \text{Var}(R_M(u)) &= \mathbb{E} \left(|T_M(u, L_M) - \mathbb{E}(T_M(u, L_M))|^2 \right) \\ &\leq \mathbb{E} \left(\frac{d}{du} T_M(u, L_M) \right) \end{aligned} \tag{5.16}$$

and:

$$C_1 \frac{M^{2(\mathcal{E}_M/\lambda)^2-1}}{\ln M} \leq \mathbb{E} \left(\frac{d}{du} T_M(u, L_M) \right) \leq C_2 \left[\frac{1}{L} + M^{2(\mathcal{E}_M/\lambda)^2-1} \right] \tag{5.17}$$

with uniform constants $0 < C_1 < C_2 < \infty$.

Proof. The first equality in (5.16) holds since $R_{M,\omega}(u)$ and $T_{M,\omega}(u, L_M)$ differ by a constant. For the variance bound, we start as in the first step in (5.5):

$$\begin{aligned} \mathbb{E} \left(\left| T_M(u, L_M) - \mathbb{E}(T_M(u, L_M)) \right|^2 \right) &\leq \mathbb{E} \left(\sum_n \frac{\mathbf{1} [|\omega_{M,n}| > L_M]}{|\omega_{M,n} - u|^2} \right) \\ &= \mathbb{E} \left(\frac{d}{du} T_M(u, L_M) \right). \end{aligned} \tag{5.18}$$

The sum in the right is estimated for any $|u| \leq W < L_M$ using the elementary inequality

$$\frac{1}{1 + W/L_M} \leq \frac{|\omega_{M,n}|}{|\omega_{M,n} - u|} \leq \frac{1}{1 - W/L_M} \tag{5.19}$$

since $|\omega_{M,n}| \geq L_M$. It hence remains to estimate $\frac{d}{du} T_{M,\omega}(0, L_M)$ which can be expressed in terms of the macroscopic variables (of (2.11) and (2.12)):

$$\mathbb{E} \left(\frac{d}{du} T_M(0, L_M) \right) = \frac{M\Delta_M^2}{\kappa_M^2} \int \frac{\mathbf{1} [|v - \mathcal{E}_M/\kappa_M| > \delta_M]}{|v - \mathcal{E}_M/\kappa_M|^2} \varrho(v) dv \tag{5.20}$$

with $\delta_M := L_M \Delta_M/\kappa_M = \sqrt{2\pi} L_M/M^{1-(\mathcal{E}_M/\lambda)^2}$. The above integral is estimated in Appendix B. Using the upper bound in (B.1) and the relations (2.10) we get:

$$\begin{aligned} \mathbb{E} \left(\frac{d}{du} T_M(0, L_M) \right) &\leq C \frac{M\Delta_M^2}{\kappa_M^2} \left[\frac{\varrho(\mathcal{E}_M/\kappa_M)}{\delta_M} + \frac{1}{(1 + |\mathcal{E}_M/\kappa_M|)^2} \right] \\ &\leq C \left[\frac{1}{L_M} + \frac{M\Delta_M^2}{\kappa_M^2} \right]. \end{aligned} \tag{5.21}$$

Likewise, the lower bound in (B.1) yields

$$\mathbb{E} \left(\frac{d}{du} T_M(0, L_M) \right) \geq c \frac{M\Delta_M^2}{(\kappa_M + |\mathcal{E}_M|)^2} \geq c \frac{M^{2(\mathcal{E}_M/\lambda)^2-1}}{\ln M}. \tag{5.22}$$

□

The variance bounds of Lemma 5.4 show a qualitative transition when $|\lambda/\mathcal{E}_M|$ crosses the value $\sqrt{2}$. For $|\lambda/\mathcal{E}_M| > \sqrt{2}$, the fluctuations of the tail function $T_M(u, L_M)$ diverge with M . Nevertheless, even there, the divergence rate is slower than the typical value of the function's derivative. This would have the significant implication that the energy at which $R_M(u, L_M)$ changes sign would be deterministic even on the microscopic scale.

5.3. Scaling Limit of the Function $R_{M,\omega}$

Considering the effects of fluctuations we arrive at the following characterization of the scaling limit of $R_{M,\omega}$ in the different regimes, as it is observed within windows $[-W, W]$ of fixed, though arbitrarily large size.

Lemma 5.5. *Let \mathcal{E}_M be a sequence of energies with $\lim_{M \rightarrow \infty} \mathcal{E}_M = \mathcal{E}$, $|\mathcal{E}| < \lambda$, and L_M a divergent sequence of cutoff values satisfying (5.1). Then, in the regimes listed below, the random function $R_{M,\omega}(u)$ (defined in (5.6)) has the following limiting behavior.*

(i) If $\mathcal{E} \in (\lambda, \lambda) \setminus \{-1, 0\}$, then the limit is $(\pm) \infty$ singular, the sign being $\pm = \text{sgn}[\mathcal{E}(\mathcal{E} + 1)]$, with the random function satisfying, for any $W < \infty$ and $b > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\min_{u \in [-W, W]} \{ -\text{sgn} [\mathcal{E}(\mathcal{E} + 1)] R_M(u) \} > b \frac{M^{(\mathcal{E}_M/\lambda)^2}}{\ln M} \right) = 1 \quad (5.23)$$

(ii) If $\mathcal{E} = 0$, and

$$\sup_M \frac{|\mathcal{E}_M|}{\kappa_M^2} < \infty \quad (5.24)$$

then for any $W < \infty$ and $\varepsilon > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{u \in [-W, W]} |R_M(u)| < \varepsilon \right) = 1 \quad (5.25)$$

(iii) In the vicinity of $\widehat{\mathcal{E}}_M^{(-1)}$ the nature of the limit depends on λ :

(a) if $\lambda < \sqrt{2}$: any sequence with

$$\tau = \lim_{M \rightarrow \infty} \left[\widehat{\mathcal{E}}_M^{(-1)} - \mathcal{E}_M \right] / \Delta_M \quad (5.26)$$

the function has a singular limit with transition at τ , and satisfies for any $b > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\min_{u: \delta_M < |u - \tau| < W} \left\{ \frac{R_M(u)}{u - \tau} \right\} > b \frac{M^{2(\mathcal{E}_M/\lambda)^2 - 1}}{\ln M} \right) = 1 \quad (5.27)$$

with $\delta_M := (\ln M)^2 / M^{(\mathcal{E}_M/\lambda)^2 - 1/2}$.

(b) If $\lambda > \sqrt{2}$: for sequences with

$$\alpha = \lim_{M \rightarrow \infty} M \Delta_M \left[\mathcal{E}_M - \widehat{\mathcal{E}}_M^{(-1)} \right] \quad (5.28)$$

$R_{M,\omega}(u)$ converges to the constant function of value α (in a sense similar to (5.25)).

Thus, conditions for resonant delocalization exist at $\mathcal{E} = 0$ at all λ , and in the vicinity of $\mathcal{E} = -1$ for $\lambda > \sqrt{2}$. Another heuristic perspective on the relevant condition is given below in Sect. 7.

Proof. (i) By (5.13), for any $\mathcal{E} \in (-\lambda, \lambda) \setminus \{-1, 0\}$, and $u \in \mathbb{R}$, at M large enough:

$$[\widehat{\varrho}_M(\mathcal{E}_M + u\Delta_M) - 1] \geq |1 + 1/\mathcal{E}|/2 > 0. \quad (5.29)$$

Using Lemma 5.2, we conclude that for any given $W < \infty$,

$$-\text{sgn}(1 + \mathcal{E}^{-1}) \mathbb{E}(R_M(\pm W)) \geq M \Delta_M \frac{|1 + \mathcal{E}^{-1}|}{2} \geq C(\mathcal{E}) \kappa_M M^{(\mathcal{E}_M/\lambda)^2}. \quad (5.30)$$

By Lemma 5.4 the mean square fluctuations are of smaller order of magnitude, by a factor which is bounded by $\ln M / \sqrt{M}$. This, combined with the monotonicity of the function over $[-W, W]$ implies (5.23).

(ii) Here (5.24) and (5.15) imply that

$$\lim_{M \rightarrow \infty} M \Delta_M (\widehat{\varrho}_M(\mathcal{E}_M) - 1) = 0. \tag{5.31}$$

The bound (5.24) on \mathcal{E}_M combined with the bounds of Lemma 5.2 and Lemma 5.4, guarantees that the fluctuations and the derivative in u vanish in the scaling limit, and thus for any $W \in \mathbb{R}$ and any $\varepsilon > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P}(|R_M(\pm W, L_M)| > \varepsilon) = 0. \tag{5.32}$$

Combined with the interpolation bound of Theorem 4.2 this implies that also $\max_{|u| \leq W} |R_M(u, L_M)|$ converges in distribution to 0.

(iii.a) ($\lambda < \sqrt{2}$) It suffices to consider the case $\mathcal{E}_M = \widehat{\mathcal{E}}_M^{(-1)}$, for which $\tau = 0$, since (5.26) differs from it by just a microscopic shift by τ . In this case,

$$\mathbb{E}[R_M(0)] = 0. \tag{5.33}$$

As in (ii), the bounds of Lemma 5.4 imply that for δ_M of (5.27) the fluctuations of $R_M(\pm\delta_M)$ are negligible (in probability) with respect to the mean, which is determined through $\frac{d}{du} \mathbb{E}[R_M(\pm\delta_M)]$. Consequently, for any $b > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P}\left(\frac{1}{\pm\delta_M} R_M(\pm\delta_M) > b \frac{M^{2(\mathcal{E}_M/\lambda)^2 - 1}}{\ln M}\right) = 1 \tag{5.34}$$

By the uniformity principle which is expressed in Theorem 4.2, this bound extends to (5.27).

(iii.b) ($\lambda > \sqrt{2}$) In this case (5.28) and (5.15) imply

$$\begin{aligned} \lim_{M \rightarrow \infty} \mathbb{E}[R_M(0)] &= \lim_{M \rightarrow \infty} M \Delta_M (\widehat{\varrho}_M(\mathcal{E}_M) - 1) \\ &= \lim_{M \rightarrow \infty} M \Delta_M \left(\widehat{\varrho}_M(\mathcal{E}_M) - \widehat{\varrho}_M(\widehat{\mathcal{E}}_M^{(-1)})\right) = \alpha. \end{aligned} \tag{5.35}$$

For this case, the bounds of Lemma 5.4 guarantee that both the variance and the mean derivative of $R_M(u)$ make no contribution to the limiting behavior of $R_M(\pm W)$. The monotonicity of the function allows to turn this into the uniform statement which was claimed in (5.28). \square

6. The Main Result

We now turn to the implications for the spectrum of H_M in (1.1).

6.1. Band of Semi-Delocalized States

Existence of localized states for this operator has been noted before, and proven in a number of works [9, 22]. In this respect the new result presented here is that the operator also has bands of semi-delocalized states. These appear in the vicinity of the spectrum of the mixing term $|\phi_0\rangle\langle\phi_0|$, which is the two point set $\{-1, 0\}$. However, at $\mathcal{E} = -1$, the formation of a band of semi-delocalized states, rather than a single extended state, requires $\lambda > \sqrt{2}$.

Theorem 6.1 (Semi-delocalization). *Let \mathcal{E}_M be a sequence with*

- (i) $\lim_{M \rightarrow \infty} \mathcal{E}_M = 0$ *satisfying (5.24) (at arbitrary $\lambda > 0$), or*

(ii) $\lim_{M \rightarrow \infty} \mathcal{E}_M = -1$ satisfying (5.28) and $\lambda > \sqrt{2}$.

Then, the behavior within the scaling window is as follows.

1. The joint process of rescaled eigenvalues and potential values converges in distribution to the Šeba point process at the level 0 in case (i) and α of (5.28) in case (ii).
2. The moments of the corresponding eigenfunctions behave as those of the Šeba process (Lemma 3.3) in the sense that for any $|n| < \infty$:
 - (a) for any $\epsilon > 0$ there is $\delta = \delta(\epsilon, n) > 0$ such that for all $M < \infty$

$$\mathbb{P} \left(1 + \delta < \frac{\|\psi_{M,n}\|_2}{\|\psi_{M,n}\|_\infty} < \frac{1}{\delta} \right) > 1 - \epsilon. \tag{6.1}$$

(b) for any $b < \infty$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\frac{\|\psi_{M,n}\|_1}{\|\psi_{M,n}\|_\infty} > b \right) = 1 \tag{6.2}$$

To facilitate comparisons of the eigenfunctions in the different regimes discussed in this section, we adapt the following conventions.

1. The spectrum is obtained from the characteristic equation (2.15), for which the cutoff parameter is set here to $L_M = \ln M$.
2. The eigenfunctions $\psi_{M,n}$ will be assigned the normalization (2.14), so that for each eigenvalue $u_{M,n}(\omega_M)$:

$$\|\psi_{M,n}\|_\infty = \frac{1}{\text{dist}(u_{M,n}(\omega_M), \omega_M)}. \tag{6.3}$$

3. The functions' ℓ^2 -norms will be split into the sum of the 'head', 'body', and 'tail' terms:

$$\|\psi_{M,n}\|_2^2 = \|\psi_{M,n}^{(H)}\|_2^2 + \|\psi_{M,n}^{(B)}\|_2^2 + \|\psi_{M,n}^{(T)}\|_2^2, \tag{6.4}$$

with:

$$\begin{aligned} \text{(i)} \quad & \|\psi_{M,n}^{(H)}\|_2^2 := \frac{1}{|u_{M,n} - \omega_{M,n^*}|^2} = \|\psi_{M,n}\|_\infty^2 \\ \text{(ii)} \quad & \|\psi_{M,n}^{(B)}\|_2^2 := \sum_{k \neq n^*} \frac{\mathbb{1}[|\omega_{M,k}| < \ln M]}{|u_{M,n} - \omega_{M,k}|^2} \\ \text{(iii)} \quad & \|\psi_{M,n}^{(T)}\|_2^2 := \sum_k \frac{\mathbb{1}[|\omega_{M,k}| \geq \ln M]}{|u_{M,n} - \omega_{M,k}|^2} = \frac{d}{du} R_{M,\omega}(u_{M,n}). \end{aligned} \tag{6.5}$$

where n^* is the value of the index k for which $\omega_{M,k}$ is closest to $u_{M,n}(\omega_M)$.

In estimating the eigenfunctions we shall use the information on the scaling limits of $T_M(u, L_M)$ provided for the different regimes by Lemma 5.5, and also the following elementary estimate.

Lemma 6.2. *Let $\delta_W^+(\omega_M)$ and $\delta_W^-(\omega_M)$ denote the the smallest, and correspondingly largest, gap between consecutive points of $\omega_M \cap (-W, W)$. Then for any $W > 1$ and sequence (ε_M) with $\lim_{M \rightarrow \infty} \varepsilon_M = 0$:*

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\delta_{-W}^- \geq \frac{\varepsilon_M}{W}, \quad \delta_W^+ \leq \frac{\ln W}{\varepsilon_M} \right) = 1. \tag{6.6}$$

Proof. For fixed W , the bounds are implied by the Poisson process calculation:

$$\begin{aligned} \lim_{M \rightarrow \infty} \mathbb{P} (\delta_W^+ > (1+t) \ln W) &\leq \int_{-W}^W dv e^{-(1+t) \ln W} = \frac{2W}{W^{1+t}}, \\ \lim_{M \rightarrow \infty} \mathbb{P} (\delta_{-W}^- < s) &\leq \int_{-W}^W dvs = 2Ws, \end{aligned} \tag{6.7}$$

for any $s, t > 0$. □

Proof of Theorem 6.1. 1. By Lemma 5.5, parts (ii) and (iii.b), for the two cases considered here $R_{M,\omega}(u)$ converges to constant functions, at the indicated values of the parameter α . By Theorem 4.4 this results in the convergence of the joint distribution of $(u_{M,n}, \omega_{M,n})$ to the corresponding Šeba processes.

2. Turning to the eigenfunction, we shall discuss separately each of the three terms in (6.4).

(i) + (ii) By the distributional convergence of the of $(u_{M,n}, \omega_{M,n})$ to the Šeba process (u_n, ω_n) , the head contribution converges in distribution to the one of the Šeba eigenfunctions $\|\psi_{M,n}\|_\infty = \|\psi_{M,n}^{(H)}\|_2 \rightarrow \|\Psi_n\|_\infty$. Lemma 3.4 and (6.6) hence imply that for any $W < \infty$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\text{for all } n \text{ with } u_{M,n} \in (-W, W): \frac{1}{\gamma_M \ln W} \leq \|\psi_{M,n}\|_\infty \leq \gamma_M W \right) = 1 \tag{6.8}$$

provided $\gamma_M \rightarrow \infty$.

To estimate the ‘body’ term, we note that for each k the sum is typically comparable to $\sum_{k \neq 0} 1/k^2$, and hence finite. For a bound on the maximum for energies in the range $u_{M,n} \in (-W, W)$:

$$\frac{1}{(\delta_W^+)^2} \leq \|\psi_{M,n}^{(B)}\|_2^2 \leq \frac{4}{(\delta_{W+1}^-)^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k^2} + \sum_k \frac{\mathbb{1}[|W| \leq |\omega_{M,k}| < \ln M]}{|\omega_{M,k}|^2}. \tag{6.9}$$

Applying (6.6) to the first sum, and a simple Chebyshev bound to the second, we conclude that for each $W < \infty$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\text{for all } n \text{ with } u_{M,n} \in (-W, W): \frac{1}{\gamma_M \ln W} \leq \|\psi_{M,n}^{(B)}\|_2 \leq \gamma_M W \right) = 1 \tag{6.10}$$

provided $\gamma_M \rightarrow \infty$.

(iii) The convergence of $R_{M,\omega}(u)$ to constant functions is accompanied (e.g., by the asymptotic linearity which is expressed in Theorem 4.2) with the convergence of $\frac{d}{du} R_{M,\omega}(u_{M,n})$ to 0, uniformly over compacta. In view of (6.5),

this in turn implies that the tail contribution to the functions vanishes, in the sense that for any $W < \infty$ and $\varepsilon > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{n: |u_{M,n}| < W} \|\psi_{M,n}^{(T)}\|_2 > \varepsilon \right) = 0 \tag{6.11}$$

The claim 2(a) is implied by the estimates (6.8), (6.10), and (6.11).

2(b) For the limiting Šeba process the ℓ^1 norms, which are given by sums comparable to $\sum_{n \neq 0} 1/|n|$, diverge (as in Lemma 3.4). By Fatou’s lemma, this implies divergence in distribution of $\|\psi_{M,n}\|_1 / \|\psi_{M,n}\|_\infty \rightarrow \infty$ for $M \rightarrow \infty$. (The more explicit bound indicated in the above remark for case (ii) is based on $M\Delta_M$.) \square

6.2. The Localization Regime

At energies which do not lie in the spectrum of the mixing term in the Hamiltonian H_M , i.e. for $\mathcal{E} \notin \{-1, 0\}$, one finds localization irrespective of the value of λ .

As was mentioned above, localization for the operator H_M was discussed previously, starting with [9]. In terms of the method presented here, localization is associated to the singular ($\pm\infty$) limiting behavior of the tail contribution $R_{M,\omega}$ to the characteristic equation, in the terminology of Definition 4.3. Following is a more quantitative analysis.

Theorem 6.3 (Localization). *For any sequence with a limiting value*

$$\mathcal{E} = \lim_{M \rightarrow \infty} \mathcal{E}_M \in (-\lambda, \lambda) \setminus \{-1, 0\}, \tag{6.12}$$

the eigenvalues within the scaling window behave as follows.

1. *Under the scaling (2.11) and (2.12) the eigenvalues coincide asymptotically, in probability, with the point process $(\omega_{M,n})$ of the rescaled potential values (the two being compared within scaling windows of fixed, but arbitrarily large size $[-W, W]$).*
2. *The eigenfunctions corresponding to energies with $|u_{M,n}| < W$ (with $W < \infty$ fixed but arbitrary) are all ℓ^2 -localized in the sense that with asymptotically full probability all satisfy, for any $\gamma > 0$:*

$$1 \leq \frac{\|\psi_{M,n}\|_2}{\|\psi_{M,n}\|_\infty} \leq 1 + \mathcal{O} \left(\frac{1}{M^{(\mathcal{E}/\lambda)^2(1-\gamma)}} \right) + \mathcal{O} \left(\frac{1}{M^{1/2-\gamma}} \right) \tag{6.13}$$

Remarks. The somewhat vague statement in (2) is made more explicit by the bounds provided in the proof.

As an immediate corollary of (1) we conclude that in the regime covered by Theorem 6.3 the rescaled eigenvalue process converges in distribution to a Poisson process.

Proof. 1. The asymptotic coalescence of the rescaled spectrum with ω is a direct consequence of the divergence of R_M , in the sense of Lemma 5.5 part (i), and the statement proven in Lemma 3.4 (for which the value of the parameter α diverges due to the singular nature of the limit of R_M).

2. In the regime discussed here, due to the asymptotic coalescence of the spectrum with ω , the ‘head’ term $\|\psi_{M,n}^{(H)}\|_\infty$ diverges, typically at the rate: $\|\psi_{M,n}^{(H)}\|_2^2 = \|\psi_{M,n}^{(H)}\|_\infty^2 \geq M^{2(\mathcal{E}_M/\lambda)^2(1-\gamma)}$, where $\gamma > 0$ can be taken arbitrarily small. More explicitly, the bounds of Lemma 5.5 combined those of with Lemma 3.4 allow to conclude that for each $W < \infty$, $\gamma > 0$ and $b < \infty$

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\min \{ \|\psi_{M,n}\|_\infty : u_{M,n} \in [-W, W] \} \geq bM^{(\mathcal{E}_M/\lambda)^2(1-\gamma)} \right) = 1. \tag{6.14}$$

For the ‘body’ term, $\|\psi_{M,n}^{(B)}\|_2$, which depends mainly on the gaps in ω , the bounds (6.9) and (6.10) apply with no change in the different spectral regimes considered in this work.

The ‘tail’ term does not vary by more than a factor $1 + \mathcal{O}(1/\ln M)$ among all the eigenfunctions within the window $[-W, W]$, cf. (5.19). One may note that

$$\|\psi_{M,n}^{(T)}\|_2^2 = \frac{d}{du} T_{M,\omega}(u_{M,n}, \ln M). \tag{6.15}$$

Applying the bounds of Lemma 5.4 and (5.19) one readily gets:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{n: u_{M,n} \in [-W, W]} \|\psi_{M,n}^{(T)}\|_2^2 \leq b \left[1 + M^{2(\mathcal{E}_M/\lambda)^2-1} \ln M \right] \right) = 1 \tag{6.16}$$

for each $W < \infty$, and $b < \infty$. The power law in this bound can be made intuitive by noting that the contribution to the sum from sites with regular values of $V(x)$ is itself about $M\Delta_M^2 \approx M^{2(\mathcal{E}_M/\lambda)^2-1}/\ln M$.

The estimates, (6.14), (6.10), and (6.16) directly imply the claim (6.13). In essence these bounds show that in the ℓ^2 -sense the ‘head’ contribution dominates throughout the regime discussed here. One may also note that among the other two terms the dominant one is the ‘tail’ for $\lambda/\mathcal{E} < \sqrt{2}$, and the ‘body’ for $\lambda/\mathcal{E} > \sqrt{2}$. \square

6.3. Localization Alongside an Isolated Extended State Near -1

The exclusion of $\lambda \leq \sqrt{2}$ in the second part of Theorem 6.1 is relevant. Intuitively, at $\lambda = 0$ the operator H_M has a single extended state (ϕ_0), which at $\lambda \approx 1$ starts to be passed, though ‘avoided crossings’ by a series of localized states. The next result shows that the picture of a single extended state embedded among localized states persists for $\lambda < \sqrt{2}$ (except for instances of hybridization during the avoided crossing which are too brief to be seen in the single λ snapshots that are discussed here). In Theorem 6.1 we saw that this picture changes at $\lambda = \sqrt{2}$, beyond which the operator acquires a band of extended states, with energies in the vicinity of $\mathcal{E} = -1$.

Theorem 6.4 (Non-resonant delocalized state). *For $\lambda < \sqrt{2}$, let \mathcal{E}_M be a sequence of energies satisfying $\lim_{M \rightarrow \infty} \mathcal{E}_M = -1$ and the condition (5.26). Then, within the scaling windows centered at \mathcal{E}_M :*

1. *There exists one eigenvalue, which occurs at (microscopic) energy*

$$u = \tau + o(1) \tag{6.17}$$

for which the corresponding eigenfunction ψ_E is ℓ^2 -delocalized, with

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\frac{\|\psi_E\|_2}{\|\psi_E\|_\infty} \geq M^{\frac{1}{\lambda^2} - \frac{1}{2} - \gamma} \right) = 1, \tag{6.18}$$

for any $\gamma > 0$.

2. All other eigenfunctions in the scaling window are ℓ^2 -localized in the sense that for each $W < \infty$ and $\gamma > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{n: |u_{M,n}| < W} \frac{\|\psi_{M,n}\|_2}{\|\psi_{M,n}\|_\infty} \leq 1 + \frac{1}{M^{\frac{1}{\lambda^2} - \frac{1}{2} - \gamma}} \right) = 1 \tag{6.19}$$

Proof. The behavior of the function $R_{M,\omega}(u)$ in this case, is described by (5.27) of Lemma 5.5:

1. the function is monotone and of high derivative, its typical order being

$$\frac{d}{du} R_{M,\omega}(u) = \mathcal{O} \left(\frac{M^{2(\mathcal{E}_M/\lambda)^2 - 1}}{\ln M} \right), \tag{6.20}$$

2. $R_{M,\omega}(u) = 0$ at $u = \tau + \mathcal{O}(\delta_M)$, with $\delta_M = (\ln M)^2 / M^{(\mathcal{E}_M/\lambda)^2 - 1/2}$,
3. the value of $S_{M,\omega}(\tau, \ln M)$ does not depend on τ , has the Cauchy distribution, and is typically of order 1.

It follows that the characteristic equation, $S_{M,\omega}(\tau, \ln M) = -R_{M,\omega}(u)$ has one solution at

$$u = \tau + \mathcal{O}(\delta_M) \tag{6.21}$$

and others at close proximity to the poles of $S_{M,\omega}(\tau, \ln M)$, as described in Theorem 4.4. The characteristics of the eigenfunctions can be read off this description by following the arguments which were used in the proofs of Theorems 6.3 and 6.1. □

7. Discussion

7.1. Relation with a Two-State Hybridization Criterion

While our main results concern the effects of resonances involving many localized approximate eigenfunctions, let us note their relation with a simple criterion for two-level eigenfunction hybridization.

In the simplest two-level system the Hamiltonian H and the corresponding resolvent operator $G(\zeta) := (H - \zeta \mathbb{1})^{-1}$ are of the form of the 2×2 matrices

$$H = \begin{pmatrix} \mathcal{E}_1 & \tau \\ \tau^* & \mathcal{E}_2 \end{pmatrix}, \quad G(\zeta) = \left[\begin{pmatrix} \mathcal{E}_1 - \zeta & \tau \\ \tau^* & \mathcal{E}_2 - \zeta \end{pmatrix} \right]^{-1}. \tag{7.1}$$

The spectrum and eigenfunctions can be found by studying the poles and residues of the resolvent matrix G . Of particular interest to us is the case where τ is small. In this situation, the system has two approximate eigenstates, or quasi-modes, corresponding to the (column) vectors $(1, 0)$ and $(0, 1)$, with τ serving as the mixing term, or the tunneling amplitude. The relevant quantity is the ratio of the quasi-modes' energy gap $\Delta\mathcal{E} = (\mathcal{E}_2 - \mathcal{E}_1)$ to the tunneling amplitude. A simple calculation shows that:

- (i) If $|\Delta\mathcal{E}| \gg |\tau|$ then the eigenfunctions of H are localized, i.e. $\Psi_1 \approx (1, 0)$, $\Psi_2 \approx (0, 1)$, to the leading order in $\tau/|\Delta\mathcal{E}|$;
- (ii) If $|\Delta\mathcal{E}| \ll |\tau|$ then the eigenfunctions are equidistributed between the two sites, and close to: $\Psi_1 \approx \frac{1}{\sqrt{2}}(1, 1)$, $\Psi_2 \approx \frac{1}{\sqrt{2}}(1, -1)$.

Turning to a system with a large configuration space and random potential, the following is a useful and established term.

Definition 7.1 (*Quasi-modes*). A *quasi-mode* for the self-adjoint operator H with discrepancy d is a pair (Ψ, \mathcal{E}) such that

$$\|(H - \mathcal{E})\Psi\| \leq d \|\Psi\|. \tag{7.2}$$

Tunneling amplitude is a regularly used term, however, its meaning is often left somewhat open, allowing for creative interpretation. In the context of operators with on-site disorder, we find the following formulation to be of relevance.

Definition 7.2 (*Tunneling amplitude*). For a collection of orthogonal quasi-modes (Ψ_j, \mathcal{E}_j) , with $\Psi_i \perp \Psi_j$ at $i \neq j$ and P_j the corresponding projections, we define the pairwise *tunneling amplitude* as $|\Sigma_{i,j}(\mathcal{E})|$ the modulus of the off diagonal term in the following representation of the operator’s resolvent, $(H - \mathcal{E})^{-1}$, restricted to the range of $P_i + P_j$:

$$\begin{pmatrix} G_{i,i}(\mathcal{E}) & G_{i,j}(\mathcal{E}) \\ G_{j,i}(\mathcal{E}) & G_{j,j}(\mathcal{E}) \end{pmatrix} = \left[\begin{pmatrix} \mathcal{E}_i + \Sigma_{i,i}(\mathcal{E}) & \Sigma_{i,j}(\mathcal{E}) \\ \Sigma_{j,i}(\mathcal{E}) & \mathcal{E}_j + \Sigma_{j,j}(\mathcal{E}) \end{pmatrix} \right]^{-1} \tag{7.3}$$

To place that in context, let us recall the Schur complement formula, which states that if for a specified pair (i, j) , the operator H is decomposed as $H = \mathcal{E}_i P_i + \mathcal{E}_j P_j + \hat{H}$, then (7.3) holds with $\Sigma(\mathcal{E}) = \begin{pmatrix} \Sigma_{i,i} & \Sigma_{i,j} \\ \Sigma_{j,i} & \Sigma_{j,j} \end{pmatrix}$ the 2×2 inverse of the restriction of $(\hat{H} - \mathcal{E})^{-1}$ to the range of $P_i + P_j$.

Consider now the situation in which an $M \times M$ matrix H_M has a large collection of quasi-modes, whose energies fluctuate with a considerable degree of independence at density $\mu_M(\mathcal{E})$. Thus in the vicinity of energy \mathcal{E} they have gaps of the order $\Delta_M(\mathcal{E}) = [M\mu_M(\mathcal{E})]^{-1}$. Assume also that the pairwise tunneling amplitudes are “mostly” of a common order of magnitude $\tau(M, \mathcal{E})$. The previous rank-two discussion suggests, as a ‘rule of thumb’ that at energies at which

$$\frac{\Delta_M(\mathcal{E})}{\tau(M, \mathcal{E})} \lesssim 1 \tag{7.4}$$

the localized quasi-modes would be unstable with respect to resonant delocalization, and the proper eigenstates will take the form of hybridized wave functions. This paper grew out of an attempt to develop further insight on the relevance of the criterion (7.4). We find that our results support its relevance in the present context.

More explicitly: from the expression (2.8) for the resolvent, we find that for the operator H_M the tunneling amplitude between a pair of the δ function quasi-modes is given by

$$\begin{aligned} \tau_{i,j}(\mathcal{E}) &= \frac{1}{M} \left| 1 + \langle \varphi_0, (\widehat{H}_M - \mathcal{E})^{-1} \varphi_0 \rangle \right| \\ &= \frac{1}{M} \left| 1 - \frac{1}{M} \sum_{n \neq i,j} \frac{1}{\kappa_M V_n - \mathcal{E}} \right|^{-1} \end{aligned} \tag{7.5}$$

where \widehat{H}_M is a modified version of H_M with $V_i = V_j = 0$. It may be noted that the ‘direct’ tunneling amplitude $1/M$ is boosted by the term $\langle \varphi_0, (\widehat{H}_M - \mathcal{E})^{-1} \varphi_0 \rangle$. Intuitively, that is so since the tunneling is through the state φ_0 .

By the estimates of Lemmas 5.4 and 5.2, typical value of the tunneling amplitude among states of energy in the vicinity of \mathcal{E} is:

$$\tau_{i,j}(\mathcal{E}) \approx \frac{1}{M} \left[1 - \widehat{\varrho}_M(\mathcal{E}) + \Theta \left(\frac{1}{M \Delta_M(\mathcal{E})} \right) \right]^{-1}. \tag{7.6}$$

Thus, the heuristic condition for resonant delocalization (7.4) corresponds to:

$$M \Delta_M(\mathcal{E}) \left| 1 - \widehat{\varrho}_M(\mathcal{E}) + \Theta \left(\frac{\sqrt{\ln M}}{\lambda M(\mathcal{E}/\lambda)^2} \right) \right| \lesssim 1. \tag{7.7}$$

which played a role in our discussion in Sect. 5.3 (see (5.11)).

To answer the question posed above we note that the two-level resonance condition is pointing at the right direction. At the same time, rank-two analysis alone does not yet address a number of relevant points, such as:

- (i) a possible stratification of quasi-mode pairs, e.g., by the distance (which in the tree graph example studied in [2] affects the tunneling amplitude) or by the efficacy of mixing channels,
- (ii) the effects of possible interactions among distinct quasi-modes, as well as other states
- (iii) the question of formation of a *band* of extended (or semi-extended) eigenstates.

The last has been tackled here through the more detailed analysis of the structure of the resolvent, and the Green function’s local scaling limit.

We also found that the hybridization among levels which resonate through a single channel, as in the present example, yields delocalization in only a partial sense: it is delocalization in the spacial distribution of the wave function and in the ℓ^1 -sense, but not in the ℓ^2 -sense, meaning that most of the state’s ℓ^2 -mass is carried by few localized sites. This point is discussed next.

7.2. Different Notions of Delocalization

Localization and delocalization of the eigenfunctions on a finite or infinite graph can be formulated in terms which may either depend on the graph’s metric or be independent of it. It is a relevant observation that the two terminologies need not coincide. The point is exemplified by functions which are the sum of few localized wave packets, which are located at large distance from each other. Judging by their spatial spread, such functions would be deemed *delocalized*, whereas judging by the number of points on which the bulk of the corresponding probability distribution (or ℓ^2 -norm) is supported, the function

may be viewed as *localized*. This distinction is of relevance in the case discussed below, and more generally wherever eigenfunctions are formed through resonances among local quasi-modes.

The two different forms of localization for functions on a graph \mathcal{G} can be quantified through the quantities:

$$\begin{aligned}
 p_{\text{diam}}(d) &= \inf \left\{ \sum_{x \in \mathcal{G} \setminus A} \frac{|\psi(x)|^2}{\|\psi\|_2^2} \mid A \subset \mathcal{G}, \text{diam}(A) \leq d \right\} \\
 p_{\text{vol}}(d) &= \inf \left\{ \sum_{x \in \mathcal{G} \setminus A} \frac{|\psi(x)|^2}{\|\psi\|_2^2} \mid A \subset \mathcal{G}, \text{card}(A) \leq d \right\}
 \end{aligned}
 \tag{7.8}$$

with $\text{diam}(A)$ the diameter of the set A , $\text{card}(A)$ the set's cardinality.

Spatial localization can be expressed through suitable bounds on the distances at which $p_{\text{diam}}(d)$ reaches small values. For example, ‘spatial exponential localization’ with localization length ξ may be expressed by a bound of the form

$$p_{\text{diam}}(d) \leq C e^{-d/\xi},
 \tag{7.9}$$

at some $C < \infty$.

ℓ^2 -localization is similarly expressed through bounds on the inverse function of $p_{\text{vol}}(d)$. Exponential ℓ^2 -localization would be expressed by a bound of the form

$$p_{\text{vol}}(d) \leq C e^{-d/\alpha},
 \tag{7.10}$$

with some $C, \alpha < \infty$. The inverse of that function shows how many sites does it take to capture all but fraction p of the function's ℓ^2 -mass.

In case the discussion concerns not a fixed graph but a sequence of graphs, of diverging diameters, the two notions of localization may be tested by whether the inverse functions of $p_{\text{diam}}(d)$, and correspondingly $p_{\text{vol}}(d)$ are uniformly bounded, or at least grow at slow rate.

In the converse direction, the term delocalization can also be given different meanings:

Spatial delocalization on scale ξ_M , can be expressed by the condition that eigenfunctions with energies in the specified range typically satisfy:

$$\sum_{x,y \in \mathcal{G}_M} |\psi(x)|^2 |\psi(y)|^2 \mathbb{1}[\text{dist}(x,y) \geq \xi_M] \geq p_0 \|\psi\|_2^2.
 \tag{7.11}$$

for some M -independent $p_0 > 0$. (Here \mathcal{G}_M are graphs of growing diameter).

ℓ^2 -delocalization for a sequence of functions on graphs \mathcal{G}_M of growing diameter, ℓ^2 -delocalization is expressed in the statement that

$$\lim_{M \rightarrow \infty} p_{\text{vol}}(d) = 1
 \tag{7.12}$$

for all $d < \infty$.

A more standard formulation of delocalization is through the function's *inverse participation ratio* (with $q = 2$ or more generally $q > 1$)

$$P_q(\psi) := \frac{\sum_x |\psi(x)|^{2q}}{[\sum_x |\psi(x)|^2]^q}. \tag{7.13}$$

These are linked to the norm ratio $r(\psi) := \|\psi\|_\infty / \|\psi\|_2$ (with $\|\psi\|_\infty := \max_{x \in \mathcal{G}_M} |\psi(x)|$) through the bounds:

$$r(\psi)^{2q} \leq P_q(\psi) \leq r(\psi)^{2(q-1)}. \tag{7.14}$$

(The upper bound is implied by a convexity argument, and the lower bound is due to the contribution of the site at which ψ is maximized.)

Thus ℓ^2 -delocalization (7.12) is equivalently expressed in the vanishing, in the suitable sense, of the inverse participation ratio P_2 , or of the norm ratio $r(\psi)$. In dynamical terms this can be viewed as the opposite of *positive recurrence* (terminology which is suggested by an analogy with a classical Markov chain term).

The example considered here is degenerate, since on the complete graph the distance between neighboring sites equaling the graph's diameter (i.e. except for the situation of total localization—when the ℓ^2 -mass is asymptotically concentrated at a single site, the function's support is of diameter comparable with that of the entire graph). Thus, in Theorem 6.1, we find that resonant delocalization occurs in the sense of spatial delocalization without meeting the standard ℓ^2 -delocalization criterion. However, this partial delocalization does coincide with the following weaker measure of the spread of the wave function

ℓ^1 -delocalization: as the size of the system is taken to infinity, for all $q \in (0, 1/2]$, the ratio $P_q(\psi)$ diverges in the distributional sense (for eigenfunctions with energies in a specified range).

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Appendix A. The Spectral Range of $H_{M,\omega}$ and the Ground State Transition

This appendix is devoted to a proof of Theorem 2.1. We abbreviate $E_0 = \min \sigma(H_M)$. Since T is of rank one, the eigenvalues of H_M and $\kappa_M V$ interlace. Since $T \leq 0$ and by the extreme-value statistics of Gaussian random variables,

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_x V(x) > \frac{\lambda}{\kappa_M} - \frac{\ln(4\pi \ln M)}{2\sqrt{2 \ln M}} + \frac{u}{\sqrt{2 \ln M}} \right) \rightarrow 1 - \exp(-e^{-u}), \tag{A.1}$$

for any $u \in \mathbb{R}$ (cf. [13]), this implies that for any $\varepsilon > 0$:

$$\lim_{M \rightarrow \infty} \mathbb{P}(d_H(\sigma(H_M) \setminus \{E_0\}, [-\lambda, \lambda]) > \varepsilon) = 0. \tag{A.2}$$

Thus all the spectrum, except for the ground state, converges as claimed. We now distinguish two cases.

The case $0 < \lambda < 1$. According to Proposition 2.2, E_0 is the smallest solution of the equation $F_M(E) = 1$. For the remainder of the proof we may assume that

$$\max_x |\kappa_M V(x)| \leq \lambda \tag{A.3}$$

since this holds with asymptotically full probability by (A.1). Under this assumption, for any $E < -\lambda$:

$$F_M(E) = \frac{1}{M} \sum_{x=1}^M \frac{\mathbb{1}[\kappa_M V(x) \geq -\lambda]}{\kappa_M V(x) - E} = T_M\left(E, -\frac{\lambda + E}{\kappa_M}\right). \tag{A.4}$$

This implies that for any $E < -\lambda$, any $\eta > 0$ and all sufficiently large M :

$$\lim_{M \rightarrow \infty} \mathbb{P}\left(|F_M(E) - \widehat{\varrho}_M(E)| \geq M^{-1/2+\eta}\right) = 0. \tag{A.5}$$

The proof of (A.5) either follows directly from Lemma 5.2 and 5.4. Alternatively, it is derived using the representation $F_M(E) = -(\lambda + E)^{-1} - \int_{-\lambda}^{\infty} \frac{1}{M} \sum_x \mathbb{1}[\kappa_M V(x) \geq t] \frac{dt}{(t-E)^2}$ and the Dvoretzky–Kiefer–Wolfowitz inequality [17]:

$$\mathbb{P}\left\{\sup_t \left| \frac{1}{M} \sum_x \mathbb{1}[\kappa_M V(x) \geq t] - \int_{t/\kappa_M}^{\infty} \frac{e^{-s^2/2}}{\sqrt{2\pi}} \frac{ds}{\sqrt{M}} \right| \geq \frac{R}{\sqrt{M}} \right\} \leq C e^{-2R^2} \tag{A.6}$$

for any $R > 0$.

Since $\widehat{\mathcal{E}}_M^{(-1)}$ is the unique solution of $\widehat{\varrho}_M(E) = 1$ for $E \leq -1 < -\lambda$, the bound (A.5) implies that for any $\eta > 0$ with asymptotically full probability $|E_0 - \widehat{\mathcal{E}}_M^{(-1)}| \leq CM^{-1/2+\eta}$. From (5.14) we hence conclude $E_0 = -1 - \kappa_M^2 + O(\kappa_M^4)$.

To prove the strong delocalization of the eigenfunction $\psi_0(x) = (\kappa_M V(x) - E_0)^{-1}$ corresponding to E_0 in this case, we use (A.3) to estimate:

$$\sqrt{M} \geq \frac{\|\psi_0\|_2}{\|\psi_0\|_{\infty}} \geq \sqrt{M} \frac{1 - \lambda}{2(1 + \lambda)}. \tag{A.7}$$

This concludes the proof in case $0 < \lambda < 1$.

The case $\lambda > 1$. In this case, the statement is contained in Theorem 6.3 applied with $\mathcal{E} = -\lambda$. Alternatively, we may directly bound $E_0 \leq \min \sigma(\kappa_M V)$. For a lower bound we may use the variational characterization and the Cauchy–Schwarz inequality:

$$E_0 \geq \inf_{\|\psi\|_2=1} \left[\sum_x \kappa_M V(x) \psi(x)^2 - \frac{1}{M} \sum_y \frac{1}{\alpha(y)} \sum_x \psi(x)^2 \alpha(x) \right] \tag{A.8}$$

for any $\alpha > 0$. We pick

$$\alpha(x) = \begin{cases} R + M(\kappa_M V(x) - \min \sigma(\kappa_M V)), & V(x) \leq -R \\ \frac{M}{1-\delta}, & V(x) > -R, \end{cases} \tag{A.9}$$

where $\delta > 0$ is such that $(1 - \delta)^{-1} < \lambda$, and $R > 0$ is a large number independent of M to be chosen shortly. One can check that

$$\sum_{V(x) \leq -R} \frac{1}{\alpha(x)} \leq \delta/2 \tag{A.10}$$

for sufficiently large $R > 0$. Therefore $\sum_x \alpha(x)^{-1} \leq 1 - \delta/2$ which by (A.8), yields

$$E_0 \geq \min \sigma(\kappa_M V) - \frac{2R}{M}. \tag{A.11}$$

By the same argument, the minimum in (A.8) is attained on a function ψ_0 which satisfies (2.5).

Appendix B. A Useful Estimate

In Lemma 5.4 use was made of the following bound.

Lemma B.1. *Let $\varrho(v) = (2\pi)^{-1/2} \exp(-v^2/2)$. Then for some $0 < c, C < \infty$ and any $0 < \delta \leq 1$ and any $v \in \mathbb{R}$:*

$$\frac{c}{(1 + |v|)^2} \leq \int_{|u-v| \geq \delta} \frac{\varrho(u) du}{|u - v|^2} \leq C \left\{ \frac{\varrho(v)}{\delta} + \frac{1}{(1 + |v|)^2} \right\}. \tag{B.1}$$

Proof. Let us assume that $|v| \geq 1$; the proof is similar and simpler for $|v| \leq 1$. For the upper bound we decompose the integral into two parts. The first part is estimated using the elementary inequality $\varrho(v + x) \leq \varrho(v)e^{|vx|}$:

$$\int_{\delta \leq |u-v| \leq C_1|v|^{-1}} \frac{\varrho(u)}{|u - v|^2} du \leq e^{C_1} \frac{\rho(v)}{\delta}. \tag{B.2}$$

The second integral is bounded by

$$\int_{|u-v| \geq C_1|v|^{-1}} \frac{e^{-u^2/4}}{\sqrt{2\pi}} \frac{e^{-u^2/4}}{|u - v|^2} du \leq C_2 \max_{|u-v| \geq C_1/|v|} \frac{e^{-u^2/4}}{|u - v|^2}. \tag{B.3}$$

For suitably chosen $C_1 > 0$, the maximum is attained inside the interval $[-1/2, 1/2]$, and is thus bounded by $C_3/|v|^2$.

For the lower bound we estimate the contribution from $|v| \leq 1$:

$$\int_{\substack{|u-v| \geq \delta \\ |u| \leq 1}} \frac{\varrho(u) du}{|u - v|^2} \geq \min_{|u| \leq 1} \varrho(u) \int_{|u-v| \geq \delta} \frac{du}{|u - v|^2} \geq \frac{c}{(1 + |v|)^2}. \tag{B.4}$$

□

Of possible interest is also the following exponential improvement of the variance bound of Lemma 5.4.

Lemma B.2. For $\lambda > 0$, $\mathcal{E} \in (-\lambda, \lambda)$, $|u| < L$, and any $\tau > 0$

$$\begin{aligned} & \mathbb{P} \left(|T_M(u, L) - \mathbb{E}(T_M(u, L))| \geq \tau \right) \\ & \leq 2 \exp \left(-c\tau \min \left\{ \frac{\tau}{\mathbb{E} \left(\left| \frac{d}{du} T_M(u, L) \right| \right)}, L \right\} \right) \end{aligned} \tag{B.5}$$

with some numerical constant $c > 0$.

Proof. The probability bound is based on the exponential Chebyshev estimate:

$$\mathbb{P} (|X| \geq \tau) \leq e^{-t\tau} (\mathbb{E} [e^{tX}] + \mathbb{E} [e^{-tX}]) \tag{B.6}$$

applied to $X = T_M(u, L) - \mathbb{E}(T_M(u, L))$, and optimized over $t \geq 0$.

In a variant of the argument which was used in the proof of Lemma 5.4, the moment generating function can be estimated by noting that $\mathbb{E} [e^{tT_M(u, L)}]$ is an average of a product of M functions of iid random variables. One obtains:

$$\begin{aligned} \mathbb{E} [e^{tT_M(u, L)}] &= \prod_n \left(1 + \mathbb{E} \left[e^{t \frac{\mathbb{1} [|\omega_{M,n}| > L]}{\omega_{M,n} - u}} - 1 \right] \right) \\ &\leq \exp \left\{ t \mathbb{E}[T_M(u, L)] + \frac{t^2}{2} e^{\frac{|t|}{L}} \mathbb{E} \left[\sum_n \frac{\mathbb{1} [|\omega_{M,n}| > L]}{|\omega_{M,n} - u|^2} \right] \right\}. \end{aligned} \tag{B.7}$$

where the inequality is based on the elementary bounds: $e^x \leq 1 + x + \frac{x^2}{2} e^{|x|}$, for $x \in \mathbb{R}$, and $1 + x \leq e^x$, for $x \geq 0$. The second expectation value in the exponential equals

$$\mathbb{E} \left[\sum_n \frac{\mathbb{1} [|\omega_{M,n}| > L]}{|\omega_{M,n} - u|^2} \right] = \mathbb{E} \left[\left| \frac{d}{du} T_M(u, L) \right| \right]. \tag{B.8}$$

The choice $t = \min \{L, \tau / (e \mathbb{E} [|\frac{d}{du} T_M(u, L)|])\} > 0$ in the Chebyshev inequality yields the claim (B.5). \square

For explicit probability bounds (B.5) may be combined with (5.17) of Lemma 5.4. Although it was not used in the present work, this exponential bound is included here since it allows to strengthen the implications on the fluctuations of $T_{M,\omega}(u, L)$ into estimates which apply uniformly over macroscopically broad ranges of \mathcal{E} .

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