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Renormalizability Conditions for Almost-Commutative Manifolds

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Abstract. We formulate conditions under which the asymptotically expanded spectral action on an almost-commutative manifold is renormalizable as a higher-derivative gauge theory. These conditions are of graph theoretical nature, involving the Krajewski diagrams that classify such manifolds. This generalizes our previous result on (super) renormalizability of the asymptotically expanded Yang–Mills spectral action to a more general class of particle-physics models that can be described geometrically in terms of a noncommutative space. In particular, it shows that the asymptotically expanded spectral action which at lowest order gives the Standard Model of elementary particles is renormalizable.

1. Introduction

Over the past few years, it has turned out that many particle-physics models can be described geometrically by modifying the internal structure of space-time, making it slightly noncommutative. Indeed, there are so-called *almost-commutative manifolds* that allow for a geometrical derivation of Yang–Mills theory [2,8], or even the full Standard Model, including Higgs potential and neutrino mass terms [9,13,24]. Theories that go beyond the Standard Model were described in [25,33–35]. Also supersymmetric models such as $N = 1$ super-Yang–Mills theory and supersymmetric QCD have been derived geometrically [4,5].

The basic idea in all these examples is that one describes an almost-commutative manifold by spectral data, and then applies a general spectral action principle to derive physical Lagrangians. This paper continues on some of our recent results on renormalizability of the asymptotically expanded spectral action considered as higher-derivative theories: in [37,38], we have shown that the Yang–Mills model is superrenormalizable as a gauge theory by observing that the asymptotically expanded spectral action contains natural higher-derivative regulators. We stress the importance of taking an asymptotic expansion, as it allows for a derivation of local Lagrangians, in contrast to eg. [23]. There, the full spectral action was considered as a non-local field theory, behaving completely differently for large momenta. This is also explained in [22].

In the present paper, we will formulate conditions for almost-commutative manifolds that render the (asymptotically expanded) spectral action renormalizable as a gauge theory, and even superrenormalizable if we take sufficiently many terms into account in the asymptotic expansion. We show that these conditions apply to the afore-mentioned physical models. A convenient way to express our conditions is in terms of cycles in Krajewski diagrams for the finite noncommutative geometries. Such diagrams were introduced in [26], following a suggestion by Connes [12]. A short partial account of the renormalizability results has already been published in [36]; here, we provide full details.

As we proceed, we note that the asymptotically expanded spectral action considered as a higher-derivative gauge theory is not multiplicatively renormalizable. This is in concordance with the interpretation of the spectral action as defining a physical theory at one particular mass scale, as already proposed in [7,8]. For the Standard Model, this mass scale is the GUT scale.

This paper is organized as follows. In Sect. 2, we recall some basic definitions from noncommutative geometry, specializing to almost-commutative manifolds of the form $M \times F$: a product of an ordinary Riemannian manifold M with a finite noncommutative space F. We recall Krajewski's diagrammatic classification and formulate for them a notion of R-connectedness; it will be related to renormalizability later on. We derive the gauge and scalar fields as a consequence of the noncommutative structure of $M \times F$. Essentially, these appear as a twist for a Dirac operator D.

In Sect. 3, we define the spectral action for $M \times F$ as

$$\mathrm{Tr}\, f(D/\Lambda)$$

for some positive function f and a cutoff parameter Λ. This is considered as an action functional in the gauge and scalar fields. We derive the lowest-order terms in an asymptotic expansion as $\Lambda \to \infty$, as well as the terms at any order in Λ but quadratic in the fields.

In Sect. 4, we introduce a gauge fixing for almost-commutative manifolds, much inspired by 't Hooft's R_ξ-gauge fixing for models with spontaneous symmetry breaking. This allows in Sect. 4.1 for a power-counting argument to show that the asymptotically expanded spectral action on $M \times F$ is (super)renormalizable. Using results from the relevant BRST cohomology, this is then completed to show renormalizability as a gauge theory, provided the Krajewski diagram for the finite space F satisfies a certain graph-theoretical property, namely, the afore-mentioned R-connectedness. In particular, this applies to the asymptotically expanded spectral action that at lowest order is the Standard Model of elementary particles.

2. Almost-Commutative Manifolds

The class of noncommutative manifolds we will be interested in is *almost-commutative manifolds*. As a motivating example, let us start with a description of ordinary, commutative manifolds in terms of purely spectral data. Suppose M is a compact Riemannian spin manifold, with a spinor bundle $\mathcal{S}$. Then, the *Hilbert space* $L^2(M,\mathcal{S})$ of its square-integrable sections sets the stage for such a spectral description. The *Dirac operator* $D_M = i\gamma^\mu \circ \nabla^{\mathcal{S}}_\mu$ associated with the metric via the Levi–Civita connection $\nabla^{\mathcal{S}}$ lifted to the spinor bundle defines a self-adjoint operator on $L^2(M,\mathcal{S})$. Ellipticity of D_M as a differential operator and compactness of M imply that the resolvents of D_M are compact operators. Finally, the Dirac operator is compatible with the action of the coordinate functions: the action of functions $f \in C^\infty(M)$ on $L^2(M,\mathcal{S})$ by pointwise multiplication has bounded commutators $[D_M, f]$.

If the manifold M is of even dimension m, there is a grading (chirality) γ_M, making D_M an odd operator. Finally, spin-manifolds are selected out of spin^c-manifolds by the charge conjugation operator: it is an anti-linear operator $J_M : L^2(M,\mathcal{S}) \to L^2(M,\mathcal{S})$.

This canonical 'triple' $(C^\infty(M), L^2(M,\mathcal{S}), D_M; \gamma_M, J_M)$ motivates the following abstract definition of a spectral triple [10,11].

Definition 1. A spectral triple $(\mathcal{A},\mathcal{H},D)$ is given by an unital $*$- algebra $\mathcal{A}$ represented faithfully as operators in a Hilbert space $\mathcal{H}$ and a self-adjoint operator D such that $(1+D^2)^{-1/2}$ is a compact operator and $[D,a]$ bounded for $a \in \mathcal{A}$.

A spectral triple is even if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading γ such that $[\gamma, a] = 0$ and $\{\gamma, D\} = 0$.

A real structure of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$ on a spectral triple is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad J\gamma = \varepsilon'' \gamma J \quad \text{(even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n \mod 8$:

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

Moreover, with $b^0 = Jb^* J^{-1}$ we impose that

$$[a, b^0] = 0, \quad [[D,a], b^0] = 0, \quad \forall\, a, b \in \mathcal{A},$$

A spectral triple with a real structure is called a real spectral triple.

In particular, the real structure gives $\mathcal{H}$ the structure of an $\mathcal{A}$-bimodule. In other words, the algebra $\mathcal{A} \otimes \mathcal{A}^\circ$ acts on $\mathcal{H}$, where $\mathcal{A}^\circ$ is the opposite algebra to $\mathcal{A}$.

Definition 2. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The $\mathcal{A}$-bimodule of Connes' differential one-forms is given by

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum_k a_k[D, b_k] : a_k, b_k \in \mathcal{A} \right\}$$

In the case of the canonical triple, Clifford multiplication establishes an isomorphism (cf. [10,28])

$$\Omega^1(M) \simeq \Omega^1_{D_M}(C^\infty(M)).$$

Besides the canonical triple for a Riemannian spin manifold M, there is the following class of simple examples.

Definition 3. A finite real spectral triple is a spectral triple for which the Hilbert space is finite dimensional. We will write such a spectral triple suggestively,

$$F := (A_F, H_F, D_F; \gamma_F, J_F)$$

Example 4. The algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices acts on itself by left and right matrix multiplication; this gives rise to a finite real spectral triple

$$(A_F = M_n(\mathbb{C}),\ H_F = M_n(\mathbb{C}),\ D_F = 0;\ J_F = (\cdot)^*).$$

This example is closely related to Yang–Mills theories (cf. [7,8])

Example 5. The noncommutative description of the Standard Model is based on the real algebra

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

It is represented on $\mathbb{C}^{96}$, where 96 is 2 (particles and anti-particles) times 3 (families) times 4 leptons plus 4 quarks with 3 colors each. Thus, the noncommutative Standard Model includes right-handed neutrinos. Finally, there is a 96×96 matrix D_F, a grading γ_F and real structure J_F, which are explicitly described in [9,13]; they constitute a real spectral triple of KO-dimension 6.

We will be interested in a combination of Riemannian spin manifolds and such finite triples.

Definition 6. An almost-commutative manifold (AC manifold) is given by the tensor product of the canonical triple and a finite spectral triple:

$$M \times F := (C^\infty(M) \otimes A_F, L^2(M, S) \otimes H_F, D_M \otimes 1 + \gamma_5 \otimes D_F)$$

The picture one should have in mind is that of Kaluza–Klein theories, where the spacetime manifold was extended by an extra dimension. In the present case, this extra dimension is the finite noncommutative space F.

2.1. Classification of Finite Spectral Triples

A first classification of finite spectral triples $(A_F, H_F, D_F; \gamma_F, J_F)$ appeared in [30]. We follow the work of Krajewski [26] where a diagrammatic approach to such a classification was introduced, generalizing it to arbitrary KO-dimension. Recall that if M is an A_F-bimodule, the *contragredient A_F-bimodule M°* is defined by

$$M^\circ = \{\overline{m} : m \in M\}$$

with action $\overline{m} \mapsto a\overline{m}b = \overline{b^* m a^*}$ for all $a, b \in A_F, m \in M$. In particular, if M is a left A_F-module, M° is a right A_F-module.

The structure of A_F can be determined explicitly from Wedderburn's Theorem:

$$A_F \simeq \bigoplus_{i=1}^{N} M_{k_i}(\mathbb{F}_i). \tag{1}$$

for some $k_1, \ldots, k_N$ and $\mathbb{F}_i = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ depending on i.

In the following, we will denote by $\Gamma^{(0)}$ and $\Gamma^{(1)}$ the vertex and edges sets of an oriented graph Γ with source and target maps $s, t : \Gamma^{(1)} \to \Gamma^{(0)}$. Also, we indicate by $(v_1 v_2)$ an edge between vertices v_1 and v_2.

Definition 7. Given a finite-dimensional algebra $A_F = \bigoplus_i M_{k_i}(\mathbb{F}_i)$, a Krajewski diagram for A_F of KO-dimension n is an oriented decorated graph Γ with the following properties

1. Edges between two vertices come in pairs with opposite orientation: if $e = (v_1 v_2)$ is an edge, then there also exists an edge $\overline{e} = (v_2 v_1)$ and these come in pairs.
2. Each vertex v is decorated by an irreducible A_F-bimodule M_v together with a choice of basis, i.e. $M_v = \mathbb{C}^{n_v} \otimes \mathbb{C}^{m_v \circ}$ for some $n_v, m_v \in \{k_1, \ldots, k_N\}$.
3. Each edge e is decorated by a non-zero first-order operator $D_e : M_{s(e)} \to M_{t(e)}$, i.e. such that

   $$D_e(amb) = aD_e(mb) + D_e(am)b - aD_e(m)b; \quad (a, b \in A_F, m \in M_v),$$

   and $D_{\overline{e}} = D_e^*$.
4. There is an involutive graph automorphism $j : \Gamma \to \Gamma$ such that $n_{j(v)} = m_v$ for all $v \in \Gamma^{(0)}$. In other words, $M_{j(v)} = M_v^\circ$. If $J_v : M_v \to M_v^\circ$ is the anti-linear map that assigns to a bimodule its contragredient bimodule,[1] we demand that for an edge $e = (v_1 v_2)$:

   $$D_{j(e)} = \epsilon' J_{j(v_2)} D_e J_{j(v_1)}^{-1}$$

In the even case, there is an additional labeling on the vertices by signs ± 1 and the edges connect only vertices of opposite signs. Moreover, if v has sign ± 1, then $j(v)$ has sign $\pm \epsilon''$.

[1] Actually, this is slightly more subtle in the case of KO-dimension 2,3,4, or 5; in that case, one needs two vertices v_1, v_2 and one has two maps $J_{v_1} : M_{v_1} \to M_{j(v_2)}$ and $J_{v_2} : M_{v_2} \to M_{j(v_1)}$ that satisfy $J_{j(v_2)} \circ J_{v_1} = -1$.

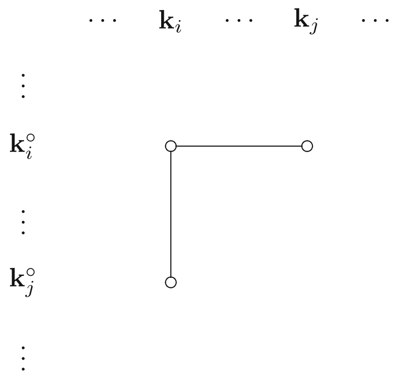


Figure 1. The lines between two nodes represent a non-zero $D_e : \mathbb{C}^{n_{s(e)}} \otimes \mathbb{C}^{m^\circ_{s(e)}} \to \mathbb{C}^{n_{t(e)}} \otimes \mathbb{C}^{m^\circ_{t(e)}}$, as well as its adjoint $D_{\overline{e}} : \mathbb{C}^{n_{t(e)}} \otimes \mathbb{C}^{m^\circ_{t(e)}} \to \mathbb{C}^{n_{s(e)}} \otimes \mathbb{C}^{m^\circ_{s(e)}}$. The non-zero components $D_{j(e)}$ and $D_{j(\overline{e})}$ are related to $\pm D_e$ and $\epsilon' D_{\overline{e}}$

Usually, one depicts a Krajewski diagram as embedded in the plane, with the columns and rows labeled by the integers k_i that appear in the decomposition (1) of A_F. One places a node at (n_v, m_v) for each vertex in Γ with $M_v = \mathbb{C}^{n_v} \otimes \mathbb{C}^{m_v \circ}$. The pairs $(e, \overline{e})$ of oriented edges in Γ are indicated by a single line in the planar diagram, which by (3) run only horizontally or vertically. The graph automorphism j translates as a reflectional symmetry of the diagram along the diagonal, with the labeling ± 1 mapped to $\pm\epsilon''$. Note that in [26], these latter signs were all $\epsilon'' = +1$, for the KO-dimension 0 case.

Given a Krajewski diagram $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)})$ for A_F, we construct a finite spectral triple for the algebra A_F as follows. We define

$$H_F = \bigoplus_{v \in \Gamma^{(0)}} M_v = \bigoplus_{v \in \Gamma^{(0)}} \mathbb{C}^{n_v} \otimes \mathbb{C}^{m_v \circ}$$

on which A_F acts on the left. The real structure J_F is the sum of operators

$$J_v : M_v \to M_{j(v)}.$$

The Dirac operator D_F is the sum of the operators

$$D_e : M_{s(e)} \to M_{t(e)}.$$

This defines a symmetric linear operator because $D_{\overline{e}} = D_e^*$. Finally, in the even case, the signs on M_v give rise to a grading γ_F on H_F for which D_F is odd and the left action of A_F on H_F is even, and such that $\gamma_F J_F = \epsilon'' J_F \gamma_F$ (Fig. 1).

Example 8. The Krajewski diagram for the Standard Model (for one generation) is depicted in Fig. 2. It indicates the precise structure of $H_F = \mathbb{C}^{96}$ as a representation space of $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The double appearance of the row and column **1** accounts for multiplicities of the corresponding representations in H_F. The nonempty blocks in the matrix D_F are indicated by the

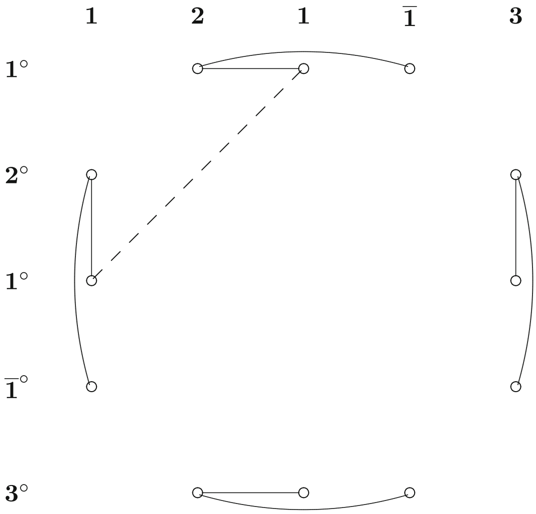


FIGURE 2. The Krajewski diagram of the Standard Model

dotted and straight lines, reflection along the diagonal gives J_F, and γ_F is of opposite sign when reflected along the diagonal (KO-dimension 6). More (eg. three) generations can be taken into account by adding multiplicities for all of the above vertices.

Krajewski has shown in [26] (at least for KO-dimension $n = 0$) that there is a one-to-one correspondence between such diagrams and finite real spectral triples modulo unitary equivalence (related to the choice of basis of M_v). We recall from *loc. cit.* a useful result for the Dirac operator.

Lemma 9. [26] Let $(A_F, H_F, D_F; \gamma_F, J_F)$ be a finite real spectral triple. There is a decomposition $D_F = D_0 + \Delta + J_F \Delta J_F^{-1}$ where D_0 commutes with the action of $A_F \otimes A_F^\circ$ and Δ commutes with the action of A_F°. Moreover,

$$\Delta = - \int_{\mathcal{U}(A_F)} g[\Delta, g^*] \mathrm{d}\mu(g), \tag{2}$$

where $\mathrm{d}\mu$ is the Haar measure on the Lie group $\mathcal{U}(A_F)$ consisting of unitaries in A_F.

Proof. If Γ is the Krajewski diagram corresponding to $(A_F, H_F, D_F; \gamma_F, J_F)$ then we can decompose $D = \sum_e D_e$ with each $D_e : M_{s(e)} \to M_{t(e)}$. As before, write $M_v = \mathbb{C}^{n_v} \otimes \mathbb{C}^{m_v \circ}$ for all v. We denote by D_0 the sum of such operators for which $n_{s(e)} = n_{t(e)}$ and $m_{s(e)} = m_{t(e)}$. If this is not the case, then since any first-order operator such as D_e splits into left A_F-linear and right A_F-linear components, we have that either $n_{s(e)} = n_{t(e)}$ or $m_{s(e)} = m_{t(e)}$, respectively. It is clear that $D_F - D_0$ is the sum of these, so we define the right A_F-linear (or left A_F°-linear) operator

$$\Delta = \sum_{e: m_{s(e)} = m_{t(e)}} D_e.$$

Then, $J_F \Delta J_F^{-1}$ gives the remaining sum over all edges e for which $n_{s(e)} = n_{t(e)}$, showing $D_F = D_0 + \Delta + J_F \Delta J_F^{-1}$.

For the integral formula (2), we compute the matrix coefficients of the difference between the left and the right-hand side in Eq. (2) in terms of a basis of the Hilbert space H_F. We decompose

$$H_F = \bigoplus_v M_v = \bigoplus_v \mathbb{C}^{n_v} \otimes \mathbb{C}^{m_v \circ},$$

and write $\{e_v^{\alpha\beta}\}$ $(\alpha = 1, \ldots, n_v, \beta = 1, \ldots m_v)$ for the corresponding basis. Then,

$$\begin{aligned} &\left\langle e_{v_1}^{\alpha_1 \beta_1}, \int_{\mathcal{U}(A_F)} g \Delta g^* \mathrm{d}\mu(g) e_{v_2}^{\alpha_2 \beta_2} \right\rangle \\ &= \int_{\mathcal{U}(A_F)} \left\langle e_{v_1}^{\alpha_1 \beta_1}, g e_{w_1}^{\alpha_1' \beta_1'} \right\rangle \Delta_{(w_2 w_1)}^{\alpha_1' \alpha_2'} \delta^{\beta_1' \beta_2'} \left\langle e_{w_2}^{\alpha_2' \beta_2'}, g^* e_{v_2}^{\alpha_2 \beta_2} \right\rangle \mathrm{d}\mu(g) \end{aligned}$$

where we sum over all repeated indices and where $\Delta_{(w_2 w_1)}^{\alpha_1' \alpha_2'} \delta^{\beta_1' \beta_2'}$ are the matrix coefficients of (the right A_F-linear) $\Delta_{(w_2 w_1)} : M_{w_2} \to M_{w_1}$. Next,

$$\left\langle e_{v_1}^{\alpha_1 \beta_1}, g e_{w_1}^{\alpha_1' \beta_1'} \right\rangle = \delta_{v_1, w_1} g_{v_1}^{\alpha_1 \alpha_1'} \delta^{\beta_1 \beta_1'}$$

in terms of the *defining* matrix coefficients $g_{v_1}^{\alpha_1 \alpha_1'}$ of $g \in \mathcal{U}(A_F)$ in the representation $\mathbb{C}^{n_{v_1}}$. Note that this is a representation of $\mathcal{U}(A_F)$, since $\mathcal{U}(A_F) \simeq \prod_i U(k_i, \mathbb{F}_i)$. This turns the above integral into

$$\Delta_{(v_2 v_1)}^{\alpha_1' \alpha_2'} \delta^{\beta_1 \beta_2} \int_{\mathcal{U}(A_F)} g_{v_1}^{\alpha_1 \alpha_1'} \overline{g_{v_2}^{\alpha_2 \alpha_2'}} \mathrm{d}\mu(g) = \Delta_{(v_2 v_1)}^{\alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} \delta_{v_1, v_2}$$

by the Peter–Weyl Theorem. However, $\Delta_{(v_2 v_1)}$ maps between different irreducible representations of $\mathcal{U}(A_F)$ which implies the vanishing of the above expression and completes the proof. □

We will now formulate a condition on Krajewski diagrams that below will turn out to characterize renormalizable models. Let Γ be a Krajewski diagram for A_F. We construct a graph $\widetilde{\Gamma}$ whose vertex set $\widetilde{\Gamma}^{(0)}$ is the set of inequivalent irreducible representations of A_F, or, of the Lie group $\mathcal{U}(A_F)$. In other words, $\widetilde{\Gamma}^{(0)}$ is the set $\{k_1, \ldots, k_N\}$ appearing in the decomposition (1). The set of edges for $\widetilde{\Gamma}$ is defined as

$$\widetilde{\Gamma}^{(1)} := \{(n, n') : \exists e \in \Gamma^{(1)} \text{ such that } n_{s(e)} = n \text{ and } n_{t(e)} = n'\}.$$

There is a map of graphs $\psi : \Gamma \to \widetilde{\Gamma}$ defined as follows. For a vertex $v \in \Gamma^{(0)}$, we set $\psi(v) = n_v \in \widetilde{\Gamma}^{(0)}$; for an edge $e \in \Gamma^{(1)}$ we set

$$\psi(e) = (n_{s(e)}, n_{t(e)}).$$

FIGURE 3. The cycle in Γ at the *left-hand side* is a lift along ψ of the cycle in $\widetilde{\Gamma}$ at the *right-hand side* (we have suppressed the loops at the two vertices)

Essentially, the graph $\widetilde{\Gamma}$ is the projection of the Krajewski diagram Γ onto the horizontal axis. By symmetry, $\widetilde{\Gamma}$ is also the projection of Γ onto the vertical axis; this corresponds to pre-composing ψ with the graph automorphism $j : \Gamma \to \Gamma$.

Adopting the usual terminology from graph theory, we will refer to an edge with the same source and target vertex as a *loop*; a *cycle* is a path which begins and starts at the same vertex, but with no other repeated vertices (i.e. it does not contain loops). In a Krajewski diagram Γ, we call a cycle *horizontal (vertical)* if it consists only of horizontal (vertical) edges.

Definition 10. In the above notation, a lift along ψ of a cycle $\gamma = \widetilde{e}_1 \cdots \widetilde{e}_m$ of length m in $\widetilde{\Gamma}$ is a cycle $e_1 \cdots e_l$ of length $l \geq m$ such that the path $\psi(e_1) \cdots \psi(e_l)$ coincides with $\widetilde{\gamma}$ modulo loops.

Figure 3 illustrates such a 'horizontal lift' of graphs; similarly, we can define a vertical lift by using the map $\psi \circ j$.

Definition 11. We say that a Krajewski diagram Γ is R-connected in dimension m if

1. every cycle $\widetilde{\gamma}$ in $\widetilde{\Gamma}$ of length $\leq m$ can be lifted along ψ to a horizontal cycle γ in Γ (necessarily of the same length),
2. every two cycles $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ of total length $\leq m$, which are not connected to a common vertex 1 or $\overline{1}$ in $\widetilde{\Gamma}$, can be lifted to a single cycle γ in Γ along ψ and $\psi \circ j$, respectively, i.e.

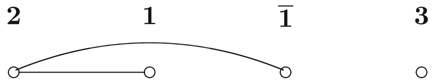
$$\psi(\gamma) \sim \widetilde{\gamma}_1; \quad \psi(j(\gamma)) \sim \widetilde{\gamma}_2$$

   where $\sim$ denotes equivalence of cycles in $\widetilde{\Gamma}$ modulo loops.
3. For $r \geq 3$, there are no tuples $\widetilde{\gamma}_1, \ldots \widetilde{\gamma}_r$ of cycles in $\widetilde{\Gamma}$ of total length $\leq m$, which are not mutually connected to a common 1 or $\overline{1}$.

Note that the last condition is trivially satisfied in the case $m \leq 4$, since every cycle has length at least 2. The case $m = 4$ happens to be our case of interest.

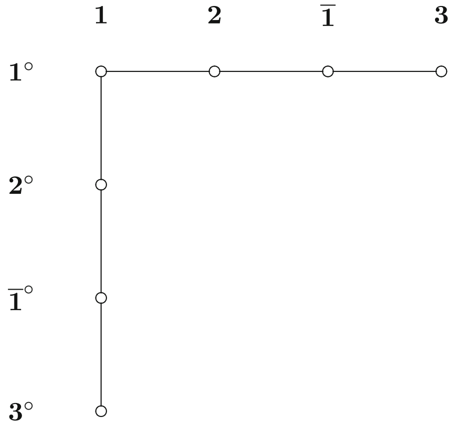
Proposition 12. The Krajewski diagram of the Standard Model (Fig. 2) is R-connected in dimension 4.

Proof. Indeed, the graph $\widetilde{\Gamma}$ is given by



where we have suppressed the loops. Every cycle and every pair of distinguished cycles in $\widetilde{\Gamma}$ of total length 4 can be lifted to a single cycle γ in the Krajewski diagram of Fig. 2. The pair consisting of two copies of the cycle $(\mathbf{12})(\mathbf{21})$ (going back-and-forth between $\mathbf{1}$ and $\mathbf{2}$) has a common vertex $\mathbf{1}$. For this reason, they do not enter in Condition (2) (which, in fact, they would not satisfy). The concatenated cycle $(\mathbf{12})(\mathbf{21})(\mathbf{12})(\mathbf{21})$ (going back-and-forth twice between $\mathbf{1}$ and $\mathbf{2}$) is of length 4, and was already treated (cf. Condition (1)). A similar argument applies to the cycle $(\overline{\mathbf{1}}\mathbf{2})(\mathbf{2}\overline{\mathbf{1}})$. □

Example 13. Let us give an example of a Krajewski diagram which is not R-connected (in dimension 4). Consider Γ given by



Then, the projected Krajewski diagram $\widetilde{\Gamma}$ is given by

so that Γ is not R-connected. Indeed, the pair of cycles $\{(\mathbf{12})(\mathbf{21}),(\overline{\mathbf{1}}\mathbf{3})(\mathbf{3}\overline{\mathbf{1}})\}$ obtained by going back and forth along the left and right edge in $\widetilde{\Gamma}$ does not lift to a single cycle in Γ.

2.2. Gauge Fields from AC Manifolds

Let us now describe how noncommutative manifolds naturally give rise to a gauge theory, following [13, Section 10.8]. For simplicity, we will restrict to almost-commutative manifolds, so that $(\mathcal{A},\mathcal{H},D)$ will always denote $M\times F$, i.e.

$$(\mathcal{A},\mathcal{H},D)=(C^\infty(M,A_F),L^2(M,\mathcal{S})\otimes H_F,D_M\otimes 1+\gamma_M\otimes D_F).$$

for some finite spectral triple $F=(A_F,H_F,D_F;\gamma_F,J_F)$.

Definition 14. Denote by $\mathcal{U}(\mathcal{A})$ the group of unitaries of $\mathcal{A}$. The gauge group of $M \times F$ is given by

$$\mathcal{SU}(\mathcal{A}) = \{u \in \mathcal{U}(\mathcal{A}) : \det_F u = 1\}$$

where the determinant is taken pointwise in the representation H_F.

The group $\mathcal{SU}(\mathcal{A})$ acts naturally on the Dirac operator D by conjugation, as well as on the representation of $\mathcal{A}$ on $\mathcal{H}$: $a \mapsto uau^*$. If there is a real structure, then we transform

$$D \mapsto UDU^*,$$

with $U = uu^{*\circ} \equiv uJuJ^{-1}$. This suggests that we should rather take the image of $\mathcal{SU}(\mathcal{A})$ under the map $u \mapsto uu^{*\circ}$. Indeed, in [18] the gauge group was defined in this way, leading to a quotient of $\mathcal{SU}(\mathcal{A})$ by an abelian group. Since in most of our examples the latter group will be finite, it will be ignored in what follows.

Proposition 15. Let $M \times F$ be an almost-commutative manifold as above and write

$$A_F = \bigoplus_{i=1}^{N} M_{k_i}(\mathbb{F}_i); \quad (\mathbb{F}_i = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}).$$

1. The gauge group of $M \times F$ is given by $\mathcal{SU}(\mathcal{A}) = C^\infty(M, \mathcal{SU}(A_F))$.
2. The Lie algebra $\mathfrak{su}(A_F)$ of $\mathcal{SU}(A_F)$ is isomorphic to

$$\mathfrak{su}(\mathcal{A}) \simeq \bigoplus_{i=1}^{N} \mathfrak{su}(k_i) \oplus \mathfrak{u}(1)^{\oplus(C-1)},$$

where C is the number of complex algebras in the above decomposition of A_F and $\mathfrak{su}(k_i)$ denotes $\mathfrak{o}(k_i), \mathfrak{su}(k_i)$ or $\mathfrak{sp}(k_i)$ depending on whether $\mathbb{F}_i = \mathbb{R}, \mathbb{C}$ or $\mathcal{H}$, respectively.

Consequently, there is a one-to-one correspondence between irreducible representations of the algebra A_F and of the Lie algebra $\mathfrak{su}(A_F)$ provided A_F contains no copies of $\mathbb{R}$, and either no complex subalgebras, or at least one non-trivial (i.e. not $\mathbb{C}$) complex subalgebra.

Proof. (1) is direct. (2). Note that $\mathfrak{u}(A)$ is a direct sum of simple Lie algebras $\mathfrak{o}(k_i), \mathfrak{u}(k_i)$ and $\mathfrak{sp}(k_i)$ according to $\mathbb{F}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. All these matrix Lie algebras have a trace, and we observe that the matrices in $\mathfrak{o}(k_i)$ and $\mathfrak{sp}(k_i)$ are already traceless. For the complex case, we can write $X_i \in \mathfrak{u}(k_i)$ as $X_i = Y_i + z_i$ where $z_i = \operatorname{Tr} X_i$, showing that:

$$\mathfrak{u}(k_i) = \mathfrak{su}(k_i) \oplus \mathfrak{u}(1).$$

The determinant condition in the definition of $\mathcal{SU}(\mathcal{A})$ translates at the infinitesimal level to the unimodularity condition $\operatorname{Tr}_{H_F} X = 0$. Explicitly, this becomes

$$\sum_i \alpha_i \operatorname{Tr}(X_i) = 0$$

where α_i are the multiplicities of the fundamental representations of $M_{k_i}(\mathbb{F}_i)$ appearing in H_F. Using the above property for the traces on simple matrix Lie algebras, we find that unimodularity is equivalent to

$$\sum_{l=1}^{C} \alpha_{i_l} z_{i_l} = 0$$

where the sum is over the complex factors (i.e. for which $\mathbb{F}_i = \mathbb{C}$) in A, labeled by $i_1, \ldots, i_C$. This reduces the C abelian factors to $C-1$ copies of $\mathfrak{u}(1)$. □

Example 16. For the Standard Model spectral triple of Example 5 (cf. Example 8) this gives $\mathfrak{su}(A_F) = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, as desired ([9, Proposition 2.16] or [13, Proposition 1.185]).

Now that we have found the gauge group of an almost-commutative manifold, let us determine the gauge fields that $M \times F$ naturally gives rise to through the differential one-forms.

Proposition 17. The differential one-forms $\Omega^1_D(\mathcal{A})$ on $M \times F$ allow for a direct sum decomposition:

$$\Omega^1_D(\mathcal{A}) \simeq \Omega^1(M, A_F) \oplus C^\infty(M, \Omega^1_{D_F}(A_F)).$$

where $\Omega^1(M, A_F) \equiv \Omega^1(M) \otimes A_F$. Moreover, the A_F-bimodule of differential one-forms $\Omega^1_{D_F}(A_F)$ is generated by Δ.

Proof. This follows directly from the splitting

$$D = D_M \otimes 1 + \gamma_M \otimes D_F$$

noting further that γ_μ and γ_M are orthogonal with respect to the Hilbert–Schmidt inner product.

The integral formula for Δ in Lemma 9 combined with the observation that $[D, a] = [\Delta, a]$ for all $a \in A_F$ shows that Δ is already a one-form; this shows that $A_F \Delta A_F \subset \Omega^1_{D_F}(A_F)$. The same observation also shows that $\Omega^1_{D_F}(A_F) \subset A_F \Delta A_F$. □

Let us describe the linearly independent components of $\Omega^1_{D_F}(A_F)$; inspired by the discussion in Krajewski [26].

An element $\phi \in \Omega^1_{D_F}(A_F)$ is given by sums of elements of the form

$$a\Delta b = \sum_{e : m_{s(e)} = m_{t(e)}} a D_e b.$$

Since some edges induce linear operators D_e between the same representations of A_F, the above summands are not independent. To turn this into a sum over linearly independent terms, the graph $\widetilde{\Gamma}$ introduced previously is quite convenient. Namely, given an edge $\widetilde{e}$ in $\widetilde{\Gamma}$ connecting different vertices, we consider the linear span $S_{\widetilde{e}}$ in $\mathrm{Hom}(\mathbb{C}^{s(\widetilde{e})}, \mathbb{C}^{t(\widetilde{e})})$ of all matrices D_e with $e \in \psi^{-1}(\widetilde{e})$. If $\{f^p_{\widetilde{e}}\}_p$ $(p = 1, \ldots, \dim S_{\widetilde{e}})$ is a basis for $S_{\widetilde{e}}$ we can write

$$D_e = \sum_p M^p_e f^p_{\psi(e)}, \quad M^p_e \in \mathbb{C}. \tag{3}$$

Note that the self-adjointness of D implies that $M^p_{\overline{e}} = \overline{M}^p_e$ and $f^p_{\psi(\overline{e})} = (f^p_{\psi(e)})^*$. Adopting this form of D_e, we can write

$$a\Delta b = \sum_{\widetilde{e}\in\widetilde{\Gamma}^{(1)}:s(\widetilde{e})\neq t(\widetilde{e})} \sum_p \left(\sum_{\substack{e\in\psi^{-1}(\widetilde{e})\\ m_{s(e)}=m_{t(e)}}} M^p_e \right) a f^p_{\widetilde{e}} b.$$

We denote the independent fields by

$$\phi^p_{\widetilde{e}} = a f^p_{\widetilde{e}} b : \quad a, b \in A_F.$$

which is an element in $\mathrm{Hom}(\mathbb{C}^{s(\widetilde{e})}, \mathbb{C}^{t(\widetilde{e})})$. Thus, we can write a general element $\phi \in \Omega^1_{D_F}(A_F)$ as

$$\phi = \sum_{e:m_{s(e)}=m_{t(e)}} \sum_p M^p_e \phi^p_{\widetilde{e}}.$$

We conclude that the number of independent components of ϕ is

$$\sum_{\widetilde{e}:s(\widetilde{e})\neq t(\widetilde{e})} s(\widetilde{e}) t(\widetilde{e}) \dim S_{\widetilde{e}}$$

A corresponding orthonormal basis (orthonormal with respect to the Hilbert–Schmidt norm on $\Omega^1_{D_F}(A_F) \subset \mathrm{End}\, H_F$) can be found by combining the indices $\widetilde{e}$ and p with the canonical bases of $\mathbb{C}^{s(\widetilde{e})}$ and $\mathbb{C}^{t(\widetilde{e})}$: we denote this orthonormal basis of $\Omega^1_{D_F}(\mathcal{A}_F)$ by $\{e_I\}_I$.

The vertices of $\widetilde{\Gamma}$ label irreducible representations of A_F, and consequently of $\mathfrak{su}(A_F)$. Thus, the fields $\phi^p_{\widetilde{e}}$ carry the induced representation, that is, by conjugation of the source and target representations $s(\widetilde{e})$ and $t(\widetilde{e})$.

Example 18. Consider the Krajewski diagram of the Standard Model (Fig. 2). The fields that appear connect the vertices 2 and 1, and 2 and $\overline{1}$ in $\widetilde{\Gamma}$: they carry the induced representation of $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. In fact, this is precisely the Higgs doublet in the electroweak model, having 2 independent degrees of freedom.

Let us end this section by describing the so-called *inner fluctuations of the metric*, induced by coupling D to gauge fields in $\Omega^1_D(\mathcal{A})$. The origin of this can also nicely be described in terms of Morita self-equivalences of the algebra $\mathcal{A}$ (cf. [13, Sect. 10.8]).

We consider a self-adjoint element $\omega + \gamma_M \phi \in \Omega^1_D(\mathcal{A})$, in terms of the splitting in Proposition 17. The unimodularity condition on the gauge group is transferred to the gauge fields by demanding that $\mathrm{Tr}_F\, \omega = 0$. Combining this with self-adjointness implies that $\omega \in \Omega^1(M, i\mathfrak{su}(A_F))$. This allows for an inner fluctuation:

$$D \rightsquigarrow D + A + \gamma_M \Phi$$

where

$$A = \omega + \epsilon' J \omega J^{-1} = i\gamma^\mu \,\mathrm{ad}\, \omega_\mu; \quad \Phi = \phi + \epsilon' J \phi J^{-1}.$$

These formulas can be checked using the splitting of Proposition 17.

Remark 19. Note that a term such as $\mathrm{Tr}_F\,\Phi^l$ (with the trace in H_F) can be easily computed from the Krajewski diagram [27]. Indeed, it corresponds to a sum over cycles in Γ of length l, for which the trace splits over a horizontal and vertical part:

$$\mathrm{Tr}_F\,\Phi^l = \sum_{\gamma=e_l\cdots e_1}\sum_{p_i} c_{p_1\ldots p_l}(\gamma)\,\mathrm{Tr}_{s(\widetilde{e}_1)}\left(\phi^{p_l}_{\widetilde{e}_l}\ldots\phi^{p_1}_{\widetilde{e}_1}\right)\mathrm{Tr}_{s(\widetilde{j(e_1)})}\left(\phi^{p_l}_{\widetilde{j(e_l)}}\ldots\phi^{p_1}_{\widetilde{j(e_1)}}\right)$$

where we have denoted $\widetilde{e}_i = \psi(e_i)$ and $\phi^p_{\widetilde{e}} \equiv 1$ if $\widetilde{e}$ is a loop in $\widetilde{\Gamma}$ (i.e., if $s(\widetilde{e}) = t(\widetilde{e})$). The coefficient is given essentially by

$$c_{p_1\ldots p_n}(\gamma) \propto M^{p_1}_{e_1}\cdots M^{p_l}_{e_l}$$

in terms of the basis coefficients of D_e in Eq. (3). Moreover, the self-adjointness of Φ implies that the components $\phi^p_{\widetilde{e}}$ satisfy:

$$\phi^p_{\widetilde{\overline{e}}} = (\phi^p_{\widetilde{e}})^*$$

where we recall that $\overline{e}$ is the edge e with reversed orientation.

This last Remark and its relation to the notion of R-connectedness of Definition 11 will play a crucial role in the subsequent discussion on renormalization of the gauge field theories that correspond to $M \times F$, which we will now define.

3. Spectral Action for Almost-Commutative Manifolds

Starting with an almost-commutative manifold $M\times F$ with Krajewski diagram Γ for F, we have now set the stage for a gauge theory on M. Summarizing, we have derived:

1. a gauge group $\mathcal{SU}(\mathcal{A}) = C^\infty(M, \mathcal{SU}(A_F)$ with reductive (local) gauge algebra $\mathfrak{su}(A_F)$,
2. gauge fields A in the adjoint representation of this gauge group,
3. scalar fields Φ, with independent components $\phi_{\widetilde{e}} \in \mathrm{Hom}(V, W)$ with V and W irreducible representations of $\mathcal{SU}(A_F)$, parametrized by the vertices $s(\widetilde{e})$ and $t(\widetilde{e})$ in the graph $\widetilde{\Gamma}$.

We search for gauge invariant action functionals. The simplest, manifestly gauge invariant one is given the trace of a function of the fluctuated Dirac operator [7,8]:

$$S[A,\Phi] := \mathrm{Tr}\, f\left(\frac{D + A + \gamma_M\Phi}{\Lambda}\right) - \mathrm{Tr}\, f\left(\frac{D}{\Lambda}\right),$$

together with a real cut-off parameter Λ. We have subtracted the purely gravitational part $\mathrm{Tr}\, f(D/\Lambda)$, being interested mostly in the gauge part of the spectral action. Locally, we have

$$D + A = i\gamma^\mu(\nabla^S_\mu + A_\mu).$$

with ∇^S_μ the spin connection on a Riemannian spin manifold M and A_μ a skew-hermitian traceless matrix. The field Φ is considered as a self-adjoint element in $C^\infty(M, \mathrm{End}\, H_F)$.

For simplicity, we take M to be flat (i.e. vanishing Riemann curvature tensor) and 4-dimensional; we, therefore, write $\gamma_5 \equiv \gamma_M$ for the grading. Furthermore, we will assume that f is a suitable Laplace transform:

$$f(x) = \int_{t>0} e^{-tx^2} g(t)\mathrm{d}t.$$

Proposition 20. [8] In the above notation, there is an asymptotic expansion (as $\Lambda \to \infty$):

$$S[A,\Phi] \sim \sum_{m>0} \Lambda^{4-m} f_{4-m} \int_M a_m(x, (D + A + \gamma_5\Phi)^2), \tag{4}$$

in terms of the Seeley–De Witt invariants of $(D + A + \gamma_5\Phi)^2$. The coefficients are defined by $f_k := \int t^{-k/2} g(t)\mathrm{d}t$.

We will denote the asymptotic expansion on the right-hand side of Eq. (4) by $S^\Lambda[A,\Phi]$. Recall that the Seeley–De Witt coefficients $a_m(x,(D + A + \gamma_5\Phi)^2)$ are gauge invariant polynomials in the fields A_μ and Φ. Indeed, the Weitzenböck formula gives

$$(D + A + \gamma_5\Phi)^2 = -\nabla_\mu\nabla^\mu - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - \gamma_5[D + A,\Phi] + \Phi^2 \tag{5}$$

in terms of the curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ of A_μ and $\nabla_\mu = \partial_\mu + A_\mu$. Consequently, a Theorem by Gilkey [20, Theorem 4.8.16] shows that (in this case) a_m are polynomial gauge invariants in $F_{\mu\nu}$ and the endomorphisms

$$E = \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \gamma_5[D + A,\Phi] - \Phi^2$$

as well as their covariant derivatives (with respect to the connection A_μ). The *order* ord of a_m is m, where we set on generators:

$$\mathrm{ord}\, A_{\mu_1;\mu_2\cdots\mu_k} = k; \quad \mathrm{ord}\, \Phi_{;\mu_1\cdots\mu_k} = k + 1.$$

Consequently, the curvature $F_{\mu\nu}$ has order 2, and $F_{\mu_1\mu_2;\mu_3\cdots\mu_k}$ has order k. For example, $a_4(x, D_A^2)$ consists of terms proportional to $\mathrm{Tr}_F\, F_{\mu\nu}F^{\mu\nu}$ and $\mathrm{Tr}_F((\nabla_\mu\Phi)^2 + \Phi^4)$. Moreover, $a_m = 0$ for all odd m. In fact, we have:

Theorem 21. The spectral action for the almost-commutative manifold $M \times F$ is given, asymptotically as $\Lambda \to \infty$, by

$$S^\Lambda[A,\Phi] = -\frac{f_2\Lambda^2}{2\pi^2}\int_M \mathrm{Tr}_F\,\Phi^2 + \frac{f_0}{8\pi^2}\int_M \mathrm{Tr}_F\left((\nabla_\mu\Phi)^2 + \Phi^4\right)$$
$$-\frac{f_0}{24\pi^2}\int_M \mathrm{Tr}_F\, F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\Lambda^{-1})$$

From Remark 19, it follows that we have in terms of the—now x-dependent—components $\phi^p_{\tilde{e}}$ of Φ:

$$\int \mathrm{Tr}_F\,\Phi^2 = \sum_{e,p} c_{p_1p_2}(e\overline{e}) \int \mathrm{Tr}_{s(\tilde{e})}(\phi^{p_1}_{\tilde{e}})^*\phi^{p_2}_{\tilde{e}}. \tag{6}$$

Similarly,

$$\int \mathrm{Tr}_F (\nabla_\mu \Phi)^2 = \sum_{e,p_1,p_2} c_{p_1 p_2}(e\overline{e}) \int \mathrm{Tr}_{s(\widetilde{e})} (\nabla_\mu \phi_{\widetilde{e}}^{p_1})^* \nabla^\mu \phi_{\widetilde{e}}^{p_2} \tag{7}$$

and finally, in terms of a sum over cycles in Γ:

$$\begin{aligned}\int \mathrm{Tr}_F\, \Phi^4 = & \sum_{\gamma=\overline{e}_1\overline{e}_2 e_2 e_1} \sum_{p_i} c_{p_1\ldots p_4}(\gamma) \int \mathrm{Tr}_{s(\widetilde{e}_1)} (\phi_{\widetilde{e}_1}^{p_4})^* (\phi_{\widetilde{e}_2}^{p_3})^* \phi_{\widetilde{e}_2}^{p_2} \phi_{\widetilde{e}_1}^{p_1} \\ & + \sum_{\gamma=\overline{j(e_1)}j(e_1)\overline{e}_2 e_2} \sum_{p_i} c_{p_1\ldots p_4}(\gamma) \int \mathrm{Tr}_{s(\widetilde{e}_2)} (\phi_{\widetilde{e}_2}^{p_4})^* \phi_{\widetilde{e}_2}^{p_2}\, \mathrm{Tr}_{s(\widetilde{e}_1)} (\phi_{\widetilde{e}_1}^{p_1})^* \phi_{\widetilde{e}_1}^{p_1} \\ & + \sum_{\gamma=\overline{j(e_1)}\overline{e}_2 j(e_1) e_2} \sum_{p_i} c_{p_1\ldots p_4}(\gamma) \int \mathrm{Tr}_{s(\widetilde{e}_2)} (\phi_{\widetilde{e}_2}^{p_4})^* \phi_{\widetilde{e}_2}^{p_2}\, \mathrm{Tr}_{s(\widetilde{e}_1)} (\phi_{\widetilde{e}_1}^{p_1})^* \phi_{\widetilde{e}_1}^{p_1},\end{aligned} \tag{8}$$

where e_1, e_2 are horizontal edges in Γ. These expressions will become useful later on.

The appearance of the Yang–Mills action and Higgs-like potential for Φ at lowest order in the spectral action on $M \times F$ is the main motivation to study this model. As a matter of fact, if we take F to be described by Fig. 2 and choosing the D_e to correspond to the physical Yukawa coupling and CKM mass matrices (3×3 for three generations), then one derives in this way the full Standard Model of elementary particles, including the spontaneous symmetry breaking potential for the Higgs field [9, 13].

In the present paper, we aim at a better understanding also of the higher-order terms in the asymptotic expansion of the spectral action and, in particular, the role they play as regulators of the quantum gauge theory defined at lowest order. The free part of $S[A,\Phi]$ is by definition the part of $S[A,\Phi]$ that is quadratic in the fields

$$S_0[A,\Phi] = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}v}\left(S[uA+vA, u\Phi+v\Phi]\right)\Big|_{u=v=0}. \tag{9}$$

Theorem 22. There is the following asymptotic expansion (as $\Lambda \to \infty$) for the free part of the spectral action on a flat background manifold M

$$\begin{aligned}S_0[A,\Phi] \sim S_0^\Lambda[A,\Phi] = \sum_{k\geq 0} (-1)^k f_{-2k}\Lambda^{-2k}\Big(& -c_k \int \mathrm{Tr}_F\, \hat{F}^{\mu\nu}\Delta^k(\hat{F}_{\mu\nu}) \\ & + c_k' \int \mathrm{Tr}_F (\partial_\mu\Phi)\Delta^k(\partial_\mu\Phi)\Big),\end{aligned}$$

where Δ is the Laplacian on (M,g), $\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and c_k, c_k' are the following positive constants:

$$c_k = \frac{1}{8\pi^2}\frac{(k+1)!}{(2k+3)(2k+1)!}; \quad c_k' = \frac{1}{8\pi^2}\frac{k!}{(2k+1)!}.$$

The free Yang–Mills part was obtained in [37]. The free contribution for the scalar field Φ can be derived along the same lines. Let us check the lowest order terms appearing in the above formula for $S_0[A]$ with the Yang–Mills action appearing in [9] (cf. Theorem 21 above).

Corollary 23. Modulo negative powers of Λ, we have

$$S_0^\Lambda[A,\Phi] = -\frac{f_2}{4\pi^2}\int_M \mathrm{Tr}_F\,\Phi^2 + \frac{1}{8\pi^2}f_0\int_M \mathrm{Tr}_F(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{f_0}{24\pi^2}\int_M \mathrm{Tr}_F\,\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \mathcal{O}(\Lambda^{-1}).$$

We see that $S_0^\Lambda[A,\Phi]$ yields the usual (free part of the) Yang–Mills action and a free scalar field action. In fact, we can write more concisely

$$S_0^\Lambda[A,\Phi] = -\frac{f_2}{4\pi^2}\int_M \mathrm{Tr}_F\,\Phi^2 + \int_M \mathrm{Tr}_F(\partial_\mu\Phi)\vartheta_\Lambda(\Delta)(\partial^\mu\Phi) - \int \mathrm{Tr}_F\,\hat{F}_{\mu\nu}\varphi_\Lambda(\Delta)(\hat{F}^{\mu\nu})$$

in terms of the following expansions (in Λ):

$$\varphi_\Lambda(x) := \sum_{k\geq 0}(-1)^k\Lambda^{-2k}f_{-2k}c_kx^k;$$
$$\vartheta_\Lambda(x) := \sum_{k\geq 0}(-1)^k\Lambda^{-2k}f_{-2k}c'_kx^k.$$

This form motivates the interpretation of the asymptotic expansion of $S_0[A,\Phi]$ (and of $S[A,\Phi]$) as a higher-derivative gauge theory. As we will see below, this indeed regularizes the theory in such a way that the asymptotic expansion of $S[A,\Phi]$ defines a superrenormalizable field theory. This comes with the usual intricacies of gauge theories with spontaneously symmetry breaking. Before proceeding with a gauge fixing and renormalization, we discuss the Higgs potential for Φ.

3.1. Higgs Mechanism and Higher Derivatives

Given the above Higgs-like form of the spectral action at lowest order in the asymptotic expansion, it is natural to expand the scalar field Φ around its vacuum expectation value $\langle\Phi\rangle_0 = v$, which we assume to be a constant minimum of the potential appearing in $S^\Lambda[A,\Phi]$. We write

$$\Phi = v + \chi$$

and refer to the fluctuations χ as the Higgs field. The constant vacuum expectation value v will appear in $S[A,\Phi]$ as generating mass terms for the Higgs and the gauge field; this is spontaneous symmetry breaking (which might also not occur when $v = 0$). Since the asymptotically expanded spectral action is considered as a higher-derivative theory, the interpretation of mass terms is not so straightforward. Still, we can asymptotically expand the free part of $S[A,\Phi]$ as above:

$$S_0^\Lambda[A,\Phi] = \frac{1}{2}\int_M \mathrm{Tr}_F(\partial_\mu\chi)\vartheta_\Lambda(\Delta)(\partial^\mu\chi) + \frac{1}{2}\int_M \mathrm{Tr}_F\, \chi\vartheta'_\Lambda(\Delta;v)(\chi)$$
$$+ \frac{1}{2}\int_M (\partial_\mu A^a_\nu)\varphi_\Lambda(\Delta)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + A^{a\mu}\varphi'_\Lambda(\Delta;v)^{ab}(A^b{}_\mu).$$

We have written $A_\mu = A^a_\mu T^a$ in terms of a Lie algebra basis $\{T^a\}$ for $\mathfrak{g}$. In addition to the expansions ϑ_Λ and φ_Λ, we now have terms involving expansions ϑ'_Λ and φ'_Λ which—as ϑ_Λ and φ_Λ do—start with a differential operator of degree 0 (i.e. a mass term). Besides derivatives, they also involve a series expansion in v.

In addition to the above free part, the splitting $\Phi = v + \chi$ induces terms in $S^\Lambda[A,\Phi]$ that are linear in both A and χ (and in v):

$$\int_M \mathrm{Tr}_F(\partial_\mu\chi)\varpi_\Lambda(\Delta;v)([A^\mu,v]) \tag{10}$$

where we have as above a expansion defined by:

$$\varpi_\Lambda(x;v) = \sum_{k\geq l\geq 0}(-1)^k\Lambda^{-2k}f_{-2k}b_{k,l}(v)x^{k-l}$$

and $b_{k,l}(v)$ acts (pointwise) on $\mathrm{End}\,H_F$ and is of order $2l \leq 2k$ in v. We also write the components of ϖ_Λ in terms of the basis $\{e_I\}_I$ of $\Omega^1_{D_F}(\mathcal{A}_F) \subset \mathrm{End}\,H_F$ introduced in the previous section:

$$\varpi_\Lambda(x;v) = (\varpi_\Lambda(x;v)_{IJ})$$

With a slight abuse of notation, we write e_0 for the identity in $\mathrm{End}\,H_F$, normalized to have Hilbert–Schmidt norm equal to 1.

For convenience, we introduce the following inner product:

$$(\phi_1,\phi_2) = \int_M \mathrm{Tr}_F\,\phi_1^*\varpi_\Lambda(\Delta;v)(\phi_2). \tag{11}$$

on endomorphisms $\phi_1,\phi_2 \in C^\infty(M,\mathrm{End}\,H_F)$. Thus, the above term (10) reads $(\partial_\mu\chi,[A^\mu,v])$.

4. R_ξ-Gauge Fixing and Renormalization

We add a R_ξ-type gauge-fixing term with higher-derivatives of the following form:

$$S^\Lambda_{\mathrm{gf}}[A,\Phi] = \frac{1}{2\xi}\int \mathrm{Tr}_F\,(\partial_\mu A^{a\mu} - \xi\chi[T^a,v])\,\varpi_\Lambda(\Delta;v)\,(\partial_\nu A^{a\nu} - \xi\chi[T^a,v]) \tag{12}$$

which is chosen so that the terms linear in both A and χ cancel the cross-terms of (10). In terms of the inner product (11), we have more concisely:

$$S^{\Lambda}_{\rm gf}[A,\Phi] = \frac{1}{2\xi}\left(\partial_\nu A^{a\nu} - \xi\chi[T^a,v], \partial_\mu A^{a\mu} - \xi\chi[T^a,v]\right)$$

where we consider $\partial_\mu A^{a\mu}(x)$ as an endomorphism of H_F (i.e. as a multiple of the identity).

As usual, the above gauge fixing requires a Jacobian, conveniently described by a Faddeev–Popov ghost Lagrangian:

$$S^{\Lambda}_{\rm gh}[A,\overline{C},C,\Phi] = \left(\overline{C}^a, \Delta C^a - \partial_\mu[A^\mu,C]^a - \xi[C,\Phi][T^a,v]\right) \tag{13}$$

Here $C,\overline{C}$ are the Faddeev–Popov ghost fields which are $\mathfrak{g}$-valued fermionic fields: $C = C^aT^a$ and $\overline{C} = \overline{C}^aT^a$. Accordingly, $[C,\Phi] := C^a[T^a,\Phi]$.

Proposition 24. The sum $S^{\Lambda}[A,\Phi] + S^{\Lambda}_{\rm gf}[A,\Phi] + S^{\Lambda}_{\rm gh}[A,\overline{C},C,\Phi]$ is invariant under the BRST transformations:

$$\begin{aligned} sA_\mu &= \partial_\mu C + [A_\mu, C]; \quad s\Phi = -[C,\Phi];\\ sC &= -\frac{1}{2}[C,C]; \quad s\overline{C}^a = \frac{1}{\xi}\partial_\mu A^{a\mu} - \chi[T^a,v]. \end{aligned} \tag{14}$$

Proof. First, $s(S^{\Lambda}) = 0$ because of gauge invariance of $S^{\Lambda}[A,\Phi]$. Indeed, sA_μ and $s\Phi$ are just gauge transformations by the (fermionic) field C.

For the gauge fixing and ghost terms, we compute

$$s(S^{\Lambda}_{\rm gf}) = \frac{1}{\xi}\left(\partial_\mu A^{a\mu} - \xi\chi[T^a,v], -\Delta C^a + \partial_\mu[A^\mu,C]^a + \xi[C,\Phi][T^a,v]\right)$$

since $s\chi = s(v+\chi) \equiv s\Phi$. On the other hand,

$$s(S^{\Lambda}_{\rm gh}) = \left(\xi^{-1}\partial_\mu A^{a\mu} - \chi[T^a,v], \Delta C^a - \partial_\mu[A^\mu,C]^a - \xi[C,\Phi][T^a,v]\right)$$

which modulo vanishing boundary terms is minus the previous expression. □

Note that $s^2 \neq 0$, which can be cured by standard homological methods: introduce an auxiliary (aka Nakanishi-Lautrup) field h so that $\overline{C}$ and h form a contractible pair in BRST cohomology. In other words, we replace the above transformation in Eq. (14) on $\overline{C}$ by $s\overline{C} = -h$ and $sh = 0$. If we replace $S^{\Lambda}_{\rm gf} + S^{\Lambda}_{\rm gh}$ by $s\Psi$ with Ψ an arbitrary *gauge fixing fermion*, it follows from gauge invariance of S^{Λ} and nilpotency of s that $S^{\Lambda} + s\Psi$ is BRST-invariant. The above special form of $S^{\Lambda}_{\rm gf} + S^{\Lambda}_{\rm gh}$ can be recovered by choosing

$$\Psi = (\overline{C}^a, \tfrac{1}{2}\xi h^a + \partial_\mu A^{a\mu} - \xi\chi[T^a,v]).$$

We derive the propagators by inverting the non-degenerate quadratic forms in the fields A, χ and C given by $S^{\Lambda}_0[A,\Phi] + S^{\Lambda}_{\rm gf}[A,\xi]$. This yields for the *gauge propagator*:

$$D^{ab}_{\mu\nu}(p,v;\Lambda) = \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \left(\frac{1}{p^2 \varphi_\Lambda(p^2) + \varphi'_\Lambda(p^2;v)} \right)^{ab} + \xi \frac{p_\mu p_\nu}{p^2} \left(\frac{1}{p^2 \varpi_\Lambda(p^2;v)_{00} + \xi \varphi'_\Lambda(p^2;v)} \right)^{ab}.$$

The *Higgs propagator* becomes:

$$D^{IJ}(p,v;\Lambda) = \left(\frac{1}{p^2 \vartheta_\Lambda(p^2) + \vartheta'_\Lambda(p^2;v) + \xi \mu(p^2;v)} \right)^{IJ}$$

where

$$\mu(p^2;v)_{IJ} := \mathrm{Tr}_F\, e_I [T^a, v] \varpi_\Lambda(p^2;v) e_J [T^a, v]$$

The *ghost propagator* is

$$\widetilde{D}^{ab}(p,v;\Lambda) = \frac{\delta^{ab}}{p^2 \varpi_\Lambda(p^2;v)_{00}}.$$

Remark 25. In [37], we argued that for the pure Yang–Mills system the function $\varphi_\Lambda(p^2)$ appearing in the denominator of the propagator is nowhere-vanishing, provided we impose the conditions $f^{(2k)}(0) \geq 0$ on the even derivatives of f. Consequently, the gauge propagator did not have other poles than a physical pole at $p^2 = 0$. In the present case, where we allow for spontaneous symmetry breaking, such a conclusion cannot be drawn. Typically, there will be unphysical poles (involving ξ) appearing in the gauge and also in the Higgs and ghost propagators. Since we will be mainly concerned with renormalizability in this paper, we will ignore these poles in what follows. Of course, a treatment of (the lack of) unitarity for this higher-derivative theory does require a careful analysis of these unphysical poles as well. At lowest order (as in Theorem 21), one expects to find a cancellation of the unphysical poles appearing in the gauge and Higgs propagator, similar to [21].

4.1. Renormalization on an Almost-Commutative Manifold

As said, we consider the asymptotic expansion (as $\Lambda \to \infty$) of the spectral action on the AC manifold $M \times F$ as a higher-derivative field theory. This means that we will use the higher derivatives of $F_{\mu\nu}$ and Φ that appear in the asymptotic expansion as natural regulators of the theory, similar to [31,32] (see also [19, Sect. 4.4]). However, note that the regularizing terms are already present in the asymptotic expansion of the spectral action and need not be introduced as such. Let us consider the expansion of Proposition 20 up to order n (which we assume to be at least 4), i.e. we set $f_{4-m} = 0$ for all $m > n$ while $f_4, \dots f_{4-n} \neq 0$. Also, assume a gauge fixing of the form (12) and (13).

Remark 26. Note that for $n = 4$, the asymptotically expanded spectral action is given by the action appearing in Theorem 21 and strictly speaking not a higher-derivative gauge theory. However, in what follows, it is convenient to also consider the case $n = 4$, giving us the physically interesting Lagrangian.

We easily derive from the structure of $\varphi_\Lambda(p^2), \vartheta_\Lambda(p^2)$ and $\varpi_\Lambda(p^2;v)$ that the propagators of the gauge field, the Higgs field χ, and the ghost field, respectively, behave as $|p|^{-n+2}$ as $|p| \to \infty$. Indeed, in this case:

$$\varphi_\Lambda(p^2) = \sum_{k=0}^{n/2-2} \Lambda^{-2k} f_{-2k} c_k p^{2k}; \quad \vartheta_\Lambda(p^2) = \sum_{k=0}^{n/2-2} \Lambda^{-2k} f_{-2k} c'_k p^{2k};$$
$$\varpi_\Lambda(p^2) = \sum_{k=0}^{n/2-2} \Lambda^{-2k} f_{-2k} c''_k(v) p^{2k}$$

which behave like $|p|^{n-4}$ as $|p| \to \infty$. Moreover, $\vartheta'_\Lambda(p^2;v)$ and $\varphi'_\Lambda(p^2;v)$ are subleading in $|p|$ since they behave as $v^2|p|^{n-2}$ as $|p| \to \infty$.

Let us now consider the weights on the vertices in a Feynman graph (not to be confused with a Krajewski diagram). For gauge-Higgs interactions involving i gauge and j Higgs fields, the maximal number of derivatives is $n-i-j$, essentially because the total order of the corresponding term in the Lagrangian is less than or equal to n. Similarly, for the gauge-ghost interaction, the maximal number of derivatives is $n-3$. Finally, the Higgs–ghost interaction behaves slightly better and has $\leq n-4$ derivatives. We adopt the following notation:

	number of ...		number of ...
I_A	internal gauge lines	V_{ij}	gauge-Higgs vertices
I_χ	internal Higgs lines	$\widetilde{V}_A$	gauge-ghost vertices
$\widetilde{I}$	internal ghost lines	$\widetilde{V}_\chi$	Higgs–ghost vertices

Let us now find an expression for the *superficial degree of divergence* ω of a Feynman graph. In 4 dimensions, we find in terms of the above notation at loop order L:

$$\omega \leq 4L-(I_A + I_\chi + \widetilde{I})(n-2) + \sum_{i+j=3}^{n} V_{ij}(n-i-j) + \widetilde{V}_A(n-3) + \widetilde{V}_\chi(n-4).$$

Lemma 27. Let E_A, E_χ and $\widetilde{E}$ denote the number of external gauge, Higgs and ghost edges, respectively. The superficial degree of divergence of the Feynman graph satisfies

$$\omega \leq (4-n)(L-1) + 4 - (E_A + E_\chi + \widetilde{E}).$$

Proof. We use the relations

$$2I_A + E_A = \sum_{i+j=3}^{n} iV_{ij} + \widetilde{V}_A; \quad 2I_\chi + E_\chi = \sum_{i+j=3}^{n} jV_{ij} + \widetilde{V}_\chi;$$
$$2\widetilde{I} + \widetilde{E} = 2\widetilde{V}_A + 2\widetilde{V}_\chi.$$

Indeed, these formulas count the number of half (gauge/Higgs/ghost) edges in a Feynman graph in two ways: from the number of edges and from the valences of the vertices. We use them to substitute for $2I_A, 2I_\chi$ and $2\widetilde{I}$ in the above expression for ω so as to obtain

$$\omega \leq 4L - I_A n - I_\chi n - \widetilde{I} n + n \left(\sum_{i,j} V_{ij} + \widetilde{V}_A + \widetilde{V}_\chi \right) - (E_A + E_\chi + \widetilde{E})$$

from which the result follows at once from Euler's formula $L = I_A + I_\chi + \widetilde{I} - \sum_{i,j} V_{ij} - \widetilde{V}_A - \widetilde{V}_\chi + 1$. □

As a consequence, $\omega < 0$ if $E + \widetilde{E} > 4$ so that the theory is powercounting renormalizable. Moreover, if $n \geq 8$ then $\omega < 0$ for all $L \geq 2$: all Feynman graphs are finite at loop order greater than 1. In this case, all divergent graphs are at one loop, and satisfy $E + \widetilde{E} \leq 4$. We conclude that the asymptotically expanded spectral action on an AC manifold is renormalizable, and if $n \geq 8$ then it is superrenormalizable.

Of course, the spectral action on an AC manifold being a gauge theory, there is more to renormalizability than just power counting: we have to establish gauge invariance of the counterterms. We already know that the counterterms needed to render the perturbative quantization of the asymptotically expanded spectral action finite are of order 4 or less in the fields and arise only from one-loop graphs. The key property of the effective action at one loop is that it is supposed to be BRST-invariant, $s(\Gamma_1) = 0$. In particular, assuming a regularization compatible with gauge invariance, the divergent part $\Gamma_{1,\infty}$ is BRST-invariant. We will use results from [3,14–17] on BRST cohomology for Yang–Mills type theories to determine the form of the BRST-closed functionals of order 4 or less in the fields. In fact, in these references, a relation is established between BRST cohomology and Lie algebra cohomology for the gauge group: BRST-closed functionals are given by integrals of gauge invariant polynomials in the fields.

First, recall that with respect to the orthogonal decomposition of $\mathfrak{su}(A_F)$ of Proposition 15 we can write the curvature $F_{\mu\nu} = \sum_i F^i_{\mu\nu}$ with $F^i_{\mu\nu} \in \mathfrak{u}(k_i, \mathbb{F}_i)$. Gauge invariant functionals are then given by

$$\int \operatorname{Tr} F^i_{\mu\nu} F^{i\mu\nu}, \tag{15}$$

for all i. These terms appear, though with a common pre-factor, in Theorem 21.

Let us then consider the field Φ with independent components $\phi^p_{\widetilde{e}}$, as labeled by the edges of the graph $\widetilde{\Gamma}$ introduced at the end of Sect. 2.1. The index p runs from $1, \ldots, \dim S_{\widetilde{e}}$ and the field $\phi_{\widetilde{e}}$ is in the representation of $\mathfrak{su}(A_F)$ induced by the irreducible representations that are given by $s(\widetilde{e})$ and $t(\widetilde{e})$ in $\widetilde{\Gamma}^{(0)}$. The most general form of a gauge invariant functional in the components of Φ of degree 2 is given by

$$\int \operatorname{Tr}(\phi^{p_1}_{\widetilde{e}})^* \phi^{p_2}_{\widetilde{e}}, \tag{16}$$

for all $\widetilde{e}, p_1, p_2$. Note that these terms appear in Theorem 21 (cf. Eq. (6)). There is also a term of second order in Φ involving covariant derivatives, it is:

$$\int \operatorname{Tr}(\nabla_\mu \phi_{\widetilde{e}}^{p_1})^* \nabla^\mu \phi_{\widetilde{e}}^{p_2}, \tag{17}$$

and is also present in Theorem 21 (cf. Eq. (7)).

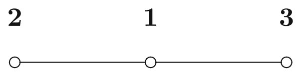
Slightly more complicated is the search for the gauge invariant functionals that are quartic in the fields $\phi_{\widetilde{e}}^{p}$. In terms of the graph $\widetilde{\Gamma}$, they are given by a combination of the following sums over cycles in $\widetilde{\Gamma}$:

$$\sum_{\widetilde{\gamma}=\widetilde{e}_1\widetilde{e}_2\widetilde{e}_3\widetilde{e}_4} \operatorname{Tr}_{s(\widetilde{e}_4)=t(\widetilde{e}_1)} \phi_{\widetilde{e}_1}^{p_1} \cdots \phi_{\widetilde{e}_4}^{p_4}; \tag{18}$$

$$\sum_{\widetilde{\gamma}=\overline{\widetilde{e}}\widetilde{e}} \operatorname{Tr}_{s(\widetilde{e})=t(\overline{\widetilde{e}})} (\phi_{\widetilde{e}}^{p_1})^* \phi_{\widetilde{e}}^{p_2} \sum_{\widetilde{\gamma}'=\overline{\widetilde{e}}'\widetilde{e}'} \operatorname{Tr}_{s(\widetilde{e}')=t(\overline{\widetilde{e}}')} (\phi_{\widetilde{e}'}^{p_1'})^* \phi_{\widetilde{e}'}^{p_2'} \tag{19}$$

That is, all gauge invariant quartic polynomials in Φ arise by taking traces along cycles of length 4 in $\widetilde{\Gamma}$, and traces along pairs of cycles of total length 4. In the latter case, we exclude the possibility that the cycles $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ both connect to the vertex $\mathbf{1}$ or the vertex $\overline{\mathbf{1}}$. In fact, the contribution arising from such cycles can be written as the trace along a single cycle of length 4, due to the fact that $\operatorname{Tr}_1 : \mathbb{C} \to \mathbb{C}$ acts as the identity.

Example 28. Consider the following graph $\widetilde{\Gamma}$:



The pair of cycles $(\mathbf{21})(\mathbf{12})$ and $(\mathbf{13})(\mathbf{31})$ gives a contribution

$$\operatorname{Tr} \phi_{(\mathbf{21})}\phi_{(\mathbf{12})} \operatorname{Tr} \phi_{(\mathbf{31})}\phi_{(\mathbf{13})} = (\phi_{(\mathbf{21})}\phi_{(\mathbf{12})})(\phi_{(\mathbf{31})}\phi_{(\mathbf{13})})$$

since $(\phi_{(\mathbf{21})}\phi_{(\mathbf{12})})$ and $(\phi_{(\mathbf{31})}\phi_{(\mathbf{13})})$ are elements in $\operatorname{Hom}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$. The concatenated cycle $(\mathbf{21})(\mathbf{12})(\mathbf{13})(\mathbf{31})$ of length 4 gives the same contribution

$$\operatorname{Tr} \phi_{(\mathbf{21})}\phi_{(\mathbf{12})}\phi_{(\mathbf{31})}\phi_{(\mathbf{13})} = (\phi_{(\mathbf{21})}\phi_{(\mathbf{12})})(\phi_{(\mathbf{31})}\phi_{(\mathbf{13})})$$

for the same reason.

Now, recalling Definition 11, if the Krajewski diagram Γ is R-connected (in dimension 4) the above traces can always be written in terms of cycles of length 4 in Γ which are precisely the terms that are present in Theorem 21 (cf. Eq. (8)). We conclude:

Theorem 29. Let $M \times F$ be an almost-commutative manifold with $\dim M = 4$; suppose that A_F contains no copies of $\mathbb{R}$, and either no complex subalgebras, or at least one non-trivial complex subalgebra (cf. Proposition 15). Consider the asymptotically expanded spectral action up to order $n \geq 4$.

If the Krajewski diagram describing the finite real spectral triple for F is R-connected in dimension 4, then the asymptotically expanded spectral action (with $f_{4-m} = 0$ for all $m > n$) for $M \times F$ is renormalizable as a gauge theory. Moreover, it is superrenormalizable as a gauge theory if $n \geq 8$.

As a corollary, we find that the asymptotically expanded Yang–Mills spectral action is superrenormalizable ($n \geq 8$), as was previously shown in [37,38]. Indeed, the spectral triple of Example 4 has Krajewski diagram

$$\begin{array}{cc} & \mathbf{N} \\ \mathbf{N} & \circ \end{array}$$

which is R-connected in a trivial way.

Similarly, Proposition 12 implies that the asymptotically expanded spectral action that at lowest order is the Standard Model, is renormalizable as a gauge theory. In particular, choosing $n = 4$ this implies that the Standard Model is renormalizable as a gauge theory.

Of course, in order to have a sensible renormalizable gauge field theory we have to make sure that no gauge anomalies appear. The cancellation of such anomalies has been discussed in terms of Krajewski diagrams already in [26], imposing further constraints on the diagrams. For the noncommutative geometry of the Standard Model, the cancellation of anomalies turns out to be equivalent to the unimodularity condition $\operatorname{Tr} A_\mu = 0$ [1].

More generally, for models beyond the Standard Model we have recently described the anomaly-free possibilities [6]. In fact, for the chiral gauge anomaly to vanish, the charges of all fermions in H_F for each of the $C-1$ factors in $\mathfrak{su}(A_F)$ isomorphic to $\mathfrak{u}(1)$ (cf. Proposition 15) should add up to zero, thus imposing restrictions on the multiplicities of the fermions.

Example 30. Let us illustrate the possible failure of renormalizability for the Krajewski diagram in Example 13 which is not R-connected. There are fields $\phi_{(\mathbf{12})}$ and $\phi_{(\overline{\mathbf{1}}\mathbf{3})}$ that could combine to give a gauge-invariant counterterms proportional to

$$\left((\phi_{(\mathbf{12})})^* \phi_{(\mathbf{12})}\right) \left((\phi_{(\overline{\mathbf{1}}\mathbf{3})})^* \phi_{(\overline{\mathbf{1}}\mathbf{3})}\right)$$

However, this term can never appear in the asymptotic expansion of the spectral action, since the edges $(\mathbf{12}), (\mathbf{21}), (\overline{\mathbf{1}}\mathbf{3})$ and $(\mathbf{3}\overline{\mathbf{1}})$ do not lift to a cycle in Γ.

Note that the asymptotically expanded spectral action is not *multiplicatively* renormalizable, since the coefficients in front of the counterterms might be different for different indices (such as i, $\widetilde{e}$, and p). This is in contrast with the classical action in Theorem 21 where there is a typical *unification* of couplings for all simple factors of the gauge group. This suggests that one takes the spectral action $S^\Lambda[A, \Phi]$ (plus gauge fixing) as a starting point for the renormalization group flow to then run the action to arbitrary energy scales.

It remains an open question whether this approach to renormalizing the asymptotically expanded spectral action using the intrinsic higher-derivative regulators is equivalent to perturbatively quantizing the gauge theory defined by the lowest-order terms (appearing in Theorem 21) using, say, dimensional regularization and minimal subtraction. Evidence that this might be true can be found in [29] and is the subject of future research.

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References

[1] Alvarez, E., Gracia-Bondía, J.M., Martin, C.P.: Anomaly cancellation and the gauge group of the standard model in NCG. Phys. Lett. B **364**, 33–40 (1995)

[2] Boeijink, J., van Suijlekom, W.D.: The noncommutative geometry of Yang–Mills fields. J. Geom. Phys. **61**, 1122–1134 (2011)

[3] Brandt, F., Dragon, N., Kreuzer, M.: Lie algebra cohomology. Nucl. Phys. B **332**, 250 (1990)

[4] van den Broek, T., van Suijlekom, W.D.: Supersymmetric QCD and noncommutative geometry. Commun. Math. Phys. **303**, 149–173 (2011)

[5] van den Broek, T., van Suijlekom, W.D.: Supersymmetric QCD from noncommutative geometry. Phys. Lett. B **699**, 119–122 (2011)

[6] van den Broek, T., van Suijlekom, W.D.: Going beyond the standard model with noncommutative geometry. JHEP **3**, 112 (2013)

[7] Chamseddine, A.H., Connes, A.: Universal formula for noncommutative geometry actions: unifications of gravity and the standard model. Phys. Rev. Lett. **77**, 4868–4871 (1996)

[8] Chamseddine, A.H., Connes, A.: The spectral action principle. Commun. Math. Phys. **186**, 731–750 (1997)

[9] Chamseddine, A.H., Connes, A., Marcolli, M.: Gravity and the standard model with neutrino mixing. Adv. Theor. Math. Phys. **11**, 991–1089 (2007)

[10] Connes, A.: Noncommutative Geometry. Academic Press, San Diego (1994)

[11] Connes, A.: Noncommutative geometry and reality. J. Math. Phys. **36**, 6194–6231 (1995)

[12] Connes, A.: Gravity coupled with matter and foundation of noncommutative geometry. Commun. Math. Phys. **182**, 155 (1996)

[13] Connes, A., Marcolli, M.: Noncommutative Geometry, Quantum Fields and Motives. AMS, Providence (2008)

[14] Dixon, J.A.: Calculation of BRS cohomology with spectral sequences. Commun. Math. Phys. **139**, 495–526 (1991)

[15] Dubois-Violette, M., Talon, M., Viallet, C.M.: BRS algebras: analysis of the consistency equations in gauge theory. Commun. Math. Phys. **102**, 105 (1985)

[16] Dubois-Violette, M., Talon, M., Viallet, C.M.: Results on BRS cohomology in gauge theory. Phys. Lett. B **158**, 231 (1985)

[17] Dubois-Violette, M., Henneaux, M., Talon, M., Viallet, C.-M.: Some results on local cohomologies in field theory. Phys. Lett. B **267**, 81–87 (1991)

[18] van den Dungen, K., van Suijlekom, W.D.: Electrodynamics from noncommutative geometry. J. Noncommut. Geom. **7**, 433–456 (2013)

[19] Faddeev, L., Slavnov, A.: Gauge Fields. Introduction to Quantum Theory. Benjaming Cummings, San Francisco (1980)

[20] Gilkey, P.B.: Invariance theory, the heat equation, and the Atiyah–Singer index theorem. Mathematics Lecture Series, vol. 11. Publish or Perish Inc., Wilmington (1984)

[21] 't Hooft, G.: Renormalizable Lagrangians for massive Yang–Mills fields. Nucl. Phys. B **35**, 167–188 (1971)

[22] Iochum, B., Levy, C., Vassilevich, D.: Global and local aspects of spectral actions. arXiv:1201.6637

[23] Iochum, B., Levy, C., Vassilevich, D.: Spectral action beyond the weak-field approximation. Commun. Math. Phys. **316**, 595–613 (2012)

[24] Jureit, J.-H., Krajewski, T., Schücker, T., Stephan, C.A.: On the noncommutative standard model. Acta Phys. Polon. B **38**, 3181–3202 (2007)

[25] Jureit, J.-H., Stephan, C.A.: On a classification of irreducible almost-commutative geometries IV. J. Math. Phys. **49**, 033502 (2008)

[26] Krajewski, T.: Classification of finite spectral triples. J. Geom. Phys. **28**, 1–30 (1998)

[27] Krajewski, T.: Constraints on the scalar potential from spectral action principle [arXiv:hep-th/9803199]

[28] Landi, G.: An Introduction to Noncommutative Spaces and their Geometry. Springer, Berlin (1997)

[29] Martin, C.P., Ruiz Ruiz, F.: Higher covariant derivative regulators and nonmultiplicative renormalization. Phys. Lett. B **343**, 218–224 (1995)

[30] Paschke, M., Sitarz, A.: Discrete spectral triples and their symmetries. J. Math. Phys. **39**, 6191–6205 (1998)

[31] Slavnov, A.A.: Invariant regularization of nonlinear chiral theories. Nucl. Phys. B **31**, 301–315 (1971)

[32] Slavnov, A.A.: Invariant regularization of gauge theories. Teor. Mat. Fiz. **13**, 174–177 (1972)

[33] Squellari, R., Stephan, C.A.: Almost-commutative geometries beyond the standard model. III. Vector doublets. J. Phys. A **40**, 10685–10698 (2007)

[34] Stephan, C.A.: Almost-commutative geometries beyond the standard model. J. Phys. A **39**, 9657 (2006)

[35] Stephan, C.A.: Almost-commutative geometries beyond the standard model. II. New Colours. J. Phys. A **40**, 9941 (2007)

[36] van Suijlekom, W.D.: Renormalizability conditions for almost-commutative geometries. Phys. Lett. B **711**, 434–438 (2012)

[37] van Suijlekom, W.D.: Renormalization of the asymptotically expanded Yang–Mills spectral action. Commun. Math. Phys. **312**, 883–912 (2012)

[38] van Suijlekom, W.D.: Renormalization of the spectral action for the Yang–Mills system. JHEP **1103**, 146 (2011)

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