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A New Asymptotic Perturbation Theory with Applications to Models of Massless Quantum Fields

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Abstract. Let H_0 and H_{I} be a self-adjoint and a symmetric operator on a complex Hilbert space, respectively, and suppose that H_0 is bounded below and the infimum E_0 of the spectrum of H_0 is a simple eigenvalue of H_0 which is *not necessarily isolated*. In this paper, we present a new asymptotic perturbation theory for an eigenvalue $E(\lambda)$ of the operator $H(\lambda) := H_0 + \lambda H_{\mathrm{I}}$ ($\lambda \in \mathbb{R}\backslash\{0\}$) satisfying $\lim_{\lambda \to 0} E(\lambda) = E_0$. The point of the theory is in that it covers also the case where E_0 is a non-isolated eigenvalue of H_0. Under a suitable set of assumptions, we derive an asymptotic expansion of $E(\lambda)$ up to an arbitrary finite order of λ as $\lambda \to 0$. We apply the abstract results to a model of massless quantum fields, called the generalized spin-boson model (Arai and Hirokawa in J Funct Anal 151:455–503, 1997) and show that the ground-state energy of the model has asymptotic expansions in the coupling constant λ as $\lambda \to 0$.

1. Introduction

As is well known, in quantum mechanics, a perturbation theory is useful to calculate approximately various quantities (e.g., energy levels of an atom) under the condition that the relevant perturbation is "small". In the standard analytic perturbation theory ([18, Chapter 7], [21, §XII.2]) and the standard asymptotic perturbation theory ([18, Chapter 8], [21, §XII.3]), however, only perturbations of an *isolated* eigenvalue are considered. But the perturbation problem of a *non-isolated* eigenvalue (typically an embedded eigenvalue) naturally appears in many-body quantum systems and models of *massless* quantum fields. For the former, dilation analytic methods (e.g., [21, §XII.6]) have been shown to be useful. For the latter, Bach et al. [10,11]) have developed a method exploring a renormalization group idea combined with the Feshbach map.

Recently, analyticity in the coupling constant for the ground-state energy[1] for some concrete models of massless quantum fields (in which the ground-state energy of the unperturbed system under consideration is an embedded eigenvalue) has been proved [1,2,14,15]. These results are very nice, but, they may be model-dependent. From a general mathematical point of view, it would be valuable to investigate to what extent it is possible to develop a model-independent general perturbation theory for non-isolated eigenvalues in such a way that, at least, it can be applied to the perturbation of a non-isolated ground-state energy in models of massless quantum fields. To our best knowledge, such a general perturbation theory is still missing. With this motivation, we present in this paper, as a first step, an *asymptotic* perturbation theory which may have applications to models of a quantum system interacting with a *massless* quantum field. In the present paper, we do not discuss possible analytic extensions of our asymptotic perturbation theory. This aspect is left for future studies.

The outline of the present paper is as follows. In Sect. 2, we define an operator for which we give a perturbation theoretical consideration and derive some facts which are bases for the asymptotic perturbation theory developed in this paper. The operator is of the following form:

$$H(\lambda) := H_0 + \lambda H_{\mathrm{I}}, \tag{1.1}$$

where H_0 (resp. H_{I}) is a symmetric (resp. linear) operator acting on a complex Hilbert space $\mathcal{H}$ and $\lambda \in \mathbb{R}\backslash\{0\}$ denotes the coupling constant (perturbation parameter). We are mainly interested in the case where H_0 has a simple eigenvalue E_0 which is *not isolated*. A basic new idea which allows one to treat such a case comes from the so-called Brillouin–Wigner perturbation theory [12,22–24], which, on an informal (heuristic) level, may be more elaborate than the Rayleigh–Schrödinger perturbation theory. We use the Brillouin–Wigner perturbation theory in a "non-perturbative" way: We derive a simultaneous (non-perturbative, closed) equation for an eigenvalue E and an eigenvector Ψ of $H(\lambda)$ $(H(\lambda)\Psi = E\Psi)$ under a condition for E (Proposition 2.1). This equation is one of the basic starting points in our analysis. Another basic fact we derive in Sect. 2 is an upper bound for the infimum $\mathcal{E}_0(\lambda)$ of the numerical range of $H(\lambda)$ in the case where H_0 is self-adjoint, bounded below and H_I is symmetric (Theorem 2.7). This aspect may have an independent interest. As a corollary to this fact, a sufficient condition for $\mathcal{E}_0(\lambda)$ to be in the resolvent set of H_0 is given (Corollary 2.11).

In Sect. 3, under additional assumptions, we derive an asymptotic expansion for the eigenvalue $E = E(\lambda)$ of $H(\lambda)$ up to the second order of λ as $\lambda \to 0$ (Theorem 3.5).

In Sect. 4, for each natural number $N \geq 2$, under stronger assumptions, we derive an asymptotic expansion for $E(\lambda)$ up to the Nth order of λ as $\lambda \to 0$ (Theorem 4.1).

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[1] For a mathematical definition, see Remark 2.4.

The last section is devoted to an application of Theorem 4.1 to the generalized spin-boson (GSB) model [5], which describes an abstract quantum system interacting with a Bose field. We first show that the infimum of the spectrum of the Hamiltonian of the GSB model is an even function of the coupling constant λ (Theorem 5.1). Then we prove that, under a set of suitable assumptions which, in the case where the Bose field is massless, requires a stronger infrared regular condition for the momentum cutoff functions of the Bose field in the interaction part, the ground-state energy of the Hamiltonian of the GSB model has an asymptotic expansion in the coupling constant λ up to an arbitrary finite order of λ with all the coefficients of odd powers of λ vanishing (Theorem 5.17).

In concluding this introduction, we mention work related to the present paper and remark in what sense the asymptotic perturbation theory presented in this paper may be novel. Hainzl and Seiringer [16] considered the Pauli–Fierz model (with spin) in non-relativistic quantum electrodynamics, a model of a non-relativistic charged particle interacting with the quantum radiation field, and derived an asymptotic expansion for the ground-state energy of the fiber Hamiltonian of the model with no external potentials as well as the one for the binding energy of the Hamiltonian with an external potential up to the first order of the coupling constant $\alpha > 0$ physically denoting the fine structure constant. Their methods, which are based on the variational principle and operator inequalities special to the model, are different from those in the present paper and seem to be difficult to extend to obtain higher order asymptotics. We note that, as for the Hamiltonians without the A^2 term (A denotes the quantum radiation field), applying Theorem 3.5 in the present paper, we can obtain results similar to those in [16] (note that A^2 term does not contribute to the energy level shift in the leading order of α). On the other hand, Bach et al. [8,9] studied the Pauli–Fierz model (without spin) and gave an iteration method to obtain an "asymptotic-like" expansion for the ground-state energy up to any finite order of α, where "asymptotic-like" means that the coefficients of the expansion depend on α in a certain manner (hence it is not an asymptotic expansion in the strict sense of the term). This is due to that the dependence of the Hamiltonian H on α takes the unusual form $\alpha^{3/2}A(\alpha x)$ with space variable $x \in \mathbb{R}^3$. As is just suggested, H does not have the form of $H(\lambda)$ given by (1.1) with $\lambda = \lambda(\alpha)$ being a function of α such that $\lim_{\alpha \to 0} \lambda(\alpha) = 0$ (even in the case without A^2 term). In addition, the methods used in [8,9] are different from those in the present paper. As the outline of the present paper described above may show, the asymptotic perturbation theory developed in the present paper is *systematic* and *general*, establishing an asymptotic expansion for the ground-state energy of $H(\lambda)$ (in the case where H_0 and $H(\lambda)$ are self-adjoint) up to *any finite order* in the coupling constant λ, even in the case where E_0 is not an isolated eigenvalue of H_0. In the present paper, we demonstrate only one application to a massless quantum field. But the theory may have a wide range of applications to (massless) quantum field models (the author plans to study applications in subsequent papers). At least, in all these senses, the asymptotic perturbation theory may be new.

2. Basic Facts

We denote the inner product and the norm of the Hilbert space $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$ (complex linear in the second variable) and $\| \cdot \|$, respectively. For a linear operator A on a Hilbert space, we use the following notation:

(i) $D(A)$: The domain of A.
(ii) $\ker A$: The kernel of A.
(iii) $\rho(A)$: The resolvent set of A.
(iv) $\sigma(A) := \mathbb{C} \backslash \rho(A)$: The spectrum of A.
(v) $\sigma_p(A)$: The point spectrum of A (the set of eigenvalues of A).

For linear operators $A_1, \ldots, A_n$ $(n \geq 2)$, $D(\sum_{j=1}^n A_j) := \cap_{j=1}^n D(A_j)$ and

$$D(A_n \ldots A_1) := \{\psi \in D(A_1) | A_{j-1} \ldots A_1 \psi \in D(A_j), j = 2, \ldots, n\}$$

as usual.

In this section, we derive some fundamental facts on the operator $H(\lambda)$ defined by (1.1). As already mentioned in Sect. 1, one of the basic hypotheses in our theory is the following:

(H.1). The operator H_0 has a simple eigenvalue E_0.

Since H_0 is symmetric, E_0 is a real number. We fix a normalized eigenvector Ψ_0 of H_0 with eigenvalue E_0:

$$H_0 \Psi_0 = E_0 \Psi_0, \quad \|\Psi_0\| = 1. \tag{2.1}$$

Let P_0 be the orthogonal projection onto the eigenspace

$$\mathcal{H}_0 := \{\alpha \Psi_0 | \alpha \in \mathbb{C}\} \tag{2.2}$$

of H_0 with eigenvalue E_0. Then

$$Q_0 := I - P_0 \tag{2.3}$$

is the orthogonal projection onto the orthogonal complement $\mathcal{H}_0^\perp$ of $\mathcal{H}_0$. Since H_0 is symmetric, H_0 is reduced by both $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$, i.e., $P_0 H_0 \subset H_0 P_0$. We denote by H_0' the reduced part of H_0 to $\mathcal{H}_0^\perp$.

2.1. Simultaneous Equation for an Eigenvalue and an Eigenvector of $H(\lambda)$

We say that two vectors Ψ and Φ in $\mathcal{H}$ overlap (or Ψ overlaps with Φ) if $\langle \Psi, \Phi \rangle \neq 0$. Also we say that a vector $\Psi \in \mathcal{H}$ overlaps with a non-empty subset $\mathcal{D} \subset \mathcal{H}$ if there exists a vector $\Phi \in \mathcal{D}$ which overlaps with Ψ.

The following proposition is a rigorous non-perturbative formulation of ideas behind the Brillouin–Wigner perturbation method [12,22,23] (cf. [24, §3.1]):

Proposition 2.1. *Let E_0, Ψ_0 and Q_0 be as above, and $\lambda \in \mathbb{R} \backslash \{0\}$ be fixed. Let E be a complex number with $E \notin \sigma_p(H_0')$ (hence the inverse $(E - H_0')^{-1}$ of $E - H_0'$ exists on $\mathcal{H}_0^\perp$, being not necessarily bounded).*

(i) *If E is an eigenvalue of $H(\lambda)$ [i.e., $E \in \sigma_p(H(\lambda))$] and Ψ_0 overlaps with the eigenspace $\ker(H(\lambda) - E)$ of $H(\lambda)$ with eigenvalue E, then there exists* *a non-zero vector* $\Psi \in \ker(H(\lambda) - E)$ *such that* $Q_0 H_{\mathrm{I}} \Psi \in D((E - H_0')^{-1})$ *and*

$$E = E_0 + \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi \rangle, \tag{2.4}$$

$$\Psi = \Psi_0 + \lambda (E - H_0')^{-1} Q_0 H_{\mathrm{I}} \Psi. \tag{2.5}$$

(ii) *Conversely, if* E *and* Ψ *satisfy* (2.4) *and* (2.5)*, then* E *is an eigenvalue of* $H(\lambda)$ *and* Ψ *is an eigenvector of* $H(\lambda)$ *with eigenvalue* E *(i.e.,* $\Psi \in \ker(H(\lambda) - E) \backslash \{0\}$*), overlapping with* Ψ_0*.*

Proof. (i) By the present assumption, there exists a vector Ξ in $\ker(H(\lambda) - E)$ such that $\langle \Psi_0, \Xi \rangle \neq 0$. We have $\langle \Psi_0, H(\lambda)\Xi \rangle = E \langle \Psi_0, \Xi \rangle$. The left hand side is equal to $E_0 \langle \Psi_0, \Xi \rangle + \lambda \langle \Psi_0, H_{\mathrm{I}} \Xi \rangle$. Hence $E \langle \Psi_0, \Xi \rangle = E_0 \langle \Psi_0, \Xi \rangle + \lambda \langle \Psi_0, H_{\mathrm{I}} \Xi \rangle$. Since $\langle \Psi_0, \Xi \rangle \neq 0$, we obtain (2.4) with $\Psi := \Xi / \langle \Psi_0, \Xi \rangle$.

To prove (2.5), we note that $\langle \Psi_0, \Psi \rangle = 1$. Hence by the projection theorem, there exists a unique vector $\Phi \in \mathcal{H}_0^{\perp}$ such that

$$\Psi = \Psi_0 + \Phi. \tag{2.6}$$

Since both Ψ and Ψ_0 are in $D(H_0)$ (note that $\Psi \in D(H(\lambda)) = D(H_0) \cap D(H_{\mathrm{I}})$), Φ is in $D(H_0)$. Hence

$$\lambda H_{\mathrm{I}} \Psi = (E - H_0)\Psi = (E - E_0)\Psi_0 + (E - H_0)\Phi.$$

By the fact that $Q_0 \Psi_0 = 0$ and $Q_0(E - H_0)\Phi = (E - H_0')\Phi$, we have

$$\lambda Q_0 H_{\mathrm{I}} \Psi = (E - H_0')\Phi.$$

This means that $\lambda Q_0 H_{\mathrm{I}} \Psi$ is in $\mathrm{Ran}\,(E - H_0')$, the range of $E - H_0'$, which is equal to $D((E - H_0')^{-1})$, and $\Phi = \lambda (E - H_0')^{-1} Q_0 H_I \Psi$. Thus (2.5) holds.

(ii) Equation (2.5) implies that $\Psi \neq 0$ and

$$(E - H_0)\Psi = (E - E_0)\Psi_0 + \lambda Q_0 H_{\mathrm{I}} \Psi.$$

Hence

$$\begin{aligned}(E - H(\lambda))\Psi &= (E - E_0)\Psi_0 + \lambda Q_0 H_{\mathrm{I}} \Psi - \lambda H_{\mathrm{I}} \Psi = (E - E_0)\Psi_0 - \lambda P_0 H_{\mathrm{I}} \Psi \\ &= (E - E_0)\Psi_0 - \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi \rangle \Psi_0.\end{aligned}$$

But, by (2.4), the last vector is zero. Hence $\Psi \in \ker(H(\lambda) - E) \backslash \{0\}$. Taking the inner product of Ψ_0 with (2.5), we have $\langle \Psi_0, \Psi \rangle = 1$. Hence Ψ overlaps with Ψ_0. □

Remark 2.2. Proposition 2.1-(ii) may be used to establish an existence theorem of an eigenvalue of $H(\lambda)$. But, in the present paper, we do not go into considerations of this aspect.

Remark 2.3. Instead of $H(\lambda)$ given by (1.1), one can also consider a more general operator of the following form:

$$H(\lambda) = H_0 + \sum_{i=1}^{n} \lambda^i H_i$$

with $n \in \mathbb{N}$ arbitrary and H_i $(i = 1, \dots, n)$ being a linear operator on $\mathcal{H}$. It is obvious that, in this case, Proposition 2.1 holds with $\lambda H_{\rm I}$ replaced by $\sum_{i=1}^n \lambda^i H_i$. Hence for a theory which is developed based on Proposition 2.1, as is done below, it may be enough to consider the case $n = 1$. Results in the case $n = 1$ would easily be translated into the case $n \geq 2$ by the replacement of $\lambda H_{\rm I}$ with $\sum_{i=1}^n \lambda^i H_i$.

2.2. Upper Bound for the Infimum of the Numerical Range of $H(\lambda)$

In this subsection, in addition to (H.1), we assume the following:

(H.2). The operator H_0 is self-adjoint and $E_0 = \inf \sigma(H_0)$.

Remark 2.4. In general, if a self-adjoint operator H on a Hilbert space is bounded below, then

$$E_{\min}(H) := \inf \sigma(H)$$

is called the *lowest* or *minimal* energy of H. If $E_{\min}(H)$ is an eigenvalue of H, then H is said to have a ground state and $E_{\min}(H)$ is called the *ground-state energy* of H. In this case, a non-zero vector of $\ker(H - E_{\min}(H))$ is called a *ground state* of H. Assumptions (H.1) and (H.2) mean that H_0 has a unique ground state Ψ_0 (up to constant multiples) with E_0 being the ground-state energy.

Under assumptions (H.1) and (H.2), $H_0' - E_0$ is an injective and non-negative self-adjoint operator on $\mathcal{H}_0^\perp$. Hence the inverse operator $(H_0' - E_0)^{-1}$ exists on $\mathcal{H}_0^\perp$ and is a non-negative self-adjoint operator.

Remark 2.5. If E_0 is an isolated eigenvalue of H_0, then $(H_0' - E_0)^{-1}$ is bounded. This case is a situation where the standard analytic perturbation theory and the standard asymptotic perturbation theory are formulated. But the case where E_0 is not an isolated eigenvalue of H_0 is out of those perturbation theories. In addition, in this case, $(H_0' - E_0)^{-1}$ is *unbounded* and hence one has to be careful about domains of operators.

If $H_{\rm I}$ is symmetric, then, for all $\Psi \in D(H(\lambda))$, $\langle \Psi, H(\lambda)\Psi \rangle$ is a real number. Hence one can define

$$\mathcal{E}_0(\lambda) := \inf_{\Psi \in D(H(\lambda)), \|\Psi\|=1} \langle \Psi, H(\lambda)\Psi \rangle \quad (\lambda \in \mathbb{R}), \tag{2.7}$$

the infimum of the numerical range of $H(\lambda)$, which is a finite real number or $-\infty$, where, for $\lambda = 0$, we set $H(0) := H_0$ so that, by the variational principle

$$\mathcal{E}_0(0) = E_0. \tag{2.8}$$

Remark 2.6. If $H(\lambda)$ is self-adjoint and bounded below, then $\mathcal{E}_0(\lambda) = E_{\min}(H(\lambda))$ by the variational principle. But, in general, $\mathcal{E}_0(\lambda)$ is not necessarily an eigenvalue of $H(\lambda)$.

Theorem 2.7. *Assume* (H.1) *and* (H.2). *Suppose that $H_{\rm I}$ is symmetric and*

$$\Psi_0 \in D\left(H_{\rm I}(H_0' - E_0)^{-1} Q_0 H_{\rm I}\right). \tag{2.9}$$

Let

$$N_0 := \|(H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0\|^2, \tag{2.10}$$

$$a := \left\langle Q_0 H_{\rm I} \Psi_0, (H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0 \right\rangle, \tag{2.11}$$

$$b := \left\langle (H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0, H_{\rm I} (H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0 \right\rangle. \tag{2.12}$$

Then, for all $\lambda \in \mathbb{R}$*,*

$$\mathcal{E}_0(\lambda) \le E_0 + \frac{1}{1 + N_0 \lambda^2} \left(\langle \Psi_0, H_{\rm I} \Psi_0 \rangle \lambda - a\lambda^2 + b\lambda^3 \right). \tag{2.13}$$

Proof. By (2.7), we have for all $\Psi \in D(H(\lambda)) \backslash \{0\}$

$$\mathcal{E}_0(\lambda) \le \frac{\langle \Psi, H(\lambda)\Psi \rangle}{\|\Psi\|^2}. \tag{2.14}$$

As the vector Ψ, we take

$$\Psi_1 := \Psi_0 - \lambda (H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0.$$

Then, by (2.9), $\Psi_1 \in D(H(\lambda))$ and we have

$$\|\Psi_1\|^2 = 1 + N_0 \lambda^2, \tag{2.15}$$

where N_0 is defined by (2.10). Since Ψ_0 is orthogonal to $\mathrm{Ran}\,(Q_0)$ (the range of Q_0) and H_0 maps $D(H_0) \cap \mathrm{Ran}\,(Q_0)$ to $\mathrm{Ran}\,(Q_0)$, it follows that

$$\langle \Psi_1, H_0 \Psi_1 \rangle = E_0 + \lambda^2 \left\langle (H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0, H_0 (H_0' - E_0)^{-1} Q_0 H_{\rm I} \Psi_0 \right\rangle.$$

Using the identity

$$H_0 (H_0' - E_0)^{-1} Q_0 = Q_0 + E_0 (H_0' - E_0)^{-1} Q_0,$$

we obtain

$$\langle \Psi_1, H_0 \Psi_1 \rangle = E_0 \|\Psi_1\|^2 + a\lambda^2$$

with a defined by (2.11). Similarly, we have

$$\langle \Psi_1, H_{\rm I} \Psi_1 \rangle = \langle \Psi_0, H_{\rm I} \Psi_0 \rangle - 2a\lambda + b\lambda^2$$

with b given by (2.12). Hence we obtain

$$\langle \Psi_1, H(\lambda) \Psi_1 \rangle = E_0 \|\Psi_1\|^2 + \langle \Psi_0, H_{\rm I} \Psi_0 \rangle \lambda - a\lambda^2 + b\lambda^3.$$

Hence by (2.14) and (2.15), we obtain (2.13). □

Remark 2.8. Theorem 2.7 is applied in Lemma 5.10 in Sect. 5.

Remark 2.9. Since $(H_0' - E_0)^{-1}$ is non-negative as mentioned above, it follows that $a \ge 0$. It is easy to see that $a > 0$ if and only if $Q_0 H_{\rm I} \Psi_0 \neq 0$.

Remark 2.10. The choice of the trial vector Ψ_1 in the proof just made is motivated by (2.4) and (2.5). Heuristically Ψ_1 may be an "approximate" solution of (2.5) in the first order of λ. Indeed, by iterating (2.5), we obtain

$$\begin{aligned} \Psi = \Psi_0 &+ \sum_{n=1}^{N} \lambda^n \left((E - H_0')^{-1} Q_0 H_{\rm I} \right)^n \Psi_0 \\ &+ \lambda^{N+1} \left((E - H_0')^{-1} Q_0 H_{\rm I} \right)^{N+1} \Psi, \end{aligned} \tag{2.16}$$

provided that $\Psi_0 \in \cap_{n=1}^{N} D\left(\left((E_0 - H_0')^{-1} Q_0 H_{\rm I}\right)^n\right)$ $(N \in \mathbb{N})$ (then, by induction in N, one can show that $\Psi \in D\left(\left((E - H_0')^{-1} Q_0 H_{\rm I}\right)^{N+1}\right)$). Hence

$$\Psi_N := \Psi_0 + \sum_{n=1}^{N} \lambda^n \left((E_0 - H_0')^{-1} Q_0 H_{\rm I}\right)^n \Psi_0$$

may be an "approximate" solution of (2.5) up to the Nth order of λ. Using the vector Ψ_N $(N \geq 2)$ as a trial vector, one may derive an inequality better than (2.13). But, in this paper, we do not go into the details.

Corollary 2.11. *Under the same assumption as in Theorem 2.7, consider the case where*

$$|\langle \Psi_0, H_{\rm I} \Psi_0 \rangle| < |\lambda|(a - b\lambda). \tag{2.17}$$

Then

$$\mathcal{E}_0(\lambda) < E_0. \tag{2.18}$$

In particular, $\mathcal{E}_0(\lambda) \in \rho(H_0)$.

Proof. By condition (2.17), we have

$$\langle \Psi_0, H_{\rm I}\Psi_0 \rangle \lambda - a\lambda^2 + b\lambda^3 = \lambda \left\{ \langle \Psi_0, H_{\rm I}\Psi_0 \rangle - \lambda(a - b\lambda) \right\} < 0.$$

Hence (2.13) implies (2.18). By (H.2), $(-\infty, E_0) \subset \rho(H_0)$. Hence $\mathcal{E}_0(\lambda) \in \rho(H_0)$ for all λ obeying (2.17). □

Remark 2.12. If

$$\langle \Psi_0, H_{\rm I}\Psi_0 \rangle = 0 \tag{2.19}$$

and

$$\lambda \neq 0, \quad b\lambda < a, \tag{2.20}$$

then (2.17) holds and hence (2.18) too.

3. Asymptotic Expansion up to the Second Order in λ

Let E_0, Ψ_0 and Q_0 be as in Sect. 2. In this section, in addition to (H.1), we assume the following:

(H.3). (i) $H_{\rm I}$ is symmetric and $\Psi_0 \in D(H_0) \cap D(H_{\rm I})$.
(ii) There exists a constant $r > 0$ such that, for all λ in the set

$$\mathbb{I}_r^{\times} := (-r, 0) \cup (0, r),$$

$H(\lambda)$ has an eigenvalue $E(\lambda)$ such that $E(\lambda) \notin \sigma_p(H_0')$ and Ψ_0 overlaps with $\ker(H(\lambda) - E(\lambda))$.

Under assumptions (H.1) and (H.3), it follows from Proposition 2.1 that, for each $\lambda \in \mathbb{I}_r^{\times}$, there exists a non-zero vector $\Psi(\lambda) \in \ker(H(\lambda) - E(\lambda))$ satisfying

$$E(\lambda) = E_0 + \lambda \langle \Psi_0, H_{\rm I}\Psi(\lambda) \rangle, \tag{3.1}$$

$$\Psi(\lambda) = \Psi_0 + \Phi(\lambda), \tag{3.2}$$

where

$$\Phi(\lambda) := \lambda \left(E(\lambda) - H_0'\right)^{-1} Q_0 H_{\mathrm{I}} \Psi(\lambda) \in \mathcal{H}_0^{\perp}. \tag{3.3}$$

Also we assume the following:

(H.4). $\lim_{\lambda\to 0} \|\Psi(\lambda)\| = 1$.

By (3.2), we have

$$\|\Psi(\lambda)\|^2 = 1 + \|\Phi(\lambda)\|^2. \tag{3.4}$$

Hence (H.4) implies that

$$\lim_{\lambda\to 0} \Phi(\lambda) = 0. \tag{3.5}$$

Therefore

$$\lim_{\lambda\to 0} \Psi(\lambda) = \Psi_0. \tag{3.6}$$

Under the framework described above, an elementary fact on the asymptotics for $E(\lambda)$ in λ is:

Theorem 3.1. *Assume* (H.1), (H.3) *and* (H.4). *Then,*

$$E(\lambda) = E_0 + \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi_0 \rangle + o(\lambda) \quad (\lambda \to 0). \tag{3.7}$$

In particular,

$$\lim_{\lambda\to 0} E(\lambda) = E_0. \tag{3.8}$$

Proof. We have by (3.1) and (3.2)

$$E(\lambda) = E_0 + \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi_0 \rangle + \lambda \langle H_{\mathrm{I}} \Psi_0, \Phi(\lambda) \rangle \tag{3.9}$$

Hence by (3.5), we have

$$\frac{E(\lambda) - E_0 - \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi_0 \rangle}{\lambda} = \langle H_{\mathrm{I}} \Psi_0, \Phi(\lambda) \rangle \to 0 \quad (\lambda \to 0).$$

Thus (3.7) holds. □

Remark 3.2. Equation (3.8) shows the continuity of $E(\lambda)$ at $\lambda = 0$. In the case where (H.1) and (H.2) hold and $H(\lambda)$ is a bounded below self-adjoint operator for all $\lambda \in \mathbb{I}_r^{\times}$, one can prove the continuity of $\mathcal{E}_0(\lambda)$ in $\lambda \in \mathbb{I}_r^{\times}$ under a condition which does not assume that $\mathcal{E}_0(\lambda)$ is an eigenvalue of $H(\lambda)$ [6, Lemma 2.1].

To develop an asymptotic perturbation theory for $E(\lambda)$ further, we still need an additional assumption:

(H.5). For all $\lambda \in \mathbb{I}_r^{\times}$, $E(\lambda) < E_0$.

Remark 3.3. If H_{I} is symmetric, then $E(\lambda)$ is real. In this case, if $E(\lambda) = \mathcal{E}_0(\lambda)$, Corollary 2.11 gives a sufficient condition for (H.5) to hold (see also Remark 2.12).

Under assumptions (H.2) and (H.5), $E(\lambda)$ is in $\rho(H_0)$. Hence $H_0 - E(\lambda)$ is bijective with bounded inverse $(H_0 - E(\lambda))^{-1}$.

Lemma 3.4. *Assume* (H.1)–(H.5)*. Then, for all* $\Xi \in D((H_0' - E_0)^{-1/2})$,

$$\lim_{\lambda \to 0} \left\langle \Xi, (H_0 - E(\lambda))^{-1} \Xi \right\rangle = \|(H_0' - E_0)^{-1/2} \Xi\|^2. \tag{3.10}$$

Proof. We denote by E_{H_0} the spectral measure of H_0. Since Ξ is in $\mathcal{H}_0^\perp$, it follows that

$$\begin{aligned}\left\langle \Xi, (H_0 - E(\lambda))^{-1} \Xi \right\rangle &= \left\langle \Xi, (H_0' - E(\lambda))^{-1} \Xi \right\rangle \\ &= \int_{(E_0,\infty)} \frac{1}{\mu - E(\lambda)} d\|E_{H_0}(\mu)\Xi\|^2.\end{aligned}$$

We have

$$0 < \frac{1}{\mu - E(\lambda)} < \frac{1}{\mu - E_0}, \quad \forall \mu > E_0$$

and $\int_{(E_0,\infty)} 1/(\mu - E_0) d\|E_{H_0}(\mu)\Xi\|^2 < \infty$, because Ξ is in $D((H_0' - E_0)^{-1/2})$. By (3.8), $\lim_{\lambda \to 0} 1/(\mu - E(\lambda)) = 1/(\mu - E_0)$ for all $\mu > E_0$. Hence by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{\lambda \to 0} \left\langle \Xi, (H_0 - E(\lambda))^{-1} \Xi \right\rangle \\ &\quad = \int_{(E_0,\infty)} \frac{1}{\mu - E_0} d\|E_{H_0}(\mu)\Xi\|^2 = \|(H_0' - E_0)^{-1/2} \Xi\|^2.\end{aligned}$$

□

For $\lambda \in \mathbb{I}_r^\times$, we define operators $K(\lambda)$ and $G(\lambda)$ by

$$K(\lambda) := (E(\lambda) - H_0)^{-1} Q_0 H_{\mathrm{I}}, \tag{3.11}$$

$$G(\lambda) := H_{\mathrm{I}} (E(\lambda) - H_0)^{-1} Q_0. \tag{3.12}$$

It follows that

$$K(\lambda)^* \supset G(\lambda),$$

where, for a densely defined linear operator A on a Hilbert space, A^* denotes the adjoint of A.

The next theorem gives the second order asymptotics of $E(\lambda)$ in λ near 0:

Theorem 3.5. *Assume* (H.1)–(H.5)*. Suppose that*

$$\Psi_0 \in D\left(G(\lambda) H_{\mathrm{I}}\right) \cap D\left((H_0' - E_0)^{-1/2} Q_0 H_{\mathrm{I}}\right) \tag{3.13}$$

for all $\lambda \in \mathbb{I}_r^\times$ *with*

$$\sup_{\lambda \in \mathbb{I}_r^\times} \|G(\lambda) H_{\mathrm{I}} \Psi_0\| < \infty. \tag{3.14}$$

Then

$$E(\lambda) = E_0 + \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi_0 \rangle - \lambda^2 \|(H_0' - E_0)^{-1/2} Q_0 H_{\mathrm{I}} \Psi_0\|^2 + o(\lambda^2) \quad (\lambda \to 0). \tag{3.15}$$

Proof. We have by (3.9), (3.3) and (3.2)

$$E(\lambda) = E_0 + \lambda \langle \Psi_0, H_{\rm I}\Psi_0\rangle - \lambda^2 \|(H_0' - E_0)^{-1/2} Q_0 H_{\rm I}\Psi_0\|^2 \\ +\lambda^2 R_\lambda + \lambda^2 \langle G(\lambda) H_{\rm I}\Psi_0, \Phi(\lambda)\rangle ,$$

where

$$R_\lambda := \langle H_{\rm I}\Psi_0, K(\lambda)\Psi_0\rangle + \|(H_0' - E_0)^{-1/2} Q_0 H_{\rm I}\Psi_0\|^2.$$

Hence

$$\frac{E(\lambda) - E_0 - \lambda \langle \Psi_0, H_{\rm I}\Psi_0\rangle + \lambda^2 \|(H_0' - E_0)^{-1/2} Q_0 H_{\rm I}\Psi_0\|^2}{\lambda^2} \\ = R_\lambda + \langle G(\lambda) H_{\rm I}\Psi_0, \Phi(\lambda)\rangle .$$

By Lemma 3.4, $\lim_{\lambda\to 0} R_\lambda = 0$. We have by the Schwarz inequality, (3.14) and (3.5)

$$|\langle G(\lambda) H_{\rm I}\Psi_0, \Phi(\lambda)\rangle| \le \left(\sup_{\lambda' \in \mathbb{I}_r^\times} \|G(\lambda') H_{\rm I}\Psi_0\|\right) \|\Phi(\lambda)\| \to 0 \ (\lambda \to 0).$$

Thus (3.15) holds. □

4. Asymptotic Expansions up to Arbitrary Finite Orders in $\boldsymbol{\lambda}$

In this section, under assumptions (H.1)–(H.5), we derive an asymptotic expansion of $E(\lambda)$ up to an arbitrary finite order in λ. Let

$$K_0 := (E_0 - H_0')^{-1} Q_0 H_{\rm I}. \tag{4.1}$$

For each $\ell \in \mathbb{N}$, we define an operator-valued function K_ℓ on $\mathbb{R}^\ell$ by

$$K_\ell(x_1, \ldots, x_\ell) := \sum_{r=1}^{\ell} (-1)^r \sum_{\substack{j_1+\ldots+j_r=\ell \\ j_1,\ldots,j_r \ge 1}} x_{j_1} \ldots x_{j_r} (E_0 - H_0')^{-(r+1)} Q_0 H_{\rm I}, \\ (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell. \tag{4.2}$$

For a natural number $N \ge 2$, we define a sequence $\{a_n\}_{n=1}^N$ as follows:

$$a_1 := \langle \Psi_0, H_{\rm I}\Psi_0\rangle , \tag{4.3}$$

$$a_n = \sum_{\substack{q+\ell=n \\ q,\ell\ge 1}} \sum_{\substack{\ell_1+\ldots+\ell_q=\ell-1 \\ \ell_1,\ldots,\ell_q \ge 0}} \left\langle H_{\rm I}\Psi_0, K_{l_1}(a_1, \ldots, a_{\ell_1}) \ldots K_{l_q}(a_1, \ldots, a_{\ell_q})\Psi_0 \right\rangle , \\ n = 2, \ldots, N, \tag{4.4}$$

provided that

$$\Psi_0 \in \cap_{n=2}^N \cap_{\substack{q+\ell=n \\ q,\ell\ge 1}} \cap_{\substack{\ell_1+\ldots+\ell_q=\ell-1 \\ \ell_1,\ldots,\ell_q\ge 0}} \cap_{r_1=0}^{\ell_1} \cdots \cap_{r_q=0}^{\ell_q} \\ \times D\left(\prod_{j=1}^q (E_0 - H_0')^{-(r_j+1)} Q_0 H_{\rm I}\right), \tag{4.5}$$

where, for (not necessarily commuting) linear operators $A_1, \ldots, A_q$ on a Hilbert space, $\prod_{j=1}^q A_j := A_1 A_2 \ldots A_q$.

Note that, on the right hand side of (4.4), only a_j ($j = 1, \ldots, n-2$) appears. Hence a_n's with $n \geq 3$ are uniquely determined.

We have

$$a_2 = -\left\langle H_{\mathrm{I}}\Psi_0, (H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}}\Psi_0 \right\rangle \leq 0, \tag{4.6}$$

$$\begin{aligned} a_3 = &\left\langle (H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}}\Psi_0, H_{\mathrm{I}}(H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}}\Psi_0 \right\rangle \\ &- \langle \Psi_0, H_{\mathrm{I}}\Psi_0 \rangle \|(H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}}\Psi_0\|^2. \end{aligned} \tag{4.7}$$

Theorem 4.1. *Let $N \geq 2$ be a natural number. Assume* (H.1)–(H.5). *Suppose that* (4.5) *holds and $\Psi_0 \in \cap_{n=1}^{N-1} D(G(\lambda)^n H_{\mathrm{I}})$ with*

$$\sup_{r \in \mathbb{I}_r^\times} \|G(\lambda)^n H_{\mathrm{I}}\Psi_0\| < \infty, \quad n = 1, \ldots, N-1. \tag{4.8}$$

Then

$$E(\lambda) = E_0 + \sum_{n=1}^N a_n \lambda^n + o(\lambda^N) \quad (\lambda \to 0). \tag{4.9}$$

Proof. We prove (4.9) by induction in $N \geq 2$. Equation (4.9) with $N = 2$ follows from Theorem 3.5.

Suppose that (4.9) holds with some $N \geq 2$. Let

$$K_r(\lambda) := (E_0 - E(\lambda))^r (E_0 - H_0')^{-(r+1)} Q_0 H_{\mathrm{I}}, \quad r = 0, 1, \ldots, N-1,$$

and $K(\lambda)$ be as in (3.11). Then, using the easily proven formula

$$\begin{aligned} (E(\lambda) - H_0)^{-1} Q_0 = &(E_0 - H_0')^{-1} Q_0 \\ &+ (E_0 - E(\lambda))(E(\lambda) - H_0)^{-1}(E_0 - H_0')^{-1} Q_0, \end{aligned}$$

we obtain

$$K(\lambda) = \sum_{r=0}^{N-1} K_r(\lambda) + (E_0 - E(\lambda))^N (E(\lambda) - H_0)^{-1} (E_0 - H_0')^{-N} Q_0 H_{\mathrm{I}}, \tag{4.10}$$

on $\cap_{r=1}^N D((E_0 - H_0')^{-r} Q_0 H_{\mathrm{I}})$. By (H.5), for all $\Psi \in D((E_0 - H_0')^{-(N+1)} Q_0 H_{\mathrm{I}})$, we have

$$\|(E(\lambda) - H_0)^{-1} (E_0 - H_0')^{-N} Q_0 H_{\mathrm{I}}\Psi\| \leq \|(E_0 - H_0')^{-(N+1)} Q_0 H_{\mathrm{I}}\Psi\|.$$

Hence by (3.7), we obtain

$$\lim_{\lambda \to 0} \frac{(E_0 - E(\lambda))^N (E(\lambda) - H_0)^{-1} (E_0 - H_0')^{-N} Q_0 H_{\mathrm{I}}\Psi}{\lambda^{N-1}} = 0,$$

i.e.,

$$(E_0 - E(\lambda))^N (E(\lambda) - H_0)^{-1} (E_0 - H_0')^{-N} Q_0 H_{\mathrm{I}}\Psi = o(\lambda^{N-1}) \quad (\lambda \to 0).$$

By the induction hypothesis (4.9), we have

$$E_0 - E(\lambda) = -\sum_{n=1}^N a_n \lambda^n + o(\lambda^N) \quad (\lambda \to 0).$$

Hence,

$$K_r(\lambda) = (-1)^r \sum_{j_1,\ldots,j_r=1}^{N} a_{j_1}\ldots a_{j_r}\lambda^{j_1+\ldots+j_r}(E_0 - H_0')^{-(r+1)}Q_0H_{\mathrm{I}} + o(\lambda^N) \quad (\lambda \to 0)$$

on $D((E_0 - H_0')^{-(r+1)}Q_0H_{\mathrm{I}})$. Putting this into $K_r(\lambda)$ on the right hand side of (4.10), we obtain

$$K(\lambda) = K_0 + \sum_{\ell=1}^{N-1} K_\ell(a_1,\ldots,a_\ell)\lambda^\ell + o(\lambda^{N-1}) \quad (\lambda \to 0) \tag{4.11}$$

on $\cap_{r=1}^{N+1} D((E_0 - H_0')^{-r}Q_0H_{\mathrm{I}})$, where K_ℓ is defined by (4.2).

On the other hand, we have by (3.9) and (3.3)

$$E(\lambda) = E_0 + \langle \Psi_0, H_{\mathrm{I}}\Psi_0\rangle \lambda + \sum_{k=1}^{N} \left\langle H_{\mathrm{I}}\Psi_0, K(\lambda)^k\Psi_0\right\rangle \lambda^{k+1} + \left\langle H_{\mathrm{I}}\Psi_0, K(\lambda)^N\Phi(\lambda)\right\rangle \lambda^{N+1}.$$

Using (4.11), we have

$$E(\lambda) = E_0 + \langle \Psi_0, H_{\mathrm{I}}\Psi_0\rangle \lambda + \sum_{k=1}^{N}\sum_{\ell_1,\ldots,\ell_k=0}^{N-1} \langle H_{\mathrm{I}}\Psi_0, K_{\ell_1}(a_1,\ldots,a_{\ell_1})\ldots K_{\ell_k}(a_1,\ldots,a_{\ell_k})\Psi_0\rangle \times\lambda^{\ell_1+\ldots+\ell_k+k+1} + o(\lambda^{N+1}) + \left\langle H_{\mathrm{I}}\Psi_0, K(\lambda)^N\Phi(\lambda)\right\rangle \lambda^{N+1} \quad (\lambda \to 0).$$

By rearranging the sums on k and $\ell_1,\ldots,\ell_k$, we obtain

$$E(\lambda) = E_0 + \langle \Psi_0, H_{\mathrm{I}}\Psi_0\rangle \lambda + \sum_{n=2}^{N+1} a_n\lambda^n + o(\lambda^{N+1}) + \left\langle H_{\mathrm{I}}\Psi_0, K(\lambda)^N\Phi(\lambda)\right\rangle \lambda^{N+1} \quad (\lambda \to 0).$$

By condition (4.8) with $N-1$ replaced by N, we have

$$|\left\langle H_{\mathrm{I}}\Psi_0, K(\lambda)^N\Phi(\lambda)\right\rangle| \le \left(\sup_{\lambda'\in\mathbb{I}_r^\times} \|G(\lambda')^N H_{\mathrm{I}}\Psi_0\|\right)\|\Phi(\lambda)\| \to 0 \quad (\lambda \to 0).$$

Hence

$$\left\langle H_{\mathrm{I}}\Psi_0, K(\lambda)^N\Phi(\lambda)\right\rangle \lambda^{N+1} = o(\lambda^{N+1}) \quad (\lambda \to 0).$$

Thus (4.9) holds with N replaced by $N+1$. □

5. Application to the GSB Model

In this section, we apply Theorem 4.1 to the GSB model mentioned in Introduction. Following [5], we first review the GSB model briefly.

The quantum filed in the GSB model is a scalar Bose field acting in the boson (symmetric) Fock space

$$\mathcal{F}_b\left(L^2(\mathbb{R}^\nu)\right) := \oplus_{n=0}^\infty \left[\otimes_s^n L^2(\mathbb{R}^\nu)\right]$$

over the Hilbert space $L^2(\mathbb{R}^\nu)\,(\nu \in \mathbb{N})$, where $\otimes_s^n L^2(\mathbb{R}^\nu)$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^\nu)$ with convention $\otimes_s^0 L^2(\mathbb{R}^\nu) := \mathbb{C}$.

We denote by $a(f)\,(f \in L^2(\mathbb{R}^\nu))$ the annihilation operator with test function f, i.e., it is a densely defined closed linear operator on $\mathcal{F}_b(L^2(\mathbb{R}^\nu))$ such that its adjoint $a(f)^*$ obeys the following properties: for all $\psi \in D(a(f)^*), (a(f)^*\psi)^{(0)} = 0$ and $(a(f)^*\psi)^{(n)} = \sqrt{n}S_n(f \otimes \psi^{(n-1)}),\ n \geq 1$, where S_n is the symmetrization operator on the n-fold tensor product $\otimes^n L^2(\mathbb{R}^\nu)$ of $L^2(\mathbb{R}^\nu)$. The time-zero field for the GSB model is defined by

$$\phi(f) := \frac{1}{2}\left(a(f)^* + a(f)\right),$$

which is called the Segal field operator with test function f.

The one-particle Hamiltonian of a free boson, which acts in $L^2(\mathbb{R}^\nu)$, is described by the multiplication operator of an energy function $\omega : \mathbb{R}^\nu \to [0,\infty); \mathbb{R}^\nu \ni k \mapsto \omega(k) \geq 0$ which is Borel measurable and $0 < \omega(k) < \infty$ for almost everywhere (a.e.) $k \in \mathbb{R}^\nu$. We denote the one-particle Hamiltonian by the same symbol ω. Then the Hamiltonian of the free quantum field with one-particle Hamiltonian ω is defined by

$$H_b := \oplus_{n=0}^\infty \omega^{(n)}$$

the second quantization of ω, where $\omega^{(0)} := 0$ as an operator on $\mathbb{C}, \omega^{(1)} := \omega$ and, for $n \geq 2, \omega^{(n)} := \sum_{j=1}^n I \otimes \ldots \otimes \overset{j\text{th}}{\omega} \otimes \ldots \otimes I$ acting in $\otimes_s^n L^2(\mathbb{R}^\nu)$ (I denotes the identity on $L^2(\mathbb{R}^\nu)$).

For more detailed descriptions on the boson Fock space theory, we refer the reader to [20, §X.7].

Let

$$\omega_0 := \text{ess.inf}_{k\in\mathbb{R}^\nu}\omega(k) \geq 0. \tag{5.1}$$

If $\omega_0 > 0$ (resp. $\omega_0 = 0$), then we say that the Bose field or the boson is *massive* (resp. *massless*).[2]

We take the Hilbert space of quantum particles interacting with the quantum scalar filed to be an abstract complex Hilbert space $\mathcal{K}$. Then a Hilbert space of state vectors for the GSB model is given by

$$\mathcal{H}_{\mathrm{GSB}} := \mathcal{K} \otimes \mathcal{F}_b(L^2(\mathbb{R}^\nu)).$$

Let $L^2_{\mathrm{sym}}((\mathbb{R}^\nu)^n; \mathcal{K})$ be the Hilbert space of $\mathcal{K}$-valued symmetric L^2-functions on $(\mathbb{R}^\nu)^n$ with convention $L^2_{\mathrm{sym}}((\mathbb{R}^\nu)^0; \mathcal{K}) := \mathcal{K}$. Then the Hilbert space $\mathcal{H}_{\mathrm{GSB}}$ has a natural identification as

[2] An example of ω for a relativistic Bose field is given by $\omega(k) = \sqrt{k^2+m^2}$ with $m \geq 0$ being the mass of the boson. In this case, we have $\omega_0 = m$, showing that ω_0 is certainly the mass of the boson. But, in general (mathematically), ω_0 is not necessarily a mass in the physical sense. For example, if the boson is non-relativistic, then one may take $\omega(k) = k^2/2m$. In this case, $\omega_0(= 0)$ is not a mass.

$$\begin{aligned}\mathcal{H}_{\mathrm{GSB}} &= \oplus_{n=0}^{\infty} L^2_{\mathrm{sym}}((\mathbb{R}^\nu)^n;\mathcal{K})\\ &= \left\{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in L^2_{\mathrm{sym}}((\mathbb{R}^\nu)^n;\mathcal{K}), n \geq 0, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty\right\}.\end{aligned}$$

For each $n \geq 0$, we call the subspace

$$\mathcal{H}^{(n)} := \{\Psi = \{0,\ldots,0,\Psi^{(n)},0,\ldots\} \in \mathcal{H}_{\mathrm{GSB}} | \Psi^{(n)} \in L^2_{\mathrm{sym}}((\mathbb{R}^\nu)^n;\mathcal{K})\} \quad (5.2)$$

the n-boson space of the GSB model. We have

$$\mathcal{H}_{\mathrm{GSB}} = \oplus_{n=0}^{\infty} \mathcal{H}^{(n)}. \quad (5.3)$$

We assume that the Hamiltonian of the system of quantum particles is given by a self-adjoint operator A on $\mathcal{K}$ which is bounded below. Then

$$E_0 := \inf \sigma(A) \quad (5.4)$$

is finite and

$$\widetilde{A} := A - E_0 \quad (5.5)$$

is a non-negative self-adjoint operator on $\mathcal{K}$.

Let $J \in \mathbb{N}$ and $B_j\,(j = 1,\ldots,J)$ be a symmetric operator on $\mathcal{K}$. Let $g_j \in L^2(\mathbb{R}^\nu)\,(j = 1,\ldots,J)$ and $\lambda \in \mathbb{R}$. In this section, we take H_0 and H_{I} as follows:

$$H_0 := A \otimes I + I \otimes H_b, \quad (5.6)$$

$$H_{\mathrm{I}} := \sum_{j=1}^{J} B_j \otimes \phi(g_j). \quad (5.7)$$

The function g_j in $\phi(g_j)$ physically means a momentum cutoff function of the boson. The Hamiltonian of the GSB model is defined by

$$H_{\mathrm{GSB}}(\lambda) := H_0 + \lambda H_{\mathrm{I}}. \quad (5.8)$$

Before going into detailed analysis on the GSB model, we take this opportunity to report an important fact on the lowest (minimal) energy of $H_{\mathrm{GSB}}(\lambda)$ in the case where $H_{\mathrm{GSB}}(\lambda)$ is self-adjoint and bounded below. Hence, let

$$\Lambda := \{\lambda \in \mathbb{R} | H_{\mathrm{GSB}}(\lambda) \text{ is self-adjoint and bounded below}\} \quad (5.9)$$

and, for each $\lambda \in \Lambda$,

$$E_0(\lambda) := \inf \sigma(H_{\mathrm{GSB}}(\lambda)). \quad (5.10)$$

Theorem 5.1. *The set Λ is reflection symmetric with respect to the origin of $\mathbb{R}$ (i.e., $\lambda \in \Lambda \Longleftrightarrow -\lambda \in \Lambda$) and $E_0(\cdot)$ is an even function on Λ:*

$$E_0(\lambda) = E_0(-\lambda), \quad \lambda \in \Lambda. \quad (5.11)$$

Proof. Let $\lambda \in \Lambda$. We denote by N_b the number operator on $\mathcal{F}_b(L^2(\mathbb{R}^\nu))$: $(N_b\Psi)^{(n)} = n\Psi^{(n)}$, $n \geq 0, \Psi \in D(N_b)$. The operator N_b is self-adjoint and non-negative. Hence $U := e^{i\pi N_b}$ is unitary. We have the operator equality

$$Ua(f)^{\#}U^{-1} = -a(f)^{\#}, \quad f \in L^2(\mathbb{R}^\nu),$$

where $a(f)^\#$ denotes either $a(f)$ or $a(f)^*$. Hence,

$$U\phi(f)U^{-1} = -\phi(f), \quad f \in L^2(\mathbb{R}^\nu).$$

Also we have

$$UH_bU^{-1} = H_b.$$

Hence we obtain

$$(I \otimes U)H_{\mathrm{GSB}}(\lambda)(I \otimes U)^{-1} = H_{\mathrm{GSB}}(-\lambda),$$

which implies that Λ is reflection symmetric with respect to the origin of $\mathbb{R}$ and

$$\sigma(H_{\mathrm{GSB}}(\lambda)) = \sigma(H_{\mathrm{GSB}}(-\lambda)).$$

Hence (5.11) holds. □

Remark 5.2. The proof given above can be applied also to a general particle-field Hamiltonian [4], which includes Pauli–Fierz type models (e.g., [10,11,17]) and Nelson type models (e.g., [2,19]), and the Dereziński–Gérard model [13] to conclude that their minimal energy is an even function of the coupling constant.

In what follows, we assume the following conditions:

(A.1) The operator A has compact resolvent.

(A.2) Each B_j $(j = 1, \ldots, J)$ is $\widetilde{A}^{1/2}$-bounded. Namely, $D(\widetilde{A}^{1/2}) \subset \cap_{j=1}^J D(B_j)$ and there exist constants $a_j \geq 0, b_j \geq 0$ such that, for all $\psi \in D(\widetilde{A}^{1/2})$,

$$\|B_j\psi\| \leq a_j\|\widetilde{A}^{1/2}\psi\| + b_j\|\psi\|, \quad j = 1, \ldots, J.$$

(A.3) $g_j, g_j/\omega \in L^2(\mathbb{R}^\nu), j = 1, \ldots, J$.

(A.4) The function ω is continuous on $\mathbb{R}^\nu$ with $\lim_{|k|\to\infty} \omega(k) = \infty$ and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma \left(1 + \omega(k) + \omega(k')\right), \quad k, k' \in \mathbb{R}^\nu.$$

Remark 5.3. Under (A.1), A has purely discrete spectrum $\{E_n\}_{n=0}^N$ with $E_0 < E_1 < \ldots < E_n < E_{n+1} < \ldots$, where $N < \infty$ or $N = \infty$. In particular, E_0 is an eigenvalue of A with a finite multiplicity. Under assumption (A.4), we have $\sigma(\omega) = [\omega_0, \infty)$, where ω_0 is given by (5.1). Hence, it follows that

$$\sigma(H_0) = \{E_0, \ldots, E_{n_0}\} \cup [E_0 + \omega_0, \infty), \tag{5.12}$$

where $n_0 := \max\{n \geq 0 | E_n < E_0 + \omega_0\}$. In particular, if the boson is massless, then $\sigma(H_0) = [E_0, \infty)$, which shows that all the eigenvalues of H_0 are embedded eigenvalues of H_0. Hence, in this case, for each E_n, one can neither apply the standard analytic perturbation theory nor the standard asymptotic perturbation theory. Thus we need a new approach in the massless case. Note that, in the massive case $\omega_0 > 0, E_n$ $(n = 0, 1, \ldots, n_0)$ is an isolated eigenvalue of H_0. We can show that H_{I} is H_0-bounded [5]. Hence, in the massive case, we can apply the standard analytic perturbation theory to conclude that, for all sufficiently small $|\lambda|$ (its smallness may depend on ω_0), $H_{\rm GSB}(\lambda)$ has a ground state and the ground-state energy is analytic in λ.

Remark 5.4. As for the spectrum of $H_{\rm GSB}(\lambda)$, the following result is known [3, Theorem 3.3]: If (A.2) holds and $g_j, g_j/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)\,(j = 1,\ldots,J)$ with $\sigma(\omega) = [0,\infty)$, then, for all $\lambda \in \Lambda$,

$$\sigma\left(H_{\rm GSB}(\lambda)\right) = [E_0(\lambda), \infty).$$

Note that, for this fact, $H_{\rm GSB}(\lambda)$ does not necessarily have a ground state.

Let

$$\Omega_0 := \{1, 0, 0, \ldots\} \in \mathcal{F}_b\left(L^2(\mathbb{R}^\nu)\right) \tag{5.13}$$

be the Fock vacuum and P_{Ω_0} be the orthogonal projections onto the one-dimensional subspace $\{\alpha\Omega_0 | \alpha \in \mathbb{C}\}$. We denote by p_0 the orthogonal projection onto $\ker \widetilde{A}$.

As for the existence of a ground state of $H_{\rm GSB}(\lambda)$, we have the following theorem [5]:

Theorem 5.5. *Assume* (A.1)–(A.4)*. Then there exists a constant $r > 0$ independent of λ such that the following hold:*

(i) $(-r, r) \subset \Lambda$.

(ii) *For all $\lambda \in (-r, r)$, $H_{\rm GSB}(\lambda)$ has a ground state $\Psi_0(\lambda)$ and there exists a constant $M > 0$ independent of $\lambda \in (-r, r)$ such that, for all $|\lambda| < r$, $\|\Psi_0(\lambda)\| \leq 1$ and*

$$\langle \Psi_0(\lambda), p_0 \otimes P_{\Omega_0}\Psi_0(\lambda)\rangle \geq 1 - \lambda^2 M^2 > 0 \tag{5.14}$$

Under the assumption of Theorem 5.5, $E_0(\lambda)$ with $\lambda \in (-r, r)$ is the ground-state energy of $H_{\rm GSB}(\lambda)$. To derive asymptotic expansions for $E_0(\lambda)$ in λ near 0, we need to check that the assumptions taken in Sects. 3 and 4 hold in the case where $H(\lambda) = H_{\rm GSB}(\lambda)$. For this purpose, however, we need additional assumptions:

(A.5) The eigenvalue of E_0 of A is simple and there exists a $j_0 \in \{1, \ldots, J\}$ such that $B_{j_0}\psi \neq 0$ for all $\psi \in \ker \widetilde{A}\backslash\{0\}$.

(A.6) The set $\{g_1, \ldots, g_J\} \subset L^2(\mathbb{R}^\nu)$ is linearly independent.

We denote by ψ_0 a normalized eigenvector of A with eigenvalue E_0:

$$A\psi_0 = E_0\psi_0, \quad \|\psi_0\| = 1. \tag{5.15}$$

Hence $\ker \widetilde{A} = \{\alpha\psi_0 | \alpha \in \mathbb{C}\}$. By (A.5), we have

$$B_{j_0}\psi_0 \neq 0. \tag{5.16}$$

It follows from the theory of tensor products of self-adjoint operators that E_0 is a simple eigenvalue of H_0 being the ground-state energy of H_0 with a normalized eigenvector

$$\Psi_0 := \psi_0 \otimes \Omega_0. \tag{5.17}$$

By the well-known fact that

$$a(f)\Omega_0 = 0, \quad f \in L^2(\mathbb{R}^\nu), \tag{5.18}$$

we have

$$\langle \Psi_0, H_{\rm I}\Psi_0\rangle = 0. \tag{5.19}$$

Hence (2.19) holds.

In the present model, the orthogonal projection P_0 onto $\ker(H_0 - E_0)$ takes the form

$$P_0 = p_0 \otimes P_{\Omega_0}.$$

Hence

$$Q_0 = I - P_0 = I - p_0 \otimes P_{\Omega_0}.$$

Lemma 5.6. *Assume* (A.1)–(A.6)*. Then* $Q_0 H_{\rm I}\Psi_0 \neq 0$*.*

Proof. By (5.18), we have

$$H_{\rm I}\Psi_0 = \sum_{j=1}^{J} \frac{1}{\sqrt{2}} B_j\psi_0 \otimes a(g_j)^*\Omega_0. \tag{5.20}$$

Since $P_{\Omega_0} a(f)^*\Omega_0 = 0$ for all $f \in L^2(\mathbb{R}^\nu)$, we have

$$Q_0 H_{\rm I}\Psi_0 = \sum_{j=1}^{J} \frac{1}{\sqrt{2}} B_j\psi_0 \otimes a(g_j)^*\Omega_0. \tag{5.21}$$

Suppose that $Q_0 H_{\rm I}\Psi_0$ were a zero vector. Then it follows that, for all $\psi \in \mathcal{K}$,

$$\sum_{j=1}^{J} \langle \psi, B_j\psi_0\rangle\, g_j = 0.$$

By (A.6), this implies that, for all $\psi \in \mathcal{K}, \langle\psi, B_j\psi_0\rangle = 0, j = 1,\dots,J$. Hence $B_j\psi_0 = 0$ for all $j = 1,\dots,J$. But this contradicts (5.16). □

As in the case of the general theory in Sect. 2, we denote by H_0' the reduced part of H_0 to $\{\alpha\Psi_0 | \alpha \in \mathbb{C}\}^\perp$.

Let H_b' be the reduced part of H_b to $\{\alpha\Omega_0|\alpha \in \mathbb{C}\}^\perp$. Then, in the GSB model, we have, under the identification (5.3),

$$H_0' - E_0 = \widetilde{A} + H_b' \tag{5.22}$$

on $\oplus_{n=1}^\infty \mathcal{H}^{(n)}$.

Lemma 5.7. *Assume* (A.1)–(A.5)*. Let* $\psi \in \mathcal{K}$ *and* $f \in L^2(\mathbb{R}^\nu)$*. Then* $\psi \otimes a(f)^*\Omega_0$ *is in* $D((H_0' - E_0)^{-1})$ *if and only if*

$$\int_{\omega(k)>0} \| \left(\widetilde{A} + \omega(k)\right)^{-1} \psi\|^2 |f(k)|^2\, dk < \infty. \tag{5.23}$$

In particular, if $f/\omega \in L^2(\mathbb{R}^\nu)$*, then* $\psi \otimes a(f)^*\Omega_0 \in D((H_0' - E_0)^{-1})$*.*

Proof. For a self-adjoint operator S, we denote by $E_S(\cdot)$ its spectral measure. We have $E_{H_0'}(X_1 \times X_2) = E_A(X_1) \otimes E_{H_b'}(X_2)$ on $\oplus_{n=1}^{\infty} \mathcal{H}^{(n)}$ for all Borel sets $X_1, X_2 \subset \mathbb{R}$ and $E_{H_b'}(\{0\}) = 0$. Hence it follows that the vector $\psi \otimes a(f)^* \Omega_0$ is in $D((H_0' - E_0)^{-1})$ if and only if

$$I_f := \int\limits_{x+y>E_0} \frac{1}{|x - E_0 + y|^2} d\|E_A(x)\psi\|^2 d\|E_{H_b'}(y)a(f)^*\Omega_0\|^2 < \infty$$

We have

$$\left(H_b' a(f)^* \Omega_0\right)^{(1)}(k) = \omega(k) f(k), \quad (H_b' a(f)^* \Omega_0)^{(n)}(k) = 0, \quad n \neq 1.$$

Hence,

$$I_f = \int\limits_{\omega(k)>0} \|(\widetilde{A} + \omega(k))^{-1}\psi\|^2 |f(k)|^2 \mathrm{d}k.$$

Thus the first statement of the present lemma holds.

Since $\widetilde{A} \geq 0$, we have $I_f \leq \|\psi\|^2 \int_{\mathbb{R}^\nu} |f(k)|^2/\omega(k)^2 \, \mathrm{d}k$. Hence if $f/\omega \in L^2(\mathbb{R}^\nu)$, then $I_f < \infty$ and thus $\psi \otimes a(f)^* \Omega_0 \in D((H_0' - E_0)^{-1})$. □

Using the Schwarz inequality for the inner product and the elementary inequality

$$\alpha\beta \leq \varepsilon^2 \alpha^2 + \frac{\beta^2}{4\varepsilon^2} \quad (\alpha \geq 0, \, \beta \geq 0, \, \varepsilon > 0),$$

we have for all $\psi \in D(\widetilde{A})$ and all $\varepsilon > 0$

$$\|\widetilde{A}^{1/2}\psi\|^2 \leq \varepsilon^2 \|\widetilde{A}\psi\|^2 + \frac{1}{4\varepsilon^2}\|\psi\|^2.$$

By this fact and (A.2), we obtain

$$\|B_j\psi\| \leq a\varepsilon\|\widetilde{A}\psi\| + b_\varepsilon\|\psi\|, \quad \psi \in D(\widetilde{A}) \quad (j = 1, \ldots, J), \tag{5.24}$$

where $a := \max\{a_1, \ldots, a_J\}, \varepsilon > 0$ is arbitrary and

$$b_\varepsilon := b + \frac{a}{2\varepsilon} \tag{5.25}$$

with $b := \max\{b_1, \ldots, b_J\}$.

Lemma 5.8. *Assume* (A.2)*. Then, for each* $j = 1, \ldots, J$ *and* $k \in \mathbb{R}^\nu$ *with* $\omega(k) > 0, B_j(\widetilde{A} + \omega(k))^{-1}$ *is bounded with*

$$\|B_j(\widetilde{A} + \omega(k))^{-1}\| \leq a\varepsilon + \frac{b_\varepsilon}{\omega(k)}, \tag{5.26}$$

where $\varepsilon > 0$ *is arbitrary.*

Proof. By (5.24), we have for all $\psi \in \mathcal{K}$

$$\|B_j(\widetilde{A} + \omega(k))^{-1}\psi\| \leq a\varepsilon\|\widetilde{A}(\widetilde{A} + \omega(k))^{-1}\psi\| + b_\varepsilon\|(\widetilde{A} + \omega(k))^{-1}\psi\|.$$

We have $\|(\widetilde{A} + \omega(k))^{-1}\psi\| \leq \omega(k)^{-1}\|\psi\|$ and $\|\widetilde{A}(\widetilde{A} + \omega(k))^{-1}\psi\| \leq \|\psi\|$. Thus the desired result follows. □

Lemma 5.9. *Assume* (A.1)–(A.6)*. Then* (2.9) *holds in the present model.*

Proof. By (5.21) and the condition $g_j/\omega\,(j = 1,\ldots,J)$, we can apply Lemma 5.7 to conclude that

$$Q_0 H_{\mathrm{I}} \Psi_0 \in D\left((H_0' - E_0)^{-1}\right).$$

Using identification (5.3), we have

$$\left((H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}} \Psi_0\right)^{(1)}(k) = \sum_{j=1}^{J} \frac{1}{\sqrt{2}} g_j(k) \left(\widetilde{A} + \omega(k)\right)^{-1} B_j \psi_0,$$
$$\omega(k) > 0, \tag{5.27}$$
$$\left((H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}} \Psi_0\right)^{(n)} = 0, \quad n \neq 1. \tag{5.28}$$

Hence $(H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}} \Psi_0$ is in $D(H_I)$ with

$$\begin{aligned}
&(H_I (H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}} \Psi_0)^{(0)} \\
&= \frac{1}{2} \sum_{\ell,j=1}^{J} \int_{\omega(k)>0} \mathrm{d}k\, g_\ell(k)^* g_j(k) B_\ell (\widetilde{A} + \omega(k))^{-1} B_j \psi_0, \\
&(H_I (H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}} \Psi_0)^{(2)}(k_1, k_2) \\
&= \frac{1}{2\sqrt{2}} \sum_{\ell,j=1}^{J} \Big\{ g_\ell(k_2) g_j(k_1) B_\ell (\widetilde{A} + \omega(k_1))^{-1} B_j \psi_0 \\
&\quad + g_\ell(k_1) g_j(k_2) B_\ell (\widetilde{A} + \omega(k_2))^{-1} B_j \psi_0 \Big\} \\
&\quad k_1, k_2 \in \mathbb{R}^\nu, \omega(k_1) > 0, \omega(k_2) > 0, \\
&(H_I (H_0' - E_0)^{-1} Q_0 H_{\mathrm{I}} \Psi_0)^{(n)} = 0, \quad n \neq 0, 2.
\end{aligned}$$

Thus the desired result follows. □

Lemma 5.10. *Assume* (A.1)–(A.6)*. Let*

$$N_{\mathrm{GSB}} := \frac{1}{2} \sum_{j,\ell=1}^{J} \int_{\omega(k)>0} \left\langle (\widetilde{A} + \omega(k))^{-1} B_j \psi_0, (\widetilde{A} + \omega(k))^{-1} B_\ell \psi_0 \right\rangle \times g_j(k)^* g_\ell(k)\, dk, \tag{5.29}$$

$$a_{\mathrm{GSB}} := \frac{1}{2} \sum_{j,\ell=1}^{J} \int_{\omega(k)>0} \left\langle B_j \psi_0, (\widetilde{A} + \omega(k))^{-1} B_\ell \psi_0 \right\rangle g_j(k)^* g_\ell(k)\, dk. \tag{5.30}$$

Then $a_{\mathrm{GSB}} > 0$ and

$$E_0(\lambda) \leq E_0 - \frac{a_{\mathrm{GSB}} \lambda^2}{1 + N_{\mathrm{GSB}} \lambda^2}. \tag{5.31}$$

In particular, for all $\lambda \in \mathbb{I}_r^\times$, $E_0(\lambda) < E_0$ and $E_0(\lambda) \in \rho(H_0)$.

Proof. As already seen, in the GSB model, (2.19) holds. Hence, by Lemmas 5.6 and 5.9, we can apply Theorem 2.7. In the present case, we have $N_0 = N_{\mathrm{GSB}}, a = a_{\mathrm{GSB}}$ and $b = 0$. Thus the desired result follows. □

Lemma 5.11. *Assume* (A.1)–(A.5) *and let* $\Psi_0(\lambda)$ *be as in Theorem* 5.5*. Then, for all* $\lambda \in \mathbb{I}_r^{\times}, \Psi_0(\lambda)$ *overlaps with* Ψ_0 *and*

$$\lim_{\lambda \to 0} |\langle \Psi_0, \Psi_0(\lambda) \rangle| = 1. \tag{5.32}$$

Proof. By (5.17), we have $p_0 \otimes P_{\Omega_0} \Psi_0(\lambda) = \langle \Psi_0, \Psi_0(\lambda) \rangle \Psi_0$. Hence by Theorem 5.5,

$$1 \geq |\langle \Psi_0, \Psi_0(\lambda) \rangle|^2 \geq 1 - \lambda^2 M^2 > 0.$$

Hence the desired results follow. □

Let (A.1)–(A.6) be satisfied. Then, by Lemma 5.11, we can define for all $\lambda \in \mathbb{I}_r^{\times}$

$$\Psi_{\mathrm{GSB}}(\lambda) := \frac{1}{\langle \Psi_0, \Psi_0(\lambda) \rangle} \Psi_0(\lambda). \tag{5.33}$$

Then, as in (3.1) and (3.2), we have for all $\lambda \in \mathbb{I}_r^{\times}$

$$E_0(\lambda) = E_0 + \lambda \langle \Psi_0, H_{\mathrm{I}} \Psi_{\mathrm{GSB}}(\lambda) \rangle, \tag{5.34}$$

$$\Psi_{\mathrm{GSB}}(\lambda) = \Psi_0 + \lambda (E_0(\lambda) - H_0)^{-1} Q_0 H_I \Psi_{\mathrm{GSB}}(\lambda). \tag{5.35}$$

Lemma 5.12. *Condition* (H.4) *with* $\Psi(\lambda) = \Psi_{\mathrm{GSB}}(\lambda)$ *holds:*

$$\lim_{\lambda \to 0} \|\Psi_{\mathrm{GSB}}(\lambda)\| = 1. \tag{5.36}$$

Proof. By Theorem 5.5-(ii) and $\|p_0 \otimes P_0\| = 1$, we have

$$1 \geq \|\Psi_0(\lambda)\| \geq \sqrt{1 - \lambda^2 M^2}.$$

Hence $\lim_{\lambda \to 0} \|\Psi_0(\lambda)\| = 1$. By this fact and (5.32), we obtain (5.36). □

Thus we have shown that, *under assumptions (A.1)–(A.6), conditions (H.1)–(H.5) in Sect.* 3 *hold.* Therefore, to establish asymptotic expansions of $E_0(\lambda)$, we need only to check assumptions in Theorem 4.1 other than (H.1)–(H.5).

For an independent interest, we first derive the asymptotic expansion of $E_0(\lambda)$ up to the second order in λ.

Theorem 5.13. *Assume* (A.1)–(A.6)*. Then*

$$E_0(\lambda) = E_0 - a_{\mathrm{GSB}} \lambda^2 + o(\lambda^2) \quad (\lambda \to 0), \tag{5.37}$$

where a_{GSB} *is given by* (5.30)*.*

Proof. We need only to check (3.14).

It follows from (5.20) that $H_{\mathrm{I}}\Psi_0 \in D(G(\lambda))$ and

$$
\begin{aligned}
&(G(\lambda)H_{\mathrm{I}}\Psi_0)^{(0)}\\
&\quad= \frac{1}{2}\sum_{\ell=1}^{J}\sum_{j=1}^{J}\int_{\mathbb{R}^\nu} \mathrm{d}k g_\ell(k)^* g_j(k) B_\ell(E(\lambda)-E_0-\widetilde{A}-\omega(k))^{-1}B_j\psi_0,\\
&\quad(G(\lambda)H_{\mathrm{I}}\Psi_0)^{(2)}(k_1,k_2) = \frac{1}{2\sqrt{2}}\sum_{\ell=1}^{J}\sum_{j=1}^{J}\\
&\qquad\qquad\times\Big\{g_\ell(k_2)g_j(k_1)B_\ell(E(\lambda)-E_0-\widetilde{A}-\omega(k_1))^{-1}B_j\psi_0\\
&\qquad\qquad+g_\ell(k_1)g_j(k_2)B_\ell(E(\lambda)-E_0-\widetilde{A}-\omega(k_2))^{-1}B_j\psi_0\Big\},\\
&\qquad\qquad(G(\lambda)H_{\mathrm{I}}\Psi_0)^{(n)} = 0, \quad n \neq 0,2.
\end{aligned}
$$

By (5.26) and $E(\lambda) < E_0$, we have

$$
\begin{aligned}
&\|(G(\lambda)H_{\mathrm{I}}\Psi_0)^{(0)}\| \le \frac{1}{2}\sum_{\ell=1}^{J}\sum_{j=1}^{J}\int_{\mathbb{R}^\nu}\mathrm{d}k|g_\ell(k)||g_j(k)|\left(a\varepsilon+b_\varepsilon\omega(k)^{-1}\right)\|B_j\psi_0\|,\\
&\|(G(\lambda)H_{\mathrm{I}}\Psi_0)^{(2)}(k_1,k_2)\| \le \frac{1}{2\sqrt{2}}\sum_{\ell=1}^{J}\sum_{j=1}^{J}\\
&\qquad\times\left\{|g_\ell(k_2)||g_j(k_1)|\left(a\varepsilon+b_\varepsilon\omega(k_1)^{-1}\right)\|B_j\psi_0\|\right.\\
&\qquad\left.+|g_\ell(k_1)||g_j(k_2)|\left(a\varepsilon+b_\varepsilon\omega(k_2)^{-1}\right)\|B_j\psi_0\|\right\}.
\end{aligned}
$$

These estimates imply (3.14). □

We next consider the asymptotic expansion of $E_0(\lambda)$ in λ up to the Nth order with $N \ge 3$.

Lemma 5.14. *Let $N \ge 2$ be a natural number and suppose that*

$$
g_j,\ \frac{g_j}{\omega^{N-1}} \in L^2(\mathbb{R}^\nu), \quad j=1,\dots,J. \tag{5.38}
$$

Then (4.5) holds in the present model.

Remark 5.15. If (5.38) holds, then $g_j/\omega^s \in L^2(\mathbb{R}^\nu)$ for all $s \in [0, N-1]$.

Proof. Let $q = 1,\dots,N-1$ and $r_\alpha = 0,1,\dots,N-2$ ($\alpha = 1,\dots,q$). By assumption (5.38) and Remark 5.15, one sees that Ψ_0 is in $D((H_0'-E_0)^{-(r_{q-1}+1)} Q_0H_{\mathrm{I}}(H_0'-E_0)^{-(r_q+1)}Q_0H_{\mathrm{I}})$. We denote by $a(\cdot)^\#$ either $a(\cdot)$ or $a(\cdot)^*$. For $q \ge 2$ and $\alpha = 1,\dots,q$, we have

$$
\begin{aligned}
&\left(\prod_{\alpha=1}^{q}(H_0'-E_0)^{-(r_\alpha+1)}Q_0H_{\mathrm{I}}\right)P_0 = \frac{1}{\sqrt{2}^q}\sum_{\#}\sum_{j_1,\dots,j_q=1}^{J}\\
&\quad\times(\widetilde{A}+H_b')^{-(r_1+1)}Q_0B_{j_1}\otimes a(g_{j_1})^\#(\widetilde{A}+H_b')^{-(r_2+1)}Q_0B_{j_2}\otimes a(g_{j_2})^\#\\
&\quad\dots Q_0B_{j_{q-2}}\otimes a(g_{j_{q-2}})^\#(H_0'-E_0)^{-(r_{q-1}+1)}Q_0B_{j_{q-1}}\otimes a(g_{j_{q-1}})^\#\\
&\quad\times(H_0'-E_0)^{-(r_q+1)}B_{j_q}\otimes a(g_{j_q})^*P_0.
\end{aligned} \tag{5.39}
$$

As in Lemma 5.8, we can show that, for all $\Phi \in D(H_b'^{-1})$ and $n \geq 1$

$$\begin{aligned} &\|(B_j(\widetilde{A}+H_b')^{-1}\Phi)^{(n)}(k_1,\ldots,k_n)\| \\ &\quad \leq a\varepsilon\|\Phi^{(n)}(k_1,\ldots,k_n)\| + b_\varepsilon \frac{1}{\sum_{i=1}^n \omega(k_i)}\|\Phi^{(n)}(k_1,\ldots,k_n)\| \end{aligned} \tag{5.40}$$

for a.e. $(k_1,\ldots,k_n) \in (\mathbb{R}^\nu)^n$, where $\|\cdot\|$ denotes the norm of $\mathcal{K}$ and $\varepsilon > 0$ is arbitrary.

For all $f \in L^2(\mathbb{R}^\nu)$, the operator $a(f)$ (resp. $a(f)^*$) maps the n-boson space $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(n-1)}$ (resp. $\mathcal{H}^{(n+1)}$) and, for all $p \geq 1, (\widetilde{A}+H_b')^{-p}$ maps $D((\widetilde{A}+H_b')^{-p}) \cap \mathcal{H}^{(n)}$ into $\mathcal{H}^{(n)}$ with

$$((\widetilde{A}+H_b')^{-p}\Psi)^{(n)}(k_1,\ldots,k_n) = \left(\widetilde{A}+\sum_{j=1}^n \omega(k_j)\right)^{-p}\Psi^{(n)}(k_1,\ldots,k_n) \in \mathcal{K}, \quad \text{a.e. } (k_1,\ldots,k_n) \in (\mathbb{R}^\nu)^n$$

for all $\Psi \in D((\widetilde{A}+H_b')^{-p}) \cap \mathcal{H}^{(n)}$. Moreover, for all $\Psi \in D((\widetilde{A}+H_b')^{-p})a(f) B_j(\widetilde{A}+H_b')^{-1}) \cap D((\widetilde{A}+H_b')^{-p})a(f)^*B_j(\widetilde{A}+H_b')^{-1}) \cap \mathcal{H}^{(n)}$, the following hold:

$$\begin{aligned} &((\widetilde{A}+H_b')^{-p}a(f)^*B_j(\widetilde{A}+H_b')^{-1}\Psi)^{(n+1)}(k_1,\ldots,k_{n+1}) \\ &\quad = \frac{1}{\sqrt{n+1}}\left(\widetilde{A}+\sum_{j=1}^{n+1}\omega(k_j)\right)^{-p}\sum_{i=1}^n f(k_i)(B_j(\widetilde{A}+H_b')^{-1}\Psi)^{(n)} \\ &\qquad \times(k_1,\ldots,\hat{k}_i,\ldots,k_{n+1}), \\ &\qquad ((\widetilde{A}+H_b')^{-p}a(f)B_j(\widetilde{A}+H_b')^{-1}\Psi)^{(n-1)}(k_1,\ldots,k_{n-1}) \\ &\quad = \sqrt{n}\left(\widetilde{A}+\sum_{i=1}^{n-1}\omega(k_i)\right)^{-p}\int_{\mathbb{R}^\nu} \mathrm{d}k_n f(k_n)^*(B_j(\widetilde{A}+H_b')^{-1}\Psi)^{(n)} \\ &\qquad \times(k_1,\ldots,k_{n-1},k_n), \end{aligned}$$

where $\hat{k}_i$ indicates the omission of $\hat{k}_i$. Hence by (5.40), we have

$$\begin{aligned} &\|((\widetilde{A}+H_b')^{-p}a(f)^*B_j(\widetilde{A}+H_b')^{-1}\Psi)^{(n+1)}(k_1,\ldots,k_{n+1})\| \\ &\quad \leq \frac{1}{\sqrt{n+1}}\left(\sum_{j=1}^{n+1}\omega(k_j)\right)^{-p}\sum_{i=1}^{n+1}|f(k_i)|\left\{a\varepsilon + b_\varepsilon\frac{1}{\sum_{h=1,h\neq i}^{n+1}\omega(k_h)}\right\} \\ &\qquad \times\|\Psi^{(n)}(k_1,\ldots,\hat{k}_i,\ldots,k_{n+1})\|, \\ &\|((\widetilde{A}+H_b')^{-p}a(f)B_j(\widetilde{A}+H_b')^{-1}\Psi)^{(n-1)}(k_1,\ldots,k_{n-1})\| \\ &\quad \leq \sqrt{n}\left(\sum_{i=1}^{n-1}\omega(k_i)\right)^{-p}\int_{\mathbb{R}^\nu}\mathrm{d}k_n|f(k_n)|\left\{a\varepsilon + b_\varepsilon\frac{1}{\sum_{h=1}^n\omega(k_h)}\right\} \\ &\qquad \times\|\Psi^{(n)}(k_1,\ldots,\ldots,k_n)\|. \end{aligned}$$

By these estimates and (5.39), one can see that Ψ_0 is in $D\left(\prod_{\alpha=1}^q (H_0'-E_0)^{-(r_\alpha+1)}Q_0H_{\mathrm{I}}\right)$. Thus the desired result follows. □

Remark 5.16. The condition $g_j/\omega \in L^2(\mathbb{R}^\nu)$ $(j = 1, \ldots, J)$ is called the infrared regular condition in the GSB model (if this condition is not satisfied for some j, then H_{GSB} may have no ground states [7]). Condition (5.38) with $N \geq 3$ means that g_j is more infrared regular than the case $N = 2$. On the other hand, (4.5) expresses a regularity of Ψ_0 in terms of H_0 and H_{I} in the sense to which operator domains it belongs. Hence Lemma 5.14 shows that the regularity of Ψ_0 in the sense just mentioned is closely related to the infrared behavior of cutoff functions g_j $(j = 1, \ldots, J)$.

Theorem 5.17. *Assume* (A.1)–(A.6) *and* (5.38) *with* $N \geq 4$ *even. Let* b_n $(n \geq 1)$ *be the number* a_n *defined by* (4.4) *with* Q_0, H_0, H_{I} *and* Ψ_0 *replaced by those in the GSB model. Then*

$$b_{2n-1} = 0, \quad n = 1, \ldots, \frac{N}{2} \tag{5.41}$$

and

$$E_0(\lambda) = E_0 + \sum_{n=1}^{N/2} b_{2n}\lambda^{2n} + o(\lambda^N) \quad (\lambda \to 0). \tag{5.42}$$

Proof. By the discussions made so far and Lemma 5.14, to prove that $E_0(\lambda)$ has an asymptotic expansion up to Nth order in λ, we need only to show that, in the present case, (4.8) holds. We first do it. We have

$$H_0' - E(\lambda) = (H_0' - E_0) + E_0 - E(\lambda)$$

and $E_0 - E(\lambda) > 0$. Hence, in the same way as in the proof of Lemma 5.14, we can show that, for all $\Psi \in D(G(\lambda)) \cap \mathcal{H}^{(n)}$,

$$\begin{aligned}
&\|(a(f)^* B_j (E(\lambda) - H_0')^{-1}\Psi)^{(n+1)}(k_1, \ldots, k_{n+1})\| \\
&\quad \leq \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} |f(k_i)| \left\{ a\varepsilon + b_\varepsilon \frac{1}{\sum_{h=1, h\neq i}^{n+1} \omega(k_h)} \right\} \\
&\qquad \times \|\Psi^{(n)}(k_1, \ldots, \hat{k}_i, \ldots, k_{n+1})\|, \\
&\|(a(f) B_j (E(\lambda) - H_0')^{-1}\Psi)^{(n-1)}(k_1, \ldots, k_{n-1})\| \\
&\quad \leq \sqrt{n} \int_{\mathbb{R}^\nu} \mathrm{d}k_n |f(k_n)| \left\{ a\varepsilon + b_\varepsilon \frac{1}{\sum_{h=1}^{n} \omega(k_h)} \right\} \|\Psi^{(n)}(k_1, \ldots, \ldots, k_n)\|.
\end{aligned}$$

By repeating this type of estimates, we can see that (4.8) holds. Therefore, by Theorem 4.1, we obtain (4.9) with a_n replaced by b_n. By Theorem 5.1, $E_0(\lambda)$ is even in λ. This implies (5.41). □

Remark 5.18. We can prove (5.41) by induction in n, too.

Remark 5.19. The methods described above would be applied also to other models [4] of a quantum system interacting with a massless quantum field.

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References

[1] Abdesselam, A.: The ground state energy of the massless spin-boson model. Ann. Henri Poincaré **12**, 1321–1347 (2011)

[2] Abdesselam, A., Hasler, D.: Analyticity of the ground state energy for massless Nelson models. Commun. Math. Phys. **310**, 511–536 (2012)

[3] Arai, A.: Essential spectrum of a self-adjoint operator on an abstract Hilbert space of Fock type and applications to quantum field Hamiltonians. J. Math. Anal. Appl. **246**, 189–216 (2000)

[4] Arai, A.: Mathematical theory of quantum particles interacting with a quantum field. In: Obata, N., Matsui, T., Hora, A. (eds.) Non-Commutativity, Infinite-Dimensionality and Probability at the Crossroads, pp. 1–50. World Scientific, Singapore (2002)

[5] Arai, A., Hirokawa, M.: On the existence and uniqueness of ground states of a generalized spin-boson model. J. Funct. Anal. **151**, 455–503 (1997)

[6] Arai, A., Hirokawa, M.: Stability of ground states in sectors and its application to the Wigner–Weisskopf model. Rev. Math. Phys. **13**, 513–527 (2001)

[7] Arai, A., Hirokawa, M., Hiroshima, F.: On the absence of eigenvalues of Hamiltonians in a class of massless quantum field models without infrared cutoff. J. Funct. Anal. **168**, 470–497 (1999)

[8] Bach, V., Fröhlich, J., Pizzo, A.: Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field. Commun. Math. Phys. **264**, 145–165 (2006)

[9] Bach, V., Fröhlich, J., Pizzo, A.: Infrared-finite algorithms in QED II. The expansion of the groundstate of an atom interacting with the quantized radiation field. Adv. Math. **220**, 1023–1074 (2009)

[10] Bach, V., Fröhlich, J., Sigal, I.M.: Renormalization group analysis of spectral problems in quantum field theory. Adv. Math. **137**, 205–298 (1998)

[11] Bach, V., Fröhlich, J., Sigal, I.M.: Quantum electrodynamics of confined nonrelativistic particles. Adv. Math. **137**, 299–395 (1998)

[12] Brillouin, L.: Champs self-consistents et electrons metalliques—III. J. Phys. Radium **4**, 1–9 (1933)

[13] Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli–Fierz Hamiltonians. Rev. Math. Phys. **11**, 383–450 (1999)

[14] Griesemer, M., Hasler, D.: Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation. Ann. Henri Poincaré **10**, 577–621 (2009)

[15] Hasler, D., Herbst, I.: Ground states in the spin boson model. Ann. Henri Poincaré **12**, 621–677 (2011)

[16] Hainzl, C., Seiringer, R.: Mass renormalization and energy level shift in nonrelativistic QED. Adv. Theor. Math. Phys. **6**, 847–871 (2002)

[17] Hiroshima, F.: Analysis of ground states of atoms interacting with a quantized radiation field. In: Araki, H., Ezawa, H. (eds.) Topics in the Theory of Schrödinger Operators. Chapter 5, World Scientific, Singapore (2004)

[18] Kato, T.: Perturbation Theory for Linear Operators. 2nd edn. Springer, Berlin (1976)

[19] Nelson, E.: Interaction of nonrelativistic particles with a quantized scalar field. J. Math. Phys. **5**, 1190–1197 (1964)

[20] Reed, M., Simon, B.: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness. Academic Press, New York (1975)

[21] Reed, M., Simon, B.: Methods of Modern Mathematical Physics IV: Analysis of Operators. Academic Press, New York (1978)

[22] Wigner, E.P.: On a modification of the Rayleigh-Schrödinger perturbation theory. Magyar Tudományos Akadémia Matematikai és Természettudományi Értesitöje **53**, 477–482 (1935)

[23] Wightman, A.S. (ed.): Collected Works of Eugene Paul Wigner Part A Volume IV, pp. 131–136, Springer, Berlin (1997)

[24] Ziman, J.M.: Elements of Advanced Quantum Theory. Cambridge University Press, Cambridge (1969)

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