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Structural Stability of a Dynamical System Near a Non-Hyperbolic Fixed Point

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Dedicated to the memory of Pierluigi Falco (1977–2014)

Abstract. We prove structural stability under perturbations for a class of discrete-time dynamical systems near a non-hyperbolic fixed point. We reformulate the stability problem in terms of the well-posedness of an infinite-dimensional nonlinear ordinary differential equation in a Banach space of carefully weighted sequences. Using this, we prove existence and regularity of flows of the dynamical system which obey mixed initial and final boundary conditions. The class of dynamical systems we study, and the boundary conditions we impose, arise in a renormalization group analysis of the 4-dimensional weakly self-avoiding walk and the 4-dimensional n-component $|\varphi|^4$ spin model.

1. Introduction and Main Result

1.1. Introduction

Let $\mathcal{V} = \mathbb{R}^3$ with elements $V \in \mathcal{V}$ written $V = (g, z, \mu)$ and considered as a column vector for matrix multiplication. For each $j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we define the *quadratic flow* $\bar{\varphi}_j : \mathcal{V} \to \mathcal{V}$ by

$$\bar{\varphi}_j(V) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta_j & \gamma_j & \lambda_j \end{pmatrix} V - \begin{pmatrix} V^{\mathrm{T}} q_j^g V \\ V^{\mathrm{T}} q_j^z V \\ V^{\mathrm{T}} q_j^\mu V \end{pmatrix}, \tag{1.1}$$

with the quadratic terms of the form

$$q_j^g = \begin{pmatrix} \beta_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_j^z = \begin{pmatrix} \theta_j & \frac{1}{2}\zeta_j & 0 \\ \frac{1}{2}\zeta_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.2}$$

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and

$$q_j^\mu = \begin{pmatrix} \upsilon_j^{gg} & \frac{1}{2}\upsilon_j^{gz} & \frac{1}{2}\upsilon_j^{g\mu} \\ \frac{1}{2}\upsilon_j^{gz} & \upsilon_j^{zz} & \frac{1}{2}\upsilon_j^{z\mu} \\ \frac{1}{2}\upsilon_j^{g\mu} & \frac{1}{2}\upsilon_j^{z\mu} & 0 \end{pmatrix}. \tag{1.3}$$

All entries in the above matrices are real numbers. Our hypotheses on the parameters of $\bar{\varphi}$ are stated precisely in Assumptions A1 and A2 below. In particular, we assume that there exists $\lambda > 1$ such that $\lambda_j \geq \lambda$ for all j.

The quadratic flow $\bar{\varphi}$ defines a time-dependent discrete-time 3-dimensional dynamical system. It is triangular, in the sense that the equation for g does not depend on z or μ, the equation for z depends only on g, and the equation for μ depends on g and z. Moreover, the equation for z is linear in z and the equation for μ is linear in μ. This makes the analysis of the quadratic flow elementary.

Our main result concerns structural stability of $\bar{\varphi}$ under a class of infinite-dimensional perturbations. Let $(\mathcal{W}_j)_{j\in\mathbb{N}_0}$ be a sequence of Banach spaces and $X_j = \mathcal{W}_j \oplus \mathcal{V}$. We write $x_j \in X_j$ as $x_j = (K_j, V_j) = (K_j, g_j, z_j, \mu_j)$. Suppose that we are given maps $\psi_j : X_j \to \mathcal{W}_{j+1}$ and $\rho_j : X_j \to \mathcal{V}$. Then, we define $\Phi_j : X_j \to X_{j+1}$ by

$$\Phi_j(K_j, V_j) = \big(\psi_j(K_j, V_j), \bar{\varphi}_j(V_j) + \rho_j(K_j, V_j)\big). \tag{1.4}$$

This is an infinite-dimensional perturbation of the 3-dimensional quadratic flow $\bar{\varphi}$, which breaks triangularity and which involves the spaces $\mathcal{W}_j$ in a nontrivial way. We impose estimates on ψ_j and ρ_j below, which make Φ a third-order perturbation of $\bar{\varphi}$.

We give hypotheses under which there exists a sequence $(x_j)_{j\in\mathbb{N}_0}$ with $x_j \in X_j$ which is a *global flow* of Φ, in the sense that

$$x_{j+1} = \Phi_j(x_j) \quad \text{for all } j \in \mathbb{N}_0, \tag{1.5}$$

obeying the boundary conditions that (K_0, g_0) is fixed, $z_j \to 0$, and $\mu_j \to 0$. Moreover, within an appropriate space of sequences, this global flow is unique.

As we discuss in more detail in Sect. 1.3 below, this result provides an essential ingredient in a renormalization group analysis of the 4-dimensional continuous-time weakly self-avoiding walk [2,3,5], where the boundary condition $\lim_{j\to\infty} \mu_j = 0$ is the appropriate boundary condition for the study of a *critical* trajectory. Similarly, our main result applies also to the analysis of the critical behaviour of the 4-dimensional n-component $|\varphi|^4$ spin model [4]. These applications provide our immediate motivation to study the dynamical system Φ, but we expect that the methods developed here will have further applications to dynamical systems arising in renormalization group analysis in statistical mechanics.

1.2. Dynamical System

We think of $\Phi = (\Phi_j)_{j\in\mathbb{N}_0}$ as the *evolution map* of a discrete time-dependent dynamical system, although it is more usual in dynamical systems to have the spaces X_j be identical. Our application in [2–5] requires the greater generality of j-dependent spaces.

In the case that Φ is a time-independent dynamical system, i.e., when $\Phi_j = \Phi$ and $X_j = X$ for all $j \in \mathbb{N}_0$, its fixed points are of special interest: $x^* \in X$ is a fixed point of Φ if $x^* = \Phi(x^*)$. The dynamical system is called *hyperbolic* near a fixed point $x^* \in X$ if the spectrum of $D\Phi(x^*)$ is disjoint from the unit circle [13]. It is a classic result that for a hyperbolic system there exists a splitting $X \cong X_s \oplus X_u$ into a *stable* and an *unstable manifold* near x^*. The stable manifold is a submanifold $X_s \subset X$ such that $x_j \to x^*$ in X, exponentially fast, when (x_j) satisfies (1.5) and $x_0 \in X_s$. On the other hand, trajectories started on the unstable manifold move away from the fixed point. This result can be generalized without much difficulty to the situation when the Φ_j and X_j are not necessarily identical, viewing "0" as a fixed point (although 0 is the origin in different spaces X_j). The hyperbolicity condition must now be imposed in a uniform way [6, Theorem 2.16].

By definition, $\bar{\varphi}_j(0) = 0$, and we will make assumptions below which can be interpreted as a weakened formulation of the fixed point equation $\Phi_j(0) = 0$ for the dynamical system defined by (1.4). Nevertheless, for simplicity we refer to 0 as a fixed point of $\Phi = (\Phi_j)$. This fixed point 0 is not hyperbolic due to the two unit eigenvalues of the matrix in the first term of (1.1). Thus, the g- and z-directions are *center* directions, which neither contract nor expand in a linear approximation. On the other hand, the hypothesis that $\lambda_j \geq \lambda > 1$ ensures that the μ-direction is *expanding*, and we will assume below that $\psi_j : X_j \to \mathcal{W}_{j+1}$ is such that the K-direction is *contractive* near the fixed point 0. The behaviour of dynamical systems near non-hyperbolic fixed points is much more subtle than for the hyperbolic case. A general classification does not exist, and a nonlinear analysis is required.

1.3. Main Result

In Sect. 2, we give an elementary proof that for $\bar{g}_0$ positive and sufficiently small, there exists a unique choice of $(\bar{z}_0, \bar{\mu}_0)$ such that the global flow $\bar{V} = (\bar{g}, \bar{z}, \bar{\mu})$ of $\bar{\varphi}$ satisfies $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$, where we write, e.g., $\bar{z}_\infty = \lim_{j\to\infty} \bar{z}_j$. Our main result is that, under the assumptions stated below, there exists a unique global flow of Φ with any small initial condition (K_0, g_0) and with final condition $(z_\infty, \mu_\infty) = (0, 0)$, and that this flow is a small perturbation of $\bar{V}$.

The sequence $\bar{g} = (\bar{g}_j)$ plays a prominent role in the analysis. Determined by the sequence (β_j), it obeys

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \quad \bar{g}_0 = g_0 > 0. \tag{1.6}$$

We regard $\bar{g}$ as a known sequence (only dependent on the initial condition g_0). The following examples are helpful to keep in mind.

Example 1.1. (i) Constant $\beta_j = b > 0$. In this case, $\bar{g}_j \sim g_0(1 + g_0 bj)^{-1} \sim (bj)^{-1}$ as $j \to \infty$ (an argument for this standard fact is outlined in the proof of (2.5) below).

(ii) Abrupt cutoff, with $\beta_j = b$ for $j \le J$ and $\beta_j = 0$ for $j > J$, with $J \gg 1$. In this case, $\bar{g}_j$ is approximately the constant $(bJ)^{-1}$ for $j > J$. In particular, $\bar{g}_j$ does not go to zero as $j \to \infty$.

Example 1.1 prompts us to make the following general definition of a cutoff time for bounded sequences β_j. Let $\|\beta\|_\infty = \sup_{j\ge 0} |\beta_j| < \infty$ and let $n_+ = n$ if $n \ge 0$ and otherwise $n_+ = 0$. Given a fixed $\Omega > 1$, we define the Ω-*cutoff time* j_Ω by

$$j_\Omega = \inf\{k \ge 0 : |\beta_j| \le \Omega^{-(j-k)_+} \|\beta\|_\infty \text{ for all } j \ge 0\}. \tag{1.7}$$

The infimum of the empty set is defined to equal ∞, e.g., if $\beta_j = b$ for all j then $j_\Omega = \infty$. By definition, $j_\Omega \le j_{\Omega'}$ if $\Omega \le \Omega'$. To abbreviate the notation, we write

$$\chi_j = \Omega^{-(j-j_\Omega)_+}. \tag{1.8}$$

The evolution maps Φ_j are specified by the real parameters η_j, γ_j, λ_j, β_j, θ_j, ζ_j, $\upsilon_j^{\alpha\beta}$, together with the maps ψ_j and ρ_j on X_j. Throughout this paper, we fix $\Omega > 1$ and make Assumptions A1 and A2 on the real parameters and Assumption A3 on the maps, all stated below. The constants in all estimates are permitted to depend on the constants in these assumptions, including Ω, but *not* on j_Ω and $g_0 > 0$. Furthermore, we consider the situation when the parameters of $\bar{\varphi}_j$ are continuous maps from a metric space M_{ext} of external parameters, $m \in M_{\text{ext}}$, into $\mathbb{R}$, that the maps ψ_j and ρ_j similarly have continuous dependence on m, and that j_Ω is allowed to depend on m. In this situation, the constants in Assumptions A1 and A3 are assumed to hold uniformly in m.

Assumption A1. *The sequence β:* The sequence (β_j) is bounded: $\|\beta\|_\infty < \infty$. There exists $c > 0$ such that $\beta_j \ge c$ for all but c^{-1} values of $j \le j_\Omega$.

Assumption A2. *The other parameters of $\bar{\varphi}$:* There exists $\lambda > 1$ such that $\lambda_j \ge \lambda$ for all $j \ge 0$. There exists $c > 0$ such that $\zeta_j \le 0$ for all but c^{-1} values of $j \le j_\Omega$. Each of η_j, γ_j, θ_j, ζ_j, $\upsilon_j^{\alpha\beta}$ is bounded in absolute value by $O(\chi_j)$, with a constant that is independent of both j and j_Ω.

Note that when $j_\Omega < \infty$, Assumption A1 permits the possibility that eventually $\beta_k = 0$ for large k. The simplest setting for the assumptions is for the case $j_\Omega = \infty$, for which $\chi_j = 1$ for all j. Our applications include situations in which β_j approaches a positive limit as $j \to \infty$, but also situations in which β_j is approximately constant in j over a long initial interval $j \le j_\Omega$ and then abruptly decays to zero.

In Sect. 2, in preparation for the proof of the main result, we prove the following elementary proposition concerning flows of the 3-dimensional quadratic dynamical system $\bar{\varphi}$.

Proposition 1.2. *Assume* (A1–A2). *If $\bar{g}_0 > 0$ is sufficiently small, then there exists a unique global flow $\bar{V} = (\bar{V})_{j\in\mathbb{N}_0} = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j\in\mathbb{N}_0}$ of $\bar{\varphi}$ with initial condition $\bar{g}_0$ and $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$. This flow satisfies $\bar{g}_j = O(\bar{g}_0)$ and*

$$\chi_j \bar{g}_j^n = O\left(\frac{\bar{g}_0}{1 + \bar{g}_0 j}\right)^n, \quad \bar{z}_j = O(\chi_j \bar{g}_j), \quad \bar{\mu}_j = O(\chi_j \bar{g}_j), \tag{1.9}$$

with constants independent of j_Ω and $\bar{g}_0$, and with the first estimate valid for real $n \in [1,\infty)$ with an n-dependent constant. Furthermore, $\bar{V}_j$ is continuously differentiable in the initial condition $\bar{g}_0$, for every $j \in \mathbb{N}_0$, and if the maps $\bar{\varphi}_j$ depend continuously on an external parameter such that (A1–A2) hold with uniform constants, then $\bar{V}_j$ is continuous in this parameter, for every $j \in \mathbb{N}_0$.

In particular, by (1.9), above scale j_Ω each of $\bar{z}_j, \bar{\mu}_j$ decays exponentially. We now define domains $D_j \subset X_j$ on which the perturbation (ψ_j, ρ_j) is assumed to be defined, and an assumption which states estimates for (ψ_j, ρ_j). The domain and estimates depend on an initial condition g_0 and the possible external parameter m. For parameters $a, h > 0$ and sufficiently small $g_0 > 0$, let $(\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j\in\mathbb{N}_0}$ be the sequence determined by Proposition 1.2 with initial condition $\bar{g}_0 = g_0$, and define the domain $D_j = D_j(g_0, a, h) \subset X_j$ by

$$\begin{aligned} D_j = \{x_j \in X_j : \|K_j\|_{\mathcal{W}_j} &\le a\chi_j \bar{g}_j^3, \\ |g_j - \bar{g}_j| &\le h\bar{g}_j^2 |\log \bar{g}_j|, \\ |z_j - \bar{z}_j| &\le h\chi_j \bar{g}_j^2 |\log \bar{g}_j|, \\ |\mu_j - \bar{\mu}_j| &\le h\chi_j \bar{g}_j^2 |\log \bar{g}_j|\}. \end{aligned} \tag{1.10}$$

Note that if β_j depends on an external parameter m, then the domain $D_j = D_j(g_0, m, a, h)$ also depends on this parameter m through $\bar{g}_j = \bar{g}_j(m)$.

Throughout the paper, we write $D_\alpha \phi$ for the Fréchet derivative of a map ϕ with respect to the component α, and $L^m(X_j, X_{j+1})$ for the space of bounded m-linear maps from X_j to X_{j+1}. The following assumption depends on positive parameters $(g_0, a, h, \kappa, \Omega, R, M)$. The norm $\|\cdot\|_{\mathcal{V}}$ is the supremum norm on $\mathbb{R}^3$.

Assumption A3. *The perturbation:* The maps $\psi_j : D_j \to \mathcal{W}_{j+1} \subset X_{j+1}$ and $\rho_j : D_j \to \mathcal{V} \subset X_{j+1}$ are three times continuously Fréchet differentiable, there exist $\kappa \in (0, \Omega^{-1})$, $R \in (0, a(1-\kappa\Omega))$, and a constant $M > 0$ such that, for all $x_j = (K_j, V_j) \in D_j$,

$$\|\psi_j(0, V_j)\|_{\mathcal{W}_{j+1}} \le R\chi_{j+1}\bar{g}_{j+1}^3, \quad \|\rho_j(x_j)\|_{\mathcal{V}} \le M\chi_{j+1}\bar{g}_{j+1}^3, \tag{1.11}$$

$$\|D_K\psi_j(x_j)\|_{L(\mathcal{W}_j,\mathcal{W}_{j+1})} \le \kappa, \quad \|D_K\rho_j(x_j)\|_{L(\mathcal{W}_j,\mathcal{V})} \le M, \tag{1.12}$$

and such that, for both $\phi = \psi$ and $\phi = \rho$ and $2 \le n + m \le 3$,

$$\|D_V\phi_j(x_j)\|_{L(\mathcal{V},X_{j+1})} \le M\chi_{j+1}\bar{g}_{j+1}^2, \tag{1.13}$$

$$\|D_V^m D_K^n \phi_j(x_j)\|_{L^{n+m}(X_j,X_{j+1})} \le M(\chi_{j+1}\bar{g}_{j+1}^3)^{1-n}(\bar{g}_{j+1}^2|\log \bar{g}_{j+1}|)^{-m}. \tag{1.14}$$

The bounds (1.11) guarantee that Φ is a third-order perturbation of $\bar{\varphi}$. Moreover, since $\kappa < 1$, the ψ-part of (1.12) ensures that the K-direction is contractive for Φ. The bounds (1.14) permit the second and third derivatives of ψ and ρ to be quite large. The restriction on R in (A3) may seem unnatural initially, but its role is seen in Lemma 1.3 below.

The following elementary lemma provides a statement of domain compatibility which shows that a sequence $(\bar{K}_j)_{j\in\mathbb{N}_0}$ can be defined inductively by

$\bar{K}_{j+1} = \psi_j(\bar{K}_j, \bar{V}_j)$. Denote by $\pi_K D_j$ the projection of D_j onto $\mathcal{W}_j$, i.e.,

$$\pi_K D_j = \{K_j \in \mathcal{W}_j : \|K_j\|_{\mathcal{W}_j} \le r\chi_j \bar{g}_j^3\}. \tag{1.15}$$

Lemma 1.3. *Assume* (A3), *let* $a^* \in (R/(1-\kappa\Omega), a]$, *and assume that* $g_0 > 0$ *is sufficiently small. Then* $\psi_j(D_j(g_0, a^*, h)) \subseteq \pi_K D_{j+1}(g_0, a^*, h)$.

Proof. The triangle inequality and the first bounds of (1.11)–(1.12) imply

$$\begin{aligned}\|\psi_j(K_j, V_j)\|_{\mathcal{W}_{j+1}} &\le \|\psi_j(0, V_j)\|_{\mathcal{W}_{j+1}} + \|\psi_j(K_j, V_j) - \psi_j(0, V_j)\|_{\mathcal{W}_{j+1}} \\ &\le R\chi_{j+1}\bar{g}_{j+1}^3 + \kappa a^* \chi_j \bar{g}_j^3. \end{aligned} \tag{1.16}$$

Therefore,

$$\begin{aligned}\|\psi_j(K_j, V_j)\|_{\mathcal{W}_{j+1}} &\le R\chi_{j+1}\bar{g}_{j+1}^3 + a^*\kappa\Omega(1 + O(g_0))\chi_{j+1}\bar{g}_{j+1}^3 \\ &\le a^* \chi_{j+1}\bar{g}_{j+1}^3, \end{aligned} \tag{1.17}$$

where the first inequality uses the facts that $\bar{g}_j^3/\bar{g}_{j+1}^3 = 1 + O(g_0)$ (verified in Lemma 2.1(i) below) and that $g_0 > 0$ is sufficiently small, and the second inequality uses the restriction on R in (A3). □

The sequence $\bar{x} = (\bar{K}_j, \bar{V}_j)_{j\in\mathbb{N}_0}$ is a flow of the dynamical system $\bar{\Phi} = (\psi, \bar{\varphi})$ in the sense of (1.5), with initial condition $(\bar{K}_0, \bar{g}_0) = (K_0, g_0)$ and final condition $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$. We consider this sequence as a function $(K_0, g_0) \mapsto \bar{x}(K_0, g_0)$ of the initial condition (K_0, g_0). Our main result is the following theorem, which shows that flows x of the dynamical system $\Phi = (\psi, \bar{\varphi} + \rho) = \bar{\Phi} + (0, \rho)$ are perturbations of the flows $\bar{x}$ of $\bar{\Phi}$.

Theorem 1.4. *Assume* (A1–A3) *with parameters* $(a, h, \kappa, \Omega, R, M)$ *and* $g_0 = g_0'$, *and let* $a_* \in (R/(1-\kappa\Omega), a)$, $b \in (0, 1)$. *There exists* $h_* > 0$ *such that for all* $h \ge h_*$, *there exists* $g_* > 0$ *such that if* $g_0' \in (0, g_*]$ *and* $\|K_0'\|_{\mathcal{W}_0} \le a_* g_0^3$, *the following conclusions hold.*

(i) *There exists a neighborhood* $\mathsf{N} = \mathsf{N}(K_0', g_0') \subset \mathcal{W}_0 \oplus \mathbb{R}$ *of* (K_0', g_0') *such that, for initial conditions* $(K_0, g_0) \in \mathsf{N}$, *there exists a global flow* x *of* $\Phi = (\psi, \bar{\varphi} + \rho)$ *with* $(z_\infty, \mu_\infty) = (0, 0)$ *such that, with* $\bar{x}$ *the unique flow of* $\bar{\Phi} = (\psi, \bar{\varphi})$ *determined by the same boundary conditions,*

$$\|K_j - \bar{K}_j\|_{\mathcal{W}_j} \le b(a - a_*)\chi_j \bar{g}_j^3, \tag{1.18}$$

$$|g_j - \bar{g}_j| \le bh\bar{g}_j^2 |\log \bar{g}_j|, \tag{1.19}$$

$$|z_j - \bar{z}_j| \le bh\chi_j \bar{g}_j^2 |\log \bar{g}_j|, \tag{1.20}$$

$$|\mu_j - \bar{\mu}_j| \le bh\chi_j \bar{g}_j^2 |\log \bar{g}_j|. \tag{1.21}$$

The sequence x *is the unique solution to* (1.5) *which obeys these boundary conditions and the bounds* (1.18)–(1.21).

(ii) *For every* $j \in \mathbb{N}_0$, *the map* $(K_j, V_j) : \mathsf{N} \to \mathcal{W}_j \oplus \mathcal{V}$ *is* C^1 *and obeys*

$$\frac{\partial z_0}{\partial g_0} = O(1), \quad \frac{\partial \mu_0}{\partial g_0} = O(1). \tag{1.22}$$

Remark 1.5. The proof of Theorem 1.4 shows that N contains a ball centered at (K'_0, g'_0) whose radius depends only on g'_0, the constants in (A1–A3), and a_*, b, h, and that this radius is bounded below away from zero uniformly in g'_0 in a compact subset of $(0, g_*]$.

Because of its triangularity, an exact analysis of the flows of $\bar{\varphi}$ with the boundary conditions of interest is straightforward: the three equations for g, z, μ can be solved successively and we do this in Sect. 2 below. Triangularity does not hold for Φ, and we prove in Sects. 3–4 below that the flows of Φ with the boundary conditions of interest nevertheless remain close to the flows of $\bar{\varphi}$ with the same boundary conditions.

Application. A fundamental element in renormalization group analysis concerns the flow of local interactions obtained via iteration of a renormalization group map [15]. The dynamical system (1.4) arises as part of renormalization group studies of the critical behavior of two different but related models: the 4-dimensional n-component $|\varphi|^4$ spin model [4], and the 4-dimensional continuous-time weakly self-avoiding walk [2,3] (see [5] for a preliminary version). The main results of [2,3] are that, for the continuous-time weakly self-avoiding walk in dimension four, the susceptibility diverges with a logarithmic correction as the critical point is approached, and the critical two-point function has $|x|^{-2}$ decay. Related results are obtained for the 4-dimensional n-component $|\varphi|^4$ spin model in [4], complementing and in some cases extending results of [7–10]. Theorem 1.4 is an essential ingredient in analyzing the flows in [2–4], and the uniformity of (1.18)–(1.21) in the cutoff time (for a given Ω) is needed. In [2–4], the index j represents an increasingly large length scale, the spaces $\mathcal{W}_j$ have a subtle definition and are of infinite dimension, and their j-dependence is an inevitable consequence of applying the renormalization group to a lattice model.

Remark 1.6. (i) For $j_\Omega = \infty$ and with (1.9), the bounds (1.18)–(1.21) imply $\|K_j\|_{\mathcal{W}_j} = O(j^{-3})$ and $V_j - \bar{V}_j = O(j^{-2}\log j)$. However, the latter bounds do not reflect that $K_j, V_j \to 0$ as $g_0 \to 0$, while the former do. Furthermore, (1.9) implies $\chi_j \bar{g}_j \to 0$ as $j \to \infty$ (also when $j_\Omega < \infty$), and thus (1.18) and (1.20)–(1.21) imply $K_j \to 0$, $z_j \to 0$, $\mu_j \to 0$ as $j \to \infty$. More precisely, these estimates imply $z_j, \mu_j = O(\chi_j \bar{g}_j)$ so that z_j and μ_j decay exponentially after the Ω-cutoff time j_Ω; we interpret this as indicating that the boundary condition $(z_\infty, \mu_\infty) = (0, 0)$ is essentially achieved already at j_Ω.

(ii) We conjecture that the error bounds in (1.18)–(1.21) have optimal decay as $j \to \infty$. Some indication of this can be found in Remark 3.2 below.

Theorem 1.4 is an analogue of a *stable manifold theorem* for the non-hyperbolic dynamical system defined by (1.4). It is inspired by [6, Theorem 2.16] which, however, holds only in the hyperbolic setting. Irwin [11] showed that the stable manifold theorem for hyperbolic dynamical systems is a consequence of the implicit function theorem in Banach spaces (see also [13,14]). Irwin's approach was inspired by Robbin [12], who showed that the local existence theorem for ordinary differential equations is a consequence of the implicit function theorem. By contrast, in our proof of Theorem 1.4, we directly apply the local existence theorem for ODEs, without explicit mention of the implicit function theorem. This turns out to be advantageous to deal with the lack of hyperbolicity.

Our choice of $\bar{\varphi}$ in (1.1) has a specific triangular form. One reason for this is that (1.1) accommodates what is required in our application in [2–5]. A second reason is that additional nonzero terms in $\bar{\varphi}$ can lead to the failure of Theorem 1.4. The condition that β_j is mainly non-negative is important for the sequence $\bar{g}_j$ of (1.6) to remain bounded. The following example shows that our sign restriction on the ζ_j term in the flow of $\bar{z}$ is also important, since positive ζ_j can lead to violation of a conclusion of Theorem 1.4.

Example 1.7. Suppose that $\zeta_j = \theta_j = \beta_j = 1$, that $\rho = 0$, and that $\bar{g}_0 > 0$ is small. For this constant β sequence, $j_\Omega = \infty$ (for any $\Omega > 1$) and hence $\chi_j = 1$ for all j. As in Example 1.1, $\bar{g}_j \sim j^{-1}$. By (1.1) and (1.6),

$$\bar{z}_{j+1} = \bar{z}_j(1 - \bar{g}_j) - \bar{g}_j^2 = \bar{z}_j \frac{\bar{g}_{j+1}}{\bar{g}_j} - \bar{g}_j^2. \tag{1.23}$$

Let $\bar{y}_j = \bar{z}_j/\bar{g}_j$. Since $\bar{g}_j/\bar{g}_{j+1} = (1 - \bar{g}_j)^{-1} \geq 1$, we obtain $\bar{y}_j \geq \bar{y}_{j+1} + \bar{g}_j$ and hence

$$\bar{y}_j \geq \bar{y}_{n+1} + \sum_{l=j}^{n} \bar{g}_l. \tag{1.24}$$

Suppose that $\bar{z}_j = O(\bar{g}_j)$, as in (1.9). Then $\bar{y}_j = O(1)$ and hence by taking the limit $n \to \infty$ we obtain

$$\bar{y}_j \geq \limsup_{n\to\infty} \left(\bar{y}_{n+1} + \sum_{l=j}^{n} \bar{g}_l \right) \geq -C + \sum_{l=j}^{\infty} \bar{g}_l. \tag{1.25}$$

However, since $\bar{g}_j \sim j^{-1}$, the last sum diverges. This contradiction implies that the conclusion $\bar{z}_j = O(\bar{g}_j)$ of (1.20) is impossible.

1.4. Continuity in External Parameter

The uniqueness statement of Theorem 1.4 implies the following corollary regarding continuous dependence on an external parameter of the global flow for (1.5) given by Theorem 1.4. In the statement of the corollary, we assume that D_j is actually the union over $m \in M_{\text{ext}}$ of the domains on the right-hand side of (1.10). Recall that the latter domains depend on m through β and $\bar{g}$.

Corollary 1.8. *Assume that $\Phi_j : D_j \times M_{\text{ext}} \to X_{j+1}$ are continuous maps and that (A1–A3) hold for $\Phi_j(\cdot, m)$, for every $m \in M_{\text{ext}}$, and with parameters independent of m. Let $x(m, u_0) = (K(m, u_0), V(m, u_0))$ be the global flow for external parameter m and initial condition $u_0 = (K_0, g_0)$ guaranteed by Theorem 1.4. Then x_j is continuous in (m, u_0) for each $j \in \mathbb{N}_0$.*

Proof. We fix $m \in M_{\text{ext}}$, $u_0 \in \mathsf{N}$ and show that x_j is continuous at this fixed (m, u_0). For any $m' \in M_{\text{ext}}$, $u_0' \in \mathsf{N}$, let $x(m', u_0') = (K(m', u_0'), V(m', u_0'))$ denote the unique global flow of Theorem 1.4; it satisfies the estimates

$$\|K_j(m') - \bar{K}_j(m')\|_{\mathcal{W}_j} \le b(a - a_*)\chi_j(m')\bar{g}_j(m')^3, \tag{1.26}$$

$$|g_j(m') - \bar{g}_j(m')| \le bh\bar{g}_j(m')^2 |\log \bar{g}_j(m')|, \tag{1.27}$$

$$|\mu_j(m') - \bar{\mu}_j(m')| \le bh\chi_j(m')\bar{g}_j(m')^2 |\log \bar{g}_j(m')|, \tag{1.28}$$

$$|z_j(m') - \bar{z}_j(m')| \le bh\chi_j(m')\bar{g}_j(m')^2 |\log \bar{g}_j(m')|. \tag{1.29}$$

By Proposition 1.2, $\bar{V}_j(m', g_0')$ is continuous in (m', g_0'), and thus in particular $\bar{V}_0(m', g_0')$ is uniformly bounded for (m', g_0') in a bounded neighbourhood I of (m, g_0). With (1.27)–(1.29), we see that $V_0(m', u_0')$ is therefore also uniformly bounded in I. Thus, for every sequence $(m', u_0') \to (m, u_0)$, $V_0(m', u_0')$ has a limit point. It suffices to show that the limit point is unique. To show this uniqueness, we fix an arbitrary limit point V_0^* and a sequence $(m', u_0') \to (m, u_0)$ such that $V_0(m', u_0') \to V_0^*$. Since $K_0' \to K_0$, we also set $K_0^* = K_0$.

Then define $x_j^* = (K_j^*, V_j^*)$ by inductive application of $\Phi_j(m, \cdot)$ starting from $x_0^* = (K_0^*, V_0^*)$, as long as $x_j^* \in D_j$. Since $(K_0', V_0(m', u_0')) \to (K_0^*, V_0^*)$, it follows by induction and the assumed continuity of ψ_j, ρ_j that $x_j(m', u_0') \to x_j^*$. By an analogous induction, using continuity of $\bar{V}_j$ and ψ_j, it follows that $\bar{K}_j(m', u_0') \to \bar{K}_j(m, u_0)$. Since $\chi_j(m') \to \chi_j(m)$, we can now take the limit of (1.26)–(1.29) along the sequence $(m', u_0') \to (m, u_0)$ and obtain

$$\|K_j^* - \bar{K}_j(m)\|_{\mathcal{W}_j} \le b(a - a_*)\chi_j(m)\bar{g}_j(m)^3, \tag{1.30}$$

$$|g_j^* - \bar{g}_j(m)| \le bh\bar{g}_j(m)^2 |\log \bar{g}_j(m)|, \tag{1.31}$$

$$|\mu_j^* - \bar{\mu}_j(m)| \le bh\chi_j(m)\bar{g}_j(m)^2 |\log \bar{g}_j(m)|, \tag{1.32}$$

$$|z_j^* - \bar{z}_j(m)| \le bh\chi_j(m)\bar{g}_j(m)^2 |\log \bar{g}_j(m)|. \tag{1.33}$$

The uniqueness assertion of Theorem 1.4 implies that $x_j^* = x_j(m, u_0)$, and we see that the above inductions can in fact be carried out indefinitely. We also conclude that $V_0^* = V_0(m, u_0)$, so there is a unique limit point of $V_0(m', u_0')$ as $(m', u_0') \to (m, u_0)$. This shows that V_0 is continuous at (m, u_0). The continuity of x_j now follows inductively from the continuity of the Φ_j. □

2. Quadratic Flow

In this section, we prove that, for the quadratic approximation $\bar{\varphi}$, there exists a unique solution $\bar{V} = (\bar{V}_j)_{j \in \mathbb{N}_0} = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)_{j \in \mathbb{N}_0}$ to the flow equation

$$\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j) \quad \text{with fixed small } \bar{g}_0 > 0 \text{ and with } (\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0). \tag{2.1}$$

Due to the triangular nature of $\bar{\varphi}$, we can obtain detailed information about the sequence $\bar{V}$. In particular, we prove Proposition 1.2.

2.1. Flow of $\bar{g}$

We start with the analysis of the sequence $\bar{g}$, which obeys the recursion

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \quad \bar{g}_0 = g_0 > 0. \tag{2.2}$$

The following lemma collects the information we need about $\bar{g}$.

Lemma 2.1. *Assume* (A1). *The following statements hold if $\bar{g}_0 > 0$ is sufficiently small, with all constants independent of j_Ω and $\bar{g}_0$.*

(i) *For all j, $\bar{g}_j > 0$,*

$$\bar{g}_j = O\Big(\inf_{k \le j} \bar{g}_k\Big), \quad \text{and} \quad \bar{g}_j \bar{g}_{j+1}^{-1} = 1 + O(\chi_j \bar{g}_j) = 1 + O(\bar{g}_0). \tag{2.3}$$

For all j and k, $\bar{g}_j$ is non-increasing in β_k.

(ii) (a) *For real $n \in [1, \infty)$ and $m \in [0, \infty)$, there exists $C_{n,m} > 0$ such that for all $k \ge j \ge 0$,*

$$\sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le C_{n,m} \begin{cases} |\log \bar{g}_k|^{m+1} & n = 1 \\ \chi_j \bar{g}_j^{n-1} |\log \bar{g}_j|^m & n > 1. \end{cases} \tag{2.4}$$

(b) *For real $n \in [1, \infty)$, there exists $C_n > 0$ such that for all $j \ge 0$,*

$$\chi_j \bar{g}_j^n \le C_n \left(\frac{\bar{g}_0}{1 + \bar{g}_0 j} \right)^n. \tag{2.5}$$

(iii) (a) *For $\gamma \ge 0$ and $j \ge 0$, there exist constants $c_j = 1 + O(\chi_j \bar{g}_j)$ (depending on γ) such that, for all $l \ge j$,*

$$\prod_{k=j}^{l} (1 - \gamma \beta_k \bar{g}_k)^{-1} = \left(\frac{\bar{g}_j}{\bar{g}_{l+1}} \right)^\gamma (c_j + O(\chi_l \bar{g}_l)). \tag{2.6}$$

The constant c_j is continuous in g_0 and if the β_j depend continuously on an external parameter such that (A1) *holds uniformly in that parameter, then c_j is also continuous in the external parameter.*

(b) *For $\zeta_j \le 0$ except for c^{-1} values of $j \le j_\Omega$, $\zeta_j = O(\chi_j)$, and $j \le l$, (with a constant independent of j and l),*

$$\prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \le O(1). \tag{2.7}$$

(iv) *Suppose that $\bar{g}$ and $\mathring{g}$ each satisfy* (2.2). *Let $\delta > 0$. If $|\mathring{g}_0 - \bar{g}_0| \le \delta \mathring{g}_0$ then $|\mathring{g}_j - \bar{g}_j| \le \delta \mathring{g}_j (1 + O(\bar{g}_0))$ for all j.*

Proof. (i) By (2.2),

$$\bar{g}_{j+1} = \bar{g}_j (1 - \beta_j \bar{g}_j). \tag{2.8}$$

Since $\beta_j = O(\chi_j)$, by (2.8), the second statement of (2.3) is a consequence of the first, so it suffices to verify the first statement of (2.3). Assume inductively that $\bar{g}_j > 0$ and that $\bar{g}_j = O(\inf_{k \le j} \bar{g}_k)$. It is then immediate from (2.8) that $\bar{g}_{j+1} > 0$ if $\bar{g}_0$ is sufficiently small depending on $\|\beta\|_\infty$, and that $\bar{g}_{j+1} \le \bar{g}_j$ if $\beta_j \ge 0$. By (A1), there are at most c^{-1} values of $j \le j_\Omega$ for which $\beta_j < 0$.

Therefore, by choosing $\bar{g}_0$ sufficiently small depending on $\|\beta\|_\infty$ and c, it follows that $\bar{g}_j \le O(\inf_{k\le j} \bar{g}_k)$ for all $j \le j_\Omega$ with a constant that is independent of j_Ω.

To advance the inductive hypothesis for $j > j_\Omega$, we use $1 - t \le e^{-t}$ and $\sum_{l=j_\Omega}^\infty |\beta_l| \le \sum_{n=1}^\infty \Omega^{-n} = O(1)$ to obtain, for $j \ge k \ge j_\Omega$,

$$\bar{g}_j \le \bar{g}_k \exp\left[-\sum_{l=k}^{j-1} \beta_l \bar{g}_l\right] \le \bar{g}_k \exp\left[C\bar{g}_k \sum_{l=k}^{j-1} |\beta_l|\right] \le O(\bar{g}_k). \tag{2.9}$$

This shows that $\bar{g}_j = O(\inf_{j_\Omega \le k \le j} \bar{g}_k)$. However, by the inductive hypothesis, $\bar{g}_{j_\Omega} = O(\inf_{k\le j_\Omega} \bar{g}_k)$ for $j \le j_\Omega$, and hence for $j > j_\Omega$ we do have $\bar{g}_j = O(\inf_{k\le j} \bar{g}_k)$ as claimed. This completes the verification of the first bound of (2.3) and thus, as already noted, also of the second.

The monotonicity of $\bar{g}_j$ in β_k can be proved as follows. Since $\bar{g}_j$ does not depend on β_k if $k \ge j$ by definition, we may assume that $k < j$. Moreover, by replacing j by $j+k$, we may assume that $k = 0$. Let $\bar{g}'_j = \partial \bar{g}_j / \partial \beta_0$. Since $\bar{g}'_0 = 0$,

$$\bar{g}'_1 = -\bar{g}_0^2 < 0. \tag{2.10}$$

Assuming that $\bar{g}'_j < 0$ by induction, it follows that for $j \ge 1$,

$$\bar{g}'_{j+1} = \bar{g}'_j(1 - 2\beta_j \bar{g}_j) = \bar{g}'_j(1 + O(g_0)) < 0, \tag{2.11}$$

and the proof of monotonicity is complete.

(ii-a) We first show that if $\psi : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then

$$\sum_{l=j}^{k} \beta_l \psi(\bar{g}_l)\bar{g}_l^2 = \int_{\bar{g}_{k+1}}^{\bar{g}_j} \psi(t)\,\mathrm{d}t + O\left(\int_{\bar{g}_{k+1}}^{\bar{g}_j} t^2|\psi'(t)|\,\mathrm{d}t\right). \tag{2.12}$$

To prove (2.12), we apply (2.2) to obtain

$$\sum_{l=j}^{k} \beta_l \psi(\bar{g}_l)\bar{g}_l^2 = \sum_{l=j}^{k} \psi(\bar{g}_l)(\bar{g}_l - \bar{g}_{l+1}) = \sum_{l=j}^{k} \int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(\bar{g}_l)\,\mathrm{d}t. \tag{2.13}$$

The integral can be written as

$$\int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(\bar{g}_l)\,\mathrm{d}t = \int_{\bar{g}_{l+1}}^{\bar{g}_l} \psi(t)\,\mathrm{d}t + \int_{\bar{g}_{l+1}}^{\bar{g}_l}\int_t^{\bar{g}_l} \psi'(s)\,\mathrm{d}s\,\mathrm{d}t. \tag{2.14}$$

The first term on the right-hand side of (2.12) is then the sum over l of the first term on the right-hand side of (2.14), so it remains to estimate the double integral. By Fubini's theorem,

$$\int_{\bar{g}_{l+1}}^{\bar{g}_l}\int_t^{\bar{g}_l} \psi'(s)\,\mathrm{d}s\,\mathrm{d}t = \int_{\bar{g}_{l+1}}^{\bar{g}_l}\int_{\bar{g}_{l+1}}^{s} \psi'(s)\,\mathrm{d}t\,\mathrm{d}s = \int_{\bar{g}_{l+1}}^{\bar{g}_l} (s - \bar{g}_{l+1})\psi'(s)\,\mathrm{d}s. \tag{2.15}$$

By (2.2) and (2.3), for s in the domain of integration we have

$$|s - \bar{g}_{l+1}| \le |\bar{g}_l - \bar{g}_{l+1}| = |\beta_l|\bar{g}_l^2 \le (1 + O(\bar{g}_0))|\beta_l|\bar{g}_{l+1}^2 \le O(s^2). \tag{2.16}$$

This permits us to estimate (2.15) and conclude (2.12).

Direct evaluation of the integrals in (2.12) with $\psi(t) = t^{n-2}|\log t|^m$ gives

$$\sum_{l=j}^{k} \beta_l \bar{g}_l^n |\log \bar{g}_l|^m \le C_{n,m} \begin{cases} |\log \bar{g}_k|^{m+1} & n = 1 \\ \bar{g}_j^{n-1} |\log \bar{g}_j|^m & n > 1. \end{cases} \tag{2.17}$$

To deduce (2.4), we only consider the case $n > 1$, as the case $n = 1$ is similar. Suppose first that $j \le j_\Omega$. Assumption A1 implies that

$$1 \le \frac{\beta_l}{c} + \left(1 + \frac{|\beta_l|}{c}\right) 1_{\beta_l < c} \le O(\beta_l) + O(1_{\beta_l < c}), \tag{2.18}$$

and therefore

$$\sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le \sum_{l=j}^{j_\Omega} O(\beta_l) \bar{g}_l^n |\log \bar{g}_l|^m + \sum_{l=j}^{j_\Omega} O(1_{\beta_l < c}) \bar{g}_l^n |\log \bar{g}_l|^m + \sum_{l=j_\Omega+1}^{k} \Omega^{-(l-j_\Omega)_+} \bar{g}_l^n |\log \bar{g}_l|^m. \tag{2.19}$$

By (2.17), the first term is bounded by $O(\bar{g}_j^{n-1} |\log \bar{g}_j|^m)$. The second term obeys the same bound, by (A1) and (2.3), as does the last term (which is only present when $j_\Omega < \infty$) due to the exponential decay. This proves (2.4) for the case $j \le j_\Omega$. On the other hand, if $j > j_\Omega$, then again using the exponential decay of χ_l and (2.3), we obtain

$$\sum_{l=j}^{k} \chi_l \bar{g}_l^n |\log \bar{g}_l|^m \le C \chi_j \bar{g}_j^n |\log \bar{g}_j|^m \le C \bar{g}_0 \chi_j \bar{g}_j^{n-1} |\log \bar{g}_j|^m. \tag{2.20}$$

This completes the proof of (2.4) for the case $n > 1$.

(ii-b) To prove (2.5), let $c > 0$ be as in Assumption A1 and set $\hat{g}_{j+1} = \hat{g}_j - c\hat{g}_j^2$ with $\hat{g}_0 = \bar{g}_0$. The sequence $(\hat{g}_j)$ satisfies the bound (2.5), since application of (2.12) with $\psi(t) = t^{-2}$ gives $(k+1)c = \hat{g}_{k+1}^{-1} - \hat{g}_0^{-1} + O(\log(\hat{g}_0/\hat{g}_{k+1})$ and hence $\hat{g}_j \sim \hat{g}_0/(1 + c\hat{g}_0 j)$. It therefore suffices to prove that $\chi_j \bar{g}_j^n \le O(\hat{g}_j^n)$.

We first show that it suffices to prove that

$$\chi_j \bar{g}_j = \bar{g}_j \le O(\hat{g}_j) \quad \text{for all } j \le j_\Omega \tag{2.21}$$

(the first inequality holds since $\chi_j = 1$ for $j \le j_\Omega$ by definition). To see this, we note that for $\bar{g}_0 > 0$ sufficiently small and for all j,

$$\Omega^{-1} \le (1 - c\bar{g}_0)^n \le (1 - c\hat{g}_j)^n = \left(\frac{\hat{g}_{j+1}}{\hat{g}_j}\right)^n. \tag{2.22}$$

For $j > j_\Omega$, by (2.21) and the fact that $\bar{g}_j = O(\bar{g}_{j_\Omega})$ by (2.3), this implies that

$$\chi_j \bar{g}_j^n \le O(\Omega^{-(j-j_\Omega)} \bar{g}_{j_\Omega}^n) \le O(\Omega^{-(j-j_\Omega)} \hat{g}_{j_\Omega}^n) \le O\left(\prod_{l=j_\Omega}^{j-1} \frac{\hat{g}_{l+1}}{\hat{g}_l}\right)^n \hat{g}_{j_\Omega}^n = O(\hat{g}_j^n). \tag{2.23}$$

For $j \le j_\Omega$, since $\chi_j = 1$, it suffices to prove (2.5) with $n = 1$, i.e., (2.21).

Let $\tilde{\beta}_j = \min\{c, \beta_j\}$, and define $\tilde{g}_j$ by the recursion $\tilde{g}_{j+1} = \tilde{g}_j - \tilde{\beta}_j \tilde{g}_j^2$ with $\tilde{g}_0 = \bar{g}_0$. By the monotonicity in β asserted in part (i),

$$\bar{g}_j \leq \tilde{g}_j, \quad \hat{g}_j \leq \tilde{g}_j. \tag{2.24}$$

Denote by $0 \leq j_1 < j_2 < \cdots < j_m$ the sequence of $j \leq j_\Omega$ such that $\beta_j < c$. By (A1), the number of elements in this sequence is indeed finite. By the first inequality of (2.24) and the definition of $\tilde{g}_j$, it follows that, for $j \leq j_\Omega$,

$$\bar{g}_j \leq \tilde{g}_j = \tilde{g}_0 \prod_{l=0}^{j-1} (1 - \tilde{\beta}_l \tilde{g}_l) = \tilde{g}_0 \prod_{l=0}^{j-1} (1 - c\tilde{g}_l) \prod_{k \leq m : j_m \leq j-1} \frac{1 - \beta_{j_k} \tilde{g}_{j_k}}{1 - c\tilde{g}_{j_k}}. \tag{2.25}$$

Thus, with $\tilde{g}_0 = \hat{g}_0$, the second inequality of (2.24), and the definition of $\hat{g}_j$,

$$\bar{g}_j \leq \hat{g}_0 \prod_{l=0}^{j-1} (1 - c\hat{g}_l) \prod_{k \leq m : j_m \leq j-1} \frac{1 - \beta_{j_k} \tilde{g}_{j_k}}{1 - c\tilde{g}_{j_k}} = \hat{g}_j \prod_{k \leq m : j_m \leq j-1} \frac{1 - \beta_{j_k} \tilde{g}_{j_k}}{1 - c\tilde{g}_{j_k}}. \tag{2.26}$$

The product on the last line is a product of at most m factors which are each $1 + O(g_0)$, and can thus be bounded by $1 + O(g_0)$. In particular,

$$\bar{g}_j \leq (1 + O(g_0))\hat{g}_j \leq O(\hat{g}_j) \quad \text{for } j \leq j_\Omega, \tag{2.27}$$

and the proof of (2.5) is complete.

(iii-a) By Taylor's theorem and (2.2), there exists $r_k = O(\beta_k \bar{g}_k)^2$ such that

$$(1 - \gamma\beta_k \bar{g}_k)^{-1} = (1 - \beta_k \bar{g}_k)^{-\gamma}(1 + r_k) = \left(\frac{\bar{g}_k}{\bar{g}_{k+1}}\right)^\gamma (1 + r_k). \tag{2.28}$$

For $l \geq j$, let

$$c_{j,l} = \prod_{k=j}^{l} (1 + r_k) = \exp\left(\sum_{k=j}^{l} \log(1 + r_k)\right). \tag{2.29}$$

Since $\log(1 + r_k) = O(\chi_k \bar{g}_k^2)$, it follows from (2.4) that the sum on the right-hand side of (2.29) is bounded by $O(\chi_j \bar{g}_j)$ uniformly in l. We can thus define

$$c_j = \exp\left(\sum_{k=j}^{\infty} \log(1 + r_k)\right) = 1 + O(\chi_j \bar{g}_j). \tag{2.30}$$

The bound on the sum also shows

$$c_j - c_{j,l} = c_j \left(1 - \exp\left(-\sum_{k=l}^{\infty} \log(1 + r_k)\right)\right) = c_j(1 + O(\chi_l \bar{g}_l)). \tag{2.31}$$

Moreover, these estimates hold uniformly in a neighborhood of g_0 and in the external parameter, by assumption. Thus, the dominated convergence theorem implies continuity of c_j, both in g_0 and in the external parameter, and the proof is complete.

(iii-b) Since $\zeta_j \le 0$ for all but c^{-1} values of $j \le j_\Omega$, by (2.3) with $\bar g_0$ sufficiently small, $\prod_{k=j}^{l}(1-\zeta_k \bar g_k)^{-1} \le O(1)$ for $l \le j_\Omega$, with a constant independent of j_Ω. For $j \ge j_\Omega$, we use $1/(1-x) \le 2e^x$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ to obtain

$$\prod_{k=j}^{l}(1-\zeta_k \bar g_k)^{-1} \le 2\exp\left[\sum_{k=j}^{l} \zeta_k \bar g_k\right] \le 2\exp\left[C\bar g_j \sum_{k=j_\Omega}^{\infty} \chi_k\right] \le O(1). \tag{2.32}$$

The bounds for $l \le j_\Omega$ and $j \ge j_\Omega$ together imply (2.7).

(iv) If $|\mathring{g}_j - \bar g_j| \le \delta_j \mathring{g}_j$ then by (2.2),

$$|\mathring{g}_{j+1} - \bar g_{j+1}| = |\mathring{g}_j - \bar g_j|(1-\beta_j(\mathring{g}_j + \bar g_j)) \le \delta_{j+1}\mathring{g}_{j+1} \tag{2.33}$$

with

$$\delta_{j+1} = \delta_j \frac{1-\beta_j(\mathring{g}_j + \bar g_j)}{1-\beta_j \mathring{g}_j} = \delta_j\left(1 - \frac{\beta_j \bar g_j}{1-\beta_j \mathring{g}_j}\right). \tag{2.34}$$

In particular, if $\beta_j \ge 0$, then $\delta_{j+1} \le \delta_j$. By (A1), there are at most c^{-1} values of $j \le j_\Omega$ for which $\beta_j < 0$, and hence $\delta_j \le \delta(1+O(\bar g_0))$ for $j \le j_\Omega$. The desired estimate therefore holds for $j \le j_\Omega$. For $j \ge l > j_\Omega$, as in (2.9) we have

$$\prod_{k=l}^{j}(1+O(\beta_k \bar g_k)) \le \exp\left[O(\bar g_l)\sum_{k=l}^{j}\chi_k\right] \le 1 + O(\bar g_0), \tag{2.35}$$

and thus the claim remains true also for $j > j_\Omega$. □

2.2. Flow of $\bar z$ and $\bar \mu$

We now establish the existence of unique solutions to the $\bar z$ and $\bar \mu$ recursions with boundary conditions $\bar z_\infty = \bar\mu_\infty = 0$ and obtain estimates on these solutions.

Lemma 2.2. *Assume (A1–A2). If $\bar g_0$ is sufficiently small then there exists a unique solution to (2.1) obeying $\bar z_\infty = \bar \mu_\infty = 0$. This solution obeys $\bar z_j = O(\chi_j \bar g_j)$ and $\bar\mu_j = O(\chi_j \bar g_j)$. Furthermore, if the maps $\bar\varphi_j$ depend continuously on $m \in M_{\rm ext}$ and (A1–A2) hold with uniform constants, then $\bar g_j$, $\bar z_j$ and $\bar \mu_j$ are continuous in $M_{\rm ext}$.*

Proof. By (1.1), $\bar z_{j+1} = \bar z_j - \zeta_j \bar g_j \bar z_j - \theta_j \bar g_j^2$, so that

$$\bar z_j = \prod_{k=j}^{n}(1-\zeta_k \bar g_k)^{-1}\bar z_{n+1} + \sum_{l=j}^{n}\prod_{k=j}^{l}(1-\zeta_k \bar g_k)^{-1}\theta_l \bar g_l^2. \tag{2.36}$$

In view of (2.7), whose assumptions are satisfied by (A2), the unique solution to the recursion for $\bar z$ which obeys the boundary condition $\bar z_\infty = 0$ is

$$\bar z_j = \sum_{l=j}^{\infty}\prod_{k=j}^{l}(1-\zeta_k \bar g_k)^{-1}\theta_l \bar g_l^2, \tag{2.37}$$

and by (A2), (2.4), and (2.7),

$$|\bar{z}_j| \le \sum_{l=j}^{\infty} O(\chi_l)\bar{g}_l^2 \le O(\chi_j \bar{g}_j). \tag{2.38}$$

Since $\bar{g}_j$ is defined by a finite recursion, its continuity in $m \in M_{\text{ext}}$ follows from the assumed continuity of each β_k in m. To verify continuity of $\bar{z}_j$ in m, let $\bar{z}_{j,n} = \sum_{l=j}^{n} \prod_{k=j}^{l} (1 - \zeta_k \bar{g}_k)^{-1} \theta_l \bar{g}_l^2$. Since $\bar{g}_j$ is continuous in M_{ext} for any $j \ge 0$, $\bar{z}_{j,n}$ is also continuous, for any $j \le n$. By (2.7) and (2.4)–(2.5), $|\bar{z}_j - \bar{z}_{j,n}| \le O(\chi_n \bar{g}_n) \to 0$ as $n \to \infty$, uniformly in m, and thus, as a uniform limit of continuous functions, $\bar{z}_j$ is continuous in $m \in M_{\text{ext}}$.

For $\bar{\mu}$, we first define

$$\sigma_j = \eta_j \bar{g}_j + \gamma_j \bar{z}_j - \upsilon_j^{gg} \bar{g}_j^2 - \upsilon_j^{gz} \bar{g}_j \bar{z}_j - \upsilon_j^{zz} \bar{z}_j^2, \quad \tau_j = \upsilon_j^{g\mu} \bar{g}_j + \upsilon_j^{z\mu} \bar{z}_j, \tag{2.39}$$

so that the recursion for $\bar{\mu}$ can be written as

$$\bar{\mu}_{j+1} = (\lambda_j - \tau_j)\bar{\mu}_j + \sigma_j. \tag{2.40}$$

Alternatively,

$$\bar{\mu}_j = (\lambda_j - \tau_j)^{-1}(\bar{\mu}_{j+1} - \sigma_j). \tag{2.41}$$

Given $\alpha \in (\lambda^{-1}, 1)$, we can choose $\bar{g}_0$ sufficiently small that

$$\tfrac{1}{2}\lambda^{-1} \le (\lambda_j - \tau_j)^{-1} \le \alpha. \tag{2.42}$$

The limit of repeated iteration of (2.41) gives

$$\bar{\mu}_j = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \sigma_l \tag{2.43}$$

as the unique solution which obeys the boundary condition $\mu_\infty = 0$. Geometric convergence of the sum is guaranteed by (2.42), together with the fact that $\sigma_j \le O(\chi_j \bar{g}_j) \le O(1)$. To estimate (2.43), we use

$$|\bar{\mu}_j| \le \sum_{l=j}^{\infty} \alpha^{l-j+1} O(\chi_l \bar{g}_l). \tag{2.44}$$

Since $\alpha < 1$, the first bound of (2.3) and monotonicity of χ imply that

$$|\bar{\mu}_j| \le O(\chi_j \bar{g}_j). \tag{2.45}$$

The proof of continuity of $\bar{\mu}_j$ in M_{ext} is analogous to that for $\bar{z}_j$. This completes the proof. □

2.3. Differentiation of Quadratic Flow

The following lemma gives estimates on the derivatives of the components of $\bar{V}_j$ with respect to the initial condition $\bar{g}_0$. We write f' for the derivative of f with respect to $g_0 = \bar{g}_0$. These estimates are an ingredient in the proof of Theorem 1.4(ii).

Lemma 2.3. *For each $j \geq 0$, $\bar{V}_j = (\bar{g}_j, \bar{z}_j, \bar{\mu}_j)$ is twice differentiable with respect to the initial condition $\bar{g}_0 > 0$, and the derivatives obey*

$$\bar{g}_j' = \left(\frac{\bar{g}_j^2}{\bar{g}_0^2}\right), \qquad \bar{z}_j' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\right), \quad \bar{\mu}_j' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\right), \tag{2.46}$$

$$\bar{g}_j'' = O\left(\frac{\bar{g}_j^2}{\bar{g}_0^3}\right), \quad \bar{z}_j'' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\right), \quad \bar{\mu}_j'' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\right). \tag{2.47}$$

Proof. Differentiation of (1.6) gives

$$\bar{g}_{j+1}' = \bar{g}_j'(1 - 2\beta_j \bar{g}_j), \tag{2.48}$$

from which we conclude by iteration and $\bar{g}_0' = 1$ that for $j \geq 1$,

$$\bar{g}_j' = \prod_{l=0}^{j-1}(1 - 2\beta_l \bar{g}_l). \tag{2.49}$$

Therefore, by (2.6),

$$\bar{g}_j' = \left(\frac{\bar{g}_j}{\bar{g}_0}\right)^2 (1 + O(\bar{g}_0)). \tag{2.50}$$

For the second derivative, we use $\bar{g}_0'' = 0$ and $\bar{g}_{j+1}'' = \bar{g}_j''(1 - 2\beta_j \bar{g}_j) - 2\beta_j \bar{g}_j'^2$ to obtain

$$\bar{g}_j'' = -2\sum_{l=0}^{j-1} \beta_l \bar{g}_l'^2 \prod_{k=l}^{j-2}(1 - 2\beta_k \bar{g}_k). \tag{2.51}$$

With the bounds of Lemma 2.1, this gives

$$\bar{g}_j'' = O\left(\frac{\bar{g}_j}{\bar{g}_0}\right)^2 \sum_{l=0}^{j-1} \beta_l \frac{\bar{g}_l^2}{\bar{g}_0^2} = O\left(\frac{\bar{g}_j^2}{\bar{g}_0^3}\right). \tag{2.52}$$

For $\bar{z}$, we define $\sigma_{j,l} = \prod_{k=j}^{l}(1 - \zeta_k \bar{g}_k)^{-1}$. Then (2.37) becomes $\bar{z}_j = \sum_{l=j}^{\infty} \sigma_{j,l}\theta_l \bar{g}_l^2$. By (2.7), $\sigma_{j,l} = O(1)$. It then follows from (A2), (2.50), and Lemma 2.1(ii,iii-b) that

$$\sigma_{j,l}' = \sigma_{j,l} \sum_{k=j}^{l}(1 - \zeta_k \bar{g}_k)^{-1} \zeta_k \bar{g}_k' = \sum_{k=j}^{l} O(\zeta_k \bar{g}_k') = O\left(\chi_j \frac{\bar{g}_j}{\bar{g}_0^2}\right). \tag{2.53}$$

We differentiate (2.37) and apply (2.50) and Lemma 2.1(ii) to obtain

$$\bar{z}_j' = \sum_{l=j}^{\infty} \sigma_{j,l}' \theta_l \bar{g}_l^2 + 2\sum_{l=j}^{\infty} \sigma_{j,l} \theta_l \bar{g}_l \bar{g}_l' = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^2}\right). \tag{2.54}$$

Similarly, $\sigma_{j,l}'' = O(\bar{g}_j / \bar{g}_0^3)$ and

$$\bar{z}_j'' = \sum_{l=j}^{\infty} \sigma_{j,l}'' \theta_l \bar{g}_l^2 + 4\sum_{l=j}^{\infty} \sigma_{j,l}' \theta_l \bar{g}_l \bar{g}_l' + 2\sum_{l=j}^{\infty} \sigma_{j,l} \theta_l (\bar{g}_l \bar{g}_l'' + \bar{g}_l'^2) = O\left(\chi_j \frac{\bar{g}_j^2}{\bar{g}_0^3}\right) \tag{2.55}$$

using the fact that $\bar{g}_j^3/\bar{g}_0^4 = O(\bar{g}_j^2/\bar{g}_0^3)$ by (2.3). It is straightforward to justify the differentiation under the sum in (2.54)–(2.55).

For $\bar{\mu}_j$, we recall from (2.42)–(2.43) that

$$\bar{\mu}_j = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \sigma_l, \tag{2.56}$$

with τ_j and σ_l given by (2.39), and with $0 \le (\lambda_j - \tau_j)^{-1} \le \alpha < 1$. This gives

$$\bar{\mu}_j' = -\sum_{l=j}^{\infty} \left(\prod_{k=j}^{l} (\lambda_k - \tau_k)^{-1} \right) \left(\sigma_l' + \sum_{i=j}^{l} (\lambda_i - \tau_i)^{-1} \tau_i' \right). \tag{2.57}$$

The first product is bounded by α^{l-j+1}, and this exponential decay, together with (2.39), (2.38), and the bounds just proved for $\bar{g}'$ and $\bar{z}'$, lead to the upper bound $|\bar{\mu}_j'| \le O(\chi_j \bar{g}_j^2 \bar{g}_0^{-2})$ claimed in (2.46). Straightforward further calculation leads to the bound on $\bar{\mu}_j''$ claimed in (2.47) (the leading behaviour can be seen from the $\bar{z}_j''$ contribution to the σ_l'' term). □

2.4. Proof of Proposition 1.2

Proof of Proposition 1.2. The estimates (1.9) follow from Lemma 2.1(ii) and Lemma 2.2. The continuity of $\bar{g}_j$, $\bar{z}_j$, and $\bar{\mu}_j$ in m follows from Lemma 2.2, and their differentiability in the initial condition $\bar{g}_0$ follows from Lemma 2.3. □

3. Proof of Main Result

In this section, we prove Theorem 1.4. We begin in Sect. 3.1 with a sketch of the main ideas, without entering into details.

3.1. Proof Strategy

Two difficulties in proving Theorem 1.4 are as follows: (i) from the point of view of dynamical systems, the evolution map Φ is not hyperbolic; and (ii) from the point of view of nonlinear differential equations, a priori bounds that any solution to (1.5) must satisfy are not readily available due to the presence of both initial and final boundary conditions.

Our strategy is to consider the one-parameter family of evolution maps $(\Phi^t)_{t\in[0,1]}$ defined by

$$\Phi^t(x) = \Phi(t,x) = (\psi(x), \bar{\varphi}(x) + t\rho(x)) \quad \text{for } t \in [0,1], \tag{3.1}$$

with the t-independent boundary conditions that (K_0, g_0) is given and that $(z_\infty, \mu_\infty) = (0,0)$. This family interpolates between the problem $\Phi^1 = \Phi$ we are interested in, and the simpler problem $\Phi^0 = \bar{\Phi} = (\psi, \bar{\varphi})$. The unique solution for $\bar{\Phi}$ is $\bar{x}_j = (\bar{K}_j, \bar{V}_j)$, where $\bar{V}$ is the unique solution of $\bar{\varphi}$ from Sect. 2, and where $\bar{K}_j$ is defined inductively for $j \ge 0$ (recall Lemma 1.3) by

$$\bar{K}_{j+1} = \psi_j(\bar{V}_j, \bar{K}_j), \quad \bar{K}_0 = K_0. \tag{3.2}$$

We refer to $\bar{x}$ as the *approximate flow.*

We seek a t-dependent global flow x which obeys the generalisation of (1.5) given by

$$x_{j+1} = \Phi_j^t(x_j). \tag{3.3}$$

Assuming that $x_j = x_j(t)$ is differentiable in t for each $j \in \mathbb{N}_0$, we set

$$\dot{x}_j = \frac{\partial}{\partial t} x_j. \tag{3.4}$$

Then differentiation of (3.3) shows that a family of flows $x = (x_j(t))_{j\in\mathbb{N}_0, t\in[0,1]}$ must satisfy the infinite nonlinear system of ODEs

$$\dot{x}_{j+1} - D_x\Phi_j(t, x_j)\dot{x}_j = \rho_j(x_j), \quad x_j(0) = \bar{x}_j. \tag{3.5}$$

Conversely, any solution $x(t)$ to (3.5), for which each x_j is continuously differentiable in t, gives a global flow for each Φ^t.

We claim that (3.5) can be reformulated as a well-posed nonlinear ODE

$$\dot{x} = F(t, x), \quad x(0) = \bar{x}, \tag{3.6}$$

in a Banach space of sequences $x = (x_0, x_1, \ldots)$ with carefully chosen weights, and for a suitable nonlinear functional F. For this, we consider the *linear* equation

$$y_{j+1} - D_x\Phi_j(t, x_j)y_j = r_j, \tag{3.7}$$

where the sequences x and r are held fixed. Its solution with the same boundary conditions as stated below (3.1) is written as $y = S(t, x)r$. Then we define F, which we consider as a map on sequences, by

$$F(t, x) = S(t, x)\rho(x). \tag{3.8}$$

Thus, $y = F(t, x)$ obeys the equation $y_{j+1} - D_x\Phi_j(t, x_j)y_j = \rho_j(x)$, and hence (3.6) is equivalent to (3.5) with the same boundary conditions.

The main work in the proof is to obtain good estimates for $S(t, x)$, in the Banach space of weighted sequences, which allow us to treat (3.6) by the standard theory of ODE. We establish bounds on the solution simultaneously with existence, via the weights in the norm. These weights are useful to obtain bounds on the solution, but they are also essential in the formulation of the problem as a well-posed ODE.

As we show in more detail in Sect. 4.1 below, the occurrence of $D_x\Phi_j(t, x_j)$ in (3.5), rather than the naive linearisation $D_x\Phi_j(0)$ at the fixed point $x = 0$, replaces the eigenvalue 1 in the upper left corner of the square matrix in (1.1) by the eigenvalue $1 - 2\beta_j g_j$, which is less than 1 except for those negligible j values for which $\beta_j < 0$. This helps address difficulty (i) mentioned above. Also, the weights guarantee that a solution in the Banach space obeys the final conditions $(z_\infty, \mu_\infty) = (0, 0)$, thereby helping to solve difficulty (ii).

3.2. Sequence Spaces and Weights

We now introduce the Banach spaces of sequences used in the reformulation of (3.5) as an ODE. These are weighted l^∞-spaces.

Definition 3.1. Let X^* be the space of sequences $x = (x_j)_{j\in\mathbb{N}_0}$ with $x_j \in X_j$. For each $\alpha = K, g, z, \mu$ and $j \in \mathbb{N}_0$, we fix a positive weight $w_{\alpha,j} > 0$. We write $x_j \in X_j = \mathcal{W}_j \oplus \mathcal{V}$ as $x_j = (x_{\alpha,j})_{\alpha=K,g,z,\mu}$. Let

$$\|x_j\|_{X_j^w} = \max_{\alpha=K,g,z,\mu} (w_{\alpha,j})^{-1}\|x_{\alpha,j}\|_{X_j}, \quad \|x\|_{X^w} = \sup_{j\in\mathbb{N}_0} \|x_j\|_{X_j^w}, \tag{3.9}$$

and

$$X^w = \{x \in X^* : \|x\|_{X^w} < \infty\}. \tag{3.10}$$

It is not difficult to check that X^w is a Banach space for any positive weight sequence w. The required weights are defined in terms of the sequence $\mathring{g} = (\mathring{g}_j)_{j\in\mathbb{N}_0}$ which is the same as the sequence $\bar{g}$ for a *fixed* $\mathring{g}_0$; i.e., given $\mathring{g}_0 > 0$, it satisfies $\mathring{g}_{j+1} = \mathring{g}_j - \beta_j \mathring{g}_j^2$. We need two different choices of weights w, defined in terms of the parameters a, h of (1.10) and the parameter a^* of Lemma 1.3. These are the weights $\mathsf{w} = \mathsf{w}(\mathring{g}_0, a, a^*, h)$ and $\mathsf{r} = \mathsf{r}(\mathring{g}_0, a, a^*, h)$ defined by

$$\mathsf{w}_{\alpha,j} = \begin{cases} (a-a_*)\chi_j \mathring{g}_j^3 & \alpha = K \\ h\mathring{g}_j^2 |\log \mathring{g}_j| & \alpha = g \\ h\chi_j \mathring{g}_j^2 |\log \mathring{g}_j| & \alpha = z, \mu, \end{cases} \qquad \mathsf{r}_{\alpha,j} = \begin{cases} (a-a_*)\chi_j \mathring{g}_j^3 & \alpha = K \\ h\chi_j \mathring{g}_j^3 & \alpha = g, z, \mu, \end{cases} \tag{3.11}$$

where (χ_j) is the Ω-dependent sequence defined by (1.8). Furthermore, let $\bar{x} = (\bar{K}, \bar{V}) = \bar{x}(K_0, g_0)$ denote the sequence in X^* that is uniquely determined from the boundary conditions $(\bar{K}_0, \bar{g}_0) = (K_0, g_0)$ and $(\bar{z}_\infty, \bar{\mu}_\infty) = (0, 0)$ via $\bar{V}_{j+1} = \bar{\varphi}_j(\bar{V}_j)$ and $\bar{K}_{j+1} = \psi_j(\bar{K}_j, \bar{V}_j)$, whenever the latter is well-defined. Given an initial condition $(\mathring{K}_0, \mathring{g}_0)$, let $\mathring{x} = \bar{x}(\mathring{K}_0, \mathring{g}_0)$.

Let $s\mathsf{B}$ denote the closed ball of radius s in X^{w}. If $\mathring{g}_0 = g_0$ and $\mathring{K}_0 = K_0$, the desired bounds (1.18)–(1.21) are equivalent to $x \in \mathring{x} + b\mathsf{B}$. Also, the projection of $\mathring{x} + \mathsf{B}$ onto the jth sequence element is contained in the domain D_j defined by (1.10). We always assume that $\mathring{g}_0$ is close to $g_0 = \bar{g}_0$, but not necessarily that they are equal. The use of $\mathring{g}$ rather than $\bar{g}$ permits us to vary the initial condition $g_0 = \bar{g}_0$ without changing the Banach spaces $X^{\mathsf{w}}, X^{\mathsf{r}}$. The use of g_0-dependent weights rather than, e.g., the weight $j^{-2}\log j$ for $j_\Omega = \infty$ [see Remark 1.6(i)] allows us to obtain estimates with good behaviour as $g_0 \to 0$. Note that the weight $\mathsf{w}_{g,j}$ does not include a factor χ_j, and thus does not go to 0 when $j_\Omega < \infty$ [see Example 1.1(ii)].

Remark 3.2. The weights w apply to the sequence $\dot{x}$ of (3.4). As motivation for their definition, consider the explicit example of $\rho_j(x_j) = \chi_j g_j^3$. In this case, the g equation becomes simply

$$g_{j+1} = g_j - \beta_j g_j^2 + t\chi_j g_j^3. \tag{3.12}$$

With the notation $\dot{g}_j = \frac{\partial}{\partial t} g_j^t$, differentiation gives

$$\dot{g}_{j+1} = \dot{g}_j(1 - 2\beta_j g_j + 3t\chi_j g_j^2) + \chi_j g_j^3. \tag{3.13}$$

Thus, by iteration, using $\dot g_0 = 0$, we obtain

$$\dot g_j = \sum_{l=0}^{j-1} \chi_l g_l^3 \prod_{k=l+1}^{j-1} (1 - 2\beta_k g_k + 3t\chi_k g_k^3). \tag{3.14}$$

For simplicity, consider the case $t = 0$, for which $g = \bar g$. In this case, it follows from (2.6), (2.3), and (2.4) that

$$\dot g_j \le O(1) \sum_{l=0}^{j-1} \left(\frac{\bar g_j}{\bar g_{l+1}}\right)^2 \chi_l \bar g_l^3 = O(1) g_j^2 \sum_{l=0}^{j-1} \chi_l \bar g_l \le O(1) \bar g_j^2 |\log \bar g_j|, \tag{3.15}$$

which produces the weight $\mathsf{w}_{g,j}$ of (3.11). (It can be verified using (2.12) that if we replace χ_j by β_j in the above then no smaller weight will work.)

3.3. Implications of Assumption A3

For ϕ equal to either of the maps ρ, ψ of (1.4), we define $\phi : \mathring{x} + \mathsf{B} \to X^*$ by

$$(\phi(x))_0 = 0, \quad (\phi(x))_{j+1} = \phi_j(x_j). \tag{3.16}$$

The next lemma expresses immediate consequences of Assumption A3 for ρ and ψ in terms of the weighted spaces.

Lemma 3.3. *Assume* (A3) *and that* $\mathring{g}_0 > 0$ *is sufficiently small. The map* ρ *obeys*

$$\|\rho(x)\|_{X^{\mathsf{r}}} \le M h^{-1}. \tag{3.17}$$

Let $\omega > \kappa\Omega$, *and let* ϕ *denote either* ψ *or* ρ. *The map* $\phi : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ *is twice continuously Fréchet differentiable, and there exists a constant* $C = C(a, a^*, h)$ *such that*

$$\begin{aligned} &\|D_K \rho(x)\|_{L(X^{\mathsf{w}}, X^{\mathsf{r}})} \le C, \quad \|D_K \psi(x)\|_{L(X^{\mathsf{w}}, X^{\mathsf{r}})} \le \omega, \\ &\|D_V \phi(x)\|_{L(X^{\mathsf{w}}, X^{\mathsf{r}})} \le O(\mathring{g}_0 |\log \mathring{g}_0|), \quad \|D_x^2 \phi(x)\|_{L^2(X^{\mathsf{w}}, X^{\mathsf{r}})} \le C. \end{aligned} \tag{3.18}$$

Proof. The bound (3.17) is equivalent to (1.11) [recall (3.16)] since

$$\|\rho_j(x_j)\|_{X^{\mathsf{r}}_{j+1}} = \mathsf{r}_{g,j+1}^{-1} \|\rho_j(x_j)\|_{\mathcal{V}} \le \mathsf{r}_{g,j+1}^{-1} M \chi_{j+1} \mathring{g}_{j+1}^3 = M/h. \tag{3.19}$$

Next, we verify the bounds on the first derivatives in (3.18). By assumptions (1.12)–(1.13), together with (2.3), the definition of the weights (3.11), and for (3.21) also the fact that $\chi_j/\chi_{j+1} \le \Omega$ by (1.8), we obtain for $x \in \mathring{x} + \mathsf{B}$,

$$\|D_V \psi_j(x_j)\|_{L(X^{\mathsf{w}}_j, X^{\mathsf{r}}_{j+1})} \le M \chi_{j+1} \mathring{g}_{j+1}^2 \mathsf{r}_{K,j+1}^{-1} \mathsf{w}_{g,j} \le O(\mathring{g}_0 |\log \mathring{g}_0|), \tag{3.20}$$

$$\|D_K \psi_j(x_j)\|_{L(X^{\mathsf{w}}_j, X^{\mathsf{r}}_{j+1})} \le \kappa \mathsf{r}_{K,j+1}^{-1} \mathsf{w}_{K,j} \le \kappa\Omega(1 + O(\mathring{g}_0)), \tag{3.21}$$

$$\|D_V \rho_j(x_j)\|_{L(X^{\mathsf{w}}_j, X^{\mathsf{r}}_{j+1})} \le M \chi_{j+1} \mathring{g}_{j+1}^2 \mathsf{r}_{g,j+1}^{-1} \mathsf{w}_{g,j} \le O(\mathring{g}_0 |\log \mathring{g}_0|), \tag{3.22}$$

$$\|D_K \rho_j(x_j)\|_{L(X^{\mathsf{w}}_j, X^{\mathsf{r}}_{j+1})} \le M \mathsf{r}_{g,j+1}^{-1} \mathsf{w}_{K,j} \le O(1), \tag{3.23}$$

which establishes the bounds on the first derivatives in (3.18), for $\mathring{g}_0$ sufficiently small.

The bounds on the second derivatives are also immediate consequences of Assumption A3. First, (1.14) and the definition of the weights (3.11) imply that, for $2 \le n + m \le 3$,

$$\|D_K^n D_V^m \phi\|_{L^{n+m}(X^{\mathsf{w}},X^{\mathsf{r}})} \le C. \tag{3.24}$$

In addition, these bounds on the second and third derivatives imply that

$$\|\phi(x+y) - \phi(x) - D\phi(x)y\|_{X^{\mathsf{r}}} \le C\|y\|_{X^{\mathsf{w}}}^2, \tag{3.25}$$

$$\|D\phi(x+y) - D\phi(x) - D^2\phi(x)y\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le C\|y\|_{X^{\mathsf{w}}}^2, \tag{3.26}$$

and hence that $\phi : \mathring{x} + \mathsf{B} \to X^{\mathsf{r}}$ is indeed twice Fréchet differentiable. The above bound on the third derivatives also implies continuity of this differentiability. This completes the proof. □

The smoothness of $\bar{x}$ is addressed in the following lemma.

Lemma 3.4. *Assume (A1–A3), and let $\delta > 0$ and $\mathring{g}_0 > 0$ both be sufficiently small. Then, there exists a neighbourhood $\bar{\mathsf{N}} = \bar{\mathsf{N}}_\delta \subset \mathcal{W}_0 \oplus \mathbb{R}_+$ of $(\mathring{K}_0, \mathring{g}_0)$ such that $\bar{x} : \bar{\mathsf{N}} \to \mathring{x} + \delta\mathsf{B}$ is continuously Fréchet differentiable, and*

$$\|D_{g_0}\bar{x}\|_{X^{\mathsf{w}}} \le O(\mathring{g}_0^{-2}|\log \mathring{g}_0|^{-1}). \tag{3.27}$$

The neighbourhood $\bar{\mathsf{N}}$ contains a ball centered at $(\mathring{K}_0, \mathring{g}_0)$ with radius depending only on $\mathring{g}_0, \delta$, and the constants in (A1–A3), which is bounded below away from 0, uniformly on compact subsets of small $\mathring{g}_0 > 0$.

Proof. Let

$$\bar{\mathsf{N}} = ([\tfrac{1}{2}\mathring{g}_0, 2\mathring{g}_0] \times \mathcal{W}_0) \cap \bar{x}^{-1}(\mathring{x} + \delta\mathsf{B}). \tag{3.28}$$

We will show that $\bar{\mathsf{N}}$ is a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$ and that $\bar{x} : \bar{\mathsf{N}} \to \mathring{x} + \delta\mathsf{B}$ is continuously Fréchet differentiable. Since $\bar{x}^{-1}(\mathring{x} + \delta\mathsf{B}) = \bar{V}^{-1}(\mathring{x} + \delta\mathsf{B}) \cap \bar{K}^{-1}(\mathring{x} + \delta\mathsf{B})$, it suffices to show that each of $\bar{V}^{-1}(\mathring{x} + \delta\mathsf{B})$ and $\bar{K}^{-1}(\mathring{x} + \delta\mathsf{B})$ is a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$, and that each of $\bar{V}$ and $\bar{K}$ is continuously Fréchet differentiable on $\bar{\mathsf{N}}$ as maps with values in subspaces of X^{w}.

We begin with $\bar{V}$. Let $\bar{V}_j'$ denote the derivative of $\bar{V}_j$ with respect to g_0, and let $\bar{V}' = (\bar{V}_j')$ denote the sequence of derivatives. It is straightforward to conclude from Lemma 2.3, Lemma 2.1(iv), and (3.11) that

$$\|\bar{V}'\|_{X^{\mathsf{w}}} \le O(\mathring{g}_0^{-2}|\log \mathring{g}_0|^{-1}). \tag{3.29}$$

In particular, this implies that $\bar{V}^{-1}(\mathring{x} + \delta\mathsf{B})$ contains a neighbourhood of $\mathring{g}_0$ satisfying the condition stated below (3.27). That $\bar{V}'$ is actually the derivative of $\bar{V}$ in the space X^{w} can be deduced from the fact that the sequence $\bar{V}''(g_0)$ is uniformly bounded in X^{w} for $g_0 \in \bar{\mathsf{N}}_g$ (though not uniform in $\mathring{g}_0$) by Lemma 2.3, using

$$\|\bar{V}_j(g_0 + \varepsilon) - \bar{V}_j(g_0) - \varepsilon\bar{V}_j'(g_0)\|_{X_j^{\mathsf{w}}} \le O(\varepsilon^2) \sup_{0<\varepsilon'<\varepsilon} \|\bar{V}_j''(g + \varepsilon')\|_{X_j^{\mathsf{w}}}. \tag{3.30}$$

The continuity of $\bar{V}'$ in X^{w} follows similarly.

For $\bar{K}$, we first note that $\|D_{K_0}\bar{K}_0\|_{L(\mathcal{W}_0,\mathcal{W}_0)} = 1$, $\|D_{g_0}\bar{K}_0\|_{\mathcal{W}_0} = 0$. By (A3) and induction,

$$\|D_{K_0}\bar{K}_{j+1}\|_{L(\mathcal{W}_0,\mathcal{W}_{j+1})} \le \kappa\|D_{K_0}\bar{K}_j\|_{L(\mathcal{W}_0,\mathcal{W}_j)} \le \kappa^{j+1}. \tag{3.31}$$

Since $\kappa < \Omega^{-1} < 1$, and since $\mathring{g}_{j+1}/\mathring{g}_j \to 1$ by (2.3), we obtain

$$\|D_{K_0}\bar{K}_{j+1}\|_{L(\mathcal{W}_0,\mathcal{W}_{j+1})} \le O(\mathring{g}_0^{-3}\mathsf{w}_{K,j+1}). \tag{3.32}$$

Similarly, by (1.13) and Lemma 2.3,

$$\begin{aligned}\|D_{g_0}\bar{K}_{j+1}\|_{\mathcal{W}_{j+1}} &\le \kappa\|D_{g_0}\bar{K}_j\|_{\mathcal{W}_j} + O(\chi_j\bar{g}_j^2)\|D_{g_0}\bar{V}_j\|_{\mathcal{V}}\\ &\le \kappa\|D_{g_0}\bar{K}_j\|_{\mathcal{W}_j} + O(\chi_j\bar{g}_j^4/\bar{g}_0^2).\end{aligned} \tag{3.33}$$

By induction, again using $\kappa < \Omega^{-1}$ and $\bar{g}_j \le \bar{g}_0$, we conclude

$$\|D_{g_0}\bar{K}_{j+1}\|_{\mathcal{W}_{j+1}} \le O(\chi_j\bar{g}_j^4/\bar{g}_0^2) \le O(\mathring{g}_0^{-1}\mathsf{w}_{K,j+1}). \tag{3.34}$$

These bounds imply that $\bar{K}^{-1}(\mathring{x} + \delta\mathsf{B})$ contains a neighbourhood of $(\mathring{K}_0, \mathring{g}_0)$ satisfying the condition stated below (3.27), and also that the component-wise derivatives of $\bar{K}$ with respect to g_0 and K_0 are respectively in $X^{\mathsf{w}} \cong L(\mathbb{R}, X^{\mathsf{w}})$ and $L(\mathcal{W}_0, X^{\mathsf{w}})$.

To verify that the component-wise derivative of the sequence $\bar{K}$ is the Fréchet derivative in the sequence space X^{w}, it again suffices to obtain bounds on the second derivatives in X^{w}, as in (3.30). For example, since $D^2_{K_0}\bar{K}_0 = 0$, $D_{K_0}\bar{V}_j = 0$, and

$$D^2_{K_0}\bar{K}_{j+1} = D_K\psi(\bar{K}_j,\bar{V}_j)D^2_{K_0}\bar{K}_j + D^2_K\psi(\bar{K}_j,\bar{V}_j)D_{K_0}\bar{K}_jD_{K_0}\bar{K}_j, \tag{3.35}$$

it follows from (3.31), (1.12)–(1.14), and induction that, for $(K_0, g_0) \in \bar{\mathsf{N}}$ with $\bar{\mathsf{N}} \subset \mathcal{W}_0 \oplus \mathbb{R}$ chosen sufficiently small, and with $\omega \in (\kappa\Omega, 1)$,

$$\|D^2_{K_0}\bar{K}_{j+1}\| \le \kappa\|D^2_{K_0}\bar{K}_j\| + O(\bar{g}_0^{-3}\kappa^j\omega^j) \le O(\mathring{g}_0^{-6}\mathsf{w}_{K,j+1}), \tag{3.36}$$

and thus that the component-wise derivative $D^2_{K_0}\bar{K}$ is bounded in the norm of $L^2(\mathcal{W}_0, X^{\mathsf{w}})$ for $(K_0, g_0) \in \bar{\mathsf{N}}$. Similarly, slightly more complicated recursion relations than (3.35) for $D^2_{g_0}\bar{K}_j$ and $D_{g_0}D_{K_0}\bar{K}_j$ show that the component-wise second derivative of $\bar{K}$ is bounded in $L^2(\mathcal{W}_0 \oplus \mathbb{R}, X^{\mathsf{w}})$ when $\bar{\mathsf{N}}$ is again chosen sufficiently small. This shows as in (3.30) that $\bar{K}$ is continuously Fréchet differentiable from $\bar{\mathsf{N}}$ to X^{w}.

We have thus shown that $\bar{x}$ is continuously Fréchet differentiable from a neighbourhood $\bar{\mathsf{N}}$ of $(\mathring{K}_0, \mathring{g}_0)$ to X^{w}, and (3.27) follows from (3.29) and (3.34). □

3.4. Reduction to a Linear Equation with Nonlinear Perturbation

For given sequences $x, r \in X^*$, we now consider the equation

$$y_{j+1} - D_x\Phi_j(t, x_j)y_j = r_j. \tag{3.37}$$

With x and r fixed, this is an inhomogeneous linear equation in y. Lemma 3.5 below, which lies at the heart of the proof of Theorem 1.4, obtains bounds on solutions to (3.37), including bounds on its x-dependence. The latter allows us to use the standard theory of ODE in Banach spaces to treat the original nonlinear equation, where x and r are both functionals of the solution y, as a perturbation of the linear equation.

In addition to the decomposition $X_j = \mathcal{W}_j \oplus \mathcal{V}$, with $x_j \in X_j$ written $x_j = (K_j, V_j)$, it is convenient to also use the decomposition $X_j = E_j \oplus F_j$ with $E_j = \mathcal{W}_j \oplus \mathbb{R}$ and $F_j = \mathbb{R} \oplus \mathbb{R}$, for which we write $x_j = (u_j, v_j)$ with $u_j = (K_j, g_j)$ and $v_j = (z_j, \mu_j)$. We denote by π_α the projection operator onto the α-component of the space in which it is applied, where α can be in any of $\{K, V\}$, $\{u, v\} = \{(K, g), (z, \mu)\}$, or $\{K, g, z, \mu\}$.

Recall that the spaces of sequences X^w are defined in Definition 3.1 and the specific weights w and r in (3.11). The following lemma is proved in Sect. 4.

Lemma 3.5. *Assume* (A1–A3). *Then there is a constant C_S, independent of the parameters a and h in* (1.10), *and a constant $C'_S = C'_S(a, h)$, such that if $\mathring{g}_0 > 0$ is sufficiently small, the following hold for all $t \in [0, 1]$, $x \in \mathring{x} + \mathsf{B}$.*

- (i) *For $r \in X^{\mathsf{r}}$, there exists a unique solution $y = S(t, x)r \in X^{\mathsf{w}}$ of* (3.37) *with boundary conditions $\pi_u y_0 = 0$, $\pi_v y_\infty = 0$.*
- (ii) *The linear solution operator $S(t, x)$ satisfies*

$$\|S(t, x)\|_{L(X^{\mathsf{r}}, X^{\mathsf{w}})} \leq C_S. \tag{3.38}$$

- (iii) *As a map $S : [0, 1] \times (\mathring{x} + \mathsf{B}) \to L(X^{\mathsf{w}}, X^{\mathsf{r}})$, the solution operator is continuously Fréchet differentiable and satisfies*

$$\|D_x S(t, x)\|_{L(X^{\mathsf{w}}, L(X^{\mathsf{r}}, X^{\mathsf{w}}))} \leq C'_S. \tag{3.39}$$

Lemma 3.5 is supplemented with the information about the perturbation ρ given by Lemma 3.3, and by the information about the initial condition $\bar{x}$ provided by Lemmas 2.2. (Note that the sequence $\bar{x}$ serves as initial condition, at $t = 0$, for the ODE (3.5), not as initial condition for the flow equation (1.4).)

Proof of Theorem 1.4(i). Let C_S be the constant of Lemma 3.5, fix $\delta > 0$ such that $b > 2\delta$ and $1 - b > 2\delta$, and define $h_* = C_S M/((b - 2\delta) \wedge (1 - b - 2\delta))$. As in the statement of the theorem, assume $h > h_*$ with this value of h_*. For $t \in [0, 1]$ and $x \in \mathring{x} + \mathsf{B}$, let

$$F(t, x) = S(t, x)\rho(x). \tag{3.40}$$

Let $(\mathring{K}_0, \mathring{g}_0) = (K'_0, g'_0)$. Lemmas 3.3 and 3.5 imply that if $\mathring{g}_0 > 0$ is sufficiently small then $F : [0, 1] \times (\mathring{x} + \mathsf{B}) \to X^{\mathsf{w}}$ is continuously Fréchet differentiable and

$$\|F(t, x)\|_{X^{\mathsf{w}}} \leq \|S(t, x)\|_{L(X^{\mathsf{r}}, X^{\mathsf{w}})} \|\rho(x)\|_{X^{\mathsf{r}}} \leq \frac{C_S M}{h} \leq (b - 2\delta) \wedge (1 - b - 2\delta). \tag{3.41}$$

Similarly, by the product rule, there exists C such that

$$\begin{aligned} \|D_x F(t, x)\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} &\leq \|[D_x S(t, x)]\rho(x)\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} \\ &\quad + \|S(t, x)[D_x \rho(x)]\|_{L(X^{\mathsf{w}}, X^{\mathsf{w}})} \leq C, \end{aligned} \tag{3.42}$$

and thus, in particular, F is Lipschitz continuous in $x \in \mathring{x} + \mathsf{B}$.

We can now apply the standard local existence theory for ODE in Banach spaces, as follows. For $y \in \mathsf{B}$, let

$$\mathring{F}(t, y) = F(t, \mathring{x} + y). \tag{3.43}$$

Let $X_0^{\mathsf{w}} = \{y \in X^{\mathsf{w}} : \pi_u y_0 = 0\}$ and $\mathsf{B}_0 = \mathsf{B} \cap X_0^{\mathsf{w}}$. Then Lemma 3.5(i) and (3.41) imply that $\mathring{F}(t,(b-2\delta)\mathsf{B}_0) \subseteq \mathring{F}(t,\mathsf{B}_0) \subseteq (b-2\delta)\mathsf{B}_0$. Let $\bar{\mathsf{N}}$ be the neighbourhood of u_0 defined by Lemma 3.4 with the same δ so that $\bar{x} : \bar{\mathsf{N}} \to \mathring{x} + \delta\mathsf{B}$. With (3.41)–(3.42), the local existence theory for ODEs on Banach spaces [1, Chapter 2, Lemma 1] implies that the initial value problem

$$\dot{y} = \mathring{F}(t, y), \quad y(0) = \bar{x}(u_0) - \mathring{x} \tag{3.44}$$

has a unique C^1-solution $y : [0,1] \to X_0^{\mathsf{w}}$ such that $y(t) \in (b - 2\delta + \delta)\mathsf{B}_0 = (b-\delta)\mathsf{B}_0$ for all $t \in [0,1]$. In particular, [1, Chapter 2, Lemma 1] implies that the length of the existence interval of the initial value problem (3.44) in $(b-\delta)\mathsf{B}$ is bounded from below by $(b-2\delta)/((b-2\delta) \wedge (1-b-2\delta)) \geq 1$ since $\|\mathring{F}(t,y)\| \leq (b-2\delta) \wedge (1-b-2\delta)$ when $\|y - y(0)\| \leq b - 2\delta$. It does not depend on the Lipschitz constant of $\mathring{F}$.

As discussed around (3.6), it follows that $x = \mathring{x} + y(1)$ is a solution to (1.5). By construction, $\pi_u x_0 = \pi_u \mathring{x}_0 + \pi_u y(0) = \mathring{u}_0 + (u_0 - \mathring{u}_0) = u_0$. Also, $\pi_v y_\infty(1) = 0$ because $y(1) \in X^{\mathsf{w}}$, and since $\pi_v \mathring{x}_\infty = \pi_v \bar{x}_\infty(u_0) = 0$, it is also true that $\pi_v x_\infty = 0$. Thus, x satisfies the required boundary conditions.

To prove the estimates (1.18)–(1.21) for $x(u_0)$ with $u_0 \in \mathsf{N} \subseteq \bar{\mathsf{N}}$, we apply $\|x(u_0) - \mathring{x}\|_{X^{\mathsf{w}}} \leq b - 2\delta$ and $\|\bar{x}(u_0) - \mathring{x}\|_{X^{\mathsf{w}}} \leq \delta$ to see that

$$\|K_j - \bar{K}_j\|_{\mathcal{W}_j} \leq \|K_j - \mathring{K}_j\|_{\mathcal{W}_j} + \|\mathring{K}_j - \bar{K}_j\|_{\mathcal{W}_j} \leq (b-\delta)(a - a_*)\mathring{g}_j^3, \tag{3.45}$$

and analogously that

$$|g_j - \bar{g}_j| \leq (b-\delta)h\mathring{g}_j^2|\log \mathring{g}_j^2|, \tag{3.46}$$

$$|z_j - \bar{z}_j| \leq (b-\delta)h\chi_j\mathring{g}_j^2|\log \mathring{g}_j^2|, \tag{3.47}$$

$$|\mu_j - \bar{\mu}_j| \leq (b-\delta)h\chi_j\mathring{g}_j^2|\log \mathring{g}_j^2|. \tag{3.48}$$

Since $b - \delta < b$, by assuming that $|\mathring{g}_0 - \bar{g}_0|$ is sufficiently small, i.e., shrinking $\bar{\mathsf{N}}$ to a smaller neighbourhood N if necessary, we obtain with (2.46) that

$$(b-\delta)\mathring{g}_j^2|\log \mathring{g}_j^2| \leq b\bar{g}_j^2|\log \bar{g}_j^2|. \tag{3.49}$$

The required shrinking is uniform on compact subsets of $g_0 > 0$. With the property of the neighbourhood $\bar{\mathsf{N}}$ stated below (3.27), this shows the assertion of Remark 1.5.

To prove uniqueness, suppose that x^* is a solution to (1.5) with boundary conditions $(K_0^*, g_0^*) = (K_0, g_0)$ and $(z_\infty^*, \mu_\infty^*) = (0,0)$ that satisfies (1.18)–(1.21) (with x replaced by x^*, and with $\bar{x}$ as before). Let $\mathring{x} = \bar{x}(K_0', g_0')$ as before. By assumption and an argument analogous to that given around (3.45)–(3.49), $x^* - \mathring{x} \in (b+2\delta)\mathsf{B}_0$. It follows that $F : [0,1] \times (x^* + (1-b-2\delta)\mathsf{B}_0) \to X^{\mathsf{w}}$ is Fréchet differentiable and $\|F(t,x)\|_{X^{\mathsf{w}}} \leq 1 - b - 2\delta$ for all $t \in [0,1]$ and for all $x \in x' + (1-b-2\delta)\mathsf{B}_0 \subset \mathring{x} + \mathsf{B}_0$, as discussed around (3.40)–(3.42). By considering the ODE backwards in time, which is equally well-posed, there is a unique solution $x^*(t)$ for $t \in [0,1]$ to $\dot{x}^* = F(t, x^*)$ with $x^*(1) = x^*$

and $x^*(t) \subset \mathring{x} + \mathsf{B}_0$. It follows that $x^*(0)$ is a flow of $\Phi^0 = \bar{\Phi}$ with the same boundary conditions as $\mathring{x}$. The uniqueness of such flows, by Lemma 2.2, implies that $x^*(0) = \mathring{x}$, and the uniqueness of solutions to the initial value problem (3.44) in $\mathring{x} + \mathsf{B}_0$ then also implies that $x = x^*$ as claimed. This completes the proof of Theorem 1.4(i). □

Proof of Theorem 1.4(ii). By Lemma 3.4, the map $\bar{x} : \mathsf{N} \subset \bar{\mathsf{N}} \to \mathring{x} + \delta\mathsf{B} \subset X^{\mathsf{w}}$ is continuously Fréchet differentiable. It therefore follows from [1, Chapter 2, Lemma 4] that the solution to the initial value problem (3.44) is continuously Fréchet differentiable in the initial condition. To denote the dependence of the solution on the latter, we write $y : [0, 1] \times \mathsf{N} \to X_0^{\mathsf{w}}$. Let $x(u_0) = \mathring{x} + y(1, u_0)$, as before.

By Proposition 1.2, $\bar{V}_j$ is continuously differentiable in g_0 for each $j \in \mathbb{N}_0$. Note also that $\bar{V}_j$ is independent of K_0. It can be concluded from the differentiability of $\bar{V}_j$ and from (A3) that $\bar{K}_j$ is continuously Fréchet differentiable in (K_0, g_0). Together with the continuous differentiability of y in the sequence space X^{w}, this implies that as elements of the spaces X_j, each $x_j = (K_j, V_j)$ is a C^1 function of u_0. To prove that the derivatives of z_0 and μ_0 with respect to g_0 are uniformly bounded, it suffices to verify this for the contributions to x due to y, by Lemma 2.3. To this end, observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}(Dy)(t) = D_x \mathring{F}(t, \bar{x} + y(t)) \circ Dy, \quad Dy(0) = \mathrm{id}. \tag{3.50}$$

Thus, by Lemma 3.5 and Gronwall's inequality [1, Chapter 2, Lemma 2],

$$\|D_{g_0} y(t, K_0, g_0)\|_{X^{\mathsf{w}}} \leq C \|D_{g_0} \bar{x}(K_0, g_0)\|_{X^{\mathsf{w}}}. \tag{3.51}$$

With Lemma 3.4, this gives

$$\|D_{g_0} y(t, K_0, g_0)\|_{X^{\mathsf{w}}} \leq O(\mathring{g}_0^{-2} |\log \mathring{g}_0|^{-1}). \tag{3.52}$$

Since $\partial \bar{z}_0 / \partial g_0 = O(1)$ and $\partial \bar{\mu}_0 / \partial g_0 = O(1)$ by Lemma 2.3, it follows from (3.52) and the definition of the weights (3.11) that

$$\frac{\partial z_0}{\partial g_0} = O(1), \quad \frac{\partial \mu_0}{\partial g_0} = O(1). \tag{3.53}$$

This completes the proof of Theorem 1.4(ii). □

4. Proof of Lemma 3.5

It now remains only to prove the key Lemma 3.5. The proof proceeds in three steps. The first two steps concern an approximate version of (3.37) and the solution of the approximate equation, and the third step treats (3.37) as a small perturbation of this approximation.

4.1. Step 1: Approximation of the Linear Equation

Define the map $\bar{\Phi}^0_j : X_j \to X_{j+1}$ by extending the map $\bar{\varphi}_j : \mathcal{V} \to \mathcal{V}$ trivially to the K-component, i.e., $\bar{\Phi}^0_j = (0, \bar{\varphi}_j)$ in the decomposition $X_{j+1} = \mathcal{W}_{j+1} \oplus \mathcal{V}$. Thus, $\Phi(t,x) = \bar{\Phi}^0(x) + (\psi(x), t\rho(x))$. Explicit computation of the derivative of $\bar{\varphi}_j$ of (1.4), using (1.1), shows that

$$D\bar{\Phi}^0_j(x_j) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1-2\beta_j g_j & 0 & 0 \\ \hline 0 & -\tilde{\xi}_j & 1-\zeta_j g_j & 0 \\ 0 & \tilde{\eta}_j & \tilde{\gamma}_j & \tilde{\lambda}_j \end{array}\right), \tag{4.1}$$

with

$$\begin{aligned} \tilde{\eta}_j &= \eta_j - 2\upsilon^{gg}_j g_j - \upsilon^{gz}_j z_j - \upsilon^{g\mu}_j \mu_j, \\ \tilde{\gamma}_j &= \gamma_j - \upsilon^{gz}_j g_j - 2\upsilon^{zz}_j z_j - \upsilon^{z\mu}_j \mu_j, \\ \tilde{\lambda}_j &= \lambda_j - \upsilon^{g\mu}_j g_j - \upsilon^{z\mu}_j z_j, \\ \tilde{\xi}_j &= 2\theta_j g_j + \zeta_j z_j. \end{aligned} \tag{4.2}$$

The block matrix structure in (4.1) is in the decomposition $X_j = E_j \oplus F_j$ introduced in Sect. 3.4. The matrix $D\bar{\Phi}^0_j(x_j)$ depends on $x_j \in X_j$, but it is convenient to approximate it by the constant matrix

$$L_j = D\bar{\Phi}^0_j(\mathring{x}_j) = \begin{pmatrix} A_j & 0 \\ B_j & C_j \end{pmatrix}, \tag{4.3}$$

where the 2×2 blocks A_j, B_j, and C_j of L_j are defined by

$$A_j = \begin{pmatrix} 0 & 0 \\ 0 & 1-2\beta_j \mathring{g}_j \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & -\mathring{\xi}_j \\ 0 & \mathring{\eta}_j \end{pmatrix}, \quad C_j = \begin{pmatrix} 1-\zeta_j \mathring{g}_j & 0 \\ \mathring{\gamma}_j & \mathring{\lambda}_j \end{pmatrix} \tag{4.4}$$

with $\mathring{\eta}_j$, $\mathring{\gamma}_j$, $\mathring{\lambda}_j$, and $\mathring{\xi}_j$ as in (4.2) with x replaced by $\mathring{x}$. Thus, we study the equation

$$y_{j+1} = L_j y_j + r_j, \tag{4.5}$$

which approximates (3.37). To analyze (4.5), and also for later purposes, we derive properties of the matrices A_j, B_j, C_j in the following lemma.

Lemma 4.1. *Assume* (A1–A2). *Let* $\alpha \in (\lambda^{-1}, 1)$. *Then for* $\mathring{g}_0 > 0$ *sufficiently small* (*depending on* α), *the following hold.*

(i) *Uniformly in all* $l \le j$,

$$A_j \cdots A_l = \begin{pmatrix} 0 & 0 \\ 0 & O(\mathring{g}^2_{j+1}/\mathring{g}^2_l) \end{pmatrix}. \tag{4.6}$$

(ii) *Uniformly in all* j,

$$B_j = \begin{pmatrix} 0 & O(\chi_j \mathring{g}_j) \\ 0 & O(\chi_j) \end{pmatrix}. \tag{4.7}$$

(iii) *Uniformly in all* $l \ge j$,

$$C_j^{-1} \cdots C_l^{-1} = \begin{pmatrix} O(1) & 0 \\ O(\chi_j) & O(\alpha^{l-j+1}) \end{pmatrix}. \tag{4.8}$$

Proof. (i) It follows immediately from (4.4) that

$$A_j \cdots A_l = \prod_{k=l}^{j} (1 - 2\beta_k \mathring{g}_k)\pi_g, \tag{4.9}$$

and thus (2.6) implies (i).

(ii) It follows directly from (4.4) and Lemma 2.2 that (4.7) holds.

(iii) Note that

$$\begin{pmatrix} c_1 & 0 \\ b_1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} c_n & 0 \\ b_n & a_n \end{pmatrix} = \begin{pmatrix} c^* & 0 \\ b^* & a^* \end{pmatrix} \tag{4.10}$$

with

$$a^* = a_1 \cdots a_n, \quad b^* = \sum_{i=1}^{n} a_1 \cdots a_{i-1} b_i c_{i+1} \cdots c_n, \quad c^* = c_1 \cdots c_n. \tag{4.11}$$

We apply this formula with the inverse matrices

$$C_j^{-1} = \begin{pmatrix} (1 - \zeta_j \mathring{g}_j)^{-1} & 0 \\ -(1 - \zeta_j \mathring{g}_j)^{-1} \mathring{\gamma}_j \mathring{\alpha}_j & \mathring{\alpha}_j \end{pmatrix} \tag{4.12}$$

where $\mathring{\alpha}_j = \mathring{\lambda}_j^{-1}$. Thus

$$C_j^{-1} \cdots C_l^{-1} = \begin{pmatrix} \mathring{\tau}_{j,l} & 0 \\ \mathring{\sigma}_{j,l} & \mathring{\alpha}_{j,l} \end{pmatrix} \tag{4.13}$$

with

$$\mathring{\alpha}_{j,l} = \mathring{\alpha}_j \cdots \mathring{\alpha}_l, \quad \mathring{\tau}_{j,l} = \prod_{k=j}^{l} (1 - \zeta_k \mathring{g}_k)^{-1}, \tag{4.14}$$

$$\mathring{\sigma}_{j,l} = \sum_{i=1}^{l-j+1} \left(\prod_{k=j+i}^{l} (1 - \zeta_k \mathring{g}_k)^{-1} \right) (-\mathring{\gamma}_{j+i-1}) \left(\prod_{k=j}^{j+i-2} \mathring{\alpha}_k \right). \tag{4.15}$$

The product defining $\mathring{\tau}_{j,l}$ is $O(1)$ by (2.7). Assume that $\mathring{g}_0$ is sufficiently small that, with Lemma 2.2 and (A2), $\mathring{\alpha}_m < \alpha$ for all m. Then $\mathring{\alpha}_{j,l} \leq O(\alpha^{l-j+1})$. Similarly, since $\mathring{\gamma}_m \leq O(\chi_m)$,

$$|\mathring{\sigma}_{j,l}| \leq \sum_{i=1}^{l-j+1} \alpha^i O(\chi_{j+i-1}) \leq O(\chi_j). \tag{4.16}$$

This completes the proof. □

The following lemma provides a solution to (4.5).

Lemma 4.2. *Assume (A1–A2) and that $\mathring{g}_0 > 0$ is sufficiently small. We write y as a column vector $y = (u, v)$. Then*

$$u_j = \sum_{l=0}^{j-1} A_{j-1} \cdots A_{l+1} \pi_u r_l \tag{4.17}$$

$$v_j = -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l) \tag{4.18}$$

is the unique solution to (4.5) *which obeys the boundary conditions* $u_0 = v_\infty = 0$ *and for which the series* (4.18) *converges.*

The lemma indeed solves (4.5): given r we first obtain u via (4.17) and then insert u into (4.18) to obtain v. The empty product in (4.17) is interpreted as the identity, so the term in the sum corresponding to $l = j-1$ is simply $\pi_u r_{j-1}$.

Proof of Lemma 4.2. The u-component of (4.5) is given by

$$u_{j+1} = A_j u_j + \pi_u r_j. \tag{4.19}$$

By induction, under the initial condition $u_0 = 0$ this recursion is equivalent to (4.17).

The v-component of (4.5) states that

$$v_{j+1} = B_j u_j + C_j v_j + \pi_v r_j, \tag{4.20}$$

which is equivalent to

$$v_j = C_j^{-1} v_{j+1} - C_j^{-1} B_j u_j - C_j^{-1} \pi_v r_j. \tag{4.21}$$

By induction, for any $k \geq j$, the latter is equivalent to

$$v_j = C_j^{-1} \cdots C_k^{-1} v_{k+1} - \sum_{l=j}^{k} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l). \tag{4.22}$$

By Lemma 4.1(iii), with some $\alpha \in (\lambda^{-1}, 1)$ and with $\mathring{g}_0$ sufficiently small, $\|C_0^{-1} \cdots C_k^{-1}\|$ is uniformly bounded. Thus, if $y_j = (u_j, v_j)$ satisfies (4.5) and $v_j \to 0$, then $C_0^{-1} \cdots C_k^{-1} v_{k+1} \to 0$ and hence

$$v_j = -\sum_{l=j}^{\infty} C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l), \tag{4.23}$$

which is (4.18). □

4.2. Step 2: Solution Operator for the Approximate Equation

We now prove existence, uniqueness, and bounds for the solution to the approximate equation (4.5).

Lemma 4.3. *Assume* (A1–A2) *and that* $\mathring{g}_0 > 0$ *is sufficiently small. For each* $r \in X^{\mathsf{r}}$ *and* $x \in \mathring{x} + \mathsf{B}$, *there exists a unique solution* $y = (u, v) = S^0 r \in X^{\mathsf{w}}$ *to* (4.5) *obeying the boundary conditions* $\pi_u y_0 = 0$, $\pi_v y_\infty = 0$. *The solution operator* S^0 *is block diagonal in the decomposition* $x = (K, V)$, *with*

$$S^0 = \begin{pmatrix} 1 & 0 \\ 0 & S^0_{VV} \end{pmatrix}. \tag{4.24}$$

There is a constant $C_{S^0} > 0$, *such that, uniformly in small* $\mathring{g}_0$,

$$\|S^0_{VV}\|_{L(X^{\mathsf{r}}, X^{\mathsf{w}})} \leq C_{S^0}. \tag{4.25}$$

The constant C_{S^0} *is independent of the parameters* a, h *which define the domain* D_j *in* (1.10).

Proof. By Lemma 4.2, it suffices to prove that the map $r \mapsto y$ defined by (4.17)–(4.18) gives a bounded map $S^0 : X^{\mathsf{r}} \to X^{\mathsf{w}}$. For this, we use Lemma 2.1 (ii), from which we recall that for all $k \geq j \geq 0$ and $m \geq 0$,

$$\sum_{l=j}^{k} \chi_l \mathring{g}_l^n |\log \mathring{g}_l|^m \leq C_{n,m} \begin{cases} |\log \mathring{g}_k|^{m+1} & n = 1 \\ \chi_j \mathring{g}_j^{n-1} |\log \mathring{g}_j|^m & n > 1. \end{cases} \tag{4.26}$$

(i) K-component. Since $\pi_K A_l = 0$, we have $\pi_K u_j = \pi_K r_{j-1}$. Therefore, by (3.11) and (2.3),

$$\|\pi_K y\|_{X^{\mathsf{w}}} \leq \sup_j \|\pi_K r_{j-1}\|_{X_j^{\mathsf{w}}} \leq \sup_j \left[\mathsf{w}_{K,j}^{-1} \mathsf{r}_{K,j-1}\right] \|r\|_{X^{\mathsf{r}}} \leq 2\|r\|_{X^{\mathsf{r}}}. \tag{4.27}$$

(ii) g-component. By Lemma 4.1(i), (3.11), (2.3), and (4.26),

$$\begin{aligned} \|\pi_g y\|_{X^{\mathsf{w}}} &\leq \sup_j \mathsf{w}_{g,j}^{-1} \sum_{l=0}^{j-1} |\pi_g A_{j-1} \cdots A_{l+1} \pi_u r_l| \\ &\leq \sup_j \mathsf{w}_{g,j}^{-1} \sum_{l=0}^{j-1} \mathsf{r}_{g,l} O(\mathring{g}_j / \mathring{g}_l)^2 \|r\|_{X^{\mathsf{r}}} \\ &\leq c\|r\|_{X^{\mathsf{r}}} \sup_j |\log \mathring{g}_j|^{-1} \sum_{l=0}^{j-1} \chi_l \mathring{g}_l \leq c\|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{4.28}$$

(iii) z-component. By (4.7)–(4.8), (4.28), (3.11), and (4.26),

$$\begin{aligned} \|\pi_z y\|_{X^{\mathsf{w}}} &\leq \sup_j \mathsf{w}_{z,j}^{-1} \sum_{l=j}^{\infty} |\pi_z C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l)| \\ &\leq \sup_j \mathsf{w}_{z,j}^{-1} \sum_{l=j}^{\infty} O(1) \left(\chi_l \mathring{g}_l \mathsf{w}_{g,l} \|r\|_{X^{\mathsf{r}}} + \chi_l \mathsf{r}_{z,l} \|r\|_{X^{\mathsf{r}}}\right) \leq c\|r\|_{X^{\mathsf{r}}}. \end{aligned} \tag{4.29}$$

(iv) μ-component. We begin with

$$\|\pi_\mu y\|_{X^{\mathsf{w}}} \leq \sup_j \mathsf{w}_{\mu,j}^{-1} \sum_{l=j}^{\infty} |\pi_\mu C_j^{-1} \cdots C_l^{-1} (B_l u_l + \pi_v r_l)|. \tag{4.30}$$

It is an exercise in matrix algebra, using (4.7)–(4.8) and (4.28), to see that

$$|\pi_\mu C_j^{-1} \cdots C_l^{-1} B_l u_l| \leq O(1) \left(\chi_l \mathring{g}_l + \alpha^{l-j+1}\right) \mathsf{w}_{g,l} \|r\|_{X^{\mathsf{r}}}, \tag{4.31}$$

$$|\pi_\mu C_j^{-1} \cdots C_l^{-1} \pi_v r_l| \leq O(1) \left(\chi_j \mathsf{r}_{z,l} + \alpha^{l-j+1} \mathsf{r}_{\mu,l}\right) \|r\|_{X^{\mathsf{r}}}. \tag{4.32}$$

Now, we use (3.11), (4.26), and also

$$\sum_{l=j}^{\infty} \alpha^{l+1-j} \chi_l \mathring{g}_l^n |\log \mathring{g}_l|^m \leq O(\chi_j \mathring{g}_j^n |\log \mathring{g}_l|^m), \tag{4.33}$$

to conclude that $\|\pi_\mu y\|_{X^{\mathsf{w}}} \leq c\|r\|_{X^{\mathsf{r}}}$. This completes the proof. □

4.3. Step 3: Solution of the Linear Equation

Now, we prove Lemma 3.5, which involves solving the Eq. (3.37) and estimating its solution operator. In preparation, we make some definitions and prove two preliminary lemmas.

We rewrite (3.37) as

$$y_{j+1} = D_x\Phi_j(t,x_j)y_j + r_j = L_j y_j + W_j(t,x_j)y_j + r_j, \tag{4.34}$$

where

$$W_j(t,x_j) = D_x\Phi_j(t,x_j) - L_j. \tag{4.35}$$

It is convenient to define an operator $W(t,x)$ on sequences via $(W(t,x))_0 = 0$ and $(W(t,x))_{j+1} = W_j(t,x)$. This operator can be written as a block matrix with respect to the decomposition $x = (K,V)$ as

$$W(t,x) = \begin{pmatrix} W_{KK} & W_{KV} \\ W_{VK} & W_{VV} \end{pmatrix}, \tag{4.36}$$

with $W_{\alpha\beta} = \pi_\alpha W(t,x)\pi_\beta$.

Lemma 4.4. *Fix* $\omega \in (\kappa\Omega, 1)$. *The map* W *obeys* $W : [0,1] \times (\mathring{x} + \mathsf{B}) \to L(X^{\mathsf{w}}, X^{\mathsf{r}})$, W *is continuously Fréchet differentiable, and if* $x \in \mathring{x} + \mathsf{B}$ *then*

$$\|W_{KK}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le \omega, \quad \|W_{VK}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le C, \tag{4.37}$$

$$\|W_{KV}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le o(1), \quad \|W_{VV}\|_{L(X^{\mathsf{w}},X^{\mathsf{r}})} \le o(1), \quad \textit{as } \mathring{g}_0 \to 0, \tag{4.38}$$

$$\|D_x W_j(t,x_j)\|_{L(X_j^{\mathsf{w}},L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}}))} \le C. \tag{4.39}$$

Proof. By definition,

$$W_j(t,x_j) = [D_x\bar\Phi_j^0(x_j) - D_x\bar\Phi_j^0(\mathring{x})] + D_x(\psi_j(x_j), t\rho_j(x_j)). \tag{4.40}$$

The first term on the right-hand side of (4.40) only depends on the V-components, and is continuously Fréchet differentiable since, by (4.1), $D^2\bar\Phi_j^0$ is a constant matrix for each j with coefficients bounded by $O(\chi_j)$. Therefore, for $x \in \mathring{x} + \mathsf{B}$ (with weights chosen maximally),

$$\begin{aligned}\|[D\bar\Phi_j^0(\mathring{x}_j) - D\bar\Phi_j^0(x_j)]\pi_V\|_{L(X_j^{\mathsf{w}},X_{j+1}^{\mathsf{r}})} &\le c\chi_j \mathsf{r}_{g,j+1}^{-1}\mathsf{w}_{g,j}^2\|\mathring{x}_j - x_j\|_{X_j^{\mathsf{w}}} \\ &= O(h\mathring{g}_0|\log\mathring{g}_0|^2).\end{aligned} \tag{4.41}$$

This contributes to the bounds (4.38), with $\mathring{g}_0$ taken small enough.

Lemma 3.3 gives bounds on the second term on the right-hand side of (4.40), as well as its derivative, and with these the proof of (4.39) is complete. □

Lemma 4.5. *For* $x \in \mathring{x} + \mathsf{B}$, *the map* $1 - S^0W(t,x)$ *has a bounded inverse in* $L(X^{\mathsf{w}}, X^{\mathsf{w}})$.

Proof. As in (4.24), we write S^0 as a block matrix with respect to the decomposition $x = (K,V)$ as

$$S^0 = \begin{pmatrix} 1 & 0 \\ 0 & S_{VV}^0 \end{pmatrix}. \tag{4.42}$$

By definition, $1 - S^0W(t,x) = A - B$ with

$$A = \begin{pmatrix} 1 - W_{KK} & 0 \\ -S^0_{VV}W_{VK} & 1 - S^0_{VV}W_{VV} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & W_{KV} \\ 0 & 0 \end{pmatrix}. \tag{4.43}$$

For any bounded operators A, B on a Banach space, with A^{-1} bounded and $\|A^{-1}B\| < 1$, the operator $A - B = A(1 - A^{-1}B)$ has a bounded inverse since $(1 - A^{-1}B)^{-1}$ is given by its Neumann series. Thus, it suffices to prove that the matrices A, B defined in (4.43) have these two properties.

By (4.37)–(4.38) (with $\mathring{g}_0$ sufficiently small), $\|W_{KK}\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} < 1$ and $\|S^0_{VV}W_{VV}\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} < 1$. Thus A is a block matrix of the form

$$A = \begin{pmatrix} A_{KK} & 0 \\ A_{VK} & A_{VV} \end{pmatrix}, \tag{4.44}$$

where A_{KK} and A_{VV} have inverses in $L(X^{\mathsf{w}}, X^{\mathsf{w}})$. It follows that A has the bounded inverse on X^{w} given by the block matrix

$$A^{-1} = \begin{pmatrix} A^{-1}_{KK} & 0 \\ A^{-1}_{VV}A_{VK}A^{-1}_{KK} & A^{-1}_{VV} \end{pmatrix}. \tag{4.45}$$

By (4.37)–(4.38) (with $\mathring{g}_0$ sufficiently small), $\|A^{-1}B\|_{L(X^{\mathsf{w}},X^{\mathsf{w}})} < 1$, and the proof is complete. □

Proof of Lemma 3.5. (i) By the assumption that $y \in X^{\mathsf{w}}$, Lemma 4.3, and (4.38), the equation (4.34) with the boundary conditions of Lemma 3.5(i) is equivalent to

$$y = S^0W(t,x)y + S^0r. \tag{4.46}$$

It follows that the solution operator is given by

$$S(t,x) = (1 - S^0W(t,x))^{-1}S^0, \tag{4.47}$$

with the existence of the inverse operator guaranteed by Lemma 4.4.

(ii) This follows from (4.47) and Lemmas 4.3 and 4.5.

(iii) By (4.47), continuous Fréchet differentiability in x of $S(t,x)$ follows from the continuous Fréchet differentiability of $S^0W(t,x)$, which itself follows from part (i) and from $D_xS^0W(t,x) = S^0D_xW(t,x)$ by linearity of S^0. Explicitly,

$$D_xS(t,x) = (1 - S^0W(t,x))^{-1}D_xS^0W(t,x)(1 - S^0W(t,x))^{-1}S^0. \tag{4.48}$$

By (4.39),

$$\|D_xS^0W(t,x)\|_{L(X^{\mathsf{w}},L(X^{\mathsf{w}},X^{\mathsf{w}}))} \le C\|D_xW(t,x)]\|_{L(X^{\mathsf{w}},L(X^{\mathsf{w}},X^{\mathsf{r}}))} \le C. \tag{4.49}$$

Together with the boundedness of the operators $(1 - S^0W(t,x))^{-1}$ and S^0, this proves (3.39) and completes the proof. □

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