

Wilson Loops in 5d $\mathcal{N} = 1$ SCFTs and AdS/CFT

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Abstract. We consider $\frac{1}{2}$ -BPS circular Wilson loops in a class of 5d superconformal field theories on S^5 . The large N limit of the vacuum expectation values of Wilson loops are computed both by localization in the field theory and by evaluating the fundamental string and D4-brane actions in the dual massive IIA supergravity background. We find agreement in the leading large N limit for a rather general class of representations, including fundamental, anti-symmetric and symmetric representations. For single-node theories the match is straightforward, while for quiver theories, the Wilson loop can be in different representations for each node. We highlight the two special cases when the Wilson loop is in either in all symmetric or all anti-symmetric representations. In the anti-symmetric case, we find that the vacuum expectation value factorizes into distinct contributions from each quiver node. In the dual supergravity description, this corresponds to probe D4-branes wrapping internal S^3 cycles. The story is more complicated in the symmetric case and the vacuum expectation value does not exhibit factorization.

1. Introduction

Wilson loops are important gauge-invariant observables in gauge theories, and provide valuable dynamical information of the system. Since the pioneering works of [1, 2], they have been studied extensively in the context of the AdS/CFT correspondence.

In this paper, we consider Wilson loops in a class of 5d $\mathcal{N} = 1$ superconformal field theories (SCFTs) and their holographic duals. There are very few quantitative statements on such Wilson loops in the literature (see however [3] where the holography of non-BPS Wilson loops in 5d maximally supersymmetric SYM are considered). Part of the reasons for this is that 5d gauge theories are non-renormalizable. There is a danger that infinitely many irrelevant operators could potentially contribute near the strongly coupled UV fixed point, hence invalidating the computation from the effective Lagrangian. The

goal of this paper is to overcome this difficulty, by first computing the vacuum expectation value of the Wilson loops in the effective theory at strong coupling using localization techniques and then comparing to the dual supergravity description, which provides a definition of the strongly coupled UV fixed point.

We consider a class of 5d $\mathcal{N} = 1$ SCFTs discovered in [4] (see also [5, 6]) and generalized recently to quiver theories in [7]. The original SCFT arises as the UV fixed point of a 5d $\mathcal{N} = 1$ supersymmetric gauge theory with a Yang–Mills vector multiplet for a $USp(2N)$ gauge group, $N_f < 8$ hypermultiplets transforming in the fundamental representation and one hypermultiplet transforming in the antisymmetric representation. The more general SCFTs arise as UV fixed points of quiver-type gauge theories labeled with an integer n . The gauge group is $G = USp(2N) \times SU(2N)^p$ for $n = 2p + 1$ and $G = USp(2N) \times SU(2N)^{p-1} \times USp(2N)$ or $SU(2N)^p$ for $n = 2p$. The matter content is given by a bifundamental hypermultiplet in each pair of adjacent gauge groups, one antisymmetric hypermultiplet in each external $SU(2N)$ (gauge factor at the beginning or the end of the line quiver picture) and N_f^a fundamental hypermultiplets in the a th gauge group factor, with $N_f = \sum_a N_f^a < 8$.

These theories are dual to warped $AdS_6 \times S^4/\mathbb{Z}_n$ compactifications in massive type IIA supergravity [7, 8] (for massive IIA supergravity see [9] and [10] for recent T-dual type IIB backgrounds), and are engineered from type I' string theory on $\mathbb{R}^{4,1} \times \mathbb{C}^2/\mathbb{Z}_n \times \mathbb{R}$ with N D4-branes, N_f D8-branes and one $O8^-$ -plane. These 5d $\mathcal{N} = 1$ theories are specified by the choice of N, N_f ¹ and n .² The existence of the fixed point requires $N_f < 8$ [6]; this is a necessary condition for the inverse square effective gauge coupling constant to stay positive everywhere on the Coulomb branch of the moduli space. In this case, the moduli space is smooth and we could take the strong coupling limit where the bare gauge coupling constant goes to infinity. One can then argue, without proof, for the existence of the UV fixed point at the intersection (origin) of the Coulomb and Higgs branches.

We consider these 5d SCFTs on the Euclidean S^5 .³ We compute the vacuum expectation value (VEV) of the $\frac{1}{2}$ -BPS circular Wilson line operator, placed on the great circle of S^5 :

$$\langle W_R \rangle = \frac{1}{\dim R} \left\langle \text{Tr}_R \mathcal{P} \exp \int (iA_\mu \dot{x}^\mu + \sigma |\dot{x}|) \right\rangle, \quad (1.1)$$

¹ The dual geometry proposed in [7] and the matrix model computations of [11] depend only on the total $N_f = \sum_a N_f^a$, however, our computations will show that the Wilson loops in quiver theories do depend on the individual $N_f^{(a)}$. This subtlety is not reflected in the gravity description in the literature. We will comment more about this later in the text.

² When n is even, there is an extra twofold choice, corresponding to a compactification with or without vector structure. However, as we will see the two choices give identical Wilson loop VEVs in the leading large N limit.

³ More details for 5d $\mathcal{N} \geq 1$ theories on S^5 , including explicit Lagrangians, can be found in [12–15].

where A_μ is the gauge field, σ is the scalar in the 5d vector multiplet, \mathcal{P} a path-ordered product, and R a representation of the gauge group.

We compute the VEV of this Wilson loop in a general representation represented by a Young diagram, both in gauge theory and gravity in the large N limit.⁴ The expression is simpler when we consider

- (1) fundamental representation
- (2) k th antisymmetric representation A_k , i.e., the anti-symmetric part of the k th tensor product of the fundamental representation
- (3) k th symmetric representation S_k , i.e., the symmetric part of the k th tensor product of the fundamental representation

of the gauge groups.

For the case $n = 1$, we find a complete agreement in the leading large N limit, with the VEVs given as:

$$\langle W_{\text{fund}} \rangle \sim \exp \left[6\pi \sqrt{\frac{N}{2(8 - N_f)}} \right], \tag{1.2}$$

$$\langle W_{A_k} \rangle = \langle W_{A_{2N-k}} \rangle \sim \exp \left[4\pi \sqrt{\frac{N^3}{2(8 - N_f)}} \left(1 - \left| 1 - \frac{k}{N} \right|^{3/2} \right) \right], \tag{1.3}$$

$$\langle W_{S_k} \rangle \sim \exp \left[9\pi \sqrt{\frac{N^3}{2(8 - N_f)}} \left(\left(1 + \frac{4k}{9N} \right)^{3/2} - 1 \right) \right]. \tag{1.4}$$

As expected, we find that for anti-symmetric representations k is bounded, while k can take arbitrary values for symmetric representations. Additionally, A_k is a reducible representation and in the leading large N limit, where k scales with N , only the largest irreducible representation gives the leading expression. In the case k is held fixed in the large N limit, all of the expressions reduce to a product of fundamental Wilson loops.

We also discuss more general representations. A representation of $USp(2N)$ is specified by a Young diagram with at most N rows. When we have a Wilson line in the representation specified by a partition (k_1, \dots, k_m) , with m held fixed in the large N limit, we find

$$\langle W_{(k_1, \dots, k_m)} \rangle \sim \exp \left[9\pi \sqrt{\frac{N^3}{2(8 - N_f)}} \sum_{i=1}^m \left(\left(1 + \frac{4k_i}{9N} \right)^{3/2} - 1 \right) \right]. \tag{1.5}$$

Similarly, when we have a Wilson line in the representation specified by a *dual* partition (l_1, \dots, l_m) , again with m held fixed in the large N limit, we find

$$\langle W_{(l_1, \dots, l_m)^T} \rangle \sim \exp \left[4\pi \sqrt{\frac{N^3}{2(8 - N_f)}} \sum_{i=1}^m \left(1 - \left| 1 - \frac{l_i}{N} \right|^{3/2} \right) \right]. \tag{1.6}$$

⁴ For similar computations for $\frac{1}{2}$ -BPS circular Wilson loops in 4d $\mathcal{N} = 4$ theories, see [16–18].

In both cases this leads to factorized expressions in the leading large N limit:

$$\langle W_{(k_1, \dots, k_m)} \rangle = \prod_i \langle W_{S_{k_i}} \rangle, \quad \langle W_{(l_1, \dots, l_m)^T} \rangle = \prod_i \langle W_{A_{l_i}} \rangle. \quad (1.7)$$

Note that when the k_i or l_i are taken to be finite in the large N limit, both expressions reduce to a product of fundamental representations, $\langle W_{\text{fund}} \rangle$, at leading order. In particular, this is consistent with self-dual partitions. The cases where m also scales with N require the back-reaction of the D4-branes to be taken into account, along the lines of [19, 20].

For $n > 1$, the expressions generalize as follows. When the Wilson loop is in the fundamental representation of a single node we have

$$\langle W_{\text{fund}} \rangle \sim \exp \left[6\pi \sqrt{\frac{nN}{2(8 - N_f)}} \right]. \quad (1.8)$$

The answer does not depend on the choice of the gauge group (node of the quiver) the Wilson loops is turned on. For arbitrary configurations of anti-symmetric representations, we find that the result factorizes into contributions from each node in the quiver

$$\langle W_{A_{k_1, A_{k_2}, \dots, A_{k_q}}} \rangle = \exp \left[4\pi \sqrt{\frac{nN}{2(8 - N_f)}} N \sum_{a=1}^q \left(1 - \left| 1 - \frac{k_a}{N} \right|^{3/2} \right) \right], \quad (1.9)$$

where the Wilson loop is in the k_a th anti-symmetric representation for the a th gauge group factor and q is the total number of gauge group factors (equal to $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ depending on the case). In contrast, for arbitrary configurations of symmetric representations, we find that the result does not factorize. We consider the special case that the flavors are distributed uniformly among the gauge groups and the Wilson loop is in the k_i th symmetric representation for each gauge group satisfying the constraint that

$$\frac{k_a}{N} + \frac{9}{4}c_a \quad \text{are independent of } a,$$

where c_a is defined to be 1 (or 2) when the a th gauge group is $USp(2N)$ ⁵ (or $SU(2N)$). We then find

$$\langle W_{S_{k_1, \dots, S_{k_q}}} \rangle = \exp \left[\frac{9\pi}{\sqrt{2(8 - N_f)}} n^{\frac{3}{2}} N^{\frac{3}{2}} \left[\left(1 + \frac{4k_{\text{tot}}}{9nN} \right)^{\frac{3}{2}} - 1 \right] \right], \quad (1.10)$$

where we have introduced $k_{\text{tot}} = \sum_{i=1}^q k_i$. The qualitative difference between symmetric and anti-symmetric representations arises in the matrix model from the fact that anti-symmetric representations do not deform the background eigenvalue distribution, while the symmetric representations do. In the symmetric case, this creates interactions among the eigenvalues and the problem becomes much more involved, except in the case where all of the parameters are distributed symmetrically.

⁵ By $USp(2N)$ we mean the compact real form of $Sp(2N)$.

On the gravity side, the representations mentioned above, respectively, correspond to

- (1) fundamental strings
- (2) D4-branes with k units of electric flux, wrapping AdS_2 and an internal S^3
- (3) D4-branes with k units of electric flux, wrapping AdS_2 and the space-time S^3 .

The latter two are the analogues of giant gravitons and dual giant gravitons. Wilson loops in more general representations correspond to multiple such D4-branes. For the case $n > 1$, the internal space has p independent 3-cycles along with their Hodge dual 3-cycles. The general anti-symmetric representations labeled by $A_{k_1}, A_{k_2}, \dots, A_{k_q}$ correspond to q D4-branes, where the a th D4-brane has k_a units of electric flux and wraps the a th 3-cycle. In the symmetric case, we expect fractional D4-branes, i.e., D6-branes wrapping space-time S^3 cycles and internal blown-up 2-cycles. The gravity description of these branes are subtle since these cycles are of vanishing size, and possibly requires one to take into account discrete holonomies of the B-field on these cycles. When there are symmetries among the different eigenvalues, as discussed above, then we have a simpler picture, where there is a single D4-brane wrapping the space-time S^3 cycle, with k_{tot} units of electric flux. This explains the formula (1.10).

The paper is organized as follows. In Sect. 2, we discuss the derivation of the CFT results. Section 3 contains the dual supergravity description. We conclude with comments and open problems in Sect. 4. We also include appendices on technical material.

2. Gauge Theory Computations

Let us first discuss the supersymmetry preserved by the Wilson loops defined in (1.1). In the conventions of [13], the SUSY variation, which is used for localization, is given by $\delta A_\mu = i\epsilon^{IJ}\xi_I\Gamma_\mu\lambda_J$, $\delta\sigma = -\epsilon^{IJ}\xi_I\lambda_J$, where $I, J = 1, 2$ are $SU(2)$ R-symmetry indices and ξ_I, λ_I are $SU(2)$ Majorana spinors. The SUSY variation of (1.1) vanishes if

$$\epsilon^{IJ}\xi_I(\Gamma_m e_\mu{}^m \dot{x}^\mu + |\dot{x}|) = 0. \quad (2.1)$$

When the Wilson loop wraps a great circle in S^5 , (2.1) is a projector equation on ξ_I and projects out half of the supersymmetries, with 8 supersymmetries remaining.⁶

In addition to the fermionic supersymmetries, the Wilson loop also preserves the $SU(2)_R \simeq Sp(2)_R$ R-symmetry and breaks the space-time symmetry to $SO(1, 2) \times SO(4)$, where the $SO(1, 2)$ is the conformal group associated

⁶ To see this explicitly, we write the metric on the S^5 as $ds_{S^5}^2 = \left[\sum_{j=1}^4 \left(\prod_{k=1}^{j-1} \sin^2 \beta_k \right) d\beta_j^2 \right] + \left(\prod_{k=1}^4 \sin^2 \beta_k \right) d\phi^2$ defining $t \equiv \prod_{i=1}^4 \sin \beta_i^0$ and taking the loop to be a great circle parametrized by ϕ with $\beta_i^0 = \text{constant}$ ($i = 1, \dots, 4$), we have $|\dot{x}| = t$ and (2.1) reduces to $\epsilon^{IJ}\xi_I t (\Gamma_5 \pm 1) = 0$. We have used the frame $e_i = \prod_{j < i} (\sin \beta_j) d\beta_i$ ($i = 1, \dots, 5$) with $\beta_5 \equiv \phi$, with the other components vanishing.

with translations in ϕ and the $SO(4)$ is the remaining unbroken rotation group which leaves invariant the point where the Wilson loop resides in the transverse space. These symmetries fit nicely into the supergroup $D(2, 1; 2) \times SU(2)$ which has exactly 8 supersymmetries and is a subgroup of $F(4)$ (see Table 2.7 in [21]). The specific real forms we are interested in are $F(4; 2)$ for Minkowski signature and $F(4; 3)$ for Euclidean signature with subgroup $D(2, 1; 2; 1) \times SU(2)$ for both cases.⁷ Additionally, we note that in the non-orbifold case, the Wilson loops also preserves an extra $SU(2)_M$ symmetry, under which the anti-symmetric hypermultiplet transforms as a doublet. Thus the full symmetry preserved by the half-BPS Wilson loops we consider in this paper is

$$\begin{aligned}
 &D(2, 1; 2) \times SU(2) \times SU(2)_M \\
 &\supset SO(1, 2) \times SO(4)_{\text{space-time}} \times SU(2)_M \times SU(2)_R.
 \end{aligned}
 \tag{2.2}$$

The orbifold action will break the $SU(2)_M$ symmetry, however, the Wilson line will remain neutral under this broken symmetry.⁸

Let us now move on to the S^5 partition function. The perturbative partition function Z_{S^5} of 5d $\mathcal{N} = 1$ Yang–Mills theory coupled to matter hypermultiplets on the 5-sphere S^5 with radius r has been computed in [14, 15] (see also [12, 13] for earlier works), building on the localization techniques developed in [23, 24] for 4d and 3d supersymmetric gauge theories. By perturbative we mean that the computation does not take into account the instanton contribution to Z_{S^5} .⁹ The result is that the partition function Z_{S^5} reduces to an integration over the Cartan subalgebra of the gauge group, divided by the order of the Weyl group $|\mathcal{W}|$:

$$Z_{S^5} = \frac{1}{|\mathcal{W}|} \int_{\text{Cartan}} d\sigma \left(\dots \right).
 \tag{2.3}$$

The integrand (the dots in 2.3) is a product of several contributions. The vector multiplet gives a factor

$$e^{-\frac{4\pi^3 r}{g_{YM}^2} \text{Tr}_F(\sigma^2)} \det_{\text{Adj}} \left(\sinh(\pi\sigma) e^{\frac{1}{2}f(i\sigma)} \right),
 \tag{2.4}$$

a hypermultiplet in a representation R of the gauge group gives a factor

$$\det_R \left(\cosh(\pi\sigma)^{\frac{1}{4}} e^{-\frac{1}{4}f(\frac{1}{2}-i\sigma) - \frac{1}{4}f(\frac{1}{2}+i\sigma)} \right),
 \tag{2.5}$$

a Chern–Simons term with level k contributes a factor

$$e^{\frac{\pi k}{3} \text{Tr}_F(\sigma^3)}.
 \tag{2.6}$$

⁷ There is a discrepancy between Table 3.75 in [21] and [22]. The real forms $F(4; 2)$ and $F(4; 3)$ are listed as having $SL(2, R)$ subgroups in [21] while in [22], they are shown to have $SU(2)$ subgroups.

⁸ It is interesting to ask if we could consider a two-parameter deformation of the Wilson line which preserves the same supersymmetry, but charged under $SU(2)_M$ symmetry and $SO(3) \subset SO(4)_{\text{space-time}}$ symmetry, which are not contained in the $D(2, 1; 2) \subset F(4)$.

⁹ See [15, 25] for the instanton part.

Here, g_{YM} is the gauge coupling and Tr_R and \det_R are the trace and the determinant in the representation R . The indices F and Adj refer to fundamental and adjoint representations, respectively, and the function $f(x)$ is defined by

$$f(x) = \frac{i\pi x^3}{3} + x^2 \log(1 - e^{2i\pi x}) + \frac{ix}{\pi} \text{Li}_2(e^{-2i\pi x}) + \frac{1}{2\pi^2} \text{Li}_3(e^{-2i\pi x}) - \frac{\zeta(3)}{2\pi^2}. \tag{2.7}$$

We can also incorporate a $\frac{1}{2}$ -BPS Wilson loop along the great circle of S^5 , in the representation R of the gauge group. In the localization computation of the partition function [14, 15], the saddle point equations imply $A = 0$ and σ constant,¹⁰ and hence the Wilson loop operator (1.1) reduces to an insertion of the following exponential factor to the integrand of the matrix integral:

$$\text{Tr}_R(e^{2\pi\sigma}). \tag{2.8}$$

The S^5 partition function depends on the value of the gauge coupling constant g_{YM} , which induces a relevant deformation of the UV fixed point. To discuss the UV fixed point, we consider the limit where such a deformation is completely turned off:

$$g_{\text{YM}}^2 \gg r. \tag{2.9}$$

Moreover, for the comparison with gravity we take the large N limit

$$N \gg 1, \tag{2.10}$$

where N is the dimension of the Cartan subalgebra (number of integration variables). In these limits the contributions from instantons and the contribution from the Yang–Mills kinetic term $e^{-\frac{4\pi^3 r}{g_{\text{YM}}^2} \text{Tr}_F(\sigma^2)}$ are subleading [11], and hence will be neglected in the rest of the computations.¹¹

After taking into account these considerations, we simplify the matrix integral as

$$Z_{S^5} = \frac{1}{|\mathcal{W}|} \int_{\text{Cartan}} d\sigma e^{-F(\sigma)}, \tag{2.11}$$

where in the large $|\sigma|$ limit we have

$$F(\sigma) = \text{Tr}_{\text{Adj}} F_V(\sigma) + \sum_j \text{Tr}_{R_j} F_H(\sigma), \tag{2.12}$$

with

$$F_V(\sigma) = \frac{\pi}{6} |\sigma|^3 - \pi |\sigma|, \quad F_H(\sigma) = -\frac{\pi}{6} |\sigma|^3 - \frac{\pi}{8} |\sigma|. \tag{2.13}$$

¹⁰ There are other saddle points with non-trivial profile of gauge fields, however, these correspond to instanton contributions which does not change the leading large N analysis in this paper.

¹¹ Note that we are not taking the 't Hooft limit; there will be no dependence on g_{YM} for the rest of the paper and we concentrate on the N dependence.

2.1. Fundamental Representation

Let us first consider the theory with $n = 1$. This theory is 5d $\mathcal{N} = 1$ $USp(2N)$ gauge theory with, in addition to the vector multiplet, N_f hypermultiplets in the fundamental representation and one hypermultiplet in the antisymmetric representation. In this case the matrix integral is over N real parameters σ_j ($j = 1, \dots, N$), parametrizing the Cartan as $\{\sigma_1, \dots, \sigma_N, -\sigma_1, -\sigma_N\}$.

The weights of the fundamental representation of $USp(2N)$ can be taken to be $\pm e_i$, the e_i forming a basis of unit vectors for \mathbb{R}^N , the antisymmetric representation then has weights $e_i \pm e_j$ with $i \neq j$, and the adjoint representation has weights $e_i \pm e_j$ with $i \neq j$ as well as $\pm 2e_i$.

We will evaluate the matrix integral in the saddle point approximation, where, as we will justify later, the saddle point value of the eigenvalues is of order $\mathcal{O}(N^{1/2})$. This means that in our large N approximation we can take the large $|\sigma_j|$ limit inside the function $F(\sigma)$, and we have

$$F(\sigma) = \sum_{i \neq j} \left(F_V(\sigma_i - \sigma_j) + F_V(\sigma_i + \sigma_j) + F_H(\sigma_i - \sigma_j) + F_H(\sigma_i + \sigma_j) \right) + \sum_j \left(F_V(2\sigma_j) + F_V(-2\sigma_j) + N_f F_H(\sigma_j) + N_f F_H(-\sigma_j) \right). \quad (2.14)$$

The Weyl group of $USp(2N)$ is given by $\mathcal{W} = \mathfrak{S}_N \ltimes \mathbb{Z}_2^N$, and hence $|\mathcal{W}| = N!2^N$.

The large N limit of the free energy $F_{S^5} = -\log |Z_{S^5}|$ in the saddle point approximation of matrix models is given in [11]:

$$F_{S^5} = -\frac{9\sqrt{2}\pi}{5\sqrt{8-N_f}} N^{5/2} + \mathcal{O}(N^{5/2}). \quad (2.15)$$

We will comment on the holographic computation of this formula in Sect. 3.4.

Here we study the fundamental $\frac{1}{2}$ -BPS Wilson loop W_{fund} , whose VEV in the limit $\frac{r}{g_{YM}^2} \ll 1$ is given by

$$\begin{aligned} \langle W_{\text{fund}} \rangle &= \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \int d^N \sigma \frac{1}{2N} \left[\sum_{j=1}^N (e^{2\pi\sigma_j} + e^{-2\pi\sigma_j}) \right] e^{-F(\sigma)} \\ &= \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \int d^N \sigma \frac{1}{N} \left[\sum_{j=1}^N e^{2\pi\sigma_j} \right] e^{-F(\sigma)}. \end{aligned} \quad (2.16)$$

We are looking for the saddle point of this integral in the large N limit. We will assume as in [11] that the saddle point is given by $\sigma_j^* = N^\alpha x_j$ with the saddle point variables x_j of order $\mathcal{O}(N^0)$. We also assume that the variables x_j at the saddle point condense into a continuous distribution, $\rho(x)$, which is smooth on an interval of finite length L and zero outside the interval. These assumptions will be justified in the computation that follows.

We can then replace the N variables by a continuous variable x with density $\rho(x)$:

$$\rho(x) = \frac{1}{N} \sum_{j=1}^N \delta(x - x_j), \quad \int dx \rho(x) = 1. \tag{2.17}$$

In this limit the Wilson loop expectation value becomes

$$\begin{aligned} \langle W_{\text{fund}} \rangle &= \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \int D\rho \left[\int dx \rho(x) e^{2\pi N^\alpha x} \right] e^{-F[\rho, \mu]}, \\ F[\rho, \mu] &= -\frac{9\pi}{8} N^{2+\alpha} \int dx dy \rho(x) \rho(y) (|x - y| + |x + y|) \\ &\quad + \frac{\pi(8 - N_f)}{3} N^{1+3\alpha} \int dx \rho(x) |x|^3 + \mu \left(1 - \int dx \rho(x) \right). \end{aligned} \tag{2.18}$$

where we have added a Lagrange multiplier μ to impose the constraint $\int dx \rho(x) = 1$. We have $\int dx \rho(x) e^{2\pi N^\alpha x} \sim \mathcal{O}(e^{2\pi N^\alpha L})$, which in the saddle point approximation is subleading compared with other contributions in $F[\rho, \mu]$. The Wilson loop therefore does not affect the saddle point equations.

The saddle point equation reduces to

$$0 = \frac{\delta F[\rho]}{\delta \rho(x)} = -\frac{9\pi}{4} N^{2+\alpha} \int dy \rho(y) (|x - y| + |x + y|) + \frac{\pi(8 - N_f)}{3} N^{1+3\alpha} |x|^3 - \mu. \tag{2.19}$$

Non-trivial solutions are obtained when the two terms are of the same order, namely when $\alpha = \frac{1}{2}$; only in this case the mutual repulsion among the eigenvalues balances the attraction from the cubic potential, giving continuous eigenvalue distributions as assumed previously. In this case the two first terms in $F[\rho, \mu]$ are both of order $N^{5/2}$, which justifies a posteriori the assumption that the Wilson loop factor does not affect the saddle point equation.

It is easy to realize that $F[\rho]$ only depends on the even part of $\rho (= \frac{1}{2}[\rho(x) + \rho(-x)])$. So the integration over ρ can be reduced to the integration over even ρ up to a factor coming out of the integration measure, which does not affect the Wilson loop computation (because of the normalization of the Wilson loop). Differentiating twice the equation (2.19) with respect to x , and assuming an even distribution ρ , we get

$$\rho(x) = \frac{|x|}{x_0^2} \quad \text{for} \quad -x_0 < x < x_0, \quad x_0^2 := \frac{9}{2(8 - N_f)}, \tag{2.20}$$

which satisfies the normalization condition (2.17). Plugging this back into the expression (2.18) we get at leading order in N :

$$\begin{aligned} \langle W_{\text{fund}} \rangle &= \frac{\exp(2\pi x_0 N^{1/2})}{2\pi x_0 N^{1/2}} \\ &= \frac{\sqrt{2(8 - N_f)}}{6\pi N^{1/2}} \exp\left(\frac{6\pi N^{1/2}}{\sqrt{2(8 - N_f)}}\right), \end{aligned} \tag{2.21}$$

which is the advertised result (1.2). It is rather simple to understand the leading contribution: the Wilson loop contribution $e^{2\pi\sigma}$ is maximized when the eigenvalues σ take the maximal possible value, which is $N^{1/2}x_0$.

2.2. Anti-Symmetric Representations

Let us next consider the k th anti-symmetric representation A_k of the $USp(2N)$ gauge group. The Wilson loop in representation A_k is given in the matrix model by

$$\langle W_{A_k} \rangle = \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \binom{2N}{k}^{-1} \times \int d^N \sigma \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq 2N} e^{2\pi(\sigma'_{j_1} + \sigma'_{j_2} + \dots + \sigma'_{j_k})} e^{-F(\sigma)},$$

where $\sigma'_j = \sigma_j$ and $\sigma'_{N+j} = -\sigma_j$ for $j = 1, \dots, N$.

In the sum $\sigma'_{j_1} + \sigma'_{j_2} + \dots + \sigma'_{j_k}$ it is possible that some terms cancel each other. In particular, this expression is invariant under exchanging k with $2N - k$, so that $\langle W_{A_k} \rangle = \langle W_{A_{2N-k}} \rangle$. Hence, we need only consider $1 \leq k \leq N$.

Let us first consider the k -plets (j_1, \dots, j_k) (with $k \leq N$) such that there is no cancellation in $\sigma'_{j_1} + \sigma'_{j_2} + \dots + \sigma'_{j_k}$ (which means all σ_j are different). These terms contribute a factor $\binom{2N}{k}^{-1} I_k$ with

$$I_k = \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \times \int d^N \sigma \sum_{\epsilon_1, \dots, \epsilon_k = \pm 1} \sum_{1 \leq j_1 < \dots < j_k \leq N} e^{2\pi(\epsilon_1 \sigma_{j_1} + \epsilon_2 \sigma_{j_2} + \dots + \epsilon_k \sigma_{j_k})} e^{-F(\sigma)}.$$

The symmetry of F implies that all terms in the sum over $\epsilon_1, \dots, \epsilon_k$ produce the same contribution, so that

$$I_k = \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} 2^k \binom{N}{k} \int d^N \sigma e^{2\pi(\sigma_1 + \sigma_2 + \dots + \sigma_k)} e^{-F(\sigma)} \tag{2.22}$$

Again we assume that the saddle point eigenvalues are distributed along an interval of length of order N^α with $\sigma_j^* = N^\alpha x_j$.

We can again argue that the Wilson loop does not modify the saddle point (2.20) in the large N limit; the Wilson loop operator contributes at most a term of order $N^{1+\alpha}$ to the saddle point equation and this is again subleading compared to the term coming from $F(N^\alpha x_j)$ with $\alpha = \frac{1}{2}$. However, this does not mean that the answer is k times the fundamental representation. This is because we need to choose k distinct eigenvalues $\sigma_1, \dots, \sigma_k$ from the eigenvalue distribution, and therefore we cannot always take the maximal value $\sigma_j = N^{1/2}x_0$ when k is large.

The dominant contribution to the integral (2.22) comes from configurations when the first k eigenvalues cover an interval $[x_0 \cos \theta_k, x_0]$ at the right end of the saddle point distribution ρ so that the factor $e^{2\pi(\sigma_1 + \sigma_2 + \dots + \sigma_k)}$ attains maximum. Here, the angle $\theta_k \in [0, \pi/2]$ is chosen such that we indeed have k eigenvalues in the interval:

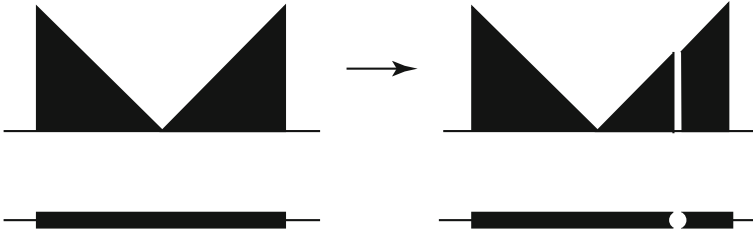


FIGURE 1. Insertion of a Wilson line in anti-symmetric representation shifts part of the eigenvalues by a constant, or equivalently an excitation of a “hole” in the eigenvalues

$$k = 2N \int_{x_0 \cos \theta_k}^{x_0} \rho(x) dx = N \sin^2 \theta_k. \tag{2.23}$$

Intuitively, the Wilson line operator corresponds to a constant electric flux for k of the eigenvalues, hence shifting the k eigenvalues and creating a “hole” in the eigenvalue distribution (cf. [16]), see Fig. 1.

The maximal value for the i th eigenvalue σ_i is $N^{1/2}x_0 \cos \theta_i$, and hence contributes $e^{2\pi N^{1/2}x_0 \cos \theta_i}$ to the integral. We then evaluate (2.22) by summing over these contributions:

$$\begin{aligned} I_k &= 2^k \binom{N}{k} \exp \left(\sum_{i=1}^k 2\pi x_0 \cos \theta_i N^{1/2} \right) \\ &\simeq 2^k \binom{N}{k} \exp \left(\int_0^{\theta_k} d\theta_k \frac{\partial k}{\partial \theta_k} 2\pi x_0 \cos \theta_i N^{1/2} \right). \end{aligned}$$

This gives

$$\begin{aligned} I_k &= 2^k \binom{N}{k} \exp \left[\frac{4\pi}{3} x_0 N^{3/2} (1 - \cos^3 \theta_k) \right] \\ &= 2^k \binom{N}{k} \exp \left[\frac{4\pi}{3} x_0 N^{3/2} \left(1 - \left(1 - \frac{k}{N} \right)^{3/2} \right) \right], \end{aligned} \tag{2.24}$$

where the prefactor $2^k \binom{N}{k}$ gives only a subleading correction of order N to the exponent.

Now we consider the terms in the sum over j_1, \dots, j_k such that two σ'_j cancel. These terms will contribute a factor $\binom{2N}{k}^{-1} (N - k + 2) I_{k-2}$. From the previous explanation it follows that this contribution is suppressed, as compared to the contribution $\binom{2N}{k}^{-1} I_k$, by a factor of order $e^{-2(2\pi N^{1/2}x_0 \cos \theta_k)}$.¹² Similarly, all the other terms left in the sum over j_1, \dots, j_k are also subleading.

¹² When $k \rightarrow N$ we have $\cos \theta_k \rightarrow 0$ and the contribution is not suppressed, however, it leads to the same contribution as I_k and the sum over all the contributions reduce to the same leading term in the exponent.

Thus the leading contribution in the large N limit is $\binom{2N}{k}^{-1} I_k$. Explicitly we have

$$\log \langle W_{A_k} \rangle = \frac{4\pi}{3} x_0 N^{3/2} \left[1 - \left| 1 - \frac{k}{N} \right|^{3/2} \right], \quad (2.25)$$

which coincides with (1.3).¹³ This expression is valid for $1 \leq k \leq 2N$, ensuring $\langle W_{A_k} \rangle = \langle W_{A_{2N-k}} \rangle$. As a consistency check, when $k \ll N$ we have

$$\log \langle W_{A_k} \rangle \sim k \left(2\pi N^{1/2} x_0 \right), \quad (2.26)$$

which could be interpreted as the k times the fundamental string contribution (2.21). Of course, this follows directly from the derivation presented above.

As explained in Appendix A, the anti-symmetric representation A_k defined in introduction is a reducible representation, and in particular (when $1 \leq k \leq N$) contains the irreducible representation defined by the Young diagram with a single column with k boxes. The arguments similar to those already explained in this subsection shows that the contributions from other irreducible representations are exponentially suppressed, and the leading contribution comes from this irreducible representation.

2.3. Symmetric Representations

Let us move onto the case of k th symmetric representation S_k of $USp(2N)$. We have

$$\begin{aligned} \langle W_{S_k} \rangle &= \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \binom{2N+k-1}{k}^{-1} \\ &\times \int d^N \sigma \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq 2N} e^{2\pi(\sigma'_{j_1} + \sigma'_{j_2} + \dots + \sigma'_{j_k})} e^{-F(\sigma)}. \end{aligned} \quad (2.27)$$

In the sum $\sigma'_{j_1} + \sigma'_{j_2} + \dots + \sigma'_{j_k}$ some of the terms could cancel out from the expression; however, these give only exponentially suppressed contributions, by the reason already explained in the case of anti-symmetric representations. If we neglect these contributions we have

$$\begin{aligned} &\frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \binom{2N+k-1}{k}^{-1} 2^k \\ &\times \int d^N \sigma \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq N} e^{2\pi(\sigma_{j_1} + \sigma_{j_2} + \dots + \sigma_{j_k})} e^{-F(\sigma)}. \end{aligned}$$

The summation here still contains several different terms. If we denote the partition of k by $\mu = (\mu_1, \dots, \mu_l)$, $\sum \mu_i = k$,¹⁴ then

$$\sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq 2N} e^{2\pi(\sigma_{j_1} + \sigma_{j_2} + \dots + \sigma_{j_k})} = \sum_{\mu} \left(e^{2\pi \sum_i \mu_i \sigma_i} + \text{cyclic} \right). \quad (2.28)$$

¹³ The subleading correction is of order N when k and N are of the same order.

¹⁴ Readers should not confuse this partition with a partition specifying a representation of $USp(2N)$. Rather, it actually corresponds to a symplectic semi-standard Young tableaux in the language of Appendix A. Here, we have avoided use of such terminologies for the minimality of the explanation.

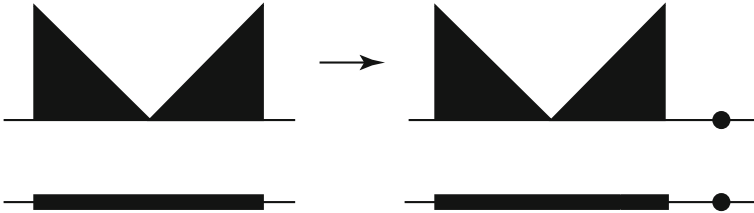


FIGURE 2. Insertion of a Wilson line in symmetric representation corresponds to exciting one of the eigenvalues to a large value, or equivalently an excitation of a “particle” in the Fermi sea

This contains many different contributions. For example, the contribution from the partition $\mu = (1, \dots, 1)$ is the same as that from the anti-symmetric representation (2.26), except for the overall factor of $\binom{2N+k-1}{k}^{-1}$:

$$\langle W_{S_k} \rangle \Big|_{\mu=(1, \dots, 1)} = \binom{2N+k-1}{k}^{-1} \binom{2N}{k} \langle W_{A_k} \rangle. \tag{2.29}$$

This is not the only contribution, however. On the other extreme, there is a contribution from $\mu = (k)$, i.e.,

$$\langle W_{S_k} \rangle \Big|_{\mu=(k)} = \frac{1}{Z_{S^5}} \frac{1}{|\mathcal{W}|} \binom{2N+k-1}{k}^{-1} 2^k N \int d^N \sigma e^{2\pi k \sigma_1} e^{-F(\sigma)}. \tag{2.30}$$

This is the contribution from the “large winding Wilson loop”.

Let us evaluate this contribution. We can replace all the σ_i by the continuum distribution determined by (2.20), except for σ_1 . Since there is a the factor of k multiplying σ_1 and since k can be large, the Wilson line does affect the saddle point for σ_1 . This happens when k is of order N or larger; the leading contribution of free energy is of order $N^{5/2}$, however they cancel out when we compute the Wilson loops (due to the normalization factor Z_{S^5}), and the subleading contribution of order $N^{3/2}$ becomes comparable with the Wilson loop contribution of order $kN^{1/2}$, when k is of order N . Intuitively, the eigenvalue σ_1 moves inside the effective potential created by the other background eigenvalues, and can be regarded as a “particle” in the eigenvalue distribution (Fig. 2).

We now have

$$\binom{2N+k-1}{k}^{-1} 2^k N \frac{\int d\sigma_1 \exp[-F_{\text{eff}}(\sigma_1; k)]}{\int d\sigma_1 \exp[-F_{\text{eff}}(\sigma_1; k=0)]}, \tag{2.31}$$

with

$$\begin{aligned}
 F_{\text{eff}}(\sigma_1; k) &= -2\pi k\sigma_1 + 2 [F_V(2\sigma_1) + N_f F_H(\sigma_1)] \\
 &\quad + 2N \int dy \rho(y) \left[F_V\left(\sigma_1 - N^{\frac{1}{2}}y\right) + F_V\left(\sigma_1 + N^{\frac{1}{2}}y\right) \right. \\
 &\quad \left. + F_H\left(\sigma_1 - N^{\frac{1}{2}}y\right) + F_H\left(\sigma_1 + N^{\frac{1}{2}}y\right) \right], \tag{2.32}
 \end{aligned}$$

where the factors 2 comes is due to the property $F_{V,H}(-\sigma) = F_{V,H}(\sigma)$. We will evaluate the integral (2.31) by the saddle point approximation with respect to $x_1 := x_0^{-1}N^{-1/2}\sigma_1$. We assume that the saddle point is given by $x_{1,*} > 1$. This will be justified a posteriori by the result of our computation. Under this assumption we have

$$F_{\text{eff}}(x_1; k) \simeq \pi x_0 N^{3/2} \left[-x_1 \left(2\frac{k}{N} + \frac{9}{2} \right) + \frac{3}{2}x_1^3 \right], \tag{2.33}$$

where we used

$$\begin{aligned}
 &\int dy \rho(y) \left[F_V\left(\sigma_1 - N^{\frac{1}{2}}y\right) + F_V\left(\sigma_1 + N^{\frac{1}{2}}y\right) \right] \\
 &= 2 \left[\frac{\pi}{6} \left(N^{\frac{1}{2}}x_0 \right)^3 \left(x_1^3 + \frac{3}{2}x_1 \right) - \pi \left(N^{\frac{1}{2}}x_0 \right) x_1 \right], \\
 &\int dy \rho(y) \left[F_H\left(\sigma_1 - N^{\frac{1}{2}}y\right) + F_H\left(\sigma_1 + N^{\frac{1}{2}}y\right) \right] \\
 &= 2 \left[-\frac{\pi}{6} \left(N^{\frac{1}{2}}x_0 \right)^3 \left(x_1^3 + \frac{3}{2}x_1 \right) - \frac{\pi}{8} \left(N^{\frac{1}{2}}x_0 \right) x_1 \right]. \tag{2.34}
 \end{aligned}$$

This is extremized by

$$x_{1,*} = \sqrt{\frac{4k}{9N} + 1}, \tag{2.35}$$

which justifies our previous assumption. Note that the eigenvalue $\sigma_{1,*} = N^{1/2}x_0x_{1,*}$ is outside the range occupied by other eigenvalues (Fig. 2).

At the saddle point we have the contribution to the free energy

$$\begin{aligned}
 \log \langle W_{S_k} \rangle \Big|_{\mu=(k)} &= - (F_{\text{eff}}(\sigma_{1,*}; k) - F_{\text{eff}}(\sigma_{1,*}; k = 0)) \\
 &= \frac{9\pi}{\sqrt{2(8 - N_f)}} N^{3/2} \left[\left(1 + \frac{4k}{9N} \right)^{3/2} - 1 \right]. \tag{2.36}
 \end{aligned}$$

In general, there are many contributions from various different choices of μ , and we need to take all of them into account. For example, when k is small all of them has the same leading contribution, with $\mu = (1, \dots, 1)$ having the largest subleading correction due to the largest multiplicity $\binom{N}{k}$. However, when k is large, of order N or larger, we can verify from the expressions above that the contribution from $\mu = (k)$ dominates.

Some readers might worry about contributions from other μ , say $\mu = (k - 1, 1)$. However when k is large this is suppressed by an exponential factor (cf. [26]):¹⁵

$$\begin{aligned} & \exp \left[-\frac{\partial}{\partial k} \frac{9\pi}{\sqrt{2(8 - N_f)}} N^{\frac{3}{2}} \left[\left(1 + \frac{4k}{9N} \right)^{\frac{3}{2}} - 1 \right] \right] \\ &= \exp \left[-\frac{6\pi}{\sqrt{2(8 - N_f)}} N^{\frac{1}{2}} \sqrt{1 + \frac{4k}{9N}} \right]. \end{aligned} \tag{2.37}$$

Similarly, we can check that contributions from other μ are likewise exponentially suppressed.

Therefore, we finally have

$$\log \langle W_{S_k} \rangle = \log \langle W_{S_k} \rangle \Big|_{\mu=(k)} = \frac{9\pi}{\sqrt{2(8 - N_f)}} N^{3/2} \left[\left(1 + \frac{4k}{9N} \right)^{3/2} - 1 \right], \tag{2.38}$$

as claimed in (1.4). Let us note that

$$\frac{\log \langle W_{S_k} \rangle}{\log \langle W_{\text{fund}} \rangle} = \frac{3N}{2} \left[\left(1 + \frac{4k}{9N} \right)^{3/2} - 1 \right]. \tag{2.39}$$

For $k \ll N$, this reduces to

$$\frac{\log \langle W_{S_k} \rangle}{\log \langle W_{\text{fund}} \rangle} = k, \tag{2.40}$$

as expected.

2.4. General Representations

We can consider more general representations (see Appendix A for representation theory of $USp(2N)$). An irreducible representation of $USp(2N)$ can be labeled by a Young diagram with at most N rows. We can label the representation by a partition $k = (k_1, \dots, k_m)$, or its dual (transpose) by $k^T =: l = (l_1, \dots, l_{k_1})$. The main question we want to answer is when does the Wilson line expectation value factorize into a product of symmetric or anti-symmetric contributions?

The Wilson line operator in representation R corresponds to an insertion of

$$sp_{(k_1, \dots, k_m)} \left(e^{2\pi\sigma_1}, \dots, e^{2\pi\sigma_N} \right), \tag{2.41}$$

where sp_k is the symplectic character for representation, introduced in Appendix A.

Let us first consider the representation described by $k = (k_1, \dots, k_m)$. We assume that m is a finite number which stays constant when N grows large. However, there are no restrictions on the size of the k_a 's, and they can grow with some power of N . The symplectic character $sp_\lambda(x)$ is a sum of

¹⁵ The presence of this factor ensures that our Wilson loop is not simply the multiple wound string, but a loop in the symmetric representation.

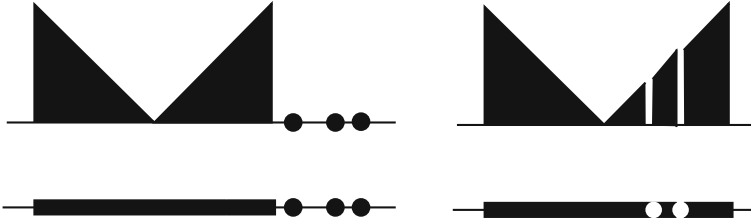


FIGURE 3. Wilson lines in general representations can be described either as the excitation of several non-interacting particles above the Fermi sea (*left*) or the excitation of non-interacting holes (*right*). The two descriptions correspond to a Young diagram and its dual, i.e., taking the transpose could be thought as a Bogoliubov-like transformation

terms labeled by symplectic semistandard Young tableaux (A.3). By arguments similar to the previous subsections, we can argue that the leading contribution comes from the tableaux

1	1	1	1	1	•	•	1
2	2	2	2	•	•	2	
3	3	•	•	3			
•	•						
•	•						
<i>m</i>	<i>m</i>						

This corresponds to an insertion of

$$e^{2\pi \sum_a k_a \sigma_a}$$

into the integrand of the matrix model. Again, there are multiplicities associated to this factor which do not alter the leading contribution and hence will be neglected for the rest of the computation. When all the k_a are large (of order N or larger), this factor excites N eigenvalues $\sigma_1, \dots, \sigma_m$ out of the Fermi sea and we can then write down the effective matrix model for $\sigma_1, \dots, \sigma_m$. Note that interactions among σ_a s are subleading of order $N^{\frac{1}{2}}$, hence at this order the eigenvalues behave independently; the excitations from the Fermi sea behave as non-interacting particles in the leading order (Fig. 3). Within these approximations, the Wilson line evaluates to

$$\log \langle W_{(k_1, \dots, k_m)} \rangle = \frac{9\pi}{\sqrt{2(8 - N_f)}} N^{3/2} \sum_{a=1}^m \left[\left(1 + \frac{4k_a}{9N} \right)^{3/2} - 1 \right]. \quad (2.42)$$

We can also consider a similar situation, where this time the transpose of the Young diagram, $l := k^T$ takes the form $l = (l_1, \dots, l_m)$ with m a finite number. Note that we have $l_a \leq N$, however, l_a can still be of order N . In the symplectic character $sp_\lambda(x)$, the leading contribution in this case comes from the tableaux

1	1	1	•	•	1
2	2	2	•	•	l_m
3	3	•			
4	•	•			
5	•	l_3			
•	l_2				
•					
l_1					

From the a th column, there is a contribution of the form

$$\sum_{j_1 < j_2 < \dots < j_{l_a}} e^{2\pi(\sigma_{j_1} + \sigma_{j_2} + \dots + \sigma_{j_{l_a}})},$$

which contributes a leading contribution which coincides with that from the l_a th anti-symmetric representation A_{l_a} . Summing up these contributions over a , we find

$$\log \langle W_{(l_1, \dots, l_m)^T} \rangle = \frac{4\pi}{2\sqrt{8 - N_f}} N^{3/2} \sum_{a=1}^m \left[\left(1 - \frac{k_a}{N} \right)^{2/3} - 1 \right]. \quad (2.43)$$

This again takes a factorized form, and has an interpretation as excitation of non-interacting holes inside the Fermi sea (Fig. 3). Note that factorization breaks down in the subleading order since the multiplicity of the leading contribution, which contributes to the subleading correction, is affected by the presence of the neighboring columns.

We thus find that for a general representation, the Wilson line expectation value factorizes into either symmetric or anti-symmetric contributions only when either the k_i or m , but not both, scale with some power of N .

In the gravity dual discussed in the next section, the two descriptions, particles or holes, correspond to¹⁶

- (1) multiple D4-branes wrapping AdS_2 and space-time S^3 ,
- (2) multiple D4-branes wrapping AdS_2 and internal S^3 .

Such an identification of a Wilson line with multiple D4-branes naturally matches the factorization found above in gauge theories. However, we should keep in mind that the factorization is far from trivial; the factorization holds only for limited representations and there are subleading corrections which lift the factorization. In the dual gravitational theory, this can be understood as a break down of the probe approximation. Here, the CFT analysis gives a quantitative prediction for when the probe approximation breaks down. Additionally, the factorization does *not* hold for the quiver theories discussed in the next subsection.

¹⁶ The related discussion for circular Wilson loops for 4d $\mathcal{N} = 4$ theory can be found in [18], where combinatorial formulas, Giambelli’s formula and Jacobi–Trudi formula, played crucial roles. Analogous formulas are known for symplectic groups.

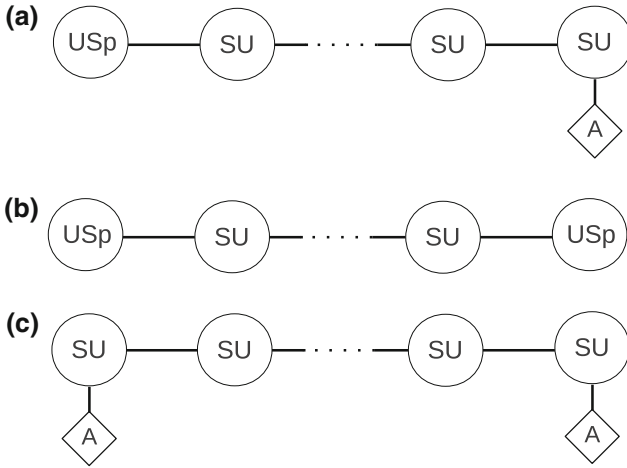


FIGURE 4. Description of the different 5d $\mathcal{N} = 1$ quiver gauge theories, without mention of the fundamental hypermultiplets. The nodes symbolize either $SU(2N)$ or $USp(2N)$ vector multiplets. *Solid lines* between two nodes symbolize bifundamental hypermultiplets and a *box* with an *A* denotes a hypermultiplet transforming in the antisymmetric representation of the node to which it is linked. **a** $n = 2p + 1$ case. There are p $SU(2N)$ nodes. **b** $n = 2p$ with vector structure. There are $p - 1$ $SU(2N)$ nodes in total. **c** $n = 2p$ without vector structure. There are p $SU(2N)$ nodes in total

2.5. Quiver Theories

In [11], more general quiver-type gauge theories were considered depending on an integer n . The gauge group is $G = USp(2N) \times SU(2N)^p$ for $n = 2p + 1$ and $G = USp(2N) \times SU(2N)^{p-1} \times USp(2N)$ or $SU(2N)^p$ for $n = 2p$. The matter content is given by a bifundamental hypermultiplet in each pair of adjacent gauge groups, one antisymmetric hypermultiplet in each external $SU(2N)$ (gauge factor at the beginning or the end of the line quiver picture) and N_f^a fundamental hypermultiplets in the a th gauge group factor, with $N_f = \sum_a N_f^a$. The quiver gauge theories are summarized in Fig. 4.

The saddle point of the corresponding matrix model has been analyzed in [11]. They assume that all the eigenvalues scale as N^α , just as in $n = 1$. The integrand of the matrix model contain terms of order $N^{2+3\alpha}$, which is extremized by the ansatz

$$\begin{aligned} \sigma_i^{(a)} &= \sigma'_i \quad (a = 1, \dots, p), \\ \sigma'_i &= -\sigma'_{N+i} \quad (i = 1, \dots, N, \quad a = 1, \dots, p). \end{aligned} \tag{2.44}$$

However, the extremal value of $N^{2+3\alpha}$ term vanishes under (2.44), and we need to discuss subleading terms, which are given by

$$\begin{aligned}
 F[\rho] = & -\frac{9\pi}{8} n N^{2+\alpha} \int dx dy \rho(x)\rho(y)(|x-y| + |x+y|) \\
 & + \frac{\pi(8 - N_f)}{3} N^{1+3\alpha} \int dx \rho(x)|x|^3,
 \end{aligned}
 \tag{2.45}$$

which is identical to the leading free energy for the $USp(2N)$ theory, up to a factor of n . This means that we again have $\alpha = 1/2$, and that at the saddle point the matrix model is the same as the matrix model of $USp(2N)^n$ gauge group without bifundamentals [11].¹⁷

We can compute the VEV of Wilson loops in these theories. For the a th gauge group (either $USp(2N)$ or $SU(2N)$) we could turn on a Wilson line in representation R_a , and compute its expectation values $\langle W_{R_1, R_2, \dots, R_q} \rangle$. Here, we take R_a to be either a anti-symmetric representation A_{k_a} or a symmetric representation S_{k_a} , and q is the total number of gauge groups, i.e., $q = p + 1$ for $n = 2p + 1$ and $q = p$ or $q = p + 1$ for $n = 2p$.

The computation is straightforward as long as the saddle point is unaffected. This is the case, for example, when there is a Wilson line on a single gauge group, which gives (1.2)–(1.4). Similarly, when all the representation are anti-symmetric we have the leading contribution

$$\langle W_{A_{k_1}, A_{k_2}, \dots, A_{k_q}} \rangle = \exp \left[4\pi \sqrt{\frac{nN}{2(8 - N_f)}} N \sum_{a=1}^q \left(1 - \left| 1 - \frac{k_a}{N} \right|^{3/2} \right) \right].
 \tag{2.46}$$

The result (2.46) is simply a product of contributions from the Wilson loops located at each gauge node. We will come back to the holographic interpretation of this result later.

The case with symmetric representations, however, is more subtle. Let us consider $\langle W_{S_{k_1}, S_{k_2}, \dots, S_{k_q}} \rangle$, for example. By the same logic as in Sect. 2.3, we find that the dominant contribution is from the large winding Wilson loops. This means that the matrix model reduces to an integral over the eigenvalues $\sigma_1^{(a)}$, while all other eigenvalues can be replaced by the smooth eigenvalue distribution. The resulting effective matrix model is similar to (2.31); however, this time $\sigma_1^{(a)}$ with different values of a interacts among themselves, and a careful analysis is required. In other words, for the $n > 1$ case there are several different species of particle-like excitations above the Fermi sea, and there are non-trivial interactions between them (Fig. 5).

Since the general case is notationally involved, let us first study the simplest non-trivial case of $n = 2$ with gauge groups $USp(2N) \times USp(2N)$. We parametrize the Cartan of the two gauge groups by ρ_i, σ_i with $i = 1, \dots, N$.

¹⁷ In the discussion above we have assumed that n is small. However, it is also possible to take n large, for example $n = n' N^\beta$ with n' finite. In this case, the leading contribution is of order $n N^{7/2}$, which vanish under (2.44). However, in the next order (2.45) gives $\alpha = (1 + \beta)/2$, and hence the free energy scales as $O(n N^{5/2}) = O(N^{(5+3\beta)/2})$.

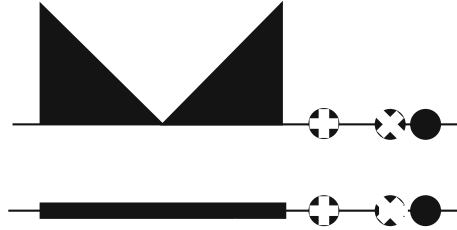


FIGURE 5. *Wilson lines* in symmetric representations for the quiver theories ($n > 1$) can be described by excitations of interacting particles above the Fermi sea. The size of the Fermi sea scales as \sqrt{n} . There are several different types of particle species, corresponding to different nodes of the quiver. In this figure the different particle species are represented by different types of *dots*

In this case, the integrand of the matrix model is $e^{-F[\rho,\sigma]}$ with

$$\begin{aligned}
 F[\rho, \sigma] = & \sum_{i \neq j} [F_V(\rho_i + \rho_j) + F_V(\rho_i - \rho_j) + F_V(\sigma_i + \sigma_j) + F_V(\sigma_i - \sigma_j)] \\
 & + 2 \sum_i [F_V(2\rho_i) + F_V(2\sigma_i)] + 2 \sum_{i,j} [F_H(\rho_i + \sigma_j) + F_H(\rho_i - \sigma_j)] \\
 & + 2 \sum_i \left[N_f^{(1)} F_H(\rho_i) + N_f^{(2)} F_H(\sigma_i) \right]. \tag{2.47}
 \end{aligned}$$

The leading contribution to the symmetric Wilson loop $\langle W_{S_k, S_l} \rangle$ comes from large winding modes, contributing $e^{2\pi(k\rho_1 + l\sigma_2)}$ to the matrix model. This is justified by arguments similar to the $n = 1$ case. By replacing $\rho_i, \sigma_i, (i \geq 2)$ with smooth eigenvalue distribution, we have an effective matrix model

$$\langle W_{S_k, S_l} \rangle = \frac{\int d\sigma_1 \exp[-F_{\text{eff}}(\rho_1, \sigma_1; k, l)]}{\int d\sigma_1 \exp[-F_{\text{eff}}(\rho_1, \sigma_1; 0, 0)]}, \tag{2.48}$$

where we neglected the multiplicity factors which does not affect the leading behavior, and defined

$$\begin{aligned}
 F_{\text{eff}}(\rho_1, \sigma_1; k, l) = & -2\pi k\rho_1 - 2\pi l\sigma_1 + 2 [F_H(\rho_1 + \sigma_1) + F_H(\rho_1 - \sigma_1)] \\
 & + 2 [F_V(2\rho_1) + F_V(2\sigma_1)] + 2 \left[N_f^{(1)} F_H(\rho_1) + N_f^{(2)} F_H(\sigma_1) \right] \\
 & + 2N \int dy \rho(y) \left(F_V \left(\rho_1 + N^{\frac{1}{2}}y \right) + F_V \left(\rho_1 - N^{\frac{1}{2}}y \right) \right) \\
 & + F_V \left(\sigma_1 + N^{\frac{1}{2}}y \right) + F_V \left(\sigma_1 - N^{\frac{1}{2}}y \right)
 \end{aligned}$$

$$\begin{aligned}
 &+2N \int dy \rho(y) \left(F_H(\rho_1 + N^{\frac{1}{2}}y) + F_H(\rho_1 - N^{\frac{1}{2}}y) \right) \\
 &+F_H(\sigma_1 + N^{\frac{1}{2}}y) + F_H(\sigma_1 - N^{\frac{1}{2}}y) \Big). \tag{2.49}
 \end{aligned}$$

This contains terms of order $kN^{\frac{1}{2}}$ and $N^{\frac{3}{2}}$ and lower; terms of order $N^{\frac{5}{2}}$ cancel between F_V and F_H . Dropping terms of order $N^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
 F_{\text{eff}}(x_1, x_2; k, l) = \pi x_0 N^{\frac{3}{2}} \Bigg[&-x_1 \left(2\frac{k}{N} + \frac{9}{2} \right) - x_2 \left(2\frac{l}{N} + \frac{9}{2} \right) \\
 &+ \frac{1}{3}x_0^2 \left(8 - N_f^{(1)} \right) x_1^3 + \frac{1}{3}x_0^2 \left(8 - N_f^{(2)} \right) x_2^3 \\
 &- \frac{1}{3}x_0^2 \left(|x_1 + x_2|^3 + |x_1 - x_2|^3 \right) \Bigg], \tag{2.50}
 \end{aligned}$$

where we defined $x_1 = N^{-1/2}x_0^{-1}\sigma_1, x_2 = N^{-1/2}x_0^{-1}\rho_1$ and we assumed $x_1, x_2 > 0$.

Carrying the same analysis for arbitrary even $n = 2p$ with symmetric representation orders $(k_1, k_2, \dots, k_{p+1})$ (still considering the case with vector structure) leads to the generalization of (2.50)

$$\begin{aligned}
 F_{\text{eff}}(x_a; k_a) = \pi x_0 N^{\frac{3}{2}} \Bigg[&\sum_{a=1}^{p+1} \left(-x_a \left(2\frac{k_a}{N} + \frac{9}{2}c_a \right) + \frac{1}{3}x_0^2 \left(8 - N_f^{(a)} \right) x_a^3 \right) \\
 &- \sum_{a=1}^p \frac{1}{3}x_0^2 \left(|x_a - x_{a+1}|^3 + |x_a + x_{a+1}|^3 \right) \Bigg], \tag{2.51}
 \end{aligned}$$

where $c_a = 1$ ($c_a = 2$) when the a th gauge group is $USp(2N)$ ($SU(2N)$).

Let us here assume that flavors are distributed evenly, i.e., $N_f^{(a)}$ are the same for all a . Let us moreover assume that $k_a/N + (9/4)c_a$ are the same for all a ; when $k_a \gg N$ this simplify means that k_a 's are the same for all a . Then by symmetry considerations it is easy to see that there are saddle points at the locus $x_a = x_1$ for all a . This ansatz kills almost all the cubic terms and we are left with

$$F_{\text{eff}}(x_1; k_a) = \pi x_0 N^{\frac{3}{2}} \left[-x_1 \left(2\frac{\sum_{a=1}^{p+1} k_a}{N} + \frac{9}{2}n \right) + \frac{1}{3}x_0^2 \left(8 - \sum_{a=1}^{p+1} N_f^{(a)} \right) x_1^3 \right], \tag{2.52}$$

where we used the relation $\sum_{a=1}^{p+1} c_a = n$. With $k_{\text{tot}} := \sum_{a=1}^{p+1} k_a, N_f = \sum_{a=1}^{p+1} N_f^{(a)}$ and the rescaling $x_1 = n^{1/2}\tilde{x}_1$, we obtain

$$F_{\text{eff}}(\tilde{x}_1; k_{\text{tot}}) = \pi x_0 n^{\frac{3}{2}} N^{\frac{3}{2}} \left[-\tilde{x}_1 \left(2\frac{k_{\text{tot}}}{nN} + \frac{9}{2} \right) + \frac{3}{2}\tilde{x}_1^3 \right]. \tag{2.53}$$

This brings us back to the non-orbifold case (2.33) with a $n^{3/2}$ prefactor and the replacement $k \rightarrow k_{\text{tot}}/n$.

We thus obtain in the end

$$\langle W_{S_1, \dots, S_q} \rangle = \exp \left[\frac{9\pi}{\sqrt{2(8 - N_f)}} n^{\frac{3}{2}} N^{\frac{3}{2}} \left[\left(1 + \frac{4k_{\text{tot}}}{9nN} \right)^{\frac{3}{2}} - 1 \right] \right]. \quad (2.54)$$

As we will see in Sect. 3.3, this result matches with the holographic computation.

The same analysis can be done for odd n and even n without vector structure and it leads, under the same assumptions, to the same result (2.54).

The more general cases, when the $k_a/N + (9/4)c_a$ are different and the numbers of flavor $N_f^{(a)}$ in each node are different, are more involved. The x_a at the saddle point are no longer equal. In this case, a match with gravity computations would require a more complete description of the type IIA background, for example, by including discrete holonomies of the B-field on the 2-cycles of the geometry and restoring the dependence on the $N_f^{(a)}$ parameters.

3. Holographic Computations

In this section, we reproduce the same results from the holographic computation in the dual massive IIA supergravity background [7, 8] (see also [27] for uniqueness). First we review the solution. The metric is given by

$$ds^2 = \frac{L^2}{(\sin \alpha)^{\frac{1}{3}}} \left[ds_{AdS_6}^2 + \frac{4}{9} \left(d\alpha^2 + \cos^2 \alpha ds_{S^3/\mathbb{Z}_n}^2 \right) \right], \quad (3.1)$$

where α ranges as $\alpha \in (0, \pi/2]$. The orbifold is realized by writing the S^3 metric as

$$ds_{S^3/\mathbb{Z}_n}^2 = \frac{1}{4} [d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + (d\theta_3 - \cos \theta_1 d\theta_2)^2], \quad (3.2)$$

and taking the angles to range as $\theta_1 \in [0, \pi)$, $\theta_2 \in [0, 2\pi)$ and $\theta_3 \in [0, 4\pi/n)$. The AdS radius L is related to the integer parameters n, N, N_f by

$$\frac{L^4}{l_s^4} = \frac{18\pi^2 n N}{8 - N_f}. \quad (3.3)$$

The dilaton ϕ , and Roman's mass F_0 , are given by

$$e^{-2\phi} = \frac{3(8 - N_f)^{\frac{3}{2}} (nN)^{\frac{1}{2}}}{2^{\frac{3}{2}} \pi} \sin^{\frac{5}{3}} \alpha, \quad F_{(0)} = \frac{8 - N_f}{2\pi l_s}. \quad (3.4)$$

Note that the dilaton diverges at $\alpha = 0$. Near this region the curvature also diverges, and the supergravity approximation breaks down.

There is also a 6-form flux corresponding to the presence of D4-branes

$$F_{(6)} = -45\pi n N L^2 l_s^3 \omega_{AdS_6}, \quad (3.5)$$

where ω_{AdS_6} is the unit volume form on AdS_6 . The number of D4-branes this flux corresponds to can be computed as follows. First, we compute $F_{(4)} = *F_{(6)}$

$$F_{(4)} = 45\pi n N l_s^3 \left(\frac{2}{3} \right)^4 \left(\sin^{\frac{1}{3}} \alpha \cos^3 \alpha \right) d\alpha \wedge \omega_{S^3/\mathbb{Z}_n}, \quad (3.6)$$

where $\omega_{S^3/\mathbb{Z}_n} = (\sin(\theta_1)/8)d\theta_1 \wedge d\theta_2 \wedge d\theta_3$. Integrating to get the charge, we have

$$Q_{D4} = \frac{1}{2\kappa^2} \int F_{(4)} = T_4 N, \tag{3.7}$$

where we have used $2\kappa^2 = (2\pi)^7 l_s^8$ and the D4-brane tension is $T_4 = 1/[(2\pi)^4 l_s^5]$. To help with computations, we introduce the notations

$$e^{-2\phi_0} := \frac{3(8 - N_f)^{\frac{3}{2}}(nN)^{\frac{1}{2}}}{2^{\frac{3}{2}}\pi}, \quad Q_4 := 45\pi n N l_s^3 \left(\frac{2}{3}\right)^4, \quad Q_6 := -45\pi n N L^2 l_s^3. \tag{3.8}$$

This solution preserves 16 supersymmetries. As discussed in Appendix C, the ten-dimensional supersymmetry parameter can be decomposed into a basis of Killing spinors as follows

$$\epsilon = \sum_{\eta=\pm} \tilde{\chi}_\eta^{(2)} \otimes \tilde{\chi}_\eta^{(3)} \otimes \tilde{\zeta}_{\eta,-\eta} \otimes \zeta_{+,+} \otimes \chi_+^{(3)}, \tag{3.9}$$

where $\tilde{\chi}_\eta^{(2)}$ is a Killing spinor on AdS_2 , $\tilde{\chi}_\eta^{(3)}$ is a Killing spinor on S^3 and $\chi_+^{(3)}$ is a Killing spinor on S^3 . The remaining components $\zeta_{\eta,-\eta}$ and $\zeta_{+,+}$ satisfy the projection conditions

$$\begin{aligned} \zeta_{+,+} &= [\cos(\alpha)\sigma^2 + \sin(\alpha)\sigma^1] \zeta_{+,+}, \\ \tilde{\zeta}_{\eta,-\eta} &= \eta [i\sigma^2 \sinh(x) + \sigma^1 \cosh(x)] \tilde{\zeta}_{\eta,-\eta}. \end{aligned} \tag{3.10}$$

Additionally, each Killing spinor and $\zeta_{\eta,-\eta}$ and $\zeta_{+,+}$ satisfy reality conditions. Note that the combination $\chi_+^{(6)} = \sum_{\eta=\pm} \tilde{\chi}_\eta^{(2)} \otimes \tilde{\chi}_\eta^{(3)} \otimes \tilde{\zeta}_{\eta,-\eta}$ yields a Killing spinor on AdS_6 . Counting degrees of freedom, we have $8 \times 2 = 16$ parameters. The 2 comes from $\chi_+^{(3)}$ while the 8 comes from $\chi_+^{(6)}$.

The gravitational dual of the Wilson loop (1.1) in the fundamental representation is the fundamental string [1, 2]. However, when we consider general anti-symmetric or symmetric representations, the fundamental string is replaced by D-branes (for the similar case of D3-branes see [16, 18, 28]). There are two possibilities:

- (1) D4-branes wrapping AdS_2 and an internal S^3 ,
- (2) D4-branes wrapping AdS_2 and the space-time S^3 .

Roughly, they respectively correspond to anti-symmetric and symmetric representations. To be more precise, they should be dual to irreducible representations. For $USp(2N)$ groups the anti-symmetric representations are reducible, as discussed in Appendix A, and the D4-branes are dual to the largest irreducible component of the anti-symmetric representations. The corresponding flat space brane configuration is given in Table 1.

TABLE 1. Supersymmetric brane configurations

	0	1	2	3	4	5	6	7	8	9
O8 ⁻ /D8	X	X	X	X	X	X	X	X	X	
D4	X	X	X	X	X					
F1	X									X
D4 _{antisymm}	X					X	X	X	X	
D4 _{symm}	X	X	X	X	X					

The two types of D4-branes, D4_{symm} and D4_{antisymm}, preserve the same $\frac{1}{2}$ -BPS supersymmetry as preserved by the fundamental string

This identification can be motivated as follows. First, these two D4-branes are the only branes which preserve the same $\frac{1}{2}$ -BPS supersymmetry as preserved by the fundamental string (Table 1).¹⁸ Secondly, if we consider fundamental strings stretched between the background stack of branes and the stack of D4-branes in case (1), we find the number of Dirichlet–Neumann directions is 8. This means the zero energy ground state of such strings is in the R sector in the NSR formalism. This behaves as a fermion and hence anti-symmetrizes the Chan–Paton indices, so that the fundamental strings are naturally anti-symmetrized. Similarly, for case (2) the ground state is in the NS sector, and correspondingly we end up with symmetric representations.

Let us comment more on the orbifold case $n > 1$. The orbifold \mathbb{Z}_n does not have a fixed point on S^3/\mathbb{Z}_n , however, it does have a fixed point on the 4d space spanned by $\alpha, \theta_1, \theta_2, \theta_3$. Locally near $\alpha = \frac{\pi}{2}$, we have an orbifold singularity of the form $\mathbb{C}^2/\mathbb{Z}_n$. Correspondingly, the geometry contains additional 2-cycles associated to the twisted sectors. The (probe) D-branes wrapping different 2-cycles will correspond to Wilson loops in representations of different gauge nodes in the quiver theory.

As detailed in [7] the various 2-cycles can be seen in the resolved geometry where the \mathbb{Z}_n orbifold is blown up to a n -centered ALE space. The orbifold corresponds to the limit when all centers merge to the same point. More precisely in our IIA geometry, there are $n - 1$ vanishing 2-cycles Σ_i wrapping the coordinates θ_1, θ_2 at the orbifold singularity $\alpha = \pi/2$ and $n - 1$ dual 2-cycles $\tilde{\Sigma}_i$ wrapping the coordinates α, θ_3 (Fig. 6). However, not all of these cycles are independent; the orientifold projection maps the i th twisted sector with the $(n - i)$ th twisted sector, this implies that the Σ_i ($\tilde{\Sigma}_i$) should be identify with Σ_{n-i} ($\tilde{\Sigma}_{n-i}$). The branes wrapping these cycles have the following interpretation.

Wilson loops in (the largest irreducible component of) the anti-symmetric representations of one of the quiver nodes are dual to D4-branes wrapping an $AdS_2 \times \tilde{S}_i^3$, with the 3-sphere $\tilde{S}_i^3 = \tilde{S}_i^1 \times S^2$, where S^2 is the 2-sphere

¹⁸ In general, the branes impose projections on the supersymmetry parameters. For each brane we introduce, there is a quantity δ_i which imposes the constraint $\delta_i \varepsilon = \varepsilon$. Generically the δ_i are traceless matrices with eigenvalues ± 1 . In order for the projection operators to be compatible, the δ_i must commute. The explicit δ_i for the above branes are given by $\delta_{O8^-/D8} = \Gamma^9 \Gamma^\sharp$, $\delta_{D4} = \delta_{D4_{\text{symm}}} = \Gamma^{56789} \Gamma^\sharp$, $\delta_{D4_{\text{antisymm}}} = \Gamma^{12349} \Gamma^\sharp$.

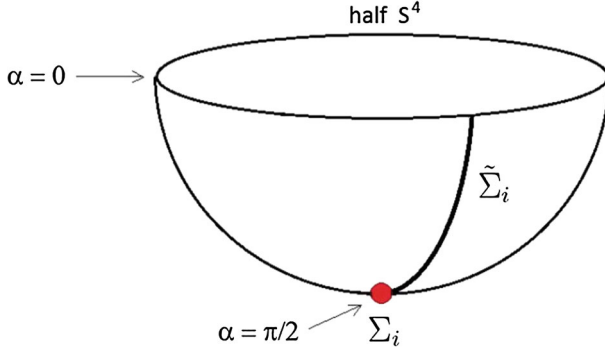


FIGURE 6. Vanishing 2-cycles Σ_i at the pole of the semi- S^4 and dual 2-cycles $\tilde{\Sigma}_i$ ($1 \leq i \leq n - 1$) spanned by coordinates α and θ_1 (S^1 transverse to the picture)

parametrized by θ_1, θ_2 and \tilde{S}_i^1 is the circle in $\tilde{\Sigma}_i$ parametrized by θ_3 (Fig. 7). To support this picture, we match the number of D4-brane embeddings to the number of Wilson loops as follows. When n is odd, there are $[n/2]$ such additional \tilde{S}_i^3 cycles. The D4-branes wrapping these cycles correspond to anti-symmetric representations of the $SU(2N)$ gauge groups, while the D4-brane wrapping the original cycle corresponds to a representation of $USp(2N)$. When n is even, there is a cycle which is mapped into itself under the orientifold projection. There are then two cases to consider. In either case, there are $[n/2] - 1$ \tilde{S}_i^3 cycles which the D4-branes can wrap yielding anti-symmetric representations of $SU(2N)$ gauge groups. In the case with vector structure, the D4-branes wrapping the remaining $\tilde{S}_{[n/2]}^3$ cycle and the original cycle yield representations of the two remaining $USp(2N)$ gauge groups. Finally in the case without vector structure, the D4-branes wrapping the remaining $\tilde{S}_{[n/2]}^3$ cycle and the original cycle must combine to yield representations of $SU(2N)$.

For Wilson loops in symmetric representations of one of the gauge factors, the holographic dual is a D4-brane wrapping the space-time $AdS_2 \times S^3$ and sitting at the point $\alpha = \pi/2$ in internal space (Fig. 7). To obtain such a configuration, we can either have a true D4-brane sitting at $\alpha = \pi/2$ or D6-branes with the same space-time embedding wrapped on the vanishing two-cycle Σ_i at $\alpha = \pi/2$.¹⁹

When we have Wilson loops in non-trivial representations for several gauge groups, we might expect the dual to be described by several D-branes wrapping distinct cycles. For anti-symmetric representations, we have several D4-branes parallel to each other at different values of α (see Fig. 8), and this

¹⁹ This is analogous to the fractional D4-branes that are D6-branes wrapped on Σ_i and which increase the rank of the corresponding gauge factor in the quiver theory. However, to determine the gravity duals of quivers with factors of different ranks, one should take into account the backreaction of the fractional D4-branes.

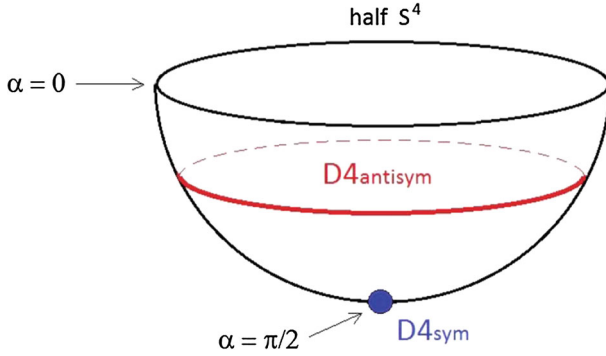


FIGURE 7. D4-branes embeddings in the internal space for symmetric and antisymmetric cases. The $D4_{\text{antisymm}}$ -brane wraps the fibered S^2 (θ_1, θ_2) in red and the S^1 (θ_3) transverse to the picture

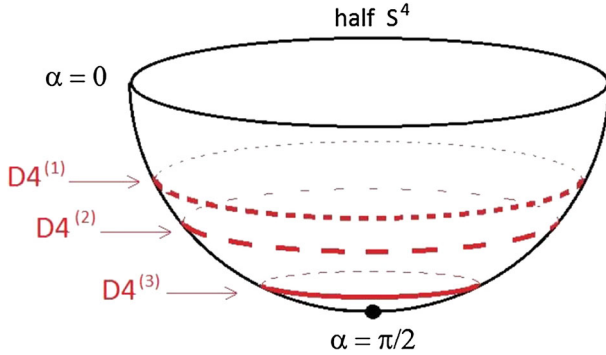


FIGURE 8. General antisymmetric representations for Wilson loops correspond to having D4-branes of different type (wrapping different cycles) at different positions in internal space. Note that each line wraps a different \tilde{S}_i^1 cycle

picture is naturally realized in the CFT result (1.9). For symmetric representations, we have seen that the factorization does not hold. This unexpected result could be related to the fact that the D4-branes corresponding to symmetric representations are really fractional D6-branes wrapping different 2-cycles whose size is vanishing in the orbifold limit. In this case, there are additional contributions one might have to take into account, for example, from discrete holonomies of the B-field on these 2-cycles. Additionally, one needs to be able to account for the $N_f^{(i)}$ dependence appearing in the quiver gauge theories.

In the rest of this section, we compute the worldsheet action of the single fundamental string and the world-volume actions of D4-branes in the antisymmetric and symmetric embeddings. We find perfect agreement with the matrix model computations of the previous section. For completeness, we also

evaluate the IIA action on the supergravity solution and match it with the free energy computation on the 5-sphere.

3.1. Fundamental Representation

The Wilson loops preserve the bosonic symmetry $SO(1, 2) \times SO(4) \times SU(2)_R \times SU(2)_M$, when $n = 1$ (Sect. 2). We will therefore choose coordinates on AdS_6 , which make this symmetry manifest

$$ds_{AdS_6}^2 = \cosh^2(x) ds_{AdS_2}^2 + \sinh^2(x) ds_{S^3}^2 + dx^2, \quad (3.11)$$

where for the symmetric spaces, we choose the coordinates (see Appendix B)

$$ds_{AdS_2}^2 = \frac{1}{\sinh^2 \rho} (d\rho^2 - d\psi^2), \quad (3.12)$$

$$ds_{S^3}^2 = d\phi_1^2 + \sin^2(\phi_1) d\phi_2^2 + \sin^2(\phi_1) \sin^2(\phi_2) d\phi_3^2.$$

When $n > 1$, the Wilson loops are not charged under the broken $SU(2)_M$ symmetry and the brane embeddings for the $n = 1$ case map, in straightforward way, to brane embeddings in the $n > 1$ case. In the following we will allow for general values of n .

We first consider a fundamental string with world-volume coordinates ξ^i with $i = 0, 1$. We take the fundamental string to wrap the AdS_2 slice. In order to preserve the $SO(4) \times SU(2)_R \times SU(2)_M$ symmetry, the string must sit at locations where the two S^3 s vanish, namely at $x = 0$ and $\alpha = \pi/2$. One can check that this choice is in fact an extremum of the Nambu-Goto action. Denoting the induced metric as G_{ij} , the on-shell action is given by

$$S_{F1} = -\frac{1}{2\pi l_s^2} \int d\xi^i \sqrt{-\det(G_{ij})} = -\frac{3\sqrt{2n\bar{N}}}{2\sqrt{8-N_f}} \int d\rho d\psi \frac{1}{\sinh^2(\rho)}. \quad (3.13)$$

This answer is divergent even after taking ψ to be compact. To get a finite answer, we compute the Legendre transformed action.²⁰ The reason for the Legendre transform is that the dual of a supersymmetric Wilson loop is a fundamental string which satisfies Dirichlet boundary conditions parallel to the boundary and Neumann boundary conditions perpendicular to the boundary [29]. For our simple string, we take $\xi^0 = \psi$ and keep ξ^1 arbitrary. The profile of the string is then given by $z = \xi^1$ and the Legendre transformed action is given by

$$A_{F1} = S_{F1} - \int d\psi \left(z \frac{\partial S_{F1}}{\partial (\partial_{\xi^1} z)} \right) \Big|_{z=0} = \frac{3\sqrt{2n\bar{N}}}{2\sqrt{8-N_f}} (2\pi R_\psi), \quad (3.14)$$

where we have taken the ψ direction to be compact with periodicity $\psi = \psi + 2\pi R_\psi$. Going to the Euclidean, we set $R_\psi = 1$, as discussed in Appendix B. We then arrive at the advertised result (1.2) for $n = 1$ and (1.8) for $n > 1$.

Next, we check the supersymmetry of the embedding. The projection corresponding to the fundamental string is given by

²⁰ Alternatively, one can use holographic renormalization, including counter-terms, to arrive at the same result.

$$\epsilon = \pm \Gamma^{01} \Gamma^\sharp \epsilon = \pm \Gamma^{23456789} \epsilon, \quad (3.15)$$

where one chooses a definite sign. Using our conventions given in Appendix C we have

$$\Gamma^{23456789} = 1_4 \otimes \sigma^1 \otimes \sigma^1 \otimes 1_2. \quad (3.16)$$

This is compatible with the projections (3.10) on $\zeta_{+,+}$ and $\tilde{\zeta}_{\eta,-\eta}$ provided $x = 0, \alpha = \pi/2$ and $\eta = \pm 1$, where the sign choice is correlated with the choice in (3.15). The restriction of η to a definite sign reduces the number of supersymmetries by half.

3.2. Anti-Symmetric Representations

We consider a D4-brane with world-volume coordinates ξ^i with $i = 0, \dots, 4$. We take the D4-brane to wrap the internal S^3 and the AdS_2 slice. In this case we can make the identification

$$\xi^0 = \psi, \quad \xi^1 = \rho, \quad \xi^2 = \theta_1, \quad \xi^3 = \theta_2, \quad \xi^4 = \theta_3. \quad (3.17)$$

We take a world-volume flux proportional to the AdS_2 volume

$$\mathcal{F} = qL^2 \frac{\cosh^2(x)}{\sin^{\frac{1}{3}}(\alpha) \sinh^2(\rho)} d\rho \wedge d\psi, \quad (3.18)$$

where q is an arbitrary coefficient, which can depend on both α and x . It will be necessary to have an explicit expression for $C_{(3)}$

$$C_{(3)} = Q_4 \frac{3}{40} \sin^{\frac{1}{3}}(\alpha) [7 \sin(\alpha) + \sin(3\alpha)] \omega_{S^3/\mathbb{Z}_n} - Q_4 \frac{18}{40} \omega_{S^3/\mathbb{Z}_n}. \quad (3.19)$$

Note that the choice of $C_{(3)}$ is not unique and in particular one can make large gauge transformations which are proportional to the unit volume form on the S^3/\mathbb{Z}_n . However, we note that the S^3/\mathbb{Z}_n is a vanishing cycle at $\alpha = 0$. In order for $C_{(3)}$ to be regular, we should then require $C_{(3)}$ to vanish at $\alpha = 0$, which then fixes the gauge freedom as above.

In order to preserve the symmetry of the space-time S^3 , the D4-brane must sit at $x = 0$. It is then consistent to take the remaining embedding coordinates, namely α , to be constant. One can check that this satisfies the general equations derived in [30]. Introducing the induced metric G_{ij} and the pullback of $C_{(3)}$ as $\hat{C}_{(3)}$, the D4-brane action is given by

$$\begin{aligned} S_{D4} &= -T_4 \int d^5 \xi e^{-\phi} \sqrt{-\det(G_{ij} + \mathcal{F}_{ij})} + T_4 \int \mathcal{F} \wedge \hat{C}_{(3)} \\ &= -T_4 \text{vol}(S^3/\mathbb{Z}_n) \int d\rho d\psi e^{-\phi_0} \frac{L^5}{\sinh^2(\rho)} \left(1 - \frac{\sinh^4(\rho) \sin^{\frac{2}{3}}(\alpha)}{L^4} (\mathcal{F}_{\rho\psi})^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\frac{2 \cos(\alpha)}{3} \right)^3 \\ &\quad - T_4 \text{vol}(S^3/\mathbb{Z}_n) \int d\rho d\psi \mathcal{F}_{\rho\psi} Q_4 \left[\frac{3}{40} \sin^{\frac{1}{3}}(\alpha) (7 \sin(\alpha) + \sin(3\alpha)) - \frac{18}{40} \right]. \end{aligned} \quad (3.20)$$

We take $d\psi \wedge d\rho \wedge \omega_{S^3/\mathbb{Z}_n}$ to be positive, which accounts for the sign in the second term. Minimizing the above action for α with fixed \mathcal{F} and then plugging in the expression for \mathcal{F} yields an equation which determines α

$$-81q\sqrt{1-q^2}Q_4 \cos(\alpha) \sin(\alpha) + 8e^{-\phi_0} L^3 [q^2(1-10\sin^2(\alpha)) + 9\sin^2(\alpha)] = 0. \tag{3.21}$$

Of course the above procedure is not necessarily consistent and we have checked that this equation can also be obtained using the general equations derived in [30]. Plugging in the explicit expressions for Q_4, ϕ_0 and L leads to

$$(9-10q^2)\sin^2(\alpha) + q(q-5\sqrt{1-q^2})\sin(2\alpha) = 0. \tag{3.22}$$

This can be solved to give q in terms of α . There are two solutions:

$$q = \pm \sin(\alpha), \quad q = \pm 9\sqrt{\frac{1-\cos(2\alpha)}{82-80\cos(2\alpha)}}. \tag{3.23}$$

The first solution is compatible with supersymmetry while the second is not. We therefore consider only the first solution. The quantization condition of the fundamental string charge is given in (E.7)

$$N_{F1} = N - N \frac{\sin^{\frac{1}{3}}(\alpha)}{6} \left(\sin(3\alpha) + 7\sin(\alpha) - \frac{4q}{\sqrt{1-q^2}} \cos^3(\alpha) \right), \tag{3.24}$$

with N_{F1} the number of fundamental strings dissolved into the D4-brane. After plugging in the expression for q , we obtain an expression giving N_{F1} in terms of α

$$N_{F1} = N - N \sin^{\frac{4}{3}}(\alpha). \tag{3.25}$$

Note this solution satisfies $N_{F1} = 0$ when $\alpha = \pi/2$ and $N_{F1} = N$ when $\alpha = 0$. This is consistent with the matching of these D4-brane embeddings to anti-symmetric representations.

Computing the on-shell action, we find

$$\begin{aligned} S_{D4} &= -T_4 \int d^5\xi e^{-\phi} \sqrt{-\det(G_{ij} + \mathcal{F}_{ij})} + T_4 \int \mathcal{F} \wedge \hat{C}_{(3)} - \frac{N_{F1}}{2\pi l_s^2} \int \mathcal{F} \\ &= \frac{2N}{3} \left[1 - \left(1 - \frac{N_{F1}}{N} \right)^{\frac{3}{2}} \right] S_{F1}. \end{aligned} \tag{3.26}$$

The last term in the first line is a boundary term resulting from the coupling of the world-volume gauge field to the boundary of the open string. As a consistency check we remark that in the limit of small N_{F1} the position of the D4-brane goes to $\alpha = \pi/2$ where the internal S^3 vanishes and we recover the fundamental string wrapped on AdS_2 , sitting at $(x, \alpha) = (0, \pi/2)$ as expected.

Surprisingly the result agrees with the gauge computation when $N_{F1} \rightarrow N$ ($k \rightarrow N$). In this limit the position of the D4-brane is pushed to $\alpha = 0$ where the orientifold sits and we might have expected that the supergravity background gets corrected in this region. The reason why the holographic computation remains valid in this region is unclear and deserves more attention.

For $n = 1$, (3.26) gives the advertised result for anti-symmetric representations (1.3). The result for more general representations, (1.6), can be interpreted as the sum over contributions from multiple D4-branes, with one D4-brane for each l_a in the representation sitting at the position α_a , as determined by the value of l_a . Similarly, the result (1.9) for anti-symmetric representations for Wilson loops in quiver theories ($n > 1$) is obtained simply by adding contributions of multiple D4-brane actions, with each D4-brane sitting at a position α_a in the internal space determined by the order k_a of the representation in the node a (Fig. 8). However, the important difference between the two is that in the latter case the D4-branes are distinct in the sense that they wrap different \tilde{S}_i^1 -cycles, as discussed at the beginning of this section. If we consider general representations for quiver theories, we have in general several D4-branes, sitting at different positions and on different cycles.

We now check the supersymmetry of this embedding. The projection matrix (D.2) reduces to

$$\begin{aligned} \Gamma &= \frac{1}{\sqrt{1-q^2}} \Gamma^{23456} - \frac{q}{\sqrt{1-q^2}} \Gamma^{789} \\ &= -\frac{1}{\sqrt{1-q^2}} (1_2 \otimes 1_2 \otimes \sigma^1 \otimes \sigma^2 \otimes 1_2) \\ &\quad - \frac{q}{\sqrt{1-q^2}} (1_2 \otimes 1_2 \otimes 1_2 \otimes i\sigma^3 \otimes 1_2). \end{aligned} \tag{3.27}$$

At $x = 0$, the project condition (3.10) on $\tilde{\zeta}_{\eta,-\eta}$ reduces to $\sigma^1 \tilde{\zeta}_{\eta,-\eta} = \eta \tilde{\zeta}_{\eta,-\eta}$. The constraint $\epsilon = \Gamma \epsilon$ then reduces to

$$\zeta_{+,+} = \eta \left(q\sigma^1 - \sqrt{1-q^2\sigma^2} \right) \zeta_{+,+}. \tag{3.28}$$

This is compatible with (3.10) for $\eta = -1$, provided we take $q = -\sin(\alpha)$. One can easily check that taking the second solution in (3.23) yields a projection on $\zeta_{+,+}$ which is incompatible with (3.10) and thus breaks all supersymmetries.

3.3. Symmetric Representations

We consider a D4-brane with world-volume coordinates ξ^i with $i = 0, \dots, 4$. We take the D4-brane to wrap the space-time S^3 and the AdS_2 slice. In this case we can make the identification

$$\xi^0 = \psi, \quad \xi^1 = \rho, \quad \xi^2 = \phi_1, \quad \xi^3 = \phi_2, \quad \xi^4 = \phi_3. \tag{3.29}$$

We again take a world-volume flux proportional to the AdS_2 volume

$$\mathcal{F} = qL^2 \frac{\cosh^2(x)}{\sin^{\frac{1}{3}}(\alpha) \sinh^2(\rho)} d\rho \wedge d\psi. \tag{3.30}$$

We will need the 5-form gauge potential. In the coordinates (3.11), $C_{(5)}$ is given by

$$\begin{aligned} C_{(5)} &= Z \frac{Q_6}{\sinh^2(\rho)} \left[\frac{\cosh^3(x)}{30} (3 \cosh(2x) - 7) + \frac{4}{30} \right] \\ &\quad \times \sin^2(\phi_1) \sin(\phi_2) d\rho \wedge d\psi \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3. \end{aligned} \tag{3.31}$$

As before, we have the freedom to make large gauge transformations, which are proportional to the unit volume forms on the AdS_2 and S^3 . Since the S^3 vanishes at $x = 0$, we require $C_{(5)}$ to vanish there as well, which fixes this gauge choice.

In order for the D4-brane to preserve the remaining $SO(4)$ symmetry of the internal S^3 , it must sit at $\alpha = \pi/2$. With this requirement, we can then again take the remaining embedding coordinates to be constant. Again, one can check that this satisfies the general equations derived in [30]. Introducing the induced metric G_{ij} and the pullback of $C_{(5)}$ as $\hat{C}_{(5)}$, the D4-brane action is given by

$$\begin{aligned}
 S_{D4} &= -T_4 \int d^5\xi e^{-\phi} \sqrt{-\det(G_{ij} + \mathcal{F}_{ij})} + T_4 \int \hat{C}_{(5)} \\
 &= -T_4 \text{vol}(S^3) \int d\rho d\psi e^{-\phi_0} \frac{L^5}{\sinh^2(\rho)} \cosh^2(x) \sinh^3(x) \\
 &\quad \times \left(1 - \frac{\sinh^4(\rho)}{L^4 \cosh^4(x)} (\mathcal{F}_{\rho\psi})^2 \right)^{\frac{1}{2}} \\
 &\quad - T_4 \text{vol}(S^3) \int d\rho d\psi \frac{Q_6}{\sinh^2(\rho)} \left[\frac{\cosh^3(x)}{30} (3 \cosh(2x) - 7) + \frac{4}{30} \right].
 \end{aligned} \tag{3.32}$$

Minimizing the above action for x with fixed \mathcal{F} and then plugging in the expression for \mathcal{F} yields an equation which determines x

$$L^5 (1 - 6q^2 \cosh^2(x) + 5 \cosh(2x)) + e^{\phi_0} \sqrt{1 - q^2} Q_6 \sinh(2x) = 0. \tag{3.33}$$

Again, this equation can also be obtained using the general equations derived in [30]. Plugging in the explicit expressions for Q_6 , ϕ_0 and L leads to

$$-1 + 3q^2 + (3q^2 - 5) \cosh(2x) + 5\sqrt{1 - q^2} \sinh(2x) = 0. \tag{3.34}$$

This can be solved to give q in terms of x . There are two solutions

$$q = \pm \frac{1}{\cosh(x)}, \quad q = \pm \sqrt{\frac{13 + 5 \cosh(2x)}{9 + 9 \cosh(2x)}}. \tag{3.35}$$

As we will see, the first solution is compatible with supersymmetry while the second is not. The quantization condition is given in Appendix (E.9)

$$N_{F1} = \frac{9}{4} nN \frac{q}{\sqrt{1 - q^2}} \sinh^3(x), \tag{3.36}$$

with N_{F1} the number of fundamental strings dissolved into the D4-brane. After plugging in for x , we obtain an expression giving N_{F1} in terms of x

$$N_{F1}^{(1)} = nN \frac{9}{4} \sinh^2(x). \tag{3.37}$$

Note that in this case, $N_{F1}^{(1)}$ is unbounded as $x \rightarrow \infty$ while it goes to zero for $x \rightarrow 0$. This is consistent with the matching of these D4-brane embeddings to symmetric representations. Computing the on-shell action gives

$$\begin{aligned}
 S_{D4} &= -T_4 \int d^5\xi e^{-\phi} \sqrt{-\det(G_{ij} + \mathcal{F}_{ij})} + T_4 \int \hat{C}_{(5)} - \frac{N_{F1}}{2\pi l_s^2} \int \mathcal{F} \\
 &= \frac{3nN}{2} \left[\left(1 + \frac{4}{9} \frac{N_{F1}}{nN} \right)^{\frac{3}{2}} - 1 \right] S_{F1}.
 \end{aligned}
 \tag{3.38}$$

Again here we remark that in the limit of small N_{F1} the position of the D4-brane goes to $x = 0$ where the space-time S^3 vanishes and we recover the fundamental string wrapped on AdS_2 , sitting at $(x, \alpha) = (0, \pi/2)$ as expected.

The result (3.38) gives the advertised result for symmetric representations (1.4), for $n = 1$, and (1.10), for $n > 1$, after we identify $N_{F1} = k_{tot}$. In the latter case, the fractional D6-branes wrapping different cycles recombine into a D4-brane, whose fundamental string charge is the sum of that of all the D6-branes.

We now check the supersymmetry of this embedding. The projection matrix (D.2) reduces to

$$\begin{aligned}
 \Gamma &= -\frac{1}{\sqrt{1-q^2}} \Gamma^{56789} - \frac{q}{\sqrt{1-q^2}} \Gamma^{234} \\
 &= \frac{1}{\sqrt{1-q^2}} (1_2 \otimes 1_2 \otimes \sigma^3 \otimes 1_2 \otimes 1_2) - \frac{q}{\sqrt{1-q^2}} (1_2 \otimes 1_2 \otimes i\sigma^2 \otimes \sigma^1 \otimes 1_2).
 \end{aligned}
 \tag{3.39}$$

At $\alpha = \pi/2$, the projection condition (3.10) on $\zeta_{+,+}$ reduces to $\sigma^1 \zeta_{+,+} = \zeta_{+,+}$. The constraint $\epsilon = \Gamma\epsilon$ then reduces to

$$\tilde{\zeta}_{\eta,-\eta} = \frac{1}{q} \sigma^1 \tilde{\zeta}_{\eta,-\eta} + \frac{\sqrt{1-q^2}}{q} i\sigma_2 \tilde{\zeta}_{\eta,-\eta}.
 \tag{3.40}$$

This is compatible with (3.10), provided we take $q = \eta/\cosh(x)$. One can easily check that taking the second solution in (3.23) yields a projection on $\zeta_{+,+}$ which is incompatible with (3.10) and thus breaks all supersymmetries. Since the solution picks a specific sign choice for η , the D4-brane preserves half of the supersymmetries.

3.4. Free Energy

In [11], the authors computed the free energy on the gravity side using holographic entanglement entropy and obtained

$$F_{\text{CFT}} = -\frac{9\sqrt{2}}{5\sqrt{8-N_f}} \pi n^{3/2} N^{5/2} + \mathcal{O}(N^{5/2}).
 \tag{3.41}$$

To complete the picture we reproduce their result by a direct computation of the gravity action, regularized appropriately. We follow the same method as in [31]. First we truncate the IIA supergravity background to pure gravity on AdS_6 and then regularize the AdS_6 infinite volume by holographic renormalization techniques [32–34]. This is a consistent truncation since we can replace AdS_6 space with any space which obeys the same Einstein equations.

In this computation, we are using the supergravity background described in the last subsection. This background contains both an orbifold singularity

at $\alpha = 0$ and an orientifold singularity at $\alpha = \pi/2$. Therefore, the supergravity description breaks down in these regions and so a priori our computation might miss an important contribution. Nevertheless, we assume the correction to our result is subdominant in the large N limit and the match with the gauge theory computation will justify a posteriori this assumption.

The effective action after the reduction reads

$$S_{\text{eff}} = -\frac{1}{2\kappa_0^2} L^8 \int_{S^4/\mathbb{Z}_n} e^{-2\phi_0} \left(\frac{4}{9}\right)^2 (\sin \alpha)^{4/3} (\cos \alpha)^3 \times \int_{AdS_6} \sqrt{g_{(6)}} (R_{(6)} - 2\Lambda_{(6)}), \tag{3.42}$$

where the subscript (6) shows that the metric, Ricci scalar and the cosmological constant are 6-dimensional, and we have $\Lambda_{(6)} = -10$.²¹ Since we want to evaluate the on-shell action, we take $R_{(6)} = 3\Lambda_{(6)} = -30$. We therefore have

$$S_{\text{eff}} = -\frac{L^2}{8\pi^5 l_s^2} n^2 N^2 (-10) \text{vol}(AdS_6) \text{vol}_4, \tag{3.43}$$

The factor vol_4 is a volume factor of the internal space, with the AdS warp factor taken into account:

$$\text{vol}_4 = \frac{\text{vol}(S^3)}{n} \int d\alpha (\sin \alpha)^{1/3} (\cos \alpha)^3 = \frac{9\pi^2}{10n}. \tag{3.44}$$

where we used $\text{vol}(S^3) = 2\pi^2$. The regularized volume of AdS_6 is given by²²

$$\text{vol}(AdS_6) = -\frac{8}{15}\pi^3. \tag{3.45}$$

Combining these results and (3.3), we can verify that (3.43) reproduces (3.41). This result matches both with the gauge theory and the holographic entanglement entropy computations, providing a non-trivial check of the concerned holographic dualities.

4. Discussion

In this paper, we have computed the large N limit of the VEVs of Wilson loops for a class of 5d $\mathcal{N} = 1$ SCFTs, both in field theory and in the dual massive IIA supergravity background. It is non-trivial and surprising that we can extract exact quantitative results about non-renormalizable gauge theories, and we

²¹ For AdS_D space-times we have $R = \frac{2D}{D-2}\Lambda$ and $\Lambda = -\frac{(D-1)(D-2)}{2}$.

²² The volume of AdS_6 is regularized by holographic renormalization techniques, see [34, Section 5] for a pedagogical introduction. The gravity action contains the bulk action plus the Gibbons–Hawking surface term. To regularize this action one needs to add (universal) covariant boundary counter-terms making the action finite. We can extract the volume of pure AdS from the renormalized gravity action. In our problem we choose Poincaré patch for the Euclidean AdS_6 so that the conformal boundary is S^5 ; in the language of [33] the coordinates are given by formula (8) with $n = 5, k = 1$. Then the action can be computed using formulas (63)–(65) of [33], where $\sigma_{k,n} = \sigma_{1,5} = \pi^3$ is the volume of the unit 5-sphere.

hope that our computation will serve as a prototypical example for a deeper understanding of more general classes of non-renormalizable theories.

For quiver theories, we have found that a complete analysis would require more information coming from the holographic background. Especially the dependence on the flavors of the different nodes is absent from the current gravity description. We suspect that it could be recovered by including discrete holonomies of the B-field on the 2-cycles of the orbifold background or perhaps by appropriate couplings of the D-brane world-volume theories to the Roman's mass F_0 . A related issue is to consider the generalization to backgrounds describing quiver theories with nodes of different ranks. This would correspond on the gravity side to having fractional D4-branes (D6-branes wrapped on vanishing 2-cycles). For this purpose, it would be useful to construct fully backreacted geometries (cf. [19, 20, 35]). Further investigations in this direction would certainly improve our understanding of AdS/CFT for quiver theories/orbifold backgrounds.

There are a number of generalizations we can consider. We can consider defects of other dimensionality, such as surface operators, or place the theory on 5-manifolds other than S^5 (cf. [36, 37]). We could also try to extend the analysis to 5d $\mathcal{N} = 1$ $USp(2N)$ theories with $N_f = 8$, or to 5d $\mathcal{N} = 2$ theories. This will lead to quantitative understanding of 6d (1, 0) theory or 6d (2, 0) theory on $S^5 \times S^1$, and the Wilson surfaces therein.

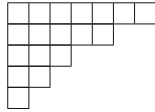
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Appendix A. Representation of $USp(2N)$

In this Appendix we summarize representation theory of the Lie algebra $USp(2N)$ needed for the main text, especially in Sect. 2.4 (see for example [38]). The representation is similar to the case of $U(N)$ gauge groups, but there are important differences.

An irreducible representation of $USp(2N)$ is specified by a Young diagram with at most N rows. This is expressed as a partition $\mu = (\mu_1, \mu_2, \dots, \mu_N)$, satisfying $\mu_1 \geq \mu_2 \geq \dots \mu_N \geq 0$, where μ_i denotes the number of boxes of the i th row. For simplicity we often drop from the notation those μ_i 's which are equal to zero. For example, $\mu = (7, 5, 3, 2, 1)$ represents



We can also represent this by the dual partition $\nu = \mu^T$. In the example above, we have $\nu = (5, 4, 3, 2, 2, 1, 1)$.

In the body of this paper we discussed k th symmetric and anti-symmetric representations, obtained by symmetrizing (or anti-symmetrizing) the k th power of the fundamental representation. For k th symmetric representation S_k is an irreducible representation, and is described by the Young diagram of the form (shown for $k = 7$),



However, the k th anti-symmetric representation A_k is not irreducible, and decomposes into several irreducible components. The component with the largest dimension is described by



For the computation of Wilson loops we need a character of the representation μ . This is given by the “symplectic character” $sp_\mu(x) = sp_\mu(x_1, \dots, x_N)$, defined by

$$sp_\mu(x) := \frac{\det_{i,j} \left(x_i^{\mu_j+n-j+1} - x_i^{-(\mu_j+n-j+1)} \right)}{\det_{i,j} \left(x_i^{n-j+1} - x_i^{-(n-j+1)} \right)}. \tag{A.1}$$

This is a generalization of the standard Schur function for $U(N)$ groups, and is invariant under the action of the Weyl group \mathcal{W} , generated by (1) permutations of x_i 's and (2) inversions $x_i \rightarrow x_i^{-1}$ for some i .

For our purposes, it is sometimes useful to use another expression for $sp_\mu(x)$, given by the “symplectic semistandard Young tableaux” [39]. This is defined by a filling of the Young diagram μ with the letters $1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}$ such that:

- (1) the entries are weakly increasing along rows and strictly increasing down the columns,
- (2) all entries in row i are larger than or equal to i .

Given such a tableaux T , we can define its weight $w(T)$ by

$$w(T) = \prod_i x_i^{\#(i) - \#(\bar{i})}. \tag{A.2}$$

Then we have

$$sp_\lambda(x) = \sum_{T: \text{shape } \lambda} w(T). \tag{A.3}$$

For example, let us consider $USp(4)$. When we have $\mu = (2) = \square\square$, there are 10 symplectic semistandard Young tableaux

$$\begin{bmatrix} 1 & 1 \\ \hline \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ \hline \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ \hline \end{bmatrix}, \begin{bmatrix} 1 & \bar{1} \\ \hline \end{bmatrix}, \begin{bmatrix} 2 & \bar{2} \\ \hline \end{bmatrix}, \begin{bmatrix} 1 & \bar{2} \\ \hline \end{bmatrix}, \begin{bmatrix} \bar{1} & 2 \\ \hline \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \hline \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{2} \\ \hline \end{bmatrix}, \begin{bmatrix} \bar{2} & \bar{2} \\ \hline \end{bmatrix},$$

giving

$$sp_{\square\square}(x) = x_1^2 + x_1x_2 + x_2^2 + 2 + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1^2} + \frac{1}{x_1x_2} + \frac{1}{x_2^2}. \tag{A.4}$$

This gives $\dim \square\square = 10$, which is consistent with fact that $\square\square$ is the symmetric part of $\square \otimes \square$.

Similarly, when we have $\mu = (1, 1) = \begin{bmatrix} \square \\ \square \end{bmatrix}$, there are 5 symplectic semistandard Young tableaux

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} \bar{1} \\ \bar{2} \end{bmatrix}, \begin{bmatrix} 1 \\ \bar{2} \end{bmatrix}, \begin{bmatrix} \bar{1} \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ \bar{2} \end{bmatrix},$$

giving

$$sp_{\begin{bmatrix} \square \\ \square \end{bmatrix}}(x) = x_1x_2 + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1x_2} + 1. \tag{A.5}$$

This gives $\dim \begin{bmatrix} \square \\ \square \end{bmatrix} = 5$. This is smaller by one than the dimension of the anti-symmetric part of $\square \otimes \square$. In fact, anti-symmetric part of $\square \otimes \square$ decomposes into $\begin{bmatrix} \square \\ \square \end{bmatrix}$ and a singlet.

Appendix B. $AdS_2 \times S^3$ Slicing of AdS_6

In this section, we discuss an AdS_2 slicing of AdS_6 suitable for our problem. To do so, we embed AdS_6 into 7-dimensional flat space, more precisely $\mathbb{R}^{2,5}$. The AdS_6 surface is described by the equation

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = -L^2, \tag{B.1}$$

where the X_i are flat coordinates on $\mathbb{R}_{2,5}$.

We first solve the constraint as follows

$$\begin{aligned} X_{-1} &= L \coth(\lambda), & X_0 &= L \frac{\sin(\varphi_1) \sinh(\varphi_2)}{\sinh(\lambda)}, \\ X_1 &= L \frac{\sin(\varphi_1) \cosh(\varphi_2)}{\sinh(\lambda)}, & X_i &= L \frac{\cos(\varphi_1)}{\sinh(\lambda)} \hat{X}_i, \quad i = 2, 3, 4, 5, \end{aligned} \tag{B.2}$$

where \hat{X}_i describes a unit S^3 . This leads to the induced metric

$$ds^2 = \frac{L^2}{\sinh^2(\lambda)} (d\lambda^2 + d\varphi_1^2 - \sin^2(\varphi_1)d\varphi_2^2 + \cos(\varphi_1)^2 ds_{S^3}^2). \tag{B.3}$$

Upon analytically continuing $\varphi_2 \rightarrow i\varphi_2$ this leads to the Euclidean metric

$$ds_E^2 = \frac{L^2}{\sinh^2(\lambda)} (d\lambda^2 + ds_{S^5}^2). \tag{B.4}$$

We will be interested in a Wilson loop which wraps a great circle in S^5 . This can be taken to be a string worldsheet whose boundary sits at $\varphi_1 = \pi/2$ and wraps φ_2 .

For computations, this metric is not the most efficient and it will be convenient to work with an $AdS_2 \times S^3$ slicing of AdS_6 . This can be introduced by solving the constraints as

$$\begin{aligned} X_{-1} &= L \coth(\rho) \cosh(x), & X_0 &= L \frac{\sinh(\psi)}{\sinh(\rho)} \cosh(x), \\ X_1 &= L \frac{\cosh(\psi)}{\sinh(\rho)} \cosh(x), & X_i &= L \sinh(x) \hat{X}_i, \quad i = 2, 3, 4, 5, \end{aligned} \quad (\text{B.5})$$

where \hat{X}_i again describes a unit S^3 . The induced metric is now given by

$$ds^2 = L^2 \left(\frac{\cosh^2(x)}{\sinh^2(\rho)} (d\rho^2 - d\psi^2) + \sinh^2(x) ds_{S^3}^2 + dx^2 \right). \quad (\text{B.6})$$

The two coordinate systems are related by first identifying the two S^3 s and then taking

$$\coth(\lambda) = \coth(\rho) \cosh(x), \quad \cot(\varphi_1) = \sinh(\rho) \tanh(x), \quad \varphi_2 = \psi. \quad (\text{B.7})$$

Reaching the boundary by taking $\rho = 0$ and x finite maps to the surface with $\varphi_1 = \pi/2$. Thus taking the string to wrap ρ and ψ gives a string whose boundary is the great circle described above. Going to the Euclidean by taking $\psi \rightarrow i\psi$, we see that ψ has periodicity 2π .

Appendix C. Supersymmetry of the Background

First we need to work out the supersymmetry of the background. The metric (3.1) is in string frame, in Einstein frame ($g_E = e^{-\phi/2} g_s$) it becomes

$$ds_E^2 = L^2 e^{-\phi_0/2} (\sin \alpha)^{\frac{1}{12}} \left[ds_{AdS_6}^2 + \frac{4}{9} \left(d\alpha^2 + \cos^2 \alpha ds_{S^3/\mathbb{Z}_n}^2 \right) \right]. \quad (\text{C.1})$$

It will be convenient to introduce the frames

$$\begin{aligned} e^m &= L e^{-\phi_0/4} (\sin \alpha)^{\frac{1}{24}} \hat{e}^m, & m &= 0, \dots, 5, \\ e^6 &= \frac{2}{3} L e^{-\phi_0/4} (\sin \alpha)^{\frac{1}{24}} d\alpha, \\ e^i &= \frac{2}{3} L e^{-\phi_0/4} (\sin \alpha)^{\frac{1}{24}} \cos(\alpha) \hat{e}^i, & i &= 7, 8, 9, \end{aligned} \quad (\text{C.2})$$

where \hat{e}^m are unit frames on AdS_6 and \hat{e}^i are unit frames on S^3/\mathbb{Z}_n . We use M to collectively denote the frame indices so that $M = 0, \dots, 9$.

In IIA supergravity, the spinor satisfies a reality condition $\epsilon^* = \mathcal{B}\epsilon$. The BPS equations in string frame, after setting $B_{(2)} = 0$, are given by [9]²³

$$\begin{aligned} \delta\lambda &= \left[(D_M\phi)\Gamma^M + \frac{5}{4}F_{(0)}e^{\frac{5}{4}\phi} + \frac{1}{96}e^{\frac{\phi}{4}} (F_{MNPQ}\Gamma^{MNPQ}) \right] \epsilon = 0, \\ \delta\psi_M &= \left[D_M - \frac{1}{32}F_{(0)}e^{\frac{5}{4}\phi}\Gamma_M + \frac{1}{128}\frac{e^{\frac{\phi}{4}}}{2}F_{NPQR} \left(\Gamma_M{}^{NPQR} - \frac{20}{3}\delta_M{}^N\Gamma^{PQR} \right) \right] \\ &\epsilon = 0. \end{aligned} \quad (\text{C.3})$$

Plugging in the solution summarized in Sect. 3, the dilatino equation reduces to the projection condition

$$\epsilon = [\cos(\alpha)\Gamma^6 - \sin(\alpha)\Gamma^{6789}] \epsilon. \quad (\text{C.4})$$

To reduce the gravitino equation, we introduce the Γ matrices as

$$\Gamma^m = \gamma^m \otimes \sigma^1 \otimes 1_2, \quad \Gamma^6 = 1_8 \otimes \sigma^2 \otimes 1_2, \quad \Gamma^i = 1_8 \otimes \sigma^3 \otimes \gamma^i, \quad (\text{C.5})$$

where γ^m satisfy $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$ and γ^i satisfy $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. Introduce $\gamma^\sharp = i\gamma^{012345}$ and $\mathcal{B}_{(6)}$ and $\mathcal{B}_{(3)}$ by

$$(\gamma^m)^* = \mathcal{B}_{(6)}\gamma^m\mathcal{B}_{(6)}^{-1}, \quad (\gamma^i)^* = -\mathcal{B}_{(3)}\gamma^i\mathcal{B}_{(3)}^{-1}, \quad (\text{C.6})$$

and so that they satisfy $\mathcal{B}_{(6)}^*\mathcal{B}_{(6)} = -1_8$ and $\mathcal{B}_{(3)}^*\mathcal{B}_{(3)} = -1_2$. In terms of these quantities, we can write \mathcal{B} as $\mathcal{B} = \mathcal{B}_{(6)} \otimes \sigma^1 \otimes \mathcal{B}_{(3)}$ and we have $(\Gamma^M)^* = \mathcal{B}\Gamma^M\mathcal{B}^{-1}$. Next we introduce Killing spinors $\chi_{\eta_1}^{(6)}$ and $\chi_{\eta_2}^{(3)}$ on AdS_6 and S^3/\mathbb{Z}_n , respectively, which satisfy the equations

$$\begin{aligned} \left(\hat{e}_m^\mu \hat{\nabla}_\mu - \frac{\eta_1}{2}\gamma_m \right) \chi_{\eta_1}^{(6)} &= 0, \\ \left(\hat{e}_i^\mu \hat{\nabla}_\mu - i\frac{\eta_2}{2}\gamma_i \right) \chi_{\eta_2}^{(3)} &= 0. \end{aligned} \quad (\text{C.7})$$

Using the symmetries of the above equations, we impose the conditions $\gamma^\sharp\chi_{\eta_1}^{(6)} = \chi_{-\eta_1}^{(6)}, \chi_{\eta_1}^{(6)*} = \mathcal{B}_{(6)}\chi_{\eta_1}^{(6)}$ and $\chi_{\eta_2}^{(3)*} = \mathcal{B}_{(3)}\chi_{\eta_2}^{(3)}$. We next decompose ϵ in the above basis of Killing spinors

$$\epsilon = \sum_{\eta_1, \eta_2} \chi_{\eta_1}^{(6)} \otimes \zeta_{\eta_1, \eta_2} \otimes \chi_{\eta_2}^{(3)}. \quad (\text{C.8})$$

The reality condition reduces to

$$\zeta^* = \sigma_1\zeta, \quad (\text{C.9})$$

²³ We have changed conventions as follows. We have inverted the sign of the dilaton and rescaled it by a factor of 2, m is identified with $F_{(0)}$ and all of the fluxes have been rescaled by a factor of 2, we have also redefined λ and ψ by multiplicative constants.

and we can express the covariant derivatives of ϵ along the symmetric spaces as

$$\begin{aligned}
 D_m \epsilon &= - \sum_{\eta_1, \eta_2} \frac{\eta_1}{2} \frac{e^{\phi_0/4}}{L \sin^{\frac{1}{24}}(\alpha)} \Gamma_m \Gamma^{6789} \left(\chi_{\eta_1}^{(6)} \otimes \zeta_{\eta_1, \eta_2} \otimes \chi_{\eta_2}^{(3)} \right) \\
 &\quad + \frac{1}{2} (e_m \cdot \omega_{n6}) \Gamma^{n6} \epsilon, \\
 D_i \epsilon &= - \sum_{\eta_1, \eta_2} \frac{\eta_2}{2} \frac{3}{2} \frac{e^{\phi_0/4}}{L \cos(\alpha) \sin^{\frac{1}{24}}(\alpha)} \Gamma_i \Gamma^{789} \left(\chi_{\eta_1}^{(6)} \otimes \zeta_{\eta_1, \eta_2} \otimes \chi_{\eta_2}^{(3)} \right) \\
 &\quad + \frac{1}{2} (e_i \cdot \omega_{j6}) \Gamma^{j6} \epsilon,
 \end{aligned} \tag{C.10}$$

where ω_{MN} is the spin-connection defined by $de^M + \omega^M_N e^N = 0$ with

$$\omega^m{}_6 = \frac{e^{\phi_0/4}}{16L} \frac{\cos(\alpha)}{\sin^{\frac{25}{24}}(\alpha)} e^m, \quad \omega^i{}_6 = \frac{3e^{\phi_0/4}}{2L \sin^{\frac{1}{24}}(\alpha)} \left(\frac{\cos(\alpha)}{24 \sin(\alpha)} - \frac{\sin(\alpha)}{\cos(\alpha)} \right) e^i. \tag{C.11}$$

The gravitino equation along AdS_6 reduces to

$$\begin{aligned}
 \epsilon &= -16 \sum_{\eta_1, \eta_2} \eta_1 \sin(\alpha) \Gamma^{6789} \left(\chi_{\eta_1}^{(6)} \otimes \zeta_{\eta_1, \eta_2} \otimes \chi_{\eta_2}^{(3)} \right) \\
 &\quad + \cos(\alpha) \Gamma^6 \epsilon + 15 \sin(\alpha) \Gamma^{6789} \epsilon.
 \end{aligned} \tag{C.12}$$

This reduces to the dilatino equation provided $\zeta_{-, \eta_2} = 0$. The gravitino equation along S^3/\mathbb{Z}_n reduces to

$$\begin{aligned}
 \epsilon &= -24 \sum_{\eta_1, \eta_2} \eta_2 \tan(\alpha) \Gamma^{789} \left(\chi_{\eta_1}^{(6)} \otimes \zeta_{\eta_1, \eta_2} \otimes \chi_{\eta_2}^{(3)} \right) \\
 &\quad + 25 \cos(\alpha) \Gamma^6 \epsilon - \frac{24}{\cos(\alpha)} \Gamma^6 \epsilon - 25 \sin(\alpha) \Gamma^{6789} \epsilon.
 \end{aligned} \tag{C.13}$$

This reduces to the dilatino equation provided $\zeta_{\eta_1, -} = 0$ and $\zeta_{+, +}$ is the only surviving component. Since $\chi_+^{(6)}$ has 8 real degrees of freedom and $\chi_+^{(3)}$ has 2 real degrees of freedom, we conclude that there are 16 real supersymmetries.

It will be convenient to further decompose AdS_6 into $AdS_2 \times S^3$ slices using the coordinates given in (3.11). We denote the directions along AdS_2 as $m_1 = 0, 1$ and the directions along S^3 as $m_2 = 2, 3, 4$. Introducing Killing spinors $\tilde{\chi}_{\eta_3}^{(2)}$ and $\tilde{\chi}_{\eta_4}^{(3)}$ on AdS_2 and S^3 , respectively, we can write $\chi_+^{(6)}$ as

$$\chi_+^{(6)} = \sum_{\eta_3, \eta_4} \tilde{\chi}_{\eta_3}^{(2)} \otimes \tilde{\chi}_{\eta_4}^{(3)} \otimes \tilde{\zeta}_{\eta_3, \eta_4}. \tag{C.14}$$

As before, we can impose reality conditions on $\tilde{\chi}_{\eta_3}^{(2)}$ and $\tilde{\chi}_{\eta_4}^{(3)}$. The reality condition on $\chi_+^{(6)}$ then leads to a reality condition on $\tilde{\zeta}_{\eta_3, \eta_4}$. We write the γ matrices as

$$\gamma^{m_1} = \tilde{\gamma}^{m_1} \otimes 1_2 \otimes \sigma^1, \quad \gamma^{m_2} = 1_2 \otimes \tilde{\gamma}^{m_2} \otimes \sigma^2, \quad \gamma^5 = 1_2 \otimes 1_2 \otimes \sigma^3, \tag{C.15}$$

where the $\tilde{\gamma}^{m_1}$ satisfy $\{\tilde{\gamma}^{m_1}, \tilde{\gamma}^{n_1}\} = 2\eta^{m_1 n_1}$ and $\tilde{\gamma}^{m_2}$ satisfy $\{\tilde{\gamma}^{m_2}, \tilde{\gamma}^{n_2}\} = 2\delta^{m_2 n_2}$. Proceeding similarly as before, we arrive at the projections

$$\begin{aligned}\tilde{\zeta}_{\eta_3, \eta_4} &= \eta_3 (i\sigma^2 \sinh(x) + \sigma^1 \cosh(x)) \tilde{\zeta}_{\eta_3, \eta_4}, \\ \tilde{\zeta}_{\eta_3, \eta_4} &= -\eta_4 (i\sigma^2 \sinh(x) + \sigma^1 \cosh(x)) \tilde{\zeta}_{\eta_3, \eta_4}.\end{aligned}\tag{C.16}$$

The first equation comes from the Killing spinor equation along AdS_2 , while the second equation comes from the Killing spinor equation along S^3 . The compatibility of these two equations sets $\tilde{\zeta}_{+,+} = \tilde{\zeta}_{-,-} = 0$. This leaves 8 real degrees of freedom for $\chi_+^{(6)}$ as expected.

Appendix D. Supersymmetry Conditions

The conditions for supersymmetry of the probe Dp-brane are derived in [40]. We summarize the results here in the conventions of [30]. A probe Dp-brane embedding preserves supersymmetries which are consistent with the projection

$$\epsilon = \Gamma \epsilon,\tag{D.1}$$

where the matrix Γ is defined by the following equation

$$d^{p+1}\xi \Gamma = -\frac{1}{\sqrt{-\det(G_{ij} + \mathcal{F}_{ij})}} e^{\mathcal{F}} \wedge X|_{\text{vol}}.\tag{D.2}$$

The quantity X is a sum of world-volume Γ -matrices:

$$X = \bigoplus_n \left(\frac{1}{(2n+1)!} d\xi^{i_{2n+1}} \wedge \dots \wedge d\xi^{i_1} \Gamma_{i_1 \dots i_{2n+1}} \right) (\Gamma_{\sharp})^{n+1},\tag{D.3}$$

where the Γ_i are pullbacks of space-time Γ -matrices so that $\Gamma_{i_1 \dots i_n} = \partial_{i_1} X^{m_1} \dots \partial_{i_n} X^{m_n} \Gamma_{m_1 \dots m_n}$ and the chirality matrix is given by $\Gamma_{\sharp} = \Gamma_{0123456789}$.

Appendix E. Quantization of World-Volume Flux

Here, we follow closely [16] and [30]. The Dp-brane action including the coupling of the world-volume gauge field to the boundary of a stack of N_{F1} fundamental strings is given by

$$S_{Dp} = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det(G_{ij} + \mathcal{F}_{ij})} + T_p \int e^{\mathcal{F}} \wedge \hat{C} + N_{F1} \int_{\partial F1} ds \cdot A,\tag{E.1}$$

where G_{ij} is the pullback of the space-time metric, in string frame, \hat{C} is the pullback of the RR-forms and $\mathcal{F} = (2\pi l_s^2)F + \hat{B}_{(2)}$, where $\hat{B}_{(2)}$ is the pullback of the NSNS two-form and F is a world volume flux with $F = dA$. Note that $\hat{C}_{(p)}$ is really defined as the gauge potential of the pullback of $F_{(p+1)}$ so that $d\hat{C}_{(p)} = \hat{F}_{(p+1)}$.

We shall restrict to the case where $B_{(2)} = 0$. It is convenient to introduce the matrix

$$M_{ij} = (\partial_i X^M \partial_j X^n g_{MN} + F_{ij}), \tag{E.2}$$

where X^M are coordinates on the space-time. We also define the inverse matrix M^{ij} , with upper indices, and the anti-symmetric part $\theta^{ij} = (M^{ij} - M^{ji})/2$. Varying with respect to the world-volume gauge field yields the equation

$$\begin{aligned} \partial_i \left(e^{-\phi} \sqrt{-M} \theta^{ij} \right) - \epsilon^{j i_2 \dots i_{p+1}} \sum_{n \geq 0} \frac{1}{n! (2!)^n (p-2n)!} (\mathcal{F}^n)_{i_2 \dots i_{2n+1}} F_{i_{2n+2} \dots i_{p+1}} \\ = \frac{1}{2\pi l_s^2} \frac{N_{F1}}{T_p} j_i^{(F1)}, \end{aligned} \tag{E.3}$$

where $j_i^{(F1)}$ is the fundamental-string current.²⁴ Introducing a flat metric and treating $e^{-\phi} \sqrt{-M} \theta$ as a two-form, this equation can be re-expressed as

$$d * \left(e^{-\phi} \sqrt{-M} \theta \right) - \sum_{n \geq 0} \frac{1}{n!} d [(\mathcal{F}^n) \wedge C_{(p-1-2n)}] = -(-1)^p \frac{N_{F1}}{2\pi l_s^2 T_p} * j^{(F1)}, \tag{E.4}$$

where we have used the fact $d\mathcal{F} = 0$ when $B_{(2)} = 0$. Integrating the above equation over a p -volume V_p which is orthogonal to the boundary of the fundamental string, we have

$$\int_{M_{p-1}} \left[* \left(e^{-\phi} \sqrt{-M} \theta \right) - \sum_{n \geq 0} \frac{1}{n!} (\mathcal{F}^n) \wedge \hat{C}_{(p-1-2n)} \right] = \frac{N_{F1}}{2\pi l_s^2 T_p}, \tag{E.5}$$

where M_{p-1} is a $p - 1$ -dimensional closed surface which encircles the fundamental string.

For the D4-brane of Sect. 3.2, which wraps the internal S^3 , the above expression reduces to

$$\int_{S^3/\mathbb{Z}_n} \left[* \left(e^{-\phi} \sqrt{-M} \theta \right) - C_{(3)} \right] = \frac{N_{F1}}{2\pi l_s^2 T_4} \tag{E.6}$$

evaluated at an arbitrary value of ρ and ψ . Plugging in the explicit quantities and solving for N_{F1} gives

$$N_{F1} = N - N \frac{\sin^{\frac{1}{3}}(\alpha)}{6} \left(\sin(3\alpha) + 7 \sin(\alpha) - \frac{4q}{\sqrt{1-q^2}} \cos^3(\alpha) \right). \tag{E.7}$$

For the D4-brane of Sect. 3.3, which wraps the space-time S^3 , the above expression reduces to

$$\int_{S^3} \left[* \left(e^{-\phi} \sqrt{-M} \theta \right) \right] = \frac{N_{F1}}{2\pi l_s^2 T_4}, \tag{E.8}$$

²⁴ Note that the extra factor of $1/(2\pi l_s^2)$ comes from restoring the factors of $2\pi l_s^2$ in [30].

evaluated at an arbitrary value of ρ and ψ . Plugging in the explicit quantities and solving for N_{F1} gives

$$N_{F1} = \frac{9}{4} n N \frac{q}{\sqrt{1-q^2}} \sinh^3(x). \quad (\text{E.9})$$

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