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A Class of Conformal Curves in the Reissner–Nordström Spacetime

Christian Lübbe and Juan Antonio Valiente Kroon

Abstract. A class of curves with special conformal properties (conformal curves) is studied on the Reissner–Nordström spacetime. It is shown that initial data for the conformal curves can be prescribed so that the resulting congruence of curves extends smoothly to future and past null infinity. The formation of conjugate points on these congruences is examined. The results of this analysis are expected to be of relevance for the discussion of the Reissner–Nordström spacetime as a solution to the conformal field equations and for the global numerical evaluation of static black hole spacetimes.

1. Introduction

Conformal methods constitute a powerful tool for the discussion of global properties of spacetimes—in particular, those representing black holes. The conformal structure of static electrovacuum black hole spacetimes is, to some extent, well understood—see e.g. [16,17]. However, the constructions involved often require several changes of variables and the introduction of some type of null coordinates. This choice of coordinates may not be the most convenient to undertake an analysis of global or asymptotic properties of a spacetime by means of the *conformal Einstein field equations*—see e.g. the discussion in [11]. A key issue in this respect, is how to construct in a systematic/canonical fashion a conformal extension of the spacetime which, in addition, eases the analysis of the underlying conformal field equations—for a review of the conformal equations and the issues involved in their analysis see e.g. [12]. In the case of vacuum spacetimes, gauges based on the use of *conformal geodesics* offer such a systematic approach—see e.g. [13,15]. Conformal geodesics are invariants of the conformal structure: a conformal transformation maps conformal geodesics into conformal geodesics—this is not the case with standard geodesics unless they are null.

One of the main advantages of the use of conformal geodesics in the construction of gauge (and coordinate) systems in a vacuum spacetime is that

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they provide an *a priori* conformal factor which can be read off directly from the data one has specified to generate the congruence of conformal geodesics. Hence, one has a canonical procedure to generate a conformal extension of the spacetime in question. In addition, gauge systems based on conformal geodesics give rise to a fairly straightforward hyperbolic reduction of the conformal Einstein field equations in which most of the evolution equations are, in fact, transport equations—see e.g. [12,14].

The useful property of having an *a priori* conformal factor is lost when one considers conformal geodesics in non-vacuum spacetimes. Nevertheless, in [20] it has been shown that this property can be recovered if one considers a more general class of curves—the *conformal curves*. These curves satisfy equations similar to the conformal geodesic equations, but with a different coupling to the curvature of the spacetime. In the vacuum case they coincide with the conformal geodesic equations. Gauges based on this class of curves have been used in [20] to revisit the stability proofs for the Minkowski and the de Sitter spacetimes first given in [9] and to obtain a stability result for purely radiative electrovacuum spacetimes. They also have been used in [21] to analyse the geodesic completeness of non-linear perturbations of Friedman–Robertson–Walker spacetimes with radiation perfect fluids.

Given the results described in the previous paragraph, a natural question to be raised is whether conformal geodesics, and more generally, the class of conformal curves introduced in [20] can be used to analyse global aspects of black hole spacetimes. A first analysis of this question has been carried out in [13] where it has been shown that the maximal extension of the Schwarzschild spacetime, the so-called Schwarzschild–Kruskal spacetime [18], can be covered with a congruence of conformal geodesics which has no conjugate points. The conformal Gaussian gauge system obtained using this congruence offers a vantage perspective for the study of conformal properties of the Schwarzschild spacetime and for its global evaluation by means of numerical methods—see e.g. [27].

In the present article we analyse to what extent a similar construction can be performed for the Reissner–Nordström spacetime. The idea of considering the Reissner–Nordström spacetime is, for several reasons, natural. The inclusion of the electromagnetic field provides a model of angular momentum—see e.g. [5,6]. Moreover, we expect our analysis to provide insights into more general (i.e. less symmetric) situations—e.g. the Kerr and Kerr–Newman spacetimes. In addition, there is an expectation that black hole spacetimes with timelike singularities could be more tractable from the point of view of the conformal geometry than black holes with spacelike singularities.[1]

The main results of our analysis is the following:

[1] This expectation is based on the analysis of the structure of spatial infinity of the Schwarzschild spacetime. In this case, the well understood divergence of the Weyl tensor at spatial infinity can also be regarded as the timelike singularity of a negative mass Schwarzschild spacetime—see e.g. [23] for a conformal diagram of this.

Theorem. *The domain of outer communication of a non-extremal Reissner–Nordström spacetime with* $q^2 \leq \frac{8}{9}m^2$ *can be covered with a timelike congruence of conformal curves which contains no conjugate points. In the extremal case* $m^2 = q^2$*, the non-existence of conjugate points for an analogous congruence can be ensured, in the worst of cases, for the region in the domain of outer communication outside a certain timelike tube intersecting the horizon. In both cases, the congruence of conformal curves extends smoothly to null infinity.*

A technical version of the main result is given in Theorem 1 of Sect. 7. The congruence obtained in this theorem is naturally parametrised in terms of the coordinates of the intersection of the curves with the time symmetric hypersurface of the domain of outer communication. In terms of this parametrisation one observes three different types of behaviour in the curves. Curves starting at points with a radial coordinate bigger than a certain critical value $r_{\circledast}$ reach (in the conformal picture) null infinity for a finite value of a (conformal) affine parameter. Curves starting at an initial radius smaller than $r_{\circledast}$ reach the horizon, again in a finite amount of the conformal affine parameter of the curve. Finally, curves starting exactly at a radius given by the critical value $r_{\circledast}$ reach timelike infinity in finite conformal time.

Numerical solutions of the conformal curve equations show that for $\frac{8}{9}m^2 < q^2 \leq m^2$ the congruence of conformal curves contains no conjugate points in the domain of outer communication. Thus, our main result can be certainly improved. Doing this, however, would increase considerably the length of our analysis. In view of future applications, the extremal case is certainly the one of the most interest. The same numerical simulations show that, generically, conjugate points in the congruence form after the curves have crossed the horizon and entered the black hole region of the spacetime. From the perspective of the Cauchy problem for the Einstein field equations, these conjugate points are not a major concern as one is mainly interested in the behaviour of the spacetime in the domain of outer communication and at the horizon. This is, in particular, the case in the problem of the so-called *non-linear stability of black hole spacetimes*—see e.g. [7].

An important insight obtained from our analysis concerns the behaviour of the conformal curves reaching timelike infinity. While in the non-extremal case the curves become null at timelike infinity, in the extremal Reissner–Nordström spacetime the curves remain timelike up to, and including, timelike infinity. This change of the causal character of the conformal curves in the non-extremal case indicates a degeneracy of the conformal structure of the spacetime—a similar behaviour in conformal curves has been identified in [12]. One concludes from this observation that the structure of timelike infinity in the extremal Reissner–Nordström spacetime is more regular than in the non-extremal case and amenable to further detailed analysis. This insight will be explored elsewhere.

Our main result provides a suitable conformal gauge to analyse the properties of the Reissner–Nordström spacetime by means of the conformal Einstein field equations. In particular, it opens the possibility of global numerical evaluations of the spacetime [26] similar to the ones carried out in [27] for the Schwarzschild spacetime.

Finally, we point out that some interesting recent work on other aspects of the Reissner–Nordström spacetime can be found in [1–3,8].

Outline of the Article

The present article is structured as follows: in Sect. 2, we present a discussion of the features of the Reissner–Nordström spacetime that will be used in our present analysis. Section 3 provides a discussion of the properties of the class of conformal curves that will be used to study the conformal properties of electrovacuum spacetimes. Section 4 particularises the expressions of Sect. 3 to the case of the Reissner–Nordström, and establishes general properties of the congruence under consideration. Section 5 contains the main results concerning the behaviour of the individual curves of the congruence. Section 6 analyses the behaviour of the deviation equation of the congruence of conformal curves and provides the proof of the fact that for the congruence under consideration the curves do not intersect in the domain of outer communication of the black hole spacetime. Finally, Sect. 7 provides some concluding remark to our analysis. The article contains an appendix in which the behaviour of conformal geodesics in the Schwarzschild spacetime is analysed in a way alternative to that of reference [13].

Notations and Conventions

In what follows μ, $\nu, \ldots$ will denote spacetime tensorial indices. The indices α, $\beta, \ldots$ are spatial tensorial indices. The signature convention for the spacetime metrics is $(+,-,-,-)$. Thus, the induced metrics on spacelike hypersurfaces are negative definite. The Latin indices i, $j, \ldots$ denote spacetime frame indices taking the values $0, \ldots, 3$, while a, $b, \ldots$ correspond to spatial frame ones ranging over 1, 2, 3.

An index-free notation will be often used. Given a 1-form $\boldsymbol{\omega}$ and a vector $\boldsymbol{v}$, we denote their contraction by $\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle$. Furthermore, $\boldsymbol{\omega}^\sharp$ and $\boldsymbol{v}^\flat$ denote, respectively, the contravariant version of $\boldsymbol{\omega}$ and the covariant version of $\boldsymbol{v}$. The metric with respect to which the operation of raising/lowering indices will be clear by the context.

In order to ease the presentation some of the notation used in [13] for the various types of coordinates has been modified.

2. The Reissner–Nordström Spacetimes

We begin by recalling that the Einstein–Maxwell field equations with vanishing Cosmological constant are given by

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\,\tilde{g}_{\mu\nu} = \tilde{F}_{\mu\lambda}\tilde{F}^{\lambda}{}_{\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\lambda\rho}\tilde{F}^{\lambda\rho}, \tag{1a}$$

$$\tilde{\nabla}^{\mu}\tilde{F}_{\mu\nu} = 0, \tag{1b}$$

$$\tilde{\nabla}_{[\mu}\tilde{F}_{\nu\lambda]} = 0, \tag{1c}$$

where $\tilde{R}_{\mu\nu}$ denotes the Ricci tensor of the Lorentzian metric $\tilde{g}_{\mu\nu}$, and $\tilde{F}_{\mu\nu}$ is the Faraday tensor. In view of *Birkhoff's theorem for electrovacuum spacetimes*— see e.g. [24] page 232— the Reissner–Norström spacetime is the only spherically symmetric solution to Eqs. (1a)–(1c).

2.1. Basic Expressions and Coordinates

In what follows, we briefly discuss various coordinate representations of the Reissner–Nordström spacetime that will be used in the sequel. Further details can be found in e.g. [16,17].

2.1.1. Standard Coordinates. In standard *spherical coordinates* (t, r, θ, φ), the line element and the Faraday tensor of the Reissner–Nordström spacetime is given by

$$\tilde{\boldsymbol{g}} = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\mathbf{d}t \otimes \mathbf{d}t - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}^2 \quad \text{(2a)}$$

$$\tilde{\boldsymbol{F}} = \frac{q}{2r^2}\mathbf{d}t \wedge \mathbf{d}r. \quad \text{(2b)}$$

where

$$\boldsymbol{\sigma} \equiv \left(\mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi\right)$$

is the standard metric of $\mathbb{S}^2$. All throughout it is assumed that

$$m > 0, \quad m^2 \geq q^2,$$

so that the solution describes a black hole. If $q = 0$, the line element (2a) reduces to the corresponding one of the Schwarzschild spacetime. As this case was analysed in detail in [13], we assume $q \neq 0$ unless explicitly stated. The *extremal case*, $q^2 = m^2$, is of particular interest. In that case the line element reduces to

$$\tilde{\boldsymbol{g}} = \left(1 - \frac{m}{r}\right)^2\mathbf{d}t \otimes \mathbf{d}t - \left(1 - \frac{m}{r}\right)^{-2}\mathbf{d}r \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}.$$

In view of our subsequent discussion we define[2]

$$D(r) \equiv \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) = \frac{1}{r^2}(r - r_+)(r - r_-), \quad (3)$$

where

$$r_\pm \equiv m \pm \sqrt{m^2 - q^2}.$$

As it is well known, the locus of points in the spacetime for which $r = r_+$ and $r = r_-$ correspond, respectively, to the *event horizon* and the *Cauchy horizon*. Notice that $D(r_\pm) = 0$. In the extremal case we have that $r_\pm = m$ so that

[2] The function $D(r)$ corresponds to the function $F(\bar{r})$ of reference [13]. A different notation has been introduced to avoid confusion with the Faraday tensor $\tilde{\boldsymbol{F}}$.

the event and Cauchy horizons coincide. Notice that in the non-extremal case $0 < r_- < r_+$, and that

$$D(r) > 0 \quad \text{if } r_- < r_+ < r,$$
$$D(r) < 0 \quad \text{if } r_- < r < r_+,$$
$$D(r) > 0 \quad \text{if } 0 < r < r_- < r_+.$$

However, in the extremal case

$$D(r) > 0 \quad \text{for} \quad 0 < r < m \quad \text{and} \quad m < r.$$

2.1.2. Isotropic Coordinates. An isotropic coordinate ϱ can be introduced in the line element (2a) via the requirement

$$\frac{\mathrm{d}\varrho}{\mathrm{d}r} = \frac{\varrho}{r\sqrt{D}}. \tag{4}$$

The latter condition implies, for $r > r_+, r_-$, the coordinate transformation

$$\varrho = \frac{1}{2}(r - m + \sqrt{r^2 - 2mr + q^2}), \quad r = \frac{1}{4\varrho}(2\varrho + m + q)(2\varrho + m - q), \tag{5}$$

so as to obtain the line element

$$\tilde{\boldsymbol{g}} = \frac{\left(1 + \frac{q^2 - m^2}{4\varrho^2}\right)^2}{\left(1 + \frac{m+q}{2\varrho}\right)^2 \left(1 + \frac{m-q}{2\varrho}\right)^2} \mathbf{d}t \otimes \mathbf{d}t$$
$$- \left(1 + \frac{m+q}{2\varrho}\right)^2 \left(1 + \frac{m-q}{2\varrho}\right)^2 (\mathbf{d}\varrho \otimes \mathbf{d}\varrho + \varrho^2 \boldsymbol{\sigma}).$$

In the extremal case, the isotropic coordinate transformation reduces to

$$\varrho = r - m, \qquad r = \varrho + m,$$

and the corresponding line element is given by

$$\tilde{\boldsymbol{g}} = \left(1 + \frac{m}{\varrho}\right)^{-2} \mathbf{d}t \otimes \mathbf{d}t - \left(1 + \frac{m}{\varrho}\right)^2 (\mathrm{d}\varrho \otimes \mathrm{d}\varrho + \varrho^2 \boldsymbol{\sigma}).$$

For future reference it is noticed that

$$D(\varrho) \equiv D(r(\varrho)) = \frac{\left(1 + \frac{q^2 - m^2}{4\varrho^2}\right)^2}{\left(1 + \frac{m+q}{2\varrho}\right)^2 \left(1 + \frac{m-q}{2\varrho}\right)^2},$$

and that

$$\varrho_\pm \equiv \varrho(r_\pm) = \pm\frac{1}{2}\sqrt{m^2 - q^2}.$$

In particular, in the extremal case one has

$$D(\varrho) = \left(1 + \frac{m}{\varrho}\right)^{-2}.$$

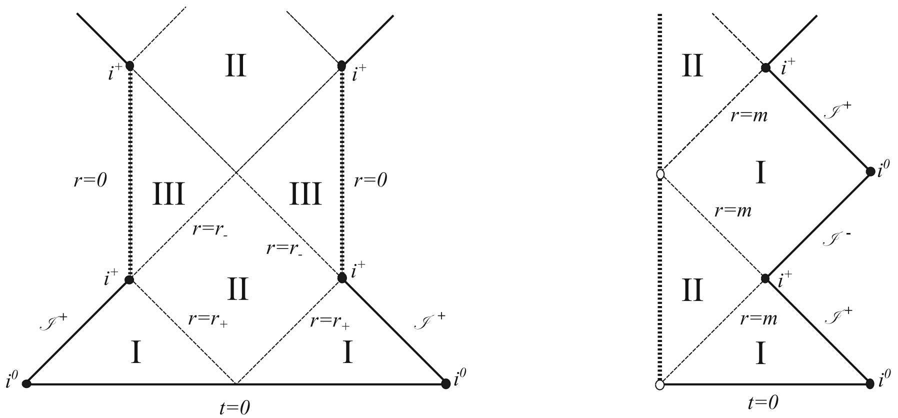

Figure 1. Conformal diagrams of the Reissner–Nordström spacetime: the non-extremal case (*left*) and the extremal case (*right*)

2.1.3. Null Coordinates. Eddington–Finkelstein-like null coordinates can be introduced in the non-extremal case via

$$u = t - \left(r + \frac{r_+^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_-^2}{r_+ - r_-} \ln |r - r_-| \right), \tag{6a}$$

$$v = t + \left(r + \frac{r_+^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_-^2}{r_+ - r_-} \ln |r - r_-| \right), \tag{6b}$$

so that one obtains the line elements

$$\tilde{\boldsymbol{g}} = D(r)\mathbf{d}u \otimes \mathbf{d}u + 2\, \mathbf{d}u \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}, \quad \tilde{\boldsymbol{g}} = D(r)\mathbf{d}v \otimes \mathbf{d}v - 2\, \mathbf{d}v \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}. \tag{7}$$

In the extremal case, the corresponding change of coordinates leading to line elements of the form given in (7) is given by

$$u=t-\left(r-\frac{m^2}{r-m}+2m\ln|r-m|\right), \quad v=t+\left(r-\frac{m^2}{r-m}+2m\ln|r-m|\right). \tag{8}$$

2.2. Conformal Diagrams of the Reissner–Nordström Spacetime

The conformal Reissner–Nordström spacetime in both the non-extremal and the extremal case are well known—see for example [4,16,17] for details on their construction. These diagrams are included in Fig. 1 for quick reference.

2.3. Time Symmetric Initial Data

In the sequel, it will be shown how to cover portions of the Reissner–Nordström spacetime by means of a congruence of curves whose initial data is prescribed on the time symmetric slice $\tilde{\mathcal{S}} = \{t = \text{constant}\}$ of the spacetime. The hypersurface $\tilde{\mathcal{S}}$ has the topology of $\mathbb{R}^3\setminus\{0\}$. The initial three-metric expressed in isotropic coordinates takes the form

$$\tilde{\boldsymbol{h}} = -\phi^2\chi^2(\mathbf{d}\varrho \otimes \mathbf{d}\varrho + \varrho^2\boldsymbol{\sigma}), \tag{9}$$

with

$$\phi \equiv \left(1 + \frac{m+q}{2\varrho}\right), \quad \chi \equiv \left(1 + \frac{m-q}{2\varrho}\right), \quad \varrho \in [0, \infty).$$

If $m^2 > q^2$ then the Riemannian three-dimensional manifold given by $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$ has two asymptotically Euclidean ends joined by a throat at $\varrho = \varrho_+$. In the extremal case one has either $\phi = 1$ or $\chi = 1$. Accordingly, $(\tilde{S}, \tilde{\boldsymbol{h}})$ has one asymptotically Euclidean end and one so-called *trumpet-like* end.

3. A Class of Conformal Curves

In what follows, we present a brief discussion of the properties of a class of *conformal curves* introduced in [20]. These curves are a generalisation of the conformal geodesics allowing to recover, for non-vacuum spacetimes, some of the properties satisfied by conformal geodesics in a vacuum spacetime. Our discussion of the properties of conformal curves is inspired by and follows closely the discussion of conformal geodesics in vacuum spacetimes given in [10,13].

Although the notion of conformal curves is applicable to spacetimes with an arbitrary matter models, in what follows, for concreteness it is assumed that $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denotes a spacetime satisfying the Einstein–Maxwell field equations (1a)–(1c). Let $(\mathcal{M}, \boldsymbol{g})$ denote a conformal extension of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Hence, there exists a scalar Θ such that the metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ are related via

$$\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}. \tag{10}$$

3.1. Basic Definitions

Given an interval $I \subseteq \mathbb{R}$, let $\boldsymbol{x}(\tau), \tau \in I$ denote a curve in $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and let $\boldsymbol{b}(\tau)$ denote a 1-form along $\boldsymbol{x}(\tau)$. Furthermore, let $\dot{\boldsymbol{x}} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ denote the tangent vector field of the curve $\boldsymbol{x}(\tau)$. The pairing between the vector $\dot{\boldsymbol{x}}$ and the 1-form $\boldsymbol{b}$ is denoted by $\langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle$. In [20] the following equations for the pair $(\boldsymbol{x}(\tau), \boldsymbol{b}(\tau))$ have been introduced:

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \dot{\boldsymbol{x}} = -2\langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle \dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \boldsymbol{b}^\sharp, \tag{11a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \boldsymbol{b} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle \boldsymbol{b} - \frac{1}{2} \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{b}, \boldsymbol{b}) \dot{\boldsymbol{x}}^\flat + \tilde{\boldsymbol{H}}(\dot{\boldsymbol{x}}, \cdot), \tag{11b}$$

where $\tilde{\nabla}_{\dot{x}}$ denotes the directional derivative of the Levi–Civita connection of the metric $\tilde{\boldsymbol{g}}$, while $\tilde{\boldsymbol{H}}$ denotes a rank 2 covariant tensor which upon the conformal transformation (10) transforms as:

$$\tilde{H}_{\mu\nu} - H_{\mu\nu} = \nabla_\mu \Upsilon_\nu + \Upsilon_\mu \Upsilon_\nu - \frac{1}{2} g^{\lambda\rho} \Upsilon_\lambda \Upsilon_\rho \, g_{\mu\nu}, \qquad \Upsilon_\mu \equiv \Theta^{-1} \nabla_\mu \Theta.$$

This transformation law is formally identical to that of the *Schouten tensor*

$$\tilde{L}_{\mu\nu} \equiv \frac{1}{2} \left(\tilde{R}_{\mu\nu} - \frac{1}{6} \tilde{R} \, \tilde{g}_{\mu\nu} \right).$$

The Eqs. (11a)–(11b) will be referred to as the *conformal curve equations*. In a slight abuse of notation, the pair $(\boldsymbol{x}(\tau), \boldsymbol{b}(\tau))$ will be called a *conformal curve*. Following [20] we set

$$\tilde{\boldsymbol{H}} = 0, \tag{12}$$

has been adopted so that Eq. (11b) reduces to

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{b} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle \boldsymbol{b} - \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{b}, \boldsymbol{b})\dot{\boldsymbol{x}}^\flat. \tag{13}$$

In what follows, the choice (12) *will be assumed.*

The choice given by Eq. (12) leads to an explicit expression for the conformal factor Θ in terms of the parameter τ. Indeed, by requiring $\boldsymbol{x}(\tau)$ to be timelike and imposing the normalisation condition

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1, \tag{14}$$

the conformal curve equations imply

$$\dot{\Theta} = \langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle \Theta, \qquad \ddot{\Theta} = \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{b}, \boldsymbol{b})\Theta^{-1}, \qquad \dddot{\Theta} = 0,$$

where $\dot{\Theta} \equiv \tilde{\nabla}_{\dot{x}}\Theta$, etc. Integrating the last of these equations one finds

$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \frac{1}{2}\ddot{\Theta}_*(\tau - \tau_*)^2, \tag{15}$$

where $\Theta_*, \dot{\Theta}_*$ and $\ddot{\Theta}_*$ are prescribed at a fiduciary value τ_* of the parameter τ. The coefficients $\dot{\Theta}_*$ and $\ddot{\Theta}_*$ satisfy the constraints

$$\dot{\Theta}_* = \langle \boldsymbol{b}_*, \dot{\boldsymbol{x}}_* \rangle \Theta_*, \qquad 2\Theta_*\ddot{\Theta}_* = \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{b}_*, \boldsymbol{b}_*), \tag{16}$$

where $\boldsymbol{b}_*$ and $\dot{\boldsymbol{x}}_*$ denote, respectively, the value of $\boldsymbol{b}$ and $\dot{\boldsymbol{x}}$ at $\tau = \tau_*$.

Finally, let $\boldsymbol{e}_i, i = 0, \ldots, 3$, denote a frame basis along $\boldsymbol{x}(\tau)$. The frame will be said to be *Weyl propagated* along the conformal curve $(\boldsymbol{x}(\tau), \boldsymbol{b}(\tau))$ if it satisfies the equation

$$\tilde{\nabla}_{\dot{x}}\boldsymbol{e}_i = -\langle \boldsymbol{b}, \boldsymbol{e}_i \rangle \dot{\boldsymbol{x}} - \langle \boldsymbol{b}, \boldsymbol{e}_i \rangle \boldsymbol{e}_i + \tilde{g}(\boldsymbol{e}_i, \dot{\boldsymbol{x}})\boldsymbol{b}^\sharp. \tag{17}$$

It can be readily seen that a Weyl propagated frame which is $\boldsymbol{g}$-orthonormal at, say, $\tau = \tau_*$ remains $\boldsymbol{g}$-orthonormal all through $\boldsymbol{x}(\tau)$—that is, $\boldsymbol{g}(\boldsymbol{e}_i, \boldsymbol{e}_j) = \eta_{ij}$. Following the discussion of [13,20], if, consistently with Eq. (14), one sets $\boldsymbol{e}_0 = \dot{\boldsymbol{x}}$ then it can be shown that

$$b_0 = \Theta^{-1}\dot{\Theta}, \qquad b_a = \langle \Theta^{-1}\boldsymbol{b}, \boldsymbol{e}_a \rangle_*, \qquad a = 1, 2, 3.$$

As a consequence, the components of the 1-form $\boldsymbol{b}$ with respect to the frame $\boldsymbol{e}_i$ can be expressed in terms of the value of various fields at $\tau = \tau_*$. Hence, the 1-form $\boldsymbol{b}$, like Θ, is known *a priori*. For full details of the computations involved see e.g. [13,20].

Remark 1. Notice that by virtue of the normalisation condition (14), τ is the $\boldsymbol{g}$-proper time of the conformal curve. We shall often refer to τ as the *unphysical proper time*.

Remark 2. With the choice $\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{L}}$, the Eqs. (11a) and (11b) yield the so-called *conformal geodesic equations*—see e.g. [10,13,15]. Notice that for a vacuum spacetime $\tilde{\boldsymbol{L}} = 0$ and the conformal geodesic equations are formally identical to Eqs. (11a)–(13). In this case, it was shown in Lemma 3.1 of [10] that a quadratic expression for the conformal factor identical to (15) can be obtained. Finally, it is observed that in the case of an electrovacuum spacetime one has that $\tilde{\boldsymbol{L}} \neq 0$ and the argument leading to Lemma 3.1 in [10] no longer holds. The desire of retaining the expression (15) is what led in [20] to the notion of conformal curves.

3.2. The $\tilde{\boldsymbol{g}}$-Adapted Equations

As already mentioned, as a consequence of the normalisation condition (14), the parameter τ is the unphysical proper time of the curve $\boldsymbol{x}(\tau)$. In some computations it is more convenient to consider a parametrisation in terms of *physical* proper time $\bar{\tau}$. The parameter transformation is given by

$$\bar{\tau} = \bar{\tau}_* + \int_{\tau_*}^{\tau} \frac{\mathrm{d}s}{\Theta(s)}, \tag{18}$$

with inverse $\tau = \tau(\bar{\tau})$. In what follows, we will write $\bar{\boldsymbol{x}} \equiv \boldsymbol{x}(\tau(\bar{\tau}))$. It can then be verified that

$$\bar{\boldsymbol{x}}' \equiv \partial_{\bar{\tau}} \bar{\boldsymbol{x}} = \Theta \dot{\boldsymbol{x}},$$

and that $\tilde{\boldsymbol{g}}(\bar{\boldsymbol{x}}', \bar{\boldsymbol{x}}') = 1$. Hence, $\bar{\tau}$ is, indeed, the $\tilde{\boldsymbol{g}}$-proper time of the curve $\bar{\boldsymbol{x}}$.

Now, in order to write the equation for the curve $\bar{\boldsymbol{x}}(\bar{\tau})$ in a convenient way, we consider the split

$$\boldsymbol{b} = \hat{\boldsymbol{b}} + \varpi \dot{\boldsymbol{x}}^\flat,$$

where the 1-form $\hat{\boldsymbol{b}}$ satisfies

$$\langle \hat{\boldsymbol{b}}, \dot{\boldsymbol{x}} \rangle = 0, \qquad \varpi = \frac{\langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle}{\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})}, \qquad \boldsymbol{g}^\sharp(\boldsymbol{b}, \boldsymbol{b}) = \langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle^2 + \boldsymbol{g}^\sharp(\hat{\boldsymbol{b}}, \hat{\boldsymbol{b}}),$$

and the indices of the vectors and forms have been moved using the metric $\tilde{\boldsymbol{g}}$. In terms of these objects the $\tilde{\boldsymbol{g}}$-*adapted equations for the conformal curves* are given by

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'} \bar{\boldsymbol{x}}' = \hat{\boldsymbol{b}}^\sharp, \tag{19a}$$

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'} \hat{\boldsymbol{b}} = \beta^2 \bar{\boldsymbol{x}}'^\flat, \tag{19b}$$

where

$$\beta^2 \equiv -\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}}, \hat{\boldsymbol{b}}) = \delta^{ab} d_a d_b = \text{constant} \tag{20}$$

is, by virtue of the discussion of Sect. 3.1, a constant along the conformal curve. Finally, it is worth noticing that as a consequence of Eq. (19a), $\hat{\boldsymbol{b}}^\sharp$ can be interpreted as the *physical acceleration* of the conformal curve.

3.3. The Deviation Equations

A crucial part of the analysis of the present article will be concerned with the question of whether a congruence of conformal curves develops conjugate points or not. To this end, let $(\boldsymbol{x}(\bar{\tau},\lambda),\boldsymbol{b}(\bar{\tau},\lambda))$ denote a family of conformal curves depending smoothly on a parameter λ. Following [13], let

$$\bar{\boldsymbol{X}} \equiv \bar{\boldsymbol{x}}', \qquad \bar{\boldsymbol{Z}} \equiv \partial_\lambda \bar{\boldsymbol{x}}, \qquad \hat{\boldsymbol{B}} \equiv \tilde{\nabla}_{\bar{\boldsymbol{Z}}} \hat{\boldsymbol{b}}.$$

One then has that

$$\tilde{\nabla}_{\bar{\boldsymbol{X}}}\tilde{\nabla}_{\bar{\boldsymbol{X}}}\bar{\boldsymbol{Z}} = \tilde{\boldsymbol{R}}(\bar{\boldsymbol{X}},\bar{\boldsymbol{Z}})\bar{\boldsymbol{X}} + \hat{\boldsymbol{B}}^\sharp, \tag{21a}$$

$$\tilde{\nabla}_{\bar{\boldsymbol{X}}}\hat{\boldsymbol{B}} = -\hat{\boldsymbol{b}}\cdot\tilde{\boldsymbol{R}}(\bar{\boldsymbol{X}},\bar{\boldsymbol{Z}}) + \tilde{\nabla}_{\bar{\boldsymbol{Z}}}\big(\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}},\hat{\boldsymbol{b}})\big)\bar{\boldsymbol{X}}^\flat + \tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}},\hat{\boldsymbol{b}})\tilde{\nabla}_{\bar{\boldsymbol{X}}}\bar{\boldsymbol{Z}}^\flat. \tag{21b}$$

A computation then shows that

$$\tilde{\nabla}_{\bar{\boldsymbol{X}}}\tilde{\nabla}_{\bar{\boldsymbol{Z}}}\big(\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}},\hat{\boldsymbol{b}})\big) = \tilde{\nabla}_{\bar{\boldsymbol{Z}}}\tilde{\nabla}_{\bar{\boldsymbol{X}}}\big(\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}},\hat{\boldsymbol{b}})\big) = 0, \tag{22}$$

as a consequence of Eq. (20). Hence, one concludes that the coefficients $\tilde{\nabla}_{\bar{\boldsymbol{Z}}}\big(\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}},\hat{\boldsymbol{b}})\big)$ and $\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}},\hat{\boldsymbol{b}})$ are constant along a conformal curve. For simplicity one can evaluate them at $\tau=\tau_*$.

3.4. Formulae in Warped Product Spaces

The Reissner–Nordström spacetime in the standard coordinates (t,r,θ,φ) of the line element of Eq. (2a) is in the form of a warped product. This structure can be exploited to simplify the analysis of the $\tilde{\boldsymbol{g}}$-adapted conformal curve equations (19a)–(19b). In this section, we adapt the discussion of [13] to the context of the conformal curves.

In what follows, we will consider spacetimes whose metric can be written as a warped product of the form

$$\tilde{\boldsymbol{g}} = l_{AB}\mathbf{d}x^A\otimes\mathbf{d}x^B + f^2 k_{cd}\mathbf{d}x^c\otimes\mathbf{d}x^d, \tag{23}$$

with

$$l_{AB}=l_{AB}(x^C), \quad k_{ab}=k_{ab}(x^c), \quad f=f(x^A)>0,$$

and A, B, $C=0$, 1 and a, b, $c=2$, 3. In addition, it is assumed that the two-dimensional metric given by the line element $\boldsymbol{l}\equiv l_{AB}\mathbf{d}x^A\otimes\mathbf{d}x^B$ is Lorentzian, while the one given by $\boldsymbol{k}=k_{cd}\mathbf{d}x^c\otimes\mathbf{d}x^d$ is a negative-definite Riemannian one. *In view of this structure it is natural to consider solutions to the conformal curve equations satisfying* $\dot{x}^a=0$ *and* $b_c=0$. In the context of the spherically symmetric Reissner–Nordström spacetime this Ansatz leads to solutions to the conformal curves whose angular coordinates are constant. One only has to consider evolution equations for the coordinates (r,t). A direct computation shows that for this type of conformal curves, the $\tilde{\boldsymbol{g}}$-adapted equations for the conformal curves imply

$$\not{D}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = \hat{\boldsymbol{b}}^\sharp, \tag{24a}$$

$$\hat{\boldsymbol{b}} = \pm\beta\boldsymbol{\epsilon}_{\boldsymbol{l}}(\bar{\boldsymbol{x}}',\,\cdot\,), \tag{24b}$$

with

$$\boldsymbol{\epsilon}_{\boldsymbol{l}} \equiv \sqrt{|\Delta|}\mathbf{d}x^0\wedge\mathbf{d}x^1, \quad \Delta\equiv\det l_{AB},$$

and where $\not{D}$ denotes the Levi–Civita covariant derivative of $\boldsymbol{l}$ and β^2 given by Eq. (20). The sign in Eq. (24b) is determined consistently with the initial conditions. A further computation shows that under the present Ansatz, the $\tilde{\boldsymbol{g}}$-adapted deviation equations (21a)–(21b) are equivalent to each other and, in turn, to the equation

$$\not{D}_{\bar{\boldsymbol{X}}} \not{D}_{\bar{\boldsymbol{X}}} \bar{\boldsymbol{Z}} = \frac{1}{2} R[\boldsymbol{l}]\, \boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Z}}) \boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \cdot)^\sharp \pm \left(\not{D}_{\bar{\boldsymbol{Z}}} \beta\, \boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \cdot)^\sharp + \beta \boldsymbol{\epsilon_l}(\not{D}_{\bar{\boldsymbol{X}}} \bar{\boldsymbol{Z}}, \cdot)^\sharp \right), \tag{25}$$

where $R[\boldsymbol{l}]$ denotes the Ricci scalar of $\boldsymbol{l}$. The sign in the last equation is chosen consistently with that of Eq. (24b).

For conformal curves satisfying $\dot{x}^a = 0$ and $b_c = 0$, the issue of whether the deviation vector field $\bar{\boldsymbol{Z}}$ degenerates can be rephrased in terms of the question of the vanishing of the scalar

$$\omega \equiv \boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Z}}). \tag{26}$$

Notice that as long as $\omega \neq 0, \bar{\boldsymbol{X}}$ and $\bar{\boldsymbol{Z}}$ are linearly independent. A computation using (25) yields

$$\not{D}_{\bar{\boldsymbol{X}}} \not{D}_{\bar{\boldsymbol{X}}} \omega = \left(\beta^2 + \frac{1}{2} R[\boldsymbol{l}] \right) \omega + \not{D}_{\bar{\boldsymbol{Z}}} \beta. \tag{27}$$

4. Basic Expressions for the Congruence of Conformal Curves in the Reissner–Nordström Spacetime

In the present section, we particularise the discussion made in Sect. 3 to a specific class of conformal curves in the Reissner–Nordström spacetime.

4.1. Initial Data for the Congruence

It follows from the discussion in Sect. 3.1 that the basic pieces of information to be prescribed in order to construct a congruence of conformal curves are the initial value of the conformal factor, Θ_*, and the initial value of the 1-form $\boldsymbol{b}_*$ at some initial hypersurface. Following the discussion of Sect. 2.3, we consider the time symmetric slice of the Reissner–Nordström spacetime. By analogy to the discussion in [13], and taking into account the three-metric of Eq. (9), we choose

$$\Theta_* = \frac{1}{r_*^2} = \frac{1}{\varrho_*^2 \phi^2 \chi^2} = \frac{\varrho_*^2}{\left(\varrho_* + \frac{m+q}{2}\right)^2 \left(\varrho_* + \frac{m-q}{2}\right)^2},$$
$$\boldsymbol{b}_* = \hat{\boldsymbol{b}}_* = \Theta_*^{-1} \mathrm{d}\Theta_* = -\frac{2}{r_*} \mathbf{d} r_* = -\frac{2\left(\varrho_*^2 - \frac{1}{4}(m^2 - q^2)\right)}{\varrho_* \left(\varrho_* + \frac{m+q}{2}\right)\left(\varrho_* + \frac{m-q}{2}\right)} \mathrm{d}\varrho_*,$$

where it is assumed that $r_* > r_+$. The symbols r_* and ϱ_* are used to denote, respectively, the radial coordinates r and ρ on the time symmetric slice. For simplicity we also set $\tau_* = 0$. In addition, one has that

$$\beta^2 \equiv -\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{b}}_*, \hat{\boldsymbol{b}}_*) = \frac{4}{r_*^2} D_*$$

where $D_* \equiv D(r_*)$ and $D(r)$ is given by Eq. (3). If, moreover, one assumes the condition

$$\langle \boldsymbol{b}, \dot{\boldsymbol{x}} \rangle_* = 0,$$

then the expression (15) for the conformal factor, together with the constraints (16) imply

$$\Theta = D_* \left(\left(\frac{2\Theta_*}{\beta} \right)^2 - \tau^2 \right), \tag{28}$$

where the coefficients D_*, Θ_* and β in this expression are taken to be constant along a given conformal curve. Moreover, the conformal curves of the congruence generated by the above conditions can be parametrised by the value of their radial coordinate on the initial hypersurface, r_* (or ϱ_*). Using expression (28) in formula (18) one finds that the parameters τ and $\bar{\tau}$ are related to each other by

$$\bar{\tau} = \frac{1}{\beta} \ln \left(\frac{2\Theta_* + \beta\tau}{2\Theta_* - \beta\tau} \right). \tag{29}$$

The inverse relation giving τ in terms of $\bar{\tau}$ is given by

$$\tau = \frac{2\Theta_*}{\beta} \tanh \left(\frac{1}{2} \beta \bar{\tau} \right). \tag{30}$$

In terms of the physical proper time, $\bar{\tau}$, the expression (28) takes the form

$$\Theta = \frac{\Theta_*}{\cosh^2 \left(\frac{1}{2} \beta \bar{\tau} \right)}. \tag{31}$$

4.2. The Conformal Curve Equations for the Standard Coordinates

In order to write the conformal curve equations (24a)–(24b) for the Reissner–Nordström metric, it is noticed that the metric $\boldsymbol{l}$ in the warped product line element (23) is given by

$$\boldsymbol{l} = D(r)\mathbf{d}t \otimes \mathbf{d}t - D^{-1}(r)\mathbf{d}r \otimes \mathbf{d}r.$$

Equations (24a)–(24b) imply

$$\bar{t}'' + \frac{\partial_{\bar{r}} D(\bar{r})}{D(\bar{r})} \bar{r}' \bar{t}' = \frac{1}{D(\bar{r})} \beta \, \bar{r}', \tag{32a}$$

$$\bar{r}'' - \frac{\partial_{\bar{r}} D(\bar{r})}{2D(\bar{r})} \bar{r}'^2 + \frac{D(\bar{r}) \partial_{\bar{r}} D(\bar{r})}{2} \bar{t}'^2 = D(\bar{r}) \beta \, \bar{t}', \tag{32b}$$

where consistent with the notation of Sect. 3.2 we have set $\bar{r} \equiv r(\bar{\tau}), \bar{t} \equiv t(\bar{\tau})$. Initial data for these equations is prescribed by observing the discussion of Sect. 4.1, and by requiring $\dot{\boldsymbol{x}}$ to be given initially by the unit normal to $\tilde{\mathcal{S}}$. It follows that

$$t_* = 0, \quad r_* > r_+, \quad \bar{t}'_* = \frac{1}{\sqrt{D_*}}, \quad \bar{r}'_* = 0, \quad (\hat{b}_t)_* = 0, \quad (\hat{b}_r)_* = -\frac{2}{r_*}, \tag{33}$$

where $\hat{\boldsymbol{b}}_t \equiv \langle \hat{\boldsymbol{b}}, \boldsymbol{\partial}_t \rangle, \hat{\boldsymbol{b}}_r \equiv \langle \hat{\boldsymbol{b}}, \boldsymbol{\partial}_r \rangle$. Notice that $r_* = \bar{r}_*, t_* = \bar{t}_*$. As a consequence of the symmetry of the hypersurface $\tilde{\mathcal{S}}$ with respect to the bifurcation sphere at $r_* = r_+$, it is only necessary to consider the case $r_* > r_+$. The Eqs. (32a)–(32b) can be decoupled by making use of the $\tilde{\boldsymbol{g}}$-normalisation condition

$$D(\bar{r})\,\bar{t}'^2 - \frac{1}{D(\bar{r})}\bar{r}'^2 = 1. \tag{34}$$

Solving the latter for $t' \geq 0$ and substituting into (32b), one obtains that

$$\bar{r}'' + \frac{1}{2}\partial_{\bar{r}} D(\bar{r}) - \beta\sqrt{D(\bar{r}) + \bar{r}'^2} = 0. \tag{35}$$

This equation can be integrated once to yield

$$\sqrt{D(\bar{r}) + \bar{r}'^2} - \beta\bar{r} = \gamma,$$

where γ is a constant given in terms of the initial data by

$$\gamma = -\sqrt{D_*}.$$

It follows that

$$\bar{r}' = \pm\sqrt{(\gamma + \beta\bar{r})^2 - D(\bar{r})}, \tag{36}$$

with the sign depending on the value of r_*.

4.3. Expressions for the Conformal Curve Equations in Null Coordinates

In order to discuss the behaviour of the conformal curves through null infinity and the horizon one needs to consider the conformal curve equations (24a)–(24b) written in terms of the Eddington–Finkelstein null coordinates of Sect. 2.1.3. A computation renders the pairs of equations

$$\bar{u}'' - \frac{1}{2}\partial_{\bar{r}} D(\bar{r})\bar{u}'^2 = -\beta\bar{u}',$$

$$\bar{r}'' + \frac{1}{2}D(\bar{r})\partial_{\bar{r}} D(\bar{r})\,\bar{u}'^2 + \partial_{\bar{r}} D(\bar{r})\,\bar{r}'\bar{u}' = \beta(\bar{r}' + D(\bar{r})\bar{u}'),$$

and

$$\bar{v}'' + \frac{1}{2}\partial_{\bar{r}} D(\bar{r})\bar{v}'^2 = \beta\bar{v}',$$

$$\bar{r}'' + \frac{1}{2}D(\bar{r})\partial_{\bar{r}} D(\bar{r})\,\bar{v}'^2 - \partial_{\bar{r}} D(\bar{r})\,\bar{r}'\bar{v}' = -\beta(\bar{r}' - D(\bar{r})\bar{v}'),$$

where $\bar{u} = u(\bar{\tau})$ and $\bar{v} = v(\bar{\tau})$—cf. [13]. As in the case of the standard Reissner–Nordström coordinates, the above expressions can be decoupled using the $\tilde{\boldsymbol{g}}$-normalisation conditions

$$D(\bar{r})\,\bar{u}'^2 + 2\bar{u}'\bar{r}' = 1, \qquad D(\bar{r})\,\bar{v}'^2 - 2\bar{v}'\bar{r}' = 1,$$

which, in turn, yield

$$\bar{u}' = \frac{1}{\sqrt{D(\bar{r}) + \bar{r}'^2} + \bar{r}'}, \tag{37a}$$

$$\bar{v}' = \frac{1}{D(\bar{r})}\left(\sqrt{D(\bar{r}) + \bar{r}'^2} + \bar{r}'\right). \tag{37b}$$

In the calculations leading to the above equations, it has been required that $u' > 0$ and $v' > 0$ at the initial hypersurface so as to have future oriented curves. The initial conditions for these equations are given, for $m^2 > q^2$ and $r_* > r_+$, by

$$u_* = -r_* - \frac{1}{r_+ - r_-} \ln \frac{(r_* - r_+)^{r_+^2}}{(r_* - r_-)^{r_-^2}}, \quad v_* = r_* + \frac{1}{r_+ - r_-} \ln \frac{(r_* - r_+)^{r_+^2}}{(r_* - r_-)^{r_-^2}}, \quad (38)$$

and in the case $m^2 = q^2$ with $r > m$ by

$$u_* = -r_* + \frac{m^2}{r_* - m} - 2m \ln(r_* - m), \quad v_* = r_* - \frac{m^2}{r_* - m} + 2m \ln(r_* - m) \quad (39)$$

—see Eqs. (6a)–(6b) and (8).

Remark. Once the function $r(\bar{\tau})$ has been determined, the expressions (37a)–(37b) allow to obtain $u(\bar{\tau})$ and $v(\bar{\tau})$ by integration.

4.4. The Polynomials $P(\bar{r})$ and $Q(\bar{r})$

Equation (36) can be written in the form

$$\bar{r}'^2 = \frac{1}{\bar{r}^2} P(\bar{r}), \quad (40)$$

where $P(\bar{r})$ is the quartic polynomial

$$P(\bar{r}) \equiv \bar{r}^2 \left((\gamma + \beta \bar{r})^2 - D(\bar{r}) \right).$$

Since $\bar{r}'_* = 0$, it can be readily verified that $\bar{r} = r_*$ is a root of $P(\bar{r})$. Hence, we write

$$P(\bar{r}) = \beta^2 (\bar{r} - r_*) Q(\bar{r}),$$

with

$$Q(\bar{r}) \equiv \bar{r}^3 + \eta \bar{r} + \xi,$$

where

$$\eta \equiv \frac{1}{\beta^2}(D_* - 1), \qquad \xi \equiv \frac{q^2}{r_* \beta^2}.$$

The discriminant of this equation is given by

$$\Delta = \frac{1}{4}\xi^2 + \frac{1}{27}\eta^3.$$

It can be verified that $D_* - 1 < 0$ if $r_* \geq r_+$ so that $\eta < 0$. Furthermore, some lengthy algebra shows that $\Delta > 0$ for $r_* \geq r_+$. Thus, $Q(\bar{r})$ has three different real roots. Now, one readily sees that $\mathrm{d}Q/\mathrm{d}\bar{r} = 0$ if $\bar{r} = \pm\frac{1}{3}\sqrt{-\eta}$. Furthermore a computation shows $Q(\frac{1}{3}\sqrt{-\eta}) < 0$. Hence, given that $Q(\bar{r}) \to \infty$ as $\bar{r} \to \infty$, it follows that $Q(\bar{r})$ has at least one positive root. Moreover, as a consequence of the Descartes rules of signs one has that $Q(\bar{r})$ has exactly two positive roots and one negative.

Let α_3 denote the root of $Q(\bar{r})$ obtained from using Cardano's formula:

$$\alpha_3 = \sqrt[3]{-\frac{1}{2}\xi + \sqrt{\Delta}} + \sqrt[3]{-\frac{1}{2}\xi - \sqrt{\Delta}}.$$

It can be verified that if $r_* > r_+$ and $m^2 \geq q^2$, one has $\alpha_3 > 0$. The remaining two roots are given in terms of α_3 and η by

$$\alpha_1 = -\frac{1}{2}\alpha_3 - \frac{1}{2}\sqrt{-3\alpha_3^2 - 4\eta}, \qquad \alpha_2 = -\frac{1}{2}\alpha_3 + \frac{1}{2}\sqrt{-3\alpha_3^2 - 4\eta}.$$

Notice, in particular, that $3\alpha_3^2 + 4\eta < 0$. In the extremal case, $q^2 = m^2$, one obtains the simpler expressions

$$\alpha_1 = -\frac{r_*}{4(r_* - m)}\left(m + \sqrt{m(8r_* - 7m)}\right), \tag{41a}$$

$$\alpha_2 = -\frac{r_*}{4(r_* - m)}\left(m - \sqrt{m(8r_* - 7m)}\right), \tag{41b}$$

$$\alpha_3 = \frac{mr_*}{2(r_* - m)}. \tag{41c}$$

For future reference it is noticed that given

$$Q_* \equiv Q(r_*) = 2r_*^2\left(2r_*^2 - 5mr_* + 3q^2\right),$$

one has that

$$\begin{aligned} Q(r_*) \geq 0 \quad &\text{if} \quad r_* \in [r_\circledast, \infty), \\ Q(r_*) < 0 \quad &\text{if} \quad r_* \in [r_+, r_\circledast), \end{aligned}$$

where

$$r_\circledast \equiv \frac{5}{4}m + \frac{1}{4}\sqrt{25m^2 - 24q^2}. \tag{42}$$

The constant $r_\circledast$ will be seen to play a central role in the subsequent analysis. In particular, one has the following lemma obtained from lengthy calculations using the expressions obtained in the previous paragraphs:

Lemma 1. *Given $m^2 > q^2$, the roots $\alpha_1, \alpha_2, \alpha_3$ of the polynomial $Q(\bar{r})$ satisfy the inequalities*

$$\begin{aligned} \alpha_1 < 0 < \alpha_2 < r_- < r_+ < r_* < r_\circledast < \alpha_3 \quad &\text{if} \quad r_* \in (r_+, r_\circledast), \\ \alpha_1 < 0 < \alpha_2 < r_- < r_+ < r_\circledast < \alpha_3 < r_* \quad &\text{if} \quad r_* \in (r_\circledast, \infty), \\ \alpha_1 < 0 < \alpha_2 < r_- < r_+ < \alpha_3 = r_\circledast \quad &\text{if} \quad r_* = r_\circledast. \end{aligned}$$

In the extremal case ($m^2 = q^2$) one has

$$\begin{aligned} \alpha_1 < 0 < \alpha_2 < m < r_* < r_\circledast < \alpha_3 \quad &\text{if} \quad r_* \in (m, r_\circledast), \\ \alpha_1 < 0 < \alpha_2 < m < r_\circledast < \alpha_3 < r_* \quad &\text{if} \quad r_* \in (r_\circledast, \infty), \\ \alpha_1 < 0 < \alpha_2 < m < \alpha_3 = r_\circledast \quad &\text{if} \quad r_* = r_\circledast. \end{aligned}$$

In terms of the roots α_1, α_2 and α_3, Eq. (40) can be conveniently rewritten as

$$\bar{r}'^2 = \frac{\beta^2}{\bar{r}^2}(\bar{r} - r_*)(\bar{r} - \alpha_1)(\bar{r} - \alpha_2)(\bar{r} - \alpha_3). \tag{43}$$

5. Analysis of the Conformal Curves

Our study of the behaviour of the conformal curves on the Reissner–Nordström spacetime will be based on an analysis of the reduced equation (36). Three qualitatively different behaviours can be identified according to whether $r_* < r_\circledast, r_* = r_\circledast$ or $r_* > r_\circledast$. As it will be seen, these cases are associated, respectively, with a periodic, constant or monotonically increasing behaviour of the function $\bar{r}$.

5.1. Conformal Curves with Constant $\bar{r}$

We start by investigating the possibility of having a conformal curve for which $\bar{r}$ is constant—that is, $\bar{r}' = \bar{r}'' = 0$. Using Eq. (35) one obtains the condition

$$\frac{1}{2}\partial_{\bar{r}} D(\bar{r}) = \beta\sqrt{D(\bar{r})}. \tag{44}$$

The latter can be solved to give

$$\bar{r} = \frac{5}{4}m \pm \frac{1}{4}\sqrt{25m^2 - 24q^2}.$$

Under the assumption $m^2 \geq q^2$, it can be readily verified that

$$\frac{5}{4}m - \frac{1}{4}\sqrt{25m^2 - 24q^2} \leq r_+ \leq \frac{5}{4}m + \frac{1}{4}\sqrt{25m^2 - 24q^2} = r_\circledast.$$

Thus, only the solution to condition (44) with the positive radicand (i.e. $\bar{r} = r_\circledast$) will be relevant for our subsequent discussion. For reasons which will become clearer in the sequel, this particular conformal curve will be known as the *critical curve*.

Some intuition can be obtained by evaluating the constant $r_\circledast$ for the particular cases of the Schwarzschild and the extremal Reissner–Nordström spacetime:

$$r_\circledast = \frac{5}{2}m, \qquad \text{for } q = 0,$$
$$r_\circledast = \frac{3}{2}m, \qquad \text{for } q^2 = m^2.$$

Using the cases $q = 0$ and $m^2 = q^2$ as boundaries one can readily check that $\frac{1}{9} \leq D_\circledast \leq \frac{1}{5}$, where $D_\circledast \equiv D(r_\circledast)$.

In order to understand the nature of the curve under consideration, one has to analyse the behaviour of the functions $\bar{t}, \bar{u}$ and $\bar{v}$. Using Eqs. (34) and (37a)–(37b) with $\bar{r}' = 0$ one obtains

$$\bar{t} = \bar{u} - u_\circledast = \bar{v} - v_\circledast = \frac{\bar{\tau}}{\sqrt{D_\circledast}}, \tag{45}$$

where $u_\circledast \equiv u_*|_{r_*=r_\circledast}, v_\circledast \equiv v_*|_{r_*=r_\circledast}$ and u_* and v_* are given by expressions (38) or (39) depending on whether one considers the non-extremal or extremal case. Equation (45) can be expressed in terms of the unphysical proper time τ using formula (29). One finds that

$$t = u - u_\circledast = v - v_\circledast = \frac{r_\circledast}{2D_\circledast} \ln\left(\frac{2\Theta_\circledast + \beta_\circledast\tau}{2\Theta_\circledast - \beta_\circledast\tau}\right),$$

where $\Theta_\circledast$ denotes the value of the conformal factor of Eq. (28) evaluated at $(\tau = 0, r_* = r_\circledast)$.

From the expressions in (45) one has that

$$\bar{t},\ \bar{u},\ \bar{v} \to \infty \text{ as } \bar{\tau} \to \infty.$$

The region of the Reissner–Nordström spacetime described by such behaviour corresponds, respectively, to the lowermost i^+ in the right-hand side of the conformal diagram of the non-extremal Reissner–Nordström spacetime, and the lowermost one of the extremal case—see Fig. 1. This computation also shows that close to future null infinity the null coordinate u tends asymptotically to an affine parameter of the generators of $\mathscr{I}^+$—see e.g. [16, 22, 25]. The unphysical proper time required for the conformal curve to reach future null infinity is given by

$$\tau_{i^+} = \frac{2\Theta_\circledast}{\beta_\circledast} = \frac{1}{r_\circledast\sqrt{D_\circledast}}.$$

Again, by looking at the cases $q = 0, q^2 = m^2$ one can estimate

$$\frac{2}{\sqrt{5}m} \leq \tau_{i^+} \leq \frac{2}{m}.$$

5.2. Conformal Curves with $r_\circledast < r_*$

The analysis of the conformal geodesics in the Schwarzschild spacetime of [13] proceeded by solving explicitly Eq. (36) in terms of elliptic functions. One could approach the analysis of the conformal geodesics in the Reissner–Nordström spacetime in a similar fashion. Notice, however, that while in the Schwarzschild case the polynomial $P(\bar{r})$ is cubic, in the present case one has to deal with a quartic polynomial. Hence, the elliptic functions one has to deal with are more complex. In view of potential extensions of the present analysis to more general classes of spacetimes it is desirable to follow a procedure which, in as much as it is possible, does not depend on explicit solutions.

If $r_\circledast < r_*$, a computation using Eq. (32b) together with the initial data (33) shows that $\bar{r}''_* > 0$. As $\bar{r}'_* = 0$, one has a local minimum at $\bar{\tau} = 0$ and one needs to consider the positive root of Eq. (40):

$$\bar{r}' = \sqrt{(\gamma + \beta\bar{r})^2 - D(\bar{r})}, \tag{46a}$$

$$= \frac{\beta}{\bar{r}}\sqrt{(\bar{r} - r_*)(\bar{r} - \alpha_1)(\bar{r} - \alpha_2)(\bar{r} - \alpha_3)}. \tag{46b}$$

Using Lemma 1, it follows that a conformal curve with $r_\circledast < \bar{r}(0) = r_*$ has no turning points—i.e. $\bar{r}' \neq 0$ for $\bar{\tau} > 0$. Accordingly, the function $\bar{r}$ is monotonically increasing if $r_* > r_\circledast$.

In order to assert the global existence of the solution to Eq. (46a)—or (46b)—one has to verify that $\bar{r}$ does not blow up for finite $\bar{\tau}$. A direct computation shows that

$$\partial_{\bar{r}} D(\bar{r}) = \frac{2}{\bar{r}^3}(m\bar{r} - q^2).$$

Thus, $\partial_{\bar{r}} D(\bar{r}) > 0$ if $q^2/m < \bar{r}$. As it is being assumed that $q^2 \geq m^2$ and one has that $\frac{3}{2}m \leq r_{\circledast}$, it follows, indeed, that $\partial_{\bar{r}} D(\bar{r}) > 0$. Hence, one has $0 < D_* \leq D(\bar{r})$. Furthermore, one finds that

$$\bar{r}' \leq \sqrt{(\gamma + \beta\bar{r})^2 - D_*} < |\gamma + \beta\bar{r}|.$$

Now, for $r_* \leq \bar{r}$, one has that

$$|\gamma + \beta\bar{r}| = D_* \left| 1 - 2\frac{\bar{r}}{r_*} \right| = D_* \left(2\frac{\bar{r}}{r_*} - 1 \right).$$

Thus, one deduces to the differential inequality

$$\bar{r}' < \frac{2D_*}{r_*}\bar{r}$$

which can be integrated to give

$$\bar{r} \leq r_* e^{2D_* \bar{\tau}/r_*}.$$

Thus, one has that $\bar{r}$ does not blow up for finite value of $\bar{\tau}$, and the solution to (46a) with $\bar{r}(0) = r_* > r_{\circledast}$ exists for all times.

In order to obtain more information about the solution we construct a lower fence. From (46b) one readily has that

$$\frac{\varkappa\beta}{\bar{r}}\sqrt{\bar{r} - r_*} < \bar{r}' \tag{47}$$

where

$$\varkappa \equiv \sqrt{(r_* - \alpha_1)(r_* - \alpha_2)(r_* - \alpha_3)}.$$

The inequality (47) can be integrated to yield

$$r_* + \left(\frac{1}{2}\left(6\varkappa\beta\bar{\tau} + 2\sqrt{16r_*^3 + 9\varkappa^2\beta^2\bar{\tau}^2} \right)^{1/3} - \frac{2r_*}{\left(6\varkappa\beta^2\bar{\tau} + \sqrt{16r_*^3 + 9\varkappa^2\beta^2\bar{\tau}^2} \right)^{1/3}} \right)^2 < \bar{r}.$$

From this last inequality it follows that $\bar{r} \to \infty$ as $\bar{\tau} \to \infty$. Recall that $\bar{\tau} \to \infty$ corresponds, following formula (29), to a finite value of τ. Accordingly, the conformal curve reaches future null infinity $\mathscr{I}^+$ for a finite value of τ.

In order to discuss the behaviour of the function $\bar{t}$, we consider the normalisation condition (34) with $\bar{r}$ being the solution discussed in the previous paragraphs. An argument similar to the one used for $\bar{r}$ shows that $\bar{t}$ with $r_* > r_{\circledast}$ does not blow up in finite time and that $\bar{t} \to \infty$ as $\bar{\tau} \to \infty$. The coordinate t, however, is not a good coordinate to discuss the behaviour of the conformal curves with respect to (future) null infinity. In order to do this, we consider Eq. (37a), with $\bar{r}$ the solution of Eq. (46a) with $r_* > r_{\circledast}$. Global existence of solutions to (46a) follows, again, by showing that the solution (and its derivative) does not blow up in finite time. We now look in more detail at the behaviour of $\bar{u}$ as $\bar{\tau} \to \infty$. As $D(\bar{r})$ is bounded for $\bar{r} \in (r_{\circledast}, \infty)$, it follows that there exists a sufficiently large positive number $\bar{\tau}_\times$ and positive constants, C_1 and C_2, for which

$$\frac{C_1}{\bar{r}'} < \bar{u}' < \frac{C_2}{\bar{r}'}, \quad \text{for } \bar{\tau} > \bar{\tau}_\times.$$

Using the chain rule in the form $\bar{u}' = \bar{r}'\mathrm{d}u/\mathrm{d}\bar{r}$ and by increasing $\bar{\tau}_\times$ if necessary, one can find constants, $\tilde{C}_1$ and $\tilde{C}_2$, for which one has

$$\frac{\tilde{C}_1}{\bar{r}^2} < \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{r}} < \frac{\tilde{C}_2}{\bar{r}^2}, \quad \text{for } \bar{r} > \bar{r}(\bar{\tau}_\times).$$

From the latter, it follows that $\bar{u}$ goes to a finite constant value (which depends on r_*) as $\bar{\tau} \to \infty$. Thus, one concludes that in the unphysical (i.e. conformally rescaled) picture, the conformal curve reaches future null infinity for a finite value of the unphysical proper time $\tau_{\mathscr{I}^+}$. This value can be read from the conformal factor (28) to be $\tau_{\mathscr{I}^+} = 2\Theta_*/\beta$.

We summarise the results of the present section in the following proposition:

Proposition 1. *The conformal curves with initial data given by* (33) *and* $r_* > r_\circledast$ *exist for all* $\bar{\tau} \in [0, \infty)$. *The curves reach future null infinity for a finite value of the parameter* τ.

5.3. Conformal Curves with $r_* < r_\circledast$

The case of conformal curves with $r_* < r_\circledast$ is, in some sense, the most interesting one. From Eq. (40), the factorisation of the quartic polynomial $P(\bar{r})$ and Lemma 1 one concludes that if $r_* < r_\circledast$, then the resulting conformal curve will have turning points at $\bar{r} = r_*$ and $\bar{r} = \alpha_2$. Using Eq. (35) a computation shows that

$$\bar{r}''|_{\bar{r}=r_*} < 0, \qquad \bar{r}''|_{\bar{r}=\alpha_2} > 0.$$

Hence, the turning points given by r_* and α_2 correspond to a maximum and a minimum of the function $\bar{r}$, respectively. As $\bar{r}$ (and $\bar{r}'$) are bounded, one concludes that the solution to Eq. (40) with $r_* < r_\circledast$ exists for all $\bar{\tau} > 0$.

5.3.1. Behaviour of the Curves in the Non-Extremal Case. As a consequence of the discussion in the previous paragraph, the function $\bar{r}$ is initially decreasing. Moreover, from Eqs. (37a)–(37b) one has that u'_*, $v'_* > 0$ so that the concavity of the curve points, initially, away from the horizon—see Fig. 2. From $\alpha_2 < r_+$, it follows there exists a value of $\bar{\tau}$ for which $\bar{r} = r_+$—implying that the conformal curve crosses the horizon. The temporal coordinate t is not appropriate for this discussion as $t \to \infty$ for any curve approaching the event horizon—this can readily be seen using the normalisation (34) and the fact that $D(r) = 0$ at the event horizon—that is, at $r = r_+$. Hence, one makes use of the null coordinate v. The evolution of this coordinate along the conformal curve is described by Eq. (37b). Initially, one has that $\bar{v}'(0) = 1/\sqrt{D(\bar{r})}$, so that $\bar{v}$ is, at least initially, increasing. As a consequence of Lemma 1 one sees that the conformal curve should cross the event and the Cauchy horizon before

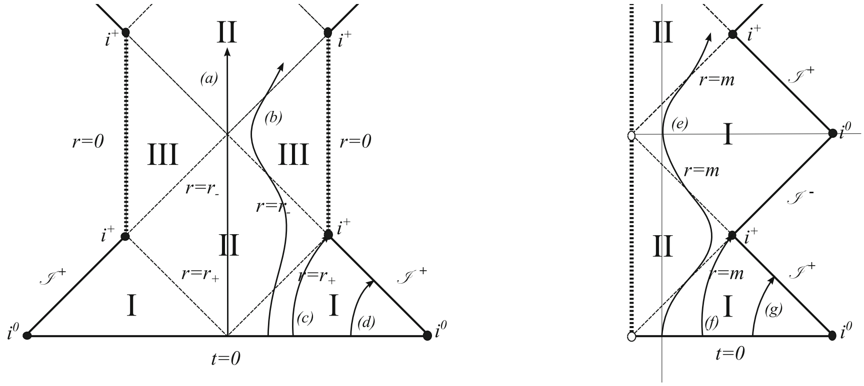

Figure 2. Schematic illustration of the behaviour of conformal curves. To the *left* the non-extremal case: **a** the curve starting at $r_* = r_+$; **b** a curve with $r_* < r_\circledast$; **c** the critical curve; **d** a curve with $r_* > r_\circledast$. To the *right* the extremal case: **e** a curve with $r_* < r_\circledast$; **f** the critical curve; **g** a curve with $r_* > r_\circledast$. The curves are not depicted on scale

reaching the minimum of $\bar{r}$ at $\bar{r} = \alpha_2$. It is just necessary to check that the various fields remain regular at the horizons.

As the curve approaches the event horizon at $\bar{r} = r_+$, both the numerator and denominator of the right-hand side of Eq. (37b) vanish—in particular, the numerator can vanish as $\bar{r}' < 0$. Using the L'Hopital rule and Eq. (35) one finds that $\bar{v}'|_{\bar{r}=r_+}$ is well defined and positive. Accordingly, the curve enters region II of conformal diagram of the non-extremal Reissner–Nordström spacetime—see Fig. 2. A similar situation occurs as $\bar{r}$ approaches r_-: the numerator and denominator of Eq. (37b) both vanish; using the L'Hopital rule one verifies that $\bar{v}'$ is well defined at r_-. Notice that as $\bar{r}' < 0$ and $\bar{v}' > 0$, the curve exits the region II of the conformal diagram through the left-hand side of the Cauchy horizon. The turning point at $\bar{r} = \alpha_2$ is located in the region III of the conformal diagram. After the curve has reached this point, one has that $\bar{r}' > 0$ and the behaviour of the curve as it approaches again the horizon at $\bar{r} = r_-$ is different. In this case, the numerator of the right-hand side of Eq. (37b) tends to a non-zero value and one has that $\bar{v} \to \infty$—and hence, also $\bar{v} \to \infty$. In order to discuss the behaviour of the curve beyond this point one would have to introduce a new set of null coordinates.

In Sect. 5.3.4 it will be shown that the points with $\bar{r} = r_+,\ r_-,\ \alpha_2$ are reached for a finite value of the physical proper time $\bar{\tau}$.

5.3.2. Behaviour of the Curves in the Extremal Case. As in the non-extremal case, conformal curves in the extremal Reissner–Nordström spacetime with $r_* < r_\circledast$ satisfy $\bar{r}'_* < 0$ and $\bar{v}'_* > 0$. Using the L'Hopital rule one can verify that $\bar{v}'$ is well defined and positive at $\bar{r} = r_+ = m$. Thus, the conformal curve penetrates in the region II of the conformal diagram of the spacetime—cf. Fig. 2. The essential difference with respect to the non-extremal case is that the turning point given by $\bar{r} = \alpha_2$ is now in region II. After the curve has passed this point one has $\bar{r}' > 0$ and $\bar{v}' > 0$. Hence, from Eq. (37b) it follows that $\bar{v},\ \bar{v}' \to \infty$ the second time the curve approaches $\bar{r} = r_+ = m$. In order to follow the behaviour of the curve beyond this point one would need a new set of null coordinates.

In Sect. 5.3.4 it will be shown that the points with $\bar{r} = m,\ \alpha_2$ are reached for a finite value of the physical proper time $\bar{\tau}$.

5.3.3. Regions of the Spacetime Not Covered by the Congruence. As already discussed, the turning point described by the condition $\bar{r} = \alpha_2$ is located in the region II in the extremal case and in region III in the non-extremal case. In these regions the curves $\bar{r} =$ constant are timelike. Regarding α_2 as a function of r_*, it follows from the expressions given in Sect. 4.4 that there exists a certain value of r_* for which α_2 attains a (non-zero) minimum. This minimum value depends only on the value of m and q.

The phenomenon described in the previous paragraph is better analysed in the extremal case where simpler analytical expressions are available. Using expression (41b) one can readily see that α_2 attains a minimum value of $(\sqrt{2}-1/2)m \approx 0.91m$ along the conformal curve with $r_* = (2-1/\sqrt{2})m \approx 1.29m$. All other conformal curves with $r_* < r_{\circledast}$ will have a higher value for the r-location of the turning point.

The discussion in the previous paragraphs shows that there exist regions in the regions III of the non-extremal Reissner–Nordström spacetime and the regions II in the extremal case that cannot be probed by means of the family of conformal curves under consideration. In particular, a conformal curve cannot get arbitrarily close to the singularities of the spacetime ($r = 0$). In this sense, one can regard our class of conformal curves as singularity avoiding.

5.3.4. Explicit Expressions in Terms of Elliptic Functions. As discussed previously, the function $\bar{r}$ is decreasing if $\bar{r} \in (\alpha_2, r_*)$. If this is the case then Eq. (40) implies

$$\bar{r}' = -\frac{\beta}{\bar{r}}\sqrt{(\bar{r}-r_*)(\bar{r}-\alpha_1)(\bar{r}-\alpha_2)(\bar{r}-\alpha_3)}.$$

The latter implies, in turn

$$\bar{\tau} = -\frac{1}{\beta}\int_{r_*}^{\bar{r}} \frac{s\mathrm{d}s}{\sqrt{(s-\alpha_1)(s-\alpha_2)(s-\bar{r}_*)(s-\alpha_3)}}.$$

The integral in the right-hand side can be evaluated in terms of elliptic functions—see e.g. [19]. For example, the physical proper time required by the curve to go from $\bar{r} = r_*$ to $\bar{r} = \alpha_2$ is given by

$$\beta\bar{\tau}(\alpha_2) = \frac{2\alpha_1}{\sqrt{(\alpha_3-\alpha_2)(\bar{r}_*-\alpha_1)}}K\left(\sqrt{\frac{(\bar{r}_*-\alpha_2)(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_2)(\bar{r}_*-\alpha_1)}}\right)$$
$$+\frac{2(\alpha_2-\alpha_1)}{\sqrt{(\alpha_3-\alpha_2)(\bar{r}_*-\alpha_1)}}\Pi\left(\frac{\bar{r}_*-\alpha_2}{\bar{r}_*-\alpha_1};\sqrt{\frac{(\bar{r}_*-\alpha_2)(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_2)(\bar{r}_*-\alpha_1)}}\right),$$

where $K(\,\cdot\,)$ and $\Pi(\,\cdot\,;\,\cdot\,)$ denote, respectively, the complete elliptic integrals of the first and third kind. The above expression is valid for both the non-extremal and extremal cases. For specific values of m and $\bar{r}_*$, these integrals can be accurately evaluated with a computer algebra system. Hence, the expression for $\bar{\tau}(\alpha_2)$ can be used to verify the accuracy of numerical solutions to the conformal curve equations. However, for analytical purposes, expressions of the type given above are too clumsy to be used. Insight into the behaviour of $\bar{\tau}$ regarded as a function of $\bar{r}$ and r_* can be obtained by means of suitable estimates.

As a consequence of Lemma 1 one has that for $\alpha_2 \leq \bar{r} \leq r_*$ one has

$$-\varkappa_1^2(\bar{r}-\alpha_2)(\bar{r}-r_*) \leq P(\bar{r}) \leq -\varkappa_2^2(\bar{r}-\alpha_2)(\bar{r}-r_*),$$

where

$$\varkappa_1^2 \equiv (\alpha_2-\alpha_1)(\alpha_3-\alpha_2), \qquad \varkappa_2^2 \equiv (r_*-\alpha_1)(\alpha_3-r_*).$$

Furthermore, one finds that

$$\frac{1}{\varkappa_1} I(\bar{r}) \leq \beta\bar{\tau} \leq \frac{1}{\varkappa_2} I(\bar{r}),$$

where

$$\begin{aligned} I(\bar{r}) &\equiv \int_{\bar{r}}^{r_*} \frac{s\mathrm{d}s}{\sqrt{(s-\alpha_2)((r_*-s))}}, \\ &= \sqrt{(\bar{r}-\alpha_2)(r_*-\bar{r})} + \frac{1}{2}(\alpha_2+r_*) \\ &\quad \times \left(\arcsin\left(\frac{3r_*+\alpha_2}{(\alpha_2-r_*)^2}\right) - \arcsin\left(\frac{2\bar{r}+r_*+\alpha_2}{(\alpha_2-r_*)^2}\right)\right). \end{aligned}$$

In particular, one has that $I(\alpha_2)$ is finite. Thus, one concludes that the conformal curves with $r_* < r_\circledast$ reach the turning point $\bar{r} = \alpha_2$ (and hence, also the horizons at $\bar{r}_\pm$) in finite physical proper time.

5.4. The Conformal Curve Starting at r_+

In the case of a non-extremal Reissner–Nordström spacetime it is also of interest to analyse the behaviour of the conformal curve starting at the bifurcation sphere ($\bar{r}_* = \bar{r}_+$). Observing that $D(r_+) = 0$, so that $\beta_+ = 0$ it follows from Eq. (35) that

$$\bar{r}'' = \frac{q^2}{\bar{r}^3} - \frac{m}{\bar{r}^2}.$$

Using the initial conditions

$$r_* = r_+, \qquad \bar{r}'_* = 0,$$

one can readily integrate to obtain

$$\bar{r} = m + \sqrt{m^2-q^2}\cos\bar{\tau} > 0.$$

Because of the reflection symmetry of the Reissner–Nordström spacetime with respect to the bifurcation sphere and the timelike character of the curve in question, the conformal curve always remains in the middle of the conformal diagram of the spacetime. The curve starts in region II where the surfaces of constant $\bar{r}$ are spacelike. The function $\bar{r}$ is decreasing in the region II, and reaches its minimum value at the Cauchy horizon—where $\bar{r} = \bar{r}_-$. From here $\bar{r}$ becomes increasing and eventually reaches its maximum (and initial value) at $\bar{r} = \bar{r}_+$. This corresponds to a second bifurcation sphere in the Penrose diagram of the maximal analytic extension of the spacetime—see Fig. 2. It is worth observing that any point along the conformal curve can be reached in a finite amount of (physical or unphysical) proper time. In particular, the distance between two consecutive bifurcation spheres in the Penrose diagram measured in terms of the parameter $\bar{\tau}$ is 2π.

5.5. Further Analysis of the Behaviour of the Critical Curve Close to Timelike Infinity

The purpose of the present section is to further discuss the behaviour of the critical curve $\bar{r} = r_\circledast$ as it approaches i^+. The reason for this analysis is motivated by the observation made in [12] that in the Schwarzschild spacetime the corresponding critical curve, which is timelike for $\tau \in [0, \tau_{i^+})$, becomes null at $\tau = \tau_{i^+}$. This observation indicates a degeneracy of the conformal structure of the spacetime.

5.5.1. The Intersection of the Critical Curve and Null Infinity. First, we consider the behaviour of conformal curves with $r_* > r_\circledast$ as $r_* \to r_\circledast$ from the right. These curves reach future null infinity in a finite amount of unphysical proper time

$$\tau_{\mathscr{I}^+} \equiv \frac{2\Theta_*}{\beta} = \frac{1}{r_*\sqrt{D_*}},$$

as it can be seen from the expression (28) for the conformal factor Θ. Setting $r_* = (1+\epsilon)r_\circledast$, for small $\epsilon > 0$, it follows that

$$\tau_{\mathscr{I}^+} = \frac{1}{r_\circledast\sqrt{D_\circledast}}\left(1 + 3\epsilon + \mathcal{O}(\epsilon^2)\right).$$

Thus, one has in particular that

$$\frac{\mathrm{d}\tau_{\mathscr{I}^+}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} = \frac{3}{r_\circledast\sqrt{D_\circledast}},$$

that is, future null infinity approaches i^+ at a finite positive angle.

5.5.2. The Intersection of the Critical Curve and the Horizon. The analysis of the behaviour of the conformal curves with $r_* < r_\circledast$ is more delicate. In what follows, let $\bar{\tau}_{\mathscr{H}^+}$ denote the value of the physical proper time for which a conformal curve reaches the horizon. From the expression (43) it follows that

$$\beta\bar{\tau}_{\mathscr{H}^+} = \int_{r_+}^{r_*} \frac{s\mathrm{d}s}{\sqrt{(s-r_*)(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)}}.$$

Now, setting $r_* = (1-\epsilon)r_\circledast$ for small $\epsilon > 0$, and observing Lemma 1 it follows that

$$\frac{s}{\sqrt{(s-r_*)(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)}} = \frac{s}{(r_\circledast - s)\sqrt{(s-\bar{\alpha}_1)(s-\bar{\alpha}_2)}} + \mathcal{O}(\epsilon), \quad s < r_\circledast,$$

where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are the values of the roots α_1 and α_2 corresponding to $r_* = r_\circledast$. A computation then shows that

$$\beta\bar{\tau}_{\mathscr{H}^+} = \kappa - \frac{r_\circledast}{\sqrt{(r_\circledast - \bar{\alpha}_1)(r_\circledast - \bar{\alpha}_2)}} \ln\epsilon + \mathcal{O}(\epsilon),$$

with κ a constant depending on m and q. One readily sees that $\bar{\tau}_{\mathscr{H}^+} \to \infty$ as $\epsilon \to 0$ consistently with the discussion of Sect. 5.1. It can be explicitly shown that

$$\eta \equiv \frac{r_\circledast}{\sqrt{(r_\circledast - \bar{\alpha}_1)(r_\circledast - \bar{\alpha}_2)}} \leq 1,$$

where the equality is achieved only for $q^2 = m^2$. Finally, noticing that

$$\frac{2\Theta_*}{\beta} = \frac{1}{r_\circledast\sqrt{D_\circledast}}\left(1 + \frac{1}{r_\circledast D_\circledast}(r_\circledast - m)\epsilon + \mathcal{O}(\epsilon^2)\right),$$

one concludes from Eq. (30) that

$$\tau_{\mathscr{H}^+} = \frac{1}{r_\circledast\sqrt{D_\circledast}}\left(1 + 2e^{-\kappa}\epsilon^\eta + \frac{1}{r_\circledast D_\circledast}(r_\circledast - m)\epsilon + \mathcal{O}\left(\epsilon^{1+\eta}\right)\right).$$

It follows that

$$\lim_{\epsilon\to 0}\frac{\mathrm{d}\tau_{\mathscr{H}^+}}{\mathrm{d}\epsilon} = \begin{cases} \infty & q^2 < m^2 \\ \frac{2}{m}(5 + 2e^{-\kappa}) < \infty & q^2 = m^2. \end{cases}.$$

Thus, the critical conformal curve and the horizon are tangent at i^+ for $q^2 < m^2$—that is, the critical curve becomes null at i^+. A similar singular behaviour for the conformal geodesic reaching i^+ in the Schwarzschild spacetime has been described in [12]. Indeed, it can be verified that $\eta = \frac{1}{\sqrt{2}}$ for $q = 0$. This behaviour indicates a degeneracy of the conformal structure at i^+ for $q^2 < m^2$. The most remarkable feature of the present analysis is the fact that the critical curve remains timelike in the extremal case $q^2 = m^2$. In this case, a more regular behaviour of the conformal structure is to be expected.

6. Analysis of the Conformal Deviation Equations

The purpose of this section is to show that the congruence of conformal curves under consideration does not form caustics in the outer domain of communication of the Reissner–Nordström spacetime.

6.1. Basic Equations

Inspired on a similar discussion in [13], we will study solutions to the reduced $\tilde{\boldsymbol{g}}$-adapted deviation equation (27) with suitable initial data. In the previous section, the value of the coordinate r on the initial hypersurface $\tilde{S}$ has been used to parametrise the conformal curves of our congruence. Thus, it would be natural to use the vector $\boldsymbol{\partial}_r$ as deviation vector. However, as we are specifically interested in the behaviour of the congruence near the event horizon, this choice is no longer adequate. Instead, we consider the vector field $\boldsymbol{\partial}_\varrho$ where ϱ is the isotropic radial coordinate given by formulae (5). Thus, in what follows we set

$$\bar{\boldsymbol{Z}}_* = (\boldsymbol{\partial}_\varrho \bar{x})_*, \qquad \chi \equiv \frac{1}{2} R[\boldsymbol{l}], \qquad \zeta \equiv \not{D}_{\bar{\boldsymbol{Z}}} \beta. \tag{48}$$

It follows that Eq. (27) can be rewritten in the form

$$\omega'' - (\beta^2 + \bar{\chi})\omega = \zeta \tag{49}$$

where the bar over χ indicates that the function is regarded as depending on $\bar{\tau}$. In the particular case of the Reissner–Nordström spacetime one has that

$$\bar{\chi} = \frac{2m}{\bar{r}^3} - \frac{3q^2}{\bar{r}^4}. \tag{50}$$

One readily sees that $\bar{\chi} > 0$ if $\bar{r} > 3q^2/2m$. By direct evaluation it can be checked that if $m^2 \geq q^2$, then $3q^2/2m \leq r_\circledast$, with the equality being achieved in the extremal case. Finally, it is noticed that

$$\frac{\mathrm{d}\bar{\chi}}{\mathrm{d}\bar{r}} = \frac{6}{\bar{r}^5}(2q^2 - m\bar{r}) > 0 \quad \text{if} \quad \bar{r} < \frac{2q^2}{m}. \tag{51}$$

Now, recalling that $\bar{X} = \bar{x}'$ and taking into account (48), one finds that the initial data for Eq. (49) on $\tilde{S}$ is given, for $\varrho_* > \varrho_+$ (i.e $r_* > r_+$), by

$$\omega_* \equiv \boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Z}})_* = \bar{t}'_* \left(\frac{\partial \bar{r}}{\partial \varrho}\right)_* - \bar{r}'_* \left(\frac{\partial \bar{t}}{\partial \varrho}\right)_* = \frac{r_*}{\varrho_*} > 1, \tag{52a}$$

$$\omega'_* = \left(\not{D}_{\bar{\boldsymbol{X}}} \boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Z}})\right)_* = 0, \tag{52b}$$

where Eqs. (4) and (33) have been used to simplify the expression for ω_*.

Following the discussion of Sect. 3, the coefficients β^2 and ζ in Eq. (49) are constant along a given conformal curve, and thus, can be conveniently be evaluated at the initial hypersurface $\tilde{\mathcal{S}}$—cf. the remark after Eq. (22). Recalling that $\beta^2 = 4D_*/r_*^2$, it follows that

$$\begin{aligned}\partial_{\varrho_*}\beta^2 &= \left(\frac{\mathrm{d}r}{\mathrm{d}\varrho}\right)_* \partial_{r_*}\beta^2 = \frac{4}{r_*^3}\left(\frac{\mathrm{d}r}{\mathrm{d}\varrho}\right)_* \left(r_* \frac{\mathrm{d}D_*}{\mathrm{d}r_*} - 2D_*\right), \\ &= -\frac{8}{r_*^5}\left(\frac{\mathrm{d}r}{\mathrm{d}\varrho}\right)_* (r_*^2 - 3mr_* + 2q^2).\end{aligned}$$

Finally, using $\mathrm{d}r/\mathrm{d}\varrho = r\sqrt{D}/\varrho$ and $\partial_\varrho\beta = (\partial_\varrho\beta^2)/2\beta$ one concludes that

$$\zeta = (\partial_\varrho \beta)_* = -\frac{2}{\varrho_* r_*^3}(r_*^2 - 3mr_* + 2q^2).$$

This expression is positive if

$$r_* \in \left(\frac{3}{2}m - \frac{1}{2}\sqrt{9m^2 - 8q^2}, \frac{3}{2}m + \frac{1}{2}\sqrt{9m^2 - 8q^2}\right).$$

In the extremal case, the expression for ζ simplifies to

$$\zeta = -\frac{2}{r_*^3}(r_* - 2m), \tag{53}$$

which is positive for $r_* < 2m$. It is also noticed that

$$\beta^2 + \chi_* = \frac{1}{\bar{r}_*^4}(4r_*^2 - 6mr_* + q^2).$$

One conclude then that

$$\beta^2 + \chi_* < 0 \quad \text{if } r_* \in \left(r_+, \frac{3}{4}m + \frac{1}{4}\sqrt{9m^2 - 4q^2}\right), \tag{54a}$$

$$\beta^2 + \chi_* \geq 0 \quad \text{if } r_* \in \left[\frac{3}{4}m + \frac{1}{4}\sqrt{9m^2 - 4q^2}, \infty\right). \tag{54b}$$

In the extremal case, the above expressions reduce to

$$\beta^2 + \chi_* < 0 \quad \text{if } r_* \in \left(m, \frac{1}{4}(3 + \sqrt{5})m\right),$$

$$\beta^2 + \chi_* \geq 0 \quad \text{if } r_* \in \left[\frac{1}{4}(3 + \sqrt{5})m, \infty\right),$$

where $\frac{1}{4}(3 + \sqrt{5}) \approx 1.309$. Finally, one has that

$$\zeta + (\beta^2 + \chi_*)\omega_* = \frac{1}{\varrho_* r_*}(2r_*^2 - 3q^2),$$

so that $\omega_*'' \geq 0$ if $r_* \geq \sqrt{\frac{3}{2}}|q|$ where $\sqrt{\frac{3}{2}} \approx 1.225$.

6.2. The Curve Deviation Equation Along the Critical Curve

The simplest situation on which to analyse the solutions of the deviation equation (49) is along the critical curve $\bar{r} = r_{\circledast}$. Letting $\chi_{\circledast} \equiv \chi(r_{\circledast})$, a direct computation shows that

$$\chi_{\circledast} > 0, \quad \zeta_{\circledast} > 0, \qquad \text{for } m^2 > q^2,$$

$$\chi_{\circledast} = 0, \quad \zeta_{\circledast} > 0, \qquad \text{for } m^2 = q^2.$$

With this information at hand, the solution to the deviation equation can be found explicitly to be given by

$$\omega(\bar{\tau}) = \left(\omega_{\circledast} + \frac{\zeta_{\circledast}}{\beta_{\circledast}^2 + \chi_{\circledast}}\right)\cosh(\sqrt{\beta_{\circledast}^2 + \chi_{\circledast}}\,\bar{\tau}) - \frac{\zeta_{\circledast}}{\beta_{\circledast}^2 + \chi_{\circledast}}.$$

This solution satisfies $\omega(\bar{\tau}) > 0, \omega'(\bar{\tau}) > 0$ for all $\bar{\tau} \geq 0$. Thus, no conjugate points arise in the critical curve when regarded as a curve on the physical Reissner–Norsdström spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. In view of future applications it is important to verify the absence of conjugate points in the conformally rescaled spacetime even at i^+. In order to do this, one has to consider the conformal curve as parametrised by the unphysical proper time τ. Following the discussion of Sect. 3.2 one has that $\bar{\boldsymbol{X}} = \Theta\dot{\boldsymbol{x}}$. The relevant deviation vector field is then given by

$$\partial_\varrho \boldsymbol{x} = \partial_\varrho \bar{\boldsymbol{x}} - \dot{\boldsymbol{x}}\partial_\rho \tau.$$

It follows that for $\bar{\boldsymbol{X}}$ and $\bar{Z}$ to remain linearly independent one requires that $\Theta\boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Z}}) \neq 0$ along the critical curve up to (and including) i^+. Using expression (31) together with the explicit expression for $\omega(\bar{\tau})$ obtained in the previous paragraph one obtains that

$$\Theta\boldsymbol{\epsilon_l}(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Z}}) \geq \Theta_\circledast \omega_\circledast > 0$$

along the critical curve up to, and including i^+ so that also no conjugate points arise on the conformal Reissner–Nordström manifold.

6.3. Curves with $r_\circledast < r_*$

The analysis of solutions of Eq. (49) with for $r_\circledast < r_*$ follows closely the discussion of [13]. It is given here for the sake of completeness. The key observation is that the solution to (49) admits the representation

$$\omega(\bar{\tau}) = \varpi(\bar{\tau}) \left(\omega_* + \left(1 - \frac{\varsigma(\bar{\tau})}{\varpi(\bar{\tau})} \right) \zeta \right), \tag{55}$$

where $\varpi(\bar{\tau})$ and $\varsigma(\bar{\tau})$ are the solutions to the auxiliary problems

$$\begin{aligned} &\varpi'' - (\beta^2 + \bar{\chi})\varpi = 0, \quad \varpi(0) = 1, \quad \varpi'(0) = 0, \\ &\varsigma'' - (\beta^2 + \bar{\chi})\varsigma = -1, \quad \varsigma(0) = 1, \quad \varsigma'(0) = 0. \end{aligned}$$

That ω as given by (55) is indeed a solution of (49) with the right initial data that can be verified by directed evaluation. As $r_* > r_\circledast$, it follows from the observation after Eq. (50) that $\beta^2 + \bar{\chi} > 0$ along the conformal curves under consideration.

Using the equation for ϖ and the initial data $\varpi(0) = 1$ one sees that $\varpi''(0) > 0$ so that ϖ has a local minimum at $\bar{\tau} = 0$. Thus, at least for positive values of $\bar{\tau}$ close to 0 one has that ϖ must be increasing. Furthermore, as $\bar{\chi} > 0$ for $r > r_\circledast$, one finds that $\beta^2\varpi \leq \varpi''$. This last differential inequality can be integrated to yield $\varpi \geq \cosh(\beta\bar{\tau})$. One concludes that ϖ is increasing for all $\bar{\tau} \geq 0$. A similar argument with the function $\eta \equiv \varpi - \varsigma$ satisfying the equation

$$\eta'' - (\beta^2 + \bar{\chi})\eta = 1, \qquad \eta(0) = 0, \qquad \eta'(0) = 0,$$

shows that $\varpi \geq \varsigma$ for all $\bar{\tau} \geq 0$.

The information obtained in the previous paragraph will be used to estimate the term $1 - \varsigma/\varpi$ in (55). Due to the monotonicity of ϖ one has that $1 - \varsigma/\varpi \geq 0$. Using that $\varpi \neq 0$ for $\bar{\tau} \geq 0$, it follows that there exists a function $f(\bar{\tau})$ such that $\varsigma = f\varpi$. It can readily be seen that f satisfies the equation

$$f'' + 2\frac{\varpi'}{\varpi} f' = -\frac{1}{\varpi}, \qquad f(0) = 1, \qquad f'(0) = 1,$$

whose solution can be written as

$$f = 1 - \int_0^{\bar{\tau}} \left(\frac{1}{\varpi^2} \int_0^s \varpi \, \mathrm{d}s' \right) \mathrm{d}s.$$

From the latter one obtains the chain of inequalities

$$\begin{aligned} 0 \leq 1 - \frac{\varsigma}{\varpi} = 1 - f = \int_0^{\bar{\tau}} \left(\frac{1}{\varpi^2} \int_0^s \varpi \, \mathrm{d}s' \right) \mathrm{d}s \\ \leq \int_0^{\bar{\tau}} \frac{s}{\varpi} \mathrm{d}s \leq \int_0^{\bar{\tau}} \frac{s}{\cosh(\beta s)} \mathrm{d}s \leq 2 \int_0^{\bar{\tau}} s e^{-2\beta s} \mathrm{d}s. \end{aligned}$$

Hence, one has that

$$0 \leq 1 - \frac{\varsigma}{\varpi} \leq \frac{2}{\beta^2} (1 - (\beta\bar{\tau} + 1) e^{-\beta\bar{\tau}}) \leq \frac{2}{\beta^2}.$$

For the values of $r_* > r_\circledast$ for which $\zeta < 0$ a direct computation shows that $-1 < 2\zeta/\beta^2 < 0$. Thus,

$$\omega_* + \left(1 - \frac{\varsigma}{\varpi}\right) \zeta \geq \omega_* + \frac{2\zeta}{\beta^2} > \omega_* - 1.$$

For the values of $r_* > r_\circledast$ for which $\zeta > 0$ one readily has that

$$\omega_* + \left(1 - \frac{\varsigma}{\varpi}\right) \zeta \geq \omega_*.$$

Hence, in both cases using Eq. (52a) one concludes that

$$\omega_* + \left(1 - \frac{v}{u}\right) \zeta \geq \omega_* - 1 = \frac{1}{\varrho_*}(m^2 - q^2 + 4\varrho_* m) > 0.$$

In order to conclude the argument we consider $\Theta\omega$ where Θ is given by Eq. (31). Putting together the discussion from the previous paragraphs one has that

$$\begin{aligned} \Theta\omega = \Theta\varpi \left(\omega_* + \left(1 - \frac{\varsigma}{\varpi}\right) \zeta \right) &\geq \frac{\Theta\varpi}{\varrho_*}(m^2 - q^2 + 4\varrho_* m) \\ &\geq \Theta_* \frac{\cosh(\bar{\tau})}{\varrho_* \cosh^2(\frac{1}{2}\beta\bar{\tau})}(m^2 - q^2 + 4\varrho_* m) \\ &\geq \frac{\Theta_*}{\varrho_*}(m^2 - q^2 + 4\varrho_* m) > 0. \end{aligned}$$

This lower bound holds even in the limit $\bar{\tau} \to \infty$ (i.e. $\tau = 0$). By the same considerations made in Sect. 6.2, it follows that the congruence of conformal curves remains free of conjugate points even at null infinity.

6.4. Curves with $r_* < r_\circledast$

The observation made in Sect. 5.3 that for $r_* < r_\circledast$, curves with a certain value of r_* can have a lower value of the turning point $\bar{r} = \alpha_2$ than curves starting closer to the horizon, shows that curves in our congruence of conformal curves must intersect at some point. This is because curves with constant coordinate value r are timelike, respectively, in the regions III of the non-extremal case and the regions II of the extremal case. The question is then: how long does this part of the congruence exists without developing conjugate points? In the sequel it will be shown that the congruence of conformal curves is free of conjugate points up to, and including, the horizon. More precisely one has that:

Proposition 2. *Assume that $q^2 \leq \frac{8}{9}m^2$ and $r_* \in (r_+, r_\circledast)$. Then, the solutions to Eq. (49) with initial data given by (52a)–(52b) satisfy $\omega > 0$ for $\bar{\tau} \in [0, \bar{\tau}_{\mathscr{H}^+}]$.*

The case $\frac{8}{9}m^2 < q^2 < m^2$ will not be considered here. Instead, we concentrate our attention on the extremal case for which one can prove the following:

Proposition 3. *Assume $q^2 = m^2$. Then there exists $r_\star \in (r_+, r_\circledast)$ such that for $r_* \in (r_\star, r_\circledast)$, the solutions to Eq. (49) with initial data given by (52a)–(52b) satisfy $\omega > 0$ for $\bar{\tau} \in [0, \bar{\tau}_{\mathscr{H}^+}]$.*

The proof of these propositions is given in Sects. 6.4.1 and 6.4.2, respectively. Clearly, the result of Proposition 3 (which includes the extremal case) is much more restrictive than that of Proposition 2. In view of the discussion of Sect. 5.5, the extremal Reissner–Nordström spacetime is the case of most relevance from the perspective of conformal geometry. In this respect, the result given in Proposition 2 will be sufficient for future applications to be considered elsewhere. Numerical evaluations of the solutions of Eq. (49) suggest, nevertheless, that the conclusions of Proposition 2 can be extended to the whole range $q^2 \leq m^2$ and $r_* \in (r_+, r_\circledast)$—so that, in particular, Proposition 3 could be superseded. This proof would require an analysis which would increase considerably the length of this article.

For simplicity of the presentation, *in the remainder of the subsection it is always assumed that $\bar{r} \in [\alpha_2, r_*]$. For these values of $\bar{r}$ one has that $\bar{r}' < 0$ with $\bar{r}' = 0$ only at $\bar{r} = r_*$, α_2.* The amount of technical details in the analysis of the solutions to Eq. (49) depends on whether the value of the charge, q, is close or not to the extremal value. In order to characterise the values of q requiring a more careful treatment, it is recalled that $\bar{\chi} \geq 0$ if $\bar{r} \geq 3q^2/2m$. As $\bar{r}$ is monotonically decreasing, it follows that for $\bar{r} \in [r_*, r_+]$, $\bar{\chi} \geq 0$ if and only if

$$\frac{3q^2}{2m} \leq r_+ = m + \sqrt{m^2 - q^2}.$$

The above inequality is saturated if $q^2 = \frac{8}{9}m^2$. Our subsequent discussion is split depending on whether q^2 is below or above the critical value found in the previous lines.

6.4.1. Proof of Proposition 2. As already discussed, if $q^2 \leq \frac{8}{9}m^2$, one has that $\bar{\chi} \geq 0$ for $\bar{r} \in [r_+, r_*]$. It follows then by an argument similar to the one used in Sect. 6.3 that $\omega > 0$ for $\bar{\tau} \in [0, \bar{\tau}_{\mathscr{H}^+}]$. The full details will not be provided, but it is noticed that, in fact, the task in this case is simpler as one is dealing with finite values of $\bar{\tau}$ and $\bar{r}$. Hence, it is only necessary to ensure the positivity of ω and not that of $\Theta\omega$.

6.4.2. Proof of Proposition 3. It is assumed throughout that $q^2 = m^2$. In this case, the analysis of the previous sections shows that there are intervals of $\bar{\tau}$ for which $\bar{\chi} < 0$, so that the arguments of Sect. 6.3, do not apply for the whole interval $[0, \bar{\tau}_{\mathscr{H}^+}]$. Hence, a more detailed analysis is required. In particular, formula (50) shows that in the extremal case $\bar{\chi}$ is always negative for $\bar{r} \in [\alpha_2, r_*]$. As before, we will restrict our attention to the behaviour of the conformal curves in the range $\bar{\tau} \in [0, \bar{\tau}_{\mathscr{H}^+}]$. In such interval $\bar{r}(\bar{\tau})$ is a monotonic decreasing function of $\bar{\tau}$ with $\bar{r}(0) = r_*$ and $\bar{r}(\bar{\tau}_{\mathscr{H}^+}) = r_+$. Hence, for $\bar{\tau} \in (0, \bar{\tau}_{\mathscr{H}^+}]$ it is convenient to reparametrise Eq. (49) in terms of $\bar{r}$. Using the chain rule to write

$$\omega' = \bar{r}'\frac{\mathrm{d}\omega}{\mathrm{d}\bar{r}}, \qquad \omega'' = \bar{r}'\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left(\bar{r}'\frac{\mathrm{d}\omega}{\mathrm{d}\bar{r}}\right)$$

one readily has that the deviation equation (49) implies the equation

$$\bar{r}'^2 \grave{\grave{\omega}} + \bar{r}'' \grave{\omega} - (\beta^2 + \bar{\chi})\omega = \zeta, \tag{56}$$

with

$$\grave{\omega} \equiv \frac{\mathrm{d}\omega}{\mathrm{d}\bar{r}}, \qquad \grave{\grave{\omega}} \equiv \frac{\mathrm{d}^2\omega}{\mathrm{d}\bar{r}^2}$$

and where $\bar{\chi}$ is now regarded as a function of $\bar{r}$, and $\bar{r}'^2$ is given by Eq. (43). Notice that as $\bar{r}' = 0$ at $\bar{r} = r_*$, α_2 Eq. (56) is formally singular. An explicit formula for $\bar{r}''$ in terms of $\bar{r}$ can be found using

$$\bar{r}'' = \frac{1}{2}\frac{1}{\bar{r}'}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}(\bar{r}'^2) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\bar{r}}(r'^2).$$

One obtains

$$\begin{aligned}\bar{r}'' = \frac{\beta^2}{2\bar{r}^3}&((\bar{r}-\alpha_1)(\bar{r}-\alpha_2)(\bar{r}-\alpha_3)(2\bar{r}_* - \bar{r}) + \bar{r}(\bar{r}-\bar{r}_*)(\bar{r}-\alpha_2)(\bar{r}-\alpha_3)\\ &+\bar{r}(\bar{r}-\bar{r}_*)(\bar{r}-\alpha_1)(\bar{r}-\alpha_3) + \bar{r}(\bar{r}-\bar{r}_*)(\bar{r}-\alpha_1)(\bar{r}-\alpha_2)).\end{aligned} \tag{57}$$

In particular, using the information from Lemma 1 one can readily conclude that

$$\bar{r}''_* < 0, \qquad \bar{r}''_2 > 0$$

where $\bar{r}''_* \equiv \bar{r}''(r_*)$ and $\bar{r}''_2 \equiv \bar{r}''(\alpha_2)$. Thus, one concludes that there exists $r_! \in (\alpha_2, r_\circledast)$ such that $\bar{r}''_! \equiv \bar{r}''(r_!) = 0$. It can be verified that this zero of $\bar{r}''$ in $(\alpha_2, r_\circledast)$ is unique. An analysis of formula (57) yields the bounds

$$m < r_! < \frac{11}{10}m.$$

The initial data for Eq. (56) is given by

$$\omega(r_*) = \omega_* \tag{58a}$$

$$\grave{\omega}_* \equiv \grave{\omega}(r_*) = \lim_{\bar{r}\to 0} \frac{\omega'}{\bar{r}'} = \lim_{\bar{r}\to 0} \frac{\omega''}{\bar{r}''} = \frac{\omega''_*}{\bar{r}''_*}, \tag{58b}$$

where

$$\omega''_* = \zeta + (\beta^2 + \chi_*)\omega_*, \qquad \bar{r}''_* = \frac{\beta^2}{2\bar{r}_*^2}(r_* - \alpha_1)(r_* - \alpha_2)(r_* - \alpha_3).$$

Using that $\bar{r}''_* < 0$, and noticing that $\omega''_* > 0$ if $r_* \geq \sqrt{\frac{3}{2}}m$ one concludes that

$$\grave{\omega}_* < 0, \quad \text{if } r_* \in \left(\sqrt{\frac{3}{2}}m, r_\circledast\right).$$

Alternatively, one can compute $\grave{\omega}_*$ directly by evaluating Eq. (56) on r_*. Similarly, differentiating (56) with respect to $\bar{r}$ and evaluating on r_* one finds that

$$\grave{\grave{\omega}}_* < 0, \qquad r_* \in (r_+, r_\circledast).$$

In particular, in the extremal case one has the following expressions:

$$\grave{\omega}_* = \frac{(3m^2 - 2r_*^2)}{(3m - 2r_*)(m - r_*)^2}, \qquad \grave{\grave{\omega}}_* = \frac{2m(r_* - 2m)}{r_*(3m - 2r_*)(m - r_*)^2}. \tag{59}$$

Notice, in particular, that both $\grave{\omega}_*$, $\grave{\grave{\omega}}_* \to \infty$ as $r_* \to r_\circledast = \frac{3}{2}m$.

Analysis of $\grave{\grave{\omega}}$. As it will be seen in the sequel, the proof of Proposition 3 requires a knowledge of the sign of $\grave{\grave{\omega}}$. In order to analyse this, it is convenient to consider a *first integral* of Eq. (56). Multiplying (56) by $\grave{\omega}$ one readily obtains that

$$\frac{1}{2}\bar{r}'^2(\grave{\omega}^2)^{\grave{}} + \bar{r}''\grave{\omega}^2 - \frac{1}{2}(\beta^2 + \bar{\chi})(\omega^2)^{\grave{}} = \zeta\grave{\omega}.$$

Integrating with respect to $\bar{r} \in [r_+, r_*]$ leads to

$$\frac{1}{2}\int_{\bar{r}}^{r_*} \bar{r}'^2(\grave{\omega}^2)^{\grave{}}\mathrm{d}s + \int_{\bar{r}}^{r_*} \bar{r}''\grave{\omega}^2\mathrm{d}s - \frac{1}{2}\int_{\bar{r}}^{r_*}(\beta^2 + \bar{\chi})(\omega^2)^{\grave{}}\mathrm{d}s = \zeta\int_{\bar{r}}^{r_*}\grave{\omega}\,\mathrm{d}s.$$

Integration by parts in the first and third terms yields

$$\frac{1}{2}\bar{r}'^2\grave{\omega}^2\Big|_{\bar{r}}^{r_*} - \frac{1}{2}\int_{\bar{r}}^{r_*}(\bar{r}'^2)^{\grave{}}\grave{\omega}^2\mathrm{d}s + \int_{\bar{r}}^{r_*}\bar{r}''\grave{\omega}^2\mathrm{d}s - \frac{1}{2}(\beta^2 + \bar{\chi})\omega^2\Big|_{\bar{r}}^{r_*}$$
$$+\frac{1}{2}\int_{\bar{r}}^{r_*}\grave{\chi}\,\omega^2\mathrm{d}s = \zeta(\omega_* - \omega).$$

Recalling that $\bar{r}'^2_* = 0$ and that $\bar{r}'' = \frac{1}{2}(\bar{r}'^2)\grave{}$, this last expression reduces to

$$-\frac{1}{2}\bar{r}'^2 \grave{\omega}^{\,2} + \frac{1}{2}(\beta^2 + \bar{\chi})\omega^2 - \frac{1}{2}(\beta^2 + \bar{\chi}_*)\omega_*^2 + \frac{1}{2}\int_{\bar{r}}^{r_*} \grave{\chi}\, \omega^2 \mathrm{d}s = \zeta(\omega_* - \omega). \quad (60)$$

This equation will be used, in the sequel to prove the following result:

Lemma 2. *The solution, ω, of Eq. (56) with initial data given by (58a) and (58b), $r_* \in (r_+, r_\circledast)$ satisfies $\grave{\grave{\omega}} < \grave{\grave{\omega}}_*$ for $r \in (\alpha_2, r_*)$.*

Proof. The proof proceeds by contradiction. Hence, assume that there exists $\bar{r} = r_\lightning$ such that $r_* \neq r_\lightning \neq r_!$ and $\grave{\grave{\omega}}_\lightning \equiv \grave{\grave{\omega}}\,(r_\lightning) = \grave{\grave{\omega}}_*$. Equation (56) implies that

$$(\grave{\omega}_\lightning)^2 = \frac{1}{(\bar{r}''_\lightning)^2}(\zeta + (\beta^2 + \chi_\lightning)\omega_\lightning - \bar{r}'^2_\lightning\, \grave{\grave{\omega}}_*)^2.$$

Substituting this expression for $(\grave{\omega}_\lightning)^2$ into the first integral (60) with $\bar{r} = r_\lightning$ and grouping terms one obtains

$$a_2\omega_\lightning^2 + a_1\omega_\lightning + a_0 = (\bar{r}''_\lightning)^2 \int_{r_\lightning}^{r_*} \grave{\chi}\, \omega^2 \mathrm{d}s, \quad (61)$$

with

$$\begin{aligned}
a_2 &\equiv (\beta^2 + \bar{\chi}_\lightning)\left((\beta^2 + \bar{\chi}_\lightning)\bar{r}'^2_\lightning - (\bar{r}''_\lightning)^2\right),\\
a_1 &\equiv \left(2(\beta^2 + \bar{\chi}_\lightning)\bar{r}'^4_\lightning\, \grave{\grave{\omega}}_* + 2\zeta(\beta^2 + \bar{\chi}_\lightning)\bar{r}'^2_\lightning - 2\zeta(\bar{r}''_\lightning)^2\right),\\
a_0 &\equiv \left((\bar{r}'^2_\lightning - 2\zeta)\bar{r}'^4_\lightning\, \grave{\grave{\omega}}_\lightning + \bar{r}'^2_\lightning\zeta^2 + 2\zeta(\bar{r}'')^2_\lightning\omega_* + (\bar{r}'')^2_\lightning(\beta^2 + \bar{\chi}_*)\omega_*^2\right).
\end{aligned}$$

The coefficients a_0, a_1, a_0 are explicitly known rational expressions of $r_\lightning$. Now, as $\grave{\chi} > 0$ for $\bar{r} \in [\alpha_2, r_*]$, it follows from Eq. (61) that

$$a_2\omega_\lightning^2 + a_1\omega_\lightning + a_0 > 0, \qquad r_\lightning \neq r_!. \quad (62)$$

A lengthy computation using the explicit expressions for a_2, a_1 and a_0 in terms of $\bar{r}$ shows that $a_2 < 0$ and $a_1^2 - 4a_2a_0 < 0$ for $r_\lightning \in (\alpha_2, r_*)$ so that the polynomial $f(x) = a_2x^2 + a_1x + a_0$ is negative for $x \in \mathbb{R}$. This is a contradiction with (62). Thus, assuming that $r_\lightning \neq r_*,\ r_!$, there is no value of $\bar{r}$ for which $\grave{\grave{\omega}} \geq \grave{\grave{\omega}}_*$. The possibility $r_\lightning = r_!$ can be excluded by continuity. □

In what follows, we restrict our attention to curves such that $r_* \in (\sqrt{\frac{3}{2}}m, r_\circledast)$ so that $\grave{\omega}_* < 0$—cf. Eq. (59). Consistent with Lemma 2 it is assumed that ω has a local maximum in (r_+, r_*)—otherwise one directly has that $\omega_+ > 0$ and there is nothing to prove. Denote by $r_\wedge$ the location of such local extremum and write $\omega_\wedge \equiv \omega(r_\wedge)$. A lower bound for $\omega_\wedge$ can be obtained from the evaluation of Eq. (56) at $\bar{r} = r_!$. As $\bar{r}''_! = 0$, using Lemma 2 one concludes that

$$\bar{r}'^2_!\, \grave{\grave{\omega}}_! = \zeta + (\beta^2 + \chi_!)\omega_! \leq \bar{r}'^2_!\, \grave{\grave{\omega}}_* < 0.$$

It follows then that

$$\frac{\bar{r}_!'^2 \grave{\grave{\omega}}_* - \zeta}{\beta^2 + \chi_!} \leq \omega_! \leq \omega_\wedge. \tag{63}$$

It can be readily verified that $\beta^2 + \chi_!$ is negative so that the lower bound of $\omega_\wedge$ given by the inequality (63) is positive. Important for the sequel is the following observation: combining inequality (63) with the value for $\grave{\grave{\omega}}_*$ given in (59) it follows that

$$\omega_\wedge \to \infty \qquad \text{as} \qquad r_* \to r_\circledast.$$

The subsequent analysis will also require of an upper bound for $r_\wedge$. Such a bound can be more easily obtained by considering Eq. (49). As already discussed, for $r_* \in (\sqrt{\frac{3}{2}}m, r_\circledast)$ one has that $\omega_*'' > 0$. As ω is assumed to have a maximum, it follows that there must exist an inflexion point at which $\omega'' = 0$. At this inflexion point equation (49) implies $-(\beta^2 + \bar{\chi})\omega = \zeta$. This last equality, together with the observation that at this point $\omega \geq \omega_*$, leads to

$$-\frac{\zeta}{\beta^2 + \bar{\chi}} \geq \omega_*. \tag{64}$$

The above inequality can be regarded as a condition on $\bar{r}$ as $\zeta > 0$ for the range of r_* under consideration—cf. (53). Notice, in particular, that $\beta^2 + \bar{\chi} > 0$ in order for (64) to make sense as $\omega_* > 1$. Some inspection shows that for $r_* \in (\sqrt{\frac{3}{2}}m, r_\circledast)$

$$r_\wedge < r_\dagger \equiv \frac{13}{10}m.$$

A final observation concerning the maximum of ω is the following: evaluating Eq. (56) at the maximum one obtains after some rearranging that

$$\frac{\bar{r}_\wedge'^2 \grave{\grave{\omega}}_\wedge - \zeta}{\beta^2 + \bar{\chi}_\wedge} = \omega_\wedge > 0.$$

However, $\bar{r}_\wedge'^2 \grave{\grave{\omega}}_\wedge - \zeta < 0$ so that necessarily $\beta^2 + \bar{\chi}_\wedge < 0$.

Estimating ω_+**.** In what follows, assume that $r_* \in (\sqrt{\frac{3}{2}}m, r_\circledast)$. Moreover, suppose that $r_+ < r_\wedge$—otherwise, as a consequence of Lemma 2 one has that $\omega_+ \neq 0$ and the result of Proposition 3 follows directly. In order to estimate ω_+ we exploit the information on the location and the size of the maximum of ω acquired in the previous section.

Clearly, $\omega > 0$ for $\bar{r}$ sufficiently close to $r_\wedge$. We make use of a bootstrap argument to show that the interval where $\omega > 0$ can be extended to include r_+. A calculation similar to the one leading to Eq. (60) yields

$$\bar{r}'^2 \grave{\omega}^2 = 2\zeta(\omega - \omega_\wedge) + (\beta^2 + \bar{\chi})\omega^2 - (\beta^2 + \bar{\chi}_\wedge)\omega_\wedge^2 + \int_{\bar{r}}^{r_\wedge} \grave{\chi}\, \omega^2 \mathrm{d}s, \qquad \bar{r} \in [r_+, r_\wedge] \tag{65}$$

where, in particular, it has been used that $\grave{\omega}_\wedge = 0$. One needs to estimate various terms in this expression. To this end it is noticed that $r_+ \leq \bar{r} \leq r_\wedge < r_\dagger$, so that using formula (43) it follows that

$$C_1^2 \grave{\omega}^2 \leq \bar{r}'^2 \grave{\omega}^2 \quad \text{for } \bar{r} \in [r_+, r_\wedge], \tag{66}$$

where

$$C_1^2 \equiv \frac{\beta^2}{r_\dagger^2}(r_+ - \alpha_1)(r_+ - \alpha_2)(r_* - r_\dagger)(\alpha_3 - r_\dagger) > 0.$$

Moreover, it can be explicitly verified that $\grave{\grave{\chi}} < 0$ so that

$$\int_{\bar{r}}^{r_\wedge} \grave{\chi}\, \omega^2 \mathrm{d}s < \grave{\chi}_+\, \omega_\wedge^2 (r_\wedge - r_+) \quad \text{as long as } \omega > 0. \tag{67}$$

Finally, one has that

$$\beta^2 + \bar{\chi} < 0 \quad \text{for } \bar{r} \in [r_+, r_\wedge]. \tag{68}$$

Making use of inequalities (66)–(68) in Eq. (65) one concludes that

$$C_1^2 \grave{\omega}^2 < 2\zeta(\omega - \omega_\wedge) - (\beta^2 + \bar{\chi}_\wedge)\omega_\wedge^2 + \grave{\chi}_+\, \omega_\wedge^2 (r_\wedge - r_+),$$
$$\text{on } [\bar{r}, r_\wedge] \quad \text{as long as} \quad \omega > 0.$$

For the convenience of the presentation let

$$C_2 \equiv \grave{\chi}_+\, \omega_\wedge^2 (r_\wedge - r_+) - (\beta^2 + \bar{\chi}_\wedge)\omega_\wedge^2 - 2\zeta\omega_\wedge,$$

so that

$$0 < C_1^2 \grave{\omega}^2 \leq 2\zeta\omega + C_2. \tag{69}$$

As $\grave{\omega} > 0$ on $[r_+, r_\wedge]$, one can consider the positive square root of inequality (69) and then integrate over $[\bar{r}, r_\wedge]$ to obtain

$$\sqrt{2\zeta\omega_\wedge + C_2} + \frac{\zeta}{C_1}(\bar{r} - r_\wedge) < \sqrt{2\zeta\omega + C_2}. \tag{70}$$

The second term of the left-hand side of this last inequality is negative as $\bar{r} < r_\wedge$. However, in view of the second equation in (59) and the bound (63) it is possible to ensure that the left-hand side is positive by choosing r_* sufficiently close to $r_\circledast$—that is, there exists $r_\star \in (\sqrt{\frac{3}{2}}m, r_\circledast)$ such that if $r_* \in (r_\star, r_\circledast)$ then

$$0 < \sqrt{2\zeta\omega_\wedge + C_2} + \frac{\zeta}{C_1}(r_+ - r_\wedge) \leq \sqrt{2\zeta\omega_\wedge + C_2} + \frac{\zeta}{C_1}(\bar{r} - r_\wedge).$$

Crucially, it can be verified that C_1 remains finite and non-zero as $r_* \to r_\circledast$. Squaring inequality (70) and simplifying one obtains the lower bound

$$\omega_\wedge + \frac{\zeta}{2C_1^2}(r_+ - r_\wedge)^2 + \frac{1}{C_1}(r_+ - r_\wedge)\sqrt{2\zeta\omega_\wedge + C_2} < \omega.$$

Using the definition of C_2, this inequality can be rewritten as

$$C_3\omega_\wedge + \frac{\zeta}{2C_1^2}(r_+ - r_\wedge)^2 < \omega, \tag{71}$$

where

$$C_3 \equiv 1 + \frac{1}{C_1}(r_+ - r_\wedge)\sqrt{\grave{\bar{\chi}}_+ \,(r_\wedge - r_+) - (\beta^2 + \bar{\chi}_\wedge)}.$$

A lengthy direct computation using the information available about the various terms in this expression shows that $C_3 > 0$ for $r_* \in (r_\star, r_\circledast)$. As $\zeta > 0$ for the range of r_* under consideration it follows from inequality (71) that $\omega > C_4 > 0$ where C_4 is independent of $\bar{r} \in [r_+, r_\circledast]$—at least for curves with r_* close enough to $r_\circledast$. In particular, one has that $\omega_+ > 0$. This proves Proposition 3.

6.5. Some Remarks

Although the analysis of the extremal Reissner–Nordström spacetime has been restricted to the case of conformal curves which start suitably close to the critical curve, numerical computations show that, in fact, no conjugate points appear in the congruence up to (and including) the horizon.

In the present discussion, no attempt has been made to analyse the behaviour of the congruence after it crosses the horizon. However, numerical evaluations of Eq. (49) show that the scalar ω goes to zero shortly after the curve has crossed the horizon, and certainly, before it reaches the turning point $\bar{r} = \alpha_2$. In any case, one knows there exists an open neighbourhood after the horizon where the congruence remains non-degenerate.

7. Conclusions

The analysis carried out in Sects. 4, 5 and 6 allows to provide the following technical version of our main Theorem:

Theorem 1. *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a Reissner–Nordström spacetime with $q^2 \leq m^2$ and let $r_\circledast$ as defined by Eq. (42). On $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ consider the congruence of timelike conformal curves defined by the initial conditions (33) and $r_* \in (r_+, \infty)$ on the time symmetric slice of the domain of outer communication. Let $\bar{\tau}$ and τ denote, respectively, the physical and conformal proper time of the curves of the congruence. For this congruence one has that:*

(a) *Each curve of this congruence exists for $\bar{\tau} \in \mathbb{R}$. Moreover:*
  - (i) *the curves with $r_* \in (r_\circledast, \infty)$ reach null infinity in an infinite amount of physical proper time but in a finite amount of conformal proper time;*
  - (ii) *the curves with $r_* = r_\circledast$ reach past and future timelike infinity in an infinite amount of physical proper time but a finite amount of conformal proper time;*
  - (iii) *the curves with $r_* \in (r_+, r_\circledast)$ reach the event horizon in a finite amount of both physical and conformal proper time.*

(b) *In addition one has that:*
  (i) *If* $q^2 \leq \frac{8}{9}m^2$ *then the congruence is free of conjugate points in the domain of outer communication.*
  (ii) *In the extremal case* $q^2 = m^2$*, there exists* $r_\star \in (r_+, r_\circledast)$ *such that the subcongruence defined by* $r_* \in (r_\star, \infty)$ *is free of conjugate points in the domain of outer communication.*

As already indicated in the main text, numerical evaluations of the congruence suggest that it should be possible to improve Theorem 1 so as to ensure that the congruence of conformal curves is free of conjugate points in the domain of outer communication for $q^2 \leq m^2$.

The analysis of this article a first step in the study of the Reissner–Nordström spacetime as a solution of the conformal field equations. In view of this programme, the results of Sect. 5.5 are specially relevant as they suggest that the conformal structure of the timelike infinity, i^+, of the extremal Reissner–Nordström spacetime may be more tractable, from an analytic point of view, than that of the non-extremal case.

Regarding the Reissner–Nordström spacetime as a spherically symmetric model of the Kerr spacetime, it is natural to wonder how much of the structure observed in the present analysis has an analogue in the Kerr solution. For example, it is natural to conjecture that the domain of outer communication of the Kerr spacetime can be covered by means of a non-singular congruence of conformal geodesics reaching beyond null infinity. It is very likely that this congruence will degenerate after it has crossed the event horizon and that the curves will have some type of singularity avoiding properties so that there may exist regions in the black hole region which cannot be probed in this way. A more tantalising possibility is that, as in the case of the extremal Reissner–Nordström spacetime, the extreme Kerr may have a more tractable structure at i^+. In any case, the analysis of conformal geodesics in the Kerr spacetime is bound to be much more complicated as the warped product structure of the line element is lost.

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Appendix A. Conformal Geodesics in the Schwarzschild Spacetime

For completeness, we include a study of the solutions to the conformal curve equations in the case of the Schwarzschild spacetime (where $q = 0$). In this case the conformal curves are, in fact, conformal geodesics. The analysis of these curves was originally done in [13] using explicit solutions in terms of elliptic functions. The discussion given here follows the strategy of Sect. 5 in the main text, and avoids the use of explicit solutions.

As in the case of the main text, essential for our analysis is the factorisation of the polynomial appearing in Eq. (40). If $q = 0$, then $P(\bar{r})$ is of degree 3 and one has the factorisation

$$P(\bar{r}) = \beta^2(\bar{r} - r_*)(\bar{r} - \alpha)(\bar{r} + \alpha), \tag{72}$$

where

$$\alpha \equiv \sqrt{\frac{mr_*}{2D_*}}.$$

The constant solution to Eq. (40) can be found to be given by $\bar{r} = r_\circledast = \frac{5}{2}m$. As discussed in [13], these curves reach the point i^+ in a finite amount of unphysical proper time, and divide the two possible regimes for the conformal geodesics.

A.1. Conformal Geodesics with $r_* > r_\circledast$

If $r_\circledast < r_*$, then the analysis of the curves is covered by the discussion in Sect. 5.2 by setting $q = 0$. These conformal geodesics reach null infinity.

A.2. Conformal Geodesics with $r_* < r_\circledast$

If $r_* < r_\circledast$, then one can readily verify that $\bar{r}''_* < 0$, so that $\bar{\tau} = 0$ is a maximum of the function $\bar{r}$ as one has that $\bar{r}'_* = 0$. Thus, $\bar{r}$ is initially decreasing. A computation shows that the following chain of inequalities hold:

$$-\alpha < 0 < 2m < r_* < \alpha \leq r_\circledast. \tag{73}$$

Thus, the curve must reach the singularity ($r = 0$) before it can reach the turning point at $\bar{r} = -\alpha$. It only remains to be seen whether the conformal geodesic reaches the singularity in finite amount of proper time.

A computation using the factorisation (72) shows that $\bar{r}'' = 0$ implies the condition

$$\bar{r}(\bar{r}-\alpha)(\bar{r}+\alpha)+\bar{r}(\bar{r}-r_*)(\bar{r}+\alpha)+\bar{r}(\bar{r}-r_*)(\bar{r}-\alpha) = 2(\bar{r}-r_*)(\bar{r}-\alpha)(\bar{r}+\alpha).$$

where it has been assumed that $\bar{r} \neq 0$. A further rearrangement yields

$$2\bar{r}^2(\bar{r} - r_*) = (\bar{r}^2 - \alpha^2)(\bar{r} - 2r_*). \tag{74}$$

Using the chain of inequalities in (73) one concludes that

$$(\bar{r}^2 - \alpha^2) < 0, \quad \bar{r} - 2r_* < 0, \quad \bar{r} - r_* < 0.$$

Thus, the left-hand side of condition (74) is negative, while the right-hand side is positive. This shows that there are no values of $\bar{r} < r_\circledast$ for which $\bar{r}'' = 0$. Hence, one concludes that the function $\bar{r}$ reaches the value $\bar{r} = 0$ in a finite value of $\bar{\tau}$—that is the conformal curves under consideration hit the singularity in a finite amount of proper time. Moreover, it is noticed that $\bar{r}' \to \infty$ as $\bar{r} \to 0$—cf. Eq. (49).

Finally, the conformal geodesic starting at the bifurcation sphere ($r_* = 2m$) is covered by the analysis of Sect. 5.4, by setting $q = 0$. One finds the explicit solution

$$\bar{r} = m(1 + \cos\bar{\tau}).$$

This conformal geodesic reaches the singularity at $\bar{\tau} = \pi$.

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Christian Lübbe
Department of Mathematics
University College London
Gower Street
London WC1E 6BT, UK

and

School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS, UK
e-mail: c.luebbe@qmul.ac.uk

Juan Antonio Valiente Kroon
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS, UK
e-mail: j.a.valiente-kroon@qmul.ac.uk

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