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On a Coordinate-Independent Description of String Worldsheet Theory

Partha Mukhopadhyay

Abstract. We rewrite the bosonic worldsheet theory in curved background in a language where it describes a single particle moving in an infinite-dimensional curved spacetime. This language is developed at a formal level without regularizing the infinite-dimensional traces. Then, we adopt DeWitt's (Phys Rev 85:653, 1952) coordinate-independent formulation of quantum mechanics in the present context. This procedure enables us to define coordinate invariant quantum analogue of classical Virasoro generators, which we call DeWitt–Virasoro generators. This framework also enables us to calculate the invariant matrix elements of an arbitrary operator constructed out of the DeWitt–Virasoro generators between two arbitrary scalar states. Using these tools, we further calculate the DeWitt–Virasoro algebra in spin-zero representation. The result is given by the Witt algebra with additional anomalous terms that vanish for Ricci-flat backgrounds. Further analysis need to be performed to precisely relate this with the beta function computation of Friedan and others. Finally, we explain how this analysis improves the understanding of showing conformal invariance for certain pp-wave that has been recently discussed using hamiltonian framework.

1. Introduction and Summary

Equations of motion (EOM) for backgrounds in string theory are derived from the condition of worldsheet conformal invariance.[1] Although this condition has mostly been studied by computing the beta functions of the nonlinear sigma model using background field method [1–3], a BRST hamiltonian approach has also been discussed in the literature [14–25] (See also [26,27]). An important issue in the latter is to find out how to define the constraint generators at the quantum level. This problem has been dealt with by considering a weak field approximation near flat space and/or using worldsheet supersymmetry.

[1] Some of the original references are [1–13].

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Recently, a similar method has been used [28–31] to study exact conformal invariance of the worldsheet theory in type IIB R-R plane-wave background [32] using Green–Schwarz formalism in semi-light-cone gauge [33]. In this case, neither we are close to flat space nor do we have worldsheet supersymmetry. However, knowing the exact string spectrum through light-cone gauge analysis [34,35] and some special properties of the background help us to fix the quantum definition of the energy-momentum (EM) tensor.[2]

The next issue is to understand how to compute Virasoro anomaly. One may naively think that such anomaly terms can be computed by directly calculating the commutators among the right and left moving EM tensor components using the basic canonical commutation relations. However, it was shown in [30] that this direct method leads to results that suffer from operator ordering ambiguity. It did not seem to be an issue in the earlier works as all the composite operators were ordered according to the normal ordering relevant to flat space. However, in our case, the exact vacuum is not close to the flat space vacuum and it is a priori not clear which ordering needs to be considered. A supersymmetry argument has been used in [31] to compute certain integrated forms of the anomaly terms indirectly. Such analysis works because there are certain relations among the supercurrents and the EM tensor components which hold true in the non-perturbative vacuum and are enough to establish Virasoro algebra.[3] However, the method itself works only for on-shell backgrounds which are supersymmetric.

Given the above discussion, one may wonder if there exists a notion of Virasoro algebra in the quantum theory which can be discussed in a vacuum-independent way. This question seems natural in the context of a background-independent formulation as a vacuum state has information about a particular background. In ordinary quantum mechanics, this question is not any special as one can indeed define all the operators of the theory before solving for the energy eigenstates. The difference is that here we have infinite number of degrees of freedom and if we sum all their contributions naively we get divergences. In this work, we will still try to formulate the problem in a background-independent manner in the sense described above by going to a particle-like description. We will find certain interesting result and it will be interesting to explore further along this direction. Below we describe our construction and result in more detail and explain why the particle-like description is well-suited for our purpose.

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[2] Authors [28] argued with certain analysis that the relevant worldsheet theory should not have a smooth flat space limit and suggested that the complete EM tensor be defined with an unusual operator ordering, called phase-space normal ordering (similar ordering was considered earlier in the literature in [25]). However, it was pointed out in [29] using the universality argument of [36,37] that such a limit should be smooth generically for any pp-wave and the definition of [28] leads to a spectrum of negative conformal dimensions. An alternative definition was suggested in [30] where such an ordering is applied only to the interaction part. It was argued that this definition leads to the correct physical spectrum.

[3] This is analogous to the case of NSR superstrings where one uses superconformal algebra. See for example [21].

At the classical level, the new description is obtained by re-writing the worldsheet theory in a language where it describes a particle moving in an infinite-dimensional curved spacetime (subject to certain potential). In this way, the infinite number of degrees of freedom of the string is given an interpretation of number of spacetime dimensions. We will define the background-independent version of the quantum Virasoro generators in this new language. In general, the expressions will have divergent terms. However, all such divergences will appear as infinite-dimensional traces and, therefore, will be formally tractable. A suitable regularization procedure needs to be developed to cure such divergences. We will leave this for future work.

To describe the quantum mechanics in a manifestly covariant manner we follow the argument of DeWitt [39] (see also [40,41])[4] and the particle-like description will provide a natural framework to adopt this idea. DeWitt [39] identified general coordinate transformations (GCT) with a subgroup of all unitary transformations in quantum mechanics. The method shows how to construct the quantum analogs of dynamical quantities in terms of the general phase-space coordinates such that the expectation values of such operators between two arbitrary states are given by covariant expressions in position space representation.

There are a few generalizations involved in our work from DeWitt's original work. The analysis of [39] considered a non-relativistic particle so that the general covariance was sought only for the spatial slice. In our case, we adopt the infinite-dimensional language for the matter part of the worldsheet theory in conformal gauge. The resulting particle-like theory looks like a worldline theory with full covariance in spacetime. A more important difference is having infinite number of Virasoro generators instead of only the hamiltonian as in DeWitt's case. Because of the presence of infinite number of dimensions, the theory possesses certain *shift* properties which look unusual from the particle point of view. These properties dictate the behavior of the theory under certain shift of the spacetime dimensions, i.e. the string modes. Since Virasoro generators relate different string modes, such shift properties are inherently related to the existence of these generators.

The quantum generators defined in the sense of DeWitt, hereafter called *DeWitt–Virasoro generators*, are ordered in a particular way and are designed to produce covariant results in spin-zero representation. Given this background-independent definition, we next proceed to compute the algebra, hereafter called *DeWitt–Virasoro algebra*, satisfied by these generators in spin-zero representation. The result is given by the Witt algebra with additional anomalous terms that vanish for Ricci-flat backgrounds. Notice that the central charge terms of the Virasoro algebra do not appear in this result. Being constructed in a background-independent way, the DeWitt–Virasoro generators are not normal ordered with respect to any particular vacuum. It is expected that the actual quantum Virasoro generators for a specific conformal background can be obtained by first specializing to that background and then

[4] DeWitt's formulation was applied to string theory earlier by Lawrence and Martinec [42].

normal ordering the DeWitt–Virasoro generators with respect to the relevant vacuum. The central charge terms are expected to arise in the algebra of such quantum Virasoro generators. This procedure is understood for flat and the pp-wave background discussed in [30]. Understanding of this in a generic sense is an important question for our construction.

Finally, as an application of the present background-independent framework, we discuss how it explains conformal invariance for the pp-wave considered in [28–30]. As mentioned earlier, the anomaly terms were shown in [30] to suffer from operator ordering ambiguity. If these terms are ordered according to the phase-space normal ordering, then the correct EOM is reproduced. However, the justification of such ordering prescription was not clear. Here we argue that the computation of [30] can be viewed as a special case of our present analysis. Therefore, the anomaly terms computed in that work are same as the DeWitt–Virasoro anomaly terms discussed here. Our present result suggests that the terms computed in [30] should vanish. However, this also implies that the Ricci-term found here, which gives the EOM for the background, was missing in the computation of [30]. This apparent discrepancy may be resolved by the following observation. The Ricci-term that we obtain here involves certain contractions with the Ricci tensor. The whole term vanishes, though the Ricci tensor itself does not, for the relevant pp-wave because of its special properties.

Given the above results, there are a number of technical issues which deserve further attention. For example, in our computation, the Ricci-flatness condition comes from terms that arise only in the left-right commutation relations. From the present analysis, it is not clear why this is so. A better understanding of this with the inclusion of other massless fields in the background is desirable. In particular, it will be interesting to investigate if the *DeWitt–Virasoro anomaly* terms contain the infinite-dimensional analogue of the low energy EOM for the massless fields as obtained in the beta function computation. Superstring extension of this result will also be interesting to find using pure spinor formalism [43]. It should also be investigated how the present analysis may be extended to higher spin representations. The main conceptual issue involved in our work is to understand how to interpret the current framework in more conventional terms. For example, how the Ricci-flatness condition found here should be related to the usual beta function computation of Friedan [1–3]. It may be possible that the present construction will offer a new method of computing α' corrections which will certainly be interesting to investigate. We hope to come back to these questions in future.

The rest of the paper is organized as follows. The infinite-dimensional language is explained in Sect. 2. The construction of the DeWitt–Virasoro generators has been discussed in Sect. 3. We summarize the results for the DeWitt–Virasoro algebra in Sect. 4. We discuss the flat and the pp-wave backgrounds as special cases of the present construction in Sect. 5. Some of the technical derivations are presented in a few Appendices.

2. Mapping to Infinite Dimensions

We consider a bosonic closed string propagating in a D dimensional curved background, hereafter called the physical spacetime, with metric $G_{\mu\nu}$. We work in the conformal gauge of the worldsheet theory so that the ghosts are given by the standard (b, c) systems. For the purpose of the present work, we will be concerned only with the matter part of the theory. The relevant classical lagrangian is given by,

$$L = \frac{1}{2}\oint \frac{\mathrm{d}\sigma}{2\pi}\ G_{\mu\nu}(X(\sigma))\left[\dot{X}^{\mu}(\sigma)\dot{X}^{\nu}(\sigma) - \partial X^{\mu}(\sigma)\partial X^{\nu}(\sigma)\right], \tag{2.1}$$

where $\oint \equiv \int_0^{2\pi}$, $\mu = 0, 1, \ldots, D-1$. A dot and a ∂ denote derivatives with respect to worldsheet time-coordinate τ and space-coordinate σ, respectively. We recast this lagrangian in a form that describes a single particle moving in an infinite-dimensional curved spacetime subject to certain potential,

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x)\left[\dot{x}^i\dot{x}^j - a^i(x)a^j(x)\right], \tag{2.2}$$

where x^i are the general coordinates of the infinite-dimensional spacetime. The index i is given by an ordered pair of indices,

$$i = \{\mu, m\}, \tag{2.3}$$

where $m \in Z$ is the string-mode-number such that,[5]

$$\begin{aligned} x^i &= \oint \frac{\mathrm{d}\sigma}{2\pi}\ X^{\mu}(\sigma)e^{-im\sigma}, \\ g_{ij}(x) &= \oint \frac{\mathrm{d}\sigma}{2\pi}\, G_{\mu\nu}(X(\sigma))e^{i(m+n)\sigma}, \\ a^i(x) &= \oint \frac{\mathrm{d}\sigma}{2\pi}\ \partial X^{\mu}(\sigma)e^{-im\sigma}. \end{aligned} \tag{2.4}$$

As mentioned in the previous section, we will mainly work using the infinite-dimensional language. Below we discuss certain properties of this language that will be relevant for our study.

1. We have claimed that the worldsheet theory (2.1) has an interpretation to be generally covariant in the infinite-dimensional sense. To see this explicitly let us consider a GCT in the physical spacetime: $X^{\mu} \rightarrow X'^{\mu}(X)$ with transition function $\Lambda^{\mu}{}_{\nu}(X) = \frac{\partial X'^{\mu}}{\partial X^{\nu}}$ and its inverse $\Lambda_{\mu}{}^{\nu}(X) = \frac{\partial X^{\nu}}{\partial X'^{\mu}}$. This induces a GCT in the infinite-dimensional spacetime: $x^i \rightarrow x'^i$ such that the Jacobian matrix $\lambda^i{}_j(x) = \frac{\partial x'^i}{\partial x^j}$ and its inverse $\lambda_i{}^j(x) = \frac{\partial x^j}{\partial x'^i}$ are given by,

$$\begin{aligned} \lambda^i{}_j(x) &= \oint \frac{\mathrm{d}\sigma}{2\pi}\Lambda^{\mu}{}_{\nu}(X(\sigma))e^{i(n-m)\sigma}, \\ \lambda_i{}^j(x) &= \oint \frac{\mathrm{d}\sigma}{2\pi}\Lambda_{\mu}{}^{\nu}(X(\sigma))e^{i(m-n)\sigma}. \end{aligned} \tag{2.5}$$

[5] Throughout the paper, we will make the following type of index identifications: $i = \{\mu, m\}$, $j = \{\nu, n\}$, $k = \{\kappa, q\}$.

One can then show (see Appendix A) that $g_{ij}(x)$ and $a^i(x)$ transform as tensors,

$$g'_{ij}(x') = \lambda_i{}^k(x)\lambda_j{}^{k'}(x)g_{kk'}(x), \quad a'^i(x') = \lambda^i{}_j(x)a^j(x) \ . \tag{2.6}$$

2. Using the map in (2.4), one can relate any field in the infinite-dimensional spacetime constructed out of the metric, its inverse, $a^i(x)$ and their derivatives to a non-local worldsheet operator. A class of examples, which will prove to be useful for us, is given by a multi-indexed object $u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x)$ constructed out of the metric, its inverse, their derivatives and $a^i(x)$ (but not its derivatives) such that $u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x)$ cannot be factored into pieces which are not contracted with each other. In this case, one can construct a local worldsheet operator $U^{\mu_1 \nu_1 \cdots}_{\mu_2 \nu_2 \ldots}(X(\sigma))$ simply by performing the following replacements in the expression of $u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x)$:

$$\begin{aligned} &g_{ij}(x) \to G_{\mu\nu}(X(\sigma)), \quad g^{ij}(x) \to G^{\mu\nu}(X(\sigma)), \\ &\partial_i \to \partial_\mu, \quad a^i(x) \to \partial X^\mu(\sigma). \end{aligned} \tag{2.7}$$

The two objects $u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x)$ and $U^{\mu_1 \nu_1 \cdots}_{\mu_2 \nu_2 \ldots}(X(\sigma))$ are related to each other by the following general rule (see Appendix A):

$$\begin{aligned} u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x) \sim [2\pi\delta(0)]^N \oint \frac{\mathrm{d}\sigma}{2\pi} \, U^{\mu_1 \nu_1 \cdots}_{\mu_2 \nu_2 \ldots}(X(\sigma)) \\ e^{i(m_2+n_2+\cdots)\sigma - i(m_1+n_1+\cdots)\sigma}, \end{aligned} \tag{2.8}$$

where N is the number of traces in u and the argument of the Dirac delta function $\delta(0)$ appearing on the right hand side is the worldsheet space direction:

$$\delta(0) = \lim_{\sigma\to\sigma'} \delta(\sigma - \sigma') = \lim_{\sigma\to\sigma'} \frac{1}{2\pi} \sum_{n\in Z} e^{in(\sigma-\sigma')} \ . \tag{2.9}$$

The way one gets N factors of $\delta(0)$ on the right hand side is as follows: Each infinite-dimensional trace breaks up into a trace in the physical spacetime which appears in the expression of U, and a sum over all the string modes which gives rise to a factor of $\sum_{n\in Z} 1 = 2\pi\delta(0)$. We relate the two sides of (2.8) by the symbol $\sim$ to indicate that such a manipulation is understood only at a formal level. The relation (2.8) implies that u enjoys the same tensorial properties in the infinite-dimensional spacetime as U does in the physical spacetime (provided g_{ij} and a^i have the right tensorial property, which is indeed the case as we have already discussed).

3. In the infinite-dimensional language, the problem at hand possesses certain shift properties which can be written as:

$$\begin{aligned} u^{i_1+ii_2\ldots}_{j_1 j_2\ldots} = u^{i_1 i_2+i\ldots}_{j_1 j_2\ldots} = u^{i_1 i_2\ldots}_{j_1-ij_2\ldots} = u^{i_1 i_2\ldots}_{j_1 j_2-i\ldots} = \cdots \ , \\ \partial_{j+l}a^{k+l}(x) = \partial_j a^k(x) + i(l)\delta^k_j, \end{aligned} \tag{2.10}$$

where the factor of i in the second term of the last equation is the imaginary number. Given the spacetime index i as in footnote 5, we have defined $(i) = m$. A shift in the infinite-dimensional index is defined to be $i + j = \{\mu, m+n\}$.[6] It is now obvious that the first relation of (2.10) is a direct consequence of (2.8). The second relation can be obtained from the following one:

$$\partial_j a^k = i(j)\delta_j^k \ . \tag{2.11}$$

The easiest way to get this is to notice that the definitions in (2.4) imply that the infinite dimensional model in (2.2) corresponds to the string worldsheet theory only for the linear profile $a^k(x) = (k)x^k$. Alternatively, one can directly calculate the left hand side of (2.11) using the third equation in (2.4) and Eq. (A.55). The result is given by $in\delta_{n,q}\delta_\nu^\kappa = i(j)\delta_j^k$.

3. DeWitt–Virasoro Generators

The goal of this section is to arrive at the background-independent version of the quantum Virasoro generators. We will start with the standard expressions for the classical EM tensor and write the classical Virasoro generators in the infinite-dimensional language. Then after quantizing the system, we will use DeWitt's argument to define the quantum DeWitt–Virasoro generators.

The right and left moving components of the classical EM tensor are given by,

$$\begin{aligned} \mathcal{T}(\sigma) &= \frac{1}{4}\left(K(\sigma) - Z(\sigma) + V(\sigma)\right) = \sum_{m\in Z} L_m e^{im\sigma}, \\ \tilde{\mathcal{T}}(\sigma) &= \frac{1}{4}\left(K(\sigma) + Z(\sigma) + V(\sigma)\right) = \sum_{m\in Z} \tilde{L}_m e^{-im\sigma}, \end{aligned} \tag{3.12}$$

respectively, where,

$$\begin{aligned} K(\sigma) &= G^{\mu\nu}(X(\sigma))P_\mu(\sigma)P_\nu(\sigma) = \sum_{m\in Z} K_m e^{im\sigma}, \\ Z(\sigma) &= 2\partial X^\mu(\sigma)P_\mu(\sigma) = \sum_{m\in Z} Z_m e^{im\sigma}, \\ V(\sigma) &= G_{\mu\nu}(X(\sigma))\partial X^\mu(\sigma)\partial X^\nu(\sigma) = \sum_{m\in Z} V_m e^{im\sigma}. \end{aligned} \tag{3.13}$$

The conjugate momentum is given by: $P_\mu = G_{\mu\nu}(X)\dot{X}^\nu$. It is related to the momentum in the infinite-dimensional language, i.e. $p_i = g_{ij}(x)\dot{x}^j$ according to the same rule in (2.8). The classical Virasoro generators L_m and $\tilde{L}_m$ can now be expressed in terms of the Fourier modes K_m, Z_m and V_m, which can, in turn, be expressed in the infinite-dimensional language. The results are as follows:

$$4L_{(i)} = K_{(i)} - Z_{(i)} + V_{(i)}, \quad 4\tilde{L}_{(i)} = K_{(\bar{i})} + Z_{(\bar{i})} + V_{(\bar{i})}, \tag{3.14}$$

[6] Notice that we choose the physical spacetime index corresponding to $i + j$ by the one associated with the first index (i.e. i) appearing in the shift. We will follow this convention in all our expressions.

where we have defined $\bar{i} = \{\mu, -m\}$ and,

$$K_{(i)} = g^{kl+i}(x) p_k p_l, \quad Z_{(i)} = 2a^{k+i}(x) p_k, \quad V_{(i)} = g_{kl}(x) a^k(x) a^{l+i}(x) \ . \tag{3.15}$$

Notice that a Virasoro generator is a scalar, but has a string mode index $(i) = m$ which, in the infinite-dimensional language, appears to be a shift of the spacetime index, as evident from Eq. (3.15).

Poisson brackets of the generators in (3.14) should satisfy the classical Virasoro algebra. Notice that since GCT is a canonical transformation which preserves the Poisson brackets, it should be possible to write such brackets in a manifestly covariant manner. We have derived these brackets in the infinite-dimensional language in Appendix B.

We now quantize the system:

$$[\hat{x}^i, \hat{p}_j] = i\alpha' \delta^i_j \ . \tag{3.16}$$

We work in the Schrödinger picture so that the operators do not have explicit τ dependence. The idea is to keep the general covariance manifest in the quantum theory. GCT of any operator independent of momenta does not suffer from any ordering ambiguity and, therefore, straightforward to compute. Transformation of the momentum operator, which preserves the canonical commutation relations, is taken to be [39–41]:

$$\hat{p}_i \to \hat{p}'_i = \frac{1}{2}\left(\lambda_i^{\ j}(\hat{x})\hat{p}_j + \hat{p}_j \lambda_i^{\ j}(\hat{x})\right) \ . \tag{3.17}$$

This defines GCT of an arbitrary operator constructed out of the phase-space variables.

Let us now introduce the position eigenbasis $|x\rangle$. The orthonormality and completeness conditions read:

$$\langle x|x'\rangle = \delta(x,x') = g^{-1/2}(x)\delta(x-x'), \quad \int \mathrm{d}w\ |x\rangle\langle x| = 1, \tag{3.18}$$

where $\delta(x-x')$ is the Dirac delta function, $\mathrm{d}w = \mathrm{d}x g^{1/2}(x)$ and $g(x) = |\det g_{ij}(x)|$. The position space representation of the momentum operator is given by [39],

$$\langle x|\hat{p}_i|x'\rangle = -i\alpha' \left[\partial_i + \frac{1}{2}\gamma_i(x)\right] \delta(x,x'), \tag{3.19}$$

where γ_i are the contracted Christoffel symbols,[7]

$$\gamma_j = \gamma^i_{ji}, \quad \gamma^i_{jk} = \frac{1}{2} g^{il}\left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\right) \ . \tag{3.21}$$

[7] Notice that γ_i has a trace and, therefore, is divergent:

$$\gamma_i(x) = 2\pi\delta(0) \oint \frac{\mathrm{d}\sigma}{2\pi}\ \Gamma_\mu(X(\sigma)) e^{im\sigma}, \tag{3.20}$$

where the expression for $\Gamma_\mu(X)$ can be read out from (3.21) using the replacement (2.7).

Using $\gamma_i^*(x) = \gamma_{\bar{i}}(x)$ and $\partial_{x^i}\delta(x,x') = -(\partial_{x'^i} + \gamma_i(x))\delta(x,x')$ (see [39]), it is straightforward to check that the position space representation in (3.19) is compatible with the following hermiticity properties:

$$(\hat{x}^i)^\dagger = \hat{x}^{\bar{i}}, \quad (\hat{p}_i)^\dagger = \hat{p}_{\bar{i}} \ . \tag{3.22}$$

To construct the DeWitt–Virasoro generators, we first define following [41]:

$$\hat{\pi}_j = \hat{p}_j + \frac{i\alpha'}{2}\gamma_j(\hat{x}), \quad \hat{\pi}_j^\star = \hat{p}_j - \frac{i\alpha'}{2}\gamma_j(\hat{x}) \ . \tag{3.23}$$

Using (3.17), it is easy to show that these objects are transformed by left and right multiplications, respectively, under GCT:

$$\hat{\pi}_i \to \hat{\pi}_i' = \lambda_i{}^j(\hat{x})\hat{\pi}_j, \quad \hat{\pi}_i^\star \to \hat{\pi}_i'^\star = \hat{\pi}_j^\star\lambda_i{}^j(\hat{x}), \tag{3.24}$$

The quantum definition of the operators in Eq. (3.15) are given by,

$$\begin{aligned} \hat{K}_{(i)} &= \hat{\pi}_k^\star g^{kl+i}(\hat{x})\hat{\pi}_l, \\ \hat{Z}_{(i)} &= \hat{Z}_{(i)}^L + \hat{Z}_{(i)}^R, \\ \hat{V}_{(i)} &= g_{kl}(\hat{x})a^k(\hat{x})a^{l+i}(\hat{x}) \end{aligned} \tag{3.25}$$

where,

$$\hat{Z}_{(i)}^L = \hat{\pi}_k^\star a^{k+i}(\hat{x}), \quad \hat{Z}_{(i)}^R = a^{k+i}(\hat{x})\hat{\pi}_k. \tag{3.26}$$

Given the transformation properties in (3.24), it is clear that all the operators in (3.25) and (3.26) are invariant under GCT and have the right classical limit. The left-right symmetric combination for $\hat{Z}_{(i)}$ considered in (3.25) gives the right hermiticity property for the DeWitt–Virasoro generators. These operators give covariant results in the following sense. Consider the matrix element of an arbitrary operator constructed out of these generators between any two scalar states. The result written in position space representation is manifestly covariant.

4. DeWitt–Virasoro Algebra

Here we will calculate the algebra satisfied by the DeWitt–Virasoro generators $\hat{L}_{(i)}$ and $\hat{\tilde{L}}_{(i)}$ defined through Eqs. (3.14, 3.25, 3.26). As mentioned earlier, for a generic background the current framework allows us to calculate this algebra only in the spin-zero representation. We have given the details of the necessary computation in Appendix C. The final results, written in the infinite-dimensional language, are as follows,

$$\langle\chi| \left\{ \begin{aligned} [\hat{L}_{(i)}, \hat{L}_{(j)}] &= (i-j)\alpha'\hat{L}_{(i+j)} + \hat{A}^R_{(i)(j)}, \\ [\hat{\tilde{L}}_{(i)}, \hat{\tilde{L}}_{(j)}] &= (i-j)\alpha'\hat{\tilde{L}}_{(i+j)} + \hat{A}^L_{(i)(j)}, \\ [\hat{L}_{(i)}, \hat{\tilde{L}}_{(j)}] &= \hat{A}_{(i)(j)} \end{aligned} \right\} |\psi\rangle, \tag{4.27}$$

where $|\chi\rangle$ and $|\psi\rangle$ are two arbitrary scalar states (τ-dependent). The above result is the Witt algebra (without the central charge terms) with additional anomalous terms given by,

$$\begin{aligned}
&\hat{A}^R_{(i)(j)} = 0,\\
&\hat{A}^L_{(i)(j)} = 0,\\
&\hat{A}_{(i)(j)} = \frac{\alpha'^2}{8}\left(\hat{\pi}^{\star k+i} r_{kl}(\hat{x}) a^{l+\bar{j}}(\hat{x}) - a^{k+i}(\hat{x}) r_{kl}(\hat{x}) \hat{\pi}^{l+\bar{j}}\right),
\end{aligned} \tag{4.28}$$

where $r_{ij}(x)$ is the Ricci tensor in the infinite-dimensional spacetime which, according to the general map (2.8), is related to the same in physical spacetime, namely $R_{\mu\nu}(X)$ in the following way,

$$r_{ij}(x) \sim 2\pi\delta(0) \oint \frac{\mathrm{d}\sigma}{2\pi}\ R_{\mu\nu}(X(\sigma)) e^{i(m+n)\sigma}\ . \tag{4.29}$$

We will now make comments on not having the central charge terms in (4.27). We have mentioned before that the DeWitt–Virasoro generators are background independent and, therefore, are different from the quantum Virasoro generators at a particular conformal background. Similarly, the algebra in (4.27, 4.28) is not expected to be same as the quantum Virasoro algebra that occurs at the conformal point. The generic procedure of obtaining the quantum Virasoro generators and algebra at an arbitrary conformal point starting from the present construction is not understood in this work. This, however, is understood for flat space and the pp-wave background considered in [30] as discussed below.

5. Flat and pp-wave Backgrounds as Special Cases

Here we will discuss how we can understand the flat and the off-shell pp-wave background considered in [29,30] as special cases of the present construction. In particular, we will explain how the central charge terms arise in the Virasoro algebra and how our background-independent construction explains the problem of computing Virasoro anomaly for the pp-wave as addressed in [30]. We will discuss the flat and the pp-wave cases separately below in Sects. (5.1) and (5.2), respectively. Before that we make some comments based on general grounds.

For both the backgrounds, the EM tensor does not involve any non-trivial ordering between fields and conjugate momenta in the chosen coordinate system. This implies that DeWitt's generally covariant formulation is, in principle, not needed.[8] Another way of seeing this is that in both the cases we have,

$$\gamma_k(x) = 0, \quad g(x) = 1. \tag{5.30}$$

This indicates that any operator calculation done using the canonical commutators without worrying about manifest general covariance should be reliable.

[8] However, we will see in Sect. (5.2) that understanding the pp-wave case as a special case of the background-independent formulation helps us resolve certain puzzles.

This justifies recognizing the computation done in [28–30] as a special case of the present analysis as will be done below. It also turns out as a consequence of (5.30) that the algebra in (4.27) can be considered as operator equations.

5.1. Flat Background

In this case, the DeWitt–Virasoro generators are given, in the usual worldsheet language, as follows,

$$\hat{L}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z}\hat{\Pi}^\mu_{m-n}\hat{\Pi}^\nu_n, \quad \hat{\tilde{L}}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z}\hat{\tilde{\Pi}}^\mu_{m-n}\hat{\tilde{\Pi}}^\nu_n, \tag{5.31}$$

where the superscript (0) refers to flat background and,

$$\begin{aligned}\hat{\Pi}^\mu_m &= \frac{1}{\sqrt{2}}\oint\frac{\mathrm{d}\sigma}{2\pi}\left(\eta^{\mu\nu}\hat{P}_\nu(\sigma) - \partial\hat{X}^\mu(\sigma)\right)e^{-im\sigma},\\ \hat{\tilde{\Pi}}^\mu_m &= \frac{1}{\sqrt{2}}\oint\frac{\mathrm{d}\sigma}{2\pi}\left(\eta^{\mu\nu}\hat{P}_\nu(\sigma) + \partial\hat{X}^\mu(\sigma)\right)e^{im\sigma},\end{aligned} \tag{5.32}$$

are the usual creation-annihilation operators,

$$[\hat{\Pi}^\mu_m, \hat{\Pi}^\nu_n] = \eta^{\mu\nu}\alpha'\delta_{m+n,0}, \quad \left[\hat{\tilde{\Pi}}^\mu_m, \hat{\tilde{\Pi}}^\nu_n\right] = \eta^{\mu\nu}\alpha'\delta_{m+n,0}\ . \tag{5.33}$$

The DeWitt–Virasoro generators differ from the actual quantum Virasoro generators $\hat{\mathcal{L}}^{(0)}_m$ and $\hat{\tilde{L}}^{(0)}_m$ by additive c-numbers,

$$\hat{L}^{(0)}_m = \hat{\mathcal{L}}^{(0)}_m + c_m, \quad \hat{\tilde{L}}^{(0)}_m = \hat{\tilde{\mathcal{L}}}^{(0)}_m + \tilde{c}_m, \tag{5.34}$$

where,

$$\hat{\mathcal{L}}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z} :\hat{\Pi}^\mu_{m-n}\hat{\Pi}^\nu_n:, \quad \hat{\tilde{\mathcal{L}}}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z} :\hat{\tilde{\Pi}}^\mu_{m-n}\hat{\tilde{\Pi}}^\nu_n:, \tag{5.35}$$

and $c_m = \tilde{c}_m$ is non-zero only when $m = 0$, in which case it is a divergent constant. The normal ordering :: used in the above equations which matter only for the Virasoro zero modes is defined as the oscillator normal ordering with respect to the vacuum $|0\rangle$ defined by,

$$\left.\begin{matrix}\hat{\Pi}^\mu_m\\ \hat{\tilde{\Pi}}^\mu_m\end{matrix}\right\}|0\rangle = 0, \quad \forall m \geq 0\ . \tag{5.36}$$

It is easy to check using (5.33) that the generators in (5.31) satisfy,

$$\begin{aligned}&[\hat{L}^{(0)}_m, \hat{L}^{(0)}_n] = (m-n)\alpha'\hat{L}^{(0)}_{m+n},\\ &[\hat{\tilde{L}}^{(0)}_m, \hat{\tilde{L}}^{(0)}_n] = (m-n)\alpha'\hat{\tilde{L}}^{(0)}_{m+n},\\ &[\hat{L}^{(0)}_m, \hat{\tilde{L}}^{(0)}_n] = 0,\end{aligned} \tag{5.37}$$

which is simply the DeWitt–Virasoro algebra in (4.27) for flat background. However, the same method of computation applied to $[\hat{\mathcal{L}}^{(0)}_m, \hat{\mathcal{L}}^{(0)}_{-m}]$ gives a result that has operator ordering ambiguity. The result is $2m\alpha'\hat{\mathcal{L}}^{(0)}_0$ up to an additive c-number contribution that cannot be calculated using this method because of the ambiguity. As indicated in [38], this c-number contribution, which turns out to be the central charge term, can be found unambiguously by calculating, for example, $\langle 0|\hat{\mathcal{L}}^{(0)}_m \hat{\mathcal{L}}^{(0)}_{-m}|0\rangle$ with $m > 0$,

$$\begin{aligned}
[\hat{\mathcal{L}}^{(0)}_m, \hat{\mathcal{L}}^{(0)}_n] &= (m-n)\alpha' \hat{\mathcal{L}}^{(0)}_{m+n} + \frac{D}{12}(m^3 - m)\alpha'^2 \delta_{m+n,0}, \\
[\hat{\tilde{\mathcal{L}}}^{(0)}_m, \hat{\tilde{\mathcal{L}}}^{(0)}_n] &= (m-n)\alpha' \hat{\tilde{\mathcal{L}}}^{(0)}_{m+n} + \frac{D}{12}(m^3 - m)\alpha'^2 \delta_{m+n,0}, \\
[\hat{\mathcal{L}}^{(0)}_m, \hat{\tilde{\mathcal{L}}}^{(0)}_n] &= 0.
\end{aligned} \tag{5.38}$$

5.2. Explaining Conformal Invariance for pp-wave

Mukhopadhyay [30], we considered a restricted ansatz for an off-shell pp-wave in type IIB string theory which includes the R-R plane-wave background. The R-R flux part of the background involves the Green–Schwarz fermions on the worldsheet. We will ignore this fermionic part and consider only the bosonic part of the computation which corresponds to switching on a metric-background where the non-trivial components of the metric (in physical space-time) are given by,

$$G_{+-} = 1, \quad G_{++} = K(\vec{X}), \quad G_{IJ} = \delta_{IJ}, \tag{5.39}$$

where the vector sign refers to the transverse part with index I. The only non-trivial component of the Ricci-tensor is,

$$R_{++} \propto \vec{\partial}^2 K. \tag{5.40}$$

The worldsheet theory is expected to be an exact CFT when R_{++} vanishes [44, 45]. We call the background in (5.39) simply as pp-wave.[9]

It was argued in [30] that the operator anomaly terms of the Virasoro algebra suffer from an ordering ambiguity and, therefore, proving conformal invariance was not completely settled. We argued below Eq. (5.30) why it should be possible to consider this computation as a special case of the present background-independent formulation. Here we would like to show how doing this explains the conformal invariance in the present case resolving the ambiguous situation in the previous work.

The first step is to relate the DeWitt–Virasoro generators specialized to the present background with the quantum Virasoro generators defined in [30]. Given the latter, this relation is precisely the same as that in (5.34),

$$\hat{L}^{pp}_m = \hat{\mathcal{L}}^{pp}_m + c_m, \quad \hat{\tilde{L}}^{pp}_m = \hat{\tilde{\mathcal{L}}}^{pp}_m + \tilde{c}_m, \tag{5.42}$$

where the superscript *pp* refers to the pp-wave being considered. According to the calculations of the present work, the algebra satisfied by $\hat{L}^{pp}_m$ and $\hat{\tilde{L}}^{pp}_m$ is given by (4.27) evaluated for the pp-wave. Following the same procedure

[9] The particular solution of $R_{++} = 0$ given by,

$$K = \sum_I s_I X^I X^I, \quad \sum_I s_I = 0, \tag{5.41}$$

is called plane-wave.

as in flat-case, which took us from Eqs. (5.37)–(5.38), one finds the following quantum Virasoro algebra in the present case,

$$\begin{aligned}
&[\hat{\mathcal{L}}^{pp}_m, \hat{\mathcal{L}}^{pp}_n] = (m-n)\alpha' \hat{\mathcal{L}}^{pp}_{m+n} + \frac{D}{12}(m^3-m)\alpha'^2 \delta_{m+n,0} + \hat{A}^R_{mn}, \\
&[\hat{\bar{\mathcal{L}}}^{pp}_m, \hat{\bar{\mathcal{L}}}^{pp}_n] = (m-n)\alpha' \hat{\bar{\mathcal{L}}}^{pp}_{m+n} + \frac{D}{12}(m^3-m)\alpha'^2 \delta_{m+n,0} + \hat{A}^L_{mn}, \\
&[\hat{\mathcal{L}}^{pp}_m, \hat{\bar{\mathcal{L}}}^{pp}_n] = \hat{A}_{mn},
\end{aligned} \tag{5.43}$$

where the operator anomaly terms $\hat{A}^R_{mn} = \hat{A}^R_{(i)(j)}$, $\hat{A}^L_{mn} = \hat{A}^L_{(i)(j)}$ and $\hat{A}_{mn} = \hat{A}_{(i)(j)}$ are given by Eq. (4.28) evaluated for the pp-wave.

We will now compare the result in (5.43) found in this work with the one in [30]. The computation of [30] was done using the local worldsheet language. The result is precisely the local version of (5.43) with the local operator anomaly terms given by,

$$\begin{aligned}
&\hat{A}^R_{\text{there}}(\sigma,\sigma') = \hat{A}^L_{\text{there}}(\sigma,\sigma') = \hat{A}_{\text{there}}(\sigma,\sigma') \\
&\quad \propto \left[\hat{\mathcal{O}}(\sigma,\sigma')\hat{P}_-(\sigma')\hat{P}_-(\sigma') - \hat{\mathcal{O}}(\sigma',\sigma)\hat{P}_-(\sigma)\hat{P}_-(\sigma)\right]\delta_\epsilon(\sigma-\sigma') \\
&\qquad - \left[\hat{\mathcal{O}}(\sigma,\sigma')\partial\hat{X}^+(\sigma')\partial\hat{X}^+(\sigma') - \hat{\mathcal{O}}(\sigma',\sigma)\partial\hat{X}^+(\sigma)\partial\hat{X}^+(\sigma)\right]\delta_\epsilon(\sigma-\sigma'),
\end{aligned} \tag{5.44}$$

where the above three anomaly terms are defined in Eq. (3.20) of [30] and

$$\hat{\mathcal{O}}(\sigma,\sigma') = \hat{P}_I(\sigma)\partial_I K(\hat{\vec{X}}(\sigma')) + \partial_I K(\hat{\vec{X}}(\sigma'))\hat{P}_I(\sigma)\ . \tag{5.45}$$

Therefore, comparing (5.43) with the corresponding result in [30] we are able to relate the operator anomaly terms in (5.43) and in (5.44),

$$\begin{aligned}
\hat{A}^R_{(i)(j)} &= \oint \frac{\mathrm{d}\sigma}{2\pi}\frac{\mathrm{d}\sigma'}{2\pi}\ \hat{A}^R_{\text{there}}(\sigma,\sigma')e^{-im\sigma-in\sigma'}, \\
\hat{A}^L_{(i)(j)} &= \oint \frac{\mathrm{d}\sigma}{2\pi}\frac{\mathrm{d}\sigma'}{2\pi}\ \hat{A}^L_{\text{there}}(\sigma,\sigma')e^{im\sigma+in\sigma'}, \\
\hat{A}_{(i)(j)} &= \oint \frac{\mathrm{d}\sigma}{2\pi}\frac{\mathrm{d}\sigma'}{2\pi}\ \hat{A}_{\text{there}}(\sigma,\sigma')e^{-im\sigma+in\sigma'}
\end{aligned} \tag{5.46}$$

It was explained in [30] that the expression in (5.44) suffers from an operator ordering ambiguity. The idea here is to resolve this ambiguity by borrowing the results found here. Therefore, using Eqs. (5.46) and (4.28), one concludes that both $\hat{A}^R_{\text{there}}(\sigma,\sigma')$ and $\hat{A}^L_{\text{there}}(\sigma,\sigma')$ should vanish. Moreover, comparing the equations in (4.28), (5.44) and (5.46) one concludes that the Ricci-term obtained in the present analysis was missing earlier. As we will show below, this is an apparent discrepancy which may be resolved by showing that the relevant term vanishes, though the Ricci tensor itself does not, for the pp-wave under consideration because of certain special properties of the background.[10]

[10] This argument, however, will rely on the scalar expectation value of the algebra in (4.27), and not operator equation.

The non-trivial components of the infinite-dimensional metric corresponding to (5.39) are,

$$g_{i_+j_-} = \delta_{m+n,0}, \quad g_{i_+j_+} = \oint \frac{\mathrm{d}\sigma}{2\pi}\, K(\vec{X}(\sigma))e^{i(m+n)\sigma}, \quad g_{i_\perp j_\perp} = \delta_{IJ}\delta_{m+n,0}, \tag{5.47}$$

where the infinite-dimensional spacetime index is divided in the following way: $i = (i_+, i_-, i_\perp)$ such that,

$$i_+ = \{+, m\}, \quad i_- = \{-, m\}, \quad i_\perp = \{I, m\}\ . \tag{5.48}$$

The only non-trivial components of the Ricci tensor are $r_{i_+j_+}(\vec{x})$ (the vector sign referring to the transverse indices $i_\perp$) which is related to $R_{++}(\vec{X})$ in (5.40) according to (4.29). Let us now go back to the last equation in (4.27). By going to the position space representation, integrating by parts and using the shift property (2.10), one can show that the Ricci-term is proportional to,

$$\int \mathrm{d}w\ \chi^* \nabla^{k+i}(r_{kl}a^{l+\bar{j}})\psi = \int \mathrm{d}x\ \chi^* g^{k_+ + ik'_-}\partial_{k'_-}(r_{k_+l_+}a^{l_+ + \bar{j}})\psi, \tag{5.49}$$

where on the right hand side we have evaluated the term for the pp-wave. This vanishes as the quantity inside the round brackets is independent of x^{i_-}.

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Appendix A. Some Technical Details Regarding the Map

Here we will discuss some technical details regarding the infinite-dimensional language described in Sect. 2. In particular, we will indicate how to get the results in (2.6) and (2.8).

The steps leading to the GCT property for the metric in (2.6) are as follows:

$$\begin{aligned} g'_{ij}(x') &= \oint \frac{\mathrm{d}\sigma}{2\pi}\ \Lambda_\mu{}^\rho(X(\sigma))\Lambda_\nu{}^\kappa(X(\sigma))G_{\rho\kappa}(X(\sigma))e^{i(m+n)\sigma}, \\ &= \sum_{r'',q'}\left[\oint \frac{\mathrm{d}\sigma''}{2\pi}\ \Lambda_\mu{}^\rho(X(\sigma''))e^{-ir''\sigma''}\right]\left[\oint \frac{\mathrm{d}\sigma'}{2\pi}\ \Lambda_\nu{}^\kappa(X(\sigma'))e^{-iq'\sigma'}\right] \end{aligned}$$

$$\oint \frac{\mathrm{d}\sigma}{2\pi}\ G_{\rho\kappa}(X(\sigma))e^{i(r''+m+q'+n)\sigma},$$
$$= \sum_{r,q}\left[\oint \frac{\mathrm{d}\sigma''}{2\pi}\ \Lambda_\mu{}^\rho(X(\sigma''))e^{i(m-r)\sigma''}\right]\left[\oint \frac{\mathrm{d}\sigma'}{2\pi}\ \Lambda_\nu{}^\kappa(X(\sigma'))e^{i(n-q)\sigma'}\right]$$
$$\oint \frac{\mathrm{d}\sigma}{2\pi}\ G_{\rho\kappa}(X(\sigma))e^{i(p+q)\sigma},$$
$$= \lambda_i{}^l(x)\lambda_j{}^k(x)g_{lk}(x), \tag{A.50}$$

where in the first step we have used the transformation of the metric tensor in physical spacetime. To go to the second step, we first write,

$$\begin{aligned}\Lambda_\mu{}^\rho(X(\sigma)) &= \oint \mathrm{d}\sigma''\ \delta(\sigma-\sigma'')\Lambda_\mu{}^\rho(X(\sigma'')),\\ \Lambda_\nu{}^\kappa(X(\sigma)) &= \oint \mathrm{d}\sigma'\ \delta(\sigma-\sigma')\Lambda_\nu{}^\kappa(X(\sigma')),\end{aligned} \tag{A.51}$$

then we write each of the delta functions $\delta(\sigma-\sigma'')$ and $\delta(\sigma-\sigma')$ as infinite sum of phases as indicated in Eq. (2.9) such that the summation indices are r'' and q', respectively. Finally, we rearrange the exponential factors suitably. In the third step, we have changed the summation indices from r'' and q' to $r = r''+m$ and $q = q'+n$, respectively. The tensorial property of $a^i(x)$ in Eq. (2.6) can also be established in a similar way.

To prove the general rule in (2.8) let us first consider $N = 0$, in which case the general form of $u^{i_1 j_1\cdots}_{i_2 j_2\ldots}(x)$ is a contraction of various factors involving derivatives of metric and its inverse. It can also have factors of $a^i(x)$, but we will consider them later. The second defining relation in (2.4) itself is the simplest example of this kind. This relation ensures that

$$g^{ij}(x) = \oint \frac{\mathrm{d}\sigma}{2\pi}\ G^{\mu\nu}(X(\sigma))e^{-i(m+n)\sigma}, \tag{A.52}$$

which also has the form (2.8), is the inverse metric:

$$g_{ij}(x)g^{jk}(x) = \sum_{n\in Z}\oint \frac{\mathrm{d}\sigma}{2\pi}\ G_{\mu\nu}(X(\sigma))e^{i(m+n)\sigma}\oint \frac{\mathrm{d}\sigma'}{2\pi}\ G^{\nu\kappa}(X(\sigma'))e^{-i(n+q)\sigma'},$$
$$= \oint \frac{\mathrm{d}\sigma}{2\pi}\oint \frac{\mathrm{d}\sigma'}{2\pi}G_{\mu\nu}(X(\sigma))G^{\nu\kappa}(X(\sigma'))e^{i(m\sigma-q\sigma')}2\pi\delta(\sigma-\sigma'),$$
$$= \delta^\kappa_\mu\delta_{m,q} = \delta^k_i, \tag{A.53}$$

where all the above steps are obvious. We now notice the following two properties of (2.8):

1. If $v^{i\cdots}_{\ldots}(x)$ and $w^{\cdots}_{i\ldots}(x)$ satisfy (2.8) with worldsheet counterparts $V^{\mu\cdots}_{\ldots}(X(\sigma))$ and $W^{\cdots}_{\mu\ldots}(X(\sigma))$, respectively, then $v^{i\cdots}_{\ldots}(x)w^{\cdots}_{i\ldots}(x)$ also satisfies (2.8) with the worldsheet counterpart given by $V^{\mu\cdots}_{\ldots}(X(\sigma))$ $W^{\cdots}_{\mu\ldots}(X(\sigma))$. Here the ellipses appearing as superscripts and subscripts in v and V indicate any number of infinite-dimensional and the corresponding physical spacetime indices, respectively.

2. If $v^{\cdots}_{\cdots}(x)$ satisfies (2.8) with its worldsheet counterpart $V^{\cdots}_{\cdots}(X(\sigma))$, then $\partial_i v^{\cdots}_{\cdots}(x)$ also satisfies (2.8) with the worldsheet counterpart given by $\partial_\mu V^{\cdots}_{\cdots}(X(\sigma))$. Here also the ellipses play similar role as before.

The first property is easy to establish using the manipulations leading to (A.52). The second one can be proved as follows:

$$\begin{aligned}\partial_i v^{\cdots}_{\cdots}(x) &= \oint \frac{\mathrm{d}\sigma}{2\pi}\, e^{im\sigma} \frac{\delta}{\delta X^\mu(\sigma)} \oint \frac{\mathrm{d}\sigma'}{2\pi}\, V^{\cdots}_{\cdots}(X(\sigma'))e^{i(\ldots)\sigma'},\\ &= \oint \frac{\mathrm{d}\sigma}{2\pi} \oint \frac{\mathrm{d}\sigma'}{2\pi}\, \partial_\mu V^{\cdots}_{\cdots}(X(\sigma'))e^{im\sigma+i(\ldots)\sigma'}2\pi\delta(\sigma-\sigma'),\\ &= \oint \frac{\mathrm{d}\sigma}{2\pi}\, \partial_\mu V^{\cdots}_{\cdots}(X(\sigma))e^{i(m+\cdots)\sigma}, \end{aligned} \tag{A.54}$$

where $\ldots$ in the phase is the appropriate sum of indices that, combined with the physical spacetime indices on V, give rise to the infinite-dimensional spacetime indices on v. In the first two steps of (A.54) we have used the following results,

$$\partial_i = \oint \frac{\mathrm{d}\sigma}{2\pi}\, e^{im\sigma} \frac{\delta}{\delta X^\mu(\sigma)}, \quad \frac{\delta X^\mu(\sigma)}{\delta X^\nu(\sigma')} = 2\pi\delta^\mu_\nu \delta(\sigma-\sigma')\ . \tag{A.55}$$

To justify these results, one may first Fourier expand: $X^\mu(\sigma) = \sum_{m\in Z} X^\mu_m e^{im\sigma}$. Then treat the set of all string modes $\{X^\mu_m\}$ and the set of all functions $\{X^\mu(\sigma)\}$ as two sets of orthonormal and complete basis for the string configurations. Using the first equation in (2.4), we identify: $x^i = X^\mu_m$ and take the orthogonality condition for these modes as:

$$\partial_j x^i = \frac{\partial X^\mu_m}{\partial X^\nu_n} = \delta^\mu_\nu \delta_{m,n} = \delta^i_j\ . \tag{A.56}$$

The results in (A.55) then follow from the above mode expansion and this orthogonality condition. Using the fact that the second equation of (2.4) and (A.52) is of the form (2.8) and the two properties below Eq. (A.52) one can argue that (2.8) holds true for $N=0$ if $u^{i_1j_1\cdots}_{i_2j_2\ldots}(x)$ is independent of $a^i(x)$. To incorporate the $a^i(x)$ dependence, we notice that the general form in which it can appear in $u^{i_1j_1\cdots}_{i_2j_2\ldots}(x)$ is as follows:

$$u^{i_1j_1\cdots}_{i_2j_2\ldots}(x) = v^{i_1j_1\cdots}_{i_2j_2\ldots k_1k_2\ldots}(x)a^{k_1}(x)a^{k_2}(x)\ldots, \tag{A.57}$$

where $v^{i_1j_1\cdots}_{i_2j_2\ldots k_1k_2\ldots}(x)$ is independent of $a(x)$ and satisfies (2.8) with $V^{\mu_1\nu_1\cdots}_{\mu_2\nu_2\ldots\kappa_1\kappa_2\ldots}(X(\sigma))$ as its worldsheet counterpart. Then using the last equation of (2.4) one can show that $u^{i_1j_1\cdots}_{i_2j_2\ldots}(x)$ satisfies (2.8) with $V^{\mu_1\nu_1\cdots}_{\mu_2\nu_2\ldots\kappa_1\kappa_2\ldots}(X(\sigma))\partial X^{\kappa_1}(\sigma)\partial X^{\kappa_2}(\sigma)\ldots$ as its worldsheet counterpart. This proves (2.8) for $N=0$.

For $N\neq 0, u^{i_1j_1\cdots}_{i_2j_2\ldots}(x)$ contains infinite-dimensional traces. Each such trace is interpreted to give a factor of $2\pi\delta(0)$ in the worldsheet language. To see what exactly we mean by this formal statement let us consider, for example, $v_{ijk\ldots}(x)$

which satisfies (2.8) for $N = 0$ with its worldsheet counterpart $V_{\mu\nu\kappa\ldots}(X(\sigma))$. Then, a single trace $u_{k\ldots}(x) \equiv g^{ij}(x)v_{ijk\ldots}(x)$ is given by,

$$\begin{aligned} u_{k\ldots}(x) &= \sum_{m,n\in Z} \oint \frac{\mathrm{d}\sigma}{2\pi}\, G^{\mu\nu}(X(\sigma))e^{-i(m+n)\sigma} \\ &\quad \times \oint \frac{\mathrm{d}\sigma'}{2\pi}\, V_{\mu\nu\kappa\ldots}(X(\sigma'))e^{i(m+n+q+\cdots)\sigma'}, \\ &= \oint \frac{\mathrm{d}\sigma}{2\pi} \oint \frac{\mathrm{d}\sigma'}{2\pi}\, [2\pi\delta(\sigma-\sigma')]^2 G^{\mu\nu}(X(\sigma))V_{\mu\nu\kappa\ldots}(X(\sigma'))e^{i(q+\cdots)\sigma'}, \\ &= 2\pi\delta(0) \oint \frac{\mathrm{d}\sigma}{2\pi}\, U_{\kappa\ldots}(X(\sigma))e^{i(q+\cdots)\sigma'}, \end{aligned} \tag{A.58}$$

where $U_{\kappa\ldots}(X) = G^{\mu\nu}(X)V_{\mu\nu\kappa\ldots}(X)$.

Appendix B. Classical Virasoro Algebra in Infinite-Dimensional Language

As we know, GCT is a point canonical transformation and, therefore, it should be possible to write classical Poisson brackets in a manifestly covariant form. Here we will discuss the results in infinite-dimensional language. Poisson bracket between any two dynamical quantities $A(x,p)$ and $B(x,p)$ is given by:

$$\{A,B\} = \sum_i \left[\frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial x^i}\frac{\partial A}{\partial p_i}\right]. \tag{B.59}$$

Below we will derive the classical algebra satisfied by the generators in (3.14) and show that the known result is obtained. The aim of this Appendix is to show how the argument goes in the infinite-dimensional language in a manifestly covariant manner.

The algebra that we wish to compute is given by,

$$\begin{aligned} &\{L_{(i)}, L_{(j)}\} \\ &\quad = \frac{1}{16}\left[T^{ZZ}_{(i)(j)} - \left(T^{KZ}_{(i)(j)} - i \leftrightarrow j\right) + \left(T^{KV}_{(i)(j)} - i \leftrightarrow j\right) - \left(T^{ZV}_{(i)(j)} - i \leftrightarrow j\right)\right], \\ &\{\tilde{L}_{(i)}, \tilde{L}_{(j)}\} \\ &\quad = \frac{1}{16}\left[T^{ZZ}_{(\bar{i})(\bar{j})} + \left(T^{KZ}_{(\bar{i})(\bar{j})} - \bar{i} \leftrightarrow \bar{j}\right) + \left(T^{KV}_{(\bar{i})(\bar{j})} - \bar{i} \leftrightarrow \bar{j}\right) + \left(T^{ZV}_{(\bar{i})(\bar{j})} - \bar{i} \leftrightarrow \bar{j}\right)\right], \\ &\{L_{(i)}, \tilde{L}_{(j)}\} \\ &\quad = \frac{1}{16}\left[-T^{ZZ}_{(i)(\bar{j})} + \left(T^{KZ}_{(i)(\bar{j})} + i \leftrightarrow \bar{j}\right) + \left(T^{KV}_{(i)(\bar{j})} - i \leftrightarrow \bar{j}\right) - \left(T^{ZV}_{(i)(\bar{j})} + i \leftrightarrow \bar{j}\right)\right], \end{aligned} \tag{B.60}$$

where,

$$T^{AB}_{(i)(j)} = \{A_{(i)}, B_{(j)}\}. \tag{B.61}$$

The terms $T^{KK}_{(i)(j)}$ and $T^{VV}_{(i)(j)}$ have not been included in Eq. (B.60) as they can be easily shown to vanish.

Let us consider the first term in Eq.(B.60). It is straightforward to show:

$$T^{ZZ}_{(i)(j)} = 4(\partial_k a^{k'+i} a^{k+j} p_{k'} - i \leftrightarrow j) \ . \tag{B.62}$$

This can also be written in a covariant form:

$$T^{ZZ}_{(i)(j)} = 4(\nabla_k a^{k'+i} a^{k+j} p_{k'} - i \leftrightarrow j), \tag{B.63}$$

where ∇_i is the covariant derivative in infinite-dimensional spacetime[11]. The reason that the expressions in (B.62) and (B.63) are same is as follows. The connection term coming from the covariant derivative does not contain any derivative of $a(x)$ and, therefore, is symmetric in i and j due to the shift property as given by the first equation in (2.10). Clearly this term drops out because of the anti-symmetrization in i and j. The expression of $T^{ZZ}_{(i)(j)}$, however, that we would like to have for calculating the Virasoro algebra is as follows:

$$T^{ZZ}_{(i)(j)} = 2i(i-j)Z_{(i+j)}, \tag{B.67}$$

which can be obtained simply using (2.11) in (B.62).

For the other $T^{AB}_{(i)(j)}$'s the analogues of Eq. (B.62) are given by:

$$\begin{aligned} T^{KZ}_{(i)(j)} &= (2\partial_k g^{ll'+i} a^{k+j} - 4g^{kl+i}\partial_k a^{l'+j}) p_l p_{l'}, \\ T^{KV}_{(i)(i)} &= -2g^{kl+i}\partial_k(g_{k'l'}a^{k'}a^{l'+j})p_l, \\ T^{ZV}_{(i)(j)} &= -2a^{k+i}\partial_k(g_{ll'}a^l a^{l'+j}) \ . \end{aligned} \tag{B.68}$$

Using the fact that the metric is covariantly constant, such expressions can be given the following covariant forms:

$$\begin{aligned} T^{KZ}_{(i)(j)} &= -4g^{kl+i}\nabla_k a^{l'+j} p_l p_{l'}, \\ T^{KV}_{(i)(j)} &= -4g^{kl+i} g_{k'l'} a^{k'} \nabla_k a^{l'+j} p_l, \\ T^{ZV}_{(i)(j)} &= -4a^{k+i} g_{ll'} a^l \nabla_k a^{l'+j} \ . \end{aligned} \tag{B.69}$$

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[11] In worldsheet language ∇_i can be written as:

$$\nabla_i = \oint \frac{\mathrm{d}\sigma}{2\pi}\, e^{im\sigma} \mathcal{D}_{X^\mu(\sigma)}, \tag{B.64}$$

where $\mathcal{D}_{X^\mu(\sigma)}$ is the functional covariant derivative which acts on a vector, for example, in the following way:

$$\mathcal{D}_{X^\nu(\sigma)} V^\mu(X(\sigma')) = \frac{\delta}{\delta X^\nu(\sigma)} V^\mu(X(\sigma')) + 2\pi\delta(\sigma-\sigma')\Gamma^\mu_{\nu\kappa}(X(\sigma))V^\kappa(X(\sigma')), \tag{B.65}$$

where $\Gamma^\mu_{\nu\kappa}(X)$ are the Christoffel symbols in physical spacetime whose expressions can be read out from those of $\gamma^i_{jk}(x)$ in (3.21) following the rules in (2.7). It can be explicitly checked that the derivative in (B.65) transforms covariantly under GCT:

$$\mathcal{D}_{X'^\nu(\sigma)} V'^\mu(X'(\sigma')) = \Lambda_\nu{}^\rho(X(\sigma))\Lambda^\mu{}_\sigma(X(\sigma'))\mathcal{D}_{X^\rho(\sigma)} V^\sigma(X(\sigma')) \ . \tag{B.66}$$

However, we need the symmetric and anti-symmetric components of the above results to use in Eq. (B.60). The anti-symmetric components are given by:

$$\begin{aligned} T^{KZ}_{(i)(j)} - i \leftrightarrow j &= 4i(i-j)K_{(i+j)}, \\ T^{KV}_{(i)(j)} - i \leftrightarrow j &= 2i(i-j)Z_{(i+j)}, \\ T^{ZV}_{(i)(j)} - i \leftrightarrow j &= 4i(i-j)V_{(i+j)}, \end{aligned} \tag{B.70}$$

which can be derived following the same way as (B.67) was found. The symmetric components, on the other hand, are as follows:

$$\begin{aligned} T^{KZ}_{(i)(j)} + i \leftrightarrow j &= 0, \\ T^{ZV}_{(i)(j)} + i \leftrightarrow j &= 0. \end{aligned} \tag{B.71}$$

To derive the first result above, we first use (2.11) and the shift symmetry (2.10) in the first equation of (B.69) to write:

$$T^{KZ}_{(i)(j)} + i \leftrightarrow j = 4 \sum_{l,l'} \left[a^k \partial_k g^{l+i+jl'} - i(l+l'+i+j) g^{l+i+jl'} \right] p_l p_{l'}. \tag{B.72}$$

Then, we use the following identity:[12]

$$a^{i+k} \partial_k u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x) = i\{(i+i_1+j_1+\cdots) - (i_2+j_2+\cdots)\} u^{i+i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x), \tag{B.75}$$

to arrive at the first equation in (B.71). Similarly, the second equation in (B.71) can be derived by first using (2.11) and the shift symmetry (2.10) in the last equation of (B.69) to arrive at:

$$T^{ZV}_{(i)(j)} + i \leftrightarrow j = -4 \sum_{l,l'} \left[a^k \partial_k g_{ll'-i-j} + i(l+l'-i-j) g_{ll'-i-j} \right] a_l a_{l'} \ . \tag{B.76}$$

Then using the identity (B.75) one shows that the right hand side of the above equation vanishes.

---

[12] To derive (B.75) we first use the map (2.8) to write:

$$\begin{aligned} & a^{i+k} \partial_k u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x) \\ & \quad \sim [2\pi\delta(0)]^N \oint \frac{\mathrm{d}\sigma}{2\pi} \ \partial X^\mu \partial_\mu U^{\mu_1 \nu_1 \cdots}_{\mu_2 \nu_2 \ldots}(X(\sigma)) e^{i(m_2+n_2+\cdots)\sigma - i(q+m_1+n_1+\cdots)\sigma}. \end{aligned} \tag{B.73}$$

Then using $\partial X^\mu \partial_\mu = \partial$ and finally integrating by parts one gets:

$$\begin{aligned} a^{i+k} \partial_k u^{i_1 j_1 \cdots}_{i_2 j_2 \ldots}(x) \sim \ & i\{(q+m_1+n_1+\cdots) - (m_2+n_2+\cdots\} \\ & [2\pi\delta(0)]^N \oint \frac{\mathrm{d}\sigma}{2\pi} \ U^{\mu_1 \nu_1 \cdots}_{\mu_2 \nu_2 \ldots}(X(\sigma)) e^{i(m_2+n_2+\cdots)\sigma - i(q+m_1+n_1+\cdots)\sigma}, \end{aligned} \tag{B.74}$$

which, in the infinite-dimensional language, reads (B.75).

Finally, one uses the results (B.67), (B.70) and (B.71) in Eq. (B.60) to establish the classical Virasoro algebra:

$$\begin{aligned} \{L_{(i)}, L_{(j)}\} &= -i(i-j)L_{(i+j)}, \\ \{\tilde{L}_{(i)}, \tilde{L}_{(j)}\} &= -i(i-j)\tilde{L}_{(i+j)}, \\ \{L_{(i)}, \tilde{L}_{(j)}\} &= 0 \ . \end{aligned} \tag{B.77}$$

Appendix C. Derivation of DeWitt–Virasoro Algebra

Here we will prove the results in (4.27, 4.28). To proceed with the computation we will first introduce certain notations. The analogues of equations in (B.60) in the present case are given by,

$$\begin{aligned} {}_\chi\left\langle\left[\hat{L}_{(i)}, \hat{L}_{(j)}\right]\right\rangle_\psi &= \frac{1}{16}{}_\chi\left\langle \hat{T}^{KK}_{(i)(j)} + \hat{T}^{ZZ}_{(i)(j)} - \left(\hat{T}^{KZ}_{(i)(j)} - i \leftrightarrow j\right) + \left(\hat{T}^{KV}_{(i)(j)} - i \leftrightarrow j\right)\right. \\ &\qquad\qquad \left. - \left(\hat{T}^{ZV}_{(i)(j)} - i \leftrightarrow j\right)\right\rangle_\psi, \\ {}_\chi\left\langle\left[\hat{\tilde{L}}_{(i)}, \hat{\tilde{L}}_{(j)}\right]\right\rangle_\psi &= \frac{1}{16}{}_\chi\left\langle \hat{T}^{KK}_{(\bar{i})(\bar{j})} + \hat{T}^{ZZ}_{(\bar{i})(\bar{j})} + \left(\hat{T}^{KZ}_{(\bar{i})(\bar{j})} - \bar{i} \leftrightarrow \bar{j}\right) + \left(\hat{T}^{KV}_{(\bar{i})(\bar{j})} - \bar{i} \leftrightarrow \bar{j}\right)\right. \\ &\qquad\qquad \left. + \left(\hat{T}^{ZV}_{(\bar{i})(\bar{j})} - \bar{i} \leftrightarrow \bar{j}\right)\right\rangle_\psi, \\ {}_\chi\left\langle\left[\hat{L}_{(i)}, \hat{\tilde{L}}_{(j)}\right]\right\rangle_\psi &= \frac{1}{16}{}_\chi\left\langle \hat{T}^{KK}_{(i)(\bar{j})} - \hat{T}^{ZZ}_{(i)(\bar{j})} + \left(\hat{T}^{KZ}_{(i)(\bar{j})} + i \leftrightarrow \bar{j}\right) + \left(\hat{T}^{KV}_{(i)(\bar{j})} - i \leftrightarrow \bar{j}\right)\right. \\ &\qquad\qquad \left. - \left(\hat{T}^{ZV}_{(i)(\bar{j})} + i \leftrightarrow \bar{j}\right)\right\rangle_\psi, \end{aligned} \tag{C.78}$$

where ${}_\chi\langle\cdots\rangle_\psi = \langle\chi|\cdots|\psi\rangle$, $|\chi\rangle$ and $|\psi\rangle$ being two arbitrary spin zero states. The operator $\hat{T}^{AB}_{(i)(j)}$ is given by the quantum version of (B.61):

$$\hat{T}^{AB}_{(i)(j)} = \left[\hat{A}_{(i)}, \hat{B}_{(j)}\right], \tag{C.79}$$

where $\hat{K}_{(i)}$, $\hat{Z}_{(i)}$ and $\hat{V}_{(i)}$ are defined in Eq. (3.25).

Below we will compute the various expectation values that appear on the right hand sides of Eq. (C.78). Such calculations are done using the following basic results extensively:

$${}_\chi\left\langle\hat{K}_{(i)}\right\rangle_x = -\alpha'^2\nabla^2_{(i)}\chi^*(x), \tag{C.80}$$

$${}_x\left\langle\hat{K}_{(i)}\right\rangle_\psi = -\alpha'^2\nabla^2_{(i)}\psi(x), \tag{C.81}$$

$${}_\chi\left\langle\hat{Z}^L_{(i)}\right\rangle_x = i\alpha'\nabla_k\chi^*(x)a^{k+i}(x), \tag{C.82}$$

$${}_x\langle\hat{Z}^L_{(i)}\rangle_\psi = -i\alpha'\nabla_k\left(a^{k+i}(x)\psi(x)\right), \tag{C.83}$$

$${}_\chi\left\langle\hat{Z}^R_{(i)}\right\rangle_x = i\alpha'\nabla_k\left(\chi^*(x)a^{k+i}(x)\right), \tag{C.84}$$

$${}_x\left\langle\hat{Z}^R_{(i)}\right\rangle_\psi = -i\alpha' a^{k+i}(x)\nabla_k\psi(x), \tag{C.85}$$

where $\nabla^2_{(i)} = g^{k+ik'}(x)\nabla_k\nabla_{k'}$ is the *shifted* Laplace-Beltrami operator. These equations can be easily derived using the basic definitions in (3.23) and (3.19).

Let us first consider ${}_\chi\langle \hat{T}^{KK}_{(i)(j)}\rangle_\psi$. Using (C.80) and (C.81) one finds:

$$
{}_\chi\left\langle \hat{K}_{(i)}\hat{K}_{(j)}\right\rangle_\psi = \alpha'^4 \int \mathrm{d}w\ \nabla^2_{(i)}\chi^*\nabla^2_{(j)}\psi = \alpha'^4 \int \mathrm{d}w\ \chi^*\nabla^2_{(i)}\nabla^2_{(j)}\psi, \tag{C.86}
$$

where in the last step we have used integrations by parts to move the derivatives from χ^* to ψ. Using the shift property in (2.10) one concludes right away that the above expression is symmetric in i and j and, therefore, the commutator ${}_\chi\langle \hat{T}^{KK}_{(i)(j)}\rangle_\psi$ must vanish. However, we would like to compute it by explicitly calculating the commutator of the shifted Laplace-Beltrami operators without imposing the shift property until the very end. This yields:

$$
\begin{aligned}
{}_\chi\left\langle \hat{T}^{KK}_{(i)(j)}\right\rangle_\psi &= \alpha'^4 \int \mathrm{d}w\ \left(\nabla^{k+i}\chi^* r_{kl}\nabla^{l+j}\psi - \nabla^{k+j}\chi^* r_{kl}\nabla^{l+i}\psi\right), \\
&= \alpha'^2 {}_\chi\left\langle \hat{\pi}^{\star k+i} r_{kl}(\hat{x})\hat{\pi}^{l+j} - i \leftrightarrow j\right\rangle_\psi, \\
&= 0,
\end{aligned} \tag{C.87}
$$

where we have used the fact that commutator of covariant derivatives acting on a scalar vanishes, but yields the following for a vector V,

$$
[\nabla_i, \nabla_j]V^k = r^k_{lij}V^l, \tag{C.88}
$$

where r^k_{lij} is the Riemann tensor of the infinite-dimensional spacetime. Notice the appearance of the Ricci tensor $r_{ij} = r^k_{ikj}$ in (C.87). We will find such contributions many times in the rest of the computations. Such terms drop out of the final result in the present computation because of the anti-symmetrization in i and j. However, as we will see, in certain other computations they will survive.

We will now consider the following term:

$$
{}_\chi\left\langle \hat{T}^{ZZ}_{(i)(j)}\right\rangle_\psi = {}_\chi\left\langle \left[\hat{T}^{Z^LZ^L}_{(i)(j)} + \hat{T}^{Z^RZ^R}_{(i)(j)} + (\hat{T}^{Z^LZ^R}_{(i)(j)} - i \leftrightarrow j)\right]\right\rangle_\psi . \tag{C.89}
$$

Using (C.82) and (C.83) one may write:

$$
\begin{aligned}
{}_\chi\left\langle \hat{Z}^L_{(i)}\hat{Z}^L_{(j)}\right\rangle_\psi &= \alpha'^2 \int \mathrm{d}w\ \nabla_k\chi^* a^{k+i}\nabla_l\left(a^{l+j}\psi\right), \\
&= -\alpha'^2 \int \mathrm{d}w\ \left(\nabla_k\nabla_l\chi^* a^{k+i}a^{l+j}\psi + \nabla_k\chi^*\nabla_l a^{k+i}a^{l+j}\psi\right) .
\end{aligned} \tag{C.90}
$$

Anti-symmetrizing the above expression and using (2.11) and the shift property one finally finds,

$$
{}_\chi\left\langle \hat{T}^{Z^LZ^L}_{(i)(j)}\right\rangle_\psi = -(i-j)\alpha'\ {}_\chi\left\langle \hat{Z}^L_{(i+j)}\right\rangle_\psi . \tag{C.91}
$$

Similarly,

$$
{}_\chi\left\langle \hat{T}^{Z^RZ^R}_{(i)(j)}\right\rangle_\psi = -(i-j)\alpha'\ {}_\chi\left\langle \hat{Z}^R_{(i+j)}\right\rangle_\psi . \tag{C.92}
$$

To compute the last term in (C.89) we first derive:

$$\chi \left\langle \hat{T}^{Z^L Z^R}_{(i)(j)} \right\rangle_\psi = -\alpha'^2 \int \mathrm{d}w \ \left[\chi^* \left(\nabla_l a^{k+i} \nabla_k a^{l+j} + a^{k+i} r_{kl} a^{l+j}\right) \psi \right. \\ \left. + \nabla_k \chi^* a^{l+j} \nabla_l a^{k+i} \psi + \chi^* \nabla_k a^{l+j} a^{k+i} \nabla_l \psi \right] . \tag{C.93}$$

Terms in the first line drop off when we anti-symmetrize between i and j. The final result can be written in the following form:

$$\chi \left\langle \hat{T}^{Z^L Z^R}_{(i)(j)} \right\rangle_\psi - i \leftrightarrow j = -(i-j)\alpha' \ \chi \left\langle \hat{Z}_{(i+j)} \right\rangle_\psi \\ -\alpha'^2 \left({}_\chi \langle a^{k+i}(\hat{x}) r_{kl}(\hat{x}) a^{l+j}(\hat{x}) \rangle_\psi - i \leftrightarrow j \right), \\ = -(i-j)\alpha' \ \chi \left\langle \hat{Z}_{(i+j)} \right\rangle_\psi , \tag{C.94}$$

where again the terms involving the Ricci tensor cancel. Accumulating all the results in (C.91), (C.92) and (C.94) one finally gets:

$$\chi \left\langle \hat{T}^{ZZ}_{(i)(j)} \right\rangle_\psi = -2(i-j)\alpha' \ \chi \left\langle \hat{Z}_{(i+j)} \right\rangle_\psi \\ -\alpha'^2 \left({}_\chi \left\langle a^{k+i}(\hat{x}) r_{kl}(\hat{x}) a^{l+j}(\hat{x}) \right\rangle_\psi - i \leftrightarrow j \right), \\ = -2(i-j)\alpha' \ {}_\chi \langle \hat{Z}_{(i+j)} \rangle_\psi \ . \tag{C.95}$$

Next we proceed to compute the symmetric and anti-symmetric parts of $\hat{T}^{KZ}_{(i)(j)}$. Following similar manipulations as above we first arrive at:

$$\chi \left\langle \hat{T}^{KZ^L}_{(i)(j)} \right\rangle_\psi = i\alpha'^3 \int \mathrm{d}w \ \left[\nabla^{k+i} \chi^* r_{kl} a^{l+j} \psi + \nabla^{l+i} \nabla_k \chi^* \nabla_l a^{k+j} \psi \right. \\ \left. - \nabla_k \chi^* \nabla_l a^{k+j} \nabla^{l+i} \psi \right], \tag{C.96}$$

Then we manipulate the last two terms in a certain way. Evaluating the covariant derivative on a in the second term one gets:

$$\nabla^{l+i} \nabla_k \chi^* \nabla_l a^{k+j} \psi = i \sum_k (k+j) \nabla_k \nabla^{k+i+j} \chi^* \psi + \gamma^k_{ll'} \nabla^{l+i+j} \nabla_k \chi^* a^{l'} \psi, \tag{C.97}$$

where we have written the summation symbol explicitly in the first term because of the k dependent pre-factor. The connection term is manipulated in the following way:

$$\gamma^k_{ll'} \nabla^{l+i+j} \nabla_k \chi^* a^{l'} \psi = \frac{1}{2} \left(\gamma^{k+i+j}_{l'\tilde{k}} g^{\tilde{k}l} + \gamma^{l+i+j}_{l'\tilde{l}} g^{k\tilde{l}} \right) \nabla_k \nabla_l \chi^* a^{l'} \psi, \\ \text{[Using shift property (2.10)]} \\ = -\frac{1}{2} \partial_{l'} g^{kl+i+j} \nabla_k \nabla_l \chi^* a^{l'} \psi, \\ \left[\text{Using: } \nabla_{l'} g^{kl} = \partial_{l'} g^{kl} + \gamma^k_{l'\tilde{k}} g^{\tilde{k}l} + \gamma^l_{l'\tilde{l}} g^{k\tilde{l}} = 0 \ . \right]$$

$$= -\frac{i}{2}\sum_{k,l}(k+l+i+j)g^{k+i+jl}\nabla_k\nabla_l\chi^*\psi,$$

[Using (B.75).]

$$= -i\sum_{k}\left(k+\frac{i+j}{2}\right)\nabla_k\nabla^{k+i+j}\chi^*\psi, \tag{C.98}$$

Substituting the result (C.98) in (C.97) one writes for the second term in (C.96):

$$\int \mathrm{d}w\ \nabla^{l+i}\nabla_k\chi^*\nabla_l a^{k+j}\psi = \frac{i}{2}(j-i)\int \mathrm{d}w\ \nabla^2_{(i+j)}\chi^*\psi. \tag{C.99}$$

Substituting this results into (C.96) and evaluating the covariant derivative on a in the last term of the same one finally gets:

$$\chi\left\langle \hat{T}^{KZ^L}_{(i)(j)}\right\rangle_\psi = i\alpha'^3\int \mathrm{d}w\left[\nabla^{k+i}\chi^* r_{kl}a^{l+j}\psi + i\sum_k\left(k+\frac{3j}{2}-\frac{i}{2}\right)\nabla_k\nabla^{k+i+j}\chi^*\psi \right.$$
$$\left. -a^{l'}\gamma^{k+i+j}_{l'l}\nabla_k\chi^*\nabla^l\psi\right]. \tag{C.100}$$

A similar manipulation gives the following result:

$$\chi\left\langle \hat{T}^{KZ^R}_{(i)(j)}\right\rangle_\psi = i\alpha'^3\int \mathrm{d}w\left[\chi^* a^{k+i}r_{kl}\nabla^{l+j}\psi + i\sum_k\left(k+\frac{3j}{2}-\frac{i}{2}\right)\nabla_k\nabla^{k+i+j}\chi^*\psi \right.$$
$$\left. -a^{l'}\gamma^{k+i+j}_{l'l}\nabla_k\psi\nabla^l\chi^*\right]. \tag{C.101}$$

Combining the results (C.100) and (C.101) we get,

$$\chi\left\langle \hat{T}^{KZ}_{(i)(j)}\right\rangle_\psi = i\alpha'^3\int \mathrm{d}w\ \left[\nabla^{k+i}\chi^* r_{kl}a^{l+j}\psi + \chi^* a^{k+i}r_{kl}\nabla^{l+j}\psi \right.$$
$$+ i\sum_k(2k+3j-i)\nabla_k\nabla^{k+i+j}\chi^*\psi$$
$$\left. - a^{l'}\gamma^{k+i+j}_{l'l}\left(\nabla_k\chi^*\nabla^l\psi + \nabla_k\psi\nabla^l\chi^*\right)\right]. \tag{C.102}$$

We further manipulate the last term in the following way:

$$a^{l'}\gamma^{k+i+j}_{l'l}\left(\nabla_k\chi^*\nabla^l\psi + \nabla_k\psi\nabla^l\chi^*\right) = a^{l'}\left(\gamma^{k+i+j}_{l'\tilde{k}}g^{\tilde{k}l} + \gamma^{l+i+j}_{l'\tilde{l}}g^{\tilde{l}k}\right)\nabla_k\chi^*\nabla_l\psi,$$
$$= -a^{l'}\partial_{l'}g^{k+i+jl}\nabla_k\chi^*\nabla_l\psi,$$
$$= -i\sum_k(2k+i+j)\nabla_k\chi^*\nabla^{k+i+j}\psi\ . \tag{C.103}$$

Substituting this result into (C.102) one finally obtains:

$$ {}_\chi\left\langle \hat{T}^{KZ}_{(i)(j)} \right\rangle_\psi = -2\alpha'(i-j)_\chi \left\langle \hat{K}_{(i+j)} \right\rangle_\psi \\ +\alpha'^2{}_\chi \left\langle \hat{\pi}^{\star k+i} r_{kl}(\hat{x}) a^{l+j}(\hat{x}) - a^{k+i}(\hat{x}) r_{kl}(\hat{x}) \hat{\pi}^{l+j} \right\rangle_\psi . \quad \text{(C.104)} $$

Notice that the first term is anti-symmetric in i and j, whereas the rest is symmetric because of the shift property. Therefore,

$$ {}_\chi\left\langle \hat{T}^{KZ}_{(i)(j)} \right\rangle_\psi - i \leftrightarrow j = -4\alpha'(i-j)_\chi \left\langle \hat{K}_{(i+j)} \right\rangle_\psi , \quad \text{(C.105)} $$

$$ {}_\chi\left\langle \hat{T}^{KZ}_{(i)(j)} \right\rangle_\psi + i \leftrightarrow j = 2\alpha'^2{}_\chi \left\langle \hat{\pi}^{\star k+i} r_{kl}(\hat{x}) a^{l+j}(\hat{x}) - a^{k+i}(\hat{x}) r_{kl}(\hat{x}) \hat{\pi}^{l+j} \right\rangle_\psi . \quad \text{(C.106)} $$

The anti-symmetric part of ${}_\chi\langle \hat{T}^{KV}_{(i)(j)} \rangle_\psi$ is straightforward to compute leading to the following result:

$$ {}_\chi\left\langle \hat{T}^{KV}_{(i)(j)} \right\rangle_\psi - i \leftrightarrow j = -2(i-j)\alpha' \ {}_\chi\langle \hat{Z}_{(i+j)} \rangle_\psi . \quad \text{(C.107)} $$

Finally, we need to compute ${}_\chi\langle \hat{T}^{ZV}_{(i)(j)} \rangle_\psi$. We will establish the following relations:

$$ {}_\chi\left\langle \hat{T}^{Z^L V}_{(i)(j)} \right\rangle_\psi = {}_\chi\left\langle \hat{T}^{Z^R V}_{(i)(j)} \right\rangle_\psi = -(i-j)\alpha' \ {}_\chi\left\langle \hat{V}_{(i+j)} \right\rangle_\psi , \quad \text{(C.108)} $$

such that we have the following results for the desired quantities:

$$ \begin{aligned} {}_\chi\left\langle \hat{T}^{ZV}_{(i)(j)} \right\rangle_\psi - i \leftrightarrow j &= -4(i-j)\alpha' \ {}_\chi\left\langle \hat{V}_{(i+j)} \right\rangle_\psi , \\ {}_\chi\left\langle \hat{T}^{ZV}_{(i)(j)} \right\rangle_\psi + i \leftrightarrow j &= 0. \end{aligned} \quad \text{(C.109)} $$

To establish (C.108) let us consider, for example, ${}_\chi\langle \hat{T}^{Z^L V}_{(i)(j)} \rangle_\psi$. It is straightforward to show:

$$ \begin{aligned} {}_\chi\left\langle \hat{T}^{Z^L V}_{(i)(j)} \right\rangle_\psi &= -2i\alpha' \int \mathrm{d}w \ \chi^* a^{k+i} g_{ll'} \nabla_k a^l a^{l'+j} \psi , \\ &= -2i\alpha' \int \mathrm{d}w \ \left[i \sum_k (k) \chi^* g_{kk'} a^{k+i} a^{k'+j} \psi \right. \\ &\quad \left. + a^{\tilde{k}} \gamma^l_{\tilde{k}k} g_{ll'} a^{k+i} a^{l'+j} \chi^* \psi \right] , \end{aligned} \quad \text{(C.110)} $$

where in the second step we have evaluated the covariant derivative. We manipulate the connection term as follows:

$$
\begin{aligned}
a^{\tilde{k}}\gamma^{l}_{\tilde{k}k}g_{ll'}a^{k+i}a^{l'+j}\chi^*\psi &= \frac{1}{2}a^{\tilde{k}}\left(\gamma^{l}_{\tilde{k}k}g_{ll'}+\gamma^{l}_{\tilde{k}l'}g_{lk}\right)a^{k+i}a^{l'+j}\chi^*\psi,\\
&\quad \left[\text{Using: } a^{k+i}a^{l'+j}=a^{l'+i}a^{k+j}\ .\right]\\
&= \frac{1}{2}a^{\tilde{k}}\partial_{\tilde{k}}g_{kl'}a^{k+i}a^{l'+j}\chi^*\psi,\\
&\quad \left[\text{Using: } \nabla_{\tilde{k}}g_{kl'}=\partial_{\tilde{k}}g_{kl'}-\left(\gamma^{l}_{\tilde{k}k}g_{ll'}+\gamma^{l}_{\tilde{k}l'}g_{lk}\right)=0\ .\right],\\
&= -\frac{i}{2}\sum_{k,k'}(k+k')\chi^*g_{kk'}a^{k+i}a^{k'+j}\psi, \qquad\qquad (\text{C.111})
\end{aligned}
$$

where in the last step we have used (B.75). Using the above result in (C.110) one gets:

$$
\begin{aligned}
{}_\chi\left\langle\hat{T}^{Z^LV}_{(i)(j)}\right\rangle_\psi &= \sum_{k,k'}(k-k')\alpha'\int \mathrm{d}w\ \chi^*g_{kk'}a^{k+i}a^{k'+j}\psi,\\
&= -(i-j)\alpha'\int \mathrm{d}w\ \chi^*g_{k-ik'-j}a^{k}a^{k'}\psi\\
&\quad +\sum_{k,k'}(k-k')\alpha'\int \mathrm{d}w\ \chi^*g_{k-ik'-j}a^{k}a^{k'}\psi, \qquad\qquad (\text{C.112})
\end{aligned}
$$

where in the second step we have shifted the summation variables $k\to k-i$ and $k'\to k'-j$. The second term in the last equation vanishes as the integrand is symmetric under $k\leftrightarrow k'$. The first term gives the desired result. The result for ${}_\chi\langle\hat{T}^{Z^RV}_{(i)(j)}\rangle_\psi$ in (C.108) can also be established using similar arguments.

Substituting the results (C.87, B.67, C.105, C.106, C.107, C.109) in (C.78) one finally establishes the results (4.27).

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Partha Mukhopadhyay
The Institute of Mathematical Sciences
C.I.T. Campus, Taramani
Chennai 600113, India
e-mail: parthamu@imsc.res.in

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