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Resonant Delocalization on the Bethe Strip

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Abstract. Recently, Aizenman and Warzel discovered a mechanism for the appearance of absolutely continuous spectrum for random Schrödinger operators on the Bethe lattice through rare resonances (resonant delocalization). We extend their analysis to operators with matrix-valued random potentials drawn from ensembles such as the Gaussian Orthogonal Ensemble. These operators can be viewed as random operators on the Bethe strip, a graph (lattice) with loops.

1. Introduction

Let $\mathcal{T}$ be a regular rooted tree with branching number $K > 1$ (Bethe lattice). We shall be interested in random Schrödinger operators on the Cartesian product $\mathcal{T} \times G$ of $\mathcal{T}$ and a finite graph G with W vertices (Bethe strip). Equivalently, these can be seen as random Schrödinger operators on $\mathcal{T}$ with matrix-valued potential. The precise definition is as follows: $H = H_{\lambda,\omega}$ is a random operator acting on

$$\ell^2(\mathcal{T} \times G) = \ell^2(\mathcal{T} \to \mathbb{R}^W),$$

and given by the matrix elements

$$H_{\lambda,\omega}(x, y) = \begin{cases} \mathbb{1}_{W\times W}, & x \sim y \ (x \text{ is adjacent to } y) \\ A + \lambda V_\omega(x). & x = y \\ 0, & \text{otherwise} \end{cases}, \quad x, y \in \mathcal{T}. \tag{1}$$

Here $\lambda \geq 0$ is a coupling constant, ω denotes an element of the probability space, A is a fixed $W \times W$ Hermitian matrix, and $V_\omega(x)$ are independent identically distributed $W \times W$ random matrices. The potential $A + \lambda V_\omega(x)$ will be denoted $U_\omega(x)$.

The question that we shall address is, what is the spectral type of H when λ is small. Before stating our results, let us review what was previously known.

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For the Bethe lattice ($W = 1, A = 0$ in our notation), the spectrum of the unperturbed operator ($\lambda = 0$) is purely absolutely continuous and fills the interval $[-2\sqrt{K}, 2\sqrt{K}]$. Under mild assumptions on the potential, Klein [9–11] showed that, for small $\lambda > 0$, the spectrum in $[-2\sqrt{K} + \epsilon, 2\sqrt{K} - \epsilon]$ is also (almost surely) absolutely continuous. Additional proofs and generalizations of this result were found by Aizenman et al. [3] and by Froese et al. [7].

On the other hand, Aizenman [1] proved that, for small λ, the spectrum of H outside $[-K - 1 - \epsilon, K + 1 + \epsilon]$ is almost surely pure point.

In the recent work, Aizenman and Warzel [4] proved the presence of absolutely continuous spectrum thorough out the interval $[-K-1+\epsilon, K+1-\epsilon]$. They found a new mechanism for the appearance of absolutely continuous spectrum, entirely different from the one appearing inside the spectrum of the unperturbed operator, and coined the term "resonant delocalization" for it. As opposed to the absolutely continuous spectrum in the interval $[-2\sqrt{K}, 2\sqrt{K}]$, which appears due to the stability of the absolutely continuous spectrum on the Bethe lattice, the absolutely continuous spectrum in $[-K - 1, K + 1] \setminus [-2\sqrt{K}, 2\sqrt{K}]$ (in the Lifshitz tails) appears due to resonances between distant sites. The interval $[K-1, K+1]$ is exactly the ℓ^1 spectrum of the unperturbed operator; the importance of the ℓ^1 spectrum is further discussed in [4] and in the survey by Warzel [15].

The goal of this present work was to extend the result of [4] to the case $W > 1$ of the Bethe strip. We make use of significant parts of the work [4]; for the reader's convenience, we denote by Statement X* the generalization of [4, Statement X].

Denote by $\{\nu_i\}_{i=1}^{W}$ the eigenvalues of A, and let

$$S_\epsilon = \bigcup_i \left[\nu_i - (K + 1) + \epsilon, \nu_i + (K + 1) - \epsilon\right].$$

Our main result is

Theorem 1.1 (Corollary 2.3*). *Assume that $V_\omega(x)$ are drawn from the Gaussian Orthogonal Ensemble (GOE). For any $\epsilon > 0$ any open interval $I \subset S_\epsilon$ almost surely has absolutely continuous spectrum of $H_{\lambda,\omega}$ in it, when $\lambda > 0$ is sufficiently small.*

Thus the mechanism of resonant delocalization from [4] may be extended to the Bethe strip, a lattice with loops. See [15, Sect. 4] for a more general discussion of possible further extensions.

Theorem 1.1 should also be compared with the result of Klein and Sadel [12] (and its ramification [13]), who proved, under weaker assumptions on the potential V_ω, that the spectrum of $H_{\lambda,\omega}$ in

$$S_\epsilon^- = \bigcap_i \left[\nu_i - 2\sqrt{K} + \epsilon, \nu_i + 2\sqrt{K} - \epsilon\right]$$

is almost surely purely absolutely continuous; the special case $K = W = 2$ was earlier considered by Froese et al. [6]. Thus we replace the intersection with union (i.e. the fastest Lyapunov exponent with the slowest one) and $2\sqrt{K}$

with $K+1$ (i.e. the ℓ^2 spectrum with the ℓ^1 spectrum) at the price of more restrictive assumptions on V_ω, and we only manage to show the existence of absolutely continuous spectrum rather than its purity. The spectrum outside the set $S_{-\epsilon}$ is pure point, as follows from the results of [1]. Thus our result provides an additional example of the appearance of absolutely continuous spectrum in the ℓ^1 spectrum of the unperturbed operator $H_{0,\omega}$, well outside the ℓ^2 spectrum.

Theorem 1.1 will follow from Theorems 1.2 and 1.3 below. Theorem 1.3 connects the presence of absolutely continuous spectrum with the (slowest) Lyapunov exponent $L = L_\lambda(E) \in \mathbb{R}_+$, which is defined in the sequel. Theorem 1.2, which holds for any (independent identically distributed) random potential U_ω with $\mathbb{E}\log^+\|U_\omega(x)\| < \infty$, guarantees that the assumptions of Theorem 1.2 are satisfied for small λ.

Theorem 1.2. *For every $\epsilon > 0$ and any interval $I \subset S_\epsilon$ one has*

$$\operatorname{mes}\{E \in I \mid L(E) < \log K\} > 0$$

for sufficiently small λ.

It is probably true that for $\lambda < \lambda_0(\epsilon)$ one has $L|_{S_\epsilon} < \log K$; this is, however, unsettled even for $W = 1$ (except for the special case of Cauchy disorder, see [4]).

In the next two theorems, we assume that $V_\omega(x)$ are drawn from the GOE. We shall comment on possible generalizations in the sequel.

Theorem 1.3 (Theorem 2.1*). *The absolutely continuous spectrum of H fills (almost surely) the set $\{E \mid L(E) < \log K\}$, meaning that the restriction of the Lebesgue measure to this set is almost surely absolutely continuous with respect to the absolutely continuous part of the spectral measure of H. In particular, this set is a subset of the absolutely continuous spectrum of H.*

Similarly to the results of [4], Theorem 1.3 is sharp in the following sense: the spectrum of $H_{\lambda,\omega}$ in $\{E \mid L_\lambda(E) > \log K\}$ is almost surely pure point, as follows from the results of [1].

For expositional reasons, we first prove

Theorem 1.4. *H has (almost surely) no pure point spectrum in the set*

$$\{E \mid L(E) < \log K\}.$$

and then the stronger Theorem 1.3.

Finally, let us comment on the generality of the results. The simplest generalization of the Bethe strip setting of [4] is the GOE potential, corresponding to $A = 0$ (and small $\lambda > 0$). In this case, only minor modifications (due to the non-commutativity of matrix product) would be required in the arguments of [4], since the Lyapunov exponents differ from one another by a quantity which vanishes in the limit $\lambda \to 0$ (at least, in the sense of Theorem 1.2).

When $A \neq 0$, additional difficulties arise, which are due to the fact that there may be a significant difference between the fastest and the slowest Lyapunov exponent. Most of the current paper is devoted to overcoming these difficulties. We state the results for the case when $V_\omega(x)$ are drawn from the GOE, but the arguments may be extended to more general potentials with off-diagonal disorder. The crucial requirement is the conditional a.c. property, stating that the conditional distribution of $V_\omega(x)_{i_0,j_0}$ given $\{V_\omega(x)_{ij} \mid (i,j) \neq (i_0,j_0),(j_0,i_0)\}$ is absolutely continuous. We try to indicate where the off-diagonal disorder assumption is used in the proof.

It would be interesting to extend the results of this paper to the case of diagonal disorder: for example, $V_\omega(x)$ is a diagonal matrix with independent identically distributed entries (which would correspond to the usual Bethe strip).

2. Preliminaries and Proof of Theorem 1.2

For

$$z \in \mathbb{C}^+ = \{z \in \mathbb{C} \mid \Im z > 0\},$$

the Green function $G_\lambda(x,y;z)$ is the xy block of the resolvent $(H_\lambda - z)^{-1}$ (from this point we suppress the dependence on ω). For a vertex u of $\mathcal{T}$, $G_\lambda^{\mathcal{T}_u}(x,y,z)$ is the xy block of the Green function associated with the restriction of H_λ to the subgraph $\mathcal{T}_u$ obtained by removing u from $\mathcal{T}$. $\mathcal{N}_u^+$ is the collection of forward neighbors of a vertex u, and $\mathcal{N}_u$ is the collection of all neighbors of u. The root of $\mathcal{T}$ is denoted 0.

Claim 2.1 (Proposition 3.1*). For any matrix-valued Schrödinger operator H on $\mathcal{T}$ with potential U, and any $z \in \mathbb{C}^+$,

$$G_\lambda(x,x;z) = \left(U(x) - z - \sum_{y \in \mathcal{N}_x} G^{\mathcal{T}_x}(y,y;z) \right)^{-1},$$

and for any ordered pair $0 \prec x \prec y$

$$\begin{aligned} G_\lambda(x,y;z) &= G_\lambda(x,x;z) G_\lambda^{\mathcal{T}_x}(x_1,y;z) = G_\lambda^{\mathcal{T}_y}(x,x_n;z) G_\lambda(y,y;z) \\ &= G_\lambda(x,x;z) G_\lambda^{\mathcal{T}_x}(x_1,x_1;z) \cdots G_\lambda^{\mathcal{T}_{x_n}}(y,y;z), \end{aligned}$$

where $xx_1x_2\cdots x_ny$ is the path from x to y.

Proof. To prove the first statement, decompose

$$\ell^2(\mathcal{T} \to \mathbb{R}^W) = \ell^2(\{x\} \to \mathbb{R}^W) \oplus \ell^2(\mathcal{T}_x \to \mathbb{R}^W),$$

and apply the Schur–Banachiewicz formula for block matrix inversion. To prove the second statement, we iterate the formula

$$G_\lambda(x,y;z) = G_\lambda(x,x;z) G_\lambda^{\mathcal{T}_x}(x_1,y;z)$$

which follows from the resolvent identity. ☐

Let $0x_1x_2x_3\cdots x_n\cdots$ be a branch of $\mathcal{T}$. Denote

$$L(z) = -\lim_{n\to\infty} \frac{1}{n+1} \ln \|G_\lambda(0,x_n;z)\|,$$

where $\|\cdot\|$ stands for the operator norm. This is the slowest Lyapunov exponent.

Claim 2.2. The Lyapunov exponent $L(z)$ is defined and non-random for any independent identically distributed matrix potential $U(x)$ which satisfies

$$\mathbb{E}\log^+ \|U(x)\| < \infty.$$

The claim follows from the Furstenberg–Kesten Theorem [8]. For $U = A + \lambda V$, we denote the Lyapunov exponent by L_λ when we need to emphasize the dependence on λ. For $E \in \mathbb{R}$, we set

$$L_\lambda(E) = \lim_{\eta\to+0} L_\lambda(E + i\eta).$$

Claim 2.3. For any matrix potential $U = A + \lambda V$, where A is fixed and $V(x)$ are independent and identically distributed with $\mathbb{E}\log^+ \|V(x)\| < \infty$, and for any $z \in \mathbb{C}^+$,

$$L_\lambda(z) \to L_0(z) \quad as \quad \lambda \to 0.$$

Claim 2.3 follows from the strong resolvent convergence outside the spectrum. From Claim 2.3 and the Fatou Lemma, we obtain

Claim 2.4 (Theorem 6.1*). For any matrix potential $U = A + \lambda V$, where A is fixed and $V(x)$ are independent and identically distributed, and for any bounded interval $I \subset \mathbb{R}$, the function

$$\lambda \mapsto \int_I L_\lambda(E)\mathrm{d}E$$

is continuous, and, in particular,

$$\lim_{\lambda\to 0}\int_I L_\lambda(E)\mathrm{d}E = \int_I L_0(E)\mathrm{d}E.$$

The argument justifying Claims 2.3 and 2.4 is identical to that of [4, Sect. 6.1]. Theorem 1.2 is a consequence of Claim 2.4 and the explicit computation of the free Lyapunov exponent L_0, which can be performed using Claim 2.1 and which shows that

$$L_0(E) < \log K \iff E \in S_0 \equiv \bigcup_i (\nu_i - (K+1), \nu_i + (K+1)).$$

3. Proof of Theorem 1.4

The proof of Theorem 1.4 makes use of the following version of the Simon–Wolff [14] criterion:

Proposition 3.1 (Matrix Simon–Wolff criterion). *Suppose an i.i.d. matrix potential $U(x)$ satisfies the following two properties:*

1. *$U(x)$ has independent entries on the diagonal,*
2. *$U(x)$ is irreducible, meaning that it has no non-trivial deterministic invariant subspace.*

Then the pure point part of the spectral measure is almost surely supported on the set

$$\Sigma = \left\{ E \in \sigma(H) \ \Big| \ \sum_{x \in \mathcal{T}} \|G(0,x;E+i0)\|^2 < \infty \quad \textit{almost surely} \right\},$$

and the continuous part is almost surely supported on its complement.

Proof. By the usual Simon–Wolff [14] criterion, the continuous spectrum is almost surely supported on the set

$$S_j = \left\{ \sum_x \sum_i |G(0,x;E+i0)_{j,i}|^2 = \infty \right\},$$

and the pure point spectrum is almost surely supported on its complement. By assumption 2, the set S_j is (almost surely) independent of j. Therefore, it coincides with

$$\left\{ \sum_x \sum_{ij} |G(0,x;E+i0)_{j,i}|^2 = \infty \right\},$$

and the latter coincides with

$$\left\{ \sum_x \|G(0,x;E+i0)\|^2 = \infty \right\}$$

due to equivalence between norms. □

Now, Claim 2.1 yields

$$\|G(0,x;z)\| = \|G(x,x;z)^* G^{\mathcal{T}_x}(0,x_-;z)^*\| \geq \|G(x,x;z)^* G^{\mathcal{T}_x}(0,x_-;z)^* w\|$$

for any unit vector w (from this point we suppress the dependence on λ, and x_- stands for the backward neighbor of a vertex x). Let $v = G^{\mathcal{T}_x}(0,x_-;z)^* w$ and $\tilde{v} = v/\|v\|$. Then

$$\begin{aligned} \|G(0,x;z)\| &\geq \|G(x,x;z)^* v\| \\ &\geq |\langle G(x,x;z)^* v, \tilde{v}\rangle| = \|v\| \, |\langle G(x,x;z)\tilde{v}, \tilde{v}\rangle|. \end{aligned} \tag{2}$$

Let

$$w = w_{\max}(G^{\mathcal{T}_x}(0,x_-;z) G^{\mathcal{T}_x}(0,x_-;z)^*)$$

be the unit eigenvector of $G^{\mathcal{T}_x}(0,x_-;z)G^{\mathcal{T}_x}(0,x_-;z)^*$ associated with the largest eigenvalue; then $\tilde{v} = w_{\max}(G^{\mathcal{T}_x}(0,x_-;z)^* G^{\mathcal{T}_x}(0,x_-;z))$. Denote

$$\begin{aligned} E_x &= \left\{ |\langle G(x,x;E+i\eta)\tilde{v}, \tilde{v}\rangle| \geq \tau \equiv \mathrm{e}^{+(L(E)+2\delta)n} \right\}, \\ R_x &= \left\{ \|G^{\mathcal{T}_x}(0,x_-;E+i\eta)\| \geq \mathrm{e}^{-(L(E)+\delta)n} \right\}, \end{aligned}$$

and

$$N = \sum_{x \in S_n} \mathbb{1}_{R_x \cap E_x},$$

where $S_n = \mathcal{N}_+^n(0)$ is the sphere of radius n about the root. According to (2),

$$\|G(0,x;E+i\eta)\| \geq \mathrm{e}^{\delta n} \quad \text{on} \quad R_x \cap E_x.$$

Proposition 3.2 (First moment bound). *For $U(x) = A + \lambda V(x)$, where $V(x)$ are drawn from the GOE,*

$$\mathbb{E}N \geq \frac{1}{C(\lambda)\tau} K^n$$

when n is large enough and $\eta > 0$ is small enough.

Proof. By continuity in $\eta \to +0$ which holds for almost every energy (cf. [4, Corollary 4.10]), it is sufficient to prove the statement for $E + i0$.

Denote by P the projection on

$$\tilde{v} = w_{\max}(G^{\mathcal{T}_x}(0,x_-;E+i\eta)^* G^{\mathcal{T}_x}(0,x_-;E+i\eta));$$

$\tilde{v}$ is independent of $V(x)$. Also set $Q = \mathbb{1} - P$. By Claim 2.1,

$$\begin{aligned}&\langle G(x,x;E+i\eta)\tilde{v},\tilde{v}\rangle\\ &\quad = P\left(A + \lambda V(x) - E - i\eta - \sum_{y\in N_x} G^{\mathcal{T}_x}(y,y;E+i\eta)\right)^{-1} P. \qquad (3)\end{aligned}$$

By the Schur–Banachiewicz formula

$$PT^{-1}P = (PTP - PTQ(QTQ)^{-1}QTP)^{-1},$$

we have

$$\langle G(x,x;E+i\eta)\tilde{v},\tilde{v}\rangle = (g-\sigma)^{-1},$$

where $g = \lambda PV(x)P$ is Gaussian, and

$$\begin{aligned}\sigma = {} & -PAP + z + \sum_{y\in N_x} PG^{\mathcal{T}_x}(y,y;E+i\eta)P\\ & + \left(PU(x)Q - \sum_{y\in N_x} PG^{\mathcal{T}_x}(y,y;E+i\eta)Q\right)\\ & \left(QU(x)Q - z - \sum_{y\in N_x} QG^{\mathcal{T}_x}(y,y;E+i\eta)Q\right)^{-1}\\ & \left(QU(x)P - \sum_{y\in N_x} QG^{\mathcal{T}_x}(y,y;E+i\eta)P\right). \qquad (4)\end{aligned}$$

Lemma 3.3. *The random variable σ is independent of g.*

Proof. (Uses off-diagonal randomness) This fact is an immediate corollary of the following property of the GOE: for every orthogonal projection P, $PV(x)P$ is independent of

$$\{(1-P)V(x)P, PV(x)(1-P), (1-P)V(x)(1-P)\}.$$

□

Lemma 3.4. *There exists* $0 < s < 1$ *so that*

$$\mathbb{E}|\sigma|^s \leq C,$$

where $C > 0$ *is a constant.*

Proof. We bound the s-moment of every term in (4). The bound on

$$\mathbb{E}\left|\sum_{y \in N_x} PG^{\mathfrak{T}_x}(y, y; E + i\eta)P\right|^s$$

follows from [4, A.1]. It, therefore, remains to bound the s-moment of the multipliers in (4) (then the $s/3$-moment of the product is bounded by Cauchy–Schwarz). The expressions

$$\mathbb{E}\|PV(x)Q\|^s, \quad \mathbb{E}\|QV(x)P\|^s$$

are estimated directly (they are finite e.g. for $s = 2$); the s-moment of the second multiplier in (4) can be bounded using an argument similar to the upper bound in Lemma 3.5 below. □

Having the two lemmata, we can conclude the proof of Proposition 3.2. By Chebyshev's inequality and Lemma 3.4,

$$\mathbb{P}\left\{|\sigma| \leq t\right\} \geq 1 - C'/t^s \tag{5}$$

can be made arbitrarily close to 1 by choosing t large enough. Now we estimate $\mathbb{E}N$ as follows: first,

$$\mathbb{E}N = \sum_{x \in S_n} \mathbb{P}(R_x \cap E_x) = K^n \mathbb{P}(R_x \cap E_x).$$

Then

$$\begin{aligned}\mathbb{P}(R_x \cap E_x) &= \mathbb{P}\left(R_x \cap \left\{|\lambda g - \sigma| \leq \tau^{-1}\right\}\right)\\ &\geq \mathbb{P}\left(R_x \cap \{|\sigma| \leq t\} \cap \left\{|\lambda g - \sigma| \leq \tau^{-1}\right\}\right)\\ &= \mathbb{E}\left(\mathbb{1}_{R_x}\mathbb{1}_{|\sigma|\leq t}\, \mathbb{P}\left\{|g - \sigma| \leq \frac{1}{\lambda\tau} \mid R_x, \sigma\right\}\right).\end{aligned}$$

From Lemma 3.3,

$$\mathbb{P}\left\{|g - \sigma| \leq \frac{1}{\lambda\tau} \,\middle|\, R_x, \sigma\right\} \geq \frac{1}{C_{\lambda,t}\tau}\mathbb{1}_{|\sigma|\leq t},$$

therefore,

$$\mathbb{P}(R_x \cap E_x) \geq \frac{1}{C_{\lambda,t}\tau}\mathbb{P}\left(R_x \cap \{|\sigma| \leq t\}\right).$$

Choosing n and t large enough, we get

$$\mathbb{P}(R_x) \geq 3/4$$

from Claim 2.2 and

$$\mathbb{P}\{|\sigma| \leq t\} \geq 3/4,$$

from (5); hence

$$\mathbb{P}\left(R_x \cap \{|\sigma| \le t\}\right) \ge 1/2$$

and

$$\mathbb{P}(R_x \cap E_x) \ge \frac{1}{2C_{t,\lambda}\tau}.$$

□

Next, we bound the second moment of N from above. The first ingredient is

Lemma 3.5. *For $s \in (0,1)$,*

$$C_-^{-1}(s,z) \le \frac{\mathbb{E}\|G^{\mathfrak{T}_x}(0,x_-;z)\|^s}{\mathbb{E}\|G^{\mathfrak{T}_{u,x}}(0,u_-;z)\|^s \mathbb{E}\|G^{\mathfrak{T}_{u,x}}(u_+,x_-;z)\|^s} \le C_+(s,z),$$

where $C_\pm(s,z)$ are uniformly bounded as $\Im z \to +0$.

Proof. We start from Claim 2.1:

$$G^{\mathfrak{T}_x}(0,x_-;z) = G^{\mathfrak{T}_{u,x}}(0,u_-;z) G^{\mathfrak{T}_x}(u,u;z) G^{\mathfrak{T}_{u,x}}(u_+,x_-;z). \tag{6}$$

Upper bound (Only requires diagonal randomness) Taking norms in (6), we obtain

$$\|G^{\mathfrak{T}_x}(0,x_-;z)\|^s \le \|G^{\mathfrak{T}_{u,x}}(0,u_-;z)\|^s \|G^{\mathfrak{T}_x}(u,u;z)\|^s \|G^{\mathfrak{T}_{u,x}}(u_+,x_-;z)\|^s.$$

By construction, $G^{\mathfrak{T}_{u,x}}(0,u_-;z), G^{\mathfrak{T}_{u,x}}(u_+,x_-;z)$, and $V(u)$ are independent. We shall show that

$$\mathbb{E}_{V(u)}\|G^{\mathfrak{T}_x}(u,u;z)\|^s \le C_+(s,z), \tag{7}$$

where $\mathbb{E}_{V(u)}$ denotes averaging over $V(u)$ (= conditioning on all the other values of the potential). Averaging (7) over $\{V(y) \mid y \ne u\}$, we obtain the upper bound in the lemma. To prove (7), note that, by the Schur–Banachiewicz formula,

$$G^{\mathfrak{T}_x}(u,u;z) = (\lambda V(u) - \sigma)^{-1},$$

where σ is independent of $V(u)$. Therefore,

$$\mathbb{E}_{V(u)}\|G^{\mathfrak{T}_x}(u,u;z)\|^s \le C\lambda^{-s} \sum_{j,k} \mathbb{E}|(V(u)-\sigma)^{-1}_{jk}|^s = C(\mathrm{I} + \mathrm{II}),$$

where I is the sum of the diagonal terms and II is the sum of the off-diagonal terms. To bound the diagonal terms, note that

$$\mathbb{E}_{V(u)}|(V(u)-\sigma)^{-1}_{jj}|^s = \mathbb{E}_{V(u)}\mathbb{E}_{V(u)_{jj}}|V(u)_{jj} - \tilde{\sigma}|^{-s},$$

where $\tilde{\sigma}$ is independent of $V(u)_{jj}$. Therefore, [by the inequality (II.2) from the paper of Aizenman–Molchanov [2]]

$$\mathbb{E}_{V(u)}|(V(u)-\sigma)^{-1}_{jj}|^s \le C(s)$$

and $\mathrm{I} \le C(s)W$.

To bound the off-diagonal terms, we use inequality (II.3) from [2]. This concludes the proof of the upper bound.

Lower bound (Uses off-diagonal randomness). We shall use

Proposition 3.6. *Let V be a random matrix drawn from GOE, and let σ be a fixed matrix. Then for any two vectors ϕ and ψ*

$$\mathbb{E}\left|\langle (V-\sigma)^{-1}\phi,\psi\rangle\right|^s \geq C_{\|\sigma\|,s}\|\phi\|^s\,\|\psi\|^s.$$

Proof. We may assume without loss of generality that $\phi = e_1$ (the first vector of the standard basis) and that $\psi = ae_1 + be_2, a^2+b^2=1$. Then

$$\langle (V-\sigma)^{-1}\phi,\psi\rangle = a(V-\sigma)^{-1}_{11} + b(V-\sigma)^{-1}_{12}.$$

By Cramer's rule,

$$a(V-\sigma)^{-1}_{11} + b(V-\sigma)^{-1}_{12} = \frac{a(g_{22}-\tilde\sigma_{22}) - b(g_{12}-\tilde\sigma_{12})}{(g_{11}-\tilde\sigma_{11})(g_{22}-\tilde\sigma_{22}) - (g_{12}-\tilde\sigma_{12})(g_{12}-\tilde\sigma_{21})},$$

where g_{ij} are Gaussian, and $\tilde\sigma$ is independent of the g_{ij}. By Hölder's inequality,

$$\begin{aligned}&\mathbb{E}_g\left|a(V-\sigma)^{-1}_{11} + b(V-\sigma)^{-1}_{12}\right|^s\\ &\quad\geq \frac{\left[\mathbb{E}_g\left|a(g_{22}-\tilde\sigma_{22}) - b(g_{12}-\tilde\sigma_{12})\right|^{s/2}\right]^2}{\mathbb{E}_g\left|(g_{11}-\tilde\sigma_{11})(g_{22}-\tilde\sigma_{22}) - (g_{12}-\tilde\sigma_{12})(g_{12}-\tilde\sigma_{21})\right|^s}\end{aligned}$$

It is easy to see that the denominator is bounded from above by a number depending only on $\tilde\sigma$. The numerator is bounded from below by a constant independent of $\tilde\sigma$. Averaging over $\tilde\sigma$ concludes the proof of Proposition 3.6. □

For any two matrices A and B one can find ϕ_0 and ψ_0 so that $\|\phi_0\| = \|\psi_0\| = 1$ and $\|A^*\psi_0\| = \|A\|, \|B\phi_0\| = \|B\|$. Then, for $S = (V-\sigma)^{-1}$,

$$\|ASB\| \geq |\langle ASB\phi_0,\,\psi_0\rangle| = |\langle SB\phi_0,\,A^*\psi_0\rangle|\,,$$

and by Proposition 3.6

$$\mathbb{E}\|ASB\|^s \geq C^{-1}\|A\|^s\|B\|^s.$$

Applying this to $A = G^{\mathcal{T}_{u,x}}(0,u_-;z), S = \lambda G^{\mathcal{T}_x}(u,u;z) = (V(x)-\sigma)^{-1}$, and $B = G^{\mathcal{T}_{u,x}}(u_+,x_-;z)$, we obtain

$$\begin{aligned}&\mathbb{E}\|G^{\mathcal{T}_{u,x}}(0,u_-;z)G^{\mathcal{T}_x}(u,u;z)G^{\mathcal{T}_{u,x}}(u_+,x_-;z)\|^s\\ &\quad\geq \mathbb{E}\mathbb{E}_{V(x)}\|G^{\mathcal{T}_{u,x}}(0,u_-;z)G^{\mathcal{T}_x}(u,u;z)G^{\mathcal{T}_{u,x}}(u_+,x_-;z)\|^s\mathbb{1}_{\|\sigma\|\leq t}\\ &\quad\geq C^{-1}_{\lambda,t}\mathbb{E}\|G^{\mathcal{T}_{u,x}}(0,u_-;z)\|^s\|G^{\mathcal{T}_{u,x}}(u_+,x_-;z)\|^s\mathbb{1}_{\|\sigma\|\leq t}\\ &\quad\geq C^{-1}_{t}\mathbb{E}\|G^{\mathcal{T}_{u,x}}(0,u_-;z)\|^s\|G^{\mathcal{T}_{u,x}}(u_+,x_-;z)\|^s\prod_{w\in\mathcal{N}_u}\mathbb{1}_{\|G^{\mathcal{T}_{u,x}}(w,w;z)\|\leq Ct},\end{aligned}$$

where we omitted the dependence on λ and W. This expression is equal to

$$C_t^{-1}\left\{\mathbb{E}\|G^{\mathfrak{T}_{u,x}}(0,u_-;z)\|^s\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(u_-,u_-;z)\|\le Ct}\right\}$$
$$\left\{\mathbb{E}\|G^{\mathfrak{T}_{u,x}}(u_+,x_-;z)\|^s\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(u_+,u_+;z)\|\le Ct}\right\}$$
$$\prod_{w\in\mathcal{N}_u\setminus u_\pm}\left\{\mathbb{E}\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\le Ct}\right\}.$$

By Chebyshev's inequality,

$$\mathbb{E}\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\le Ct}\ge 1-C't^{-s}$$

can be made arbitrarily close to 1 by choosing t large enough. It remains to show that

$$\mathbb{E}_{V(w)}\|G^{\mathfrak{T}_{u,x}}(w,w';z)\|^s\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\ge Ct}\le\epsilon(t)\mathbb{E}_{V(w)}\mathbb{E}\|G^{\mathfrak{T}_{u,x}}(w,w';z)\|^s,$$

where $\epsilon(t)\to 0$ as $t\to\infty$. We will prove a stronger statement:

$$\begin{aligned}&\mathbb{E}_{V(w)^{\mathrm{diag}}}\|G^{\mathfrak{T}_{u,x}}(w,w';z)\|^s\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\ge Ct}\\&\quad\le\epsilon(t)\mathbb{E}_{V(w)^{\mathrm{diag}}}\mathbb{E}\|G^{\mathfrak{T}_{u,x}}(w,w';z)\|^s,\end{aligned}$$

where $\mathbb{E}_{V(w)^{\mathrm{diag}}}$ denotes the expectation over the diagonal elements of $V(w)$. Since the dependence on W is not important for us, it is sufficient to show that, for every j and k,

$$\begin{aligned}&\mathbb{E}_{V(w)^{\mathrm{diag}}}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^s\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\ge Ct}\\&\quad\le\epsilon(t)\mathbb{E}_{V(w)^{\mathrm{diag}}}\mathbb{E}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^s.\end{aligned}$$

Choose $p,q>1$ so that $1/p+1/q=1$ and $sp<1$. By Hölder's inequality,

$$\begin{aligned}&\mathbb{E}_{V(w)^{\mathrm{diag}}}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^s\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\ge Ct}\\&\quad\le\left\{\mathbb{E}_{V(w)^{\mathrm{diag}}}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^{sp}\right\}^{1/p}\left\{\mathbb{E}\mathbb{1}_{\|G^{\mathfrak{T}_{u,x}}(w,w;z)\|\ge Ct}\right\}^{1/q}\\&\quad\le C't^{-s/q}\left\{\mathbb{E}_{V(w)^{\mathrm{diag}}}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^{sp}\right\}^{1/p}.\end{aligned}$$

It remains to show that

$$\begin{aligned}&\left\{\mathbb{E}_{V(w)^{\mathrm{diag}}}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^{sp}\right\}^{1/(sp)}\\&\quad\le C\left\{\mathbb{E}_{V(w)^{\mathrm{diag}}}|G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)|^{s}\right\}^{1/s}.\end{aligned}\tag{8}$$

The expression $G^{\mathfrak{T}_{u,x}}(w,w';z)(j,k)$ is a fractional-linear function of every diagonal element of $V(w)$. Therefore, (8) follows from the following decoupling lemma:

Proposition 3.7. *Let $X_j, 1\le j\le W$, be independent identically distributed random variables with bounded density and finite moments. Then, for every function $f(x_1,\ldots,x_W)$ which is fractional-linear as a function of every variable, and every $0<\alpha<\beta<1$,*

$$(\mathbb{E}|f(X_1,\ldots,X_W)|^\beta)^{1/\beta}\le C(\mathbb{E}|f(X_1,\ldots,X_W)|^\alpha)^{1/\alpha},$$

where $C>0$ may depend on α and β but not on f.

The proof is given (in more general setting) in [5, Proposition 3.2]. This concludes the proof of Lemma 3.5. □

Similar considerations allow to extend the arguments leading to two more statements from [4] to our matrix setting:

Lemma 3.8 (Lemma 3.4*). *For $s \in (0,1)$,*

$$\frac{1}{C(s,z)} \leq \frac{\mathbb{E}\|G^{\mathcal{T}_x}(0,x_-;z)\|^s}{\|G^{\mathcal{T}_{x_-}}(0,x_{--};z)\|^s} \leq C(s,z),$$

and

$$\frac{1}{C(s,z)} \leq \frac{\mathbb{E}\|G(0,x_-;z)\|^s}{\mathbb{E}\|G^{\mathcal{T}_x}(0,x_-;z)\|^s} \leq C(s,z),$$

where $C(s,z)$ remains bounded (for fixed $\Re z$) as $\Im z \to +0$.

Proposition 3.9 (Theorem 3.2*). *Let*

$$\phi_\lambda(s;z) = \lim_{\mathrm{dist}(x,0)\to\infty} \log \mathbb{E}\|G_\lambda(0,x;z)\|^s.$$

For any $z \in \mathbb{C}^+$ the function $(0,\infty) \ni s \mapsto \phi_\lambda(s;z)$ has the following properties:

1. *$\phi_\lambda(\cdot,z)$ is convex and non-increasing;*
2. *for $s \in (0,2]$,*
$$-sL(z) \leq \phi_\lambda(s;z) \leq -s\log\sqrt{K};$$
3. *for any $s \in (0,1)$ and $x \in \mathcal{T}$,*
$$\frac{1}{C(s,z)}\mathrm{e}^{\phi_\lambda(s;z)\mathrm{dist}(x,0)} \leq \mathbb{E}\|G_\lambda(0,x;z)\|^s \leq C(s,z)\mathrm{e}^{\phi_\lambda(s;z)\mathrm{dist}(x,0)},$$
where $C(s,z) \in (0,\infty)$; if $s \in (0,1), C(s,z)$ remains bounded as $\Im z \to +0$.

Definition 3.10. The no-a.c. hypothesis holds at energy $E \in \mathbb{R}$ if, for a fixed vector v,

$$\Im\langle (H - E - i0)^{-1}v, v\rangle = 0$$

almost surely.

Note that the definition does not depend on the choice of the vector v.

Claim 3.11. Under the no-ac hypothesis $G(0,0;E+i0)$ is almost surely real symmetric.

Proof. Let us show that

$$G(0,0;E+i0)_{kj} = \overline{G(0,0;E+i0)_{jk}}. \tag{9}$$

For $j = k$ this follows directly from the definition (applied to $v = (0,j)$). For $j \neq k$, apply the definition to

$$v_1 = \delta(0,j) + \delta(0,k), \quad v_2 = \delta(0,j) + i\delta(0,k).$$

We obtain that

$$\langle G(0,0;E+i0)v_1, v_1\rangle = G(0,0;E+i0)_{jj} + G(0,0;E+i0)_{kk} \\ +G(0,0;E+i0)_{jk} + G(0,0;E+i0)_{kj}$$

is real; hence

$$G(0,0;E+i0)_{jk} + G(0,0;E+i0)_{kj}$$

is real; also,

$$\langle G(0,0;E+i0)v_2, v_2\rangle \\ = G(0,0;E+i0)_{jj}+G(0,0;E+i0)_{kk} - iG(0,0;E+i0)_{jk}+iG(0,0;E+i0)_{kj}$$

is real; hence

$$G(0,0;E+i0)_{jk} - G(0,0;E+i0)_{kj}$$

is pure imaginary. To conclude the proof of (9), note that if $a+b$ is real and $a-b$ is pure imaginary, then $a=\bar{b}$.

G is always symmetric; hence (9) implies that $G(0,0;E+i0)$ is real symmetric. □

Claim 3.12. For any real symmetric $W\times W$ matrix A,

$$\|A\| \leq C_W \max\left\{\max_j |\langle Ae_j, e_j\rangle|, \max_{j\neq k} |\langle A(e_j+e_k), (e_j+e_k)\rangle|\right\}.$$

Proof. Denote $\|A\| = R$. Then $\|A\|_\infty \geq R/B_W$ (where $\|\cdot\|_\infty$ stands for the maximum of the absolute values of the matrix entries). There are two cases:

1. There exists j so that $|A_{jj}| \geq \frac{R}{3B_W}$ for some j (then the conclusion of the claim is obvious)
2. There exist j and k so that $|A_{jk}| \geq \frac{R}{B_W}$, and $|A_{jj}|, |A_{kk}| < \frac{R}{3B_W}$. Then

$$|\langle A(e_j+e_k), (e_j+e_k)\rangle| \\ = |a_{jj} + a_{kk} + 2a_{jk}| \geq 2|a_{jk}| - |a_{kk}| - |a_{jj}| \geq \frac{R}{B_W}.$$

□

Proposition 3.13. *Under the no-ac assumption, there exists $C>0$ so that for any $n\geq 1$ and $\eta>0$*

$$\mathbb{E}N(N-1) \leq C\tau^{-2}K^{2n}.$$

Proof. Recall that

$$E_x = \{|\langle G(x,x;E+i\eta)\tilde{v}, \tilde{v}\rangle| \geq \tau\},$$

therefore, (by Claim 3.12)

$$E_x \subset \tilde{E}_x = \{\|G(x,x;E+i\eta)\| \geq \tau\} \subset \bigcup_j \tilde{E}_x^j \cup \bigcup_{jk} \tilde{E}_x^{jk},$$

where

$$\tilde{E}_x^j = \{|\langle G(x,x;E+i\eta)e_j, e_j\rangle| \geq \tau/C\}$$

and

$$\tilde{E}_x^{jk} = \{|\langle G(x,x;E+i\eta)(e_j+e_k),(e_j+e_k)\rangle| \geq \tau/C\}.$$

Therefore,

$$\begin{aligned}\mathbb{E}N(N-1) &= \sum_{x,y\in S_n, x\neq y} \mathbb{P}(R_x\cap E_x\cap R_y\cap E_y)\\ &\leq \sum \mathbb{P}(E_x\cap E_y)\\ &\leq \sum\Big\{\sum_{jj'}\mathbb{P}(\tilde{E}_x^j\cap\tilde{E}_y^{j'})+\sum_{jj'k'}\mathbb{P}(\tilde{E}_x^j\cap\tilde{E}_y^{j'k'})\\ &\qquad +\sum_{jkj'}\mathbb{P}(\tilde{E}_x^{jk}\cap\tilde{E}_y^{j'})+\sum_{jkj'k'}\mathbb{P}(\tilde{E}_x^{jk}\cap\tilde{E}_y^{j'k'})\Big\}\\ &= \sum(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}).\end{aligned}$$

Let us estimate the terms I (the other terms are estimated in the same way). We apply [4, Theorem A.2]. It yields

$$\mathbb{P}(\tilde{E}_x^j\cap\tilde{E}_y^{j'}) \leq \frac{C}{\tau}\left\{\frac{C}{\tau}+\mathbb{E}\min\right.$$
$$\left.\times\left(1,\sum_{u\sim(x,j),v\sim(y,j')}\left|H(x,j;u)G^{(x,j;y,j')}(u,v;E+i\eta)H(v;y,j)\right|\right)\right\}.$$

Here $H(x,j;u)$ and $H(v;y,j)$ are Gaussian random variables, independent of each other and of $G^{(x,j;y,j')}$, the Green function corresponding to the operator obtained by erasing the vertices (x,j) and (y,j') of $\mathcal{T}\times G$. The first term is of the desired form since the number of addends is bounded by $C_W K^{2n}$. For the second term we use the inequality

$$\min(1,|x|)\leq |x|^s, \quad 0\leq s\leq 1,$$

and then estimate

$$\begin{aligned}&\mathbb{E}\sum_{u\sim(x,j),v\sim(y,j')}\left|H(x,j;u)G^{(x,j;y,j')}(u,v;E+i\eta)H(v;y,j)\right|^s\\ &\quad=\sum_{u\sim(x,j),v\sim(y,j')}\mathbb{E}|H(x,j;u)|^s\,\mathbb{E}|G^{(x,j;y,j')}(u,v;E+i\eta)|^s\,\mathbb{E}|H(v;y,j)|^s\\ &\quad\leq C\sum_{u\sim(x,j),v\sim(y,j')}\mathbb{E}|G^{(x,j;y,j')}(u,v;E+i\eta)|^s.\end{aligned}$$

If $u=(x,k), v=(y,k')$ (where $k\neq j, k'\neq j'$), repeated application of Lemma 3.8 and Proposition 3.9 yields

$$\mathbb{E}|G^{(x,j;y,j')}(u,v;E+i\eta)|^s\leq C\mathbb{E}\|G(x,y;E+i\eta)\|^s\leq C'K^{-\frac{s}{2}\operatorname{dist}(x,y)}.$$

Combining these estimates and taking $s=\frac{L(E)+2\delta}{\log K}\in(0,1)$, we obtain the desired bound. This completes the proof of Proposition 3.13. ☐

Proposition 3.14 (Modified Theorem 4.6*). *For almost all*

$$E \in \sigma(H) \cap \{L(E) < \log K\} \cap \{\textit{no-ac holds}\},$$

there exist $\delta, p_0 > 0$ *and* $n_0 \geq 0$ *so that for all* $n \geq n_0$

$$\liminf_{\eta \to 0} \mathbb{P}\left\{\max_{x \in S_n} \|G(0, x; E + i\eta\| \geq \mathrm{e}^{\delta n}\right\} \geq p_0.$$

Proof. By Propositions 3.2 and 3.13 there exist C, η_0 and n_0 so that for $n \geq n_0$ and $\eta \in (0, \eta_0)$

$$\frac{\mathbb{E}N^2}{\{\mathbb{E}N\}^2} = \frac{1}{\mathbb{E}N} + \frac{\mathbb{E}N(N-1)}{\{\mathbb{E}N\}^2} \leq C.$$

Therefore,

$$\mathbb{P}\{N \geq 1\} \geq \frac{\{\mathbb{E}N\}^2}{\mathbb{E}N^2} \geq \frac{1}{C}$$

uniformly in $n \geq n_0$ and $\eta \in (0, \eta_0)$. ☐

Proof of Theorem 1.4. We argue by contradiction: if the no-ac hypothesis holds for a given $E \in \sigma(H)$, the conclusion of Proposition 3.14 implies that

$$\sum \|G(0, x; E + i0)\|^2 = \infty$$

with positives probability and hence almost surely. Proposition 3.1 concludes the proof. ☐

4. Proof of Theorem 1.3

Denote

$$\Gamma(y) = \Gamma(y; E + i\eta) = G^{\mathcal{T}_{y-}}(y, y; E + i\eta); \quad \tilde{\Gamma}(y) = \frac{\Gamma(y) - \Gamma(y)^*}{2i}$$

(the latter is the matrix analogue of $\Im\Gamma$ from [4]). Theorem 1.3 will follow from the following statements:

Lemma 4.1 (Lemma 4.4*). *For any* $A > 0$, *if*

$$\mathbb{P}\left\{\|\tilde{\Gamma}\| \geq A\right\} \geq q > 0$$

for some $q \in (0, 1)$, *then*

$$\mathbb{P}\left\{\|\tilde{\Gamma}\| \geq \frac{A}{R}\right\} \to 1$$

as $R \to \infty$, *uniformly in* $\eta > 0$.

The proof is identical to that of [4, Lemma 4.4] (note, however, that, unlike the rest of the current paper, one has to work with the fastest Lyapunov exponent rather than the slowest one).

Proposition 4.2 (Theorem 4.6*). *For almost all*

$$E \in \sigma(H) \cap \{L(E) < \log K\} \cap \{\textit{no-ac holds}\}\,,$$

there exist $\delta, p_0 > 0$ *and* $n_0 \geq 0$ *so that for all* $n \geq n_0$

$$\liminf_{\eta \to +0} \mathbb{P}\Big\{\exists x \in S_n, y \in \mathcal{N}_x^+ \big| \|G^{\mathcal{T}_x}(0, x_-, E+i\eta)\| \geq \mathrm{e}^{-(L(E)+\delta)n}, \|\tilde{\Gamma}\| \geq \xi(p),$$
$$\Big|\Big\langle G(x,x;E+i\eta) w_{\max}(\tilde{\Gamma}(y)), w_{\max}(G^{\mathcal{T}_x}(0,x_-;E+i\eta)^* G^{\mathcal{T}_x}(0,x_-;E+i\eta))\Big\rangle\Big|$$
$$\geq \mathrm{e}^{+(L(E)+2\delta)n}\Big\} \geq q > 0,$$

where

1. q *may depend on* δ *and* p*, but not on* η *and* n*;*
2. $\xi(p) = \inf\Big\{t \mid \mathbb{P}\{\|\tilde{\Gamma}\| \geq t\} \geq p\Big\}$ *is the pth quantile of* $\|\tilde{\Gamma}\|$*;*
3. $w_{\max}$ *denotes the eigenvector associated with the maximal eigenvalue.*

The following lemma will be used both in the proof and in the application of Proposition 4.2:

Lemma 4.3. *The (self-adjoint) matrix* $\tilde{\Gamma}(0)$ *admits the lower bound*

$$\tilde{\Gamma}(0) \geq \sum_{x \in S_n} \sum_{y \in \mathcal{N}_x^+} G(0,x;E+i\eta)\tilde{\Gamma}(y)G(0,x;E+i\eta)^*$$

in the sense of quadratic forms.

Proof. From the resolvent identity,

$$\begin{aligned}
\tilde{\Gamma}(0) &= \frac{\Gamma(0) - \Gamma(0)^*}{2i} \\
&= \frac{1}{2i}\Gamma(0)\left\{\eta + \sum_{y \in \mathcal{N}_x^+} (\Gamma(y) - \Gamma(y)^*)\right\}\Gamma(0)^* \\
&\geq \sum_{y \in \mathcal{N}_x^+} \Gamma(0)(\Gamma(y) - \Gamma(y)^*)\Gamma(0)^* \\
&= \sum_{y \in \mathcal{N}_x^+} G(0,0;E+i\eta)(\Gamma(y) - \Gamma(y)^*)G(0,0;E+i\eta)^*.
\end{aligned}$$

This yields the statement for $n = 0$. The statement for larger n follows by iteration. □

Proof of Proposition 4.2. Denote

$$\begin{aligned}
I_x &= \Big\{\|\tilde{\Gamma}(x)\| \geq \xi(p)\Big\}\,, \\
R_x &= \Big\{\|G^{\mathcal{T}_x}(0,x_-;E+i\eta)\| \geq \mathrm{e}^{-(L(E)+\delta)n}\Big\}\,, \\
E_x &= \{|\langle G(x,x;E+i\eta)v; w\rangle| \geq \tau\}\,,
\end{aligned}$$

where

$$v = w_{\max}(\tilde{\Gamma}(y)), \quad w = w_{\max}(G^{\mathcal{T}_x}(0,x_-;E+i\eta)^* G^{\mathcal{T}_x}(0,x_-;E+i\eta)).$$

Then Proposition 4.2 states that

$$\liminf_{\eta\to+0} \mathbb{P}\left\{\bigcup_x I_x \cap R_x \cap E_x\right\} \geq q > 0.$$

Denote

$$N = \sum_{x\in S_n} \mathbb{1}_{I_x\cap R_x\cap E_x}.$$

As in the proof of Theorem 1.4, we shall prove that

$$\frac{\mathbb{E}N^2}{\{\mathbb{E}N\}^2} \leq C.$$

The upper bound on $\mathbb{E}N(N-1)$ follows from the argument of Proposition 3.13. Indeed, in the notation of the proof of Proposition 3.13,

$$\mathbb{P}(I_x \cap R_x \cap E_x \cap I_y \cap R_y \cap E_y) \leq \mathbb{P}(E_x \cap E_y) \leq \mathbb{P}(\tilde{E}_x \cap \tilde{E}_y);$$

hence

$$\mathbb{E}N(N-1) \leq C\tau^{-2}K^{2n}.$$

To bound $\mathbb{E}N$ from below, we need to show that

$$\mathbb{P}(I_x \cap R_x \cap E_x) \geq C\tau^{-1}.$$

By the parallelogram law,

$$\langle G(x,x)v, w\rangle = \frac{1}{4}\big[\langle G(x,x)(v+w),(v+w)\rangle - \langle G(x,x)(v-w),(v-w)\rangle\big].$$

In our case, $\|v\| = \|w\| = 1$; hence $v+w \perp v-w$. Without loss of generality we may assume that $\|v+w\| \geq \|v-w\|$, then $\|v+w\| \geq 2 \geq \|v-w\|$. Set $e_1 = (v+w)/\|v+w\|, e_2 = (v-w)/\|v-w\|$. Then

$$\{|\langle G(x,x)v,w\rangle| \geq \tau\} \supset \{|\langle G(x,x)e_1,e_1\rangle| \geq 2\tau,\ |\langle G(x,x)e_2,e_2\rangle| \leq \tau\}.$$

No generality is lost if we assume that e_1 and e_2 are the first two vectors of the standard basis. Let P be the projection onto e_1, e_2. Then

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22}\end{pmatrix} = PG(x,x)P = \left(\lambda\begin{pmatrix} V_{11} & 0 \\ 0 & V_{22}\end{pmatrix} - X\right)^{-1},$$

where

$$X = \begin{pmatrix} a & b \\ b & c\end{pmatrix}$$

is independent of V_{11} and V_{22}. Consider two cases:

1. $|b| \leq 1/\sqrt{\tau}$. Then the argument of Proposition 3.2 yields

$$\mathbb{P}_{V_{11},V_{22}}\{|G_{11}| \geq 2\tau\} \geq \frac{1}{C\tau},$$

whereas [4, Theorem A.2] yields

$$\mathbb{P}_{V_{11},V_{22}}\{|G_{11}| \geq 2\tau,\ |G_{22}| \geq \tau\} \leq \frac{C}{\tau^{3/2}}.$$

Therefore,

$$\mathbb{P}_{V_{11},V_{22}}\left\{|G_{11}| \geq 2\tau,\, |G_{22}| \leq \tau\right\} \geq \frac{1}{C'\tau}.$$

2. $|b| > 1/\sqrt{\tau}$. If $|G_{22}| \geq \tau$, then

$$|(V_{11} - a) - b^2/(V_{22} - c)| = |G_{22}|^{-1} \leq \frac{1}{\tau};$$

therefore,

$$|V_{11} - a| \leq \frac{1}{\tau} + \left|\frac{b^2}{V_{22} - c}\right|.$$

If in addition $|V_{22} - c| > 2b$, then

$$|V_{11} - a| \leq \frac{b}{2} + \frac{1}{\tau} \leq \frac{2b}{3}.$$

Therefore,

$$\left|\frac{V_{22} - c}{V_{11} - a}\right| \geq \frac{2b}{2b/3} = 3.$$

This implies

$$\begin{aligned}\frac{1}{|G_{22}|} &= \left|V_{22} - c - \frac{b^2}{V_{11} - a}\right| \\ &= \left|\frac{V_{22} - c}{V_{11} - a}\right| \left|V_{11} - a - \frac{b^2}{V_{22} - c}\right| \geq \frac{3}{|G_{11}|}.\end{aligned}$$

Hence in this case

$$\begin{aligned}&\{V_{11}, V_{22} \mid |G_{11}| \geq 2\tau, |G_{22}| \leq \tau\} \\ &\quad\supset \left\{V_{11}, V_{22} \mid \frac{1}{3\tau} < \frac{1}{|G_{11}|} < \frac{1}{2\tau}, |V_{22}| > 2b\right\},\end{aligned}$$

and the probability of this event is again $\geq C^{-1}(b)\tau^{-1}$. The rest of the argument follows the proof of Proposition 3.2. □

Proof of Theorem 1.3. Theorem 1.3 follows immediately from Lemma 4.1, Proposition 4.2, and Lemma 4.3. □

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