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A Limit Equation Criterion for Solving the Einstein Constraint Equations on Manifolds with Ends of Cylindrical Type

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Abstract. We prove that in a certain class of conformal data on a manifold with ends of cylindrical type, if the conformally decomposed Einstein constraint equations do not admit a solution, then one can always find a nontrivial solution to the limit equation first explored by Dahl et al. (Duke Math J 161(14):2669–2798, 2012). We also give an example of a Ricci curvature condition on the manifold which precludes the existence of a solution to this limit equation. This shows that the limit equation criterion can be a useful tool for proving the existence of solutions to the Einstein constraint equations on manifolds with ends of cylindrical type.

1. Introduction

It has been known for over 60 years that Einstein's equations in vacuo can be decomposed into a well-posed initial value problem in which a Riemannian n-manifold $(M,\overline{g})$ along with a symmetric 2-tensor $\overline{K}$ can be isometrically embedded as a spacelike slice of a spacetime with extrinsic curvature tensor $\overline{K}$ if and only if the Einstein constraint equations

$$R_{\overline{g}} - |\overline{K}|^2_{\overline{g}} + (\mathrm{tr}_{\overline{g}}\overline{K})^2 = 2\Lambda \tag{1.1}$$

$$\mathrm{div}_{\overline{g}}\overline{K} - \nabla \mathrm{tr}_{\overline{g}}\overline{K} = 0 \tag{1.2}$$

are satisfied. Here, Λ is the cosmological constant.

The study of these equations has been an active field of research in recent years, and several approaches have been taken to solve (1.1–1.2). The most common is the conformal method, in which one specifies a background metric g and looks for a solution of the constraints in the conformal class of g. In particular, we seek a scalar function ϕ and a vector field W such that $(\overline{g},\overline{K})$ solves (1.1–1.2) where this pair is of the form:

$$\overline{g}_{ij} = \phi^{N-2} g_{ij} \tag{1.3}$$

$$\overline{K}_{ij} = \frac{\tau}{n}\overline{g}_{ij} + \phi^{-2}(\sigma_{ij} + (LW)_{ij}). \tag{1.4}$$

Here, $\tau = \mathrm{tr}_{\overline{g}}\overline{K}$, σ is a specified transverse (i.e., divergence free) traceless tensor, L is the conformal Killing operator which we define below, and N is a dimensional constant we will frequently use throughout this paper, given by

$$N = \frac{2n}{n-2}.$$

It is well known (see, for example, [8]) that the constraint equations are thus reduced to solving a particular semilinear elliptic system in ϕ and W, which is often called the LCBY equations in honor of Lichnerowicz, Choquet-Bruhat, and York who first studied them:

$$\Delta_g\phi - c_n R_g\phi = \beta_\Lambda\phi^{N-1} - c_n|\sigma + LW|_g^2\phi^{-N-1} \tag{1.5}$$

$$\mathrm{div}LW = \frac{n-1}{n}\phi^N d\tau. \tag{1.6}$$

Equation (1.5) is known as the *Lichnerowicz equation* and in it we have introduced the dimensional constant

$$c_n = \frac{n-2}{4(n-1)}$$

and defined the function β_Λ as

$$\beta_\Lambda = \beta_\Lambda(\tau) = \frac{n-2}{4n}\tau^2 - \frac{n-2}{2(n-1)}\Lambda.$$

We will refer to Eq. (1.6) as the vector equation.

Many of the efforts aimed at finding solutions to the constraint equations have been focused on determining the solvability of the system (1.5–1.6). One of the earliest systematic studies of these equations was the 1995 paper of J. Isenberg [13] which considered the case of constant mean curvature (CMC) τ on closed manifolds, in which case the LCBY system decouples since the vector equation becomes trivial. In the years since, many authors have sought to generalize Isenberg's results on closed manifolds with near-CMC conditions [1,12], and more recently these near-CMC conditions have been relaxed by Holst, Nagy, and Tsogtgerel in [11] in the non-vacuum case, and by Maxwell in [16]. Their results required a Schauder fixed point argument to find a solution to the LCBY equations under the condition that the tensor σ have sufficiently small norm.

These methods for finding solutions have been extended to other geometries including asymptotically Euclidean [7], asymptotically hyperbolic [9], and compact with boundary [6,10]. General results for the solvability of the LCBY equations on manifolds with asymptotically cylindrical or periodic geometries (which we collectively call *geometries of cylindrical type*) appeared in 2012 with the pair of papers [2] and [3] which analyzed the Lichnerowicz equation and vector equation separately. It was noted in the first of these that several known examples of black hole spacetimes admit CMC hypersurfaces with ends of cylindrical type, including the extreme Kerr metrics. Solutions to the fully coupled LCBY equations were recently constructed on such manifolds in [15] using an adaptation of the Schauder fixed point technique.

Another approach to finding solutions of the LCBY equations on closed manifolds is the relatively recent result of Dahl, Gicquaud, and Humbert [4]. In that paper, the authors showed that in the absence of conformal Killing fields, so long as the mean curvature τ is non-vanishing and $\sigma \not\equiv 0$ for the non-negative Yamabe classes, then the system (1.5–1.6) fails to have a solution only if there is a nontrivial solution W to

$$\mathrm{div} LW = \alpha_0 \sqrt{\frac{c_n}{\beta_\Lambda}} |LW| d\tau \tag{1.7}$$

for some $\alpha_0 \in (0, \frac{n-1}{n}]$ [only the case $\Lambda = 0$ was addressed in [4], but their method extends trivially to prove this result for non-zero Λ]. This is known as the *limit equation* for reasons that will become clear in the proof of the theorem. We will give a summary of their proof in the next section.

This result has thus far proven to be quite adaptable to other (i.e., non-compact) geometries. Indeed, a limit equation result has very recently been shown to hold in the asymptotically Euclidean near-CMC setting in [5] and in the asymptotically hyperbolic setting in [9], but difficulties arose in proving it in the compact with boundary case, c.f. [6]. It is thus natural to wonder in which geometries such a result holds. In this paper, we prove that an analogous limit equation result holds on a manifold with asymptotically cylindrical ends, thus shedding more light on solvability of the constraint equations in such a geometry. We will also give an example of (conformal) initial data which does *not* admit any solutions to (1.7), and thus necessarily admits a solution to the LCBY system (1.5–1.6). We will do this by imposing a particular reasonable bound on the Ricci curvature.

We mention here a recent work of David Maxwell in [17]. In that paper he discusses the conformal method along with two formulations of the related conformal thin sandwich method. He shows that these methods are all equivalent and lead to the same solutions of the constraints. In the standard conformal method which we use in this paper, the solutions are not conformally covariant in the sense that conformally related data (including proper scaling of the non-metric data) do not lead to the same solution. However, Maxwell shows that there exists a conformally covariant interpretation of this method which is equivalent to our own.

In the context of this paper, Maxwell's results imply that if we replaced the conformal Killing operator L in (1.5–1.6) with $\frac{1}{2f}L$ for some positive function f, we could treat our method as conformally covariant. Thus, we can work with any metric in the same conformal class as g. In particular, assumptions on scalar curvature, such as $c_n R_g + \beta_\Lambda > c > 0$, can be replaced by the assumption that there is some metric g' in the conformal class of g that satisfies (all of) the postulated inequalities. However, since this paper was prepared before Maxwell's work was released, we did not make these adjustments.

1.1. Notation and Definitions

We will assume that our manifold (M, g) is asymptotically cylindrical. This means that there is some compact set $\mathcal{K} \subset M$ whose complement in M decomposes as

$$M \setminus \mathcal{K} = \bigsqcup_{\ell=1}^{m} E_\ell.$$

Here, for each ℓ, there is some closed $(n-1)$-manifold N_ℓ such that E_ℓ is diffeomorphic to the half-cylinder $\mathbb{R}_+ \times N_\ell$. Moreover, on each end the metric g decays exponentially to the exactly cylindrical metric $dt^2 + \mathring{g}_\ell$, where $\mathring{g}_\ell$ is some metric on N_ℓ and t is some C^3 coordinate function on the ends which is "radial" in the sense that there are positive constants $C_1 \leq C_2 \leq C_3$ such that

$$C_1 + t \leq C_2 + \mathrm{dist}_g(\cdot, \partial E_\ell) \leq C_3 + t.$$

We impose a similar decay condition on its derivatives, which can be made precise by requiring that on each end

$$|\mathring{\nabla}^k[g - (dt^2 + \mathring{g}_\ell)]| = \mathcal{O}(e^{-\omega t})$$

for some positive ω for all applicable k. Here, $\mathring{\nabla}$ is the covariant derivative associated with the exactly cylindrical metric. We will call a metric $\check{g}$ *conformally asymptotically cylindrical* if we may write it in the form $\check{g} = w^{N-2} g$ where g is an asymptotically cylindrical metric and w is a positive function such that, on each end E_ℓ, $w \to \mathring{w}_\ell$ at the rate $\mathcal{O}(e^{-\omega t})$ along with its derivatives, where $\mathring{w}_\ell$ is a smooth positive function on N_ℓ. ("Smooth" in this paper will mean "as smooth as the metric.") Note that this condition implies that the metric $\check{g}$ does not decay on the ends, and so metrics with conic or cusp singularities do not belong to the class of conformally asymptotically cylindrical metrics according to our definition.

We would like the class of conformal initial data (g, σ, τ) we consider to have certain nice asymptotic properties like the metric. We will thus restrict to *tame* initial data, defined as follows.

Definition 1.1. Conformal initial data (g, σ, τ) is said to be tame if g is (conformally) asymptotically cylindrical and both of the following are satisfied:

- $\beta_\Lambda(\tau) \geq \beta_\Lambda(\tau_0) > 0$ and $\tau^2 \to \mathring{\tau}_\ell^2 > 0$ on the end N_ℓ at the rate $\mathcal{O}(e^{-\omega t})$, where $\mathring{\tau}_\ell^2$ and $\beta_\Lambda(\tau_0)$ are constants. Let $\mathring{\tau}$ be a smooth function with $\mathring{\tau} = \mathring{\tau}_\ell$ on each end E_ℓ.
- $|\sigma|_g^2 \to \mathring{\sigma}^2$ at the rate $\mathcal{O}(e^{-\omega t})$ where $\mathring{\sigma}^2 \not\equiv 0$ is some smooth nonnegative function on N_ℓ, or any smooth function on N_ℓ if $R < 0$ on the ends.

Now that we have defined our admissible initial data, we need to define the function spaces in which we will perform our analysis. These will be defined as in [15]. For any $1 \leq p < \infty$, define the weighted L^p-Sobolev space $W_\delta^{k,p}$ to be the space of all functions which are finite with respect to the norm:

$$\|X\|_{W_\delta^{k,p}} = \left(\sum_{j \leq k} \int_M |\nabla^j X|^p e^{-p\delta t} dV_g \right)^{1/p}. \tag{1.8}$$

As in [3], we will denote the function space $W^{k,2}_\delta$ by H^k_δ. Next, assume without loss of generality that the radius of injectivity for (M, g) is >1. Then if $B_1(q)$ is the ball of radius 1 about the point $q \in M$, we define local and global weighted Hölder norms by

$$\|X\|_{k,\mu;B_1(q)} = \sum_{i=0}^{k} \sup_{B_1(q)} |\nabla^i X| + \sup_{x,y\in B_1(q)} \frac{|\nabla^k X(x) - \nabla^k X(y)|}{d_g(x,y)^\mu}$$

$$\|X\|_{k,\mu} = \sup_{q\in M} \|X\|_{k,\mu;B_1(q)},$$

and also the weighted Hölder norms by

$$\|X\|_{k,\mu,\delta} = \|e^{-\delta t} X\|_{k,\mu}.$$

The space of all functions (or tensor fields) for which the global Hölder norm is finite will be denoted by $C^{k,\mu}$, and the space of functions (or tensor fields) which are finite with respect to the weighted Hölder norm will be denoted by $C^{k,\mu}_\delta$. We similarly denote the standard sup-norm by $\|\cdot\|_\infty$ (or by $\|\cdot\|_0$ if restricted to C^0) and the corresponding weighted sup-norm by $\|\cdot\|_{\infty,\delta}$ (or $\|\cdot\|_{0,\delta}$).

1.2. Preliminary Results

The conformal Killing operator L which appears in the LCBY equations is given by

$$(LX)_{ij} = \nabla_i X_j + \nabla_j X_i - \frac{2}{n}\text{div} X g_{ij}.$$

This is merely the trace-free part of the deformation tensor associated with X. Any vector field in the nullspace of L is called a *conformal Killing field*, and the operator div$L$ is known as the *conformal vector Laplacian*. The mapping properties of this operator acting between weighted Sobolev spaces were studied extensively in [3]. The most important mapping property for us will be [3, Thm 6.1]. Before we state the theorem, we first define $\mathscr{Y}_\ell$ to be the set of all globally bounded conformal Killing fields with respect to the *exactly* cylindrical metric on E_ℓ. If we then define a smooth cutoff function χ_ℓ which vanishes outside of E_ℓ and is identically 1 where $t \geq 1$ on E_ℓ, we set $\mathscr{Y} = \oplus_\ell \{\chi_\ell Y : Y \in \mathscr{Y}_\ell\}$. The theorem then gives us the mapping properties of div$L$ acting as an operator between weighted Sobolev spaces:

Theorem 1.2. *Let (M, g) be a Riemannian n-manifold with a finite number of ends which are asymptotically cylindrical. Suppose further that there are no global L^2 conformal Killing fields. Then there exists a number $\delta_* > 0$ such that if $0 < \delta < \delta_*$, then*

$$\text{div}L : H^{k+2}_\delta(TM) \to H^k_\delta(TM)$$

is surjective and the map

$$\text{div}L : H^{k+2}_{-\delta}(TM) \to H^k_{-\delta}(TM)$$

is injective for every $k \geq 0$. Moreover, if $0 < \delta < \delta_*$, then for all $k \geq 0$ the map

$$\mathrm{div}L : H^{k+2}_{-\delta}(TM) \oplus \mathscr{Y} \to H^k_{-\delta}(TM)$$

is surjective with finite dimensional nullspace.

This theorem is proven using a parametrix construction found in [18] to build a generalized inverse $G : H^k_{-\delta} \to H^{k+2}_{-\delta} \oplus \mathscr{Y}$ (the latter space being a subspace of H^{k+2}_{δ}) which satisfies $\mathrm{div}L \circ G = Id$. As shown in that paper, this parametrix is also a bounded map between weighted L^p-Sobolev spaces and weighted Hölder spaces, and so Theorem 1.2 also holds when one considers $\mathrm{div}L$ as a map between such spaces. The observation that $\mathrm{div}L : W^{k+2,p}_{-\delta} \oplus \mathscr{Y} \to W^{k,p}_{-\delta}$ is surjective will be crucial in the analysis below.

The final analytical tools we will need are the Sobolev embedding theorems for function spaces on asymptotically cylindrical manifolds. In general, we will assume $p > n$ and that $\delta < \delta_*$, where δ_* is as in Theorem 1.2. We also need that the coefficients in the Lichnerowicz equation are regular enough to imply that the solution $\phi \in C^2$. This could be achieved by assuming the coefficients are in $W^{1,p}$ or in $C^{0,\beta}$ for some $1 > \beta > 0$, and thus we make the metric $W^{3,p}$ or $C^{2,\beta}$ to guarantee this regularity of the scalar curvature. The standard Sobolev embedding theorems hold with a subscript δ. In particular, we have the following theorem.

Theorem 1.3. *Assume the radial variable t lies in C^k with bounded derivatives up to k-th order.*

- *If $l - \frac{n}{q} = k - \frac{n}{p}$ and $k \geq l$, then $W^{k,p}_{\delta}(M) \subseteq W^{l,q}_{\delta}(M)$, with a continuous embedding.*
- *If $k - \frac{n}{p} = r + \alpha$, then $W^{k,p}_{\delta}(M) \subseteq C^{r,\alpha}_{\delta}(M)$, with a continuous embedding.*
- *If $k - \frac{n}{p} > r$, then the embedding $W^{k,p}_{\delta}(M) \hookrightarrow C^{r}_{\delta'}(M)$ is compact for any $\delta' > \delta$.*

One proves this theorem using an exhaustion of M by compact sets and standard embedding arguments. The details are left to the reader.

2. Summary of the Main Results

In what follows, we will first use a strategy very similar to that in [15] to prove that there is a solution to the subcritical LCBY equations:

Theorem 2.1. *Let (g, σ, τ) be tame conformal data on a complete Riemannian n-manifold (M, g) with a finite number of asymptotically cylindrical ends. Assume that there are no global L^2 conformal Killing fields. If $\sigma \in W^{1,p}$, $\tau - \mathring{\tau} \in W^{1,p}_{-\delta}$, and the scalar curvature R satisfies $c_n R + \beta_\Lambda > c > 0$, then for any $0 < \epsilon < 1$ the subcritical LCBY equations*

$$\Delta_g \phi - c_n R_g \phi = \beta_\Lambda \phi^{N-1} - c_n |\sigma + LW|^2_g \phi^{-N-1} \tag{2.1}$$

$$\mathrm{div}LW = \frac{n-1}{n} \phi^{N-\epsilon} d\tau \tag{2.2}$$

have a solution (ϕ, W) *with* $\phi - \mathring{\phi} \in W^{3,p}_{-\nu}$ *for some* $\nu > 0$ *and* $\phi > 0$ *and* $W \in W^{2,p}_{-\delta} \oplus \mathscr{Y}$. *The function* $\mathring{\phi}$ *is defined below in* (3.2).

Once we know that the system (2.1–2.2) can always be solved, we next establish that the scalar part of such a solution ϕ can be uniformly (i.e., independent of ϵ) bounded above by the energy $\|LW\|_{L^2}$. Thus to any sequence of sufficiently small positive numbers $\epsilon_i \to 0$, we can associate a sequence of energies

$$\gamma_i = \int_M |LW_i|^2 \tag{2.3}$$

where (ϕ_i, W_i) is some solution to (2.1–2.2) with $\epsilon = \epsilon_i$. It is natural to ask what happens to the sequence $\{\gamma_i\}$ as $i \to \infty$. As was shown by [4] in the closed manifold case, this limit gives us valuable information about the solvability of the LCBY Eqs. (1.5–1.6). In particular, if this sequence stays bounded, then there exists a solution to the LCBY equations with the same hypotheses on conformal data as in Theorem 2.1. On the other hand, if $\gamma_i \to \infty$, we show that the limit Eq. (1.7) admits a nontrivial solution. The precise statement of this result is our main theorem:

Theorem 2.2. *Let* (g, σ, τ) *be conformal data on an asymptotically cylindrical manifold satisfying the conditions of Theorem* 2.1*. Then at least one of the following is true:*

- *The system* (1.5–1.6) *admits a solution* (ϕ, W) *with regularity as in Theorem* 2.1*. Also, the set of these solutions is compact.*
- *There exists a non-zero solution* $W \in W^{2,p}_{-\delta} \oplus \mathscr{Y}$ *of the limit equation*

$$\mathrm{div} LW = \alpha_0 \sqrt{\frac{c_n}{\beta_\Lambda}} |LW| d\tau$$

for some $\alpha_0 \in (0, \frac{n-1}{n}]$ *such that* $|LW| \leq Ce^{-\delta t}$ *for some* C *independent of* ϕ_i, W_i *and* W.

Note that we have no reason to suspect that this result is a dichotomy. That is, it may hold that the LCBY equations and the limit equation *both* admit nontrivial solutions. The utility of the main theorem is that we may show the existence of solutions to the LCBY equations whenever we can show that the limit Eq. (1.7) admits no solutions. The idea behind the proof is that given any sequence $\{\gamma_i\}$ as above, if the limit equation admits no solutions then the sequence stays bounded, which in turn implies that the LCBY equations admit a solution (these facts are proven in Sect. 4 below). From the perspective of one trying to understand the constraint equations on asymptotically cylindrical manifolds, this result is only useful if we can find examples of conformal data which admit no solutions to the limit equation. In Sect. 5, we show that this set is nonempty by proving the following corollary.

Corollary 2.3. *Let* (g, σ, τ) *be conformal data on an asymptotically cylindrical manifold satisfying the conditions of Theorem* 2.1, *and suppose* $\mathrm{Ric} \leq (c_1\chi^2 -$

$c_2e^{-2\mu t})g$ for some constants $c_i, \mu > 0$ and a smooth compactly supported function χ. Then there is some $C > 0$ such that if

$$\left\| \frac{d\tau}{\sqrt{\beta_\Lambda}} \right\|_{C^0_{-\mu}} < C,$$

there is a solution to the LCBY Eq. (1.5–1.6).

Finally, in section 6 we discuss how these results can be extended to the case where the metric g is only assumed to be *conformally* asymptotically cylindrical.

3. Global Barriers

In this section we prove Theorem 2.1 by constructing a global supersolution and a global subsolution (defined below) with desirable asymptotics. Throughout this section, we assume for notational simplicity that our manifold has a single asymptotically cylindrical end. It is easy to see that this assumption results in no loss of generality; the extension of our analysis to multiple ends is completely trivial. We shall make use of the following lemma, whose proof is nearly identical to one found in [15]:

Lemma 3.1. *Let ϕ be a bounded continuous function on a manifold M which does not admit any L^2 conformal Killing fields and $d\tau \in C^0_{-\delta'}(TM)$ for some positive $\delta' < \delta_*$. If W_ϕ is the solution of the subcritical vector* Eq. (2.2) *associated with a conformally asymptotically cylindrical metric, then for any $\delta < \delta'$, there exists some constant K which does not depend on ϕ or τ such that the following pointwise estimate holds:*

$$|LW_\phi| \leq K\|d\tau\|_{0,-\delta'}\|\phi\|_0^{N-\epsilon}e^{-\delta t}. \tag{3.1}$$

We define the constant $K_\tau = K\|d\tau\|_{0,-\delta'}$ for brevity in our analysis below. Note in particular that the lemma implies that $|\sigma + LW|^2_g \to \mathring{\sigma}^2$ on the end. Hence, we expect a solution which has an asymptotic limit on the cylindrical end to approach a solution to the *reduced Lichnerowicz equation*, which is

$$\Delta_{\mathring{g}}\mathring{\phi} - c_n\mathring{R}\mathring{\phi} = \beta_\Lambda(\mathring{\tau})\mathring{\phi}^{N-1} - c_n\mathring{\sigma}^2\mathring{\phi}^{-N-1}. \tag{3.2}$$

Given tame data (g, σ, τ), this equation admits a constant supersolution. Moreover, one easily checks that if $\mathring{\phi}_1$ is a positive solution to the equation

$$\Delta_{\mathring{g}}u - (c_n\mathring{R} + \beta_\Lambda(\mathring{\tau}))u = -\mathring{\sigma}^2,$$

then $\rho\mathring{\phi}_1$ is a subsolution of the reduced equation for any sufficiently small $\rho > 0$. Hence, (3.2) admits a solution by [2, Proposition A.4]. We note that this is the only place where we use the requirement that $\sigma \not\equiv 0$ unless $R < 0$ on the ends. We extend this solution to a smooth function on all of M which is t-independent on the end, and we will also label this function $\mathring{\phi}$.

Next, we define the nonlinear operator Lich_ϕ on C^2 by

$$\mathrm{Lich}_\phi(u) := \Delta_g u - c_nRu - \beta_\Lambda u^{N-1} + c_n|\sigma + LW_\phi|^2_g u^{-N-1} \tag{3.3}$$

where W_ϕ is the solution of the subcritical vector Eq. (2.2) (such a solution exists by Theorem 1.2). We call ϕ_+ a *global supersolution* if for any $0 < \phi \leq \phi_+$, $\mathrm{Lich}_\phi(\phi_+) \leq 0$. If ϕ_+ is of lower regularity so that this inequality only holds in the weak sense, we call it a *weak global supersolution*. Given a (weak) global supersolution ϕ_+, we define an associated *(weak) global subsolution* $\phi_- > 0$ which, for any $\phi_- \leq \phi \leq \phi_+$, satisfies $\mathrm{Lich}_\phi(\phi_-) \geq 0$. Having constructed a pair of weak global sub/supersolutions $\phi_- \leq \phi_+$ for the system (2.1–2.2) which both approach $\mathring{\phi}$ asymptotically on the end, one argues as in [15] and applies the Schauder fixed point theorem to produce a solution to the system.

In what follows, we construct a pair of weak global sub/supersolutions for the subcritical LCBY equations. The construction below is independent of the Yamabe invariant of the background metric, though we assume as in [2] that $c_n R + \beta_\Lambda > c > 0$ everywhere. If one drops this assumption in the case where the Yamabe invariant is positive, the construction of a pair of weak global sub/supersolutions proceeds almost exactly as in [15].

3.1. Global Supersolution

We first show that a sufficiently large constant B is a global supersolution for the subcritical LCBY system. Suppose $0 < \phi \leq B$. Using Lemma 3.1, dropping all subscripts "g,"

$$\begin{aligned}\mathrm{Lich}_\phi(B) &= -c_n RB - \beta_\Lambda B^{N-1} + c_n|\sigma + LW_\phi|^2 B^{-N-1}\\ &\leq -c_n RB - \beta_\Lambda B^{N-1} + c_n B^{-N-1}(|\sigma|^2 + 2|\sigma||LW_\phi| + |LW_\phi|^2)\\ &\leq -c_n RB - \beta_\Lambda B^{N-1} + C_1 B^{-N-1} + C_2 B^{-1-\epsilon} + C_3 B^{N-1-2\epsilon}\end{aligned}$$

where we have used Lemma 3.1 and defined $C_1 = c_n \sup_M |\sigma|^2$, $C_2 = 2c_n \sup_M |\sigma| K_\tau$, and $C_3 = c_n K_\tau^2$. Since $\beta_\Lambda \geq \beta_\Lambda(\tau_0) > 0$, this final expression is negative for sufficiently large B and hence such a constant is a global supersolution. Notice that subcriticality was crucial in establishing this fact.

We would like to construct a supersolution which approaches the function $\mathring{\phi}$ on the ends asymptotically. This way, the solution obtained from the fixed point theorem will also approach $\mathring{\phi}$. For this, we prove the following proposition whose proof is similar to Proposition 3.5 in [15] and Theorem 4.3 in [2]:

Proposition 3.2. *Choose some positive $\nu < \delta/(2N+2)$ so small that*

$$\mathrm{Lich}_0(\mathring{\phi}) = \mathcal{O}(e^{-2\nu t}).$$

With the initial data above, there exists some $T > 0$ such that the function $\mathring{\phi}(1 + be^{-\nu t})$ is a global supersolution for any sufficiently large b on the set where $t \geq T$.

Once we establish this proposition, one easily sees that we can choose b so large that $\mathring{\phi}(1 + be^{-\nu t}) > B$ whenever $t \leq T$ and therefore $\inf[B, \mathring{\phi}(1 + be^{-\nu t})]$ is continuous and thus a weak global supersolution with the desired asymptotics.

Proof. Define a new metric $\tilde{g} = \mathring{\phi}^{N-2} g$, and denote with a tilde all operators and quantities associated with this metric. The Lichnerowicz equation is well known to be conformally covariant, which means that $\mathring{\phi} u$ is a solution to

$$\Delta_g(\mathring{\phi} u) - c_n R_g \mathring{\phi} u = \beta_\Lambda(\mathring{\phi} u)^{N-1} - c_n|\sigma|_g^2(\mathring{\phi} u)^{-N-1}$$

if and only if u is a solution to

$$\Delta_{\tilde{g}} u - c_n R_{\tilde{g}} u = \beta_\Lambda u^{N-1} - c_n|\mathring{\phi}^{-2}\sigma|_{\tilde{g}}^2 u^{-N-1}.$$

It follows from this pointwise property that for any $\phi \leq \mathring{\phi}(1 + be^{-\nu t})$,

$$\begin{aligned}
\mathrm{Lich}_\phi(\mathring{\phi}(1 + be^{-\nu t})) &= (\Delta - c_n R)(\mathring{\phi}(1 + be^{-\nu t})) - \beta_\Lambda(\mathring{\phi}(1 + be^{-\nu t}))^{N-1} \\
&\quad + c_n|\sigma + LW_\phi|^2(\mathring{\phi}(1 + be^{-\nu t}))^{-N-1} \\
&= (\tilde{\Delta} - c_n \tilde{R})(1 + be^{-\nu t}) - \beta_\Lambda(1 + be^{-\nu t})^{N-1} \\
&\quad + c_n|\mathring{\phi}^{-2}(\sigma + LW_\phi)|_{\tilde{g}}^2(1 + be^{-\nu t})^{-N-1} \\
&= (\tilde{\Delta} - c_n \tilde{R})(1 + be^{-\nu t}) - \beta_\Lambda(1 + be^{-\nu t})^{N-1} \\
&\quad + c_n|\tilde{\sigma} + \mathring{\phi}^{-2} LW_\phi|_{\tilde{g}}^2(1 + be^{-\nu t})^{-N-1}.
\end{aligned}$$

Next, observe by the covariance of the conformal Laplacian that

$$(\tilde{\Delta} - c_n \tilde{R})u = \mathring{\phi}^{-(N-1)}(\Delta - c_n R)(\mathring{\phi} u) \tag{3.4}$$

for any function $u \in C^2$, and so taking $u \equiv 1$ gives

$$\begin{aligned}
-c_n \tilde{R} &= \mathring{\phi}^{-(N-1)}(\beta_\Lambda \mathring{\phi}^{N-1} - c_n|\sigma|^2 \mathring{\phi}^{-N-1} + \mathcal{O}(e^{-2\nu t})) \\
&= \beta_\Lambda - c_n|\tilde{\sigma}|_{\tilde{g}}^2 + s
\end{aligned}$$

where s is some function satisfying $|s| \leq Ce^{-2\nu t}$. Thus, $\mathrm{Lich}_\phi(\mathring{\phi}(1 + be^{-\nu t}))$ is given by the expression

$$\begin{aligned}
&\left(\tilde{\Delta} + \beta_\Lambda - c_n|\tilde{\sigma}|_{\tilde{g}}^2 + s\right)(1 + be^{-\nu t}) - \beta_\Lambda(1 + be^{-\nu t})^{N-1} \\
&\quad + c_n|\tilde{\sigma} + \mathring{\phi}^{-2} LW_\phi|_{\tilde{g}}^2(1 + be^{-\nu t})^{-N-1}
\end{aligned}$$

which, after adding and subtracting $(N-1)\beta_\Lambda be^{-\nu t}$, we may rewrite as

$$\begin{aligned}
&\left(\tilde{\Delta} - \left(c_n|\tilde{\sigma}|_{\tilde{g}}^2 + (N-2)\beta_\Lambda - s\right)\right)(be^{-\nu t}) + \beta_\Lambda\left((N-1)be^{-\nu t}\right. \\
&\quad \left. + 1 - (1 + be^{-\nu t})^{N-1}\right) + c_n|\tilde{\sigma} + \mathring{\phi}^{-2} LW_\phi|_{\tilde{g}}^2(1 + be^{-\nu t})^{-N-1} - c_n|\tilde{\sigma}|_{\tilde{g}}^2 + s.
\end{aligned}$$

We now make a few observations about this expression. First, since $\beta_\Lambda(\tau_0) > 0$, for some large T_0,

$$h := c_n|\tilde{\sigma}|_{\tilde{g}}^2 + (N-2)\beta_\Lambda - s \geq c > 0 \tag{3.5}$$

for all $t \geq T_0$. Note too that the expression $(N-1)be^{-\nu t} + 1 - (1 + be^{-\nu t})^{N-1}$ is negative, as may be seen by differentiating the function $r(x) = (N-1)x + 1 - (1 + x)^{N-1}$ and observing that $r(0) = 0$.

Having chosen T_0 as above, note that for any $\rho > 0$ satisfying $4\rho^2 < c$, if $\nu < \rho$ then $\tilde{\Delta} e^{-\nu t} \leq 2\rho^2 e^{-\nu t}$ and hence $(\tilde{\Delta} - h)e^{-\nu t} \leq -2\rho^2 e^{-\nu t}$. From this, we conclude that for $t \geq T_0$, $\mathrm{Lich}_\phi(\mathring{\phi}(1 + be^{-\nu t}))$ is bounded above by

$$-2b\rho^2 e^{-\nu t} + s + \beta_\Lambda b^{N-1} f(b,t) + 2c_n |\tilde{\sigma}|_{\tilde{g}} |\mathring{\phi}^{-2} LW_\phi|_{\tilde{g}} b^{-N-1} e^{(N+1)\nu t}$$
$$+ c_n |\mathring{\phi}^{-2} LW_\phi|^2_{\tilde{g}} b^{-N-1} e^{(N+1)\nu t}$$

where we have simultaneously used the basic inequalities

$$|\tilde{\sigma} + \mathring{\phi}^{-2} LW|^2_{\tilde{g}} \leq |\tilde{\sigma}|^2_{\tilde{g}} + 2|\tilde{\sigma}|_{\tilde{g}} |\mathring{\phi}^{-2} LW|_{\tilde{g}} + |\mathring{\phi}^{-2} LW|^2_{\tilde{g}} \tag{3.6}$$
$$(1 + be^{-\nu t})^{-N-1} \leq \min(1, b^{-N-1} e^{(N+1)\nu t}) \tag{3.7}$$

and set

$$f(b,t) = \frac{1}{b^{N-1}} + \frac{N-1}{b^{N-2}} e^{-\nu t} - \left(\frac{1}{b} + e^{-\nu t}\right)^{N-1}.$$

Noting that $\tilde{L}W = \mathring{\phi}^{-(N-2)} LW$, one applies Lemma 3.1 and the fact that $\phi \leq \mathring{\phi}(1 + be^{-\nu t})$ to find that the bound for $\mathrm{Lich}_\phi(\mathring{\phi}(1 + be^{-\nu t})$ we obtained can itself be bounded above by

$$\begin{aligned} &-2b\rho^2 e^{-\nu t} + s + \beta_\Lambda b^{N-1} f(b,t) \\ &\quad + k_1 ||\mathring{\phi}(1 + be^{-\nu t})||_\infty^{N-\epsilon} b^{-N-1} e^{((N+1)\nu - \delta)t} \\ &\quad + k_2 ||\mathring{\phi}(1 + be^{-\nu t})||^{2N-2\epsilon} b^{-N-1} e^{((N+1)\nu - 2\delta)t}. \end{aligned} \tag{3.8}$$

where $k_1 = 2c_n K_\tau \|\tilde{\sigma}\|_\infty \|\mathring{\phi}\|_\infty^{N-4}$ and $k_2 = c_n K_\tau^2 \|\mathring{\phi}\|_\infty^{2N-8}$ are constants. Finally, when $b \gg 1$ we may factor out the b from the expression $||\mathring{\phi}(1+be^{-\nu t})||_\infty$ to find that (3.8) is bounded above by

$$\begin{aligned} &-2b\rho^2 e^{-\nu t} + s + \beta_\Lambda b^{N-1} f(b,t) + k_1' b^{-1-\epsilon} e^{((N+1)\nu - \delta)t} \\ &\quad + k_2' b^{N-1-2\epsilon} e^{((N+1)\nu - 2\delta)t} \end{aligned} \tag{3.9}$$

where $k_1' = (2\|\mathring{\phi}\|_\infty)^{N-\epsilon} k_1$ and $k_2' = (2\|\mathring{\phi}\|_\infty)^{2N-2\epsilon} k_2$. We may thus choose some b_0 so large that, for all $b \geq b_0$,

$$-\beta_\Lambda(\tau_0) b^{N-1} + k_1' b^{-1-\epsilon} + k_2' b^{N-1-2\epsilon} < 0. \tag{3.10}$$

Now to prove the proposition, we show that there is some choice of $T > 0$ such that (3.9) is negative for any $t \geq T$ and $b \geq b_0$. Clearly, there is some $T_1 \geq T_0$ such that $-2\rho^2 e^{-\nu t} + s < 0$ for all $t \geq T_1$, so our task is reduced to proving that the sum of the final three terms in (3.9) is negative for such a choice of t and b. The analysis of these terms differs slightly depending on whether $N \geq 3$ (i.e., $n \leq 6$) or $N < 3$, so we consider these cases separately.

First, suppose $N \geq 3$. For any fixed t, we see that $f(b,t) \to -e^{-(N-1)\nu t}$ as $b \to \infty$. On the other hand, we find by differentiating that f is increasing in b, so we thus conclude that $f(b,t) < -e^{(N-1)\nu t}$ for *all* b and t. The final three terms in (3.9) are thus bounded above by

$$\begin{aligned}&-\beta_\Lambda(\tau_0)b^{N-1}e^{-(N-1)\nu t}+k_1'b^{-1-\epsilon}e^{((N+1)\nu-\delta)t}+k_2'b^{N-1-2\epsilon}e^{((N+1)\nu-2\delta)t}\\&\quad\le e^{-(N-1)\nu t}(-\beta_\Lambda(\tau_0)b^{N-1}+k_1'b^{-1-\epsilon}+k_2'b^{N-1-2\epsilon})\\&\quad<0\end{aligned}$$

where we have used our smallness condition on ν in the first inequality and the fact that $b \ge b_0$ in the second. This completes the proposition in the case where $N \ge 3$.

The case in which $N < 3$ is not much more difficult. The key difference is that we now have $\partial f/\partial b < 0$. Note first that for any fixed b, one computes the limit (using a Taylor expansion, for example)

$$\lim_{t\to\infty}\frac{f(b,t)}{e^{-2\nu t}}=-\frac{b^{3-N}(N-1)(N-2)}{2},\tag{3.11}$$

so we may thus choose some $b_1 \ge b_0$ such that this limit is less than -2 for all $b \ge b_1$. Hence, there exists some $T_2 \ge T_1$ such that $f(b_1,t) < -e^{-2\nu t}$ for all $t \ge T_2$. But since $\partial f/\partial b < 0$, we see that $f(b,t) < -e^{-2\nu t}$ for *any* $b \ge b_1$ and hence for any such b the final three terms in (3.9) are bounded above by

$$\begin{aligned}&-\beta_\Lambda(\tau_0)b^{N-1}e^{-2\nu t}+k_1'b^{-1-\epsilon}e^{((N+1)\nu-\delta)t}+k_2'b^{N-1-2\epsilon}e^{((N+1)\nu-2\delta)t}\\&\quad\le e^{-2\nu t}(-\beta_\Lambda(\tau_0)b^{N-1}+k_1'b^{-1-\epsilon}+k_2'b^{N-1-2\epsilon})\\&\quad<0\end{aligned}$$

for any $t \ge T_2$. This proves the case and the proposition. □

3.2. Global Subsolution

Based on our construction of a global supersolution, we may suspect that the function $\mathring{\phi}(1-ae^{-\nu t})$ will provide a global subsolution far out on the end. We will show this to be the case. First observe, again by the conformal covariance of the Lichnerowicz equation, that for any $\phi \ge \mathring{\phi}(1-ae^{-\nu t})$, $\mathrm{Lich}_\phi(\mathring{\phi}(1-ae^{-\nu t}))$ is given by

$$\begin{aligned}&(\tilde\Delta-c_n\tilde R)(1-ae^{-\nu t})-\beta_\Lambda(1-ae^{-\nu t})^2\\&\quad+c_n|\tilde\sigma+\mathring{\phi}^{-2}LW_\phi|^2_{\tilde g}(1-ae^{-\nu t})^{-N-1}\\&=(\tilde\Delta+\beta_\Lambda-c_n|\tilde\sigma|^2_{\tilde g}+s)(1-ae^{-\nu t})-\beta_\Lambda(1-ae^{-\nu t})^{N-1}\\&\quad+c_n|\tilde\sigma+\mathring{\phi}^{-2}LW_\phi|^2_{\tilde g}(1-ae^{-\nu t})^{-N-1}\\&=-a\tilde\Delta e^{-\nu t}+s(1-ae^{-\nu t})+(\beta_\Lambda-c_n|\tilde\sigma|^2_{\tilde g})(1-ae^{-\nu t})-\beta_\Lambda(1-ae^{-\nu t})^{N-1}\\&\quad+c_n|\tilde\sigma+\mathring{\phi}^{-2}LW_\phi|^2_{\tilde g}(1-ae^{-\nu t})^{-N-1}.\end{aligned}$$

As noted in [2], given any positive number $\mu < 1$, there is some $\mu' > 0$ such that for all $y \in [\mu, 1)$,

$$\frac{y-y^{N-1}}{1-y}\ge\mu'.\tag{3.12}$$

Hence, fixing a and choosing t so large that $\mu < 1-ae^{-\nu t} < 1$, we find that

$$\beta_\Lambda(1-ae^{-\nu t})-\beta_\Lambda(1-ae^{-\nu t})^{N-1}\ge\beta_\Lambda a\mu' e^{-\nu t}.\tag{3.13}$$

We thus conclude that, for such a choice of t, $\mathrm{Lich}_\phi(\mathring{\phi}(1-ae^{-\nu t}))$ is bounded below by

$$-a\tilde{\Delta}e^{-\nu t}+s(1-ae^{-\nu t})+\beta_\Lambda(\tau_0)a\mu' e^{-\nu t}-c_n|\tilde{\sigma}|^2_{\tilde{g}}(1-ae^{-\nu t})\\ +c_n|\tilde{\sigma}+\mathring{\phi}^{-2}LW_\phi|^2_{\tilde{g}}(1-ae^{-\nu t})^{-N-1}.$$

Now using that $(1-ae^{-\nu t})^{-N-1}>1>1-ae^{-\nu t}$ and that $|\tilde{\sigma}+\mathring{\phi}^{-2}LW_\phi|^2_{\tilde{g}}\geq(|\tilde{\sigma}|_{\tilde{g}}-\mathring{\phi}^{-2}|LW_\phi|_{\tilde{g}})^2$, we see that the previous expression is bounded below by

$$-a\tilde{\Delta}e^{-\nu t}+s(1-ae^{-\nu t})+\beta_\Lambda(\tau_0)a\mu' e^{-\nu t}+c_n|\mathring{\phi}^{-2}LW_\phi|_{\tilde{g}}(|\mathring{\phi}^{-2}LW_\phi|_{\tilde{g}}-2|\tilde{\sigma}|_{\tilde{g}})\\ +c_n|\tilde{\sigma}|^2_{\tilde{g}}[(1-ae^{-\nu t})^{-N-1}-(1-ae^{-\nu t})]\\ \geq -a\tilde{\Delta}e^{-\nu t}+s(1-ae^{-\nu t})+\beta_\Lambda(\tau_0)a\mu' e^{-\nu t}-2c_n|\mathring{\phi}^{-2}LW_\phi|_{\tilde{g}}|\tilde{\sigma}|_{\tilde{g}}.$$

At this point we use the fact that $\phi\leq\phi_+:=\mathring{\phi}(1+ae^{-\nu t})$ and Lemma 3.1 to conclude

$$\mathrm{Lich}_\phi(\mathring{\phi}(1-ae^{-\nu t}))\geq -a\tilde{\Delta}e^{-\nu t}+s(1-ae^{-\nu t})\\ +\beta_\Lambda(\tau_0)a\mu' e^{-\nu t}-k_1\|\phi_+\|_\infty^{N-\epsilon}e^{-\delta t},$$

where the constant k_1 is defined in the previous section. Now, there exists a constant $C>0$ such that $\tilde{\Delta}e^{-\nu t}\leq C\nu^2e^{-\nu t}$, so the previous expression is bounded below by

$$-aC\nu^2e^{-\nu t}+s(1-ae^{-\nu t})+\beta_\Lambda(\tau_0)a\mu' e^{-\nu t}-C'e^{-\delta t}. \tag{3.14}$$

We thus see that if we choose $\nu\leq\sqrt{\beta_\Lambda(\tau_0)\mu'/C}$, the total contribution of the first and third term is positive. Since the second term decays like $\mathcal{O}(e^{-2\nu t})$, we conclude that, with this choice of ν, for some $T'>0$ the expression (3.14) is positive for all $t\geq T'$.

Our next objective is then to find a global subsolution on the compact piece $\mathcal{K}=\{t\leq T'\}$ which is positive, yet sufficiently small on the boundary of $\mathcal{K}$. Calling such a function η, the function $\sup(\eta,\mathring{\phi}(1+ae^{-\nu t}))$ is then continuous and thus a weak global subsolution. To accomplish this, we merely define a slightly larger compact set $\mathcal{K}'=\{t\leq T''\}$, where $T''>T'$ is so large that $ae^{-\nu t}<1/2$ for all $t\geq T''$, and solve the Dirichlet problem

$$\begin{cases}(\Delta-c_nR-\beta_\Lambda)\eta=0\\ \eta|_{\partial\mathcal{K}'}=\frac{1}{2}\inf(1,\inf_M\mathring{\phi}).\end{cases} \tag{3.15}$$

The function η is positive and less than 1 on the boundary of $\mathcal{K}'$ and hence on all of $\mathcal{K}'$ by the maximum principle. One easily checks that η is a global subsolution on $\mathcal{K}'$, and it is less than $\mathring{\phi}(1-ae^{-\nu t})$ near the boundary of $\mathcal{K}'$ by our choice of T''. Therefore, if we extend η to be zero identically outside of $\mathcal{K}'$, we conclude that $\sup(\eta,\mathring{\phi}(1+ae^{-\nu t}))$ is a weak global subsolution. We note that this is the only place where we used the condition that $c_nR+\beta_\Lambda>0$.

3.3. Continuity of the Solution Maps

Having constructed global sub and supersolutions of the subcritical system (2.1–2.2), one finds a solution to this system with an application of the Schauder fixed point theorem as in [11], [16], or [15]. The fixed point theorem we need, a proof of which can be found in [14], is the following:

Theorem 3.3. *Let X be a Banach space, and let $U \subset X$ be a non-empty, convex, closed, bounded subset. If $T : U \to U$ is a compact operator, then there exists a fixed point $u \in U$ such that $T(u) = u$.*

To apply this theorem in the present context, we look for a solution (ϕ, W) where $\phi = \mathring{\phi} + \psi$ and $\psi \in W^{3,p}_{-\nu}$. In this way, we think of the subcritical system as having a solution $(\psi, W) \in W^{3,p}_{-\nu}(M) \times (W^{2,p}_{-\delta}(TM) \oplus \mathscr{Y})$, and ψ shall be found as a fixed point of a particular function on the set

$$U = \{\psi \in L^{\infty}_{-\nu'} : \phi_- - \mathring{\phi} \le \psi \le \phi_+ - \mathring{\phi}\} \tag{3.16}$$

where we choose some positive $\nu' < \nu$. The set U clearly meets all the criteria of Theorem 3.3 as a subset of the space $L^{\infty}_{-\nu'}$.

Let $\mathcal{W}_\epsilon : U \to W^{2,p}_{-\delta}(TM) \oplus \mathscr{Y}$ be the map which sends $\psi \in U$ to the vector field $G(n^{-1}(n-1)(\mathring{\phi} + \psi)^{N-\epsilon} d\tau)$, where G is the bounded generalized inverse defined above for the operator div$L$. One can easily see that the map $\mathcal{W}_\epsilon$ is continuous. For any traceless 2-tensor $\Sigma \in C^0$ satisfying $|\Sigma|^2 \to \mathring{\sigma}^2$ on the ends at the rate $e^{-\delta t}$, we define $\mathcal{Q}(\Sigma)$ to be the unique solution of the Lichnerowicz equation (1.5), with $\sigma + LW$ replaced by Σ, which satisfies $\phi_- \le \mathcal{Q}(\Sigma) \le \phi_+$. This map is shown to be well defined in [15]. We thus define a map $\mathcal{S}_\sigma : C^0_{-\delta}(S^2_0(M)) \to W^{2,p}_{-\nu}(M)$ by

$$\pi \mapsto \mathcal{Q}(\sigma + \pi) - \mathring{\phi}. \tag{3.17}$$

If we could show this map to be continuous, then the map $\mathcal{N}_{\sigma,\epsilon} = \mathcal{S}_\sigma \circ L \circ \mathcal{W}_\epsilon$ would itself be continuous. The range of this function lies in U by definition, and the composition of $\mathcal{N}_{\sigma,\epsilon}$ with the compact embedding $W^{2,p}_{-\nu} \hookrightarrow C^0_{-\nu'}$ gives us a map T which satisfies the hypothesis of Theorem 3.3. If $\tilde{\psi}$ is this fixed point, then $(\mathring{\phi} + \tilde{\psi}, W_{\mathring{\phi}+\tilde{\psi}})$ is a solution of the subcritical system by construction. We thus need only show that $\mathcal{S}_\sigma$ is continuous, and to do this we require the following lemma.

Lemma 3.4. *The map $\mathcal{S}_\sigma : C^0_{-\delta}(S^2_0(M)) \to W^{2,p}_{-\nu}(M)$ is continuous.*

The proof of this lemma is an implicit function theorem argument which goes through exactly as the proof of [15, Lem 4.2] with the obvious modifications.

4. Convergence of Solutions

In this section we show that any solution of the subcritical equations has an L^∞ bound depending only on the L^2-norm of LW. We essentially follow the proof of this for closed manifolds found in [4], though the geometry of the ends clearly necessitates several modifications to their argument. Let $\epsilon \in [0, 1)$ be arbitrary, and let (ϕ, W) be a solution to the subcritical Eqs. (2.1–2.2). We define the energy of the solution by

$$\gamma(\phi, W) = \int_M |LW|^2 \, dv,$$

and let $\tilde{\gamma} = \max\{\gamma, 1\}$. Note that $\tilde{\gamma}$ is finite by Lemma 3.1. We rescale ϕ, W, and σ as

$$\tilde{\phi} = \tilde{\gamma}^{-\frac{1}{2N}}\phi, \quad \tilde{W} = \tilde{\gamma}^{-\frac{1}{2}}W, \quad \tilde{\sigma} = \tilde{\gamma}^{-\frac{1}{2}}\sigma.$$

The deformed equations can then be renormalized as

$$\frac{1}{\tilde{\gamma}^{1/n}}(\Delta\tilde{\phi} - c_n R\tilde{\phi}) = \beta_\Lambda \tilde{\phi}^{N-1} - c_n|\tilde{\sigma} + L\tilde{W}|^2\tilde{\phi}^{-N-1}, \tag{4.1}$$

$$\mathrm{div} L\tilde{W} = \frac{n-1}{n}\tilde{\gamma}^{-\frac{\epsilon}{2N}}\tilde{\phi}^{N-\epsilon}d\tau. \tag{4.2}$$

Notice that because of our rescaling,

$$\int_M |L\tilde{W}|^2 \, dv \leq 1.$$

Throughout this section, "bounded" will mean "bounded independent of ϵ, ϕ and W", and all constants C or C_i will be similarly independent of ϵ, ϕ, and W. We first prove an important lemma.

Lemma 4.1. *Suppose that $k \geq 0$. Then, for any solution $\tilde{\phi}$ of the renormalized subcritical equations (4.1–4.2), and any $\delta > 0$,*

$$\begin{aligned} -C_1\left(\int_M e^{-\delta t}\tilde{\phi}^{2N+Nk}\right)^{\frac{N+2+Nk}{2N+Nk}} &+ \beta_\Lambda(\tau_0)\int_M e^{-\delta t}\tilde{\phi}^{2N+Nk} \\ &\leq 2c_n\int_M e^{-\delta t}|\sigma|^2\tilde{\phi}^{Nk} + C_2\int_M |L\tilde{W}|^2\tilde{\phi}^{Nk}. \end{aligned} \tag{4.3}$$

Proof. We multiply Eq. (4.1) by $e^{-\delta t}\tilde{\phi}^{N+1+Nk}$ and integrate over M to get

$$\begin{aligned} &\frac{1}{\tilde{\gamma}^{1/n}}\int_M \left(-e^{-\delta t}\tilde{\phi}^{N+1+Nk}\Delta\tilde{\phi} + c_n e^{-\delta t}R\tilde{\phi}^{N+2+Nk}\right)dv \\ &\quad + \int_M \beta_\Lambda e^{-\delta t}\tilde{\phi}^{2N+Nk}dv = c_n\int_M e^{-\delta t}|\tilde{\sigma} + L\tilde{W}|^2\tilde{\phi}^{Nk}dv. \end{aligned} \tag{4.4}$$

Consider the first integral on the left:

$$\begin{aligned} \int_M -e^{-\delta t}\tilde{\phi}^{N+1+Nk}\Delta\tilde{\phi} &= \int_M c_1 d(e^{-\delta t})d(\tilde{\phi}^{N+2+Nk}) + \int c_2 e^{-\delta t}\tilde{\phi}^{N+Nk}|d\tilde{\phi}|^2 \\ &\geq -C\int_M \Delta(e^{-\delta t})\tilde{\phi}^{N+2+Nk} \\ &\geq -C\int_M e^{-\delta t}\tilde{\phi}^{N+2+Nk} \end{aligned}$$

where $c_1 = \frac{1}{N+2+Nk}$ and $c_2 = N+1+Nk$. The first and second lines are by integration by parts. These integration by parts are valid because of the exponential falloff term $e^{-\delta t}$. The third line follows from the inequality $\Delta e^{-\delta t} \leq Ce^{-\delta t}$, since $\|t\|_{C^2} < \infty$. Using this, and combining with the R term, the first integral in Eq. (4.4) is greater than or equal to

$$\int_M (c_n R - C) e^{-\delta t} \tilde{\phi}^{N+2+Nk}.$$

Let $v = \frac{2N+Nk}{N+2+Nk}$, $u = \frac{2N+Nk}{N-2}$. Note that $\frac{1}{u} + \frac{1}{v} = 1$. Let $\mu = 1/u$. We then use Hölder's inequality to see that

$$\begin{aligned}\int_M (c_n R - C) e^{-\delta t} \tilde{\phi}^{N+2+Nk} &= \int_M (c_n R - C) e^{-\delta t \mu} \tilde{\phi}^{N+2+Nk} e^{-\delta t(1-\mu)} \\ &\geq -\| |c_n R - C| e^{-\delta t \mu} \|_{L^u} \| \tilde{\phi}^{N+2+Nk} e^{-\delta t(1-\mu)} \|_{L^v} \\ &\geq -\| c_n R - C \|_{u,\delta/u} \left(\int_M e^{-\delta t} \tilde{\phi}^{2N+Nk} \right)^{\frac{N+2+Nk}{2N+Nk}}.\end{aligned}$$

Note that $\|c_n R - C\|_{u,\delta/u} < \infty$, since $R \in L^\infty$ and $\delta/u > 0$. After using $\tilde{\gamma}^{-1} \leq 1$, this gives us the first term of the desired inequality.

The second term is easily found by pulling out the infimum of β_Λ. The right hand term is found by the inequalities $|\tilde{\sigma} + L\tilde{W}|^2 \leq 2|\tilde{\sigma}|^2 + 2|L\tilde{W}|^2$, $e^{-\delta t} \leq C$, and $|\tilde{\sigma}|^2 \leq |\sigma|^2$. This completes the lemma. □

Proposition 4.2. *Suppose ϕ is a positive solution of the subcritical Eqs. (2.1–2.2) for tame initial data which satisfies $\phi \to \mathring{\phi}$ on the ends. If $\epsilon \in [0, 1)$, then*

$$\phi < C\tilde{\gamma}^{\frac{1}{2N}}.$$

Proof. As in [4], we will prove this proposition in four steps.

Step 1. *L^1_δ bound on $\tilde{\phi}^{2N}$*

Suppose $d\tau \in L^p_{-\delta}$. Then using Lemma 4.1 with $k = 0$, $\tilde{\phi}^{2N}$ is clearly bounded in L^1_δ as long as the right hand side of (4.3) is bounded. However, since $k = 0$ and $\sigma \to \mathring{\sigma}$, it is easy to see that both integrals are bounded. Thus, $\tilde{\phi}^{2N}$ is bounded in L^1_δ.

Step 2. *Bounds for LW.*

Suppose by induction $\tilde{\phi}^{p_i N}$ is bounded in L^1_δ for some $2 \leq p_i$. Let $\frac{1}{q_i} = \frac{1}{p_i} + \frac{1}{p}$. If $q_i > n$, we continue on to step 4. Otherwise, define $\frac{1}{r_i} = \frac{1}{q_i} - \frac{1}{n}$. We will show at the end of step 3 that we can ensure q_i is never n.

Let $\alpha = \max\{\delta\left(\frac{1}{p_i} - 1\right), -\delta_*/2\}$. Note that

$$\tilde{\phi}^{N-\epsilon} \leq \frac{N-\epsilon}{N}\tilde{\phi}^N + \frac{\epsilon}{N} \leq \tilde{\phi}^N + \frac{1}{N}.$$

Using this and Eq. (4.2),

$$\begin{aligned}
\|\mathrm{div} L\tilde{W}\|_{L^{q_i}_\alpha} &= C\tilde{\gamma}^{-\frac{\epsilon}{2N}}\|\tilde{\phi}^{N-\epsilon}|d\tau|\|_{L^{q_i}_\alpha}\\
&\le C\left\|\left(\tilde{\phi}^N+\frac{1}{N}\right)|d\tau|\right\|_{L^{q_i}_{\delta\left(\frac{1}{p_i}-1\right)}}\\
&\le C\left\|\left(\tilde{\phi}^N+\frac{1}{N}\right)|d\tau|e^{-\frac{\delta}{p_i}t}e^{\delta t}\right\|_{L^{q_i}}\\
&\le C\left\|\left(\tilde{\phi}^N+\frac{1}{N}\right)e^{-\frac{\delta}{p_i}t}\right\|_{L^{p_i}}\||d\tau|e^{\delta t}\|_{L^p}\\
&\le C\left(\|\tilde{\phi}^{Np_i}\|^{1/p_i}_{L^1_\delta}+\|1/N\|_{L^{p_i}_{\delta/p_i}}\right)\|d\tau\|_{L^p_{-\delta}}.
\end{aligned} \tag{4.5}$$

The second line holds because $\tilde{\gamma}\ge 1$. The fourth line is Hölder's inequality with p_i and p. The last line follows from the definitions of the norms and the triangle inequality.

This last inequality shows that $\|\mathrm{div} L\tilde{W}\|_{L^{q_i}_\alpha}$ is bounded since the first norm on the right is bounded by hypothesis and the second is bounded since $d\tau\in L^p_{-\delta}$. We also calculate, since $q_i<n$,

$$\begin{aligned}
\|L\tilde{W}\|_{L^{r_i}_{\delta/r_i}} &\le C\|L\tilde{W}\|_{W^{1,q_i}_\alpha}\\
&\le C\|\tilde{W}\|_{W^{2,q_i}_\alpha\oplus\mathscr{Y}}\\
&\le C\|\mathrm{div} L\tilde{W}\|_{L^{q_i}_\alpha}.
\end{aligned}$$

The first line follows from Theorem 1.3 and the fact that $\delta/r_i>\alpha$. The second line is because L maps $\mathscr{Y}$ to an exponentially decaying piece. The last line is by the existence of generalized inverse of $\mathrm{div}L$ implied by Theorem 1.2.

Thus, $\|L\tilde{W}\|_{L^{r_i}_{\delta/r_i}}$ is bounded.

Step 3. *Induction on p_i*

Define k_i by $\frac{2}{r_i}+\frac{k_i}{p_i}=1$. Lemma 4.1 implies that we can show that $\tilde{\phi}^{2N+Nk_i}$ is bounded in L^1_δ as long as

$$2c_n\int_M e^{-\delta t}|\sigma|^2\tilde{\phi}^{Nk_i}+C_2\int_M|L\tilde{W}|^2\tilde{\phi}^{Nk_i}$$

is bounded. For both integrals, we use Hölder's inequality with $\frac{r_i}{2}$ and $\frac{p_i}{k_i}$ to bound this above by

$$C\|\tilde{\phi}^{p_iN}\|^{k_i/p_i}_{L^1_\delta}\left(\|\sigma\|^2_{L^{r_i}_{\delta/r_i}}+\|L\tilde{W}\|^2_{L^{r_i}_{\delta/r_i}}\right).$$

The norm on $\tilde{\phi}$ is bounded by assumption. The norm on σ is bounded since $\delta/r_i>0$ and by our conditions on σ. The norm on $L\tilde{W}$ is bounded by the previous step.

All of this shows that $\tilde{\phi}^{2N+Nk_i}$ is bounded in L^1_{δ}. Let $p_{i+1} = 2 + k_i$. With this,

$$\frac{p_{i+1}}{p_i} = \frac{2+k_i}{p_i} = \frac{2}{p_i} + 1 - \frac{2}{r_i} = \frac{2}{p_i} + 1 - 2\left(\frac{1}{q_i} - \frac{1}{n}\right)$$
$$= \frac{2}{p_i} + 1 - 2\left(\frac{1}{p_i} + \frac{1}{p} - \frac{1}{n}\right) = 1 + \frac{2}{n} - \frac{2}{p} > 1$$

since $p > n$. Hence $p_i \to \infty$, and so $q_i \to p$. We continue steps 2 and 3 a finite number of times until some k, such that $q_k > n$. We can avoid the case that $q_i = n$ by slightly decreasing p and δ at the beginning of the proposition, since $L^p_{-\delta} \subset L^{p-\epsilon}_{-\delta+\epsilon}$ for small $\epsilon > 0$.

Step 4. *L^∞ bound on $\tilde{\phi}$.*

Since $q_k > n$, similar to (4.5) in step 2,

$$\|L\tilde{W}\|_{L^\infty_\alpha} \le C\|\tilde{W}\|_{W^{2,q_k}_\alpha} \le C\|\mathrm{div} L\tilde{W}\|_{L^{q_k}_\alpha}$$
$$\le C\left(\|\tilde{\phi}^{Np_i}\|^{1/p_i}_{L^1_\delta} + \|1/N\|_{L^{p_i}_{\delta/p_i}}\right)\|d\tau\|_{L^p_{-\delta}}$$

where the right hand side is again bounded. Since $\alpha < 0$, this implies that $|L\tilde{W}|$ is bounded as well.

From the fact that the Laplacian acting on functions only involves first-order derivatives of the metric, and since the coefficients of the Lichnerowicz equation (4.1) are at least in $C^{0,\beta}$ for some $\beta > 0$ (since $p > n$), it can be easily seen that the function $\tilde{\phi}$ is in $C^{2,\beta}$. We can thus apply the maximum principle. Let $x \in M$ be where $\tilde{\phi}$ reaches its maximum value, if it has one. At such a point,

$$\frac{c_n}{\tilde{\gamma}^{1/n}} R\tilde{\phi} + \beta_\Lambda \tilde{\phi}^{N-1} \le c_n|\tilde{\sigma} + L\tilde{W}|^2 \tilde{\phi}^{-N-1}$$

which simplifies to

$$\frac{c_n}{\tilde{\gamma}^{1/n}} R\tilde{\phi}^{N+2} + \beta_\Lambda \tilde{\phi}^{2N} \le c_n|\tilde{\sigma} + L\tilde{W}|^2.$$

Since $R \in L^\infty$ and $\tilde{\gamma} \ge 1$, $\tilde{\phi}$ is bounded.

If $\phi > \sup_M \mathring{\phi}$ at some point, it (and thus $\tilde{\phi}$) has a maximum. Thus, if ϕ does not have a maximum, $\phi \le \sup_M \mathring{\phi}$, which is an even stronger upper bound than the proposition requires.

By recalling that $\tilde{\phi} = \tilde{\gamma}^{-\frac{1}{2N}}\phi$, we have proven the proposition. ☐

Now that we have the bound, let us consider what happens as $\epsilon \to 0$.

Lemma 4.3. *Assume that there exist sequences ϵ_i and (ϕ_i, W_i) such that $\epsilon_i \searrow 0$ and (ϕ_i, W_i) is a solution of the deformed Eqs. (2.1–2.2) with $\epsilon = \epsilon_i$. Also, assume that $\gamma(\phi_i, W_i)$ is bounded. Then there exists a constant $\nu > 0$ and a sequence of the (ϕ_i, W_i) which converges in the $W^{2,p}_{-\delta} \oplus \mathscr{Y}$ norm to a solution (ϕ_∞, W_∞) of the original conformal constraint equations.*

Proof. From the previous proposition, we know that the ϕ_i are uniformly bounded in the L^∞ norm. As in (4.5),

$$\|W_i\|_{W^{2,p}_{-\delta}\oplus\mathscr{Y}} \leq C\left(\|\phi_i\|_0^N + \|1/N\|_0\right)\|d\tau\|_{L^p_{-\delta}}, \tag{4.6}$$

the sequence W_i is uniformly bounded in $W^{2,p}_{-\delta}\oplus\mathscr{Y}$. The operator $L : W^{2,p}_{-\delta}\oplus\mathscr{Y} \to C^0_{-\delta'}$ is compact by Theorem 1.3 and the fact that $\mathscr{Y}$ is finite dimensional. Thus, up to selecting a subsequence, we can assume that the sequence LW_i converges in $C^0_{-\delta'}$ to some LW_∞.

Thus by Lemma 3.4, the functions $\psi_i := \phi_i - \mathring{\phi}$ converge in $W^{2,p}_{-\nu}$ (and thus in $L^\infty_{-\nu}$) for some $\nu > 0$ to a function ψ_∞. We must assume we picked δ' close enough to δ in the previous paragraph, such that $\nu < \frac{\delta'}{2N}$. Since $\phi_i \to \mathring{\phi}$ on the ends, this implies that $\phi_\infty := \psi_\infty + \mathring{\phi}$ approaches $\mathring{\phi}$ on the ends exponentially fast. Since the right hand side of the vector Eq. (1.6) converges in $L^p_{-\delta}$, the sequence W_i converges in the $W^{2,p}_{-\delta}\oplus\mathscr{Y}$ norm as well. The regularity of ϕ_∞ and W_∞ guarantee that they are solutions of the conformal constraint equations (with $\epsilon = 0$). □

Lemma 4.4. *Assume there exists sequences ϵ_i and (ϕ_i, W_i) such that $\epsilon_i \searrow 0$ and (ϕ_i, W_i) is a solution of the subcritical Eqs. (2.1–2.2) with $\epsilon = \epsilon_i$. Also, assume that $\gamma(\phi_i, W_i) \to \infty$. Then there exists a non-zero solution $W \in W^{2,p}_{-\delta}\oplus\mathscr{Y}$ of the limit equation*

$$\operatorname{div} LW = \alpha_0\sqrt{\frac{c_n}{\beta_\Lambda}}|LW|d\tau$$

for some $\alpha_0 \in (0, \frac{n-1}{n}]$ such that $|LW| \leq Ce^{-\delta t}$ for some C independent of ϕ_i, W_i, and W.

Proof. Arguing as in the previous lemma, the $\tilde{W}_i$ are uniformly bounded in $W^{2,p}_{-\delta}\oplus\mathscr{Y}$. Without loss of generality, we can assume that $\gamma > 1$, and so

$$\int_M |L\tilde{W}_i|^2 = 1.$$

Up to selecting a subsequence, we can then assume that $L\tilde{W}_i$ converges in $C^0_{-\delta'}$ for some $0 < \delta' < \delta$ to some $L\tilde{W}_\infty$.

We can show the falloff of $L\tilde{W}_\infty$ by considering

$$\begin{aligned}\|LW_i\|_{L^\infty_{-\delta}} &\leq C\|\phi_i^{N-\epsilon_i}|d\tau|\|_{L^p_{-\delta}}\\ &\leq C\left\|(\phi_i^N + 1/N)|d\tau|\right\|_{L^p_{-\delta}}\\ &\leq C\tilde{\gamma}_i^{1/2}\|d\tau\|_{L^p_{-\delta}}\end{aligned}$$

as in (4.5), but using Proposition 4.2 and $\tilde{\gamma} \geq 1$. Thus,

$$\|L\tilde{W}_i\|_{L^\infty_{-\delta}} \leq C$$

for some C independent of ϵ, W_i and ϕ_i. Since $L\tilde{W}_i$ converges weakly in $W^{1,p}_{-\delta}$, and thus also weakly in $L^\infty_{-\delta}$,

$$|L\tilde{W}_\infty| \leq Ce^{-\delta t}$$

for the same C as for $L\tilde{W}_i$.

Let $\tilde{\phi}_\infty$ be defined by

$$\tilde{\phi}^N_\infty = \sqrt{\frac{c_n}{\beta_\Lambda}}|L\tilde{W}_\infty|$$

i.e., $\tilde{\phi}_\infty$ satisfies

$$\beta_\Lambda \tilde{\phi}^{N-1}_\infty = c_n |L\tilde{W}_\infty|^2 \tilde{\phi}^{-N-1}_\infty.$$

If we can show that $\tilde{\phi}_i \to \tilde{\phi}_\infty$ in L^∞, then the continuity of the vector equation implies that $\tilde{W}_\infty$ is a $W^{2,p}_{-\delta} \oplus \mathscr{Y}$ solution to the limit equation with $\alpha_0 = \frac{n-1}{n} \lim \gamma(\phi_i, W_i)^{-\frac{\epsilon_i}{2N}}$. Since $\gamma(\phi_i, W_i) \to \infty$, $\alpha_0 \in [0, \frac{n-1}{n}]$. Note that

$$\int_M |L\tilde{W}_\infty|^2 = 1$$

since $L\tilde{W}_i$ converges in $C^0_{-\delta'}$, and so $\tilde{W}_\infty \not\equiv 0$, and so the solution is nontrivial. Since we assumed there are no global conformal Killing fields in L^2, we cannot have the case $\alpha_0 = 0$.

To show this convergence, we will show that for any $\epsilon > 0$ that $|\tilde{\phi}_\infty - \tilde{\phi}_i| < \epsilon$ for large enough i. Take a C^2 function with bounded derivatives $\tilde{\phi}_+$ such that

$$\tilde{\phi}_\infty + \frac{\epsilon}{2} \leq \tilde{\phi}_+ \leq \tilde{\phi}_\infty + \epsilon.$$

We show that $\tilde{\phi}_+$ is a supersolution of the rescaled Lichnerowicz equation (4.1) if i is large enough. Multiplying the rescaled Lichnerowicz equation (4.1) by $\tilde{\phi}^{N+1}_+$, we need to show that

$$\frac{\tilde{\phi}^{N+1}_+}{\tilde{\gamma}^{1/n}}\left(-\Delta\tilde{\phi}_+ + c_n R\tilde{\phi}_+\right) + \beta_\Lambda \tilde{\phi}^{2N}_+ \geq c_n|\tilde{\sigma} + L\tilde{W}_i|^2.$$

Since

$$\tilde{\phi}^{2N}_+ \geq \left(\tilde{\phi}_\infty + \frac{\epsilon}{2}\right)^{2N} \geq \tilde{\phi}^{2N}_\infty + \left(\frac{\epsilon}{2}\right)^{2N},$$

the previous inequality will be satisfied provided that

$$\frac{\tilde{\phi}^{N+1}_+}{\tilde{\gamma}^{1/n}}\left(-\Delta\tilde{\phi}_+ + c_n R\tilde{\phi}_+\right) + \beta_\Lambda \left(\frac{\epsilon}{2}\right)^{2N} \geq c_n|\tilde{\sigma} + L\tilde{W}_i|^2 - c_n|L\tilde{W}_\infty|^2.$$

Note that everything goes to zero as $i \to \infty$ except for the ϵ term. Since $\beta_\Lambda \geq \beta_\Lambda(\tau_0)$, there exists an i_0 such that for all $i \geq i_0$, $\tilde{\phi}_+$ is a supersolution.

Note that $\tilde{\phi}_+ \geq \frac{\epsilon}{2}$. Also, $\tilde{\phi}_i \to \tilde{\mathring{\phi}} = \tilde{\gamma}_i^{-1/2N} \mathring{\phi}$ on the ends for every i. Thus, for i large enough, $\tilde{\phi}_i < \frac{\epsilon}{2}$ outside some compact set K_i. Inside K_i, since $\tilde{\phi}_\infty$ is a supersolution and $\tilde{\phi}_i$ is regular enough, we can apply the maximum principle to show that $\tilde{\phi}_+$ remains larger than $\tilde{\phi}_i$. Thus, $\tilde{\phi}_i \leq \tilde{\phi}_+ \leq \tilde{\phi}_\infty + \epsilon$ for large enough i.

We proceed similarly with a $\tilde{\phi}_- \in C^2$ with

$$\tilde{\phi}_\infty - \epsilon \le \tilde{\phi}_- \le \tilde{\phi}_\infty - \frac{\epsilon}{2}.$$

Since $LW_\infty \to 0$ on the ends, $\tilde{\phi}_-$ is negative on the ends. On the set where it is positive, however, we can show that it is also a subsolution to the rescaled Lichnerowicz equation (4.1). By the same argument as before, $\tilde{\phi}_i \ge \tilde{\phi}_- \ge \tilde{\phi}_\infty - \epsilon$. This completes the theorem. □

We are now able to prove Theorem 2.2:

Proof of Theorem 2.2. Assume that the limit equation admits no such solution for any $\alpha_0 \in (0, \frac{n-1}{n}]$. From Theorem 2.1, we know there exists a sequence of solutions (ϕ_i, W_i) with appropriate regularity of the deformed constraints (2.1–2.2) with $\epsilon_i = 1/i$. If the sequence $\gamma(\phi_i, W_i)$ was unbounded, there would be a non-zero solution to the limit equation by Lemma 4.4, a contradiction. Thus the sequence is bounded, and so by Lemma 4.3 there exists a solution (ϕ_∞, W_∞) with appropriate regularity of the conformal constraint equations (1.5–1.6).

For compactness, let (ϕ_i, W_i) be an arbitrary sequence of solutions to the conformal constraint equations. Using Lemma 4.4 with $\epsilon_i = 0$, $\gamma(\phi_i, W_i)$ is bounded. Lemma 4.3 then says that a subsequence of (ϕ_i, W_i) converges. This completes the proof. □

5. Existence Results

Theorem 2.2 says that if we can show that the limit Eq. (1.7) has no solutions with particular properties, then there is a solution to the full constraint Eqs. (1.5–1.6). In this section, we will prove Corollary 2.3 which uses this result to show that for certain tame near-CMC seed data, there is no solution to the limit equation.

Proof of Corollary 2.3. Assume $W \in W^{2,p}_{-\delta} \oplus \mathscr{Y}$ is a solution of the limit equation. We claim that for any such W,

$$\int |LW|^2 \ge C \int |W|^2 e^{-2\mu t}$$

for some positive $\mu \le \delta, C > 0$. If so, we can then take the limit equation, multiply by W and integrate by parts to get

$$\begin{aligned} \int |LW|^2 &\le -\left(\frac{n-1}{n}\sqrt{c_n}\right) \int |LW| W \frac{d\tau}{\sqrt{\beta_\Lambda}} \\ &\le \left\| \frac{d\tau}{\sqrt{\beta_\Lambda}} \right\|_{C^0_{-\mu}} \|LW\|_{L^2} \left(\int |W|^2 e^{-2\mu t} \right)^{1/2}. \end{aligned}$$

We then immediately see that there is a constant C (the one used in the hypotheses) such that

$$\left\| \frac{d\tau}{\sqrt{\beta_\Lambda}} \right\|_{C^0_{-\mu}} \geq C.$$

This contradicts our hypotheses, and so there is a solution to the constraint equations.

We prove the claimed inequality first for compactly supported vector fields $V \in W^{2,p}_{-\delta}$. First, recall the pointwise Bochner-type formula

$$\frac{1}{2}\mathrm{div}LV = \Delta V + \left(1 - \frac{2}{n}\right) \nabla(\mathrm{div}V) + \mathrm{Ric}(V, \cdot),$$

which is shown, for example, in [9, App B]. Multiplying both sides by V and integrating by parts, we get

$$\frac{1}{2}\int |LV|^2 = \int |\nabla V|^2 + \left(1 - \frac{2}{n}\right)(\mathrm{div}V)^2 - \mathrm{Ric}(V, V).$$

Dropping the div$V$ term and using the Ricci curvature bound, we get

$$\begin{aligned} \frac{1}{2}\int |LV|^2 + c_1 \int \chi^2 |V|^2 &\geq \int |\nabla V|^2 + c_2 \int |V|^2 e^{-2\mu t} \\ &\geq C\|V\|^2_{W^{1,2}(\mathrm{supp}\chi)} \end{aligned} \tag{5.1}$$

where the C depends on χ and μ.

Next, we want to show that $\int |LV|^2 \geq C \int \chi^2 |V|^2$. Assume this were not true. Then there exists a sequence V_i such that

$$\int |LV_i|^2 \leq \frac{1}{i}\int \chi^2 |V_i|^2.$$

We normalize the V_i such that $\int \chi^2 |V_i|^2 = 1$. Because of inequality (5.1), $\|V_i\|_{W^{1,2}(\mathrm{supp}\chi)}$ is bounded. Thus, V_i converge strongly to some V in $L^2(\mathrm{supp}\chi)$ by the Rellich–Kondrachov theorem. In particular, $\int \chi^2 |V|^2 = 1$ and so it is non-zero. Also, since $\int |\nabla V_i|^2$ is bounded, we get weak convergence of $|LV_i|$, and so $\int |LV|^2 = 0$. This implies V is a nontrivial global L^2 conformal Killing field, contradicting our assumptions. Therefore,

$$\int |LV|^2 \geq C \int \chi^2 |V|^2,$$

and this immediately gives that

$$\int |LV|^2 \geq C \int |V|^2 e^{-2\mu t}$$

for compactly supported V. We claim the same inequality holds for $W \in W^{2,p}_{\delta} \oplus \mathscr{Y}$. Indeed, there are smooth cutoff functions η_i such that for $W_i = \eta_i W$,

$$\left| \int |LW|^2 - \int |LW_i|^2 \right| < \frac{1}{i} \quad \text{and} \quad \left| \int |W|^2 e^{-2\mu t} - \int |W_i|^2 e^{-2\mu t} \right| < \frac{1}{i}.$$

This is because LW decays exponentially fast outside a compact set and because $|W|$ is bounded. Thus,

$$\begin{aligned} C\int |W|^2e^{-2\mu t} &\leq C\int |W_i|^2e^{-2\mu t}+\frac{C}{i}\\ &\leq (1+\epsilon)\int |LW_i|^2-C\epsilon\int |W_i|^2e^{-2\mu t}+\frac{C}{i}\\ &\leq (1+\epsilon)\int |LW|^2-C\epsilon\int |W_i|^2e^{-2\mu t}+\frac{C+1+\epsilon}{i} \end{aligned}$$

for some small fixed $\epsilon > 0$. Thus, for large enough i, the last two terms add together to be negative, and so

$$C\int |W|^2e^{-2\mu t}\leq (1+\epsilon)\int |LW|^2$$

for any small enough $\epsilon > 0$. Thus, the desired inequality holds. This completes the proof. □

6. Extension of Main Results to Conformally Asymptotically Cylindrical and Asymptotically Periodic Metrics

While the results in this paper have been proven only for asymptotically cylindrical manifolds, analogous results hold for conformally asymptotically cylindrical manifolds (see Subsect. 1.1) as well with only a few changes in the proof. First, as explained in [15], the L^p-Sobolev version of Theorem 1.2 holds even for conformally AC metrics, and this observation is the basis for the proof of Lemma 3.1. Now, let w be any conformal factor (as in Subsection 1.1) such that $\breve{g} = w^{N-2}g$, where g is an asymptotically cylindrical manifold. It follows from the covariance of the Lichnerowicz equation that ϕ is a global sub/supersolution of the LCBY Eqs. (1.5–1.6) for $\breve{g}$ if and only if $w\phi$ is a global sub/supersolution of the equation

$$\Delta_g\theta - c_nR_g\theta = \beta_\Lambda\theta^{N-1} - c_n|w^2(\sigma + L_{\breve{g}}W)|_g^2\theta^{-N-1} \tag{6.1}$$

coupled with the vector Eq. (1.6) for $\breve{g}$. Notice that only the L operator is defined with respect to $\breve{g}$, while the rest are with respect to g.

Because of this fact, we can find global sub/supersolutions ϕ_-, ϕ_+ to Eq. (6.1) as in Sect. 3, since that metric is asymptotically cylindrical. Then, $\phi_-/w, \phi_+/w$ are global sub and supersolutions to the original conformal constraint equations for $\breve{g}$. Note that these will produce a solution ϕ to the Lichnerowicz equation that asymptotes to $\mathring{\phi}/\mathring{w}_\ell$, which is still a valid asymptote since $\mathring{w}_\ell$ is also a function on N_ℓ. The rest of the proof proceeds in the same way.

In the discussion of the conformal method on manifolds with ends of cylindrical type, it is often of interest to consider metrics which are *asymptotically periodic* as well. This is to say that the metric and its derivatives decay exponentially on each end E_ℓ to a metric $\mathring{g}_\ell$ which is periodic in t. If the period is given by some $T_\ell > 0$, we may think of this metric as a lift to

$\mathbb{R} \times N_\ell$ of a metric on the closed manifold $(\mathbb{R}/T_\ell) \times N_\ell$. These metrics are of particular interest because they arise when one wants to conformally change a cylindrical end to a metric with constant scalar curvature. In particular, one may quotient an exactly cylindrical end by any period and apply the complete theory of the Yamabe problem on closed manifolds to find a conformal factor which gives rise to a constant scalar curvature metric on the quotient. One then lifts this conformal factor to the cylinder to produce a periodic end with constant scalar curvature.

To prove that Theorem 2.2 holds for a manifold with an asymptotically periodic end, we first define the asymptote $\mathring{\phi}$ to be a solution of the reduced Lichnerowicz equation (3.2) on $(\mathbb{R}/T_\ell) \times N_\ell$. Note that since this manifold is also of dimension n, the reduced Lichnerowicz equation is, in fact, precisely the Lichnerowicz equation. The existence of solutions to the subcritical LCBY Eqs. (2.1)-(2.2) which have $\mathring{\phi}$ as an asymptote for the conformal factor is proved in precisely the same way as Theorem 2.1, for one can show that the central theorems used in Sect. 3 all apply in the asymptotically periodic case except for a minor change in the definition of $\mathscr{Y}$ (see [3], [15]). As in the conformally asymptotically cylindrical case, the arguments in Sect. 4 carry over essentially unchanged to the asymptotically periodic case.

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