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Exponential Decay of Equal-Time Four-Point Correlation Functions in the Hubbard Model on the Copper-Oxide Lattice

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Abstract. For the Hubbard model on the two-dimensional copper-oxide lattice, equal-time four-point correlation functions at positive temperature are proved to decay exponentially in the thermodynamic limit if the magnitude of the on-site interactions is smaller than some power of temperature. This result especially implies that the equal-time correlation functions for singlet Cooper pairs of various symmetries decay exponentially in the distance between the Cooper pairs in high temperatures or in low-temperature weak-coupling regimes. The proof is based on a multi-scale integration over the Matsubara frequency.

Notation

Parameters and constants

Notation	*Description*
L	Size of lattice of the position variable
t	Hopping amplitude
U_{c}	Coupling constant on the Cu sites
U_{o}	Coupling constant on the O sites
$\epsilon_{\text{c}}^{\sigma}$, $\epsilon_{\text{o}}^{\sigma}$ $(\sigma \in \{\uparrow,\downarrow\})$	Spin-dependent on-site energies
β	Proportional to the inverse of temperature
$E_{\max}$	$\max_{\sigma\in\{\uparrow,\downarrow\}}\{1,\lvert t\rvert,\lvert\epsilon_{\text{c}}^{\sigma}\rvert,\lvert\epsilon_{\text{o}}^{\sigma}\rvert\}$
$\hat{\mathcal{X}}_j$, $\hat{\mathcal{Y}}_j$ $(j=1,2)$	Same as $(\hat{\rho}_j,\hat{\mathbf{x}}_j,\hat{\sigma}_j)$, $(\hat{\eta}_j,\hat{\mathbf{y}}_j,\hat{\tau}_j)$ $(j=1,2)$, Fixed sites to define the correlation function
h	Step size of the discretization of $[0,\beta)$, $[-\beta,\beta)$
$N_{L,h}$	$6L^2\beta h$, Cardinality of $I_{L,h}$
λ_1, λ_{-1}	Used to modify the interaction
c	Generic constant depending only on a fixed smooth function

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M	Parameter to control the size of the support of the cut-off function
N_h	$\lfloor \log(2h)/\log(M)\rfloor$
N_β	$\max\{\lfloor \log(1/\beta)/\log(M)\rfloor+1,1\}$
c_0	Constant depending on M and β
α	Additional parameter used in the multi-scale integration

Sets

Notation	*Description*
Γ	$(\mathbb{Z}/L\mathbb{Z})^2$
$[0,\beta)_h$	$\{0,1/h,\dots,\beta-1/h\}$
$[-\beta,\beta)_h$	$\{-\beta,-\beta+1/h,\dots,-1/h\}\cup[0,\beta)_h$
Γ^*	$(\frac{2\pi}{L}\mathbb{Z}/(2\pi\mathbb{Z}))^2$
$\mathcal{M}_h$	$\{\omega\in\pi(2\mathbb{Z}+1)/\beta \mid \lvert\omega\rvert<\pi h\}$
$I_{L,h}$	$\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)_h$
$\tilde{I}_{L,h}$	$I_{L,h}\times\{1,-1\}$
$(I_{L,h})_o^m$	Subset of $I_{L,h}^m$
D_{small}	Subset of $\mathbb{C}^4$
D_R	Subset of $\mathbb{C}$

Functions

Notation	*Description*
$\mathcal{F}_{t,\beta}(\cdot)$	Used to specify the domain of analyticity of the covariance
$\hat{s}(\cdot)$	Fixed function of spin
$\mathcal{C}(\cdot,\cdot)$	Covariance of full scale
$\chi_l(\cdot)$	Cut-off function of l-th scale
$\mathcal{C}_l(\cdot,\cdot)$	Covariance of l-th scale

1. Introduction

1.1. Introductory Remarks

In order to explain high-temperature superconductivity in ceramic copper oxide materials, several tight-binding models for the charge carriers in two-dimensional plane have been proposed with the consensus that the superconducting pairing mechanism should be understood by focusing on the conducting CuO_2 plane first. In the hierarchy of the well-known 2D models (see, e.g., [3]) the three-band Hubbard model on the copper-oxide lattice [4], or the CuO Hubbard model in short, is believed to be the closest to the reality since it explicitly distinguishes one relevant electron orbital of the copper and those of the oxygens surrounding the copper in the unit cell. Being more realistic also means being more complex. Rigorous mathematical methods need to be developed to explore the relatively involved structure of the CuO Hubbard model in depth.

In this paper we prove that equal-time 4-point correlation functions in the CuO Hubbard model at positive temperature decay exponentially in the thermodynamic limit if the coupling constants on both the copper and the oxygen sites are smaller than some power of temperature. The result will be fully stated in Sect. 1.3. One direct consequence of this theorem is the exponential decay of pairing–pairing correlation functions in the distance between the center of two electrons and that of two holes, excluding long range correlations between singlet Cooper pairs in high temperatures or in low-temperature weak-coupling regimes.

It has been proved in [11] that finite-temperature equal-time correlation functions for many-electron models, including the Hubbard model as one instance, on the hyper-cubic lattice of arbitrary dimension decay exponentially if the interaction is smaller than some power of temperature. The proof of [11] essentially uses the volume-, temperature-independent determinant bound on the covariance matrix established by Pedra and Salmhofer [13]. The exponential decay of the correlation functions in the CuO Hubbard model cannot be deduced as an immediate corollary of the theorems in [11], since Pedra–Salmhofer's determinant bound in its original form [13, Theorem 2.4] does not apply to the covariance for multi-band many-Fermion models such as the CuO Hubbard model. Thus one has to alter the way to achieve the goal. As a way out we expand the covariance over the Matsubara frequency through the Fourier transform this time and try to control the correlation function analytically by means of a multi-scale expansion along the segments of the large Matsubara frequency. The dispersion relation for the free particle hopping to the nearest neighbor sites on the CuO lattice can be a square root of cosine of the momentum variable, which is, unlike in the single-band models treated in [10,11], non-analytic. Once transformed into the Matsubara sum, however, the covariance appears to contain only the square of the dispersion relation. Thus the covariance in the Matsubara sum representation explicitly shows its analytic property with respect to the momentum variable. As in [11] the analyticity of the covariance enables us to reformulate the correlation function multiplied by the distance between the electrons and the holes into a multi-contour integral of the correlation function with respect to new complex variables inserted in the covariance. The practical role of the multi-scale integration over the Matsubara frequency in this paper is to establish a volume-independent upper bound on the perturbed correlation function inside the multi-contour integral. Due to a self-contained nature of the multi-scale Matsubara expansion, the proofs in this paper merely rely on the repeated use of the tree formula for logarithm of the Grassmann Gaussian integral.

More precisely speaking, the correlation function of our original interest is expressed as a well-defined finite-dimensional Grassmann integral during the intermediate technical construction. In the major part of this paper we deal with the Grassmann integral formulation, which is flexible to mathematical manipulations, as the rigorous counterpart of the correlation function. This is the same stance as taken in [10,11], or more generally in the constructive Fermionic quantum field theory (see, e.g., [6]). Finally, by sending the finite-dimensional formulation to the limit we withdraw the conclusion on the original correlation function defined by trace operations over the Fermionic Fock space.

This paper is not the first to consider multi-scale analysis over the large Matsubara frequency. On the contrary, a number of papers have already discussed qualitatively similar problems to the Matsubara ultraviolet problem posed in this paper, see, e.g., [1,2,7,8] by one of the pioneering groups of the subject. One of the purposes of this paper is set to provide readers with an alternative method to solve the Matsubara ultraviolet problem. In order to help the readers to properly comprehend the purpose of this paper, let us summarize the main differences between the methods used in this article and those in the preceding papers. First, this paper uses a version of the finite-dimensional Grassmann integral formulation reported in [10]. The reduction to the finite dimensionality in this formulation is based on the discretization of the interval of temperature in the perturbative expansion of the partition function. Accordingly the basis of Grassmann algebra is indexed by the finite space–time variables and the step size of the discretization explicitly appears in the characterization of the covariance as a parameter, changing the face of the covariance from the well-known free propagator. This paper does not introduce Grassmann algebra indexed by the momentum variables. In [1,7,8] the derivation of the finite-dimensional Grassmann integral formulation is based on the cut-off on the Matsubara frequency. As a result the basis of Grassmann algebra is indexed by the finite momentum variables. Second, the multi-scale analysis in this paper is completed by the induction on the scale level, which assumes a norm bound on the input and then proves the relevant norm bound on the output produced by the single-scale integration. The papers [1,2,7,8] use a family of trees, called the Gallavotti–Nicolò trees, to organize the multi-scale integration process, achieving collective descriptions of the theory all through the integration levels. This paper's concept of finding a norm bound on the output of the integration at one scale is closer to the rigorous analysis on finite-dimensional Grassmann algebra established in [5,6]. However, the paper [5] and the book [6] apply a representation theorem developed by themselves to expand logarithm of the Grassmann Gaussian integral, while this paper as well as the papers [1,2,7,8] use the tree expansion for the same purpose. Third, this paper derives equal-time 4-point correlation functions by substituting an artificial quartic term into the original Hamiltonian and differentiating the free energy governed by the modified Hamiltonian with respect to the coefficient of the artificial term. The papers [1,2,7,8] derive correlation functions by inserting the source Grassmann variables into the Grassmann integral formulation and then letting the Grassmann derivatives act on the modified Grassmann integral formulation, called the generating function.

Though this paper involves a multi-scale analysis concerning the Matsubara sum as the main technical ingredient, it does not treat any infrared multi-scale analysis around zero points of the dispersion relation. Accordingly, this paper has no improvement on the temperature dependency of the allowed magnitude of the interaction over the single-scale analysis [10,11] and cannot study the behavior of correlation functions at zero temperature. In recent years, infrared multi-scale integration techniques have been intensively applied to describe the zero-temperature limit of thermal expectation values of various observables in the Hubbard model on the honeycomb lattice by Giuliani and Mastropietro [8] and by Giuliani et al. [9]. In connection with the main result of this article we should remark that the many-electron model of graphene studied in [8,9] also has a matrix-valued kinetic energy, so the single-scale analysis previously reported in [11] does not prove the exponential decay of the finite-temperature correlation functions in the system. However, it is straightforward to adapt the proofs in this article to conclude the same result for the Hubbard model on the honeycomb lattice as claimed for the CuO Hubbard model.

This paper is outlined as follows: in the following subsections we define the CuO Hubbard model and state the main result of this paper. In Sect. 2 we characterize the correlation function as a limit of the finite-dimensional Grassmann integral and derive the contour integral formulation. In Sect. 3 we prepare some necessary tools for the multi-scale integration such as the cut-off function and the sliced covariances. In Sect. 4 we carry out the multi-scale integration over the Matsubara frequency and prove the main theorem. In Appendix A we derive the covariance governed by the free Hamiltonian on the CuO lattice. Appendix B provides a sketch of how to prove the convergence property of the Grassmann integral formulation. In Appendix C we prove a general formula for logarithm of Grassmann polynomials, which is necessary for the multi-scale integration. Finally, Appendix D shows that the correlation function converges to a finite value in the thermodynamic limit if the coupling constants obey the smallness condition under which the multi-scale analysis is performed.

1.2. The Hubbard Model on the CuO Lattice

Here we define the model Hamiltonian operator. For $L \in \mathbb{N}$ let $\Gamma := (\mathbb{Z}/L\mathbb{Z})^2$. The CuO lattice consists of three separate lattices, each of which is isomorphic to Γ (see Fig. 1). For $\mathbf{x} \in \Gamma$ let $(1, \mathbf{x})$ represent a Cu site, $(2, \mathbf{x})$ represent the O site right to $(1, \mathbf{x})$, and $(3, \mathbf{x})$ denote the O site above $(1, \mathbf{x})$ (see Fig. 2). The CuO lattice is viewed as the union of $\{(\rho, \mathbf{x}) \mid \mathbf{x} \in \Gamma\}$ $(\rho = 1, 2, 3)$.

The model Hamiltonian is defined as a self-adjoint operator on the Fermionic Fock space $F_f(L^2(\{1, 2, 3\} \times \Gamma \times \{\uparrow, \downarrow\}))$. See, e.g, [10, Appendix A] for a brief description of the Fermionic Fock space defined on a finite lattice. The CuO Hubbard model was originally designed to govern the total energy of

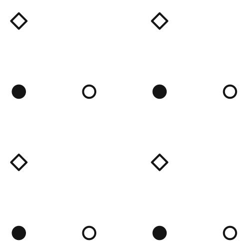

FIGURE 1. The CuO lattice for $L = 2$, where *bullet symbols* denotes Cu sites, *circles* denotes O sites, and *diamond symbols* denote the other O sites

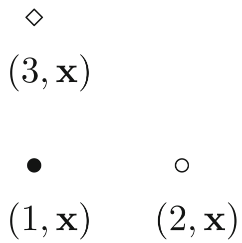

Figure 2. Labeling each site

holes moving and interacting on the CuO_2 plane (see [4]). Thus the vacuum of $F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}))$ should be interpreted as the state where every site of $\{1,2,3\}\times\Gamma$ is occupied by an electron pair.

For $(\rho,\mathbf{x},\sigma)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}$ let $\psi_{\rho\mathbf{x}\sigma}$ be the annihilation operator defined on $F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}))$. The physical role of $\psi_{\rho\mathbf{x}\sigma}$ is to annihilate a hole with spin σ at the site $(\rho,\mathbf{x})$. We write the adjoint operator of $\psi_{\rho\mathbf{x}\sigma}$ as $\psi^*_{\rho\mathbf{x}\sigma}$. The operator $\psi^*_{\rho\mathbf{x}\sigma}$ is called the creation operator and physically considered to be creating a hole with spin σ at the site $(\rho,\mathbf{x})$. The CuO Hubbard model H is defined as follows:

$$
\begin{aligned}
H &:= H_0+V,\\
H_0 &:= t\sum_{(\mathbf{x},\sigma)\in\Gamma\times\{\uparrow,\downarrow\}}(\psi^*_{1\mathbf{x}\sigma}\psi_{2\mathbf{x}\sigma}+\psi^*_{1\mathbf{x}\sigma}\psi_{2(\mathbf{x}-\mathbf{e}_1)\sigma}+\psi^*_{1\mathbf{x}\sigma}\psi_{3\mathbf{x}\sigma}\\
&\qquad\qquad +\psi^*_{1\mathbf{x}\sigma}\psi_{3(\mathbf{x}-\mathbf{e}_2)\sigma}+\text{h.c})\\
&\quad+\sum_{(\mathbf{x},\sigma)\in\Gamma\times\{\uparrow,\downarrow\}}\epsilon^\sigma_{\mathrm{c}}\psi^*_{1\mathbf{x}\sigma}\psi_{1\mathbf{x}\sigma}+\sum_{(\rho,\mathbf{x},\sigma)\in\{2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}\epsilon^\sigma_{\mathrm{o}}\psi^*_{\rho\mathbf{x}\sigma}\psi_{\rho\mathbf{x}\sigma},\\
V &:= U_{\mathrm{c}}\sum_{\mathbf{x}\in\Gamma}\psi^*_{1\mathbf{x}\uparrow}\psi^*_{1\mathbf{x}\downarrow}\psi_{1\mathbf{x}\downarrow}\psi_{1\mathbf{x}\uparrow}+U_{\mathrm{o}}\sum_{(\rho,\mathbf{x})\in\{2,3\}\times\Gamma}\psi^*_{\rho\mathbf{x}\uparrow}\psi^*_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow},
\end{aligned}
$$

where $\mathbf{e}_1:=(1,0),\mathbf{e}_2:=(0,1)\in\mathbb{Z}^2$ and the terminology "Hermitian conjugate" is shortened to "h.c", meaning that the adjoint operators of the operators in front are placed. The parameters t, U_{c}, U_{o}, $\epsilon^\sigma_{\mathrm{c}}$, $\epsilon^\sigma_{\mathrm{o}}$ ($\sigma\in\{\uparrow,\downarrow\}$) are initially set to be real. The parameter t is the hopping amplitude between a Cu site and the neighboring O sites. The parameters $\epsilon^\sigma_{\mathrm{c}}$ and $\epsilon^\sigma_{\mathrm{o}}$ represent the on-site energy minus the hole chemical potential for the Cu sites and the O sites, respectively. We assume that the quadratic Hamiltonian H_0 may contain the contribution from the magnetic field such as $h_{\mathrm{c}}\sum_{\mathbf{x}\in\Gamma}S^z_{1,\mathbf{x}}+h_{\mathrm{o}}\sum_{(\rho,\mathbf{x})\in\{2,3\}\times\Gamma}S^z_{\rho,\mathbf{x}}$ with $h_{\mathrm{c}},h_{\mathrm{o}}\in\mathbb{R}$, $S^z_{\rho,\mathbf{x}}:=\frac{1}{2}(\psi^*_{\rho\mathbf{x}\uparrow}\psi_{\rho\mathbf{x}\uparrow}-\psi^*_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\downarrow})$ ($\rho\in\{1,2,3\},\mathbf{x}\in\Gamma$). This is the reason why $\epsilon^\sigma_{\mathrm{c}}$ and $\epsilon^\sigma_{\mathrm{o}}$ are defined to be spin-dependent. The strength of the on-site interaction is expressed by U_{c} on the Cu sites and by U_{o} on the O sites.

Let $\beta>0$ denote the inverse of temperature times the Boltzmann constant. The thermal expectation value of an observable O is defined as $\mathrm{Tr}(e^{-\beta H}O)/\mathrm{Tr}\,e^{-\beta H}$, where the trace is taken over the Fock space $F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}))$. For conciseness we write $\langle O\rangle_L$ in place of $\mathrm{Tr}(e^{-\beta H}O)/\mathrm{Tr}\,e^{-\beta H}$.

1.3. Exponential Decay Property of the Correlation Functions

Let $\|\cdot\|_{\mathbb{R}^2}$ denote the Euclidean norm of $\mathbb{R}^2$ and $e(\approx 2.71828)$ be the base of the natural logarithms. This paper is devoted to establish the following theorem:

Theorem 1.1. *There exist non-decreasing positive functions $f_1(\cdot)$, $f_2(\cdot) : \mathbb{R}_{\ge 1} \to \mathbb{R}_{>0}$ such that if*

$$|U_{\mathrm{c}}|, |U_{\mathrm{o}}| \le \frac{1}{f_1\left(\max_{\sigma\in\{\uparrow,\downarrow\}}\{1, |t|, |\epsilon_{\mathrm{c}}^{\sigma}|, |\epsilon_{\mathrm{o}}^{\sigma}|\}\right) \max\{1, \beta^{16}\}\beta}, \tag{1.1}$$

$\lim_{L\to\infty, L\in\mathbb{N}} \langle \psi^*_{\hat{\rho}_1\hat{\mathbf{x}}_1\hat{\sigma}_1}\psi^*_{\hat{\rho}_2\hat{\mathbf{x}}_2\hat{\sigma}_2}\psi_{\hat{\eta}_2\hat{\mathbf{y}}_2\hat{\tau}_2}\psi_{\hat{\eta}_1\hat{\mathbf{y}}_1\hat{\tau}_1} + h.c\rangle_L$ *exists and satisfies that*

$$\begin{aligned} &\left|\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \langle \psi^*_{\hat{\rho}_1\hat{\mathbf{x}}_1\hat{\sigma}_1}\psi^*_{\hat{\rho}_2\hat{\mathbf{x}}_2\hat{\sigma}_2}\psi_{\hat{\eta}_2\hat{\mathbf{y}}_2\hat{\tau}_2}\psi_{\hat{\eta}_1\hat{\mathbf{y}}_1\hat{\tau}_1} + h.c\rangle_L\right| \\ &\quad\le f_2\left(\max_{\sigma\in\{\uparrow,\downarrow\}}\{1, |t|, |\epsilon_{\mathrm{c}}^{\sigma}|, |\epsilon_{\mathrm{o}}^{\sigma}|\}\right)\max\{1,\beta^{16}\} \\ &\quad\quad\cdot\left(\frac{1}{\max\{1,t^2\}\max\{\beta,\beta^2\}}+1\right)^{-\frac{1}{8e}\|\sum_{j=1}^{2}(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j - \hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j)\|_{\mathbb{R}^2}}, \end{aligned} \tag{1.2}$$

for any $(\hat{\rho}_j, \hat{\mathbf{x}}_j, \hat{\sigma}_j), (\hat{\eta}_j, \hat{\mathbf{y}}_j, \hat{\tau}_j) \in \{1,2,3\}\times\mathbb{Z}^2\times\{\uparrow,\downarrow\}$ $(j=1,2)$, $t, \epsilon_{\mathrm{c}}^{\sigma}, \epsilon_{\mathrm{o}}^{\sigma} \in \mathbb{R}$ $(\sigma\in\{\uparrow,\downarrow\})$, $\beta\in\mathbb{R}_{>0}$ and any map $\hat{s}(\cdot) : \{\uparrow,\downarrow\}\to\{1,-1\}$.

Remark 1.2. The correlation function $\langle \psi^*_{\hat{\rho}_1\hat{\mathbf{x}}_1\hat{\sigma}_1}\psi^*_{\hat{\rho}_2\hat{\mathbf{x}}_2\hat{\sigma}_2}\psi_{\hat{\eta}_2\hat{\mathbf{y}}_2\hat{\tau}_2}\psi_{\hat{\eta}_1\hat{\mathbf{y}}_1\hat{\tau}_1} + \mathrm{h.c}\rangle_L$ is defined for $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2 \in \mathbb{Z}^2$ by considering $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2$ as the corresponding sites in Γ by periodicity.

Remark 1.3. As a result of our proof, the growth rates of $f_1(\cdot)$, $f_2(\cdot)$ are estimated as $f_1(x) = O(x^{44})$, $f_2(x) = O(x^{36})$ $(x\to\infty)$. However, since it is not the main aim of our analysis, these orders are not quantitatively optimized.

Remark 1.4. The theorem provides decay bounds on the thermodynamic limit of the correlation functions for singlet Cooper pairs. For instance, let us define the s-wave pairing operator $\Delta_{\mathrm{s}}(\rho,\mathbf{x})$, the extended s-wave pairing operator $\Delta_{\mathrm{s}^*}(\rho,\mathbf{x})$, and the $d_{x^2-y^2}$-wave pairing operator $\Delta_{d_{x^2-y^2}}(\rho,\mathbf{x})$ as follows: for $(\rho,\mathbf{x})\in\{1,2,3\}\times\Gamma$,

$$\begin{aligned} \Delta_{\mathrm{s}}(\rho,\mathbf{x}) &:= \psi_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow}, \\ \Delta_{\mathrm{s}^*}(\rho,\mathbf{x}) &:= \frac{1}{2}(\psi_{\rho(\mathbf{x}+\mathbf{e}_1)\downarrow}\psi_{\rho\mathbf{x}\uparrow} + \psi_{\rho(\mathbf{x}-\mathbf{e}_1)\downarrow}\psi_{\rho\mathbf{x}\uparrow} + \psi_{\rho(\mathbf{x}+\mathbf{e}_2)\downarrow}\psi_{\rho\mathbf{x}\uparrow} \\ &\quad + \psi_{\rho(\mathbf{x}-\mathbf{e}_2)\downarrow}\psi_{\rho\mathbf{x}\uparrow}), \\ \Delta_{d_{x^2-y^2}}(\rho,\mathbf{x}) &:= \frac{1}{2}(\psi_{\rho(\mathbf{x}+\mathbf{e}_1)\downarrow}\psi_{\rho\mathbf{x}\uparrow} + \psi_{\rho(\mathbf{x}-\mathbf{e}_1)\downarrow}\psi_{\rho\mathbf{x}\uparrow} - \psi_{\rho(\mathbf{x}+\mathbf{e}_2)\downarrow}\psi_{\rho\mathbf{x}\uparrow} \\ &\quad - \psi_{\rho(\mathbf{x}-\mathbf{e}_2)\downarrow}\psi_{\rho\mathbf{x}\uparrow}). \end{aligned}$$

If the map $\hat{s}(\cdot) : \{\uparrow,\downarrow\}\to\{1,-1\}$ is identically 1, the theorem shows that $|\lim_{L\to\infty,L\in\mathbb{N}}\langle\Delta_a(\hat{\rho},\hat{\mathbf{x}})^*\Delta_a(\hat{\eta},\hat{\mathbf{y}}) + \mathrm{h.c}\rangle_L|$ decays exponentially with $\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^2}$ for $\hat{\rho},\hat{\eta}\in\{1,2,3\}$, $a = s, s^*, d_{x^2-y^2}$. If we take $\hat{s}(\cdot)$ to obey $\hat{s}(\uparrow) = -\hat{s}(\downarrow)$, on the other hand, the theorem also implies exponential decay of spin–spin correlation functions of the form $\lim_{L\to\infty,L\in\mathbb{N}}\langle S^x_{\hat{\rho},\hat{\mathbf{x}}}S^x_{\hat{\eta},\hat{\mathbf{y}}} + S^y_{\hat{\rho},\hat{\mathbf{x}}}S^y_{\hat{\eta},\hat{\mathbf{y}}}\rangle_L$ with $\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^2}$, where the spin operators $S^x_{\rho,\mathbf{x}}$, $S^y_{\rho,\mathbf{x}}$ are defined by $S^x_{\rho,\mathbf{x}} := \frac{1}{2}(\psi^*_{\rho\mathbf{x}\uparrow}\psi_{\rho\mathbf{x}\downarrow} + \psi^*_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow})$, $S^y_{\rho,\mathbf{x}} := \frac{1}{2}(-i\psi^*_{\rho\mathbf{x}\uparrow}\psi_{\rho\mathbf{x}\downarrow} + i\psi^*_{\rho\mathbf{x}\downarrow}\psi_{\rho\mathbf{x}\uparrow})$ $((\rho,\mathbf{x}) \in \{1,2,3\} \times \Gamma)$.

Remark 1.5. The coupling constants U_{c}, U_{o} satisfying (1.1) can be taken arbitrarily large as $\beta \searrow 0$. This means that the theorem generally proves exponential decay of the correlation functions in high temperatures.

Remark 1.6. Consider the case that $\epsilon^\sigma_{\mathrm{c}} = -\frac{1}{2}U_{\mathrm{c}}$ and $\epsilon^\sigma_{\mathrm{o}} = -\frac{1}{2}U_{\mathrm{o}}$ $(\forall \sigma \in \{\uparrow,\downarrow\})$. The Hamiltonian H becomes invariant under the transform $\psi_{1\mathbf{x}\sigma} \to \psi^*_{1\mathbf{x}\sigma}$, $\psi^*_{1\mathbf{x}\sigma} \to \psi_{1\mathbf{x}\sigma}$, $\psi_{\rho\mathbf{x}\sigma} \to -\psi^*_{\rho\mathbf{x}\sigma}$, $\psi^*_{\rho\mathbf{x}\sigma} \to -\psi_{\rho\mathbf{x}\sigma}$ $(\rho \in \{2,3\}, (\mathbf{x},\sigma) \in \Gamma \times \{\uparrow,\downarrow\})$. This invariance implies that $\langle \psi^*_{\rho\mathbf{x}\sigma}\psi_{\rho\mathbf{x}\sigma}\rangle_L = \frac{1}{2}$ $(\forall(\rho,\mathbf{x},\sigma) \in \{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\})$ and thus the system is half-filled. According to our construction, $f_1(1) > 1$. If $\beta > 1$, the constraint (1.1) implies $|U_{\mathrm{c}}|$, $|U_{\mathrm{o}}| < 1$. Therefore, we can claim the theorem for $\beta > 1$ by eliminating $\epsilon^\sigma_{\mathrm{c}}, \epsilon^\sigma_{\mathrm{o}}$ $(\sigma \in \{\uparrow,\downarrow\})$ in the right-hand sides of (1.1) and (1.2). On the other hand, for arbitrarily large $|U_{\mathrm{c}}|$, $|U_{\mathrm{o}}|$ there exists $\beta \le 1$ such that (1.1) holds. Thus, the theorem concludes the exponential decay of correlation functions with the strong couplings if the temperature is high enough.

Remark 1.7. A power-law decay property of equal-time 4-point correlation functions can be proved by exactly following the argument of [12]. One result is that

$$\limsup_{\substack{L\to\infty\\ L\in\mathbb{N}}} |\langle \psi^*_{\hat{\rho}_1\hat{\mathbf{x}}\hat{\sigma}_1}\psi^*_{\hat{\rho}_2\hat{\mathbf{x}}\hat{\sigma}_2}\psi_{\hat{\eta}_2\hat{\mathbf{y}}\hat{\tau}_2}\psi_{\hat{\eta}_1\hat{\mathbf{y}}\hat{\tau}_1} + \mathrm{h.c}\rangle_L| \le 2\|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|_{\mathbb{R}^2}^{-\tilde{c}f(\beta)},$$

for any $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{Z}^2$ with sufficiently large $\|\hat{\mathbf{x}}-\hat{\mathbf{y}}\|_{\mathbb{R}^2}$, $(\hat{\rho}_j,\hat{\sigma}_j), (\hat{\eta}_j,\hat{\tau}_j) \in \{1,2,3\} \times \{\uparrow,\downarrow\}$ $(j = 1,2)$ and $\beta \in \mathbb{R}_{>0}$, where $\tilde{c} > 0$ is a constant, the function $f(\cdot) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is decreasing, and asymptotically behaves as $f(\beta) = O(\beta^{-1})$ $(\beta \to \infty)$, $O(|\log\beta|)$ $(\beta \searrow 0)$. An advantage of the framework [12], apart from its conciseness, is that it requires no constraint on the magnitude of the interactions. However, it has not been applied to prove exponential decay of correlations in 2D many-electron systems, to the author's knowledge.

2. Formulation

In this section we formulate the correlation function using the notion of Grassmann integral and show that the Grassmann integral representation of the correlation function multiplied by the distance between the holes and the electrons is transformed into a contour integral of the Grassmann integral. This procedure is essentially the same as we did in [10,11]. In order to avoid unnecessary repetition we present the proofs at a minimum.

Let us introduce notations which are used throughout the paper. For simplicity set $E_{\max} := \max_{\sigma\in\{\uparrow,\downarrow\}}\{1, |t|, |\epsilon^\sigma_{\mathrm{c}}|, |\epsilon^\sigma_{\mathrm{o}}|\}$. The sites on which the 4-point correlation function is defined are fixed to be $(\hat{\rho}_1,\hat{\mathbf{x}}_1,\hat{\sigma}_1)$, $(\hat{\rho}_2,\hat{\mathbf{x}}_2,\hat{\sigma}_2)$, $(\hat{\eta}_1,\hat{\mathbf{y}}_1,\hat{\tau}_1)$, $(\hat{\eta}_2,\hat{\mathbf{y}}_2,\hat{\tau}_2) \in \{1,2,3\} \times \mathbb{Z}^2 \times \{\uparrow,\downarrow\}$. We simply write $\hat{\mathcal{X}}_j$, $\hat{\mathcal{Y}}_j$ $(j = 1,2)$ instead of $(\hat{\rho}_j,\hat{\mathbf{x}}_j,\hat{\sigma}_j)$, $(\hat{\eta}_j,\hat{\mathbf{y}}_j,\hat{\tau}_j)$ $(j = 1,2)$, respectively. We also fix a map $\hat{s}(\cdot) : \{\uparrow,\downarrow\} \to \{1,-1\}$. Let us accept that a site of $\mathbb{Z}^2$ is identified as the corresponding site of Γ whenever we consider a problem in Γ. For $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n$, $\langle \mathbf{x}, \mathbf{y}\rangle := \sum_{j=1}^n x_j y_j$, $\langle \mathbf{x}, \mathbf{y}\rangle_{\mathbb{C}^n} := \sum_{j=1}^n x_j \overline{y_j}$ and $\|\mathbf{x}\|_{\mathbb{C}^n} := \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle_{\mathbb{C}^n}}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer which does not exceed x. Let $1_P := 1$ if the proposition P is true, $1_P := 0$ otherwise. For any subset $\mathcal{O}$ of a topological space let $\mathcal{O}^i$ denote the interior of $\mathcal{O}$. Let $\mathbb{S}_n$ be the set of all permutations over $\{1, 2, \dots, n\}$ ($n \in \mathbb{N}$). It will be convenient to use the function $\mathcal{F}_{t,\beta}(\cdot) : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{F}_{t,\beta}(x) := \frac{1}{2}\sinh^{-1}\left(\frac{x\pi^2}{8\max\{1,t^2\}\max\{\beta,\beta^2\}}\right).$$

Here recall that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$.

The correlation function will be formulated as a limit of Grassmann integration over a finite-dimensional Grassmann algebra. The reduction to the finite-dimensional problem is done by discretizing the integrals over the interval $[0,\beta)$ in the perturbative expansion of the partition function. For this purpose, take a parameter $h \in 2\mathbb{N}/\beta$ and set $[0,\beta)_h := \{0, 1/h, 2/h, \dots, \beta - 1/h\}$, $[-\beta,\beta)_h := \{-\beta, -\beta + 1/h, \dots, -1/h\} \cup [0,\beta)_h$. Note that $\sharp[0,\beta)_h = \beta h$, $\sharp[-\beta,\beta)_h = 2\beta h$. We have seen in [10, Appendix C] that taking the parameter h from $2\mathbb{N}/\beta$ rather than from $\mathbb{N}/\beta$ is convenient for the discretization of $[0,\beta)$ and $[-\beta,\beta)$. Set $I_{L,h} := \{1,2,3\} \times \Gamma \times \{\uparrow,\downarrow\} \times [0,\beta)_h$ and $N_{L,h} := \sharp I_{L,h} = 6L^2\beta h$. We define the lattice of the momentum variable Γ^* and the subset of the Matsubara frequency $\mathcal{M}_h$ by $\Gamma^* := (\frac{2\pi}{L}\mathbb{Z}/(2\pi\mathbb{Z}))^2$ and $\mathcal{M}_h := \{\omega \in \pi(2\mathbb{Z}+1)/\beta \mid |\omega| < \pi h\}$.

2.1. The Grassmann Gaussian Integral

Here let us summarize the notion of Grassmann Gaussian integral. For a finite-dimensional complex vector space W and $n \in \mathbb{N}$, let $\bigwedge^n W$ denote the n-fold anti-symmetric tensor product of W and $\bigwedge^0 W := \mathbb{C}$. Moreover, set $\bigwedge W := \bigoplus_{n=0}^{\dim W} \bigwedge^n W$.

Let $\mathcal{V}$, $\mathcal{V}^+$, $\mathcal{V}^-$, $\mathcal{V}_p$ ($p \in \mathbb{N}$) be the complex vector spaces spanned by the basis $\{\overline{\psi}_X, \psi_X\}_{X\in I_{L,h}}$, $\{\overline{\psi}_X\}_{X\in I_{L,h}}$, $\{\psi_X\}_{X\in I_{L,h}}$, $\{\overline{\psi}^p_X, \psi^p_X\}_{X\in I_{L,h}}$ ($p \in \mathbb{N}$), respectively. This paper concerns various problems formulated in the Grassmann algebras $\bigwedge\mathcal{V}$, $\bigwedge\mathcal{V}^+$, $\bigwedge\mathcal{V}^-$, $\bigwedge\mathcal{V}_p$ ($p \in \mathbb{N}$). Remark that there is a vector space isomorphism between $\bigwedge\mathcal{V}$ and $(\bigwedge\mathcal{V}^+)\otimes(\bigwedge\mathcal{V}^-)$, the tensor product of $\bigwedge\mathcal{V}^+$ and $\bigwedge\mathcal{V}^-$. Then, let $\mathcal{P}_n : \bigwedge\mathcal{V} \to (\bigwedge^n\mathcal{V}^+)\otimes(\bigwedge^n\mathcal{V}^-)$ denote the standard projection ($n \in \{0, 1, \dots, N_{L,h}\}$).

Let us give a number from 1 to $N_{L,h}$ to each element of $I_{L,h}$ so that we can write $I_{L,h} = \{X_{o,j}\}_{j=1}^{N_{L,h}}$. Set $\psi := (\overline{\psi}_{X_{o,1}}, \dots, \overline{\psi}_{X_{o,N_{L,h}}}, \psi_{X_{o,1}}, \dots, \psi_{X_{o,N_{L,h}}})$, $\psi^p := (\overline{\psi}^p_{X_{o,1}}, \dots, \overline{\psi}^p_{X_{o,N_{L,h}}}, \psi^p_{X_{o,1}}, \dots, \psi^p_{X_{o,N_{L,h}}})$ ($p \in \mathbb{N}$). Take $p, q_1, \dots, q_n \in \mathbb{N}$ with $p \neq q_j$ ($\forall j \in \{1, \dots, n\}$). The Grassmann Gaussian integral $\int \cdot d\mu_C(\psi^p)$ with a covariance $(C(X,Y))_{X,Y\in I_{L,h}}$ is a linear map from $\bigwedge\left(\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)\bigoplus\mathcal{V}_p\right)$ to $\bigwedge\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)$ defined as follows. For $f \in \bigwedge\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)$ and $X_1, \dots, X_a$,

$Y_1,\dots,Y_b \in I_{L,h}$,

$$\int f\, \mathrm{d}\mu_C(\psi^p) := f,$$

$$\int f\overline{\psi}^p_{X_1}\cdots\overline{\psi}^p_{X_a}\psi^p_{Y_b}\cdots\psi^p_{Y_1}\,\mathrm{d}\mu_C(\psi^p) := \begin{cases}\det(C(X_j,Y_k))_{1\le j,k\le a} f & \text{if } a=b,\\ 0 & \text{if } a\neq b.\end{cases}$$

Then for any $g \in \bigwedge\left(\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)\bigoplus \mathcal{V}_p\right)$, $\int g\,\mathrm{d}\mu_C(\psi^p)$ can be defined by linearity and anti-symmetry.

Though it is not used during the formulation in this section, let us recall the notion of left derivative at this stage for later use. For $X' \in I_{L,h}$ the left derivative $\partial/\partial\psi^p_{X'}$ is a linear operator on $\bigwedge\left(\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)\bigoplus \mathcal{V}_p\right)$. By letting $\mathcal{V}'_p$ be the vector space with the basis $\{\overline{\psi}^p_X,\psi^p_X\}_{X\in I_{L,h}}\backslash\{\psi^p_{X'}\}$,

$$\frac{\partial}{\partial\psi^p_{X'}}(f\psi^p_{X'}g) := (-1)^m fg,\quad \frac{\partial}{\partial\psi^p_{X'}}g := 0,$$

for $f \in \bigwedge^m\left(\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)\bigoplus \mathcal{V}'_p\right)$ $(m\in\mathbb{N}\cup\{0\})$, $g \in \bigwedge\left(\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)\bigoplus \mathcal{V}'_p\right)$. Then, $(\partial/\partial\psi^p_{X'})g$ can be defined for any $g \in \bigwedge\left(\left(\bigoplus_{j=1}^n \mathcal{V}_{q_j}\right)\bigoplus \mathcal{V}_p\right)$ by linearity. The definition of the left derivative $\partial/\partial\overline{\psi}^p_{X'}$ is parallel to that of $\partial/\partial\psi^p_{X'}$.

2.2. The Covariance

In our formulation the covariance is given as a 2-point correlation function governed by the free Hamiltonian H_0. For $(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)$,

$$\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y) := \frac{\mathrm{Tr}(\mathrm{e}^{-\beta H_0}T(\psi^*_{\rho\mathbf{x}\sigma}(x)\psi_{\eta\mathbf{y}\tau}(y)))}{\mathrm{Tr}\,\mathrm{e}^{-\beta H_0}},$$

where $\psi^*_{\rho\mathbf{x}\sigma}(x) := \mathrm{e}^{xH_0}\psi^*_{\rho\mathbf{x}\sigma}\mathrm{e}^{-xH_0}$, $\psi_{\eta\mathbf{y}\tau}(y) := \mathrm{e}^{yH_0}\psi_{\eta\mathbf{y}\tau}\mathrm{e}^{-yH_0}$, $T(\psi^*_{\rho\mathbf{x}\sigma}(x)\psi_{\eta\mathbf{y}\tau}(y)) := 1_{x\ge y}\psi^*_{\rho\mathbf{x}\sigma}(x)\psi_{\eta\mathbf{y}\tau}(y) - 1_{x<y}\psi_{\eta\mathbf{y}\tau}(y)\psi^*_{\rho\mathbf{x}\sigma}(x)$.

The following characterization of $\mathcal{C}$ is done in Appendix A. For any $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y)\in I_{L,h}$,

$$\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y) = \frac{\delta_{\sigma,\tau}}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{i(x-y)\omega}\mathcal{B}^\sigma_{\rho,\eta}(\mathbf{k},\omega), \tag{2.1}$$

where for $\mathbf{k}=(k_1,k_2)\in\Gamma^*$, $\omega\in\mathcal{M}_h$, $\sigma\in\{\uparrow,\downarrow\}$,

$$\left(\mathcal{B}^\sigma_{\rho,\eta}(\mathbf{k},\omega)\right)_{1\le\rho,\eta\le3} := \begin{pmatrix} \frac{\mathcal{N}^\sigma_{1,1}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)} & \frac{\mathcal{N}^\sigma_{1,2}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)} & \frac{\mathcal{N}^\sigma_{1,3}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)} \\ \frac{\mathcal{N}^\sigma_{2,1}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)} & \frac{1}{h(1-\mathrm{e}^{-i\omega/h+\epsilon^\sigma_o/h})}\left(1+\frac{\mathcal{N}^\sigma_{2,2}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)}\right) & \frac{\mathcal{N}^\sigma_{2,3}(\mathbf{k},\omega)}{h(1-\mathrm{e}^{-i\omega/h+\epsilon^\sigma_o/h})\mathcal{D}^\sigma(\mathbf{k},\omega)} \\ \frac{\mathcal{N}^\sigma_{3,1}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)} & \frac{\mathcal{N}^\sigma_{3,2}(\mathbf{k},\omega)}{h(1-\mathrm{e}^{-i\omega/h+\epsilon^\sigma_o/h})\mathcal{D}^\sigma(\mathbf{k},\omega)} & \frac{1}{h(1-\mathrm{e}^{-i\omega/h+\epsilon^\sigma_o/h})}\left(1+\frac{\mathcal{N}^\sigma_{3,3}(\mathbf{k},\omega)}{\mathcal{D}^\sigma(\mathbf{k},\omega)}\right) \end{pmatrix},$$

$$
\begin{aligned}
\mathcal{D}^\sigma(\mathbf{k},\omega) &:= h^2\left(1-\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}\right)^2\\
&\quad -\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}\left(\frac{(\epsilon_{\mathrm{c}}^\sigma-\epsilon_{\mathrm{o}}^\sigma)^2}{4}+2t^2\sum_{j=1}^{2}(1+\cos k_j)\right)\\
&\quad +\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}O_1^\sigma(\mathbf{k}),\\
\mathcal{N}_{1,1}^\sigma(\mathbf{k},\omega) &:= h\left(1-\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}\right)+\frac{\epsilon_{\mathrm{c}}^\sigma-\epsilon_{\mathrm{o}}^\sigma}{2}\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}\\
&\quad +\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}O_2^\sigma(\mathbf{k}),\\
\mathcal{N}_{1,2}^\sigma(\mathbf{k},\omega) &:= t(1+\mathrm{e}^{ik_1})\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}(1+O_3^\sigma(\mathbf{k})),\\
\mathcal{N}_{1,3}^\sigma(\mathbf{k},\omega) &:= \mathcal{N}_{1,2}^\sigma((k_2,k_1),\omega),\\
\mathcal{N}_{2,1}^\sigma(\mathbf{k},\omega) &:= \mathcal{N}_{1,2}^\sigma(-\mathbf{k},\omega),\\
\mathcal{N}_{2,2}^\sigma(\mathbf{k},\omega) &:= 2t^2(1+\cos k_1)\Big(\frac{1}{2}\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}+\frac{1}{2}\mathrm{e}^{-\frac{2i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+3\epsilon_{\mathrm{o}}^\sigma)}\\
&\quad +\left(\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}+\mathrm{e}^{-\frac{2i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+3\epsilon_{\mathrm{o}}^\sigma)}\right)O_4^\sigma(\mathbf{k})\\
&\quad +\left(\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}-\mathrm{e}^{-\frac{2i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+3\epsilon_{\mathrm{o}}^\sigma)}\right)O_5^\sigma(\mathbf{k})\Big),\\
\mathcal{N}_{2,3}^\sigma(\mathbf{k},\omega) &:= t^2(1+\mathrm{e}^{-ik_1})(1+\mathrm{e}^{ik_2})\Big(\frac{1}{2}\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}+\frac{1}{2}\mathrm{e}^{-\frac{2i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+3\epsilon_{\mathrm{o}}^\sigma)}\\
&\quad +\left(\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}+\mathrm{e}^{-\frac{2i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+3\epsilon_{\mathrm{o}}^\sigma)}\right)O_4^\sigma(\mathbf{k})\\
&\quad +\left(\mathrm{e}^{-\frac{i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+\epsilon_{\mathrm{o}}^\sigma)}-\mathrm{e}^{-\frac{2i}{h}\omega+\frac{1}{2h}(\epsilon_{\mathrm{c}}^\sigma+3\epsilon_{\mathrm{o}}^\sigma)}\right)O_5^\sigma(\mathbf{k})\Big),\\
\mathcal{N}_{3,1}^\sigma(\mathbf{k},\omega) &:= \mathcal{N}_{1,2}^\sigma(-(k_2,k_1),\omega),\quad \mathcal{N}_{3,2}^\sigma(\mathbf{k},\omega):=\mathcal{N}_{2,3}^\sigma(-\mathbf{k},\omega),\\
\mathcal{N}_{3,3}^\sigma(\mathbf{k},\omega) &:= \mathcal{N}_{2,2}^\sigma((k_2,k_1),\omega).
\end{aligned}
\tag{2.2}
$$

The functions $O_j^\sigma(\cdot):\mathbb{C}^2\to\mathbb{C}$ ($j\in\{1,\dots,5\},\sigma\in\{\uparrow,\downarrow\}$) are entirely analytic and satisfy that $O_j^\sigma(\mathbf{k}+2\pi m\mathbf{e}_1+2\pi n\mathbf{e}_2)=O_j^\sigma(\mathbf{k})$ ($\forall\mathbf{k}\in\mathbb{C}^2,m,n\in\mathbb{Z}$). Moreover, for any compact set $K\subset\mathbb{C}^2$,

$$
\sup_{\mathbf{k}\in K,j\in\{1,\dots,5\},\sigma\in\{\uparrow,\downarrow\}}|O_j^\sigma(\mathbf{k})|\le\frac{C_{K,E_{\max}}}{h},\tag{2.3}
$$

where $C_{K,E_{\max}}$ is a positive constant depending only on K and $E_{\max}$. Though these information about O_j^σ are sufficient for our analysis to proceed, the functions O_j^σ are made explicit in (A.7) in Appendix A.

Remark 2.1. The functions $\mathcal{D}^\sigma(\mathbf{k},\omega)$, $\mathcal{N}_{\rho,\eta}^\sigma(\mathbf{k},\omega)$ ($\rho,\eta\in\{1,2,3\},\sigma\in\{\uparrow,\downarrow\}$) are analytic with respect to $\mathbf{k}$. This property is one essential requirement of our method to prove exponential decay of the correlation functions. As shown in Appendix A, in the preliminary form before being expanded over $\mathcal{M}_h$ the covariance $\mathcal{C}(X,Y)$ contains a square root of $(\epsilon_{\mathrm{c}}^\sigma-\epsilon_{\mathrm{o}}^\sigma)^2/t^2+8\sum_{j=1}^2(1+\cos k_j)$, which is not analytic. In order to make the analyticity with $\mathbf{k}$ apparent, we choose to transform the covariance into the sum over $\Gamma^*\times\mathcal{M}_h$.

Remark 2.2. The dispersion relation for the free particle hopping to the nearest neighbor sites on the CuO lattice is given by (A.1) in Appendix A. As discussed in Remark 1.6, taking ϵ_{c}^{σ}, ϵ_{o}^{σ} to be $-U_{c}/2$, $-U_{o}/2$, respectively, makes the system half-filled. If we shift the on-site quadratic term to the interacting part of the Hamiltonian, one of the dispersion relation denoted by $A_1^{\sigma}(t,\mathbf{k})$ in (A.1) is changed into 0. The formulation including the quadratic term in the interacting part is parallel to the formulation of the half-filled honeycomb lattice model in [8], though in [8] the quadratic term is eventually erased by the non-corresponding property of the covariance at equal space-time. One remarkable fact is that the zero set of the free particle dispersion relation in the half-filled formulation of the CuO Hubbard model is, thus, the whole momentum space, while that is the contour of a square in the half-filled Hubbard model on the square lattice (see, e.g, [14]) and that consists of two distinct points in the half-filled Hubbard model on the honeycomb lattice (see [8]). This suggests that trying to improve the temperature dependency of the convergence theory in the half-filled CuO Hubbard model would require a qualitatively different method from the infrared integration regimes for the half-filled 2D Hubbard model developed so far, in which the degeneracy of the zero set of the dispersion relation is crucial.

2.3. The Grassmann Integral Formulation

In order to relate the correlation function to the Grassmann Gaussian integral, we introduce parameters λ_1, $\lambda_{-1} \in \mathbb{C}$ and define $U_{(\lambda_1,\lambda_{-1})}(\cdot,\cdot,\cdot,\cdot) : (\{1,2,3\} \times \Gamma \times \{\uparrow,\downarrow\})^4 \to \mathbb{C}$ by

$$\begin{aligned}
&U_{(\lambda_1,\lambda_{-1})}(\rho_1\mathbf{x}_1\sigma_1,\rho_2\mathbf{x}_2\sigma_2,\eta_1\mathbf{y}_1\tau_1,\eta_2\mathbf{y}_2\tau_2)\\
&\quad := \frac{1}{4}(1_{(\sigma_1,\sigma_2)=(\uparrow,\downarrow)} - 1_{(\sigma_1,\sigma_2)=(\downarrow,\uparrow)})(1_{(\tau_1,\tau_2)=(\downarrow,\uparrow)} - 1_{(\tau_1,\tau_2)=(\uparrow,\downarrow)})1_{\mathbf{x}_1=\mathbf{x}_2=\mathbf{y}_1=\mathbf{y}_2}\\
&\qquad \cdot(U_c 1_{\rho_1=\rho_2=\eta_1=\eta_2=1} + U_o 1_{\rho_1=\rho_2=\eta_1=\eta_2=2 \text{ or } 3})\\
&\qquad +\frac{1}{4}\lambda_1(1_{(\rho_1\mathbf{x}_1\sigma_1,\rho_2\mathbf{x}_2\sigma_2)=(\hat{\mathcal{X}}_1,\hat{\mathcal{X}}_2)} - 1_{(\rho_1\mathbf{x}_1\sigma_1,\rho_2\mathbf{x}_2\sigma_2)=(\hat{\mathcal{X}}_2,\hat{\mathcal{X}}_1)})\\
&\qquad \cdot(1_{(\eta_1\mathbf{y}_1\tau_1,\eta_2\mathbf{y}_2\tau_2)=(\hat{\mathcal{Y}}_2,\hat{\mathcal{Y}}_1)} - 1_{(\eta_1\mathbf{y}_1\tau_1,\eta_2\mathbf{y}_2\tau_2)=(\hat{\mathcal{Y}}_1,\hat{\mathcal{Y}}_2)})\\
&\qquad +\frac{1}{4}\lambda_{-1}(1_{(\rho_1\mathbf{x}_1\sigma_1,\rho_2\mathbf{x}_2\sigma_2)=(\hat{\mathcal{Y}}_1,\hat{\mathcal{Y}}_2)} - 1_{(\rho_1\mathbf{x}_1\sigma_1,\rho_2\mathbf{x}_2\sigma_2)=(\hat{\mathcal{Y}}_2,\hat{\mathcal{Y}}_1)})\\
&\qquad \cdot(1_{(\eta_1\mathbf{y}_1\tau_1,\eta_2\mathbf{y}_2\tau_2)=(\hat{\mathcal{X}}_2,\hat{\mathcal{X}}_1)} - 1_{(\eta_1\mathbf{y}_1\tau_1,\eta_2\mathbf{y}_2\tau_2)=(\hat{\mathcal{X}}_1,\hat{\mathcal{X}}_2)}).
\end{aligned} \tag{2.4}$$

For another application in Sect. 4 we purposely defined $U_{(\lambda_1,\lambda_{-1})}(\cdot,\cdot,\cdot,\cdot)$ to satisfy $U_{(\lambda_1,\lambda_{-1})}(X_2,X_1,Y_1,Y_2) = U_{(\lambda_1,\lambda_{-1})}(X_1,X_2,Y_2,Y_1) = -U_{(\lambda_1,\lambda_{-1})}(X_1,X_2,Y_1,Y_2)$. Define the Grassmann polynomial $V_{(\lambda_1,\lambda_{-1})}(\psi) \in \bigwedge \mathcal{V}$ by

$$\begin{aligned}
&V_{(\lambda_1,\lambda_{-1})}(\psi)\\
&\quad := -\frac{1}{h}\sum_{x\in[0,\beta)_h}\sum_{\substack{X_1,X_2,Y_1,Y_2\\ \in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}} U_{(\lambda_1,\lambda_{-1})}(X_1,X_2,Y_1,Y_2)\overline{\psi}_{X_1x}\overline{\psi}_{X_2x}\psi_{Y_1x}\psi_{Y_2x}.
\end{aligned}$$

The Grassmann integral formulation of the correlation function is summarized as follows:

Lemma 2.3. (i) *For any* $U > 0$ *there exists* $N_U \in \mathbb{N}$ *such that* $\mathrm{Re} \int e^{V(\lambda,\lambda)(\psi)} d\mu_{\mathcal{C}}(\psi) > 0$ *for any* $h \in 2\mathbb{N}/\beta$ *with* $h \geq 2N_U/\beta$, $\lambda, U_c, U_o \in \mathbb{R}$ *with* $|\lambda|, |U_c|, |U_o| \leq U$.

(ii)

$$\langle \psi^*_{\hat{\mathcal{X}}_1} \psi^*_{\hat{\mathcal{X}}_2} \psi_{\hat{\mathcal{Y}}_2} \psi_{\hat{\mathcal{Y}}_1} + h.c \rangle_L = -\frac{1}{\beta} \lim_{\substack{h\to\infty \\ h\in 2\mathbb{N}/\beta}} \frac{\partial}{\partial\lambda} \log \left(\int e^{V(\lambda,\lambda)(\psi)} d\mu_{\mathcal{C}}(\psi) \right) \Big|_{\lambda=0},$$

where for $z \in \mathbb{C}$ *with* $\mathrm{Re}\, z > 0$, $\log z := \log |z| + i \,\mathrm{Arg}\, z$, $\mathrm{Arg}\, z \in (-\pi/2, \pi/2)$.

Lemma 2.3 can be proved in a way similar to [11, Section 3]. For the readers' convenience we outline the proof in Appendix B.

The analysis in the following sections treats the perturbed covariance containing complex momentum variables inside. For $\mathbf{p} \in \mathbb{C}^2$,

$$\mathcal{C}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)(\mathbf{p}) := \frac{\delta_{\sigma,\tau}}{\beta L^2} \sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h} e^{-i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle} e^{i(x-y)\omega} \mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k} + \hat{s}(\sigma)\mathbf{p}, \omega),$$

$\mathcal{C}(\mathbf{p}) := (\mathcal{C}(X,Y)(\mathbf{p}))_{X,Y\in I_{L,h}}$. By admitting a few facts proved in Sect. 3, we can show the next lemma. The equality in Lemma 2.4 (iii) will be estimated in Sect. 4 as the main objective.

Lemma 2.4. *For any* $L \in \mathbb{N}$, $R \in (\mathcal{F}_{t,\beta}(8/\pi^2), \infty)$, $\varepsilon \in (8/\pi^2, 1)$ *and sufficiently large* $h \in 2\mathbb{N}/\beta$ *there exists* $U_{\mathrm{small}} > 0$ *such that the following statements hold true:*

(i) $\mathrm{Re} \int e^{V(\lambda_1,\lambda_{-1})(\psi)} d\mu_{\mathcal{C}(w\mathbf{e}_p)}(\psi) > 0$ *for any* $p \in \{1,2\}$ *and all* $(\lambda_1, \lambda_{-1}, U_c, U_o, w) \in \mathbb{C}^5$ *with* $|\lambda_1|$, $|\lambda_{-1}|$, $|U_c|$, $|U_o| \leq U_{\mathrm{small}}$, $|\mathrm{Re}\, w| \leq R$, $|\mathrm{Im}\, w| \leq \mathcal{F}_{t,\beta}(\varepsilon)$.

(ii) *For any* $p \in \{1,2\}$ *the function*

$$(\lambda_1, \lambda_{-1}, U_c, U_o, w) \mapsto \log \left(\int e^{V(\lambda_1,\lambda_{-1})(\psi)} d\mu_{\mathcal{C}(w\mathbf{e}_p)}(\psi) \right)$$

is analytic in

$$\left\{ (\lambda_1, \lambda_{-1}, U_c, U_o, w) \in \mathbb{C}^5 \;\middle|\; \begin{array}{l} |\lambda_1|, |\lambda_{-1}|, |U_c|, |U_o| < U_{\mathrm{small}}, \\ |\mathrm{Re}\, w| < R, |\mathrm{Im}\, w| < \mathcal{F}_{t,\beta}(\varepsilon) \end{array} \right\}.$$

(iii) *For any* $n \in \mathbb{N}$ *with* $2\pi n/L + \mathcal{F}_{t,\beta}(8/\pi^2) < R$, $U_c, U_o \in \mathbb{C}$ *with* $|U_c|, |U_o| < U_{\mathrm{small}}$ *and* $p \in \{1,2\}$,

$$\left(\frac{L}{2\pi}\left(e^{i\frac{2\pi}{L}\langle\sum_{j=1}^{2}(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}-1\right)\right)^n$$

$$\cdot\frac{\partial}{\partial\lambda}\log\left(\int e^{V_{(\lambda,\lambda)}(\psi)}\,d\mu_{\mathcal{C}}(\psi)\right)\Bigg|_{\lambda=0}$$

$$=\sum_{a\in\{1,-1\}}\prod_{j=1}^{n}\Bigg(\frac{L}{2\pi}\int_0^{2\pi a/L}d\theta_{a,j}\frac{1}{2\pi i}$$

$$\cdot\oint_{|w_{a,j}-\theta_{a,j}|=\mathcal{F}_{t,\beta}(8/\pi^2)/n}dw_{a,j}\frac{1}{(w_{a,j}-\theta_{a,j})^2}\Bigg)$$

$$\cdot\frac{\partial}{\partial\lambda_a}\log\left(\int e^{V_{(\lambda_1,\lambda_{-1})}(\psi)}\,d\mu_{\mathcal{C}(\sum_{j=1}^{n}w_{a,j}\mathbf{e}_p)}(\psi)\right)\Bigg|_{\lambda_1=\lambda_{-1}=0},$$

where $\oint_{|w_{a,j}-\theta_{a,j}|=\mathcal{F}_{t,\beta}(8/\pi^2)/n}dw_{a,j}$ *represents the contour integral along the contour* $\{w_{a,j}\in\mathbb{C}\mid|w_{a,j}-\theta_{a,j}|=\mathcal{F}_{t,\beta}(8/\pi^2)/n\}$ *oriented counter clock-wise.*

Proof. (i): It follows from Lemma 3.3 (i), Lemma 3.4 (i) and (3.5) that the function $w\mapsto\mathcal{C}(X,Y)(w\mathbf{e}_p)$ is analytic in $\{w\in\mathbb{C}\mid|\mathrm{Im}\,w|<\mathcal{F}_{t,\beta}(\varepsilon')\}$ for any $\varepsilon'\in(0,1)$, sufficiently large $h\in2\mathbb{N}/\beta$ and $X,Y\in I_{L,h}$. Thus, for any fixed large $h\in2\mathbb{N}/\beta$, $|\mathcal{C}(X,Y)(w\mathbf{e}_p)|$ is uniformly bounded with respect to $X,Y\in I_{L,h}$ and $w\in\mathbb{C}$ with $|\mathrm{Re}\,w|\le R$, $|\mathrm{Im}\,w|\le\mathcal{F}_{t,\beta}(\varepsilon)$. Note that by definition $\int e^{V_{(\lambda_1,\lambda_{-1})}(\psi)}d\mu_{\mathcal{C}(w\mathbf{e}_p)}(\psi)$ is a polynomial of λ_1, λ_{-1}, U_c, U_o, whose constant term is 1 and higher order terms have finite sums and products of $\mathcal{C}(X,Y)(w\mathbf{e}_p)$ $(X,Y\in I_{L,h})$ in their coefficients. Thus, the uniform boundedness of $\mathcal{C}(X,Y)(w\mathbf{e}_p)$ ensures that

$$\lim_{U\searrow0}\sup_{\substack{(\lambda_1,\lambda_{-1},U_c,U_o,w)\in\mathbb{C}^5\\|\lambda_1|,|\lambda_{-1}|,|U_c|,|U_o|\le U,|\mathrm{Re}\,w|\le R,|\mathrm{Im}\,w|\le\mathcal{F}_{t,\beta}(\varepsilon)}}\cdot\left|\int e^{V_{(\lambda_1,\lambda_{-1})}(\psi)}\,d\mu_{\mathcal{C}(w\mathbf{e}_p)(\psi)}-1\right|=0,$$

which implies the claim (i).

(ii): The claim (i) and the analyticity of $\mathcal{C}(X,Y)(w\mathbf{e}_p)$ with respect to w verify the statement.

(iii): Set

$$S_1(\mathcal{C}):=-\frac{1}{h}\sum_{x\in[0,\beta)_h}\int\overline{\psi}_{\hat{\mathcal{X}}_1x}\overline{\psi}_{\hat{\mathcal{X}}_2x}\psi_{\hat{\mathcal{Y}}_2x}\psi_{\hat{\mathcal{Y}}_1x}e^{V_{(0,0)}(\psi)}\,d\mu_{\mathcal{C}}(\psi)\Big/\int e^{V_{(0,0)}(\psi)}\,d\mu_{\mathcal{C}}(\psi),$$

$$S_{-1}(\mathcal{C}):=-\frac{1}{h}\sum_{x\in[0,\beta)_h}\int\overline{\psi}_{\hat{\mathcal{Y}}_1x}\overline{\psi}_{\hat{\mathcal{Y}}_2x}\psi_{\hat{\mathcal{X}}_2x}\psi_{\hat{\mathcal{X}}_1x}e^{V_{(0,0)}(\psi)}\,d\mu_{\mathcal{C}}(\psi)\Big/\int e^{V_{(0,0)}(\psi)}\,d\mu_{\mathcal{C}}(\psi).$$

Note the equality that

$$\begin{aligned}&\mathrm{e}^{ai\frac{2\pi}{L}\langle\sum_{j=1}^{n}(\hat{s}(\sigma_j)\mathbf{x}_j-\hat{s}(\tau_j)\mathbf{y}_j),\mathbf{e}_p\rangle}\det\left(\mathcal{C}(\rho_j\mathbf{x}_j\sigma_jx_j,\eta_k\mathbf{y}_k\tau_ky_k)\right)_{1\le j,k\le n}\\&\quad=\det\left(\mathcal{C}(\rho_j\mathbf{x}_j\sigma_jx_j,\eta_k\mathbf{y}_k\tau_ky_k)\left(a\frac{2\pi}{L}\mathbf{e}_p\right)\right)_{1\le j,k\le n}\qquad(\forall a\in\{1,-1\})\end{aligned}$$

and the fact that $V_{(0,0)}(\psi)$ is invariant under the scaling $\overline{\psi}_{\rho\mathbf{x}\sigma x}\to\mathrm{e}^{ia\hat{s}(\sigma)\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_p\rangle}\overline{\psi}_{\rho\mathbf{x}\sigma x}$, $\psi_{\rho\mathbf{x}\sigma x}\to\mathrm{e}^{-ia\hat{s}(\sigma)\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_p\rangle}\psi_{\rho\mathbf{x}\sigma x}$ $(a\in\{1,-1\},(\rho,\mathbf{x},\sigma,x)\in I_{L,h})$. Then, by remarking the definition of the Grassmann Gaussian integral and the claim (ii) we can justify the following transformations:

$$\begin{aligned}&\mathrm{e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^{2}(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}\frac{\partial}{\partial\lambda}\log\left(\int\mathrm{e}^{V_{(\lambda,\lambda)}(\psi)}\,\mathrm{d}\mu_{\mathcal{C}}(\psi)\right)\Big|_{\lambda=0}\\&\quad=\sum_{a\in\{1,-1\}}\mathrm{e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^{2}(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}S_a(\mathcal{C})=\sum_{a\in\{1,-1\}}S_a\left(\mathcal{C}\left(a\frac{2\pi}{L}\mathbf{e}_p\right)\right).\\&\frac{L}{2\pi}\left(\mathrm{e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^{2}(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}-1\right)\frac{\partial}{\partial\lambda}\log\left(\int\mathrm{e}^{V_{(\lambda,\lambda)}(\psi)}\,\mathrm{d}\mu_{\mathcal{C}}(\psi)\right)\Big|_{\lambda=0}\\&\quad=\sum_{a\in\{1,-1\}}\frac{L}{2\pi}\left(S_a\left(\mathcal{C}\left(a\frac{2\pi}{L}\mathbf{e}_p\right)\right)-S_a(\mathcal{C}(\mathbf{0}))\right)\\&\quad=\sum_{a\in\{1,-1\}}\frac{L}{2\pi}\int_0^{2\pi a/L}\mathrm{d}\theta_a\frac{\mathrm{d}}{\mathrm{d}\theta_a}S_a\left(\mathcal{C}\left(\theta_a\mathbf{e}_p\right)\right)\\&\quad=\sum_{a\in\{1,-1\}}\frac{L}{2\pi}\int_0^{2\pi a/L}\mathrm{d}\theta_a\frac{1}{2\pi i}\\&\qquad\cdot\oint_{|w_a-\theta_a|=\mathcal{F}_{t,\beta}(8/\pi^2)/n}\mathrm{d}w_a\frac{1}{(w_a-\theta_a)^2}S_a(\mathcal{C}(w_a\mathbf{e}_p)).\end{aligned}$$

Repeating this procedure n times results in the equality claimed in (iii). □

3. Preliminaries

In this section we show some lemmas concerning the cut-off function and the sliced covariance, which are the necessary tools for the forthcoming multi-scale analysis. To begin with, let us fix a function $\phi\in C_0^\infty(\mathbb{R})$ with the following properties: (i) $\phi(x)=1$ if $|x|\le1$. (ii) $\phi(x)=0$ if $|x|\ge2$. (iii) $\phi(x)\in(0,1)$ if $1<|x|<2$ and is strictly increasing in $(-2,-1)$, strictly decreasing in $(1,2)$. See, e.g, [6, Problem II.6] for a concrete construction of such a function. From now let the notation 'c' stand for a generic positive constant which depends only on ϕ and is independent of any other parameters.

3.1. The Cut-Off Function

With a parameter $M \in \mathbb{R}_{>2}$ define the function $\chi \in C_0^\infty(\mathbb{R})$ by $\chi(x) := \phi((x-M)/(M^2-M)+1)$. We can see that $\chi(x)=1$ $(\forall x \in [0,M])$, $\chi(x)=0$ $(\forall x \in [M^2,\infty))$, $\chi(x) \in (0,1)$ $(\forall x \in (M,M^2))$, $\chi(\cdot)$ is strictly decreasing in (M,M^2) and

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \chi(x)\right| \le cM^{-2m} \quad (\forall x \in [0,\infty), \forall m \in \{0,\dots,4\}). \tag{3.1}$$

In the next subsection χ will be differentiated at most four times. Thus, it suffices to prepare the bound (3.1) only up to $m=4$.

Set $N_h := \lfloor \log(2h)/\log(M) \rfloor$ for $h \in 2\mathbb{N}/\beta$, $N_\beta := \max\{\lfloor \log(1/\beta)/\log(M)\rfloor + 1, 1\}$. For large $h \in 2\mathbb{N}/\beta$ satisfying $N_h \ge N_\beta + 1$ we have that

$$\frac{1}{\beta} < M^{N_\beta} \le \max\left\{1, \frac{1}{\beta}\right\} M, \tag{3.2}$$

$$M^l \le 2h \quad (\forall l \in \{N_\beta, N_\beta+1,\dots,N_h\}). \tag{3.3}$$

Define the functions $\chi_l(\cdot) : \mathbb{R} \to \mathbb{R}$ $(l \in \{N_\beta, N_\beta+1,\dots,N_h\})$ by

$$\begin{aligned}
\chi_{N_\beta}(\omega) &:= \chi(M^{-N_\beta} h|1-\mathrm{e}^{i\omega/h}|),\\
\chi_l(\omega) &:= \chi(M^{-l} h|1-\mathrm{e}^{i\omega/h}|) - \chi(M^{-(l-1)} h|1-\mathrm{e}^{i\omega/h}|)\\
&\quad (\forall l \in \{N_\beta+1,\dots,N_h\}).
\end{aligned}$$

Since $h|1-\mathrm{e}^{i\omega/h}| \le 2h \le M^{N_h+1}$, $\chi(M^{-N_h} h|1-\mathrm{e}^{i\omega/h}|) = 1$ $(\forall \omega \in \mathbb{R})$. This implies that

$$\sum_{l=N_\beta}^{N_h} \chi_l(\omega) = 1 \quad (\forall \omega \in \mathbb{R}). \tag{3.4}$$

The support property of these functions is described as follows: for any $\omega \in \mathbb{R}$,

$$\chi_{N_\beta}(\omega) = \begin{cases} 1 & \text{if } h|1-\mathrm{e}^{i\omega/h}| \le M^{N_\beta+1},\\ \in (0,1) & \text{if } M^{N_\beta+1} < h|1-\mathrm{e}^{i\omega/h}| < M^{N_\beta+2},\\ 0 & \text{if } h|1-\mathrm{e}^{i\omega/h}| \ge M^{N_\beta+2}, \end{cases}$$

$$\chi_l(\omega) = \begin{cases} 0 & \text{if } h|1-\mathrm{e}^{i\omega/h}| \le M^{l},\\ \in (0,1] & \text{if } M^{l} < h|1-\mathrm{e}^{i\omega/h}| < M^{l+2},\\ 0 & \text{if } h|1-\mathrm{e}^{i\omega/h}| \ge M^{l+2}, \end{cases}$$

$$(\forall l \in \{N_\beta+1,\dots,N_h\}).$$

The role of $\chi_l(\cdot)$ is a cut-off in the Matsubara frequency. The support of $\chi_l(\cdot)$ can be estimated as follows:

Lemma 3.1. *For any* $l \in \{N_\beta, N_\beta+1,\dots,N_h\}$, $\frac{1}{\beta}\sum_{\omega \in \mathcal{M}_h} 1_{\chi_l(\omega)\neq 0} \le cM^{l+2}$.

3.2. Properties of the Sliced Covariances

By using the cut-off function χ_l we define the covariance $\mathcal{C}_l$ of l-th scale ($l \in \{N_\beta, N_\beta+1, \dots, N_h\}$) by

$$\mathcal{C}_l(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)(\mathbf{p}) := \frac{\delta_{\sigma,\tau}}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h} \mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{i(x-y)\omega}\chi_l(\omega)\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+\hat{s}(\sigma)\mathbf{p},\omega)$$

for $(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y) \in I_{L,h}$, $\mathbf{p}\in\mathbb{C}^2$. Let $\mathcal{C}_l(\mathbf{p}) := (\mathcal{C}_l(X,Y)(\mathbf{p}))_{X,Y\in I_{L,h}}$. We will specify a domain where $\mathcal{C}_l(\cdot)$ is well defined later in this subsection. On such a domain the equality (3.4) implies that

$$\mathcal{C}(\mathbf{p}) = \sum_{l=N_\beta}^{N_h}\mathcal{C}_l(\mathbf{p}). \tag{3.5}$$

In this subsection we study various properties of $\mathcal{C}_l$. For this purpose set

$$E(t,\mathbf{k}) := 2t^2\sum_{j=1}^{2}(1+\cos k_j) : \mathbb{C}^2 \to \mathbb{C}, \tag{3.6}$$

and let us estimate $E(t,\mathbf{k})$, first of all.

Lemma 3.2. *For any $\mathbf{k}\in\mathbb{R}^2$, $j,p,q\in\{1,2\}$, $m\in\mathbb{N}\cup\{0\}$ and $w,z\in\mathbb{C}$ with $|\operatorname{Im} w|, |\operatorname{Im} z| \le r$,*

$$\left|\left(\frac{\partial}{\partial k_j}\right)^m E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q)\right| \le 8t^2 + 8t^2\sinh(2r), \tag{3.7}$$

$$|\operatorname{Im} E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q)| \le 4t^2\sinh(2r), \tag{3.8}$$

$$\operatorname{Re} E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q) \ge -4t^2\sinh(2r). \tag{3.9}$$

Proof. Note that

$$\begin{aligned}
&E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q)\\
&\quad = 4t^2 + 2t^2\sum_{j=1}^{2}\cos(k_j+\operatorname{Re} w\delta_{p,j}+\operatorname{Re} z\delta_{q,j})\cosh(\operatorname{Im} w\delta_{p,j}+\operatorname{Im} z\delta_{q,j})\\
&\qquad -i2t^2\sum_{j=1}^{2}\sin(k_j+\operatorname{Re} w\delta_{p,j}+\operatorname{Re} z\delta_{q,j})\sinh(\operatorname{Im} w\delta_{p,j}+\operatorname{Im} z\delta_{q,j}),
\end{aligned}$$

which leads to $|E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q)| \le 8t^2+8t^2\sinh(2r)$. The upper bounds on $|(\partial/\partial k_j)^m E(\omega,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q)|$, $|\operatorname{Im} E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q)|$ can be obtained similarly. Moreover, $\operatorname{Re} E(t,\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q) \ge 4t^2-4t^2\cosh(2r) \ge -4t^2\sinh(2r)$. □

The following lemma summarizes properties of $\mathcal{C}_{N_\beta}$. The β-dependency of Theorem 1.1 in low temperatures mainly stems from these upper bounds on $\mathcal{C}_{N_\beta}$. From now we assume that

$$M \ge 78E_{\max}^2. \tag{3.10}$$

Lemma 3.3. *For any $\varepsilon \in (0,1)$ there exists $N_\varepsilon \in \mathbb{N}$ such that for any $h \in 2\mathbb{N}/\beta$ with $h \geq 2N_\varepsilon/\beta$ the following statements hold true:*

(i) *The function $(w,z) \mapsto \mathcal{C}_{N_\beta}(X,Y)(w\mathbf{e}_p + z\mathbf{e}_q)$ is analytic in $\{(w,z) \in \mathbb{C}^2 \mid |\operatorname{Im} w|, |\operatorname{Im} z| < \mathcal{F}_{t,\beta}(\varepsilon)\}$ for any $X, Y \in I_{L,h}$, $p, q \in \{1,2\}$.*

(ii)

$$\frac{1}{h}\sum_{(\mathbf{x},x)\in\Gamma\times[-\beta,\beta)_h} |\mathcal{C}_{N_\beta}(\rho\mathbf{x}\sigma x, \eta\mathbf{0}\sigma 0)(w\mathbf{e}_p)| \leq \frac{c}{(1-\varepsilon)\varepsilon^2} M^{9-N_\beta}\max\{1,\beta\}^8$$

for any $\rho, \eta \in \{1,2,3\}$, $\sigma \in \{\uparrow,\downarrow\}$, $p \in \{1,2\}$ and $w \in \mathbb{C}$ with $|\operatorname{Im} w| < \mathcal{F}_{t,\beta}(\varepsilon)$.

(iii)

$$|\det(\langle \mathbf{u}_j, \mathbf{v}_k\rangle_{\mathbb{C}^m} \mathcal{C}_{N_\beta}(X_j, Y_k)(w\mathbf{e}_p))_{1\leq j,k\leq n}| \leq \left(\frac{c}{1-\varepsilon} M^6 \max\{1,\beta\}^3\right)^n$$

for any $m, n \in \mathbb{N}$, $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{C}^m$ with $\|\mathbf{u}_j\|_{\mathbb{C}^m}, \|\mathbf{v}_j\|_{\mathbb{C}^m} \leq 1$, $X_j, Y_j \in I_{L,h}$ $(j = 1,\dots,n)$, $p \in \{1,2\}$ and $w \in \mathbb{C}$ with $|\operatorname{Im} w| < \mathcal{F}_{t,\beta}(\varepsilon)$.

Proof. First note that $\chi_{N_\beta}(\omega) \neq 0$ implies $2|\omega|/\pi \leq M^{N_\beta+2}$, since $2|\theta|/\pi \leq |1 - e^{i\theta}|$ $(\forall \theta \in [-\pi,\pi])$. This inequality coupled with (3.2) proves that if $\chi_{N_\beta}(\omega) \neq 0$,

$$|\omega| \leq c\max\left\{1, \frac{1}{\beta}\right\} M^3. \tag{3.11}$$

(i): From the definition (2.2), (2.3) and (3.11) we observe that

$$\begin{aligned}\mathcal{D}^\sigma(\mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q, \omega) &= -\omega^2 + \epsilon_{\mathrm{c}}^\sigma\epsilon_{\mathrm{o}}^\sigma - \operatorname{Re} E(t, \mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q)\\ &\quad + i(-\omega(\epsilon_{\mathrm{c}}^\sigma + \epsilon_{\mathrm{o}}^\sigma) - \operatorname{Im} E(t, \mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q)) + O(h^{-1}),\end{aligned}$$

where $O(h^{-1})$ represents terms of order h^{-1}. Moreover, if $|\operatorname{Im} w|, |\operatorname{Im} z| < r$, by (3.8) and (3.9),

$$\begin{aligned}&|\mathcal{D}^\sigma(\mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q, \omega)|\\ &\quad\geq \max\{\omega^2 - \epsilon_{\mathrm{c}}^\sigma\epsilon_{\mathrm{o}}^\sigma + \operatorname{Re} E(t, \mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q),\\ &\qquad |\omega(\epsilon_{\mathrm{c}}^\sigma + \epsilon_{\mathrm{o}}^\sigma)| - |\operatorname{Im} E(t, \mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q)|\} + O(h^{-1})\\ &\quad\geq \max\left\{\frac{\pi^2}{\beta^2} - \frac{1}{2}(\epsilon_{\mathrm{c}}^\sigma + \epsilon_{\mathrm{o}}^\sigma)^2 - 4t^2\sinh(2r), \frac{\pi}{\beta}|\epsilon_{\mathrm{c}}^\sigma + \epsilon_{\mathrm{o}}^\sigma| - 4t^2\sinh(2r)\right\}\\ &\qquad + O(h^{-1})\\ &\quad\geq 1_{|\epsilon_{\mathrm{c}}^\sigma + \epsilon_{\mathrm{o}}^\sigma| \leq \frac{\pi}{\beta}}\left(\frac{\pi^2}{2\beta^2} - 4t^2\sinh(2r)\right) + 1_{|\epsilon_{\mathrm{c}}^\sigma + \epsilon_{\mathrm{o}}^\sigma| > \frac{\pi}{\beta}}\left(\frac{\pi^2}{\beta^2} - 4t^2\sinh(2r)\right)\\ &\qquad + O(h^{-1})\\ &\quad\geq \frac{\pi^2}{2\beta^2} - 4t^2\sinh(2r) + O(h^{-1}).\end{aligned}$$

If $r = \mathcal{F}_{t,\beta}(\varepsilon)$ and h is large enough,

$$|\mathcal{D}^\sigma(\mathbf{k} + w\mathbf{e}_p + z\mathbf{e}_q, \omega)| \geq \frac{(1-\varepsilon)\pi^2}{4\beta^2} > 0. \tag{3.12}$$

Therefore, the denominator of $\chi_{N_\beta}(\omega)\mathcal{B}^\sigma_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q,w)$ does not vanish for any $\rho,\eta\in\{1,2,3\}$, which ensures the analyticity of $\mathcal{C}_{N_\beta}(X,Y)(w\mathbf{e}_p+z\mathbf{e}_q)$ in the claimed domain.

(ii): Fix $w,z\in\mathbb{C}$ with $|\operatorname{Im} w|$, $|\operatorname{Im} z|<\mathcal{F}_{t,\beta}(\varepsilon)$ and $p,q\in\{1,2\}$. We will use the following bounds. For any $(k_1,k_2)\in\mathbb{R}^2$,

$$\begin{aligned}&|\sin(k_j+w\delta_{j,p}+z\delta_{j,q})|,|\cos(k_j+w\delta_{j,p}+z\delta_{j,q})|,\\&|\sin(k_1+w\delta_{1,p}+z\delta_{1,q}-k_2-w\delta_{2,p}-z\delta_{2,q})|,\\&|\cos(k_1+w\delta_{1,p}+z\delta_{1,q}-k_2-w\delta_{2,p}-z\delta_{2,q})|\le c\left(1+\frac{1}{\max\{\beta,\beta^2\}}\right)\end{aligned}\tag{3.13}$$

($\forall j\in\{1,2\}$). By keeping (2.3), (3.10), (3.11) and (3.13) in mind, one can deduce the following: for any $\omega\in\mathcal{M}_h$ with $\chi_{N_\beta}(\omega)\neq 0$ and large enough $h\in 2\mathbb{N}/\beta$,

$$\begin{aligned}&|\mathcal{N}^\sigma_{1,1}(\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q,\omega)|\le cM^3\max\left\{1,\frac{1}{\beta}\right\},\\&|\mathcal{N}^\sigma_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q,\omega)|\le cM\left(1+\frac{1}{\max\{\beta,\beta^2\}}\right)\\&\quad(\forall(\rho,\eta)\in\{1,2,3\}^2\backslash\{(1,1)\}),\\&|h(1-e^{iw/h+\epsilon^\sigma_o/h})|\ge|\omega|+O(h^{-1})\ge\frac{c}{\beta}.\end{aligned}$$

It follows from these inequalities and (3.12) that

$$\begin{aligned}&|\mathcal{B}^\sigma_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q,\omega)|\\&\quad\le\begin{cases}\frac{c}{1-\varepsilon}M^3\beta\max\{1,\beta\} & \text{if } (\rho,\eta)=(1,1),\\ \frac{c}{1-\varepsilon}M\beta\max\{1,\beta\} & \text{if } (\rho,\eta)\in\{(1,2),(1,3),(2,1),(3,1)\},\\ c\beta+\frac{c}{1-\varepsilon}M\beta^2\max\{1,\beta\} & \text{if } (\rho,\eta)\in\{(2,2),(2,3),(3,2),(3,3)\},\end{cases}\end{aligned}$$

which results in

$$|\mathcal{B}^\sigma_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p+z\mathbf{e}_q,\omega)|\le\frac{c}{1-\varepsilon}M^3\beta\max\{1,\beta\}^2\quad(\forall\rho,\eta\in\{1,2,3\}).\tag{3.14}$$

Then, by using Lemma 3.1 and (3.2) we have for any $X,Y\in I_{L,h}$ that

$$|\mathcal{C}_{N_\beta}(X,Y)(w\mathbf{e}_p+z\mathbf{e}_q)|\le\frac{c}{1-\varepsilon}M^{N_\beta+5}\beta\max\{1,\beta\}^2\le\frac{c}{1-\varepsilon}M^6\max\{1,\beta\}^3.\tag{3.15}$$

The rest of the proof of (ii) proceeds in the same way as in [11, Subsection 5.2]. By noting the domain of analyticity proved in (i) and the periodicity of

$\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k},\omega)$ with respect to $\mathbf{k}$ one can derive the following equality: for $n \in \mathbb{N}$,

$$\left(\frac{L}{2\pi}\left(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_q\rangle}-1\right)\right)^n \mathcal{C}_{N_\beta}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)(w\mathbf{e}_p)$$
$$=\prod_{j=1}^{n}\left(\frac{L}{2\pi}\int_0^{2\pi/L} d\theta_j \frac{1}{2\pi i}\oint_{|z_j-\theta_j|=\mathcal{F}_{t,\beta}(\varepsilon/2)/n} dz_j \frac{1}{(z_j-\theta_j)^2}\right)$$
$$\cdot\mathcal{C}_{N_\beta}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)\left(w\mathbf{e}_p+\hat{s}(\sigma)\sum_{j=1}^{n} z_j\mathbf{e}_q\right). \tag{3.16}$$

By taking the absolute value of both sides of (3.16) and using the inequality $n^n \le n!e^n$ and (3.15) we obtain

$$\left|\frac{L}{2\pi}\left(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_q\rangle}-1\right)\right|^n |\mathcal{C}_{N_\beta}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)(w\mathbf{e}_p)|$$
$$\le \frac{c}{1-\varepsilon}M^6\max\{1,\beta\}^3\frac{n!e^n}{\mathcal{F}_{t,\beta}(\varepsilon/2)^n}$$

for any $n \in \mathbb{N}\cup\{0\}$, which leads to

$$|\mathcal{C}_{N_\beta}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)(w\mathbf{e}_p)| \le \frac{c}{1-\varepsilon}M^6\max\{1,\beta\}^3$$
$$\cdot\left(\frac{\varepsilon\pi^2}{16\max\{1,t^2\}\max\{\beta,\beta^2\}}+1\right)^{-\frac{1}{8e}\sum_{q=1}^{2}\left|\frac{e^{i2\pi\langle \mathbf{x}-\mathbf{y},\mathbf{e}_q\rangle/L}-1}{2\pi/L}\right|}.$$

Then, using the inequality that $|(e^{i2\pi m/L}-1)/(2\pi/L)| \ge 2|m|/\pi$ ($\forall m \in \mathbb{Z}$ with $|m| \le L/2$) and (3.2) we can deduce that

$$\frac{1}{h}\sum_{(\mathbf{x},x)\in\Gamma\times[-\beta,\beta)_h}|\mathcal{C}_{N_\beta}(\rho\mathbf{x}\sigma x,\eta\mathbf{0}\tau 0)(w\mathbf{e}_p)|$$
$$\le \frac{c}{1-\varepsilon}M^6\beta\max\{1,\beta\}^3\left(\frac{\left(\frac{\varepsilon\pi^2}{16\max\{1,t^2\}\max\{\beta,\beta^2\}}+1\right)^{1/(4\pi e)}+1}{\left(\frac{\varepsilon\pi^2}{16\max\{1,t^2\}\max\{\beta,\beta^2\}}+1\right)^{1/(4\pi e)}-1}\right)^2$$
$$\le \frac{c}{1-\varepsilon}M^6\beta\max\{1,\beta\}^3\left(1+\frac{\max\{1,t^2\}\max\{\beta,\beta^2\}}{\varepsilon}\right)^2$$
$$\le \frac{c}{(1-\varepsilon)\varepsilon^2}M^8\beta\max\{1,\beta\}^7 \le \frac{c}{(1-\varepsilon)\varepsilon^2}M^{9-N_\beta}\max\{1,\beta\}^8.$$

(iii): Define the complex Hilbert space $\mathcal{H}$ by $\mathcal{H} := \mathbb{C}^m \otimes L^2(\{1,2,3\}\times\Gamma^*\times\{\uparrow,\downarrow\}\times\mathcal{M}_h)$ with the inner product

$$\langle \mathbf{u}\otimes f,\mathbf{v}\otimes g\rangle_{\mathcal{H}} := \langle \mathbf{u},\mathbf{v}\rangle_{\mathbb{C}^m}\frac{1}{\beta L^2}\sum_{\substack{(\rho,\mathbf{k},\sigma,\omega)\\ \in\{1,2,3\}\times\Gamma^*\times\{\uparrow,\downarrow\}\times\mathcal{M}_h}} f(\rho,\mathbf{k},\sigma,\omega)\overline{g(\rho,\mathbf{k},\sigma,\omega)}.$$

Moreover, define the vectors $f_{l,X}$, $g_{l,X} \in L^2(\{1,2,3\} \times \Gamma^* \times \{\uparrow,\downarrow\} \times \mathcal{M}_h)$ ($X \in I_{L,h}$, $l \in \{N_\beta, \dots, N_h\}$) by

$$f_{l,\rho\mathbf{x}\sigma x}(\eta, \mathbf{k}, \tau, \omega) := \delta_{\sigma,\tau} e^{-i\langle \mathbf{x},\mathbf{k}\rangle} e^{ix\omega} \chi_l(\omega)^{1/2} \mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k} + \hat{s}(\sigma) w \mathbf{e}_p, \omega), \quad (3.17)$$

$$g_{l,\rho\mathbf{x}\sigma x}(\eta, \mathbf{k}, \tau, \omega) := \delta_{\rho,\eta}\delta_{\sigma,\tau} e^{-i\langle \mathbf{x},\mathbf{k}\rangle} e^{ix\omega} \chi_l(\omega)^{1/2}. \quad (3.18)$$

The vectors $f_{l,X}$, $g_{l,X}$ for $l \geq N_\beta + 1$ will be used in the proof of the next lemma. We see that $\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{C}^m} \mathcal{C}_{N_\beta}(X,Y)(w\mathbf{e}_p) = \langle \mathbf{u} \otimes f_{N_\beta,X}, \mathbf{v} \otimes g_{N_\beta,Y}\rangle_{\mathcal{H}}$. By Lemma 3.1, (3.2) and (3.14),

$$\|\mathbf{u} \otimes f_{N_\beta,X}\|_{\mathcal{H}} \leq (M^3 \max\{1,\beta^{-1}\})^{1/2} \frac{c}{1-\varepsilon} M^3 \beta \max\{1,\beta\}^2,$$
$$\|\mathbf{v} \otimes g_{N_\beta,X}\|_{\mathcal{H}} \leq c(M^3 \max\{1,\beta^{-1}\})^{1/2},$$

if $\|\mathbf{u}\|_{\mathbb{C}^m}$, $\|\mathbf{v}\|_{\mathbb{C}^m} \leq 1$. Therefore, Gram's inequality guarantees that if $\|\mathbf{u}_j\|_{\mathbb{C}^m}$, $\|\mathbf{v}_j\|_{\mathbb{C}^m} \leq 1$ ($\forall j \in \{1,\dots,n\}$),

$$\begin{aligned} &|\det(\langle \mathbf{u}_j, \mathbf{v}_k\rangle_{\mathbb{C}^m} \mathcal{C}_{N_\beta}(X_j,Y_k)(w\mathbf{e}_p))_{1\leq j,k\leq n}| \\ &\leq \prod_{j=1}^{n} \|\mathbf{u}_j \otimes f_{N_\beta,X_j}\|_{\mathcal{H}} \|\mathbf{v}_j \otimes g_{N_\beta,Y_j}\|_{\mathcal{H}} \\ &\leq \left(\frac{c}{1-\varepsilon} M^6 \max\{1,\beta\}^3\right)^n. \end{aligned}$$

□

The following lemma gives upper bounds on $\mathcal{C}_l$ ($l \in \{N_\beta+1,\dots,N_h\}$), which are essentially independent of β in low temperatures:

Lemma 3.4. *For any $\varepsilon \in (0,1)$ there exists $N_\varepsilon \in \mathbb{N}$ such that for any $h \in 2\mathbb{N}/\beta$ with $h \geq 2N_\varepsilon/\beta$ and $l \in \{N_\beta+1,\dots,N_h\}$ the following statements hold true:*

(i) *The function $w \mapsto \mathcal{C}_l(X,Y)(w\mathbf{e}_p)$ is analytic in $\{w \in \mathbb{C} \mid |\operatorname{Im} w| < \mathcal{F}_{t,\beta}(\varepsilon)\}$ for any $X, Y \in I_{L,h}$, $p \in \{1,2\}$.*

(ii)
$$\frac{1}{h} \sum_{(\mathbf{x},x)\in\Gamma\times[-\beta,\beta)_h} |\mathcal{C}_l(\rho\mathbf{x}\sigma x, \eta\mathbf{0}\sigma 0)(w\mathbf{e}_p)| \leq cM^{8-l}$$
for any $\rho,\eta \in \{1,2,3\}$, $\sigma \in \{\uparrow,\downarrow\}$, $p \in \{1,2\}$ and $w \in \mathbb{C}$ with $|\operatorname{Im} w| < \mathcal{F}_{t,\beta}(\varepsilon)$.

(iii)
$$|\det(\langle \mathbf{u}_j, \mathbf{v}_k\rangle_{\mathbb{C}^m} \mathcal{C}_l(X_j,Y_k)(w\mathbf{e}_p))_{1\leq j,k\leq n}| \leq (cM^4)^n$$
for any $m,n \in \mathbb{N}$, $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{C}^m$ with $\|\mathbf{u}_j\|_{\mathbb{C}^m}, \|\mathbf{v}_j\|_{\mathbb{C}^m} \leq 1$, $X_j, Y_j \in I_{L,h}$ ($j = 1,\dots,n$), $p \in \{1,2\}$ and $w \in \mathbb{C}$ with $|\operatorname{Im} w| < \mathcal{F}_{t,\beta}(\varepsilon)$.

(iv)
$$|\mathcal{C}_l(\rho\hat{\mathbf{x}}\sigma x, \eta\hat{\mathbf{y}}\tau y)(w\mathbf{e}_p)| \leq cM^{3+N_\beta-l}$$
for any $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{Z}^2$ with $1 \leq \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|_{\mathbb{R}^2} \leq L/2$, (ρ,σ,x), $(\eta,\tau,y) \in \{1,2,3\} \times \{\uparrow,\downarrow\} \times [0,\beta)_h$, $p \in \{1,2\}$ and $w \in \mathbb{C}$ with $|\operatorname{Im} w| < \mathcal{F}_{t,\beta}(\varepsilon)$.

(v)

$$|\mathcal{C}_l(\rho\mathbf{0}\sigma 0,\eta\mathbf{0}\tau 0)(w\mathbf{e}_p)|\le cM^3(M^{l-N_h}+M^{N_\beta-l})$$

for any $(\rho,\sigma),(\eta,\tau)\in\{1,2,3\}\times\{\uparrow,\downarrow\}$, $p\in\{1,2\}$ *and* $w\in\mathbb{C}$ *with* $|\operatorname{Im} w|<\mathcal{F}_{t,\beta}(\varepsilon)$.

Proof. (i): For any $\omega\in\mathbb{R}$ with $\chi_l(\omega)\neq 0$ and sufficiently large h,

$$\frac{1}{2}M^l-E_{\max}\le |h(1-e^{i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)})|\le 2M^{l+2}+E_{\max}. \tag{3.19}$$

The condition (3.10) implies that $10+\frac{1}{2}E_{\max}+9E_{\max}^2\le\frac{39}{2}E_{\max}^2\le\frac{1}{4}M$, or

$$9E_{\max}^2+10M^{l-1}\le\frac{1}{2}\left(\frac{1}{2}M^l-E_{\max}\right)\le\frac{1}{2}\left(\frac{1}{2}M^l-E_{\max}\right)^2. \tag{3.20}$$

Note that by (3.2) and (3.7),

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial k_j}\right)^m\left(\frac{(\epsilon_c^\sigma-\epsilon_o^\sigma)^2}{4}+E(t,\mathbf{k}+w\mathbf{e}_p)\right)\right|\\&\quad\le 9E_{\max}^2+\frac{\pi^2}{\max\{\beta,\beta^2\}}\le 9E_{\max}^2+\pi^2M^{l-1}\end{aligned} \tag{3.21}$$

P for any $m\in\{0,\dots,4\}$. Then, using (3.19), (3.20) and (3.21) we have for any $\mathbf{k}\in\mathbb{R}^2$ that

$$\begin{aligned}|\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)|&\ge\left(\frac{1}{2}M^l-E_{\max}\right)^2-e^{E_{\max}/h}(9E_{\max}^2+\pi^2M^{l-1})+O(h^{-1})\\&\ge\left(\frac{1}{2}M^l-E_{\max}\right)^2-9E_{\max}^2-10M^{l-1}\ge\frac{1}{2}\left(\frac{1}{2}M^l-E_{\max}\right)^2\ge\frac{1}{16}M^{2l}.\end{aligned} \tag{3.22}$$

Thus, the denominator of $\chi_l(\omega)\mathcal{B}_{\rho,\eta}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)$ is non-zero for any $\rho,\eta\in\{1,2,3\}$, which proves the claim (i).

(ii), (iv): Take $\omega\in\mathbb{R}$ with $\chi_l(\omega)\neq 0$, $p\in\{1,2\}$, $\mathbf{k}\in\mathbb{R}^2$, $\sigma\in\{\uparrow,\downarrow\}$ and $w\in\mathbb{C}$ with $|\operatorname{Im} w|<\mathcal{F}_{t,\beta}(\varepsilon)$. Estimating $|\chi_l(\omega)(\partial/\partial k_j)^m\mathcal{B}_{\rho,\eta}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)|$, $|(\partial/\partial\omega)^m(\chi_l(\omega)\mathcal{B}_{\rho,\eta}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega))|$ $(m=0,\dots,4)$ provides sufficient information to bound the sum of $\mathcal{C}_l(w\mathbf{e}_p)$ over $\Gamma\times[0,\beta)_h$. By using the inequalities (3.2), (3.3) and (3.19) we obtain

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial\omega}\right)^m\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right|\le cM^{4+(2-m)l}\quad(\forall m\in\{0,\dots,4\}),\\&\left|\left(\frac{\partial}{\partial k_j}\right)^n\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right|\le cM^{N_\beta}\quad(\forall n\in\{1,\dots,4\},j\in\{1,2\}),\end{aligned}$$

which, combined with (3.22), yields

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial\omega}\right)^m\frac{1}{\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)}\right|\le cM^{4m-(2+m)l}\quad(\forall m\in\{0,\dots,4\}),\\&\left|\left(\frac{\partial}{\partial k_j}\right)^n\frac{1}{\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)}\right|\le cM^{N_\beta-4l}\quad(\forall n\in\{1,\dots,4\},j\in\{1,2\}).\end{aligned} \tag{3.23}$$

One can similarly derive the following inequalities: For any $m \in \{0,\dots,4\}$, $n \in \{1,\dots,4\}$, $j \in \{1,2\}$ and $(\rho,\eta) \in \{1,2,3\}^2 \backslash \{(1,1)\}$,

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial \omega}\right)^m \mathcal{N}^{\sigma}_{1,1}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{2+(1-m)l},\\
&\left|\left(\frac{\partial}{\partial k_j}\right)^n \mathcal{N}^{\sigma}_{1,1}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le c,\\
&\left|\left(\frac{\partial}{\partial \omega}\right)^m \mathcal{N}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{N_\beta+1-ml},\\
&\left|\left(\frac{\partial}{\partial k_j}\right)^n \mathcal{N}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{N_\beta+1}.
\end{aligned} \tag{3.24}$$

These imply that for any $m \in \{0,\dots,4\}$, $n \in \{1,\dots,4\}$, $j \in \{1,2\}$, $\rho,\eta \in \{1,2,3\}$,

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial \omega}\right)^m \mathcal{N}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{2+(1-m)l},\\
&\left|\left(\frac{\partial}{\partial k_j}\right)^n \mathcal{N}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{N_\beta+1}.
\end{aligned} \tag{3.25}$$

As in (3.19), $|h(1-\mathrm{e}^{i\omega/h+\epsilon_o^\sigma/h})| \ge \frac{1}{2}M^l - E_{\max} \ge cM^l$. Thus,

$$\left|\left(\frac{\partial}{\partial \omega}\right)^m \frac{1}{h(1-\mathrm{e}^{i\omega/h+\epsilon_o^\sigma/h})}\right| \le cM^{-(m+1)l} \quad (\forall m \in \{0,\dots,4\}). \tag{3.26}$$

One can also check that

$$\left|\left(\frac{\partial}{\partial \omega}\right)^m \chi_l(\omega)\right| \le cM^{m(1-l)} \quad (\forall m \in \{0,\dots,4\}). \tag{3.27}$$

Then by using (3.23), (3.25), (3.26), (3.27) and Leibniz' formula, we have for any $\rho,\eta \in \{1,2,3\}$, $j \in \{1,2\}$ that

$$1_{\chi_l(\omega)\neq 0}\left|\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{2-l}, \tag{3.28}$$

$$\left|\left(\frac{\partial}{\partial \omega}\right)^4 (\chi_l(\omega)\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega))\right| \le cM^{18-5l}, \tag{3.29}$$

$$\left|\chi_l(\omega)\left(\frac{\partial}{\partial k_j}\right)^4 \mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+w\mathbf{e}_p,\omega)\right| \le cM^{1+N_\beta-2l}. \tag{3.30}$$

It follows from (3.28) and Lemma 3.1 that

$$|\mathcal{C}_l(X,Y)(w\mathbf{e}_p)| \le cM^4 \quad (\forall X,Y \in I_{L,h}). \tag{3.31}$$

For a function $f:\mathbb{C}\to\mathbb{C}$, let $d_\beta f(\omega) := \frac{\beta}{2\pi}(f(\omega+2\pi/\beta)-f(\omega))$. By remarking the periodicity that $\chi_l(\omega+2\pi hm)\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p,\omega+2\pi hm) =$

$\chi_l(\omega)\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p,\omega)$ $(\forall m\in\mathbb{Z})$, we observe that

$$\begin{aligned}&\left(\frac{\beta}{2\pi}\left(\mathrm{e}^{-i\frac{2\pi}{\beta}(x-y)}-1\right)\right)^4\mathcal{C}_l(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)(w\mathbf{e}_p)\\&\quad=\frac{\delta_{\sigma,\tau}}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{i(x-y)\omega}d_\beta^4\left(\chi_l(\omega)\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p,\omega)\right)\\&\quad=\frac{\delta_{\sigma,\tau}}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{i(x-y)\omega}\prod_{m=1}^{4}\left(\frac{\beta}{2\pi}\int_0^{2\pi/\beta}\mathrm{d}v_m\right)\\&\qquad\cdot\left(\frac{\partial}{\partial\omega}\right)^4\left(\chi_l\left(\omega+\sum_{m=1}^4v_m\right)\mathcal{B}^{\sigma}_{\rho,\eta}\left(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p,\omega+\sum_{m=1}^4v_m\right)\right).\end{aligned}$$

Then, the bound (3.29) and Lemma 3.1 lead to

$$\left|\frac{\beta}{2\pi}\left(\mathrm{e}^{-i\frac{2\pi}{\beta}(x-y)}-1\right)\right|^4|\mathcal{C}_l(X,Y)(w\mathbf{e}_p)|\le cM^{20-4l}.\tag{3.32}$$

Similarly, using the periodicity that $\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p+2\pi n\mathbf{e}_j,\omega)=\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p,\omega)$ $(\forall n\in\mathbb{Z})$ we obtain

$$\begin{aligned}&\left(\frac{L}{2\pi}\left(\mathrm{e}^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1\right)\right)^4\mathcal{C}_l(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)(w\mathbf{e}_p)\\&\quad=\frac{\delta_{\sigma,\tau}}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{i(x-y)\omega}\\&\qquad\cdot\prod_{n=1}^{4}\left(\frac{L}{2\pi}\int_0^{2\pi/L}\mathrm{d}u_n\right)\chi_l(\omega)\left(\frac{\partial}{\partial k_j}\right)^4\mathcal{B}^{\sigma}_{\rho,\eta}\left(\mathbf{k}+\hat{s}(\sigma)w\mathbf{e}_p+\sum_{n=1}^4u_n\mathbf{e}_j,\omega\right),\end{aligned}$$

which, combined with (3.30) and Lemma 3.1, yields

$$\left|\frac{L}{2\pi}\left(\mathrm{e}^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1\right)\right|^4|\mathcal{C}_l(X,Y)(w\mathbf{e}_p)|\le cM^{3+N_\beta-l}\quad(\forall j\in\{1,2\}).\tag{3.33}$$

The inequalities (3.31), (3.32) and (3.33) result in

$$\begin{aligned}&|\mathcal{C}_l(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)(w\mathbf{e}_p)|\\&\quad\le\frac{cM^4}{1+M^{l-N_\beta+1}\sum_{j=1}^2\left|\frac{L}{2\pi}(\mathrm{e}^{i2\pi\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle/L}-1)\right|^4+M^{4l-16}\left|\frac{\beta}{2\pi}(\mathrm{e}^{i2\pi(x-y)/\beta}-1)\right|^4}\end{aligned}\tag{3.34}$$

for all $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y)\in I_{L,h}$. The decay bound (3.34) implies the claim (ii) and the claim (iv).

(iii): The proof of (iii) is parallel to that of Lemma 3.3 (iii). Recall (3.17) and (3.18).Using Lemma 3.1 and (3.28) one can show that for any $\mathbf{u}$, $\mathbf{v}\in\mathbb{C}^m$ with $\|\mathbf{u}\|_{\mathbb{C}^m}$, $\|\mathbf{v}\|_{\mathbb{C}^m}\le 1$, $\|\mathbf{u}\otimes f_{l,X}\|_{\mathcal{H}}\le c(M^{l+2})^{1/2}M^{2-l}$, $\|\mathbf{v}\otimes g_{l,X}\|_{\mathcal{H}}\le$

$c(M^{l+2})^{1/2}$. Thus, we can apply Gram's inequality to conclude that

$$\begin{aligned}&|\det(\langle \mathbf{u}_j, \mathbf{v}_k\rangle_{\mathbb{C}^m} \mathcal{C}_l(X_j, Y_k)(w\mathbf{e}_p))_{1\le j,k\le n}|\\ &\quad \le \prod_{j=1}^{n} \|\mathbf{u}_j \otimes f_{l,X_j}\|_{\mathcal{H}} \|\mathbf{v}_j \otimes g_{l,X_j}\|_{\mathcal{H}} \le (cM^{l+2}\cdot M^{2-l})^n \le (cM^4)^n.\end{aligned}$$

(v): Take $\omega \in \mathcal{M}_h$ with $\chi_l(\omega) \neq 0$. Since

$$|\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega) - h^2(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)})^2| \le 9E_{\max}^2 + 10M^{l-1}$$

by (3.21), the inequalities (3.19) and (3.20) justify that

$$\begin{aligned}&\frac{1}{\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)}\\ &\quad = \frac{1}{h^2\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)^2} + \frac{1}{h^3\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)^3}\\ &\qquad \cdot \sum_{m=1}^{\infty} \frac{\left(h^2\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)^2 - \mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right)^m}{\left(h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)\right)^{2m-1}},\\ &\quad \sum_{m=1}^{\infty} \left|\frac{\left(h^2\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)^2 - \mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right)^m}{\left(h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)\right)^{2m-1}}\right| \le 1.\end{aligned}$$

This particularly implies that

$$\begin{aligned}&\left|\frac{1}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h} \chi_l(\omega)\mathcal{B}_{1,1}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right|\\ &\quad \le \left|\frac{1}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h} \frac{\chi_l(\omega)h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)}{\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)}\right|\\ &\qquad + \frac{1}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h} \frac{\chi_l(\omega)\left|\mathcal{N}_{1,1}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega) - h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)\right|}{|\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)|}\\ &\quad \le \left|\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h} \frac{\chi_l(\omega)}{h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)}\right| + cM^{3-l},\end{aligned}\tag{3.35}$$

where Lemma 3.1, (3.19) and (3.22) were also used. Note that

$$\begin{aligned}&\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h} \frac{\chi_l(\omega)}{h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)}\\ &\quad = \frac{1}{2\beta h}\sum_{\omega\in\mathcal{M}_h}\chi_l(\omega)\\ &\qquad + \frac{1}{2\beta}\sum_{\omega\in\mathcal{M}_h}\chi_l(\omega)\frac{h\left(1-\mathrm{e}^{(\epsilon_c^\sigma+\epsilon_o^\sigma)/h}\right)}{h^2\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)\left(1-\mathrm{e}^{i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)}.\end{aligned}$$

Then again by using Lemma 3.1, (3.3) and (3.19) we have

$$\left|\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h}\frac{\chi_l(\omega)}{h\left(1-\mathrm{e}^{-i\omega/h+(\epsilon_c^\sigma+\epsilon_o^\sigma)/(2h)}\right)}\right|\le cM^{l-N_h+2}+cM^{3-l}. \quad (3.36)$$

Substituting (3.36) into (3.35) gives

$$\left|\frac{1}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\chi_l(\omega)\mathcal{B}_{1,1}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right|\le cM^{l-N_h+2}+cM^{3-l}. \quad (3.37)$$

It follows from (3.24) that

$$\frac{1}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\frac{\chi_l(\omega)|\mathcal{N}_{\rho,\eta}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)|}{|\mathcal{D}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)|}\le cM^{3+N_\beta-l}$$
$$(\forall(\rho,\eta)\in\{1,2,3\}^2\backslash\{(1,1)\}). \quad (3.38)$$

The procedure to derive (3.36) similarly shows that

$$\left|\frac{1}{\beta}\sum_{\omega\in\mathcal{M}_h}\frac{\chi_l(\omega)}{h\left(1-\mathrm{e}^{-i\omega/h+\epsilon_o^\sigma/h)}\right)}\right|\le cM^{l-N_h+2}+cM^{3-l}. \quad (3.39)$$

The bounds (3.38) and (3.39) yield that for any $(\rho,\eta)\in\{1,2,3\}^2\backslash\{(1,1)\}$,

$$\left|\frac{1}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\chi_l(\omega)\mathcal{B}_{\rho,\eta}^\sigma(\mathbf{k}+w\mathbf{e}_p,\omega)\right|\le cM^{l-N_h+2}+cM^{3+N_\beta-l}. \quad (3.40)$$

By (3.37) and (3.40) we can confirm the inequality claimed in (v). □

4. Multi-Scale Integration

In this section we will find an h-,L-independent upper bound on

$$\left(\frac{L}{2\pi}\left(\mathrm{e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^2(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}-1\right)\right)^n$$
$$\cdot\frac{\partial}{\partial\lambda}\log\left(\int\mathrm{e}^{V_{(\lambda,\lambda)}(\psi)}\,\mathrm{d}\mu_{\mathcal{C}}(\psi)\right)\Big|_{\lambda=0} \quad (4.1)$$

($n\in\mathbb{N}\cup\{0\}$, $p\in\{1,2\}$) by estimating the right-hand side of Lemma 2.4 (iii) by means of a multi-scale integration over the Matsubara frequency $\mathcal{M}_h$. Using the upper bound on (4.1) we will complete the proof of Theorem 1.1 in the end of this section.

4.1. Notations for the Multi-Scale Expansion

Let us decide some notational rules to systematically handle Grassmann polynomials during the multi-scale expansion, in addition to those already introduced in Sect. 2.1.

For $\mathbf{X}^m=(X_1^m,X_2^m,\ldots,X_m^m)\in I_{L,h}^m$ $(m\in\mathbb{N})$ let $(\overline{\psi})_{\mathbf{X}^m}:=\overline{\psi}_{X_1^m}\overline{\psi}_{X_2^m}\cdots\overline{\psi}_{X_m^m}$, $(\psi)_{\mathbf{X}^m}:=\psi_{X_1^m}\psi_{X_2^m}\cdots\psi_{X_m^m}\in\bigwedge^m\mathcal{V}$. Define the extended index set $\tilde{I}_{L,h}$ by $\tilde{I}_{L,h}:=I_{L,h}\times\{1,-1\}$. The index set $\tilde{I}_{L,h}$ is used in the following way: For $(X,a)\in\tilde{I}_{L,h}$, $\psi_{(X,a)}:=\overline{\psi}_X$ if $a=1$, $\psi_{(X,a)}:=\psi_X$ if $a=-1$. For $\tilde{\mathbf{X}}^m=(\tilde{X}_1^m,\tilde{X}_2^m,\ldots,\tilde{X}_m^m)\in\tilde{I}_{L,h}^m$ let $(\psi)_{\tilde{\mathbf{X}}^m}:=\psi_{\tilde{X}_1^m}\psi_{\tilde{X}_2^m}\cdots\psi_{\tilde{X}_m^m}\in\bigwedge^m\mathcal{V}$.

For $\mathbf{X}^m\in I_{L,h}^m$, $\mathbf{X}^n=(X_1^n,X_2^n,\ldots,X_n^n)\in I_{L,h}^n$ with $m\le n$, we write $\mathbf{X}^m\subset\mathbf{X}^n$ if there exist $j_1,j_2,\ldots,j_m\in\{1,2,\ldots,n\}$ such that $j_1<j_2<\cdots<j_m$ and $\mathbf{X}^m=(X_{j_1}^n,X_{j_2}^n,\ldots,X_{j_m}^n)$. Moreover, in this case we define $\mathbf{X}^n\backslash\mathbf{X}^m\in I_{L,h}^{n-m}$ by $\mathbf{X}^n\backslash\mathbf{X}^m:=(X_{k_1}^n,X_{k_2}^n,\ldots,X_{k_{n-m}}^n)$, where $1\le k_1<k_2<\cdots<k_{n-m}\le n$ and $k_q\notin\{j_1,j_2,\ldots,j_m\}$ $(\forall q\in\{1,2,\ldots,n-m\})$.

For $\tilde{\mathbf{X}}^m\in\tilde{I}_{L,h}^m$, $\tilde{\mathbf{X}}^n\in\tilde{I}_{L,h}^n$ with $m\le n$ the notations $\tilde{\mathbf{X}}^m\subset\tilde{\mathbf{X}}^n$ and $\tilde{\mathbf{X}}^n\backslash\tilde{\mathbf{X}}^m$ are defined in the same way as above. For $\mathbf{X}^m=(X_1^m,X_2^m,\ldots,X_m^m)\in I_{L,h}^m$ and $a\in\{1,-1\}$ let $\tilde{\mathbf{X}}(a)^m:=((X_1^m,a),(X_2^m,a),\ldots,(X_m^m,a))\in\tilde{I}_{L,h}^m$.

For a function $f_m:I_{L,h}^m\times I_{L,h}^m\to\mathbb{C}$ $(m\in\mathbb{N})$ let

$$\|f_m\|_1:=\left(\frac{1}{h}\right)^{2m}\sum_{\mathbf{X}^m,\mathbf{Y}^m\in I_{L,h}^m}|f_m(\mathbf{X}^m,\mathbf{Y}^m)|,$$

$$\begin{aligned}\|f_m\|_{1,\infty}:=\max\Bigg\{&\max_{\substack{j\in\{0,\ldots,m-1\},\\ X\in I_{L,h}}}\Bigg\{\left(\frac{1}{h}\right)^{2m-1}\sum_{\mathbf{X}^j\in I_{L,h}^j}\sum_{\mathbf{X}^{m-1-j}\in I_{L,h}^{m-1-j}}\\ &\cdot\sum_{\mathbf{Y}^m\in I_{L,h}^m}|f_m((\mathbf{X}^j,X,\mathbf{X}^{m-1-j}),\mathbf{Y}^m)|\Bigg\},\\ &\max_{\substack{j\in\{0,\ldots,m-1\},\\ Y\in I_{L,h}}}\Bigg\{\left(\frac{1}{h}\right)^{2m-1}\sum_{\mathbf{Y}^j\in I_{L,h}^j}\sum_{\mathbf{Y}^{m-1-j}\in I_{L,h}^{m-1-j}}\\ &\cdot\sum_{\mathbf{X}^m\in I_{L,h}^m}|f_m(\mathbf{X}^m,(\mathbf{Y}^j,Y,\mathbf{Y}^{m-1-j}))|\Bigg\}\Bigg\}.\end{aligned}$$

We see that $\|\cdot\|_1$, $\|\cdot\|_{1,\infty}$ are norms in the complex vector space of all functions on $I_{L,h}^m\times I_{L,h}^m$. For notational consistency we also set $\|f_0\|_1$, $\|f_0\|_{1,\infty}:=|f_0|$ for any complex number f_0.

Let us call a function $f_m:I_{L,h}^m\times I_{L,h}^m\to\mathbb{C}$ bi-anti-symmetric if

$$\begin{aligned}&f_m((X_{\nu(1)},X_{\nu(2)},\ldots,X_{\nu(m)}),(Y_{\xi(1)},Y_{\xi(2)},\ldots,Y_{\xi(m)}))\\ &\quad=\mathrm{sgn}(\nu)\,\mathrm{sgn}(\xi)f_m((X_1,X_2,\ldots,X_m),(Y_1,Y_2,\ldots,Y_m))\end{aligned}$$

for any $(X_1, X_2, \dots, X_m)$, $(Y_1, Y_2, \dots, Y_m) \in I_{L,h}^m$ and $\nu, \xi \in \mathbb{S}_m$. Recalling the numbering that $I_{L,h} = \{X_{o,j}\}_{j=1}^{N_{L,h}}$, let

$$(I_{L,h})_o^m := \{(X_{o,j_1}, X_{o,j_2}, \dots, X_{o,j_m}) \in I_{L,h}^m \mid j_1 < j_2 < \cdots < j_m\} \quad (\forall m \in \mathbb{N}).$$

It holds for any bi-anti-symmetric function $f_m(\cdot,\cdot) : I_{L,h}^m \times I_{L,h}^m \to \mathbb{C}$ that

$$\|f_m\|_1 = \left(\frac{1}{h}\right)^{2m} (m!)^2 \sum_{\mathbf{X}^m, \mathbf{Y}^m \in (I_{L,h})_o^m} |f_m(\mathbf{X}^m, \mathbf{Y}^m)|. \tag{4.2}$$

Bi-anti-symmetric functions appear as kernels of Grassmann polynomials. Note that $f(\psi) \in \bigoplus_{n=0}^{N_{L,h}} \mathcal{P}_n \bigwedge \mathcal{V}$ can be uniquely written as

$$f(\psi) = \sum_{m=0}^{N_{L,h}} \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} f_m(\mathbf{X}^m, \mathbf{Y}^m) (\overline{\psi})_{\mathbf{X}^m} (\psi)_{\mathbf{Y}^m}$$

with bi-anti-symmetric kernels $f_m(\cdot,\cdot) : I_{L,h}^m \times I_{L,h}^m \to \mathbb{C}$ $(m \in \{0, \dots, N_{L,h}\})$. Moreover, if

$$\left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} f_m(\mathbf{X}^m, \mathbf{Y}^m) (\overline{\psi})_{\mathbf{X}^m} (\psi)_{\mathbf{Y}^m} = \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} g_m(\mathbf{X}^m, \mathbf{Y}^m) (\overline{\psi})_{\mathbf{X}^m} (\psi)_{\mathbf{Y}^m}$$

and $f_m(\cdot,\cdot)$ is bi-anti-symmetric, then the inequalities

$$\|f_m\|_1 \le \|g_m\|_1 \quad \text{and} \quad \|f_m\|_{1,\infty} \le \|g_m\|_{1,\infty} \tag{4.3}$$

hold.

Assume that $f_{l,m}(\cdot,\cdot) : I_{L,h}^m \times I_{L,h}^m \to \mathbb{C}$ is bi-anti-symmetric $(\forall l, m \in \mathbb{N} \cup \{0\})$ and $\lim_{l\to\infty} f_{l,m}(\mathbf{X}^m, \mathbf{Y}^m)$ exists in $\mathbb{C}$ $(\forall m \in \mathbb{N} \cup \{0\}, \mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m)$. Set

$$f_l(\psi) := \sum_{m=0}^{N_{L,h}} \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} f_{l,m}(\mathbf{X}^m, \mathbf{Y}^m) (\overline{\psi})_{\mathbf{X}^m} (\psi)_{\mathbf{Y}^m}.$$

In this case we define $\lim_{l\to\infty} f_l(\psi) \in \bigoplus_{n=0}^{N_{L,h}} \mathcal{P}_n \bigwedge \mathcal{V}$ by

$$\lim_{l\to\infty} f_l(\psi) := \sum_{m=0}^{N_{L,h}} \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} \lim_{l\to\infty} f_{l,m}(\mathbf{X}^m, \mathbf{Y}^m) (\overline{\psi})_{\mathbf{X}^m} (\psi)_{\mathbf{Y}^m}.$$

We call $f_z(\psi) \in \bigoplus_{n=0}^{N_{L,h}} \mathcal{P}_n \bigwedge \mathcal{V}$ analytic with respect to z in a domain $\mathcal{O}(\subset \mathbb{C})$ if so is every bi-anti-symmetric kernel of $f_z(\psi)$. Under this condition we define $(\mathrm{d}/\mathrm{d}z) f_z(\psi) \in \bigoplus_{n=0}^{N_{L,h}} \mathcal{P}_n \bigwedge \mathcal{V}$ by replacing each bi-anti-symmetric kernel of $f_z(\psi)$ by its derivative. Moreover, the following Taylor expansion holds true: For any $\hat{z} \in \mathcal{O}$,

$$f_z(\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n f_z(\psi)\Big|_{z=\hat{z}} (z-\hat{z})^n$$

in a neighbor of $\hat{z}$.

4.2. A Multi-Scale Integration Over the Matsubara Frequency

Here let us describe the multi-scale integration process. From now until the proof of Theorem 1.1 in Sect. 4.4 we fix arbitrary $R \in (\mathcal{F}_{t,\beta}(8/\pi^2), \infty)$, $\varepsilon \in (9/\pi^2, 1)$, $p \in \{1,2\}$, $L \in \mathbb{N}$ satisfying $\max_{j,k\in\{1,2\}} \|\hat{\mathbf{x}}_j - \hat{\mathbf{y}}_k\|_{\mathbb{R}^2} \le L/2$ and sufficiently large $h \in 2\mathbb{N}/\beta$. There exists $U_{\mathrm{small}} > 0$ such that all the statements of Lemmas 2.4, 3.3, and 3.4 hold true for these fixed parameters. Set

$$D_{\mathrm{small}} := \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid |z_j| \le U_{\mathrm{small}} \quad (\forall j \in \{1,2,3,4\})\},$$
$$D_R := \{z \in \mathbb{C} \mid |\operatorname{Re} z| \le R,\ |\operatorname{Im} z| \le \mathcal{F}_{t,\beta}(9/\pi^2)\}.$$

By taking U_{small} smaller if necessary we may assume that $\operatorname{Re} \int \mathrm{e}^{V_{(\lambda_1,\lambda_{-1})}(\psi)}\, \mathrm{d}\mu_{\sum_{j=l}^{N_h} \mathcal{C}_j(w\mathbf{e}_p)}(\psi) > 0$ for all $(\lambda_1, \lambda_{-1}, U_{\mathrm{c}}, U_{\mathrm{o}}) \in D_{\mathrm{small}}$, $w \in D_R$ and $l \in \{N_\beta, \dots, N_h\}$. This property allows us to define $G^{\ge l}(\psi) \in \bigwedge \mathcal{V}$ ($l \in \{N_\beta, \dots, N_h + 1\}$) by

$$G^{\ge l}(\psi) := \log\left(\int \mathrm{e}^{V_{(\lambda_1,\lambda_{-1})}(\psi+\psi^0)}\, \mathrm{d}\mu_{\sum_{j=l}^{N_h} \mathcal{C}_j(w\mathbf{e}_p)}(\psi^0)\right) \ (l \in \{N_\beta, \dots, N_h\}),$$
$$G^{\ge N_h+1}(\psi) := V_{(\lambda_1,\lambda_{-1})}(\psi)$$

for any $(\lambda_1, \lambda_{-1}, U_{\mathrm{c}}, U_{\mathrm{o}}) \in D_{\mathrm{small}}$, $w \in D_R$. The definition of logarithm of Grassmann polynomials is provided in Definition C.1 in Appendix C.

By noting the equality that

$$G^{\ge l}(\psi) = \log\left(\int \left(\int \mathrm{e}^{V_{(\lambda_1,\lambda_{-1})}(\psi+\psi^1+\psi^0)}\, \mathrm{d}\mu_{\sum_{j=l+1}^{N_h} \mathcal{C}_j(w\mathbf{e}_p)}(\psi^0)\right) \mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^1)\right)$$

(see, e.g, [6, Proposition I.21] to justify this equality), Lemma C.2 proved in Appendix C ensures that for any $(\lambda_1, \lambda_{-1}, U_{\mathrm{c}}, U_{\mathrm{o}}) \in D_{\mathrm{small}}$, $w \in D_R$, $l \in \{N_\beta, \dots, N_h\}$,

$$G^{\ge l}(\psi) = \log\left(\int \mathrm{e}^{G^{\ge l+1}(\psi+\psi^1)}\, \mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^1)\right). \tag{4.4}$$

Since

$$\lim_{U_{\mathrm{small}} \searrow 0} \sup_{\substack{(\lambda_1,\lambda_{-1},U_{\mathrm{c}},U_{\mathrm{o}}) \in D_{\mathrm{small}},\\ w\in D_R, z\in\mathbb{C} \text{ with } |z|\le 2}} \left| \int \mathrm{e}^{zG^{\ge l+1}(\psi^0)}\, \mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0) - 1 \right| = 0,$$

one can see from Definition C.1 that $z \mapsto \log\left(\int \mathrm{e}^{zG^{\ge l+1}(\psi+\psi^0)}\, \mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\right)$ is analytic in $\{z \in \mathbb{C} \mid |z| < 2\}$ for any $(\lambda_1, \lambda_{-1}, U_{\mathrm{c}}, U_{\mathrm{o}}) \in D_{\mathrm{small}}$, $w \in D_R$ if U_{small} is small enough. Thus, the Taylor expansion around $z = 0$ reads

$$G^{\ge l}(\psi) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \log\left(\int \mathrm{e}^{zG^{\ge l+1}(\psi+\psi^0)}\, \mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\right)\Bigg|_{z=0} \tag{4.5}$$

for any $(\lambda_1, \lambda_{-1}, U_{\mathrm{c}}, U_{\mathrm{o}}) \in D_{\mathrm{small}}$, $w \in D_R$, $l \in \{N_\beta, \dots, N_h\}$.

Each term of (4.5) can be characterized further. It follows from Definition C.1 and (C.1) that

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\left(\int e^{zG^{\geq l+1}(\psi+\psi^0)}\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\right)\Bigg|_{z=0} = \int G^{\geq l+1}(\psi+\psi^0)\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0).$$

The higher order derivatives can be expanded by means of the tree formula. We especially apply the version clearly proved in [15, Theorem 3]. For $n\in\mathbb{N}_{\geq 2}$,

$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\log\left(\int e^{zG^{\geq l+1}(\psi+\psi^0)}\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\right)\Bigg|_{z=0} = T_{ree}(n,\mathcal{C}_l(w\mathbf{e}_p),G^{\geq l+1}), \tag{4.6}$$

where for $n\in\mathbb{N}_{\geq 2}$, a matrix $Q=(Q(X,Y))_{X,Y\in I_{L,h}}$ and $f(\psi)\in\bigwedge\mathcal{V}$,

$$T_{ree}(n,Q,f) := \frac{1}{n!}\sum_{T\in\mathbb{T}_n}\prod_{\{q,r\}\in T}(\Delta_{q,r}(Q)+\Delta_{r,q}(Q))\int_{[0,1]^{n-1}}\mathrm{d}\mathbf{s}\sum_{\xi\in\mathbb{S}_n(T)}\varphi(T,\xi,\mathbf{s}) \cdot e^{\sum_{u,v=1}^{n}M_{at}(T,\xi,\mathbf{s})_{u,v}\Delta_{u,v}(Q)}\prod_{j=1}^{n}f(\psi^j+\psi)\Bigg|_{\substack{\psi^j=\mathbf{0}\\ \forall j\in\{1,\dots,n\}}}. \tag{4.7}$$

The new notations in (4.7) are defined as follows: $\mathbb{T}_n$ is the set of all trees over the vertices $\{1,2,\dots,n\}$, for $q,r\in\{1,\dots,n\}$,

$$\Delta_{q,r}(Q) := -\sum_{X,Y\in I_{L,h}}Q(X,Y)\frac{\partial}{\partial\overline{\psi}_X^q}\frac{\partial}{\partial\psi_Y^r} : \bigwedge\left(\bigoplus_{j=1}^{n}\mathcal{V}_j\right)\to\bigwedge\left(\bigoplus_{j=1}^{n}\mathcal{V}_j\right),$$

$\mathbb{S}_n(T)$ is a T-dependent subset of $\mathbb{S}_n$, the function $\varphi(T,\xi,\cdot):[0,1]^{n-1}\to\mathbb{R}_{\geq 0}$ depends on $T\in\mathbb{T}_n$, $\xi\in\mathbb{S}_n(T)$ and satisfies

$$\int_{[0,1]^{n-1}}\mathrm{d}\mathbf{s}\sum_{\xi\in\mathbb{S}_n(T)}\varphi(T,\xi,\mathbf{s}) = 1 \quad (\forall T\in\mathbb{T}_n), \tag{4.8}$$

and $(M_{at}(T,\xi,\mathbf{s})_{u,v})_{1\leq u,v\leq n}$ is a $(T,\xi,\mathbf{s})$-dependent real symmetric non-negative matrix satisfying $M_{at}(T,\xi,\mathbf{s})_{u,u}=1$ ($\forall u\in\{1,\dots,n\}$).

Our strategy is to introduce a counterpart of $G^{\geq l}$ via the tree formula inductively without assuming that $(\lambda_1,\lambda_{-1},U_{\mathrm{c}},U_{\mathrm{o}})\in D_{\mathrm{small}}$ and prove that the counterpart is well defined for larger $(\lambda_1,\lambda_{-1},U_{\mathrm{c}},U_{\mathrm{o}})$. Consequently, by the identity theorem for analytic functions we will be able to find an upper bound on (4.1) with the enlarged coupling constants U_{c}, U_{o} in the end of this section.

4.3. Estimation by Induction

Let us start the concrete analysis. In the following we fix arbitrary $w\in D_R$ unless otherwise stated. Define $J^{\geq l}(\psi)$, $F^{\geq l}(\psi)$, $T^{\geq l}(\psi)\in\bigoplus_{n=0}^{N_{L,h}}\mathcal{P}_n\bigwedge\mathcal{V}$ ($l\in$

$\{N_\beta, \dots, N_h + 1\}$) inductively as follows:

$$F^{\ge N_h+1}(\psi) := V_{(\lambda_1,\lambda_{-1})}(\psi), \quad T^{\ge N_h+1}(\psi) := 0,$$
$$J^{\ge N_h+1}(\psi) := F^{\ge N_h+1}(\psi) + T^{\ge N_h+1}(\psi).$$

For $l \in \{N_\beta, \dots, N_h\}$,

$$F^{\ge l}(\psi) := \int J^{\ge l+1}(\psi + \psi^0)\, \mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0),$$
$$T_n^{\ge l}(\psi) := T_{ree}(n, \mathcal{C}_l(w\mathbf{e}_p), J^{\ge l+1}) \ (\forall n \in \mathbb{N}_{\ge 2}), \quad T^{\ge l}(\psi) := \sum_{n=2}^{\infty} T_n^{\ge l}(\psi),$$
$$J^{\ge l}(\psi) := F^{\ge l}(\psi) + T^{\ge l}(\psi).$$

We will later make sure that $\sum_{n=2}^{\infty} T_n^{\ge l}(\psi)$ is well defined in $\bigoplus_{n=0}^{N_{L,h}} \mathcal{P}_n \bigwedge \mathcal{V}$ if the input $J^{\ge l+1}$ satisfies a certain smallness condition. For $m \in \mathbb{N} \cup \{0\}$, $l \in \{N_\beta, \dots, N_h + 1\}$ let

$$\begin{aligned}
F_m^{\ge l}(\psi) &:= \mathcal{P}_m F^{\ge l}(\psi) \\
&= \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} F_m^{\ge l}(\mathbf{X}^m, \mathbf{Y}^m)(\overline{\psi})_{\mathbf{X}^m}(\psi)_{\mathbf{Y}^m}, \\
T_{n,m}^{\ge l}(\psi) &:= \mathcal{P}_m T_n^{\ge l}(\psi) \\
&= \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} T_{n,m}^{\ge l}(\mathbf{X}^m, \mathbf{Y}^m)(\overline{\psi})_{\mathbf{X}^m}(\psi)_{\mathbf{Y}^m} \quad (\forall n \in \mathbb{N}_{\ge 2}), \\
T_m^{\ge l}(\psi) &:= \mathcal{P}_m T^{\ge l}(\psi) \\
&= \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} T_m^{\ge l}(\mathbf{X}^m, \mathbf{Y}^m)(\overline{\psi})_{\mathbf{X}^m}(\psi)_{\mathbf{Y}^m},
\end{aligned}$$

where $F_m^{\ge l}(\cdot,\cdot)$, $T_{n,m}^{\ge l}(\cdot,\cdot)$, $T_m^{\ge l}(\cdot,\cdot) : I_{L,h}^m \times I_{L,h}^m \to \mathbb{C}$ are bi-anti-symmetric. It will be convenient to set $J_{\mathrm{free},m}^{\ge l}(\psi) := F_m^{\ge l}(\psi)$, $J_{\mathrm{tree},m}^{\ge l}(\psi) := T_m^{\ge l}(\psi)$ and write

$$J_{b,m}^{\ge l}(\psi) = \left(\frac{1}{h}\right)^{2m} \sum_{\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m} J_{b,m}^{\ge l}(\mathbf{X}^m, \mathbf{Y}^m)(\overline{\psi})_{\mathbf{X}^m}(\psi)_{\mathbf{Y}^m}$$

with the bi-anti-symmetric kernel $J_{b,m}^{\ge l}(\cdot,\cdot)$ for $b \in \{\mathrm{free}, \mathrm{tree}\}$. Moreover, set

$$c_0 := \frac{\max\{c, 1\}}{(1-\varepsilon)\varepsilon^2} M^9 \max\{1, \beta\}^8, \tag{4.9}$$

where the constant c is taken to be the largest one among those appearing in the upper bounds of Lemmas 3.3 and 3.4. We observe that $c_0 \ge 1$ and

$$\|\mathcal{C}_l(w\mathbf{e}_p)\|_{1,\infty} \le c_0 M^{-l} \quad (\forall l \in \{N_\beta, \dots, N_h\}), \tag{4.10}$$

$$|\det(\langle \mathbf{u}_j, \mathbf{v}_k\rangle_{\mathbb{C}^m} \mathcal{C}_l(X_j, Y_k)(w\mathbf{e}_p))_{1\le j,k\le n}| \le c_0^n \tag{4.11}$$
$$(\forall l \in \{N_\beta, \dots, N_h\}, m, n \in \mathbb{N}, \mathbf{u}_j, \mathbf{v}_j \in \mathbb{C}^m \text{ with } \|\mathbf{u}_j\|_{\mathbb{C}^m}, \|\mathbf{v}_j\|_{\mathbb{C}^m} \le 1,$$
$$X_j, Y_j \in I_{L,h} (j = 1, \dots, n)),$$

$$|\mathcal{C}_l(\rho\hat{\mathbf{x}}\sigma 0, \eta\hat{\mathbf{y}}\tau 0)(w\mathbf{e}_p)| \le c_0(M^{l-N_h} + M^{N_\beta - l}) \tag{4.12}$$
$$(\forall l \in \{N_\beta + 1, \dots, N_h\}, \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{Z}^2$$
$$\text{with } 0 \le \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|_{\mathbb{R}^2} \le L/2, \rho, \eta \in \{1,2,3\}, \sigma, \tau \in \{\uparrow, \downarrow\}).$$

Let us introduce a parameter $\alpha \in \mathbb{R}_{>0}$. As the main objective in this subsection we will prove the following:

Proposition 4.1. *Assume that*

$$\begin{aligned} &M \ge \max\{78E_{\max}^2, 2^8\}, \quad \alpha \ge 2^{10}M^2, \\ &|\lambda_1|, |\lambda_{-1}|, |U_{\mathrm{c}}|, |U_{\mathrm{o}}| < 2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}. \end{aligned} \tag{4.13}$$

Then for any $l \in \{N_\beta + 1, \dots, N_h + 1\}$ the following inequalities hold:

$$\begin{aligned} &M^{-N_\beta}\alpha c_0 \sum_{b\in\{\text{free},\text{tree}\}} \|J_{b,1}^{\ge l}\|_{1,\infty} < 1, \\ &M^{-N_\beta}\sum_{m=1}^{N_{L,h}} \alpha^m c_0^m M^{(l-N_\beta)(m-2)} \sum_{b\in\{\text{free},\text{tree}\}} \|J_{b,m}^{\ge l}\|_{1,\infty} < 1. \end{aligned} \tag{4.14}$$

The core part of the proof of Proposition 4.1 is the estimation of $\|T_{n,m}^{\ge l}\|_{1,\infty}$, which needs the next lemma.

Lemma 4.2. *For any $\tilde{\mathbf{X}}^{m_j} \in \tilde{I}_{L,h}^{m_j}$ $(j = 1, \dots, n)$, $T \in \mathbb{T}_n$, $\xi \in \mathbb{S}_n(T)$, $\mathbf{s} \in [0,1]^{n-1}$ and $l \in \{N_\beta, \dots, N_h\}$,*

$$\left| e^{\sum_{q,r=1}^n M_{at}(T,\xi,\mathbf{s})_{q,r}\Delta_{q,r}(\mathcal{C}_l(w\mathbf{e}_p))} \prod_{j=1}^n (\psi^j)_{\tilde{\mathbf{X}}^{m_j}} \Big|_{\substack{\psi^j = \mathbf{0} \\ \forall j\in\{1,\dots,n\}}} \right| \le c_0^{\frac{1}{2}\sum_{j=1}^n m_j}.$$

Proof. This can be proved by using (4.11) and the properties of $M_{at}(T, \xi, \mathbf{s})$ and by repeating the same argument as in [10, Lemma 4.5]. □

Lemma 4.3. *For any $m \in \{0, \dots, N_{L,h}\}$, $n \in \mathbb{N}_{\ge 2}$ and $l \in \{N_\beta, \dots, N_h\}$,*

$$\begin{aligned} \|T_{n,m}^{\ge l}\|_{1,\infty} &\le (1_{m=0}N_{L,h}/h + 1_{m\ge 1})2^{-3m}c_0^{-m}\frac{1}{n(n-1)}M^{-l(n-1)} \\ &\quad \cdot \prod_{j=1}^n \left(\sum_{m_j=1}^{N_{L,h}} 2^{5m_j} c_0^{m_j} \|J_{m_j}^{\ge l+1}\|_{1,\infty} \right) 1_{\sum_{j=1}^n m_j - n + 1 \ge m}. \end{aligned}$$

Proof. For $T \in \mathbb{T}_n$ and a matrix $Q = (Q(X,Y))_{X,Y\in I_{L,h}}$ define the operator $O_{pe}(T,Q)$ on $\bigwedge\left(\left(\bigoplus_{j=1}^n \mathcal{V}_j\right)\bigoplus\mathcal{V}\right)$ by

$$O_{pe}(T,Q) := \int_{[0,1]^{n-1}} d\mathbf{s} \sum_{\xi\in\mathbb{S}_n(T)} \varphi(T,\xi,\mathbf{s}) e^{\sum_{q,r=1}^n M_{at}(T,\xi,\mathbf{s})_{q,r}\Delta_{q,r}(Q)} \cdot \prod_{\{q,r\}\in T} (\Delta_{q,r}(Q)+\Delta_{r,q}(Q)). \tag{4.15}$$

It follows from the definition that

$$T_{n,m}^{\ge l}(\psi) = \prod_{j=1}^n \left(\sum_{m_j=1}^{N_{L,h}} \left(\frac{1}{h}\right)^{2m_j} \sum_{\mathbf{X}^{m_j},\mathbf{Y}^{m_j}\in I_{L,h}^{m_j}} J_{m_j}^{\ge l+1}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})\right) 1_{\sum_{j=1}^n m_j-n+1\ge m} \cdot\frac{1}{n!}\sum_{T\in\mathbb{T}_n}\mathcal{P}_m\left(O_{pe}(T,\mathcal{C}_l(w\mathbf{e}_p))\prod_{q=1}^n(\overline{\psi}^q+\overline{\psi})_{\mathbf{X}^{m_q}}(\psi^q+\psi)_{\mathbf{Y}^{m_q}}\Big|_{\substack{\psi^j=\mathbf{0}\\ \forall j\in\{1,\dots,n\}}}\right).$$

The constraint $1_{\sum_{j=1}^n m_j-n+1\ge m}$ is due to the fact that the operator $\prod_{\{q,r\}\in T}$ $(\Delta_{q,r}(\mathcal{C}_l(w\mathbf{e}_p)) \ +\Delta_{r,q}(\mathcal{C}_l(w\mathbf{e}_p)))$ erases $n-1$ fields from $\prod_{j=1}^n(\overline{\psi}^j)_{\mathbf{X}^{m_j}}$ and from $\prod_{j=1}^n(\psi^j)_{\mathbf{Y}^{m_j}}$, respectively. Using anti-symmetry,

$$\begin{aligned}
T_{n,m}^{\ge l}(\psi) = \prod_{j=1}^n &\left(\sum_{m_j=1}^{N_{L,h}} \left(\frac{1}{h}\right)^{2m_j} \sum_{\mathbf{X}^{m_j},\mathbf{Y}^{m_j}\in I_{L,h}^{m_j}} \sum_{k_j,l_j=0}^{m_j} \sum_{\mathbf{W}^{k_j}\subset\mathbf{X}^{m_j}} \sum_{\mathbf{Z}^{l_j}\subset\mathbf{Y}^{m_j}} \right.\\
&\left.\cdot J_{m_j}^{\ge l+1}((\mathbf{W}^{k_j},\mathbf{X}^{m_j}\backslash\mathbf{W}^{k_j}),(\mathbf{Z}^{l_j},\mathbf{Y}^{m_j}\backslash\mathbf{Z}^{l_j}))\right)\\
&\cdot 1_{\sum_{j=1}^n m_j-n+1\ge m} 1_{\sum_{j=1}^n k_j=\sum_{j=1}^n l_j=m}\\
&\cdot\frac{1}{n!}\sum_{T\in\mathbb{T}_n}\left(O_{pe}(T,\mathcal{C}_l(w\mathbf{e}_p))\prod_{q=1}^n(\overline{\psi})_{\mathbf{W}^{k_q}}(\overline{\psi}^q)_{\mathbf{X}^{m_q}\backslash\mathbf{W}^{k_q}}\right.\\
&\left.\cdot(\psi)_{\mathbf{Z}^{l_q}}(\psi^q)_{\mathbf{Y}^{m_q}\backslash\mathbf{Z}^{l_q}}\Big|_{\substack{\psi^j=\mathbf{0}\\ \forall j\in\{1,\dots,n\}}}\right)\\
= \prod_{j=1}^n &\left(\sum_{m_j=1}^{N_{L,h}}\sum_{k_j,l_j=0}^{m_j}\binom{m_j}{k_j}\binom{m_j}{l_j}\right) 1_{\sum_{j=1}^n m_j-n+1\ge m}1_{\sum_{j=1}^n k_j=\sum_{j=1}^n l_j=m}\\
&\cdot\frac{1}{n!}\sum_{T\in\mathbb{T}_n}\prod_{q=1}^n\left(\left(\frac{1}{h}\right)^{k_q+l_q}\sum_{\mathbf{W}^{k_q}\in I_{L,h}^{k_q}}\sum_{\mathbf{Z}^{l_q}\in I_{L,h}^{l_q}}\right)\\
&\cdot J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}((\mathbf{W}^{k_1},\dots,\mathbf{W}^{k_n}),(\mathbf{Z}^{l_1},\dots,\mathbf{Z}^{l_n}))\\
&\cdot\prod_{r=1}^n(\overline{\psi})_{\mathbf{W}^{k_r}}\prod_{s=1}^n(\psi)_{\mathbf{Z}^{l_s}},
\end{aligned}$$

where

$$J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}((\mathbf{W}^{k_1},\dots,\mathbf{W}^{k_n}),(\mathbf{Z}^{l_1},\dots,\mathbf{Z}^{l_n}))$$
$$:= \prod_{j=1}^{n}\Bigg(\left(\frac{1}{h}\right)^{2m_j-k_j-l_j}\sum_{\mathbf{X}^{m_j-k_j}\in I_{L,h}^{m_j-k_j}}\sum_{\mathbf{Y}^{m_j-l_j}\in I_{L,h}^{m_j-l_j}} \cdot J_{m_j}^{\ge l+1}((\mathbf{W}^{k_j},\mathbf{X}^{m_j-k_j}),(\mathbf{Z}^{l_j},\mathbf{Y}^{m_j-l_j}))\Bigg)$$
$$\cdot\varepsilon_{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}O_{pe}(T,\mathcal{C}_l(w\mathbf{e}_p))$$
$$\cdot\prod_{q=1}^{n}(\overline{\psi}^q)_{\mathbf{X}^{m_q-k_q}}(\psi^q)_{\mathbf{Y}^{m_q-l_q}}\Big|_{\substack{\psi^j=\mathbf{0}\\ \forall j\in\{1,\dots,n\}}}, \tag{4.16}$$

with the factor $\varepsilon_{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)} \in \{1,-1\}$ depending only on $(m_1,\dots,m_n)$, $(k_1,\dots,k_n)$, $(l_1,\dots,l_n)$. By (4.3) and the triangle inequality of the norm $\|\cdot\|_{1,\infty}$,

$$\|T_{n,m}^{\ge l}\|_{1,\infty} \le \prod_{j=1}^{n}\left(\sum_{m_j=1}^{N_{L,h}}\sum_{k_j,l_j=0}^{m_j}\binom{m_j}{k_j}\binom{m_j}{l_j}\right)$$
$$\cdot 1_{\sum_{j=1}^n m_j-n+1\ge m}1_{\sum_{j=1}^n k_j=\sum_{j=1}^n l_j=m}$$
$$\cdot\frac{1}{n!}\sum_{T\in\mathbb{T}_n}\|J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_{1,\infty}. \tag{4.17}$$

Let us find an upper bound on $\|J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_{1,\infty}$. Let d_j denote the incidence number of the vertex j in T. If $d_j > 2m_j - k_j - l_j$ for some $j\in\{1,\dots,n\}$, $\|J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_{1,\infty}=0$, since in this case

$$\prod_{\{q,r\}\in T}(\Delta_{q,r}(\mathcal{C}_l(w\mathbf{e}_p))+\Delta_{r,q}(\mathcal{C}_l(w\mathbf{e}_p)))\prod_{s=1}^{n}(\overline{\psi}^s)_{\mathbf{X}^{m_s-k_s}}(\psi^s)_{\mathbf{Y}^{m_s-l_s}}=0$$

for any $\mathbf{X}^{m_s-k_s}\in I_{L,h}^{m_s-k_s}$, $\mathbf{Y}^{m_s-l_s}\in I_{L,h}^{m_s-l_s}$ $(s=1,\dots,n)$.

Assume that $d_j \le 2m_j-k_j-l_j$ $(\forall j\in\{1,\dots,n\})$. First consider the case that $m\neq 0$. Let $q_0\in\{1,\dots,n\}$ be a vertex with $k_{q_0}\neq 0$. For $q,r\in\{1,\dots,n\}$ let $\mathrm{dis}_T(q,r)(\in\mathbb{N}\cup\{0\})$ denote the distance between the vertex q and the vertex r along the unique path connecting q with r in T. Define $L_r^q(T)\subset T$ by

$$L_r^q(T) := \{\{r,s\}\in T \mid \mathrm{dis}_T(q,s)=\mathrm{dis}_T(q,r)+1\}. \tag{4.18}$$

Note that if $d_r=1$ and $r\neq q_0$, then $L_r^{q_0}(T)=\emptyset$. If $d_r\neq 1$ or $r=q_0$, we can number each line of $L_r^{q_0}(T)$ so that

$$L_{q_0}^{q_0}(T)=\{\{q_0,s_1^{q_0}\},\{q_0,s_2^{q_0}\},\dots,\{q_0,s_{d_{q_0}}^{q_0}\}\},$$
$$L_r^{q_0}(T)=\{\{r,s_1^r\},\{r,s_2^r\},\dots,\{r,s_{d_r-1}^r\}\}$$
$$(\forall r\in\{1,\dots,n\}\backslash\{q_0\} \text{ with } d_r\neq 1).$$

For any $\{q_0,s\}\in L_{q_0}^{q_0}(T)$ there uniquely exists $j\in\{1,2,\dots,d_{q_0}\}$ such that $\{q_0,s\}=\{q_0,s_j^{q_0}\}$. For $\nu\in\mathbb{S}_{d_{q_0}}$ set $\nu(\{q_0,s\}):=\nu(j)$. Similarly, for $r\in$

$\{1,\dots,n\}\backslash\{q_0\}$ with $d_r \neq 1$, $\{r,s\} \in L_r^{q_0}(T)$ and $\nu \in \mathbb{S}_{d_r-1}$ let $\nu(\{r,s\}) \in \{1,2,\dots,d_r-1\}$ be defined by $\nu(\{r,s\}) := \nu(j)$, where $s = s_j^r$.

Moreover, define $\tilde{\mathcal{C}} : \tilde{I}_{L,h} \times \tilde{I}_{L,h} \to \mathbb{C}$ by

$$\tilde{\mathcal{C}}((X,u),(Y,v)) := \begin{cases} 0 & \text{if } u = v, \\ -\mathcal{C}_l(X,Y)(w\mathbf{e}_p) & \text{if } u = 1, v = -1, \\ \mathcal{C}_l(Y,X)(w\mathbf{e}_p) & \text{if } u = -1, v = 1. \end{cases}$$

By considering q_0 as the root of T we see that

$$\begin{aligned}
&\prod_{\{q,r\}\in T} (\Delta_{q,r}(\mathcal{C}_l(w\mathbf{e}_p)) + \Delta_{r,q}(\mathcal{C}_l(w\mathbf{e}_p))) \prod_{j=1}^{n} (\overline{\psi}^j)_{\mathbf{X}^{m_j-k_j}} (\psi^j)_{\mathbf{Y}^{m_j-l_j}} \\
&= \prod_{\{q,r\}\in T} \left(\sum_{\tilde{X},\tilde{Y}\in \tilde{I}_{L,h}} \tilde{\mathcal{C}}(\tilde{X},\tilde{Y}) \frac{\partial}{\partial \psi^q_{\tilde{X}}} \frac{\partial}{\partial \psi^r_{\tilde{Y}}} \right) \prod_{j=1}^{n} (\psi^j)_{\tilde{\mathbf{X}}(1)^{m_j-k_j}} (\psi^j)_{\tilde{\mathbf{Y}}(-1)^{m_j-l_j}} \\
&= \prod_{\substack{j=1\\ j\neq q_0}}^{n} \Bigg(\sum_{\substack{\tilde{X}_j \in \tilde{I}_{L,h} \text{ with} \\ \tilde{X}_j \subset (\tilde{\mathbf{X}}(1)^{m_j-k_j}, \tilde{\mathbf{Y}}(-1)^{m_j-l_j})}} \Bigg) \\
&\quad \cdot \sum_{\substack{\tilde{\mathbf{X}}^{d_{q_0}} \in \tilde{I}_{L,h}^{d_{q_0}} \text{ with} \\ \tilde{\mathbf{X}}^{d_{q_0}} \subset (\tilde{\mathbf{X}}(1)^{m_{q_0}-k_{q_0}}, \tilde{\mathbf{Y}}(-1)^{m_{q_0}-l_{q_0}})}} \sum_{\nu_{q_0}\in \mathbb{S}_{d_{q_0}}} \prod_{\{q_0,r\}\in L_{q_0}^{q_0}(T)} \tilde{\mathcal{C}}(\tilde{X}^{d_{q_0}}_{\nu_{q_0}(\{q_0,r\})}, \tilde{X}_r) \\
&\quad \cdot \prod_{\substack{q=1\\ q\neq q_0 \text{ and } d_q\neq 1}}^{n} \Bigg(\sum_{\substack{\tilde{\mathbf{X}}^{d_q-1} \in \tilde{I}_{L,h}^{d_q-1} \text{ with} \\ \tilde{\mathbf{X}}^{d_q-1} \subset (\tilde{\mathbf{X}}(1)^{m_q-k_q}, \tilde{\mathbf{Y}}(-1)^{m_q-l_q})\backslash \tilde{X}_q}} \sum_{\nu_q \in \mathbb{S}_{d_q-1}} \\
&\quad \cdot \prod_{\{q,r\}\in L_q^{q_0}(T)} \tilde{\mathcal{C}}(\tilde{X}^{d_q-1}_{\nu_q(\{q,r\})}, \tilde{X}_r) \Bigg) \\
&\quad \cdot \varepsilon_{\pm} \cdot (\psi^{q_0})_{(\tilde{\mathbf{X}}(1)^{m_{q_0}-k_{q_0}}, \tilde{\mathbf{Y}}(-1)^{m_{q_0}-l_{q_0}})\backslash \tilde{\mathbf{X}}^{d_{q_0}}} \\
&\quad \cdot \prod_{\substack{s=1\\ s\neq q_0}}^{n} \big(1_{d_s\neq 1} (\psi^s)_{((\tilde{\mathbf{X}}(1)^{m_s-k_s}, \tilde{\mathbf{Y}}(-1)^{m_s-l_s})\backslash \tilde{X}_s)\backslash \tilde{\mathbf{X}}^{d_s-1}} \\
&\quad + 1_{d_s=1} (\psi^s)_{(\tilde{\mathbf{X}}(1)^{m_s-k_s}, \tilde{\mathbf{Y}}(-1)^{m_s-l_s})\backslash \tilde{X}_s} \big),
\end{aligned} \tag{4.19}$$

where $\varepsilon_{\pm} = 1$ or -1. In the following, let $\prod_{\substack{u=1\\ \text{ordered}}}^{v} g_u$ denote $g_1 g_2 \dots g_v$ for $v \in \mathbb{N}$. One finds this notation useful when each term g_u depends on g_1, $g_2, \dots, g_{u-1}$. Moreover, set $d(T, q_0) := \max_{1\le j\le n} \mathrm{dis}_T(q_0, j)$. By substituting (4.19) into (4.16) and using (4.8), (4.10), and Lemma 4.2 we have for any $W_{fixed} \in I_{L,h}$ that

$$\begin{aligned}
&\left(\frac{1}{h}\right)^{2m-1} \sum_{\mathbf{W}^{k_{q_0}-1} \in I_{L,h}^{k_{q_0}-1}} \sum_{\mathbf{Z}^{l_{q_0}} \in I_{L,h}^{l_{q_0}}} \prod_{\substack{j=1\\ j\neq q_0}}^{n} \left(\sum_{\mathbf{W}^{k_j} \in I_{L,h}^{k_j}} \sum_{\mathbf{Z}^{l_j} \in I_{L,h}^{l_j}} \right) \\
&\quad \cdot |J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}((\mathbf{W}^{k_1}, \dots, \mathbf{W}^{k_{q_0}-1}, (W_{fixed}, \mathbf{W}^{k_{q_0}-1}),
\end{aligned}$$

$$\mathbf{W}^{k_{q_0+1}},\dots,\mathbf{W}^{k_n}),(\mathbf{Z}^{l_1},\dots,\mathbf{Z}^{l_n}))|$$

$$\le\left(\frac{1}{h}\right)^{2m_{q_0}-1}\sum_{\mathbf{W}^{k_{q_0}-1}\in I_{L,h}^{k_{q_0}-1}}\sum_{\mathbf{Z}^{l_{q_0}}\in I_{L,h}^{l_{q_0}}}\sum_{\mathbf{X}^{m_{q_0}-k_{q_0}}\in I_{L,h}^{m_{q_0}-k_{q_0}}}\sum_{\mathbf{Y}^{m_{q_0}-l_{q_0}}\in I_{L,h}^{m_{q_0}-l_{q_0}}}$$

$$\cdot|J_{m_{q_0}}^{\ge l+1}((W_{fixed},\mathbf{W}^{k_{q_0}-1},\mathbf{X}^{m_{q_0}-k_{q_0}}),(\mathbf{Z}^{l_{q_0}},\mathbf{Y}^{m_{q_0}-l_{q_0}}))|$$

$$\cdot\sum_{\substack{\tilde{\mathbf{X}}^{d_{q_0}}\in\tilde{I}_{L,h}^{d_{q_0}}\text{ with}\\ \tilde{\mathbf{X}}^{d_{q_0}}\subset(\tilde{\mathbf{X}}(1)^{m_{q_0}-k_{q_0}},\tilde{\mathbf{Y}}(-1)^{m_{q_0}-l_{q_0}})}}\sum_{\nu_{q_0}\in\mathbb{S}_{d_{q_0}}}\prod_{\{q_0,r\}\in L_{q_0}^{q_0}(T)}$$

$$\cdot\Bigg(\left(\frac{1}{h}\right)^{2m_r}\sum_{\mathbf{W}^{k_r}\in I_{L,h}^{k_r}}\sum_{\mathbf{Z}^{l_r}\in I_{L,h}^{l_r}}\sum_{\mathbf{X}^{m_r-k_r}\in I_{L,h}^{m_r-k_r}}\sum_{\mathbf{Y}^{m_r-l_r}\in I_{L,h}^{m_r-l_r}}$$

$$\cdot\sum_{\substack{\tilde{X}_r\in\tilde{I}_{L,h}\text{ with}\\ \tilde{X}_r\subset(\tilde{\mathbf{X}}(1)^{m_r-k_r},\tilde{\mathbf{Y}}(-1)^{m_r-l_r})}}$$

$$\cdot|J_{m_r}^{\ge l+1}((\mathbf{W}^{k_r},\mathbf{X}^{m_r-k_r}),(\mathbf{Z}^{l_r},\mathbf{Y}^{m_r-l_r}))||\tilde{\mathcal{C}}(\tilde{X}_{\nu_{q_0}(\{q_0,r\})}^{d_{q_0}},\tilde{X}_r)|\Bigg)$$

$$\cdot\prod_{\substack{u=1\\ \text{ordered}}}^{d(T,q_0)-1}\Bigg(\prod_{\substack{j\in\{1,\dots,n\}\text{ with}\\ \mathrm{dis}_T(q_0,j)=u\text{ and }d_j\neq 1}}\Bigg(\sum_{\substack{\tilde{\mathbf{X}}^{d_j-1}\in\tilde{I}_{L,h}^{d_j-1}\text{ with}\\ \tilde{\mathbf{X}}^{d_j-1}\subset(\tilde{\mathbf{X}}(1)^{m_j-k_j},\tilde{\mathbf{Y}}(-1)^{m_j-l_j})\backslash\tilde{X}_j}}$$

$$\cdot\sum_{\nu_j\in\mathbb{S}_{d_j-1}}\prod_{\{j,r\}\in L_j^{q_0}(T)}$$

$$\cdot\Bigg(\left(\frac{1}{h}\right)^{2m_r}\sum_{\mathbf{W}^{k_r}\in I_{L,h}^{k_r}}\sum_{\mathbf{Z}^{l_r}\in I_{L,h}^{l_r}}\sum_{\mathbf{X}^{m_r-k_r}\in I_{L,h}^{m_r-k_r}}\sum_{\mathbf{Y}^{m_r-l_r}\in I_{L,h}^{m_r-l_r}}$$

$$\cdot\sum_{\substack{\tilde{X}_r\in\tilde{I}_{L,h}\text{ with}\\ \tilde{X}_r\subset(\tilde{\mathbf{X}}(1)^{m_r-k_r},\tilde{\mathbf{Y}}(-1)^{m_r-l_r})}}$$

$$\cdot|J_{m_r}^{\ge l+1}((\mathbf{W}^{k_r},\mathbf{X}^{m_r-k_r}),(\mathbf{Z}^{l_r},\mathbf{Y}^{m_r-l_r}))||\tilde{\mathcal{C}}(\tilde{X}_{\nu_j(\{j,r\})}^{d_j-1},\tilde{X}_r)|\Bigg)\Bigg)\Bigg)$$

$$\cdot\Bigg|\int_{[0,1]^{n-1}}\mathrm{d}\mathbf{s}\sum_{\xi\in\mathbb{S}_n(T)}\varphi(T,\xi,\mathbf{s})\mathrm{e}^{\sum_{q,r=1}^n M_{at}(T,\xi,\mathbf{s})_{q,r}\Delta_{q,r}(\mathcal{C}_l(w\mathbf{e}_p))}$$

$$\cdot(\psi^{q_0})_{(\tilde{\mathbf{X}}(1)^{m_{q_0}-k_{q_0}},\tilde{\mathbf{Y}}(-1)^{m_{q_0}-l_{q_0}})\backslash\tilde{\mathbf{X}}^{d_{q_0}}}\prod_{\substack{s=1\\ s\neq q_0}}^{n}$$

$$\cdot\Big(1_{d_s\neq 1}(\psi^s)_{((\tilde{\mathbf{X}}(1)^{m_s-k_s},\tilde{\mathbf{Y}}(-1)^{m_s-l_s})\backslash\tilde{X}_s)\backslash\tilde{\mathbf{X}}^{d_s-1}}$$

$$+1_{d_s=1}(\psi^s)_{(\tilde{\mathbf{X}}(1)^{m_s-k_s},\tilde{\mathbf{Y}}(-1)^{m_s-l_s})\backslash\tilde{X}_s}\Big)\Big|_{\substack{\psi^j=\mathbf{0}\\ \forall j\in\{1,\dots,n\}}}\Bigg|$$

$$\le \|J_{m_{q_0}}^{\ge l+1}\|_{1,\infty}\binom{2m_{q_0}-k_{q_0}-l_{q_0}}{d_{q_0}}d_{q_0}!$$
$$\cdot\prod_{\{q_0,r\}\in L_{q_0}^{q_0}(T)}\left((2m_r-k_r-l_r)\|J_{m_r}^{\ge l+1}\|_{1,\infty}c_0M^{-l}\right)$$
$$\cdot\prod_{u=1}^{d(T,q_0)-1}\Bigg(\prod_{\substack{j\in\{1,\dots,n\}\text{ with}\\ \mathrm{dis}_T(q_0,j)=u\text{ and }d_j\neq 1}}\Bigg(\binom{2m_j-k_j-l_j-1}{d_j-1}(d_j-1)!$$
$$\cdot\prod_{\{j,r\}\in L_j^{q_0}(T)}((2m_r-k_r-l_r)\|J_{m_r}^{\ge l+1}\|_{1,\infty}c_0M^{-l})\Bigg)\Bigg)c_0^{\frac{1}{2}\sum_{q=1}^n(2m_q-k_q-l_q-d_q)}$$
$$=(c_0M^{-l})^{n-1}\prod_{j=1}^n\Bigg(c_0^{\frac{1}{2}(2m_j-k_j-l_j-d_j)}\|J_{m_j}^{\ge l+1}\|_{1,\infty}(2m_j-k_j-l_j)$$
$$\cdot\binom{2m_j-k_j-l_j-1}{d_j-1}(d_j-1)!\Bigg). \tag{4.20}$$

By arbitrariness of q_0 and the fixed variable W_{fixed}, $\|J_T^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_{1,\infty}$ can be bounded by the right-hand side of (4.20).

In the case that $m=0$ we fix any $q_0\in\{1,\dots,n\}$ and repeat the same calculation as above by setting k_j, l_j to be 0 for all $j\in\{1,\dots,n\}$. The only difference in the consequence is that $\|J_{m_{q_0}}^{\ge l+1}\|_1$ comes in place of $\|J_{m_{q_0}}^{\ge l+1}\|_{1,\infty}$. Since $\|J_{m_{q_0}}^{\ge l+1}\|_1\le(N_{L,h}/h)\|J_{m_{q_0}}^{\ge l+1}\|_{1,\infty}$, we only need to multiply the right-hand side of (4.20) by the extra factor $N_{L,h}/h$ in this case.

By substituting these results into (4.17), replacing the sum over trees by the sum over possible incidence numbers and using Cayley's theorem on the number of trees with fixed incidence numbers, we can deduce that

$$\|T_{n,m}^{\ge l}\|_{1,\infty}$$
$$\le\prod_{j=1}^n\left(\sum_{m_j=1}^{N_{L,h}}\sum_{k_j,l_j=0}^{m_j}\binom{m_j}{k_j}\binom{m_j}{l_j}\right)1_{\sum_{j=1}^n m_j-n+1\ge m}1_{\sum_{j=1}^n k_j=\sum_{j=1}^n l_j=m}$$
$$\cdot\frac{1}{n!}\prod_{q=1}^n\left(\sum_{d_q=1}^{2m_q-k_q-l_q}\right)1_{\sum_{q=1}^n d_q=2(n-1)}\frac{(n-2)!}{\prod_{s=1}^n(d_s-1)!}$$
$$\cdot(1_{m=0}N_{L,h}/h+1_{m\ge 1})(c_0M^{-l})^{n-1}$$
$$\cdot\prod_{r=1}^n\Bigg(c_0^{\frac{1}{2}(2m_r-k_r-l_r-d_r)}\|J_{m_r}^{\ge l+1}\|_{1,\infty}(2m_r-k_r-l_r)$$
$$\cdot\binom{2m_r-k_r-l_r-1}{d_r-1}(d_r-1)!\Bigg)$$

$$= (1_{m=0}N_{L,h}/h + 1_{m\ge 1})c_0^{-m}\frac{1}{n(n-1)}M^{-l(n-1)}$$
$$\cdot\prod_{j=1}^{n}\left(\sum_{m_j=1}^{N_{L,h}}\sum_{k_j,l_j=0}^{m_j}\binom{m_j}{k_j}\binom{m_j}{l_j}c_0^{m_j}\|J_{m_j}^{\ge l+1}\|_{1,\infty}\right)$$
$$\cdot 1_{\sum_{j=1}^{n}m_j-n+1\ge m}1_{\sum_{j=1}^{n}k_j=\sum_{j=1}^{n}l_j=m}$$
$$\cdot\prod_{q=1}^{n}\left((2m_q-k_q-l_q)\sum_{d_q=1}^{2m_q-k_q-l_q}\binom{2m_q-k_q-l_q-1}{d_q-1}\right)1_{\sum_{q=1}^{n}d_q=2(n-1)}. \tag{4.21}$$

By using the inequality that $2m_q - k_q - l_q \le 2^{m_q-(k_q+l_q)/2+1}$ one has

$$\prod_{q=1}^{n}\left((2m_q-k_q-l_q)\sum_{d_q=1}^{2m_q-k_q-l_q}\binom{2m_q-k_q-l_q-1}{d_q-1}\right)\le 2^{3\sum_{q=1}^{n}m_q-3m}. \tag{4.22}$$

By combining (4.22) with (4.21), dropping the constraints $1_{\sum_{j=1}^{n}k_j=\sum_{j=1}^{n}l_j=m}$, $1_{\sum_{q=1}^{n}d_q=2(n-1)}$ and summing over k_j, l_j $(j = 1,\dots,n)$ we obtain the claimed upper bound. □

The following lemma will not be used until Sect. 4.4. Since its proof is close to the proof of Lemma 4.3, let us show at this point

Lemma 4.4. *For any $m, m' \in \{0,\dots,N_{L,h}\}$, $n\in\mathbb{N}_{\ge 2}$, $b\in\{\text{free},\text{tree}\}$, $\mathbf{X}^{m'}, \mathbf{Y}^{m'} \in I_{L,h}^{m'}$ and $l\in\{N_\beta,\dots,N_h\}$,*

$$\left\|\frac{\partial T_{n,m}^{\ge l}}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right\|_1 \le 1_{m'\ge 1}(m'!)^2h^{-2m'}2^{5m'-3m}c_0^{m'-m}M^{-l(n-1)}$$
$$\cdot\prod_{j=1}^{n-1}\left(\sum_{m_j=1}^{N_{L,h}}2^{5m_j}c_0^{m_j}\|J_{m_j}^{\ge l+1}\|_{1,\infty}\right)1_{\sum_{j=1}^{n-1}m_j+m'-n+1\ge m}.$$

Proof. By using anti-symmetry,

$$\frac{\partial T_{n,m}^{\ge l}(\psi)}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}$$
$$= h^{-2m'}(m'!)^2\sum_{j_0=1}^{n}\prod_{\substack{j=1\\ j\neq j_0}}^{n}\left(\sum_{m_j=1}^{N_{L,h}}\left(\frac{1}{h}\right)^{2m_j}\sum_{\mathbf{X}^{m_j},\mathbf{Y}^{m_j}\in I_{L,h}^{m_j}}J_{m_j}^{\ge l+1}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})\right)$$
$$\cdot 1_{\sum_{j=1,j\neq j_0}^{n}m_j+m'-n+1\ge m}\frac{1}{n!}\sum_{T\in\mathbb{T}_n}\mathcal{P}_m\Bigg(O_{pe}(T,\mathcal{C}_l(w\mathbf{e}_p))$$

$$\cdot(\overline{\psi}^{j_0}+\overline{\psi})_{\mathbf{X}^{m'}}(\psi^{j_0}+\psi)_{\mathbf{Y}^{m'}}$$
$$\times\prod_{\substack{q=1\\q\neq j_0}}^{n}(\overline{\psi}^{q}+\overline{\psi})_{\mathbf{X}^{m_q}}(\psi^{q}+\psi)_{\mathbf{Y}^{m_q}}\Big|_{\substack{\psi^j=\mathbf{0}\\\forall j\in\{1,\dots,n\}}}\Bigg). \tag{4.23}$$

We can see from (4.23) that $\partial T^{\geq l}_{n,m}(\psi)/\partial J^{\geq l+1}_{b,m'}(\mathbf{X}^{m'},\mathbf{Y}^{m'})=0$ if $m'=0$, since $\prod_{\{q,r\}\in T}(\Delta_{q,r}(\mathcal{C}_l(w\mathbf{e}_p))+\Delta_{r,q}(\mathcal{C}_l(w\mathbf{e}_p)))\prod_{j=1,j\neq j_0}^{n}(\overline{\psi}^{j}+\overline{\psi})_{\mathbf{X}^{m_j}}(\psi^{j}+\psi)_{\mathbf{Y}^{m_j}}=0$. The equality (4.23) leads to

$$\frac{\partial T^{\geq l}_{n,m}(\psi)}{\partial J^{\geq l+1}_{b,m'}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}=h^{-2m'}(m'!)^2\sum_{j_0=1}^{n}\sum_{m_{j_0}=1}^{N_{L,h}}\sum_{k_{j_0},l_{j_0}=0}^{m_{j_0}}1_{m_{j_0}=m'}$$
$$\cdot\prod_{\substack{j=1\\j\neq j_0}}^{n}\left(\sum_{m_j=1}^{N_{L,h}}\sum_{k_j,l_j=0}^{m_j}\binom{m_j}{k_j}\binom{m_j}{l_j}\right)$$
$$\cdot 1_{\sum_{j=1}^{n}m_j-n+1\geq m}1_{\sum_{j=1}^{n}k_j=\sum_{j=1}^{n}l_j=m}\frac{1}{n!}\sum_{T\in\mathbb{T}_n}$$
$$\cdot\prod_{q=1}^{n}\left(\left(\frac{1}{h}\right)^{k_q+l_q}\sum_{\mathbf{W}^{k_q}\in I_{L,h}^{k_q}}\sum_{\mathbf{Z}^{l_q}\in I_{L,h}^{l_q}}\right)$$
$$\cdot J_{T,j_0}^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}((\mathbf{W}^{k_1},\dots,\mathbf{W}^{k_n}),(\mathbf{Z}^{l_1},\dots,\mathbf{Z}^{l_n}))$$
$$\cdot\prod_{r=1}^{n}(\overline{\psi})_{\mathbf{W}^{k_r}}\prod_{s=1}^{n}(\psi)_{\mathbf{Z}^{l_s}},$$

where

$$J_{T,j_0}^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}((\mathbf{W}^{k_1},\dots,\mathbf{W}^{k_n}),(\mathbf{Z}^{l_1},\dots,\mathbf{Z}^{l_n}))$$
$$:=h^{k_{j_0}+l_{j_0}}1_{\mathbf{W}^{k_{j_0}}\subset\mathbf{X}^{m'}}1_{\mathbf{Z}^{l_{j_0}}\subset\mathbf{Y}^{m'}}\prod_{\substack{j=1\\j\neq j_0}}^{n}\Bigg(\left(\frac{1}{h}\right)^{2m_j-k_j-l_j}$$
$$\cdot\sum_{\mathbf{X}^{m_j-k_j}\in I_{L,h}^{m_j-k_j}}\sum_{\mathbf{Y}^{m_j-l_j}\in I_{L,h}^{m_j-l_j}}$$
$$\cdot J^{\geq l+1}_{m_j}((\mathbf{W}^{k_j},\mathbf{X}^{m_j-k_j}),(\mathbf{Z}^{l_j},\mathbf{Y}^{m_j-l_j}))\Bigg)$$
$$\cdot\varepsilon_{\mathbf{W}^{k_{j_0}},\mathbf{X}^{m'},\mathbf{Z}^{l_{j_0}},\mathbf{Y}^{m'},j_0,(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}$$
$$\cdot O_{pe}(T,\mathcal{C}_l(w\mathbf{e}_p))(\overline{\psi}^{j_0})_{\mathbf{X}^{m'}\backslash\mathbf{W}^{k_{j_0}}}(\psi^{j_0})_{\mathbf{Y}^{m'}\backslash\mathbf{Z}^{l_{j_0}}}$$
$$\cdot\prod_{\substack{q=1\\q\neq j_0}}^{n}(\overline{\psi}^{q})_{\mathbf{X}^{m_q-k_q}}(\psi^{q})_{\mathbf{Y}^{m_q-l_q}}\Big|_{\substack{\psi^j=\mathbf{0}\\\forall j\in\{1,\dots,n\}}},$$

with the factor $\varepsilon_{\mathbf{W}^{k_{j_0}},\mathbf{X}^{m'},\mathbf{Z}^{l_{j_0}},\mathbf{Y}^{m'},j_0,(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)} \in \{1,-1\}$ depending only on $\mathbf{W}^{k_{j_0}},\mathbf{X}^{m'},\mathbf{Z}^{l_{j_0}},\mathbf{Y}^{m'}$, $j_0,(m_1,\dots,m_n),(k_1,\dots,k_n)$, $(l_1,\dots,l_n)$. It follows from (4.3) and the triangle inequality of the norm $\|\cdot\|_1$ that

$$\begin{aligned}\left\|\frac{\partial T^{\ge l}_{n,m}}{\partial J^{\ge l+1}_{b,m'}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right\|_1 &\le h^{-2m'}(m'!)^2\sum_{j_0=1}^{n}\sum_{m_{j_0}=1}^{N_{L,h}}\sum_{k_{j_0},l_{j_0}=0}^{m_{j_0}}1_{m_{j_0}=m'}\\ &\cdot\prod_{\substack{j=1\\ j\neq j_0}}^{n}\left(\sum_{m_j=1}^{N_{L,h}}\sum_{k_j,l_j=0}^{m_j}\binom{m_j}{k_j}\binom{m_j}{l_j}\right)\\ &\cdot 1_{\sum_{j=1}^n m_j-n+1\ge m}1_{\sum_{j=1}^n k_j=\sum_{j=1}^n l_j=m}\frac{1}{n!}\\ &\cdot\sum_{T\in\mathbb{T}_n}\|J_{T,j_0}^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_1. \end{aligned}\tag{4.24}$$

The estimation of $\|J_{T,j_0}^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_1$ is parallel to that of $\|J_{T}^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_{1,\infty}$ in the proof of Lemma 4.3. Here we consider j_0 as the root of T, while this role was played by the vertex q_0 in the previous lemma. By noting that

$$\begin{aligned}&\left(\frac{1}{h}\right)^{k_{j_0}+l_{j_0}}\sum_{\mathbf{W}^{k_{j_0}}\in I_{L,h}^{k_{j_0}}}\sum_{\mathbf{Z}^{l_{j_0}}\in I_{L,h}^{l_{j_0}}}(h^{k_{j_0}+l_{j_0}}1_{\mathbf{W}^{k_{j_0}}\subset\mathbf{X}^{m'}}1_{\mathbf{Z}^{l_{j_0}}\subset\mathbf{Y}^{m'}})\\ &\quad=\binom{m_{j_0}}{k_{j_0}}\binom{m_{j_0}}{l_{j_0}}\end{aligned}$$

and letting $d_1,\dots,d_n$ be the incidence numbers of T we have

$$\begin{aligned}\|J_{T,j_0}^{(m_1,\dots,m_n),(k_1,\dots,k_n),(l_1,\dots,l_n)}\|_1 &\le 1_{d_j\le 2m_j-k_j-l_j\ (\forall j\in\{1,\dots,n\})}\\ &\cdot\binom{m_{j_0}}{k_{j_0}}\binom{m_{j_0}}{l_{j_0}}(c_0M^{-l})^{n-1}\prod_{\substack{q=1\\ q\neq j_0}}^{n}\|J^{\ge l+1}_{m_q}\|_{1,\infty}\\ &\cdot\prod_{j=1}^{n}\Bigg(c_0^{\frac{1}{2}(2m_j-k_j-l_j-d_j)}(2m_j-k_j-l_j)\\ &\cdot\binom{2m_j-k_j-l_j-1}{d_j-1}(d_j-1)!\Bigg). \end{aligned}\tag{4.25}$$

By returning the right-hand side of (4.25) to (4.24) and replacing the sum over trees by the sum over possible incidence numbers we obtain

$$\left\| \frac{\partial T^{\geq l}_{n,m}}{\partial J^{\geq l+1}_{b,m'}(\mathbf{X}^{m'},\mathbf{Y}^{m'})} \right\|_1 \leq 1_{m'\geq 1} h^{-2m'}(m'!)^2 \sum_{j_0=1}^{n} c_0^{-m} \frac{1}{n(n-1)} M^{-l(n-1)}$$
$$\cdot \prod_{j=1}^{n} \left(\sum_{m_j=1}^{N_{L,h}} \sum_{k_j,l_j=0}^{m_j} \binom{m_j}{k_j}\binom{m_j}{l_j} c_0^{m_j} \right)$$
$$\cdot 1_{m_{j_0}=m'} 1_{\sum_{j=1}^n m_j - n+1 \geq m} 1_{\sum_{j=1}^n k_j = \sum_{j=1}^n l_j = m} \prod_{\substack{r=1\\ r\neq j_0}}^{n} \| J^{\geq l+1}_{m_r} \|_{1,\infty}$$
$$\cdot \prod_{q=1}^{n} \left((2m_q - k_q - l_q) \sum_{d_q=1}^{2m_q-k_q-l_q} \binom{2m_q-k_q-l_q-1}{d_q-1} \right)$$
$$\cdot 1_{\sum_{q=1}^n d_q = 2(n-1)}.$$

Then, the same calculation as in the last part of the proof of Lemma 4.3 yields the claimed upper bound. □

For compactness of the argument we assume the condition (4.13) throughout this section. The following lemma itself, however, can be proved under a weaker condition:

Lemma 4.5. *Fix any* $l \in \{N_\beta, \dots, N_h\}$. *Assume that* (4.13) *and* (4.14) *for* $l+1$ *hold. Then,* $T^{\geq l}(\psi)$ *is well defined in* $\bigoplus_{n=0}^{N_{L,h}} \mathcal{P}_n \bigwedge \mathcal{V}$. *Moreover, the following inequalities hold true:*

$$\|T^{\geq l}_m\|_{1,\infty} \leq (1_{m=0} N_{L,h}/h + 1_{m\geq 1}) 2^{-3m} c_0^{-m} M^{N_\beta} (2^6\alpha^{-1})^2 M^{-(l-N_\beta)} \quad (\forall m \in \{0,\dots,N_{L,h}\}). \tag{4.26}$$

$$M^{-N_\beta} \sum_{m=1}^{N_{L,h}} \alpha^m c_0^m M^{(l-N_\beta)(m-2)} \|T^{\geq l}_m\|_{1,\infty} \leq 2^8 \alpha^{-1} M^2. \tag{4.27}$$

$$M^{-N_\beta} \sum_{m=3}^{N_{L,h}} \alpha^m c_0^m M^{(l-N_\beta)(m-2)} \|F^{\geq l}_m\|_{1,\infty} \leq 2^7 M^{-1}. \tag{4.28}$$

$$M^{-N_\beta} \sum_{m=3}^{N_{L,h}} \binom{m}{j}^2 c_0^m \|J^{\geq l+1}_m\|_{1,\infty} \leq (2^2\alpha^{-1})^3 M^{-(l+1-N_\beta)} \quad (\forall j \in \{1,2,3\}). \tag{4.29}$$

Proof. Proof of (4.26): The assumptions ensure that

$$M^{-N_\beta} 2^5 c_0 \|J^{\geq l+1}_1\|_{1,\infty} \leq 2^5\alpha^{-1}, \; M^{-N_\beta} \sum_{m=2}^{N_{L,h}} 2^{5m} c_0^m \|J^{\geq l+1}_m\|_{1,\infty} \leq (2^5\alpha^{-1})^2,$$

which result in

$$M^{-N_\beta} \sum_{m=1}^{N_{L,h}} 2^{5m} c_0^m \|J^{\geq l+1}_m\|_{1,\infty} \leq 2^6 \alpha^{-1}. \tag{4.30}$$

By substituting (4.30) into the upper bound obtained in Lemma 4.3 we have

$$\sum_{n=2}^{\infty} \|T_{n,m}^{\geq l}\|_{1,\infty} \leq (1_{m=0}N_{L,h}/h + 1_{m\geq 1})2^{-3m}c_0^{-m}M^{N_\beta} \cdot \sum_{n=2}^{\infty} \frac{1}{n(n-1)} M^{-(l-N_\beta)(n-1)}(2^6\alpha^{-1})^n$$
$$\leq (1_{m=0}N_{L,h}/h + 1_{m\geq 1})2^{-3m}c_0^{-m}M^{N_\beta}M^{-(l-N_\beta)}(2^6\alpha^{-1})^2,$$

where we used that $\sum_{n=2}^{\infty}\frac{1}{n(n-1)} = 1$. This implies the well-definedness of $T^{\geq l}(\psi)$ and (4.26).

Proof of (4.27): It follows from Lemma 4.3 and the inequalities $2^{-3}\alpha M^{l-N_\beta}(2^{-3}\alpha M^{l-N_\beta} - 1)^{-1} \leq 2$ and $2^{2m}M^{-m} \leq 2^2M^{-1}$ $(\forall m \in \mathbb{N})$ that

$$M^{-N_\beta}\sum_{m=1}^{N_{L,h}} \alpha^m c_0^m M^{(l-N_\beta)(m-2)}\|T_m^{\geq l}\|_{1,\infty}$$
$$\leq M^{-N_\beta}M^{-2(l-N_\beta)}\sum_{n=2}^{\infty}\frac{1}{n(n-1)}M^{-l(n-1)} \cdot \prod_{j=1}^{n}\left(\sum_{m_j=1}^{N_{L,h}} 2^{5m_j}c_0^{m_j}\|J_{m_j}^{\geq l+1}\|_{1,\infty}\right)\sum_{m=1}^{\sum_{q=1}^{n}m_q-n+1} 2^{-3m}\alpha^m M^{(l-N_\beta)m}$$
$$\leq 2\sum_{n=2}^{\infty}\frac{1}{n(n-1)}M^{-(l-N_\beta)(n+1)} \cdot \prod_{j=1}^{n}\left(M^{-N_\beta}\sum_{m_j=1}^{N_{L,h}} 2^{5m_j}c_0^{m_j}\|J_{m_j}^{\geq l+1}\|_{1,\infty}\right)(2^{-3}\alpha M^{l-N_\beta})^{\sum_{q=1}^{n}m_q-n+1}$$
$$= 2^{-2}\alpha\sum_{n=2}^{\infty}\frac{1}{n(n-1)} \cdot \left(2^3\alpha^{-1}M^{-N_\beta}\sum_{m=1}^{N_{L,h}} 2^{2m}\alpha^m c_0^m M^{(l-N_\beta)(m-2)}\|J_m^{\geq l+1}\|_{1,\infty}\right)^n$$
$$\leq 2^{-2}\alpha\sum_{n=2}^{\infty}\frac{1}{n(n-1)} \cdot \left(2^5\alpha^{-1}M^{1-N_\beta}\sum_{m=1}^{N_{L,h}} \alpha^m c_0^m M^{(l+1-N_\beta)(m-2)}\|J_m^{\geq l+1}\|_{1,\infty}\right)^n$$
$$\leq 2^{-2}\alpha\sum_{n=2}^{\infty}\frac{1}{n(n-1)}(2^5\alpha^{-1}M)^n \leq 2^8\alpha^{-1}M^2.$$

Proof of (4.28): One can see from the definition of $F_m^{\ge l}(\psi)$ and (4.11) that

$$\|F_m^{\ge l}\|_{1,\infty} \le \sum_{j=m}^{N_{L,h}} \begin{pmatrix} j \\ m \end{pmatrix}^2 c_0^{j-m} \|J_j^{\ge l+1}\|_{1,\infty} \le \sum_{j=m}^{N_{L,h}} 2^{2j} c_0^{j-m} \|J_j^{\ge l+1}\|_{1,\infty}.$$

Substituting this inequality and using the inequalities $\alpha M^{l-N_\beta}(\alpha M^{l-N_\beta} - 1)^{-1} \le 2$ and $2^{2j}M^{-j} \le 2^6 M^{-3}$ $(\forall j \in \mathbb{N}_{\ge 3})$ yield that

$$\begin{aligned}
&M^{-N_\beta} \sum_{m=3}^{N_{L,h}} \alpha^m c_0^m M^{(l-N_\beta)(m-2)} \|F_m^{\ge l}\|_{1,\infty} \\
&\quad\le M^{-N_\beta} \sum_{j=3}^{N_{L,h}} \sum_{m=3}^{j} 2^{2j} \alpha^m c_0^j M^{(l-N_\beta)(m-2)} \|J_j^{\ge l+1}\|_{1,\infty} \\
&\quad\le 2M^{-N_\beta} \sum_{j=3}^{N_{L,h}} 2^{2j} \alpha^j c_0^j M^{(l-N_\beta)(j-2)} \|J_j^{\ge l+1}\|_{1,\infty} \\
&\quad\le 2^7 M^{-1-N_\beta} \sum_{j=3}^{N_{L,h}} \alpha^j c_0^j M^{(l+1-N_\beta)(j-2)} \|J_j^{\ge l+1}\|_{1,\infty} \le 2^7 M^{-1}.
\end{aligned}$$

Proof of (4.29): By using the inequalities that $2^{2m}\alpha^{-m} \le (2^2\alpha^{-1})^3$ and $M^{l+1-N_\beta} \le M^{(l+1-N_\beta)(m-2)}$ $(\forall m \in \mathbb{N}_{\ge 3})$,

$$\begin{aligned}
&M^{-N_\beta} \sum_{m=3}^{N_{L,h}} \begin{pmatrix} m \\ j \end{pmatrix}^2 c_0^m \|J_m^{\ge l+1}\|_{1,\infty} \le M^{-N_\beta} \sum_{m=3}^{N_{L,h}} 2^{2m} c_0^m \|J_m^{\ge l+1}\|_{1,\infty} \\
&\le (2^2\alpha^{-1})^3 M^{-(l+1-N_\beta)} M^{-N_\beta} \sum_{m=3}^{N_{L,h}} \alpha^m c_0^m M^{(l+1-N_\beta)(m-2)} \|J_m^{\ge l+1}\|_{1,\infty} \\
&\le (2^2\alpha^{-1})^3 M^{-(l+1-N_\beta)}.
\end{aligned}$$

□

Proposition 4.1 can be proved by repeatedly using the inequalities of Lemma 4.5.

Proof of Proposition 4.1. The proof is made by induction on $l \in \{N_\beta + 1, \dots, N_h + 1\}$. Set $U_{\max} := \max\{|\lambda_1|, |\lambda_{-1}|, |U_{\mathrm{c}}|, |U_{\mathrm{o}}|\}$. The bi-anti-symmetric kernel $F_2^{\ge N_h+1}(\cdot,\cdot)$ can be written as follows:

$$\begin{aligned}
&F_2^{\ge N_h+1}((\rho_1, \mathbf{x}_1, \sigma_1, x_1), (\rho_2, \mathbf{x}_2, \sigma_2, x_2), (\eta_1, \mathbf{y}_1, \tau_1, y_1), (\eta_2, \mathbf{y}_2, \tau_2, y_2)) \\
&\quad = -h^3 1_{x_1=x_2=y_1=y_2} U_{(\lambda_1,\lambda_{-1})}((\rho_1, \mathbf{x}_1, \sigma_1), (\rho_2, \mathbf{x}_2, \sigma_2), (\eta_1, \mathbf{y}_1, \tau_1), (\eta_2, \mathbf{y}_2, \tau_2)),
\end{aligned}$$

where $U_{(\lambda_1,\lambda_{-1})}(\cdot,\cdot,\cdot,\cdot)$ is defined in (2.4). This implies that

$$\|F_2^{\ge N_h+1}\|_{1,\infty} \le \frac{3}{2} U_{\max}, \tag{4.31}$$

and thus by (4.13),

$$M^{-N_\beta}\sum_{m=1}^{N_{L,h}}\alpha^m c_0^m M^{(N_h+1-N_\beta)(m-2)}\sum_{b\in\{\text{free},\text{tree}\}}\|J_{b,m}^{\ge N_h+1}\|_{1,\infty}$$
$$\le M^{-N_\beta}\alpha^2c_0^2\frac{3}{2}U_{\max}<1.$$

Hence, (4.14) holds for $l=N_h+1$.

Take any $l\in\{N_\beta+1,\dots,N_h\}$ and assume that (4.14) holds true for all $\hat{l}\in\{l+1,\dots,N_h+1\}$. Note the following equalities:

$$\begin{aligned}F_1^{\ge l}(\psi)&=F_1^{\ge l+1}(\psi)+\mathcal{P}_1\int J_2^{\ge N_h+1}(\psi+\psi^0)\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\\&\quad+\mathcal{P}_1\int(F_2^{\ge l+1}(\psi+\psi^0)-J_2^{\ge N_h+1}(\psi+\psi^0))\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\\&\quad+T_1^{\ge l+1}(\psi)+\mathcal{P}_1\int T_2^{\ge l+1}(\psi+\psi^0)\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\\&\quad+\mathcal{P}_1\int\sum_{j=3}^{N_{L,h}}J_j^{\ge l+1}(\psi+\psi^0)\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0),\end{aligned}\tag{4.32}$$

$$\begin{aligned}F_2^{\ge l}(\psi)&=F_2^{\ge l+1}(\psi)+T_2^{\ge l+1}(\psi)\\&\quad+\mathcal{P}_2\int\sum_{j=3}^{N_{L,h}}J_j^{\ge l+1}(\psi+\psi^0)\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0).\end{aligned}\tag{4.33}$$

Since

$$\begin{aligned}&\mathcal{P}_1\int J_2^{\ge N_h+1}(\psi+\psi^0)\,\mathrm{d}\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\\&=-\frac{1}{h}\sum_{(\rho,\mathbf{x},\sigma,x)\in I_{L,h}}(1_{\rho=1}U_{\mathrm{c}}+1_{\rho\in\{2,3\}}U_{\mathrm{o}})\mathcal{C}_l(\rho\mathbf{0}\sigma0,\rho\mathbf{0}\sigma0)(w\mathbf{e}_p)\overline{\psi}_{\rho\mathbf{x}\sigma x}\psi_{\rho\mathbf{x}\sigma x}\\&\quad-\frac{\lambda_1}{h}\sum_{x\in[0,\beta)_h}\big(\mathcal{C}_l(\hat{\mathcal{X}}_10,\hat{\mathcal{Y}}_10)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{X}}_2x}\psi_{\hat{\mathcal{Y}}_2x}-\mathcal{C}_l(\hat{\mathcal{X}}_10,\hat{\mathcal{Y}}_20)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{X}}_2x}\psi_{\hat{\mathcal{Y}}_1x}\\&\quad-\mathcal{C}_l(\hat{\mathcal{X}}_20,\hat{\mathcal{Y}}_10)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{X}}_1x}\psi_{\hat{\mathcal{Y}}_2x}+\mathcal{C}_l(\hat{\mathcal{X}}_20,\hat{\mathcal{Y}}_20)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{X}}_1x}\psi_{\hat{\mathcal{Y}}_1x}\big)\\&\quad-\frac{\lambda_{-1}}{h}\sum_{x\in[0,\beta)_h}\big(\mathcal{C}_l(\hat{\mathcal{Y}}_10,\hat{\mathcal{X}}_10)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{Y}}_2x}\psi_{\hat{\mathcal{X}}_2x}-\mathcal{C}_l(\hat{\mathcal{Y}}_10,\hat{\mathcal{X}}_20)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{Y}}_2x}\psi_{\hat{\mathcal{X}}_1x}\\&\quad-\mathcal{C}_l(\hat{\mathcal{Y}}_20,\hat{\mathcal{X}}_10)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{Y}}_1x}\psi_{\hat{\mathcal{X}}_2x}+\mathcal{C}_l(\hat{\mathcal{Y}}_20,\hat{\mathcal{X}}_20)(w\mathbf{e}_p)\overline{\psi}_{\hat{\mathcal{Y}}_1x}\psi_{\hat{\mathcal{X}}_1x}\big),\end{aligned}$$

the Eq. (4.32), coupled with (4.12), leads to

$$\begin{aligned}\|F_1^{\ge l}\|_{1,\infty}&\le\|F_1^{\ge l+1}\|_{1,\infty}+9U_{\max}c_0(M^{l-N_h}+M^{N_\beta-l})\\&\quad+2^2c_0\|F_2^{\ge l+1}-J_2^{\ge N_h+1}\|_{1,\infty}+\|T_1^{\ge l+1}\|_{1,\infty}+2^2c_0\|T_2^{\ge l+1}\|_{1,\infty}\\&\quad+\sum_{j=3}^{N_{L,h}}j^2c_0^{j-1}\|J_j^{\ge l+1}\|_{1,\infty}.\end{aligned}\tag{4.34}$$

By the induction hypothesis we can apply (4.26), (4.29) to derive the following from (4.33):

$$\begin{aligned}
\|F_2^{\geq l} - J_2^{\geq N_h+1}\|_{1,\infty} &\leq \|F_2^{\geq l+1} - J_2^{\geq N_h+1}\|_{1,\infty} + \|T_2^{\geq l+1}\|_{1,\infty} \\
&\quad + \sum_{j=3}^{N_{L,h}} \binom{j}{2}^2 c_0^{j-2} \|J_j^{\geq l+1}\|_{1,\infty} \\
&\leq \|F_2^{\geq l+1} - J_2^{\geq N_h+1}\|_{1,\infty} \\
&\quad + 2^{-6} c_0^{-2} M^{N_\beta} (2^6\alpha^{-1})^2 M^{-(l+1-N_\beta)} \\
&\quad + c_0^{-2} M^{N_\beta} (2^2\alpha^{-1})^3 M^{-(l+1-N_\beta)} \\
&\leq (2^{-6} c_0^{-2} (2^6\alpha^{-1})^2 + c_0^{-2} (2^2\alpha^{-1})^3) M^{N_\beta} \sum_{j=l}^{N_h} M^{-(j+1-N_\beta)} \\
&\leq 2^{-4} c_0^{-2} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+1-N_\beta)}, \qquad (4.35)
\end{aligned}$$

where we have also used that $M(M-1)^{-1} \leq 2$. Similarly, we have

$$\|F_2^{\geq l+1} - J_2^{\geq N_h+1}\|_{1,\infty} \leq 2^{-4} c_0^{-2} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+2-N_\beta)}. \qquad (4.36)$$

Then, by inserting (4.26), (4.29) and (4.36) into (4.34),

$$\begin{aligned}
\|F_1^{\geq l}\|_{1,\infty} &\leq \|F_1^{\geq l+1}\|_{1,\infty} + 9U_{\max} c_0 (M^{l-N_h} + M^{N_\beta - l}) \\
&\quad + 2^{-2} c_0^{-1} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+2-N_\beta)} \\
&\quad + 2^{-3} c_0^{-1} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+1-N_\beta)} \\
&\quad + 2^{-4} c_0^{-1} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+1-N_\beta)} \\
&\quad + c_0^{-1} (2^2\alpha^{-1})^3 M^{N_\beta} M^{-(l+1-N_\beta)} \\
&\leq \|F_1^{\geq l+1}\|_{1,\infty} + 9U_{\max} c_0 (M^{l-N_h} + M^{N_\beta - l}) \\
&\quad + 2^{-1} c_0^{-1} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+1-N_\beta)} \\
&\leq 9U_{\max} c_0 \sum_{j=l}^{N_h} (M^{j-N_h} \\
&\quad + M^{N_\beta - j}) + 2^{-1} c_0^{-1} (2^6\alpha^{-1})^2 M^{N_\beta} \sum_{j=l}^{N_h} M^{-(j+1-N_\beta)} \\
&\leq 36 U_{\max} c_0 + c_0^{-1} (2^6\alpha^{-1})^2 M^{N_\beta} M^{-(l+1-N_\beta)}. \qquad (4.37)
\end{aligned}$$

It follows from (4.26) and (4.37) that

$$\begin{aligned}
&M^{-N_\beta} \alpha c_0 \sum_{b \in \{\text{free},\text{tree}\}} \|J_{b,1}^{\geq l}\|_{1,\infty} \\
&\quad \leq 36 U_{\max} \alpha c_0^2 M^{-N_\beta} + 2^{12}\alpha^{-1} M^{-(l+1-N_\beta)} + 2^9 \alpha^{-1} M^{-(l-N_\beta)} \\
&\quad \leq 36 U_{\max} \alpha c_0^2 M^{-N_\beta} + 2^{12}\alpha^{-1} M^{-2} + 2^9 \alpha^{-1} M^{-1}. \qquad (4.38)
\end{aligned}$$

Moreover, using (4.27), (4.28), (4.31), (4.35) and (4.37),

$$
\begin{aligned}
& M^{-N_\beta}\sum_{m=1}^{N_{L,h}}\alpha^m c_0^m M^{(l-N_\beta)(m-2)}\sum_{b\in\{\text{free},\text{tree}\}}\|J_{b,m}^{\ge l}\|_{1,\infty}\\
&\quad\le M^{-N_\beta}\alpha c_0 M^{-(l-N_\beta)}\|F_1^{\ge l}\|_{1,\infty}+M^{-N_\beta}\alpha^2c_0^2\|F_2^{\ge l}\|_{1,\infty}\\
&\qquad+M^{-N_\beta}\sum_{m=3}^{N_{L,h}}\alpha^m c_0^m M^{(l-N_\beta)(m-2)}\|F_m^{\ge l}\|_{1,\infty}\\
&\qquad+M^{-N_\beta}\sum_{m=1}^{N_{L,h}}\alpha^m c_0^m M^{(l-N_\beta)(m-2)}\|T_m^{\ge l}\|_{1,\infty}\\
&\quad\le 36U_{\max}\alpha c_0^2M^{-l}+2^{12}\alpha^{-1}M^{2N_\beta-2l-1}+2^8M^{N_\beta-l-1}\\
&\qquad+\frac{3}{2}U_{\max}\alpha^2c_0^2M^{-N_\beta}+2^7M^{-1}+2^8\alpha^{-1}M^2\\
&\quad\le 2U_{\max}\alpha^2c_0^2M^{-N_\beta}+2^{12}\alpha^{-1}M^{-3}+2^8M^{-2}+2^7M^{-1}+2^8\alpha^{-1}M^2. \qquad (4.39)
\end{aligned}
$$

One can check that the right-hand sides of (4.38) and (4.39) are less than 1 under the assumption (4.13) and conclude the proof. □

4.4. An Upper Bound on the Final Integration

Later in this subsection we will see that (4.1) is equal to the multi-contour integral of $\partial J_0^{\ge N_\beta}/\partial\lambda_a|_{(\lambda_1,\lambda_{-1})=(0,0)}$ $(a=1,-1)$ if the coupling constants U_{c}, U_{o} obey the sufficient condition for Proposition 4.1 to hold. Keeping this fact in mind, let us try to find an h-,L-independent upper bound on $|\partial J_0^{\ge N_\beta}/\partial\lambda_a|_{(\lambda_1,\lambda_{-1})=(0,0)}|$ using the results obtained in the previous subsection. This will enable us to bound (4.1), too. We need the following lemma:

Lemma 4.6. *Assume* (4.13). *For any* $b\in\{free,tree\}$, $m'\in\{0,\dots,N_{L,h}\}$, $\mathbf{X}^{m'},\mathbf{Y}^{m'}\in I_{L,h}^{m'}$ *and* $l\in\{N_\beta+1,\dots,N_h\}$ *the following inequalities hold:*

$$
\begin{aligned}
&\sum_{m=0}^{N_{L,h}}\left\|\frac{\partial T_m^{\ge l}}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right\|_1(Ac_0)^m\\
&\quad\le 1_{m'\ge1}h^{-2m'}(m'!)^2(2^2Ac_0)^{m'}2^8\alpha^{-1}M^{-(l-N_\beta)}A\quad(\forall A\in[2^4,M^{l+1-N_\beta}]).
\end{aligned}\tag{4.40}
$$

$$
\sum_{d\in\{\text{free},\text{tree}\}}\left|\frac{\partial J_{d,0}^{\ge N_\beta}}{\partial J_{b,m'}^{\ge N_\beta+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right|\le h^{-2m'}(m'!)^2(2^5c_0)^{m'}.\tag{4.41}
$$

$$
\begin{aligned}
&\sum_{m=0}^{N_{L,h}}\sum_{d\in\{\text{free},\text{tree}\}}\left\|\frac{\partial J_{d,m}^{\ge l}}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right\|_1(Ac_0)^m\le h^{-2m'}(m'!)^2(2^2Ac_0)^{m'}\\
&\quad(\forall A\in[2^4,M^{l+1-N_\beta}]).
\end{aligned}\tag{4.42}
$$

$$\sum_{m=0}^{N_{L,h}} \left\| \frac{\partial F_{m}^{\geq l}}{\partial F_{\hat{m}}^{\geq l+1}(\hat{\mathbf{X}}^{\hat{m}}, \hat{\mathbf{Y}}^{\hat{m}})} \right\|_1 m!(Ac_0)^m$$
$$\leq h^{-2\hat{m}}(\hat{m}!)^3((M^{l-N_h} + M^{N_\beta - l})c_0 + Ac_0)^{\hat{m}}$$
$$(\forall \hat{\mathbf{X}}^{\hat{m}}, \hat{\mathbf{Y}}^{\hat{m}} \in I_{L,h}^{\hat{m}} \text{ with } \hat{\mathbf{X}}^{\hat{m}} \subset ((\hat{\mathcal{X}}_1, s), (\hat{\mathcal{X}}_2, s)), \hat{\mathbf{Y}}^{\hat{m}} \subset ((\hat{\mathcal{Y}}_1, s), (\hat{\mathcal{Y}}_2, s))$$
$$\text{for some } s \in [0, \beta)_h, \ \forall A \geq 0). \tag{4.43}$$

$$\frac{1}{\beta} \sum_{m=0}^{N_{L,h}} \left\| \frac{\partial F_{m}^{\geq N_h+1}}{\partial \lambda_a} \right\|_1 m!(Ac_0)^m \leq 2(Ac_0)^2 \quad (\forall a \in \{1, -1\}). \tag{4.44}$$

Proof. Proof of (4.40): By Lemma 4.4 and the inequality that $2^{-3}A(2^{-3}A - 1)^{-1} \leq 2$,

$$\sum_{m=0}^{N_{L,h}} \left\| \frac{\partial T_{m}^{\geq l}}{\partial J_{b,m'}^{\geq l+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})} \right\|_1 (Ac_0)^m$$
$$\leq 1_{m' \geq 1} h^{-2m'}(m'!)^2(2^5 c_0)^{m'} \sum_{n=2}^{\infty} M^{-l(n-1)} \prod_{j=1}^{n-1} \left(\sum_{m_j=1}^{N_{L,h}} 2^{5m_j} c_0^{m_j} \|J_{m_j}^{\geq l+1}\|_{1,\infty} \right)$$
$$\cdot \sum_{m=0}^{\sum_{j=1}^{n-1} m_j + m' - n + 1} (2^{-3}A)^m$$
$$\leq 1_{m' \geq 1} h^{-2m'}(m'!)^2 2(2^2 Ac_0)^{m'}$$
$$\cdot \sum_{n=2}^{\infty} \left(2^3 A^{-1} M^{-l} \sum_{m=1}^{N_{L,h}} 2^{2m} A^m c_0^m \|J_{m}^{\geq l+1}\|_{1,\infty} \right)^{n-1}. \tag{4.45}$$

By Proposition 4.1 and the assumption that $A \leq M^{l+1-N_\beta}$ we have

$$2^3 A^{-1} M^{-l} \sum_{m=1}^{N_{L,h}} 2^{2m} A^m c_0^m \|J_{m}^{\geq l+1}\|_{1,\infty} = 2^3 A^{-1} M^{-(l-N_\beta)}$$
$$\cdot \left(2^2 M^{-N_\beta} Ac_0 \|J_{1}^{\geq l+1}\|_{1,\infty} + A^2 M^{-N_\beta} \sum_{m=2}^{N_{L,h}} 2^{2m} c_0^m A^{m-2} \|J_{m}^{\geq l+1}\|_{1,\infty} \right)$$
$$\leq 2^3 A^{-1} M^{-(l-N_\beta)}(2^2 A\alpha^{-1} + 2^4 A^2 \alpha^{-2}) = M^{-(l-N_\beta)}(1 + 2^2 A\alpha^{-1})2^5 \alpha^{-1}. \tag{4.46}$$

By giving (4.46) back to (4.45) and remarking that $M^{-(l-N_\beta)}(1 + 2^2 A\alpha^{-1}) \leq 1$ and $1 + 2^2 A\alpha^{-1} \leq 2A$, we obtain

$$\sum_{m=0}^{N_{L,h}} \left\| \frac{\partial T_{m}^{\geq l}}{\partial J_{b,m'}^{\geq l+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})} \right\|_1 (Ac_0)^m$$
$$\leq 1_{m' \geq 1} h^{-2m'}(m'!)^2(2^2 Ac_0)^{m'} M^{-(l-N_\beta)} 2^2 A \sum_{n=2}^{\infty} (2^5 \alpha^{-1})^{n-1},$$

which gives the bound (4.40).

Proof of (4.41): By Lemma 4.4 and (4.30),

$$\left|\frac{\partial T_0^{\ge N_\beta}}{\partial J_{b,m'}^{\ge N_\beta+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right| \le 1_{m'\ge 1} h^{-2m'}(m'!)^2(2^5c_0)^{m'}\sum_{n=2}^{\infty}(2^6\alpha^{-1})^{n-1}$$
$$\le 1_{m'\ge 1} h^{-2m'}(m'!)^2(2^5c_0)^{m'}2^7\alpha^{-1}. \quad (4.47)$$

Let us characterize the derivative of $F_m^{\ge l}(\psi)$ with respect to $J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})$, as it will be useful in the rest of the proof of (4.41) as well as in the proofs of (4.42), (4.43). If $m \le m'$,

$$\frac{\partial F_m^{\ge l}(\psi)}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})} = h^{-2m}\sum_{\mathbf{X}^m,\mathbf{Y}^m\in I_{L,h}^m}\Big(h^{2m-2m'}(m'!)^2 1_{\mathbf{X}^m\subset\mathbf{X}^{m'}}1_{\mathbf{Y}^m\subset\mathbf{Y}^{m'}}$$
$$\cdot\varepsilon_{\mathbf{X}^m,\mathbf{X}^{m'},\mathbf{Y}^m,\mathbf{Y}^{m'}}\int(\overline{\psi}^0)_{\mathbf{X}^{m'}\backslash\mathbf{X}^m}(\psi^0)_{\mathbf{Y}^{m'}\backslash\mathbf{Y}^m}\,d\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\Big)(\overline{\psi})_{\mathbf{X}^m}(\psi)_{\mathbf{Y}^m},$$

where the factor $\varepsilon_{\mathbf{X}^m,\mathbf{X}^{m'},\mathbf{Y}^m,\mathbf{Y}^{m'}} \in \{1,-1\}$ depends only on $\mathbf{X}^m, \mathbf{X}^{m'}, \mathbf{Y}^m, \mathbf{Y}^{m'}$. This equality and (4.3) imply that

$$\left\|\frac{\partial F_m^{\ge l}}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right\|_1$$
$$\le h^{-2m'}(m'!)^2\sum_{\substack{\mathbf{X}^m\in I_{L,h}^m\\ \mathbf{X}^m\subset\mathbf{X}^{m'}}}\sum_{\substack{\mathbf{Y}^m\in I_{L,h}^m\\ \mathbf{Y}^m\subset\mathbf{Y}^{m'}}}\left|\int(\overline{\psi}^0)_{\mathbf{X}^{m'}\backslash\mathbf{X}^m}(\psi^0)_{\mathbf{Y}^{m'}\backslash\mathbf{Y}^m}\,d\mu_{\mathcal{C}_l(w\mathbf{e}_p)}(\psi^0)\right|. \quad (4.48)$$

By using (4.11) one can derive from (4.48) that

$$\left|\frac{\partial F_0^{\ge N_\beta}}{\partial J_{b,m'}^{\ge N_\beta+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right| \le h^{-2m'}(m'!)^2c_0^{m'},$$

which, coupled with (4.47), yields the bound (4.41).

Proof of (4.42): By using (4.48) and the inequality that

$$\sum_{m=0}^{m'}\binom{m'}{m}^2 \le (1_{m'=0}+1_{m'\ge 1}2^{-1})2^{2m'},$$

$$\sum_{m=0}^{N_{L,h}}\left\|\frac{\partial F_m^{\ge l}}{\partial J_{b,m'}^{\ge l+1}(\mathbf{X}^{m'},\mathbf{Y}^{m'})}\right\|_1 (Ac_0)^m$$
$$\le h^{-2m'}(m'!)^2\sum_{m=0}^{m'}\binom{m'}{m}^2 c_0^{m'-m}(Ac_0)^m$$
$$\le h^{-2m'}(m'!)^2(Ac_0)^{m'}\sum_{m=0}^{m'}\binom{m'}{m}^2$$
$$\le (1_{m'=0}+1_{m'\ge 1}2^{-1})h^{-2m'}(m'!)^2(2^2Ac_0)^{m'}. \quad (4.49)$$

By combining (4.49) with (4.40) one can obtain (4.42).

Proof of (4.43): By applying (4.12) to (4.48) we have

$$\sum_{m=0}^{N_{L,h}} \left\| \frac{\partial F_{m}^{\geq l}}{\partial F_{\hat{m}}^{\geq l+1}(\hat{\mathbf{X}}^{\hat{m}}, \hat{\mathbf{Y}}^{\hat{m}})} \right\|_1 m!(Ac_0)^m$$
$$\leq h^{-2\hat{m}}(\hat{m}!)^2 \sum_{m=0}^{\hat{m}} \binom{\hat{m}}{m}^2 (\hat{m}-m)!m!((M^{l-N_h}+M^{N_\beta-l})c_0)^{\hat{m}-m}(Ac_0)^m$$
$$= h^{-2\hat{m}}(\hat{m}!)^3((M^{l-N_h}+M^{N_\beta-l})c_0+Ac_0)^{\hat{m}}.$$

Proof of (4.44): The claimed inequality follows from (4.3) and the equality

$$\frac{\partial F_{m}^{\geq N_h+1}(\psi)}{\partial \lambda_a} = -1_{m=2}\frac{1}{h}\sum_{x\in[0,\beta)_h}(1_{a=1}\overline{\psi}_{\hat{\mathcal{X}}_1 x}\overline{\psi}_{\hat{\mathcal{X}}_2 x}\psi_{\hat{\mathcal{Y}}_2 x}\psi_{\hat{\mathcal{Y}}_1 x}$$
$$+1_{a=-1}\overline{\psi}_{\hat{\mathcal{Y}}_1 x}\overline{\psi}_{\hat{\mathcal{Y}}_2 x}\psi_{\hat{\mathcal{X}}_2 x}\psi_{\hat{\mathcal{X}}_1 x}).$$

□

Corollary 4.7. *Assume (4.13). Take any $l \in \{N_\beta, \dots, N_h\}$, $d \in \{\text{free}, \text{tree}\}$, $m \in \{0, 1, \dots, N_{L,h}\}$ and $\mathbf{X}^m, \mathbf{Y}^m \in I_{L,h}^m$. Moreover, assume that $m = 0$ if $l = N_\beta$. The following statements hold true:*

- (i) *$J_{d,m}^{\geq l}(\mathbf{X}^m, \mathbf{Y}^m)$ is analytic with respect to the independent variables $J_{b,m'}^{\geq l+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})$ $(b\in\{\text{free,tree}\}, m'\in\{0,\dots,N_{L,h}\}, \mathbf{X}^{m'}, \mathbf{Y}^{m'}\in(I_{L,h})_o^{m'})$ in the domain characterized by (4.14) for $l+1$.*
- (ii) *The function $(\lambda_1, \lambda_{-1}, U_c, U_o, w) \mapsto J_{d,m}^{\geq l}(\mathbf{X}^m, \mathbf{Y}^m)$ is analytic in the domain*

$$\{(\lambda_1, \lambda_{-1}, U_c, U_o, w) \in \mathbb{C}^5 |\ |\lambda_1|, |\lambda_{-1}|, |U_c|, |U_o| < 2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}, w \in D_R^i\}. \quad (4.50)$$

Remark 4.8. Since the inequality (4.14) for $l+1$ is independent of $J_{b,0}^{\geq l+1}$ ($b \in \{\text{free}, \text{tree}\}$), the claim (i) implies that $J_{d,m}^{\geq l}(\mathbf{X}^m, \mathbf{Y}^m)$ is entirely analytic with respect to the variables $J_{b,0}^{\geq l+1}$ ($b \in \{\text{free}, \text{tree}\}$).

Proof of Corollary 4.7. The inequalities (4.41), (4.42) imply the claim (i). It is trivial that $(\lambda_1, \lambda_{-1}, U_c, U_o, w) \mapsto J_{b,m'}^{\geq N_h+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})$ is analytic in (4.50) for all $b \in \{\text{free}, \text{tree}\}$, $m' \in \{0, \dots, N_{L,h}\}$, $\mathbf{X}^{m'}, \mathbf{Y}^{m'} \in I_{L,h}^{m'}$. Then the analyticity of $J_{d,m}^{\geq l}(\mathbf{X}^m, \mathbf{Y}^m)$ with respect to $(\lambda_1, \lambda_{-1}, U_c, U_o)$ follows from the claim (i), Proposition 4.1 and the analyticity of composition of analytic functions. Assume that for some $l' \in \{l, \dots, N_h\}$, $J_{b,m'}^{\geq l'+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})$ ($b \in \{\text{free}, \text{tree}\}, m' \in \{0, \dots, N_{L,h}\}, \mathbf{X}^{m'}, \mathbf{Y}^{m'} \in I_{L,h}^{m'}$) are analytic with respect to $w \in D_R^i$. For any $q \in \{0, \dots, N_{L,h}\}$, $\mathbf{X}^q, \mathbf{Y}^q \in I_{L,h}^q$ the analyticity of $F_q^{\geq l'}(\mathbf{X}^q, \mathbf{Y}^q)$, $T_{n,q}^{\geq l'}(\mathbf{X}^q, \mathbf{Y}^q)$ ($n \in \mathbb{N}_{\geq 2}$) is clear since these consist of finite sums and products of $J_{m'}^{\geq l'+1}(\mathbf{X}^{m'}, \mathbf{Y}^{m'})$ and $\mathcal{C}_{l'}(w\mathbf{e}_p)$, which are analytic in D_R^i. Moreover, the proof of the inequality (4.26) shows that $\sum_{n=2}^{j} T_{n,q}^{\geq l'}(\mathbf{X}^q, \mathbf{Y}^q)$ converges to $T_q^{\geq l'}(\mathbf{X}^q, \mathbf{Y}^q)$ uniformly with respect to $w \in D_R$ as $j \to \infty$. This implies that $T_q^{\ge l'}(\mathbf{X}^q,\mathbf{Y}^q)$ is analytic in D_R^i. Thus, the induction concludes that $w \mapsto J_{d,m}^{\ge l}(\mathbf{X}^m,\mathbf{Y}^m)$ is analytic in D_R^i. □

Proposition 4.9. *Assume* (4.13). *The following inequality holds for any* $a \in \{1,-1\}$ *and* $(\lambda_1,\lambda_{-1},U_{\mathrm{c}},U_{\mathrm{o}},w)$ *contained in the domain* (4.50)*:*

$$\frac{1}{\beta}\left|\frac{\partial J_0^{\ge N_\beta}}{\partial \lambda_a}\right| \le 2^{12}c_0^2.$$

Proof. Let us assume that $a=1$. The proof for $a=-1$ is essentially the same. By Corollary 4.7 we can apply the chain rule to derive the following:

$$\begin{aligned}\frac{1}{\beta}\left|\frac{\partial J_0^{\ge N_\beta}}{\partial \lambda_1}\right| &\le \frac{1}{\beta}\prod_{l=N_\beta}^{N_h+1}\left(\sum_{m_l=0}^{N_{L,h}}\sum_{b_l\in\{free,tree\}}\sum_{\mathbf{X}^{m_l},\mathbf{Y}^{m_l}\in(I_{L,h})_o^{m_l}}\right)1_{m_{N_\beta}=0}\\ &\quad\cdot\left|\frac{\partial J_{b_{N_h+1},m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\\ &\quad\times\prod_{j=N_\beta}^{N_h}\left|\frac{\partial J_{b_j,m_j}^{\ge j}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})}{\partial J_{b_{j+1},m_{j+1}}^{\ge j+1}(\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_{j+1}})}\right|\\ &= \sum_{\hat{l}=N_\beta}^{N_h+1} C_R(\hat{l}),\end{aligned} \tag{4.51}$$

where

$$\begin{aligned}C_R(\hat{l}) &:= \frac{1}{\beta}\prod_{l=N_\beta}^{N_h+1}\left(\sum_{m_l=0}^{N_{L,h}}\sum_{b_l\in\{\text{free},\text{tree}\}}\sum_{\mathbf{X}^{m_l},\mathbf{Y}^{m_l}\in(I_{L,h})_o^{m_l}}\right)\\ &\cdot 1_{m_{N_\beta}=0}1_{b_l=\text{free}(\forall l\in\{\hat{l},\hat{l}+1,\dots,N_h+1\}),b_{\hat{l}-1}=\text{tree}}\\ &\cdot\left|\frac{\partial J_{b_{N_h+1},m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\prod_{j=N_\beta}^{N_h}\left|\frac{\partial J_{b_j,m_j}^{\ge j}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})}{\partial J_{b_{j+1},m_{j+1}}^{\ge j+1}(\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_{j+1}})}\right|.\end{aligned}$$

Let us decompose $\sum_{\hat{l}=N_\beta}^{N_h+1}C_R(\hat{l})$ into $\sum_{\hat{l}=N_\beta}^{N_\beta+1}C_R(\hat{l})$ and $\sum_{\hat{l}=N_\beta+2}^{N_h+1}C_R(\hat{l})$ and estimate each part separately. In the following calculation we use the equality (4.2) repeatedly. By using (4.41), (4.43), (4.44) in this order,

$$\begin{aligned}&\sum_{\hat{l}=N_\beta}^{N_\beta+1}C_R(\hat{l})\\ &= \frac{1}{\beta}\prod_{l=N_\beta+1}^{N_h+1}\left(\sum_{m_l=0}^{N_{L,h}}\sum_{\mathbf{X}^{m_l},\mathbf{Y}^{m_l}\in(I_{L,h})_o^{m_l}}\right)\left|\frac{\partial F_{m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\\ &\quad\cdot 1_{\exists x\in[0,\beta)_h,\exists\nu,\xi\in\mathbb{S}_2\text{ s.t. }\mathbf{X}^{m_{N_h+1}}=((\hat{\mathcal{X}}_{\nu(1)},x),(\hat{\mathcal{X}}_{\nu(2)},x)),\mathbf{Y}^{m_{N_h+1}}=((\hat{\mathcal{Y}}_{\xi(1)},x),(\hat{\mathcal{Y}}_{\xi(2)},x))}\end{aligned}$$

$$
\begin{aligned}
&\cdot \prod_{j=N_\beta+1}^{N_h}\left(\left|\frac{\partial F_{m_j}^{\ge j}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})}{\partial F_{m_{j+1}}^{\ge j+1}(\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_{j+1}})}\right| 1_{\mathbf{X}^{m_j}\subset\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_j}\subset\mathbf{Y}^{m_{j+1}}}\right)\\
&\cdot \sum_{d\in\{\text{free},\text{tree}\}}\left|\frac{\partial J_{d,0}^{\ge N_\beta}}{\partial F_{m_{N_\beta+1}}^{\ge N_\beta+1}(\mathbf{X}^{m_{N_\beta+1}},\mathbf{Y}^{m_{N_\beta+1}})}\right|\\
&\le \frac{1}{\beta}\prod_{l=N_\beta+1}^{N_h+1}\left(\sum_{m_l=0}^{N_{L,h}}\sum_{\mathbf{X}^{m_l},\mathbf{Y}^{m_l}\in(I_{L,h})_o^{m_l}}\right)\left|\frac{\partial F_{m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\\
&\cdot 1_{\exists x\in[0,\beta)_h,\exists\nu,\xi\in\mathbb{S}_2\text{ s.t. }\mathbf{X}^{m_{N_h+1}}=((\hat{\mathcal{X}}_{\nu(1)},x),(\hat{\mathcal{X}}_{\nu(2)},x)),\mathbf{Y}^{m_{N_h+1}}=((\hat{\mathcal{Y}}_{\xi(1)},x),(\hat{\mathcal{Y}}_{\xi(2)},x))}\\
&\cdot \prod_{j=N_\beta+1}^{N_h}\left(\left|\frac{\partial F_{m_j}^{\ge j}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})}{\partial F_{m_{j+1}}^{\ge j+1}(\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_{j+1}})}\right| 1_{\mathbf{X}^{m_j}\subset\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_j}\subset\mathbf{Y}^{m_{j+1}}}\right)\\
&\cdot h^{-2m_{N_\beta+1}}(m_{N_\beta+1}!)^2(2^5c_0)^{m_{N_\beta+1}}\\
&\le \frac{1}{\beta}\sum_{m_{N_h+1}=0}^{N_{L,h}}\sum_{\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}}\in(I_{L,h})_0^{m_{N_h+1}}}\left|\frac{\partial F_{m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\\
&\cdot h^{-2m_{N_h+1}}(m_{N_h+1}!)^3\left(\sum_{l=N_\beta+1}^{N_h}(M^{l-N_h}+M^{N_\beta-l})c_0+2^5c_0\right)^{m_{N_h+1}}\\
&\le 2\left(\sum_{l=N_\beta+1}^{N_h}(M^{l-N_h}+M^{N_\beta-l})c_0+2^5c_0\right)^2. \qquad (4.52)
\end{aligned}
$$

By applying (4.41), (4.42), (4.40), (4.43), (4.44) in this order and recalling the condition (4.13) we observe that

$$
\begin{aligned}
&\sum_{\hat{l}=N_\beta+2}^{N_h+1}C_R(\hat{l})\\
&\le \sum_{\hat{l}=N_\beta+2}^{N_h+1}\frac{1}{\beta}\prod_{l=\hat{l}-1}^{N_h+1}\left(\sum_{m_l=0}^{N_{L,h}}\sum_{\mathbf{X}^{m_l},\mathbf{Y}^{m_l}\in(I_{L,h})_o^{m_l}}\right)\left|\frac{\partial F_{m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\\
&\cdot\prod_{j=\hat{l}}^{N_h}\left|\frac{\partial F_{m_j}^{\ge j}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})}{\partial F_{m_{j+1}}^{\ge j+1}(\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_{j+1}})}\right|\cdot\left|\frac{\partial T_{m_{\hat{l}-1}}^{\ge\hat{l}-1}(\mathbf{X}^{m_{\hat{l}-1}},\mathbf{Y}^{m_{\hat{l}-1}})}{\partial F_{m_{\hat{l}}}^{\ge\hat{l}}(\mathbf{X}^{m_{\hat{l}}},\mathbf{Y}^{m_{\hat{l}}})}\right|\\
&\cdot h^{-2m_{\hat{l}-1}}(m_{\hat{l}-1}!)^2(2^{2(\hat{l}-2-N_\beta)}2^5c_0)^{m_{\hat{l}-1}}\\
&\le \sum_{\hat{l}=N_\beta+2}^{N_h+1}\frac{1}{\beta}\prod_{l=\hat{l}}^{N_h+1}\left(\sum_{m_l=0}^{N_{L,h}}\sum_{\mathbf{X}^{m_l},\mathbf{Y}^{m_l}\in(I_{L,h})_o^{m_l}}\right)\left|\frac{\partial F_{m_{N_h+1}}^{\ge N_h+1}(\mathbf{X}^{m_{N_h+1}},\mathbf{Y}^{m_{N_h+1}})}{\partial\lambda_1}\right|\\
&\cdot\prod_{j=\hat{l}}^{N_h}\left|\frac{\partial F_{m_j}^{\ge j}(\mathbf{X}^{m_j},\mathbf{Y}^{m_j})}{\partial F_{m_{j+1}}^{\ge j+1}(\mathbf{X}^{m_{j+1}},\mathbf{Y}^{m_{j+1}})}\right|
\end{aligned}
$$

$$\cdot h^{-2m_{\hat{l}}}(m_{\hat{l}}!)^2(2^{2(\hat{l}-1-N_\beta)}2^5c_0)^{m_{\hat{l}}}2^{13}\alpha^{-1}M^{-(\hat{l}-1-N_\beta)}2^{2(\hat{l}-2-N_\beta)}$$
$$\le 2\sum_{\hat{l}=N_\beta+2}^{N_h+1}\left(\sum_{l=\hat{l}}^{N_h}(M^{l-N_h}+M^{N_\beta-l})c_0+2^{2(\hat{l}-1-N_\beta)}2^5c_0\right)^2(2^2M^{-1})^{\hat{l}-1-N_\beta}. \tag{4.53}$$

Finally, by putting (4.51),(4.52), (4.53) together,

$$\begin{aligned}&\frac{1}{\beta}\left|\frac{\partial J_0^{\ge N_\beta}}{\partial\lambda_1}\right|\\ &\le 2\sum_{\hat{l}=N_\beta+1}^{N_h+1}\left(\sum_{l=\hat{l}}^{N_h}(M^{l-N_h}+M^{N_\beta-l})c_0+2^{2(\hat{l}-1-N_\beta)}2^5c_0\right)^2(2^2M^{-1})^{\hat{l}-1-N_\beta}\\ &\le 2(4c_0+2^5c_0)^2\sum_{\hat{l}=N_\beta+1}^{N_h+1}(2^6M^{-1})^{\hat{l}-1-N_\beta}\\ &\le 2(4+2^5)^2c_0^2\left(1-\frac{1}{4}\right)^{-1}\\ &\le 2^{12}c_0^2.\end{aligned}$$

□

Here we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that $M=\max\{78E_{\max}^2,2^8\}$ and $\alpha=2^{10}M^2$. Then, if $|\lambda_1|,|\lambda_{-1}|,|U_c|,|U_o|<2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}$, the condition (4.13) holds.

By Lemma 2.3 (i), for any sufficiently large $h\in 2\mathbb{N}/\beta$ there exists a domain $\mathcal{O}_h\subset\mathbb{C}$ containing the interval $[-2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta},\,2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}]$ inside such that $(U_c,U_o)\mapsto(\partial/\partial\lambda)\log(\int e^{V_{(\lambda,\lambda)}(\psi)}\,d\mu_{\mathcal{C}}(\psi))|_{\lambda=0}$ is analytic in $\mathcal{O}_h\times\mathcal{O}_h$. Let us fix such a large $h\in 2\mathbb{N}/\beta$.

By the construction of $G_0^{\ge N_\beta}$ and $J_0^{\ge N_\beta}$ and Corollary 4.7 (ii) there exists $U_{\text{small}}>0$ such that $J_0^{\ge N_\beta}=G_0^{\ge N_\beta}$ holds and $(\lambda_1,\lambda_{-1},U_c,U_o,w)\mapsto J_0^{\ge N_\beta}$ is analytic in $D_{\text{small}}^i\times D_R^i$. In order to indicate the dependency on the variable w, let us write $J_0^{\ge N_\beta}(w\mathbf{e}_p)$, $G_0^{\ge N_\beta}(w\mathbf{e}_p)$ instead of $J_0^{\ge N_\beta}$, $G_0^{\ge N_\beta}$. Then for any $n\in\mathbb{N}$ with $2\pi n/L+\mathcal{F}_{t,\beta}(8/\pi^2)<R$ and $(\lambda_1,\lambda_{-1},U_c,U_o)\in D_{\text{small}}^i$,

$$\begin{aligned}&\sum_{a\in\{1,-1\}}\prod_{j=1}^{n}\\ &\quad\cdot\left(\frac{L}{2\pi}\int_0^{2\pi a/L}d\theta_{a,j}\frac{1}{2\pi i}\oint_{|w_{a,j}-\theta_{a,j}|=\mathcal{F}_{t,\beta}(8/\pi^2)/n}dw_{a,j}\frac{1}{(w_{a,j}-\theta_{a,j})^2}\right)\\ &\quad\cdot\left(\frac{\partial}{\partial\lambda_a}G_0^{\ge N_\beta}\left(\sum_{j=1}^n w_{a,j}\mathbf{e}_p\right)-\frac{\partial}{\partial\lambda_a}J_0^{\ge N_\beta}\left(\sum_{j=1}^n w_{a,j}\mathbf{e}_p\right)\right)=0.\end{aligned}$$

On the other hand, Corollary 4.7 (ii) implies that

$$(U_{\rm c},U_{\rm o})\mapsto\sum_{a\in\{1,-1\}}\prod_{j=1}^{n}\cdot\left(\frac{L}{2\pi}\int_0^{2\pi a/L}{\rm d}\theta_{a,j}\frac{1}{2\pi i}\oint_{|w_{a,j}-\theta_{a,j}|=\mathcal{F}_{t,\beta}(8/\pi^2)/n}{\rm d}w_{a,j}\frac{1}{(w_{a,j}-\theta_{a,j})^2}\right)\cdot\frac{\partial}{\partial\lambda_a}J_0^{\ge N_\beta}\left(\sum_{j=1}^{n}w_{a,j}\mathbf{e}_p\right)\Big|_{\lambda_1=\lambda_{-1}=0}$$

is analytic in $\{(U_{\rm c},U_{\rm o})\in\mathbb{C}^2\mid|U_{\rm c}|,|U_{\rm o}|<2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}\}$. Therefore, by Lemma 2.4 (iii), the identity theorem for analytic functions ensures that

$$\begin{aligned}&\left(\frac{L}{2\pi}\left({\rm e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^2(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}-1\right)\right)^n\cdot\frac{1}{\beta}\frac{\partial}{\partial\lambda}\log\left(\int{\rm e}^{V_{(\lambda,\lambda)}(\psi)}{\rm d}\mu_{\mathcal{C}}(\psi)\right)\Big|_{\lambda=0}\\&=\sum_{a\in\{1,-1\}}\prod_{j=1}^{n}\cdot\left(\frac{L}{2\pi}\int_0^{2\pi a/L}{\rm d}\theta_{a,j}\frac{1}{2\pi i}\oint_{|w_{a,j}-\theta_{a,j}|=\mathcal{F}_{t,\beta}(8/\pi^2)/n}dw_{a,j}\frac{1}{(w_{a,j}-\theta_{a,j})^2}\right)\\&\quad\cdot\frac{1}{\beta}\frac{\partial}{\partial\lambda_a}J_0^{\ge N_\beta}\left(\sum_{j=1}^{n}w_{a,j}\mathbf{e}_p\right)\Big|_{\lambda_1=\lambda_{-1}=0}\end{aligned}\tag{4.54}$$

for all $U_{\rm c},U_{\rm o}\in\mathbb{R}$ with $|U_{\rm c}|,|U_{\rm o}|<2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}$. Then using Proposition 4.9 and $n^n\le n!{\rm e}^n$ we can estimate (4.54) as follows:

$$\left|\frac{L}{2\pi}\left({\rm e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^2(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}-1\right)\right|^n\cdot\left|\frac{1}{\beta}\frac{\partial}{\partial\lambda}\log\left(\int{\rm e}^{V_{(\lambda,\lambda)}(\psi)}{\rm d}\mu_{\mathcal{C}}(\psi)\right)\Big|_{\lambda=0}\right|\le2^{13}c_0^2n!{\rm e}^n\mathcal{F}_{t,\beta}(8/\pi^2)^{-n}.\tag{4.55}$$

Note that the inequality (4.55) for $n=0$ can be derived in the same way. By Lemma 2.3 (ii) we can send $h\to\infty$ in (4.55) so that

$$\left|\frac{L}{2\pi}\left({\rm e}^{i\frac{2\pi}{L}\langle\sum_{j=1}^2(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle}-1\right)\right|^n|\langle\psi^*_{\hat{\mathcal{X}}_1}\psi^*_{\hat{\mathcal{X}}_2}\psi_{\hat{\mathcal{Y}}_2}\psi_{\hat{\mathcal{Y}}_1}+\text{ h.c}\rangle_L|\le2^{13}c_0^2n!{\rm e}^n\mathcal{F}_{t,\beta}(8/\pi^2)^{-n}.\tag{4.56}$$

As we have fixed the parameters arbitrarily in the beginning of Sect. 4.2, we can claim (4.56) for all $n \in \mathbb{N} \cup \{0\}$, $p \in \{1,2\}$, $U_c, U_o \in \mathbb{R}$ with $|U_c|, |U_o| < 2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}$ and sufficiently large $L \in \mathbb{N}$.

Set

$$f_1(E_{\max}) := 2^5\alpha^2\left(\frac{\max\{c,1\}}{(1-\varepsilon)\varepsilon^2}M^9\right)^2, \quad f_2(E_{\max}) := 2^{14}\left(\frac{\max\{c,1\}}{(1-\varepsilon)\varepsilon^2}M^9\right)^2.$$

By noting (3.2) and (4.9) we can confirm that

$$f_1(E_{\max})\max\{1,\beta^{16}\}\beta > 2^4\alpha^2c_0^2M^{-N_\beta}, \quad f_2(E_{\max})\max\{1,\beta^{16}\} = 2^{14}c_0^2,$$

$f_1(E_{\max})$, $f_2(E_{\max})$ are non-decreasing with respect to $E_{\max} \in \mathbb{R}_{\ge 1}$ and $f_1(E_{\max}) = O(E_{\max}^{44})$, $f_2(E_{\max}) = O(E_{\max}^{36})$ as $E_{\max} \to \infty$.

It is straightforward to derive the following inequality from (4.56):

$$|\langle \psi^*_{\hat{\mathcal{X}}_1}\psi^*_{\hat{\mathcal{X}}_2}\psi_{\hat{\mathcal{Y}}_2}\psi_{\hat{\mathcal{Y}}_1} + \text{ h.c } \rangle_L| \le f_2(E_{\max})\max\{1,\beta^{16}\}$$
$$\cdot\left(\frac{1}{\max\{1,t^2\}\max\{\beta,\beta^2\}}+1\right)^{-\frac{1}{8e}\sum_{p=1}^{2}\left|\frac{e^{i2\pi\langle\sum_{j=1}^{2}(\hat{s}(\hat{\sigma}_j)\hat{\mathbf{x}}_j-\hat{s}(\hat{\tau}_j)\hat{\mathbf{y}}_j),\mathbf{e}_p\rangle/L}-1}{2\pi/L}\right|}$$

for any $U_c, U_o \in \mathbb{R}$ with $|U_c|, |U_o| \le (f_1(E_{\max})\max\{1,\beta^{16}\}\beta)^{-1}$ and sufficiently large $L \in \mathbb{N}$. Finally by Lemma D.2 proved in Appendix D we can take the limit $L \to \infty$ and complete the proof. □

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Appendix A. Derivation of the Covariance

In this part of Appendix we derive the representation of the covariance (2.1), (2.2). Define the 3×3 matrix $M^\sigma_{t,\mathbf{k}} = (M^\sigma_{t,\mathbf{k}}(\rho,\eta))_{1\le\rho,\eta\le 3}$ ($\mathbf{k} = (k_1,k_2) \in \Gamma^*, t \in \mathbb{R}, \sigma \in \{\uparrow,\downarrow\}$) by

$$M^\sigma_{t,\mathbf{k}} := \begin{pmatrix} \epsilon^\sigma_c & t(1+e^{-ik_1}) & t(1+e^{-ik_2}) \\ t(1+e^{ik_1}) & \epsilon^\sigma_o & 0 \\ t(1+e^{ik_2}) & 0 & \epsilon^\sigma_o \end{pmatrix}.$$

We see that

$$H_0 = \sum_{\substack{(\rho,\mathbf{x}),(\eta,\mathbf{y})\\ \in\{1,2,3\}\times\Gamma}}\sum_{\sigma\in\{\uparrow,\downarrow\}}\frac{1}{L^2}\sum_{\mathbf{k}\in\Gamma^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}M^\sigma_{t,\mathbf{k}}(\rho,\eta)\psi^*_{\rho\mathbf{x}\sigma}\psi_{\eta\mathbf{y}\sigma}.$$

For $\mathbf{k} \in \Gamma^*$, $t \in \mathbb{R}$, $\sigma \in \{\uparrow,\downarrow\}$ and $\rho \in \{2,3\}$ set

$$\begin{aligned} &A_1^{\sigma}(t,\mathbf{k}) := \begin{cases} \epsilon_{\mathrm{c}}^{\sigma} & \text{if } \mathbf{k} = (\pi,\pi) \text{ in } \Gamma^* \text{ or } t = 0, \\ \epsilon_{\mathrm{o}}^{\sigma} & \text{otherwise,} \end{cases} \\ &A_{\rho}^{\sigma}(t,\mathbf{k}) \\ &:= \begin{cases} \epsilon_{\mathrm{o}}^{\sigma} & \text{if } \mathbf{k} = (\pi,\pi) \text{ in } \Gamma^* \text{ or } t = 0, \\ \frac{1}{2}(\epsilon_{\mathrm{c}}^{\sigma} + \epsilon_{\mathrm{o}}^{\sigma}) + (-1)^{\rho}\frac{t}{2}\sqrt{\frac{1}{t^2}(\epsilon_{\mathrm{c}}^{\sigma} - \epsilon_{\mathrm{o}}^{\sigma})^2 + 8\sum_{j=1}^{2}(1+\cos k_j)} & \text{otherwise.} \end{cases} \end{aligned} \tag{A.1}$$

Recall (3.6), i.e., $E(t,\mathbf{k}) = 2t^2\sum_{j=1}^{2}(1+\cos k_j)$ $(t \in \mathbb{R}, \mathbf{k} \in \Gamma^*)$, and define the 3×3 matrix $\mathcal{U}_{t,\mathbf{k}}^{\sigma} = (\mathcal{U}_{t,\mathbf{k}}^{\sigma}(\rho,\eta))_{1\le\rho,\eta\le 3}$ by $\mathcal{U}_{t,\mathbf{k}}^{\sigma}(\rho,\eta) := \delta_{\rho,\eta}$ if $\mathbf{k} = (\pi,\pi)$ in Γ^* or $t = 0$,

$$\mathcal{U}_{t,\mathbf{k}}^{\sigma} := \begin{pmatrix} 0 & \frac{A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma}}{((A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))^{1/2}} & \frac{A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma}}{((A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))^{1/2}} \\ \frac{t(1+\mathrm{e}^{-ik_2})}{E(t,\mathbf{k})^{1/2}} & \frac{t(1+\mathrm{e}^{ik_1})}{((A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))^{1/2}} & \frac{t(1+\mathrm{e}^{ik_1})}{((A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))^{1/2}} \\ \frac{-t(1+\mathrm{e}^{-ik_1})}{E(t,\mathbf{k})^{1/2}} & \frac{t(1+\mathrm{e}^{ik_2})}{((A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))^{1/2}} & \frac{t(1+\mathrm{e}^{ik_2})}{((A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))^{1/2}} \end{pmatrix}$$

otherwise. One can check that $\mathcal{U}_{t,\mathbf{k}}^{\sigma}$ is unitary and

$$(\mathcal{U}_{t,\mathbf{k}}^{\sigma})^* M_{t,\mathbf{k}}^{\sigma} \mathcal{U}_{t,\mathbf{k}}^{\sigma} = \begin{pmatrix} A_1^{\sigma}(t,\mathbf{k}) & 0 & 0 \\ 0 & A_2^{\sigma}(t,\mathbf{k}) & 0 \\ 0 & 0 & A_3^{\sigma}(t,\mathbf{k}) \end{pmatrix}. \tag{A.2}$$

By using $\mathcal{U}_{t,\mathbf{k}}^{\sigma}$ let us define the matrix $W_t = (W_t(\rho\mathbf{x}\sigma,\eta\mathbf{y}\tau))_{(\rho,\mathbf{x},\sigma),(\eta,\mathbf{y},\tau)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}$ by

$$W_t(\rho\mathbf{x}\sigma,\eta\mathbf{y}\tau) := \frac{\delta_{\sigma,\tau}}{L^2}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\overline{\mathcal{U}_{t,\mathbf{k}}^{\sigma}(\rho,\eta)}.$$

One can also verify that $(W_t^* W_t)(\rho\mathbf{x}\sigma,\eta\mathbf{y}\tau) = 1_{(\rho,\mathbf{x},\sigma)=(\eta,\mathbf{y},\tau)}$. With the matrix W_t define the operator $G(W_t) : F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\})) \to F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}))$ by

$$\begin{aligned} &G(W_t)\Omega := \Omega, \\ &G(W_t)\psi_{\rho_1\mathbf{x}_1\sigma_1}^*\psi_{\rho_2\mathbf{x}_2\sigma_2}^*\cdots\psi_{\rho_n\mathbf{x}_n\sigma_n}^*\Omega \\ &\quad := (W_t\psi^*)_{\rho_1\mathbf{x}_1\sigma_1}(W_t\psi^*)_{\rho_2\mathbf{x}_2\sigma_2}\cdots(W_t\psi^*)_{\rho_n\mathbf{x}_n\sigma_n}\Omega \\ &\quad (n\in\mathbb{N}, (\rho_j,\mathbf{x}_j,\sigma_j)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\ (j=1,\ldots,n)), \end{aligned}$$

and by linearity. Here the notation Ω represents the vacuum of $F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}))$ and $(W_t\psi^*)_{\rho\mathbf{x}\sigma} := \sum_{(\eta,\mathbf{y},\tau)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}} W_t(\rho\mathbf{x}\sigma,\eta\mathbf{y}\tau)\psi_{\eta\mathbf{y}\tau}^*$. The operator $G(W_t)$ is unitary. By letting $(\overline{W_t}\psi)_{\rho\mathbf{x}\sigma}$ denote $\sum_{(\eta,\mathbf{y},\tau)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}} \overline{W_t(\rho\mathbf{x}\sigma,\eta\mathbf{y}\tau)}\psi_{\eta\mathbf{y}\tau}$, we observe that $G(W_t)H_0\phi = \tilde{H}_0 G(W_t)\phi$ for any $\phi\in F_f(L^2(\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}))$, where

$$\tilde{H}_0 := \sum_{\substack{(\rho,\mathbf{x}),(\eta,\mathbf{y})\\ \in\{1,2,3\}\times\Gamma}}\sum_{\sigma\in\{\uparrow,\downarrow\}}\frac{1}{L^2}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{e}^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}M_{t,\mathbf{k}}^{\sigma}(\rho,\eta)(W_t\psi^*)_{\rho\mathbf{x}\sigma}(\overline{W_t}\psi)_{\eta\mathbf{y}\sigma}.$$

By using (A.2) we have

$$\tilde{H}_0 = \sum_{\rho\in\{1,2,3\}} \sum_{\mathbf{x},\mathbf{y}\in\Gamma} \sum_{\sigma\in\{\uparrow,\downarrow\}} \left(\frac{1}{L^2} \sum_{\mathbf{k}\in\Gamma^*} \mathrm{e}^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle} A_\rho^\sigma(t,\mathbf{k}) \right) \psi^*_{\rho\mathbf{x}\sigma}\psi_{\rho\mathbf{y}\sigma}.$$

For $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)$ let

$$\tilde{\psi}^*_{\rho\mathbf{x}\sigma}(x) := \mathrm{e}^{x\tilde{H}_0}\psi^*_{\rho\mathbf{x}\sigma}\mathrm{e}^{-x\tilde{H}_0}, \quad \tilde{\psi}_{\eta\mathbf{y}\tau}(y) := \mathrm{e}^{y\tilde{H}_0}\psi_{\eta\mathbf{y}\tau}\mathrm{e}^{-y\tilde{H}_0},$$
$$T(\tilde{\psi}^*_{\rho\mathbf{x}\sigma}(x)\tilde{\psi}_{\eta\mathbf{y}\tau}(y)) := 1_{x\ge y}\tilde{\psi}^*_{\rho\mathbf{x}\sigma}(x)\tilde{\psi}_{\eta\mathbf{y}\tau}(y) - 1_{x<y}\tilde{\psi}_{\eta\mathbf{y}\tau}(y)\tilde{\psi}^*_{\rho\mathbf{x}\sigma}(x).$$

The unitary property of $G(W_t)$ implies that

$$\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y) = \sum_{\substack{(\rho',\mathbf{x}',\sigma'),(\eta',\mathbf{y}',\tau')\\ \in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}} W_t(\rho\mathbf{x}\sigma,\rho'\mathbf{x}'\sigma')\overline{W_t(\eta\mathbf{y}\tau,\eta'\mathbf{y}'\tau')}$$
$$\cdot\frac{\mathrm{Tr}(\mathrm{e}^{-\beta\tilde{H}_0}T(\tilde{\psi}^*_{\rho'\mathbf{x}'\sigma'}(x)\tilde{\psi}_{\eta'\mathbf{y}'\tau'}(y)))}{\mathrm{Tr}\,\mathrm{e}^{-\beta\tilde{H}_0}}. \quad (A.3)$$

Since $\tilde{H}_0$ is diagonal with respect to $\rho\in\{1,2,3\}$, the characterization of $\mathrm{Tr}(\mathrm{e}^{-\beta\tilde{H}_0}T(\tilde{\psi}^*_{\rho'\mathbf{x}'\sigma'}(x)\tilde{\psi}_{\eta'\mathbf{y}'\tau'}(y)))/\,\mathrm{Tr}\,\mathrm{e}^{-\beta\tilde{H}_0}$ can be carried out by a standard argument. See, e.g., [10, Appendix B] for the derivation of the covariance governed by a free Hamiltonian defined on $F_f(L^2(\Gamma\times\{\uparrow,\downarrow\}))$. As the result we obtain

$$\frac{\mathrm{Tr}(\mathrm{e}^{-\beta\tilde{H}_0}T(\tilde{\psi}^*_{\rho'\mathbf{x}'\sigma'}(x)\tilde{\psi}_{\eta'\mathbf{y}'\tau'}(y)))}{\mathrm{Tr}\,\mathrm{e}^{-\beta\tilde{H}_0}} = \frac{\delta_{\rho',\eta'}\delta_{\sigma',\tau'}}{L^2}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{e}^{-i\langle\mathbf{x}'-\mathbf{y}',\mathbf{k}\rangle}\mathrm{e}^{(x-y)A^{\sigma'}_{\rho'}(t,\mathbf{k})}$$
$$\cdot\left(\frac{1_{x\ge y}}{1+\mathrm{e}^{\beta A^{\sigma'}_{\rho'}(t,\mathbf{k})}} - \frac{1_{x<y}}{1+\mathrm{e}^{-\beta A^{\sigma'}_{\rho'}(t,\mathbf{k})}}\right). \quad (A.4)$$

Substituting (A.4) into (A.3) yields that for $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)$,

$$\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y) = \frac{\delta_{\sigma,\tau}}{L^2}\sum_{\gamma\in\{1,2,3\}}\sum_{\mathbf{k}\in\Gamma^*}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{(x-y)A^\sigma_\gamma(t,\mathbf{k})}$$
$$\cdot\left(\frac{1_{x\ge y}}{1+\mathrm{e}^{\beta A^\sigma_\gamma(t,\mathbf{k})}} - \frac{1_{x<y}}{1+\mathrm{e}^{-\beta A^\sigma_\gamma(t,\mathbf{k})}}\right)\overline{\mathcal{U}^\sigma_{t,\mathbf{k}}(\rho,\gamma)}\mathcal{U}^\sigma_{t,\mathbf{k}}(\eta,\gamma). \quad (A.5)$$

Moreover, by applying [10, Lemma C.3] to the right-hand side of (A.5) one reaches the equality that for $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y)\in I_{L,h}$,

$$\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)$$
$$= \frac{\delta_{\sigma,\tau}}{\beta L^2}\sum_{(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h}\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\mathrm{e}^{i(x-y)\omega}\sum_{\gamma\in\{1,2,3\}}\frac{\overline{\mathcal{U}^\sigma_{t,\mathbf{k}}(\rho,\gamma)}\mathcal{U}^\sigma_{t,\mathbf{k}}(\eta,\gamma)}{h(1-\mathrm{e}^{-i\omega/h+A^\sigma_\gamma(t,\mathbf{k})/h})}.$$

We need to show that for any $(\mathbf{k},\omega)\in\Gamma^*\times\mathcal{M}_h$, $\rho,\eta\in\{1,2,3\}$, $t\in\mathbb{R}$, $\sigma\in\{\uparrow,\downarrow\}$,

$$\sum_{\gamma\in\{1,2,3\}}\frac{\overline{\mathcal{U}^{\sigma}_{t,\mathbf{k}}(\rho,\gamma)}\mathcal{U}^{\sigma}_{t,\mathbf{k}}(\eta,\gamma)}{h(1-\mathrm{e}^{-i\omega/h+A^{\sigma}_{\gamma}(t,\mathbf{k})/h})}=\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k},\omega), \tag{A.6}$$

where $\mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k},\omega)$ is written in (2.2). The equality (A.6) can be confirmed by direct calculation. To assist the readers' verification, we present some intermediate results appearing in the calculation. The functions $O^{\sigma}_j(\cdot):\mathbb{C}^2\to\mathbb{C}$ $(j\in\{1,\dots,5\},\sigma\in\{\uparrow,\downarrow\})$ in (2.2) are in fact given as follows:

$$\begin{aligned}
O^{\sigma}_1(\mathbf{k}) &:= -2\sum_{n=2}^{\infty}\frac{1}{(2n)!h^{2n-2}}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{4}+E(t,\mathbf{k})\right)^n,\\
O^{\sigma}_2(\mathbf{k}) &:= -\sum_{n=1}^{\infty}\frac{1}{(2n)!h^{2n-1}}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{4}+E(t,\mathbf{k})\right)^n\\
&\quad+\frac{\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}}}{2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)!h^{2n}}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{4}+E(t,\mathbf{k})\right)^n,\\
O^{\sigma}_3(\mathbf{k}) &:= \sum_{n=1}^{\infty}\frac{1}{(2n+1)!h^{2n}}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{4}+E(t,\mathbf{k})\right)^n,\\
O^{\sigma}_4(\mathbf{k}) &:= \sum_{n=2}^{\infty}\frac{1}{(2n)!h^{2n-2}}\sum_{m=1}^{n}\binom{n}{m}E(t,\mathbf{k})^{m-1}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{4}\right)^{n-m},\\
O^{\sigma}_5(\mathbf{k}) &:= -\frac{\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}}}{2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)!h^{2n-1}}\sum_{m=1}^{n}\binom{n}{m}\\
&\quad\cdot E(t,\mathbf{k})^{m-1}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{4}\right)^{n-m}.
\end{aligned}\tag{A.7}$$

From (A.7) one can see that (2.3) holds.

First assume that $\mathbf{k}=(\pi,\pi)$ in Γ^* or $t=0$. In this case $E(t,\mathbf{k})=0$ and thus $\mathcal{D}^{\sigma}(\mathbf{k},\omega)$ and $\mathcal{N}^{\sigma}_{\rho,\eta}(\mathbf{k},\omega)$ given in (2.2) are simplified as follows:

$$\begin{aligned}
\mathcal{D}^{\sigma}(\mathbf{k},\omega)&=h^2(1-\mathrm{e}^{-i\omega/h+\epsilon^{\sigma}_{\mathrm{c}}/h})(1-\mathrm{e}^{-i\omega/h+\epsilon^{\sigma}_{\mathrm{o}}/h}),\\
\mathcal{N}^{\sigma}_{1,1}(\mathbf{k},\omega)&=h(1-\mathrm{e}^{-i\omega/h+\epsilon^{\sigma}_{\mathrm{o}}/h}),\\
\mathcal{N}^{\sigma}_{\rho,\eta}(\mathbf{k},\omega)&=0\quad(\forall(\rho,\eta)\in\{1,2,3\}^2\backslash\{(1,1)\}).
\end{aligned}$$

By using these, the equality (A.6) can be confirmed in this case.

Next consider the case that $\mathbf{k}\neq(\pi,\pi)$ in Γ^* and $t\neq 0$. To organize the calculation, set $f(w,A):=h(1-\mathrm{e}^{-i\omega/h+A/h})$. Remark that for $\rho\in\{2,3\}$,

$$\begin{aligned}
&f(\omega,A^{\sigma}_{\rho}(t,\mathbf{k}))=h-h\mathrm{e}^{-\frac{i}{h}\omega+\frac{\epsilon^{\sigma}_{\mathrm{c}}+\epsilon^{\sigma}_{\mathrm{o}}}{2h}}\\
&\cdot\left(\sum_{n=0}^{\infty}\frac{1}{(2n)!h^{2n}}\left(\frac{t}{2}\left(\frac{(\epsilon^{\sigma}_{\mathrm{c}}-\epsilon^{\sigma}_{\mathrm{o}})^2}{t^2}+8\sum_{j=1}^{2}(1+\cos k_j)\right)^{1/2}\right)^{2n}\right.
\end{aligned}$$

$$+(-1)^{\rho}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!h^{2n+1}}\left(\frac{t}{2}\left(\frac{(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2}{t^2}+8\sum_{j=1}^{2}(1+\cos k_j)\right)^{1/2}\right)^{2n+1}\Bigg),$$

$$f(\omega,A_2^{\sigma}(t,\mathbf{k}))f(\omega,A_3^{\sigma}(t,\mathbf{k}))=\mathcal{D}^{\sigma}(\mathbf{k},\omega),$$

$$((A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))((A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))$$
$$=4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2.$$

By using these equalities we observe that

$$\begin{aligned}
&(\text{The left-hand side of (A.6) for } (\rho,\eta)=(1,1))\\
&=\big(f(\omega,A_3^{\sigma}(t,\mathbf{k}))(A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2((A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))\\
&\quad+f(\omega,A_2^{\sigma}(t,\mathbf{k}))(A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2((A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))\big)\\
&\quad\cdot/\big((4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2)\mathcal{D}^{\sigma}(\mathbf{k},\omega)\big)\\
&=\frac{(4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2)\mathcal{N}_{1,1}^{\sigma}(\mathbf{k},\omega)}{(4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2)\mathcal{D}^{\sigma}(\mathbf{k},\omega)}\\
&=\mathcal{B}_{1,1}^{\sigma}(\mathbf{k},\omega),\\
&(\text{The left-hand side of (A.6) for } (\rho,\eta)=(1,2))\\
&=t(1+e^{ik_1})\big(f(\omega,A_3^{\sigma}(t,\mathbf{k}))(A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})((A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))\\
&\quad+f(\omega,A_2^{\sigma}(t,\mathbf{k}))(A_3^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})((A_2^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))\big)\\
&\quad\cdot/\big((4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2)\mathcal{D}^{\sigma}(\mathbf{k},\omega)\big)\\
&=\frac{(4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2)\mathcal{N}_{1,2}^{\sigma}(\mathbf{k},\omega)}{(4E(t,\mathbf{k})^2+E(t,\mathbf{k})(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2)\mathcal{D}^{\sigma}(\mathbf{k},\omega)}\\
&=\mathcal{B}_{1,2}^{\sigma}(\mathbf{k},\omega),\\
&(\text{The left-hand side of (A.6) for } (\rho,\eta)=(2,2))\\
&=\frac{1}{f(\omega,A_1^{\sigma}(t,\mathbf{k}))}+2t^2(1+\cos k_1)\\
&\cdot\left(\frac{-1}{E(t,\mathbf{k})f(\omega,A_1^{\sigma}(t,\mathbf{k}))}+\sum_{j=2}^{3}\frac{1}{((A_j^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))f(\omega,A_j^{\sigma}(t,\mathbf{k}))}\right).
\end{aligned}\tag{A.8}$$

Note that

$$\begin{aligned}
&\frac{-1}{E(t,\mathbf{k})f(\omega,A_1^{\sigma}(t,\mathbf{k}))}+\sum_{j=2}^{3}\frac{1}{((A_j^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))f(\omega,A_j^{\sigma}(t,\mathbf{k}))}\\
&\quad=\frac{-1}{E(t,\mathbf{k})f(\omega,A_1^{\sigma}(t,\mathbf{k}))}\\
&\qquad+\Bigg(h-he^{-\frac{i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma}}{2h}}\sum_{n=0}^{\infty}\frac{1}{(2n)!h^{2n}}\left(\frac{(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2}{4}+E(t,\mathbf{k})\right)^n\\
&\qquad-\frac{\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma}}{2}e^{-\frac{i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma}}{2h}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!h^{2n}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\frac{(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2}{4}+E(t,\mathbf{k})\right)^n\Bigg)/(E(t,\mathbf{k})\mathcal{D}^{\sigma}(t,\mathbf{k}))\\
&=\Bigg(\mathrm{e}^{-\frac{i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma}}{2h}}\\
&\cdot\Bigg(\sum_{n=0}^{\infty}\left(\frac{1}{(2n)!h^{2n-2}}-\frac{\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma}}{2(2n+1)!h^{2n-1}}\right)\left(\frac{(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2}{4}+E(t,\mathbf{k})\right)^n\\
&-h^2\mathrm{e}^{-\frac{\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma}}{2h}}\Bigg)+\mathrm{e}^{-\frac{2i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+3\epsilon_{\mathrm{o}}^{\sigma}}{2h}}\\
&\cdot\Bigg(\sum_{n=0}^{\infty}\left(\frac{1}{(2n)!h^{2n-2}}+\frac{\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma}}{2(2n+1)!h^{2n-1}}\right)\left(\frac{(\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma})^2}{4}+E(t,\mathbf{k})\right)^n\\
&-h^2\mathrm{e}^{\frac{\epsilon_{\mathrm{c}}^{\sigma}-\epsilon_{\mathrm{o}}^{\sigma}}{2h}}\Bigg)\Bigg)/(E(t,\mathbf{k})f(\omega,A_1^{\sigma}(t,\mathbf{k}))\mathcal{D}^{\sigma}(t,\mathbf{k}))\\
&=\Big(\frac{1}{2}\mathrm{e}^{-\frac{i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma}}{2h}}+\frac{1}{2}\mathrm{e}^{-\frac{2i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+3\epsilon_{\mathrm{o}}^{\sigma}}{2h}}+(\mathrm{e}^{-\frac{i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma}}{2h}}+\mathrm{e}^{-\frac{2i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+3\epsilon_{\mathrm{o}}^{\sigma}}{2h}})O_4^{\sigma}(\mathbf{k})\\
&+(\mathrm{e}^{-\frac{i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma}}{2h}}-\mathrm{e}^{-\frac{2i}{h}\omega+\frac{\epsilon_{\mathrm{c}}^{\sigma}+3\epsilon_{\mathrm{o}}^{\sigma}}{2h}})O_5^{\sigma}(\mathbf{k})\Big)\\
&\cdot/(f(\omega,A_1^{\sigma}(t,\mathbf{k}))\mathcal{D}^{\sigma}(t,\mathbf{k})). \qquad \text{(A.9)}
\end{aligned}
$$

By inserting (A.9) into (A.8) we obtain (A.6) for $(\rho,\eta)=(2,2)$. Moreover, by using (A.9),

$$
\begin{aligned}
&(\text{The left-hand side of (A.6) for } (\rho,\eta)=(2,3))=t^2(1+\mathrm{e}^{-ik_1})(1+\mathrm{e}^{ik_2})\\
&\cdot\left(\frac{-1}{E(t,\mathbf{k})f(\omega,A_1^{\sigma}(t,\mathbf{k}))}+\sum_{j=2}^{3}\frac{1}{((A_j^{\sigma}(t,\mathbf{k})-\epsilon_{\mathrm{o}}^{\sigma})^2+E(t,\mathbf{k}))f(\omega,A_j^{\sigma}(t,\mathbf{k}))}\right)\\
&=\frac{\mathcal{N}_{2,3}^{\sigma}(\mathbf{k},\omega)}{f(\omega,A_1^{\sigma}(t,\mathbf{k}))\mathcal{D}^{\sigma}(t,\mathbf{k})}\\
&=\mathcal{B}_{2,3}^{\sigma}(\mathbf{k},\omega).
\end{aligned}
$$

By using the results for $(\rho,\eta)=(1,1),(1,2),(2,2),(2,3)$ and symmetries, (A.6) for $(\rho,\eta)=(1,3),(2,1),(3,1),(3,2),(3,3)$ can be immediately proved. Thus, the representations (2.1), (2.2) have been derived.

Appendix B. Convergence of the Grassmann Integral Formulation

In this section we sketch how to prove Lemma 2.3. With a parameter $\lambda\in\mathbb{C}$ let us introduce the modified Hamiltonian H_λ by

$$H_\lambda:=H+\lambda(\psi_{\hat{\mathcal{X}}_1}^{*}\psi_{\hat{\mathcal{X}}_2}^{*}\psi_{\hat{\mathcal{Y}}_2}\psi_{\hat{\mathcal{Y}}_1}+\text{ h.c}).$$

It follows that

$$H_\lambda = H_0 + \sum_{\substack{X_1,X_2,Y_1,Y_2 \\ \in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}} U_{(\lambda,\lambda)}(X_1,X_2,Y_1,Y_2)\psi^*_{X_1}\psi^*_{X_2}\psi_{Y_1}\psi_{Y_2},$$

where $U_{(\lambda,\lambda)}$ is introduced in (2.4). The partition function $\mathrm{Tr}\, e^{-\beta H_\lambda}/\mathrm{Tr}\, e^{-\beta H_0}$ can be expanded as a perturbation series by straightforwardly following [10, Appendix B].

$$\frac{\mathrm{Tr}\, e^{-\beta H_\lambda}}{\mathrm{Tr}\, e^{-\beta H_0}} = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{m=1}^{n} \cdot\left(\sum_{\substack{X_{2m-1},X_{2m},Y_{2m-1},Y_{2m} \\ \in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}}\int_0^\beta ds_{2m-1}U_{(\lambda,\lambda)}(X_{2m-1},X_{2m},Y_{2m-1},Y_{2m})\right) \cdot\det(\mathcal{C}(X_p s_p, Y_q s_q))_{1\le p,q\le 2n}\Big|_{\substack{s_{2j}=s_{2j-1} \\ \forall j\in\{1,\dots,n\}}}. \tag{B.1}$$

Let the function $P(\lambda, U_c, U_o)$ $(: \mathbb{C}^3 \to \mathbb{C})$ be defined by the right-hand side of (B.1). Moreover, by replacing the integral over $[0,\beta)$ in the right-hand side of (B.1) by the Riemann sum we can define the discrete analogue of P.

$$P_h(\lambda, U_c, U_o) := 1 + \sum_{n=1}^{N_{L,h}/2}\frac{1}{n!}\prod_{m=1}^{n} \cdot\left(\sum_{\substack{X_{2m-1},X_{2m},Y_{2m-1},Y_{2m} \\ \in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}}}\frac{1}{h}\sum_{s_{2m-1}\in[0,\beta)_h}U_{(\lambda,\lambda)}(X_{2m-1},X_{2m},Y_{2m-1},Y_{2m})\right) \cdot\det(\mathcal{C}(X_p s_p, Y_q s_q))_{1\le p,q\le 2n}\Big|_{\substack{s_{2j}=s_{2j-1} \\ \forall j\in\{1,\dots,n\}}}.$$

The function P_h uniformly converges to P in the following sense. For any $U > 0$,

$$\lim_{\substack{h\to\infty \\ h\in 2\mathbb{N}/\beta}}\ \sup_{\substack{(\lambda,U_c,U_o)\in\mathbb{C}^3 \\ |\lambda|,|U_c|,|U_o|\le U}} |P_h(\lambda,U_c,U_o) - P(\lambda,U_c,U_o)| = 0. \tag{B.2}$$

To prove the convergence property (B.2) we need to use the determinant bound of the following form:

$$|\det(\mathcal{C}(\rho_p\mathbf{x}_p\sigma_p x_p, \eta_q\mathbf{y}_q\tau_q y_q))_{1\le p,q\le n}| \le C_1(L)\cdot C_2(L)^n, \tag{B.3}$$

where the constants $C_1(L)$, $C_2(L) > 0$ may depend on L, but are independent of n and how to choose $(\rho_p, \mathbf{x}_p, \sigma_p, x_p)$, $(\eta_p, \mathbf{y}_p, \tau_p, y_p) \in \{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)$ $(p = 1,\dots,n)$. The bound (B.3) can be verified as follows: we can choose the operators $A_1, A_2, \dots, A_{2n}$ from $\{e^{x_p H_0}\psi^*_{\rho_p\mathbf{x}_p\sigma_p}e^{-x_p H_0}, e^{y_p H_0}\psi_{\eta_p\mathbf{y}_p\tau_p} e^{-y_p H_0}\}_{p=1}^n$ so that

$$
\begin{aligned}
|\det(\mathcal{C}(\rho_p\mathbf{x}_p\sigma_p x_p,\eta_q\mathbf{y}_q\tau_q y_q))_{1\le p,q\le n}| &= \left|\operatorname{Tr}(\mathrm{e}^{-\beta H_0}A_1A_2\dots A_{2n})\right|/\operatorname{Tr}\mathrm{e}^{-\beta H_0}\\
&\le \frac{2^{6L^2}}{\operatorname{Tr}\mathrm{e}^{-\beta H_0}}\left(\mathrm{e}^{\beta\|H_0\|}\right)^{2n+1},
\end{aligned}
$$

where $\|H_0\|$ denotes the operator norm of H_0.

Let us recall that in [11, Lemma 3.4] Pedra–Salmhofer's determinant bound [13, Theorem 2.4] was applied to prove the essentially same statements as Lemma 2.3. Though we do not have a volume-independent determinant bound like [13, Theorem 2.4] on our covariance $\mathcal{C}$ at hand, the crude bound (B.3) sufficiently works to show (B.2) in the argument parallel to the proof of [11, Lemma 3.4].

The following equality directly follows from the definition of the Grassmann Gaussian integral and P_h.

$$
\int \mathrm{e}^{V_{(\lambda,\lambda)}(\psi)}\,\mathrm{d}\mu_{\mathcal{C}}(\psi) = P_h(\lambda,U_{\mathrm{c}},U_{\mathrm{o}}) \quad (\forall(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})\in\mathbb{C}^3). \tag{B.4}
$$

Since $\inf_{(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})\in\mathbb{R}^3,|\lambda|,|U_{\mathrm{c}}|,|U_{\mathrm{o}}|\le U} P(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})>0$, the uniform convergence property (B.2) and the equality (B.4) ensure the claim (i) of Lemma 2.3.

By using [10, Lemma 2.3] and (B.2) we have for any U_{c}, $U_{\mathrm{o}}\in\mathbb{R}$ and $\delta>0$,

$$
\begin{aligned}
\langle\psi^*_{\hat{\mathcal{X}}_1}\psi^*_{\hat{\mathcal{X}}_2}\psi_{\hat{\mathcal{Y}}_2}\psi_{\hat{\mathcal{Y}}_1}+\mathrm{h.c}\rangle_L &= -\frac{1}{\beta}\frac{\partial}{\partial\lambda}\log P(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})\Big|_{\lambda=0}\\
&= -\frac{1}{\beta}\frac{1}{P(0,U_{\mathrm{c}},U_{\mathrm{o}})}\frac{1}{2\pi i}\oint_{|\lambda|=\delta}\mathrm{d}\lambda\frac{P(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})}{\lambda^2}\\
&= -\frac{1}{\beta}\lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}}\frac{1}{P_h(0,U_{\mathrm{c}},U_{\mathrm{o}})}\frac{1}{2\pi i}\oint_{|\lambda|=\delta}\mathrm{d}\lambda\frac{P_h(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})}{\lambda^2}\\
&= -\frac{1}{\beta}\lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}}\frac{\partial}{\partial\lambda}\log P_h(\lambda,U_{\mathrm{c}},U_{\mathrm{o}})\Big|_{\lambda=0}.
\end{aligned} \tag{B.5}
$$

Substituting (B.4) into the right-hand side of (B.5) yields the claim (ii) of Lemma 2.3.

Appendix C. Logarithm of Grassmann Polynomials

The aim of this section is to extend the notion of logarithm of Grassmann polynomials summarized in [6] to be available for Grassmann polynomials with complex constant terms. In the following let $f_0, g_0\in\mathbb{C}$ denote the constant term of $f, g\in\bigwedge\mathcal{V}$, respectively:

Definition C.1. For $f\in\bigwedge\mathcal{V}$ with $\operatorname{Re} f_0>0$, $\log f\in\bigwedge\mathcal{V}$ is defined by

$$
\log f := \log(f_0) + \sum_{n=1}^{2N_{L,h}}\frac{(-1)^{n-1}}{n}\left(\frac{f-f_0}{f_0}\right)^n,
$$

where $\log z := \log|z| + i\operatorname{Arg} z$, $\operatorname{Arg} z\in(-\pi/2,\pi/2)$ for $z\in\mathbb{C}$ with $\operatorname{Re} z>0$.

Recall that for $f \in \bigwedge \mathcal{V}$, $\mathrm{e}^f \in \bigwedge \mathcal{V}$ is defined by

$$\mathrm{e}^f := \mathrm{e}^{f_0} \sum_{n=0}^{2N_{L,h}} \frac{1}{n!}(f - f_0)^n. \tag{C.1}$$

It was proved in [6, Problem I.2] that for any $f, g \in \bigwedge \mathcal{V}$ satisfying $fg = gf$,

$$\mathrm{e}^f \cdot \mathrm{e}^g = \mathrm{e}^g \cdot \mathrm{e}^f = \mathrm{e}^{f+g}. \tag{C.2}$$

The following equality was also shown in [6, Problem I.4 b)]. For any $f \in \bigwedge \mathcal{V}$ with $f_0 \in \mathbb{R}_{>0}$,

$$\mathrm{e}^{\log f} = f. \tag{C.3}$$

The multi-scale analysis in this paper needs an extension of (C.3).

Lemma C.2. *For any $f \in \bigwedge \mathcal{V}$ with $\mathrm{Re}\, f_0 \in \mathbb{R}_{>0}$, $\mathrm{e}^{\log f} = f$.*

Proof. Take $f \in \bigwedge \mathcal{V}$ with $\mathrm{Re}\, f_0 > 0$. Since $\log(|f_0|^2) = \log(f_0) + \log(\overline{f_0})$,

$$\log(\overline{f_0} \cdot f) = \log(|f_0|^2) + \sum_{n=1}^{2N_{L,h}} \frac{(-1)^{n-1}}{n} \left(\frac{\overline{f_0} \cdot f - |f_0|^2}{|f_0|^2} \right)^n = \log(\overline{f_0}) + \log f. \tag{C.4}$$

It follows from (C.3) that

$$\mathrm{e}^{\log(\overline{f_0} \cdot f)} = \overline{f_0} \cdot f. \tag{C.5}$$

By using (C.2), (C.4) and (C.5) we observe that

$$\mathrm{e}^{\log f} = \mathrm{e}^{-\log(\overline{f_0}) + \log(\overline{f_0} \cdot f)} = \mathrm{e}^{-\log(\overline{f_0})} \cdot \mathrm{e}^{\log(\overline{f_0} \cdot f)} = \frac{1}{\overline{f_0}} \cdot \overline{f_0} \cdot f = f.$$

□

Appendix D. Existence of the Thermodynamic Limit

Here we show that the correlation function $\langle \psi^*_{\hat{\mathcal{X}}_1} \psi^*_{\hat{\mathcal{X}}_2} \psi_{\hat{\mathcal{Y}}_2} \psi_{\hat{\mathcal{Y}}_1} + \mathrm{h.c} \rangle_L$ converges to a finite value as $L \to \infty$ if $|U_{\mathrm{c}}|$, $|U_{\mathrm{o}}|$ are smaller than certain value. The idea of the proof is similar to [11, Appendix B] and based on the perturbative expansion of logarithm of the Grassmann Gaussian integral. We also use the following lemma:

Lemma D.1. (i) *For any $(\rho, \mathbf{x}, \sigma, x)$, $(\eta, \mathbf{y}, \tau, y) \in \{1,2,3\} \times \mathbb{Z}^2 \times \{\uparrow, \downarrow\} \times [0, \beta)$ with $x \neq y$,*

$$|\mathcal{C}(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)| \le \frac{c(E_{\max}, \beta)}{1 + \sum_{p=1}^{2} \left(\frac{L}{2\pi}\right)^3 \left| \mathrm{e}^{i2\pi \langle \mathbf{x} - \mathbf{y}, \mathbf{e}_p \rangle / L} - 1 \right|^3},$$

where the constant $c(E_{\max}, \beta) > 0$ depends only on $E_{\max}$ and β.

(ii) *For any $(\rho, \mathbf{x}, \sigma, x)$, $(\eta, \mathbf{y}, \tau, y) \in \{1,2,3\} \times \mathbb{Z}^2 \times \{\uparrow, \downarrow\} \times [0, \beta)$, $\lim_{L \to \infty, L \in \mathbb{N}} \mathcal{C}(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)$ exists.*

Proof. (i): Take any $(\rho, \mathbf{x}, \sigma, x)$, $(\eta, \mathbf{y}, \tau, y) \in \{1,2,3\} \times \mathbb{Z}^2 \times \{\uparrow, \downarrow\} \times [0, \beta)$. By using the notations introduced in Appendix A, set

$$g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(\mathbf{k}) := \delta_{\sigma,\tau} \sum_{\gamma\in\{1,2,3\}} \mathrm{e}^{(x-y)A^{\sigma}_{\gamma}(t,\mathbf{k})} \cdot \left(\frac{1_{x\ge y}}{1+\mathrm{e}^{\beta A^{\sigma}_{\gamma}(t,\mathbf{k})}} - \frac{1_{x<y}}{1+\mathrm{e}^{-\beta A^{\sigma}_{\gamma}(t,\mathbf{k})}} \right) \overline{\mathcal{U}^{\sigma}_{t,\mathbf{k}}(\rho,\gamma)} \mathcal{U}^{\sigma}_{t,\mathbf{k}}(\eta,\gamma).$$

By (A.5), $\mathcal{C}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) = \frac{1}{L^2}\sum_{\mathbf{k}\in\Gamma^*} \mathrm{e}^{-i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle} g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(\mathbf{k})$. Since $|\mathcal{U}^{\sigma}_{t,\mathbf{k}}(\rho,\gamma)|$, $|\mathcal{U}^{\sigma}_{t,\mathbf{k}}(\eta,\gamma)| \le 1$, $|g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(\mathbf{k})| \le 3$. This implies that $|\mathcal{C}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)| \le 3$.

Let us additionally assume that $x \ne y$. In this case we can expand $g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(\mathbf{k})$ as a sum over $\pi(2\mathbb{Z}+1)/\beta$ so that

$$\mathcal{C}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) = \frac{\delta_{\sigma,\tau}}{\beta L^2} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\pi(2\mathbb{Z}+1)/\beta} \mathrm{e}^{-i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle} \mathrm{e}^{i(x-y)\omega} \mathcal{B}^{\sigma,\infty}_{\rho,\eta}(\mathbf{k},\omega),$$

where

$$\mathcal{B}^{\sigma,\infty}_{\rho,\eta}(\mathbf{k},\omega) := \sum_{\gamma\in\{1,2,3\}} \frac{\overline{\mathcal{U}^{\sigma}_{t,\mathbf{k}}(\rho,\gamma)} \mathcal{U}^{\sigma}_{t,\mathbf{k}}(\eta,\gamma)}{i\omega - A^{\sigma}_{\gamma}(t,\mathbf{k})}.$$

We can see from (A.6) that $\mathcal{B}^{\sigma,\infty}_{\rho,\eta}(\mathbf{k},\omega) = \lim_{h\to\infty, h\in 2\mathbb{N}/\beta} \mathcal{B}^{\sigma}_{\rho,\eta}(\mathbf{k},\omega)$. Thus by setting

$$\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega) := \left(i\omega - \frac{1}{2}(\epsilon^{\sigma}_{\mathrm{c}} + \epsilon^{\sigma}_{\mathrm{o}}) \right)^2 - \frac{1}{4}(\epsilon^{\sigma}_{\mathrm{c}} - \epsilon^{\sigma}_{\mathrm{o}})^2 - 2t^2 \sum_{j=1}^{2} (1+\cos k_j),$$

it follows from (2.2) that for any $\mathbf{k} = (k_1, k_2) \in \Gamma^*$ and $\omega \in \pi(2\mathbb{Z}+1)/\beta$,

$$\begin{aligned}
&\mathcal{B}^{\sigma,\infty}_{1,1}(\mathbf{k},\omega) = \frac{i\omega - \epsilon^{\sigma}_{\mathrm{o}}}{\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)},\ \mathcal{B}^{\sigma,\infty}_{1,2}(\mathbf{k},\omega) = \frac{t(1+\mathrm{e}^{ik_1})}{\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)},\\
&\mathcal{B}^{\sigma,\infty}_{1,3}(\mathbf{k},\omega) = \mathcal{B}^{\sigma,\infty}_{1,2}((k_2,k_1),\omega),\\
&\mathcal{B}^{\sigma,\infty}_{2,1}(\mathbf{k},\omega) = \mathcal{B}^{\sigma,\infty}_{1,2}(-\mathbf{k},\omega),\ \mathcal{B}^{\sigma,\infty}_{2,2}(\mathbf{k},\omega) = \frac{1}{i\omega - \epsilon^{\sigma}_{\mathrm{o}}} + \frac{2t^2(1+\cos k_1)}{(i\omega - \epsilon^{\sigma}_{\mathrm{o}})\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)},\\
&\mathcal{B}^{\sigma,\infty}_{2,3}(\mathbf{k},\omega) = \frac{t^2(1+\mathrm{e}^{-ik_1})(1+\mathrm{e}^{ik_2})}{(i\omega - \epsilon^{\sigma}_{\mathrm{o}})\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)},\ \mathcal{B}^{\sigma,\infty}_{3,1}(\mathbf{k},\omega) = \mathcal{B}^{\sigma,\infty}_{1,2}(-(k_2,k_1),\omega),\\
&\mathcal{B}^{\sigma,\infty}_{3,2}(\mathbf{k},\omega) = \mathcal{B}^{\sigma,\infty}_{2,3}(-\mathbf{k},\omega),\ \mathcal{B}^{\sigma,\infty}_{3,3}(\mathbf{k},\omega) = \mathcal{B}^{\sigma,\infty}_{2,2}((k_2,k_1),\omega).
\end{aligned}$$

Periodicity with respect to $\mathbf{k} \in \Gamma^*$ guarantees that for $p \in \{1,2\}$,

$$\begin{aligned}
&\left(\frac{L}{2\pi} \left(\mathrm{e}^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_p\rangle} - 1 \right) \right)^3 \mathcal{C}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&= \frac{\delta_{\sigma,\tau}}{\beta L^2} \sum_{\mathbf{k}\in\Gamma^*} \sum_{\omega\in\pi(2\mathbb{Z}+1)/\beta} \mathrm{e}^{-i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle} \mathrm{e}^{i(x-y)\omega}
\end{aligned}$$

$$\cdot \prod_{j=1}^{3}\left(\frac{L}{2\pi}\int_{0}^{2\pi/L} \mathrm{d}\theta_j\right)\left(\frac{\partial}{\partial k_p}\right)^3 \mathcal{B}_{\rho,\eta}^{\sigma,\infty}\left(\mathbf{k}+\sum_{j=1}^{3}\theta_j\mathbf{e}_p,\omega\right). \quad \text{(D.1)}$$

Note that for any $\mathbf{k}\in\mathbb{R}^2$, $\omega\in\pi(2\mathbb{Z}+1)/\beta$,

$$\begin{aligned}|\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)| &\geq \max\{|\operatorname{Re}\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)|,|\operatorname{Im}\mathcal{D}^{\sigma,\infty}(\mathbf{k},\omega)|\}\\ &\geq \max\{\omega^2-\epsilon_{\mathrm{c}}^{\sigma}\epsilon_{\mathrm{o}}^{\sigma},|\omega(\epsilon_{\mathrm{c}}^{\sigma}+\epsilon_{\mathrm{o}}^{\sigma})|\}\geq \frac{1}{2}\omega^2. \end{aligned} \quad \text{(D.2)}$$

By using (D.2) we can estimate the equality (D.1) and deduce that

$$\begin{aligned}&\left|\frac{L}{2\pi}\left(\mathrm{e}^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_p\rangle}-1\right)\right|^3|\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)|\\ &\quad\leq \frac{1}{\beta}\sum_{\omega\in\pi(2\mathbb{Z}+1)/\beta}\frac{c(E_{\max},\beta)}{\omega^2}\leq c(E_{\max},\beta). \end{aligned} \quad \text{(D.3)}$$

By coupling (D.3) with the bound $|\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)|\leq 3$ we obtain the inequality in (i).

(ii): Note that for any $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y)\in\{1,2,3\}\times\Gamma\times\{\uparrow,\downarrow\}\times[0,\beta)$,

$$\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)=\frac{1}{(2\pi)^2}\int_{[-\pi,\pi)}\mathrm{d}p_1\int_{[-\pi,\pi)}\mathrm{d}p_2\tilde{g}_{L,(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)}(p_1,p_2),$$

where $\tilde{g}_{L,(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)}(p_1,p_2):=\mathrm{e}^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(k_1,k_2)$ with $k_j\in\{-\pi,-\pi+2\pi/L,\dots,\pi-2\pi/L\}$ satisfying that $p_j\in[k_j,k_j+2\pi/L)$ $(j=1,2)$. Since $\mathbf{k}\mapsto g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(\mathbf{k})$ is continuous in $(-\pi,\pi)^2$ by definition, $\lim_{L\to\infty,L\in\mathbb{N}}\tilde{g}_{L,(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)}(\mathbf{p})$ exists for any $\mathbf{p}\in(-\pi,\pi)^2$. As we have seen above, $|\tilde{g}_{L,(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)}(\mathbf{p})|=|g_{L,(\rho,\sigma,x),(\eta,\tau,y)}(\mathbf{k})|\leq 3$. Therefore, the dominated convergence theorem concludes that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)=\frac{1}{(2\pi)^2}\int_{[-\pi,\pi)^2}\mathrm{d}\mathbf{p}\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\tilde{g}_{L,(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)}(\mathbf{p}).$$

□

Lemma D.2. *Assume that $U_{\mathrm{c}},U_{\mathrm{o}}\in\mathbb{R}$ and (4.13) holds with c_0 defined in (4.9). Then, $\langle\psi_{\hat{\mathcal{X}}_1}^*\psi_{\hat{\mathcal{X}}_2}^*\psi_{\hat{\mathcal{Y}}_2}\psi_{\hat{\mathcal{Y}}_1}+h.c\rangle_L$ converges to a finite value as $L\to\infty$ ($L\in\mathbb{N}$).*

Proof. Fix $U_{\mathrm{c}},U_{\mathrm{o}}\in\mathbb{R}$ with $|U_{\mathrm{c}}|,|U_{\mathrm{o}}|<2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}$. It follows from Lemma 2.3 (ii) and (4.54) for $n=0$ that

$$\langle\psi_{\hat{\mathcal{X}}_1}^*\psi_{\hat{\mathcal{X}}_2}^*\psi_{\hat{\mathcal{Y}}_2}\psi_{\hat{\mathcal{Y}}_1}+\mathrm{h.c}\rangle_L=-\frac{1}{\beta}\lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}}\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}.$$

Thus, it suffices to prove the convergence of

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}}\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}. \quad \text{(D.4)}$$

In order to make clear the dependency on U_{c}, U_{o} we write

$$\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(U_{\mathrm{c}},U_{\mathrm{o}})$$

in place of $(\partial/\partial\lambda_a)J_0^{\geq N_\beta}(\mathbf{0})|_{(\lambda_1,\lambda_{-1})=(0,0)}$. We can take $\varepsilon > 0$ such that $(1+\varepsilon)|U_{\mathrm{c}}|$, $(1+\varepsilon)|U_{\mathrm{o}}| < 2^{-4}\alpha^{-2}c_0^{-2}M^{N_\beta}$. By Corollary 4.7 (ii) there is a domain $D_o \subset \mathbb{C}$ containing the disk $\{z\in\mathbb{C} \mid |z|\leq 1+\varepsilon\}$ inside such that

$$z\mapsto \sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(zU_{\mathrm{c}},zU_{\mathrm{o}})$$

is analytic in D_{o}. Thus,

$$\begin{aligned}&\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(U_{\mathrm{c}},U_{\mathrm{o}})\\ &\quad=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left(\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(zU_{\mathrm{c}},zU_{\mathrm{o}})\right)\Bigg|_{z=0}.\end{aligned}$$

Moreover, by Proposition 4.9, for any $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}&\left|\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left(\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(zU_{\mathrm{c}},zU_{\mathrm{o}})\right)\Bigg|_{z=0}\right|\\ &\quad=\left|\frac{1}{2\pi i}\oint_{|z|=1+\varepsilon}\mathrm{d}z\cdot z^{-n-1}\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(zU_{\mathrm{c}},zU_{\mathrm{o}})\right|\\ &\quad\leq 2^{13}\beta c_0^2(1+\varepsilon)^{-n}.\end{aligned}$$

Since $(1+\varepsilon)^{-n}$ is summable over $\mathbb{N}\cup\{0\}$, the dominant convergence theorem guarantees that (D.4) converges if

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\ in2\mathbb{N}/\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left(\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(zU_{\mathrm{c}},zU_{\mathrm{o}})\right)\Bigg|_{z=0} \tag{D.5}$$

exists for all $n\in\mathbb{N}\cup\{0\}$.

Again by (4.54) for $n=0$ we can write for any $x\in\mathbb{R}$ with $|x|\leq 1+\varepsilon$ that

$$\begin{aligned}&\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\geq N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(xU_{\mathrm{c}},xU_{\mathrm{o}})\\ &\quad=\frac{\partial}{\partial\lambda}\log\left(\int \mathrm{e}^{V_{(\lambda,\lambda)}(\psi)}\,\mathrm{d}\mu_{\mathcal{C}}(\psi)\right)\Bigg|_{\lambda=0}(xU_{\mathrm{c}},xU_{\mathrm{o}}),\end{aligned} \tag{D.6}$$

which implies that

$$\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\ge N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(0,0)$$
$$=-\beta(\det(\mathcal{C}(\hat{\mathcal{X}}_p0,\hat{\mathcal{Y}}_q0))_{1\le p,q\le 2}+\det(\mathcal{C}(\hat{\mathcal{Y}}_p0,\hat{\mathcal{X}}_q0))_{1\le p,q\le 2}).$$

Thus, Lemma D.1 (ii) proves the existence of (D.5) for $n=0$.

It follows from (D.6) that for any $n\in\mathbb{N}$,

$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left(\sum_{a\in\{1,-1\}}\frac{\partial}{\partial\lambda_a}J_0^{\ge N_\beta}(\mathbf{0})\Big|_{(\lambda_1,\lambda_{-1})=(0,0)}(zU_{\mathrm{c}},zU_{\mathrm{o}})\right)\Bigg|_{z=0}$$
$$=\frac{\partial}{\partial\lambda}\left(\frac{1}{(n+1)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n+1}\log\left(\int \mathrm{e}^{xV_{(\lambda,\lambda)}(\psi)}\,\mathrm{d}\mu_{\mathcal{C}}(\psi)\right)\Bigg|_{x=0}\right)\Bigg|_{\lambda=0}$$
$$=\frac{\partial}{\partial\lambda}\mathcal{P}_0T_{\mathrm{ree}}(n+1,\mathcal{C},V_{(\lambda,\lambda)})\Big|_{\lambda=0}.$$

Recall that $T_{\mathrm{ree}}(\cdot,\cdot,\cdot)$ is defined in (4.7). In the expansion of $\mathcal{P}_0T_{\mathrm{ree}}(n+1,\mathcal{C},V_{(\lambda,\lambda)})$ we apply the operator $\prod_{\{q,r\}\in T}(\Delta_{q,r}(\mathcal{C})+\Delta_{r,q}(\mathcal{C}))$ first and then erase the rest of Grassmann polynomials by the operator $\mathrm{e}^{\sum_{q,r=1}^{n+1}M_{at}(T,\xi,\mathbf{s})_{q,r}\Delta_{q,r}(\mathcal{C})}$. By recalling the notation (4.18) we observe that

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\mathcal{P}_0T_{\mathrm{ree}}(n+1,\mathcal{C},V_{(\lambda,\lambda)})\Big|_{\lambda=0}\\
&=\sum_{a\in\{1,-1\}}\sum_{T\in\mathbb{T}_{n+1}}\frac{1}{h}\sum_{x_1\in[0,\beta)_h}\\
&\quad\cdot\prod_{j=2}^{n+1}\Bigg(\sum_{\rho_j\in\{1,2,3\}}(1_{\rho_j=1}U_{\mathrm{c}}+1_{\rho_j=2,3}U_{\mathrm{o}})\sum_{\substack{\sigma_1^j,\sigma_2^j,\tau_1^j,\tau_2^j\\ \in\{\uparrow,\downarrow\}}}1_{(\sigma_1^j,\sigma_2^j,\tau_1^j,\tau_2^j)=(\uparrow,\downarrow,\downarrow,\uparrow)}\\
&\quad\cdot\frac{1}{h}\sum_{(\mathbf{x}_j,x_j)\in\Gamma\times[0,\beta)_h}\Bigg)\\
&\quad\cdot\prod_{\{1,r\}\in L_1^1(T)}\left(\sum_{k_{\{1,r\}}=1}^{2}\sum_{l_{\{1,r\}}=1}^{2}\sum_{b_{\{1,r\}}\in\{1,-1\}}\mathcal{C}_{\{1,r\},a}^{k_{\{1,r\}},l_{\{1,r\}},b_{\{1,r\}}}(x_1,\mathbf{x}_rx_r)\right)\\
&\quad\cdot\prod_{q=2}^{n+1}\prod_{\{q,r\}\in L_q^1(T)}\Bigg(\sum_{k_{\{q,r\}}=1}^{2}\sum_{l_{\{q,r\}}=1}^{2}\sum_{b_{\{q,r\}}\in\{1,-1\}}\\
&\quad\cdot\mathcal{C}_{\{q,r\}}^{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}}(\mathbf{x}_qx_q,\mathbf{x}_rx_r)\Bigg)f(T,a,\{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}\}_{\{q,r\}\in T},\mathcal{C}),\end{aligned}\tag{D.7}$$

where

$$
\begin{aligned}
&\mathcal{C}_{\{1,r\},a}^{k_{\{1,r\}},l_{\{1,r\}},b_{\{1,r\}}}(x_1,\mathbf{x}_r x_r)\\
&\quad:=\begin{cases}
\mathcal{C}(\hat{\mathcal{X}}_{k_{\{1,r\}}}x_1,\rho_r\mathbf{x}_r\tau^r_{l_{\{1,r\}}}x_r) & \text{if } a=1,\ b_{\{1,r\}}=1,\\
\mathcal{C}(\rho_r\mathbf{x}_r\sigma^r_{l_{\{1,r\}}}x_r,\hat{\mathcal{Y}}_{k_{\{1,r\}}}x_1) & \text{if } a=1,\ b_{\{1,r\}}=-1,\\
\mathcal{C}(\hat{\mathcal{Y}}_{k_{\{1,r\}}}x_1,\rho_r\mathbf{x}_r\tau^r_{l_{\{1,r\}}}x_r) & \text{if } a=-1,\ b_{\{1,r\}}=1,\\
\mathcal{C}(\rho_r\mathbf{x}_r\sigma^r_{l_{\{1,r\}}}x_r,\hat{\mathcal{X}}_{k_{\{1,r\}}}x_1) & \text{if } a=-1,\ b_{\{1,r\}}=-1,
\end{cases}\\
&\mathcal{C}_{\{q,r\}}^{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}}(\mathbf{x}_q x_q,\mathbf{x}_r x_r)\\
&\quad:=\begin{cases}
\mathcal{C}(\rho_q\mathbf{x}_q\sigma^q_{k_{\{q,r\}}}x_q,\rho_r\mathbf{x}_r\tau^r_{l_{\{q,r\}}}x_r) & \text{if } b_{\{q,r\}}=1,\\
\mathcal{C}(\rho_r\mathbf{x}_r\sigma^r_{l_{\{q,r\}}}x_r,\rho_q\mathbf{x}_q\tau^q_{k_{\{q,r\}}}x_q) & \text{if } b_{\{q,r\}}=-1,
\end{cases}\\
&f(T,a,\{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}\}_{\{q,r\}\in T},\mathcal{C})\\
&\quad:=\frac{1}{n!}\int_{[0,1]^n}d\mathbf{s}\sum_{\xi\in\mathbb{S}_{n+1}(T)}\varphi(T,\xi,\mathbf{s})e^{\sum_{u,v=1}^{n+1}M_{at}(T,\xi,\mathbf{s})_{u,v}\Delta_{u,v}(\mathcal{C})}\\
&\qquad\cdot\prod_{\{1,r\}\in L_1^1(T)}\mathcal{L}_{\{1,r\},a}^{k_{\{1,r\}},l_{\{1,r\}},b_{\{1,r\}}}(x_1,\mathbf{x}_r x_r)\prod_{q=2}^{n+1}\prod_{\{q,r\}\in L_q^1(T)}\\
&\qquad\cdot\mathcal{L}_{\{q,r\}}^{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}}(\mathbf{x}_q x_q,\mathbf{x}_r x_r)\\
&\qquad\cdot(-1_{a=1}\overline{\psi}^1_{\hat{\mathcal{X}}_1x_1}\overline{\psi}^1_{\hat{\mathcal{X}}_2x_1}\psi^1_{\hat{\mathcal{Y}}_2x_1}\psi^1_{\hat{\mathcal{Y}}_1x_1}-1_{a=-1}\overline{\psi}^1_{\hat{\mathcal{Y}}_1x_1}\overline{\psi}^1_{\hat{\mathcal{Y}}_2x_1}\psi^1_{\hat{\mathcal{X}}_2x_1}\psi^1_{\hat{\mathcal{X}}_1x_1})\\
&\qquad\cdot\prod_{s=2}^{n+1}(-\overline{\psi}^s_{\rho_s\mathbf{x}_s\sigma_1^sx_s}\overline{\psi}^s_{\rho_s\mathbf{x}_s\sigma_2^sx_s}\psi^s_{\rho_s\mathbf{x}_s\tau_1^sx_s}\psi^s_{\rho_s\mathbf{x}_s\tau_2^sx_s})\Big|_{\substack{\psi^j=\mathbf{0}\\ \forall j\in\{1,\dots,n+1\}}},\\
&\mathcal{L}_{\{1,r\},a}^{k_{\{1,r\}},l_{\{1,r\}},b_{\{1,r\}}}(x_1,\mathbf{x}_r x_r)\\
&\quad:=\begin{cases}
-(\partial/\partial\overline{\psi}^1_{\hat{\mathcal{X}}_{k_{\{1,r\}}}x_1})(\partial/\partial\psi^r_{\rho_r\mathbf{x}_r\tau^r_{l_{\{1,r\}}}x_r}) & \text{if } a=1,\ b_{\{1,r\}}=1,\\
-(\partial/\partial\overline{\psi}^r_{\rho_r\mathbf{x}_r\sigma^r_{l_{\{1,r\}}}x_r})(\partial/\partial\psi^1_{\hat{\mathcal{Y}}_{k_{\{1,r\}}}x_1}) & \text{if } a=1,\ b_{\{1,r\}}=-1,\\
-(\partial/\partial\overline{\psi}^1_{\hat{\mathcal{Y}}_{k_{\{1,r\}}}x_1})(\partial/\partial\psi^r_{\rho_r\mathbf{x}_r\tau^r_{l_{\{1,r\}}}x_r}) & \text{if } a=-1,\ b_{\{1,r\}}=1,\\
-(\partial/\partial\overline{\psi}^r_{\rho_r\mathbf{x}_r\sigma^r_{l_{\{1,r\}}}x_r})(\partial/\partial\psi^1_{\hat{\mathcal{X}}_{k_{\{1,r\}}}x_1}) & \text{if } a=-1,\ b_{\{1,r\}}=-1,
\end{cases}\\
&\mathcal{L}_{\{q,r\}}^{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}}(\mathbf{x}_q x_q,\mathbf{x}_r x_r)\\
&\quad:=\begin{cases}
-(\partial/\partial\overline{\psi}^q_{\rho_q\mathbf{x}_q\sigma^q_{k_{\{q,r\}}}x_q})(\partial/\partial\psi^r_{\rho_r\mathbf{x}_r\tau^r_{l_{\{q,r\}}}x_r}) & \text{if } b_{\{q,r\}}=1,\\
-(\partial/\partial\overline{\psi}^r_{\rho_r\mathbf{x}_r\sigma^r_{l_{\{q,r\}}}x_r})(\partial/\partial\psi^q_{\rho_q\mathbf{x}_q\tau^q_{k_{\{q,r\}}}x_q}) & \text{if } b_{\{q,r\}}=-1.
\end{cases}
\end{aligned}
$$

By the translation invariance and the periodicity of $\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)$ with respect to $\mathbf{x},\mathbf{y}\in\Gamma$,

$$\frac{\partial}{\partial\lambda}\mathcal{P}_0 T_{\rm ree}(n+1,\mathcal{C},V_{(\lambda,\lambda)})\Bigg|_{\lambda=0}$$
$$=\frac{1}{h}\sum_{x_1\in[0,\beta)_h}\prod_{j=2}^{n+1}\left(\frac{1}{h}\sum_{(\mathbf{x}_j,x_j)\in\Gamma\times[0,\beta)_h}\right)F_L(x_1,\mathbf{x}_2x_2,\dots,\mathbf{x}_{n+1}x_{n+1}),\quad \text{(D.8)}$$

where

$$F_L(x_1,\mathbf{x}_2x_2,\dots,\mathbf{x}_{n+1}x_{n+1}):=\sum_{a\in\{1,-1\}}\sum_{T\in\mathbb{T}_{n+1}}\prod_{j=2}^{n+1}$$
$$\cdot\left(\sum_{\rho_j\in\{1,2,3\}}(1_{\rho_j=1}U_{\rm c}+1_{\rho_j=2,3}U_{\rm o})\sum_{\substack{\sigma_1^j,\sigma_2^j,\tau_1^j,\tau_2^j\\ \in\{\uparrow,\downarrow\}}}1_{(\sigma_1^j,\sigma_2^j,\tau_1^j,\tau_2^j)=(\uparrow,\downarrow,\downarrow,\uparrow)}\right)$$
$$\cdot\prod_{\{1,r\}\in L_1^1(T)}\left(\sum_{k_{\{1,r\}}=1}^{2}\sum_{l_{\{1,r\}}=1}^{2}\sum_{b_{\{1,r\}}\in\{1,-1\}}\tilde{\mathcal{C}}_{\{1,r\},a}^{k_{\{1,r\}},l_{\{1,r\}},b_{\{1,r\}}}(x_1,\mathbf{x}_rx_r)\right)$$
$$\cdot\prod_{q=2}^{n+1}\prod_{\{q,r\}\in L_q^1(T)}\left(\sum_{k_{\{q,r\}}=1}^{2}\sum_{l_{\{q,r\}}=1}^{2}\sum_{b_{\{q,r\}}\in\{1,-1\}}\mathcal{C}_{\{q,r\}}^{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}}(\mathbf{0}x_q,\mathbf{x}_rx_r)\right)$$
$$\cdot f(T,a,\{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}\}_{\{q,r\}\in T},\mathcal{C}),\quad \text{(D.9)}$$
$$\tilde{\mathcal{C}}_{\{1,r\},a}^{k_{\{1,r\}},l_{\{1,r\}},b_{\{1,r\}}}(x_1,\mathbf{x}_rx_r)$$
$$:=\begin{cases}\mathcal{C}(\hat{\rho}_{k_{\{1,r\}}}\mathbf{0}\hat{\sigma}_{k_{\{1,r\}}}x_1,\rho_r\mathbf{x}_r\tau^r_{l_{\{1,r\}}}x_r) & \text{if } a=1,\quad b_{\{1,r\}}=1,\\ \mathcal{C}(\rho_r\mathbf{x}_r\sigma^r_{l_{\{1,r\}}}x_r,\hat{\eta}_{k_{\{1,r\}}}\mathbf{0}\hat{\tau}_{k_{\{1,r\}}}x_1) & \text{if } a=1,\quad b_{\{1,r\}}=-1,\\ \mathcal{C}(\hat{\eta}_{k_{\{1,r\}}}\mathbf{0}\hat{\tau}_{k_{\{1,r\}}}x_1,\rho_r\mathbf{x}_r\tau^r_{l_{\{1,r\}}}x_r) & \text{if } a=-1,\quad b_{\{1,r\}}=1,\\ \mathcal{C}(\rho_r\mathbf{x}_r\sigma^r_{l_{\{1,r\}}}x_r,\hat{\rho}_{k_{\{1,r\}}}\mathbf{0}\hat{\sigma}_{k_{\{1,r\}}}x_1) & \text{if } a=-1,\quad b_{\{1,r\}}=-1.\end{cases}$$

Though we do not explicitly write for simplicity, we should remark that the dependency of $f(T,a,\{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}\}_{\{q,r\}\in T},\mathcal{C})$ on the variables $x_1\in[0,\beta)_h$, $(\mathbf{x}_j,x_j)\in\Gamma\times[0,\beta)_h$ $(j=2,\dots,n+1)$ in (D.9) is different from that in (D.7).

For $s_1\in[0,\beta)$, $(\mathbf{x}_j,s_j)\in\mathbb{Z}^2\times[0,\beta)$ $(j=2,\dots,n+1)$ set

$$F_{L,h}(s_1,\mathbf{x}_2s_2,\dots,\mathbf{x}_{n+1}s_{n+1}):=F_L(x_1,\mathbf{x}_2x_2,\dots,\mathbf{x}_{n+1}x_{n+1}),$$

where $x_j\in[0,\beta)_h$ satisfies that $s_j\in[x_j,x_j+h^{-1})$ $(\forall j\in\{1,\dots,n+1\})$. Since $(x,y)\mapsto\mathcal{C}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)$ is continuous a.e. in $[0,\beta)^2$,

$$\lim_{\substack{h\to\infty\\ h\in2\mathbb{N}/\beta}}F_{L,h}(s_1,\mathbf{x}_2s_2,\dots,\mathbf{x}_{n+1}s_{n+1})=F_L(s_1,\mathbf{x}_2s_2,\dots,\mathbf{x}_{n+1}s_{n+1})$$

for a.e. $(s_1,s_2,\dots,s_{n+1})\in[0,\beta)^{n+1}$, and thus

$$\lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}} \frac{\partial}{\partial\lambda}\mathcal{P}_0 T_{ree}(n+1,\mathcal{C},V_{(\lambda,\lambda)})\Big|_{\lambda=0}$$
$$= \lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}} \int_0^\beta \mathrm{d}s_1 \prod_{j=2}^{n+1}\left(\int_0^\beta \mathrm{d}s_j \sum_{\mathbf{x}_j\in\Gamma}\right) F_{L,h}(s_1,\mathbf{x}_2 s_2,\ldots,\mathbf{x}_{n+1}s_{n+1})$$
$$= \int_0^\beta ds_1 \prod_{j=2}^{n+1}\left(\int_0^\beta \mathrm{d}s_j \sum_{\mathbf{x}_j\in\Gamma}\right) F_L(s_1,\mathbf{x}_2 s_2,\ldots,\mathbf{x}_{n+1}s_{n+1}).$$

Lemma D.1 (ii) implies that $\lim_{L\to\infty,L\in\mathbb{N}} F_L(s_1,\mathbf{x}_2 s_2,\ldots,\mathbf{x}_{n+1}s_{n+1})$ exists for any $s_1\in[0,\beta)$, $(\mathbf{x}_j,s_j)\in\mathbb{Z}^2\times[0,\beta)$ $(j=2,\ldots,n+1)$.

By Lemma D.1 (i), (4.8) and the fact that $|M_{at}(T,\xi,\mathbf{s})_{u,v}|\le 1$ $(\forall u,v\in\{1,\ldots,n+1\})$ (see the proof of [10, Lemma 4.5]) there exists $c(n,T,E_{\max},\beta)>0$ depending only on n, T, $E_{\max}$ and β such that $|f(T,a,\{k_{\{q,r\}},l_{\{q,r\}},b_{\{q,r\}}\}_{\{q,r\}\in T},\mathcal{C})|\le c(n,T,E_{\max},\beta)$ for a.e. $(s_1,\ldots,s_{n+1})\in[0,\beta)^{n+1}$. Therefore, by setting $U_{\max}:=\max\{|U_{\mathrm{c}}|,|U_{\mathrm{o}}|\}$ and using Lemma D.1 (i),

$$\begin{aligned}
&1_{\mathbf{x}_j\in\{-\lfloor L/2\rfloor,-\lfloor L/2\rfloor+1,\ldots,-\lfloor L/2\rfloor+L-1\}^2\ (\forall j\in\{2,\ldots,n+1\})}\\
&\quad\cdot|F_L(s_1,\mathbf{x}_2 s_2,\ldots,\mathbf{x}_{n+1}s_{n+1})|\\
&\le U_{\max}^n\sum_{T\in\mathbb{T}_{n+1}} c(n,T,E_{\max},\beta)\prod_{\{1,r\}\in L_1^1(T)}\frac{1}{1+\left(\frac{2}{\pi}\right)^3\sum_{p=1}^2|\langle\mathbf{x}_r,\mathbf{e}_p\rangle|^3}\\
&\quad\cdot\prod_{q=2}^{n+1}\prod_{\{q,r\}\in L_q^1(T)}\frac{1}{1+\left(\frac{2}{\pi}\right)^3\sum_{p=1}^2|\langle\mathbf{x}_r,\mathbf{e}_p\rangle|^3}\\
&=U_{\max}^n\prod_{j=2}^{n+1}\frac{1}{1+\left(\frac{2}{\pi}\right)^3\sum_{p=1}^2|\langle\mathbf{x}_j,\mathbf{e}_p\rangle|^3}\sum_{T\in\mathbb{T}_{n+1}} c(n,T,E_{\max},\beta)
\end{aligned}\tag{D.10}$$

for a.e. $(s_1,\ldots,s_{n+1})\in[0,\beta)^{n+1}$ and any $\mathbf{x}_j\in\mathbb{Z}^2$ $(j=2,\ldots,n+1)$. The right-hand side of (D.10) is in $L^1([0,\beta)\times(\mathbb{Z}^2\times[0,\beta))^n)$. Thus, the dominated convergence theorem proves that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in 2\mathbb{N}/\beta}} \frac{\partial}{\partial\lambda}\mathcal{P}_0 T_{\mathrm{ree}}(n+1,\mathcal{C},V_{(\lambda,\lambda)})\Big|_{\lambda=0}$$
$$\begin{aligned}
&= \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\int_0^\beta \mathrm{d}s_1\prod_{j=2}^{n+1}\left(\int_0^\beta \mathrm{d}s_j\sum_{\mathbf{x}_j\in\mathbb{Z}^2}\right)\\
&\quad\cdot 1_{\mathbf{x}_j\in\{-\lfloor L/2\rfloor,-\lfloor L/2\rfloor+1,\ldots,-\lfloor L/2\rfloor+L-1\}^2\ (\forall j\in\{2,\ldots,n+1\})}\\
&\quad\cdot F_L(s_1,\mathbf{x}_2 s_2,\ldots,\mathbf{x}_{n+1}s_{n+1})\\
&= \int_0^\beta \mathrm{d}s_1\prod_{j=2}^{n+1}\left(\int_0^\beta \mathrm{d}s_j\sum_{\mathbf{x}_j\in\mathbb{Z}^2}\right)\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} F_L(s_1,\mathbf{x}_2 s_2,\ldots,\mathbf{x}_{n+1}s_{n+1}).
\end{aligned}$$

This implies the existence of (D.5) for $n\in\mathbb{N}$ and completes the proof. □

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