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Symmetries of Quantum Lax Equations for the Painlevé Equations

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Abstract. Based on the fact that the Painlevé equations can be written as Hamiltonian systems with affine Weyl group symmetries, a canonical quantization of the Painlevé equations preserving such symmetries has been studied recently. On the other hand, since the Painlevé equations can also be described as isomonodromic deformations of certain second-order linear differential equations, a quantization of such Lax formalism is also a natural problem. In this paper, we introduce a canonical quantization of Lax equations for the Painlevé equations and study their symmetries. We also show that our quantum Lax equations are derived from Virasoro conformal field theory.

1. Introduction

It is known that the Painlevé equations are Hamiltonian systems and, except for the first one, admit the affine Weyl group actions, as Bäcklund transformations [19–22]. For example, the second Painlevé equation $\mathrm{P_{II}}(\alpha)$ $(\alpha \in \mathbb{C})$ is the Hamiltonian system:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H_{\mathrm{II}}}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_{\mathrm{II}}}{\partial q},$$

where

$$H_{\mathrm{II}}(q,p,t,\alpha) = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right)p - \alpha q.$$

Let (q,p) be a solution to $\mathrm{P_{II}}(\alpha)$. Then, birational canonical transformations defined by

$$s(q,p) = \left(q + \frac{\alpha}{p}, p\right),$$
$$\pi(q,p) = (-q, -p + 2q^2 + t),$$

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give solutions to $P_{II}(-\alpha)$, $P_{II}(1-\alpha)$, respectively. The Bäcklund transformation group generated by s, π is equivalent to the extended affine Weyl group of type $A_1^{(1)}$.

Since the Painlevé equations are Hamiltonian systems, their quantization can be considered naturally. A canonical quantization of the Painlevé equations preserving the affine Weyl group actions have been studied [12,16,17] (see also [7,13,14]). For example, the quantum second Painlevé equation QP_{II} can be written as the time-dependent Schrödinger equation:

$$\kappa\frac{\partial}{\partial t}\Psi(t,x) = H_{II}\left(x,\frac{\partial}{\partial x},t,\alpha\right)\Psi(t,x)$$
$$= \left(\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 - x\frac{\partial}{\partial x}x - \frac{t}{2}\frac{\partial}{\partial x} - \alpha x\right)\Psi(t,x).$$

Bäcklund transformations of QP_{II} are realized by the Euler transformation (or the Riemann–Liouville integral) and a gauge transformation. Let $\Psi(t,x)$ be a solution to $QP_{II}(\alpha)$. Then, transformations of a solution $\Psi(t,x)$ defined by

$$s\left(\Psi(t,x)\right) = \int_{\Delta}(x-u)^{\alpha-1}\Psi(t,u)\mathrm{d}u,$$
$$\pi\left(\Psi(t,x)\right) = \exp\left(\frac{2}{3}x^3 + xt\right)\Psi(t,-x),$$

with an appropriate cycle Δ, are solutions to $QP_{II}(-\alpha)$, $QP_{II}(-\kappa-\alpha)$, respectively. Similarly, the affine Weyl group symmetries for the quantum Painlevé equations QP_{III}–QP_{VI} were realized using gauge transformations and the Laplace transformation [17]. In both the classical and quantum cases, the affine Weyl group symmetries play an important role to study special solutions to the systems.

On the other hand, the Painlevé equations describe the isomonodromic deformation for certain second-order linear differential equations [6]. Since this fact is crucial for the Painlevé equations, it will be important to study its quantization. In the present paper, we introduce quantum Lax equations[1] and study their symmetries. In doing this, a useful fact is that the classical Lax equation can be written concisely in terms of the quantum and classical Hamiltonians. For example, the Lax equation for the second Painlevé equation $P_{II}(\alpha+1)$ can be written as

$$\left(H_{II}\left(x,\frac{\partial}{\partial x},t,\alpha\right) - H_{II}(q,p,t,\alpha+1) - \frac{1}{2(x-q)}\left(\frac{\partial}{\partial x} - p\right)\right)y(x) = 0,$$

[1] We call the linear differential equations (the Lax auxiliary linear problems) simply as Lax equations.

and a natural quantization of this gives the following quantum Lax equation:

$$\left(H_{\mathrm{II}}\left(x,\epsilon_1\frac{\partial}{\partial x},t,\alpha\right)-H_{\mathrm{II}}\left(q,\epsilon_2\frac{\partial}{\partial q},t,\alpha+\epsilon_1-\epsilon_2\right)\right.$$
$$\left.-\frac{\epsilon_1-\epsilon_2}{2(x-q)}\left(\epsilon_1\frac{\partial}{\partial x}-\epsilon_2\frac{\partial}{\partial q}\right)\right)\Phi(x,q)=0.$$

The symmetry of these quantum Lax equations can be derived using the symmetry properties of the quantum Hamiltonians studied in [17]. Taking the classical limit of the quantum Lax equations as $\epsilon_2\to 0$ with $\epsilon_2\partial/\partial q\to p$, we recover the classical Lax equations and symmetries of them. On realization of symmetries of the classical Lax equations, see [9,26] and references therein, for example. We also derive the quantum Lax equations from Virasoro conformal field theory with two null fields at x and q. Note that the quantum Painlevé equations are derived from the conformal field theory with one null field [2,3,16].

Similarly in the case of the quantum Painlevé equations [17], symmetries constructed in this paper generate solutions to the quantum Lax equations. We shall investigate solutions to the quantum Lax equations in the forthcoming paper.

The remainder of this paper is organized as follows. In Sect. 2, we introduce quantum Lax equations for the Painlevé equations. After recalling symmetries of the quantum Painlevé equations, we define transformations and show that those are Bäcklund transformations for the quantum Lax equations. In Sect. 3, we derive quantum Lax equations introduced in Sect. 2 from Virasoro conformal field theory. In appendix, we summarize the known results for the classical case.

Remark 1.1. It is known that the quantum Painlevé equations with $\kappa=1$ have a relation to corresponding classical Lax equations [18,25,28]. More precisely, the wave functions of the classical Lax equations multiplied by the tau functions of the Painlevé equations are solutions to the quantum Painlevé equations with $\kappa=1$. This means that the classical Lax equations are related to the conformal field theory with the central charge $c=1$. In [4], the tau function of the classical sixth Painlevé equation is interpreted as a four points correlation function in the conformal field theory with $c=1$.

2. Symmetry

In this section, we introduce the quantum Lax equations for the Painlevé equations and describe symmetries of them. To construct Bäcklund transformations of the quantum Lax equations, we use Bäcklund transformations of the quantum Painlevé equations.

2.1. $\mathrm{P_{VI}}$ Case

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ defined by the generators x, y, q, p, t, d, α_i $(0 \le i \le 4)$, ϵ_1, ϵ_2, and the commutation relations

$$[y, x] = \epsilon_1, \quad [p, q] = \epsilon_2, \quad [d, t] = 1,$$

and the other commutation relations are zero, and a relation $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = -\epsilon_1 + \epsilon_2$.

Let $H^x_{\mathrm{VI}}(\alpha)$ $(\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4))$ be the Hamiltonian for the quantum sixth Painlevé equation defined by

$$\begin{aligned} H^x_{\mathrm{VI}}(\alpha) = x(x-1)(x-t)\left(y - \frac{\alpha_4 - \epsilon_1}{x} - \frac{\alpha_3 - \epsilon_1}{x-1} - \frac{\alpha_0 - \epsilon_2}{x-t}\right) y \\ + (\alpha_2 + \epsilon_1)(\alpha_1 + \alpha_2 + \epsilon_1)x. \end{aligned}$$

Let $H^q_{\mathrm{VI}}(\alpha)$ be defined by replacing x, y, ϵ_1, ϵ_2 in $H^x_{\mathrm{VI}}(\alpha)$ with q, p, ϵ_2, ϵ_1, respectively.

Let us introduce the quantum Lax operators $L_{\mathrm{VI}}(\alpha)$ and $B_{\mathrm{VI}}(\alpha)$ for the sixth Painlevé equation defined by

$$\begin{aligned} L_{\mathrm{VI}}(\alpha) &= H^x_{\mathrm{VI}}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) - H^q_{\mathrm{VI}}(\alpha_0, \alpha_1, \alpha_2 + \kappa, \alpha_3, \alpha_4) \\ &\quad - \frac{\kappa}{x-q}\left(x(x-1)(q-t)y - q(q-1)(x-t)p\right), \qquad (2.1) \\ B_{\mathrm{VI}}(\alpha) &= \epsilon_2 H^x_{\mathrm{VI}}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) - \epsilon_1 H^q_{\mathrm{VI}}(\alpha_0, \alpha_1, \alpha_2 + \kappa, \alpha_3, \alpha_4) \\ &\quad - \kappa\epsilon_1\epsilon_2 t(t-1)d. \qquad (2.2) \end{aligned}$$

Here $\kappa = \epsilon_1 - \epsilon_2$. We use this notation throughout the paper.

Let us recall the extended affine Weyl group $\widetilde{W}(D^{(1)}_4)$ symmetry of the quantum sixth Painlevé equation. Here, $\widetilde{W}(D^{(1)}_4) = W(D^{(1)}_4) \rtimes G$, where $W(D^{(1)}_4) = \langle s_0, s_1, s_2, s_3, s_4 \rangle$ is the affine Weyl group of type $D^{(1)}_4$ and $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is the automorphism group of the Dynkin diagram of type $D^{(1)}_4$.

Definition 2.1 (cf. [14]). Let the automorphisms s^q for $s \in \{s_0, s_1, s_2, s_3, s_4, \sigma_1, \sigma_2, \sigma_3\}$ on $\mathcal{K}$ be defined by the following table:

z	α_0	α_1	α_2	α_3	α_4	q	p	t	d
$s^q_0(z)$	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4	q	$p - \frac{\alpha_0}{q-t}$	t	$d + \frac{\alpha_0/\epsilon_2}{q-t}$
$s^q_1(z)$	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4	q	p	t	d
$s^q_2(z)$	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$	$q + \frac{\alpha_2}{p}$	p	t	d
$s^q_3(z)$	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4	q	$p - \frac{\alpha_3}{q-1}$	t	d
$s^q_4(z)$	α_0	α_1	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$	q	$p - \frac{\alpha_4}{q}$	t	d
$\sigma^q_1(z)$	α_0	α_1	α_2	α_4	α_3	$1 - q$	$-p$	$1 - t$	$-d$
$\sigma^q_2(z)$	α_0	α_4	α_2	α_3	α_1	$\frac{1}{q}$	$-q(pq + \alpha_2)$	$\frac{1}{t}$	$-t^2 d$
$\sigma^q_3(z)$	α_4	α_1	α_2	α_3	α_0	$\frac{t-q}{t-1}$	$-(t-1)p$	$\frac{t}{t-1}$	$(1-t)(q-1)p$ $-(t-1)^2 d$

Let s_i $(0 \le i \le 4)$ be the automorphisms on $\mathcal{K}$ defined by $s_i(\alpha_j) = s^q_i(\alpha_j)$ for $j = 0, \ldots, 4$, and $s_i(f) = f$, for $f = x, y, q, p, t, d$, and let σ_i $(1 \le i \le 3)$

be the automorphisms on $\mathcal{K}$ defined by $\sigma_i(\alpha_j) = \sigma_i^q(\alpha_j)$ for $j = 0, \ldots, 4$, and $\sigma_i(f) = f$ for $f = x, y, q, p, t, d$.

The automorphisms s^q for $s \in \{s_0, s_1, s_2, s_3, s_4, \sigma_1, \sigma_2, \sigma_3\}$ are expressed as compositions of transformations s for parameters and transformations R_s^q for variables, that is, $s^q = s \circ R_s^q$ [17, Theorem 2.4]. The automorphism R_s^q is a Bäcklund transformation for the quantum sixth Painlevé equation, which transforms a solution with the parameter α to a solution with the parameter $s(\alpha)$. As for birational actions of the Weyl group of any symmetrizable generalized Cartan matrix, see [10] and reference therein.

Let $\mathcal{L}_x$, $\mathcal{L}_q$ be the Laplace transformations on $\mathcal{K}$ with respect to x, q, respectively, defined by

$$\mathcal{L}_x\,(y) = x, \quad \mathcal{L}_x\,(x) = -y, \quad \mathcal{L}_q\,(p) = q, \quad \mathcal{L}_q\,(q) = -p.$$

Let $\mathrm{Ad}((x-c)^{\beta/\epsilon_1})$ for $(c \in \mathbb{C},\, \beta \in \mathbb{C})$ be the gauge transformations on $\mathcal{K}$ defined by

$$\mathrm{Ad}((x-c)^{\beta})\,(y) = y - \frac{\beta}{x-c}.$$

Let $\mathrm{Ad}((x-t)^{\beta/\epsilon_1})$ for $(\beta \in \mathbb{C})$ be the gauge transformations on $\mathcal{K}$ defined by

$$\mathrm{Ad}((x-t)^{\beta/\epsilon_1})\,(y) = y - \frac{\beta}{x-t}, \quad \mathrm{Ad}((x-t)^{\beta/\epsilon_1})\,(d) = d + \frac{\beta/\epsilon_1}{x-t}.$$

Here, we have omitted to write the transformation on the variables if it acts identically. The automorphisms $\mathrm{Ad}\left((q-c)^{\beta/\epsilon_2}\right)$, $\mathrm{Ad}\left((q-t)^{\beta/\epsilon_2}\right)$ are defined in the same way above.

Definition 2.2 (cf. [17]). Let the automorphisms $R^x_{s_i}(\alpha_i)$ $(i = 0, 1, 2, 3, 4)$ and R_{σ_i} $(i = 1, 3)$, $R^x_{\sigma_2}(\alpha_2)$ on $\mathcal{K}$ be defined by

$$\begin{aligned}
R^x_{s_0}(\alpha_0) &= \mathrm{Ad}\left((x-t)^{-\frac{\alpha_0}{\epsilon_1}}\right), \quad R^x_{s_1}(\alpha_1) = \mathrm{id},\\
R^x_{s_2}(\alpha_2) &= \mathcal{L}_x^{-1} \circ \mathrm{Ad}\left(x^{-\frac{\alpha_2}{\epsilon_1}}\right) \circ \mathcal{L}_x,\\
R^x_{s_3}(\alpha_3) &= \mathrm{Ad}\left((x-1)^{-\frac{\alpha_3}{\epsilon_1}}\right), \quad R^x_{s_4}(\alpha_4) = \mathrm{Ad}\left(x^{-\frac{\alpha_4}{\epsilon_1}}\right),\\
R_{\sigma_1} &= (x \mapsto 1-x, q \mapsto 1-q, t \mapsto 1-t)\,,\\
R^x_{\sigma_2}(\alpha_2) &= R^x_{s_4}(\alpha_2+\epsilon_1) \circ \left(x \mapsto \frac{1}{x}, q \mapsto \frac{1}{q}, t \mapsto \frac{1}{t}\right),\\
R_{\sigma_3} &= \left(x \mapsto \frac{t-x}{t-1}, q \mapsto \frac{t-q}{t-1}, t \mapsto \frac{t}{t-1}\right).
\end{aligned}$$

Here, $(x \mapsto f(x,t), t \mapsto g(x,t))$ stands for a transformation of variables. The automorphisms $R^q_{s_i}(\alpha_i)$ $(i = 0, 1, 2, 3, 4)$, $R^q_{\sigma_2}(\alpha_2)$ are defined by replacing x, ϵ_1 in $R^x_{s_i}(\alpha_i)$, $R^x_{\sigma_2}(\alpha_2)$ with q, ϵ_2, respectively.

Proposition 2.3 ([14]). *The automorphisms* $R^x_{s_i}(\alpha_i)(i=0,1,2,3,4), R_{\sigma_i}(i=1,3), R^x_{\sigma_2}(\alpha_2)$ *preserve the Hamiltonian* $H^x_{\mathrm{VI}}(\alpha)$ *in the following sense:*

$$\begin{aligned} R^x_{s_i}(\alpha_i)(H^x_{\mathrm{VI}}(\alpha)) &= H^x_{\mathrm{VI}}(s_i(\alpha)) + C_{s_i},\\ R_{\sigma_1}(H^x_{\mathrm{VI}}(\alpha)) &= -H^x_{\mathrm{VI}}(\sigma_1(\alpha)) + C_{\sigma_1},\\ R^x_{\sigma_2}(\alpha_2)(H^x_{\mathrm{VI}}(\alpha)) &= \frac{1}{t}H^x_{\mathrm{VI}}(\sigma_2(\alpha)) + C_{\sigma_2},\\ R_{\sigma_3}(H^x_{\mathrm{VI}}(\alpha)) &= \frac{1}{1-t}H^x_{\mathrm{VI}}(\sigma_3(\alpha)) + C_{\sigma_3}, \end{aligned}$$

where

$$\begin{aligned} C_{s_0} &= \alpha_0\left(\alpha_4 - \epsilon_1 + \kappa x + \kappa\frac{x(x-1)}{t-x}\right),\\ C_{s_1} &= 0,\\ C_{s_2} &= \alpha_2(\alpha_3+\alpha_1+\alpha_2+\epsilon_1+(\alpha_0+\alpha_1+\alpha_2+\epsilon_1+\kappa)t),\\ C_{s_3} &= \alpha_3((\alpha_4-\epsilon_1)t-\kappa x),\\ C_{s_4} &= \alpha_4(\alpha_0-\epsilon_2+(\alpha_3-\epsilon_1)t),\\ C_{\sigma_1} &= (\alpha_2+\epsilon_1)(\alpha_1+\alpha_2+\epsilon_1),\\ C_{\sigma_2} &= \frac{1}{t}(\alpha_2+\epsilon_1)(\alpha_0+\alpha_1+\alpha_2+\kappa+t(\alpha_1+\alpha_2+\alpha_3),\\ C_{\sigma_3} &= \frac{t}{t-1}\left((\alpha_2+\epsilon_1)(\alpha_1+\alpha_2+\epsilon_1)-\kappa(x-1)y\right). \end{aligned}$$

By definition, the automorphisms $R^q_{s_i}(\alpha_i)$ $(i=0,1,2,3,4)$ and R_{σ_i} $(i=1,3)$, $R^q_{\sigma_2}(\alpha_2)$ act the Hamiltonian $H^q_{\mathrm{VI}}(\alpha)$ in the same way above.

Let $D(\alpha_2)$ be defined by

$$D(\alpha_2) = yp + \frac{\alpha_2+\epsilon_1}{x-q}y + \frac{\alpha_2+\epsilon_1}{q-x}p. \tag{2.3}$$

We use this notation throughout the paper.

Definition 2.4. Let the automorphisms R_{s_i} $(i{=}0,1,3,4)$, R_{σ_2}, $T_{s_0s_1s_3s_4s_2}$ and S on $\mathcal{K}$ be defined by

$$\begin{aligned} R_{s_i} &= R^x_{s_i}(\alpha_i)R^q_{s_i}(\alpha_i), \quad R_{\sigma_2} = R^q_{s_4}(\alpha_2+\epsilon_1)R^x_{\sigma_2},\\ T_{s_0s_1s_3s_4s_2} &= R^x_{s_2}(-\alpha_2-\kappa)R^x_{s_0}(\alpha_0)R^x_{s_1}(\alpha_1)R^x_{s_3}(\alpha_3)R^x_{s_4}(\alpha_4)R^q_{s_0}(\bar{\alpha_0})R^q_{s_1}(\bar{\alpha_1})\\ &\quad R^q_{s_3}(\bar{\alpha_3})R^q_{s_4}(\bar{\alpha_4})R^q_{s_2}(\alpha_2+\kappa),\\ S &= \mathrm{Ad}(D(\alpha_2)^{-1})R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2+\kappa), \end{aligned}$$

where $\bar{\alpha_i} = -s_0s_1s_3s_4s_2(\alpha_0) = \alpha_i+\alpha_2+\kappa$ for $i=0,1,3,4$.

These automorphisms R_{s_i} $(i=0,1,3,4)$, R_{σ_2} and $T_{s_0s_1s_3s_4s_2}$ are naturally given by looking at the change of parameters when the automorphisms $R^x_{s_i}(\alpha_i)$, $R^q_{s_i}(\alpha_i)$ act the quantum Lax operators.

Theorem 2.5. *The automorphisms* $R_{s_i}(i=0,1,3,4), T_{s_0s_1s_3s_4s_2}$ *and* S *act the quantum Lax operators* $L_{\mathrm{VI}}(\alpha)$ *and* $B_{\mathrm{VI}}(\alpha)$ *as follows.*

For $s \in \{s_0, s_1, s_3, s_4, \sigma_1, \sigma_2, \sigma_3\}$,

$$R_s\left(L_{\mathrm{VI}}(\alpha), B_{\mathrm{VI}}(\alpha)\right) = c_s\left(L_{\mathrm{VI}}(s(\alpha)), B_{\mathrm{VI}}(s(\alpha)) + f_s\right),$$

where

$$c_{s_i} = 1 \quad (i = 0, 1, 3, 4), \quad c_{\sigma_1} = -1, \quad c_{\sigma_2} = \frac{1}{t}, \quad c_{\sigma_3} = \frac{1}{1-t},$$

and

$$\begin{aligned}
f_{s_0} &= -\kappa\alpha_0(\alpha_4 + (t-1)(\epsilon_1 + \epsilon_2)),\\
f_{s_1} &= 0,\\
f_{s_3} &= -\kappa\alpha_3\alpha_4 t,\\
f_{s_4} &= -\kappa\alpha_4(\alpha_0 - \epsilon_1 - \epsilon_2 + \alpha_3 t),\\
f_{\sigma_1} &= \kappa(\alpha_2 + \epsilon_1)(\alpha_1 + \alpha_2 + \epsilon_1),\\
f_{\sigma_2} &= -\kappa(\alpha_2 + \epsilon_1)(\alpha_0 + \alpha_1 + \alpha_2 - \epsilon_2 + (\alpha_1 + \alpha_2 + \alpha_3 + \epsilon_1)t),\\
f_{\sigma_3} &= \kappa(\alpha_2 + \epsilon_1)(\alpha_1 + \alpha_2 + \epsilon_1)t.
\end{aligned}$$

For the automorphism $T_{s_0s_1s_3s_4s_2}$,

$$\begin{aligned}
l_{T_{s_0s_1s_3s_4s_2}} T_{s_0s_1s_3s_4s_2}\left((x-q)L_{\mathrm{VI}}(\alpha)\right) &= (x-q)L_{\mathrm{VI}}\left(s_0s_1s_3s_4s_2(\alpha)\right),\\
T_{s_0s_1s_3s_4s_2}\left(B_{\mathrm{VI}}(\alpha)\right) &= B_{\mathrm{VI}}\left(s_0s_1s_3s_4s_2(\alpha)\right) + f_{T_{s_0s_1s_3s_4s_2}},
\end{aligned} \tag{2.4}$$

where

$$f_{T_{s_0s_1s_3s_4s_2}} = -\kappa\left((\alpha_2 + \epsilon_1)(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_0 t) + (\alpha_2 + \kappa)(\alpha_1 + \alpha_2 + \epsilon_1)t\right),$$

and $l_{T_{s_0s_1s_3s_4s_2}}$ *is some element in* $\mathcal{K}$ *whose explicit form is given in the proof.*

For the automorphism S,

$$\begin{aligned}
&yp\left(R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2+\kappa)\left((x-q)L_{\mathrm{VI}}(\alpha)\right)\right)D(\alpha_2)\\
&= ((x-q)yp + (\alpha_2 + \kappa - \epsilon_2)y + (\epsilon_1 - \alpha_2)p)\\
&\quad\times\left(D(\alpha_2) - \frac{(\alpha_2+\epsilon_1)(\epsilon_1+\epsilon_2)}{(x-q)^2}\right)L_{\mathrm{VI}}(\tilde{\alpha_0}, \tilde{\alpha_1}, -\alpha_2 - 2\epsilon_1, \tilde{\alpha_3}, \tilde{\alpha_4}),\\
&S\left(B_{\mathrm{VI}}(\alpha) + f_S\right) = B_{\mathrm{VI}}(\tilde{\alpha_0}, \tilde{\alpha_1}, -\alpha_2 - 2\epsilon_1, \tilde{\alpha_3}, \tilde{\alpha_4})\\
&\qquad\qquad - D(\alpha_2)^{-1}\frac{2(\alpha_2+\epsilon_1)\epsilon_1\epsilon_2}{(x-q)^2}L_{\mathrm{VI}}(\tilde{\alpha_0}, \tilde{\alpha_1}, -\alpha_2 - 2\epsilon_1, \tilde{\alpha_3}, \tilde{\alpha_4}),
\end{aligned}$$

where $\tilde{\alpha_i} = \alpha_i + \alpha_2 + \epsilon_1 (i = 0, 1, 3, 4)$ *and*

$$f_S = \kappa(\alpha_2 + \epsilon_1)(\alpha_1 + \alpha_2 + \alpha_3 + \epsilon_1 + (\alpha_0 + \alpha_1 + \alpha_2 - \epsilon_2)t).$$

Proof. A proof follows from direct computation. As an example, we compute (2.4) whose precise form is

$$\begin{aligned}
&((x-q)p - \alpha_2 - \kappa)\, yR^x_{s_2}(-\alpha_2-\kappa)R^x_{s_0}(\alpha_0)R^x_{s_3}(\alpha_3)R^x_{s_4}(\alpha_4)\left((x-q)L_{\mathrm{VI}}(\alpha)\right)\\
&\quad - ((x-q)y + \alpha_2 + \kappa)\, pR^2_{s_2}(-\alpha_2-\kappa)R^q_{s_0}(-\bar{\alpha_0})R^q_{s_3}(-\bar{\alpha_3})R^q_{s_4}(-\bar{\alpha_4})\left((x-q)\right.\\
&\quad \left.\times L_{\mathrm{VI}}(s_0s_1s_3s_4s_2(\alpha))\right) = 0.
\end{aligned} \tag{2.5}$$

From Proposition 2.3, we have

$$
\begin{aligned}
&R^x_{s_0}(\alpha_3)R^x_{s_3}(\alpha_3)R^x_{s_4}(\alpha_0)\left(H^x_{\mathrm{VI}}(\alpha)-\frac{\kappa}{x-q}x(x-1)(q-t)y\right)\\
&\quad = H^x_{\mathrm{VI}}(s_0s_1s_3s_4(\alpha)) - \epsilon_1((\alpha_4+\alpha_3)t+\alpha_4+\alpha_0)\\
&\qquad -\frac{\kappa}{x-q}\left(x(x-1)(q-t)y+\alpha_4 t+(\alpha_4(q-t-1)\right.\\
&\qquad \left.+\alpha_3(q-t)+\alpha_0(q-1))\,x\right).
\end{aligned}
$$

From Proposition 2.3 and above, we have

$$
\begin{aligned}
&R^x_{s_2}(-\alpha_2-\kappa)R^x_{s_0}(\alpha_0)R^x_{s_3}(\alpha_3)R^x_{s_4}(\alpha_4)\left((x-q)H^x_{\mathrm{VI}}(\alpha)-\kappa x(x-1)(q-t)y\right)\\
&\quad =(x-q)\left(H^x_{\mathrm{VI}}(s_0s_1s_3s_4s_2(\alpha))+A_1\right)\\
&\qquad -\kappa((q-t)(x(x-1)y+(\alpha_2+\kappa)(2x-1))+\alpha_4 t+B_1 x)\\
&\qquad +\frac{\alpha_2+\kappa}{y}\left(H^x_{\mathrm{VI}}(s_0s_1s_3s_4s_2(\alpha))+A_1-\kappa(\alpha_2-\epsilon_2)(q-t)-\kappa B_1\right), \quad (2.6)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= -\epsilon_1((\alpha_4+\alpha_3)t+\alpha_4+\alpha_0)\\
&\quad +(\alpha_2+\kappa)(\alpha_3+\alpha_1+\alpha_2+\epsilon_2+(\alpha_0+\alpha_1+\alpha_2+\epsilon_1)t),\\
B_1 &= \alpha_4(q-t-1)+\alpha_3(q-t)+\alpha_0(q-1).
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
&R^q_{s_2}(-\alpha_2-\kappa)R^q_{s_0}(-\bar{\alpha_0})R^q_{s_3}(-\bar{\alpha_3})R^q_{s_4}(-\bar{\alpha_4})\\
&\quad \times\left((x-q)H^q_{\mathrm{VI}}(s_0s_1s_3s_4s_2(\alpha))-\kappa q(q-1)(x-t)p\right)\\
&\quad =(x-q)\left(H^q_{\mathrm{VI}}(\alpha_0,\alpha_1,\alpha_2+\kappa,\alpha_3,\alpha_4)+A_2\right)\\
&\qquad -\kappa((x-t)(q(q-1)p+(\alpha_2+\kappa)(2q-1))-(\alpha_4+\alpha_2+\kappa)t-B_2 q)\\
&\qquad -\frac{\alpha_2+\kappa}{p}\left(H^q_{\mathrm{VI}}(\alpha_0,\alpha_1,\alpha_2+\kappa,\alpha_3,\alpha_4)+A_2+\kappa(\alpha_2+\kappa-\epsilon_2)(x-t)-\kappa B_2\right),
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
A_2 &= -\epsilon_2((\alpha_3+\alpha_1-\kappa+(\alpha_0+\alpha_1-\kappa)t\\
&\quad -(\alpha_2+\kappa)(\alpha_3+\alpha_1+\alpha_2+\epsilon_1+(\alpha_0+\alpha_1+\alpha_2+\epsilon_2)t),\\
B_2 &= \alpha_1(t+1-x)+\alpha_0 t+\alpha_3+\alpha_2 x+\kappa(2x+t+1).
\end{aligned}
$$

We substitute (**??**) and (2.7) into the left-hand side of (2.5) and then we compute it directly using the commutation relations. After straightforward calculations, we obtain the relation (2.5). □

2.2. $\mathrm{P_V}$ Case

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ defined by the generators x, y, q, p, t, d, α_i $(0\le i\le 3)$, ϵ_1,ϵ_2, and the commutation relations:

$$
[y,x]=\epsilon_1,\quad [p,q]=\epsilon_2,\quad [d,t]=1,
$$

and the other commutation relations are zero, and a relation $\alpha_0+\alpha_1+\alpha_2+\alpha_3 = -\epsilon_1+\epsilon_2$.

Let $H^x_{\mathrm{V}}(\alpha)$ $(\alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3))$ be the Hamiltonian for the quantum fifth Painlevé equation defined by

$$H^x_{\mathrm{V}}(\alpha) = (x-1)(y+t)xy - (\alpha_1+\alpha_3-\epsilon_1)xy + \alpha_1 y + (\alpha_2+\epsilon_1)tx.$$

Let $H^q_{\mathrm{V}}(\alpha)$ be defined by replacing x, y, ϵ_1 in $H^x_{\mathrm{V}}(\alpha)$ with q, p, ϵ_2, respectively.

Let us introduce the quantum Lax operators $L_{\mathrm{V}}(\alpha)$ and $B_V(\alpha)$ for the fifth Painlevé equation defined by

$$\begin{aligned} L_{\mathrm{V}}(\alpha) &= H^x_{\mathrm{V}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3) - H^q_{\mathrm{V}}(\alpha_0+\kappa,\alpha_1,\alpha_2+\kappa,\alpha_3)\\ &\quad - \frac{\kappa}{x-q}(x(x-1)y - q(q-1)p),\\ B_V(\alpha) &= \epsilon_2 H^x_{\mathrm{V}}(\alpha_0,\alpha_1,\alpha_2,\alpha_3) - \epsilon_1 H^q_{\mathrm{V}}(\alpha_0+\kappa,\alpha_1,\alpha_2+\kappa,\alpha_3) - \kappa\epsilon_1\epsilon_2 td. \end{aligned}$$

Let us recall the extended affine Weyl group $\widetilde{W}(A^{(1)}_3)$ symmetry of the quantum fifth Painlevé equation. Here, $\widetilde{W}(A^{(1)}_3) = W(A^{(1)}_3) \rtimes G$, where $W(A^{(1)}_3) = \langle s_0, s_1, s_2, s_3\rangle$ is the affine Weyl group of type $A^{(1)}_3$ and $G = \langle \pi,\sigma\rangle$ is the automorphism group of the Dynkin diagram of type $A^{(1)}_3$.

Definition 2.6 (cf. [12]). Let the automorphisms s^q for $s \in \{s_0, s_1, s_2, s_3, \pi, \sigma\}$ on $\mathcal{K}$ be defined by the following table:

z	α_0	α_1	α_2	α_3	q	p	t	d
$s^q_0(z)$	$-\alpha_0$	$\alpha_1+\alpha_0$	α_2	$\alpha_3+\alpha_0$	$q+\frac{\alpha_0}{p+t}$	p	t	$d-\frac{\alpha_0/\epsilon_2}{p+t}$
$s^q_1(z)$	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	q	$p-\frac{\alpha_1}{q}$	t	d
$s^q_2(z)$	α_0	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$q+\frac{\alpha_2}{p}$	p	t	d
$s^q_3(z)$	$\alpha_0+\alpha_3$	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	q	$p-\frac{\alpha_3}{q-1}$	t	d
$\pi^q(z)$	α_1	α_2	α_3	α_0	$-\frac{p}{t}$	$t(q-1)$	t	$d+\frac{(1-q)}{\epsilon_2 t}p$
$\sigma^q(z)$	α_2	α_1	α_0	α_3	q	$p+t$	$-t$	$-d-q/\epsilon_2$

Definition 2.7. Let the automorphisms $R^x_{s_i}(\alpha_i)$ $(i = 0,1,2,3)$, R^x_π, R_σ on $\mathcal{K}$ be defined by

$$\begin{aligned} &R^x_{s_0}(\alpha_0) = \mathcal{L}_x^{-1}\circ \mathrm{Ad}\left((x+t)^{-\frac{\alpha_0}{\epsilon_1}}\right)\circ\mathcal{L}_x, \quad R^x_{s_1}(\alpha_1) = \mathrm{Ad}\left(x^{-\frac{\alpha_1}{\epsilon_1}}\right),\\ &R^x_{s_2}(\alpha_2) = \mathcal{L}_x^{-1}\circ \mathrm{Ad}\left(x^{-\frac{\alpha_2}{\epsilon_1}}\right)\circ\mathcal{L}_x, \quad R^x_{s_3}(\alpha_3) = \mathrm{Ad}\left((x-1)^{-\frac{\alpha_3}{\epsilon_1}}\right),\\ &\quad R^x_\pi = (x\mapsto t(x-1))\circ\mathcal{L}_x,\\ &\quad R_\sigma = (t\mapsto -t)\circ \mathrm{Ad}\left(\exp\left(\frac{xt}{\epsilon_1}\right)\right)\circ \mathrm{Ad}\left(\exp\left(\frac{qt}{\epsilon_2}\right)\right). \end{aligned}$$

The automorphisms $R^q_{s_i}(\alpha_i)$ $(i = 0,1,2,3)$, R^q_π are defined by replacing x, ϵ_1 in $R^x_{s_i}(\alpha_i)$, R^x_π with q, ϵ_2, respectively.

Proposition 2.8 [12,17]. *The automorphisms $R^x_{s_i}(\alpha_i)(i = 0,1,2,3), R^x_\pi$, R_σ preserve the Hamiltonian $H^x_{\mathrm{V}}(\alpha)$ in the following sense.*

$$\begin{aligned} &R^x_{s_i}(\alpha_i)\left(H^x_V(\alpha)\right) = H^x_V(s_i(\alpha)) + C_{s_i},\\ &\quad R^x_\pi\left(H^x_V(\alpha)\right) = H^x_V(\pi^{-1}(\alpha)) + C_\pi, \quad R_\sigma\left(H^x_V(\alpha)\right) = H^x_V(\sigma(\alpha)) + C_\sigma, \end{aligned}$$

where

$$\begin{aligned}
C_{s_0} &= -\alpha_0(\alpha_2 + 2\epsilon_1 - \epsilon_2) + \kappa t \frac{\alpha_0}{y+t},\\
C_{s_1} &= -\alpha_1(\alpha_3 + t - \epsilon_1),\\
C_{s_2} &= -\alpha_2(\alpha_0 + 2\epsilon_1 - \epsilon_2 + t),\\
C_{s_3} &= -\alpha_3(\alpha_1 - \epsilon_1),\\
C_{\pi} &= \alpha_3\epsilon_1 + \alpha_1(\epsilon_1 - t) - \kappa\epsilon_1(x-1),\\
C_{\sigma} &= (\alpha_1 - \epsilon_1 + \kappa x)t.
\end{aligned}$$

By definition, the automorphisms $R^q_{s_i}(\alpha_i)$ $(i = 0, 1, 2, 3)$, R^q_π and R_σ act the Hamiltonian $H^q_{\mathrm{V}}(\alpha)$ in the same way above.

Definition 2.9. Let the automorphisms R_{s_i} $(i = 1, 3)$, R_π, $T_{\sigma s_1 s_3 s_2}$, $T_{s_1 s_2 s_3 \pi^{-1}}$, S on $\mathcal{K}$ be defined by

$$\begin{aligned}
R_{s_i} &= R^x_{s_i}(\alpha_i)R^q_{s_i}(\alpha_i), \quad R_\pi = R^x_\pi R^q_\pi,\\
T_{\sigma s_1 s_3 s_2} &= R^x_{s_2}(-\alpha_2 - \kappa)R^x_{s_1}(\alpha_1)R^x_{s_3}(\alpha_3)R_\sigma R^q_{s_1}(\bar{\alpha_1})R^q_{s_3}(\bar{\alpha_3})R^q_{s_2}(\alpha_2 + \kappa),\\
T_{s_1 s_2 s_3 \pi^{-1}} &= R^x_\pi R^x_{s_3}(s_1 s_2(\alpha_3))R^x_{s_2}(s_1(\alpha_2))R^x_{s_1}(\alpha_1)R^q_{s_1}(-\alpha_1 + \kappa)\\
&\quad \times R^q_{s_2}(-s_1(\alpha_2))R^q_{s_3}(-s_1 s_2(\alpha_3))\left(R^q_\pi\right)^{-1},\\
S &= \mathrm{Ad}(D(\alpha_2)^{-1})R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2 + \kappa),
\end{aligned}$$

where $\bar{\alpha_i} = -\sigma s_1 s_3 s_2(\alpha_i) = \alpha_i + \alpha_2 + \kappa$ for $i = 1, 3$, and $D(\alpha_2)$ is given in (**??**).

Theorem 2.10. *The automorphisms* $R_{s_i}(i = 1, 3), R_\sigma$, R^2_π, $T_{\sigma s_1 s_3 s_2}, T_{s_1 s_2 s_3 \pi^{-1}}$ *and* S *act the quantum Lax operators* $L_{\mathrm{V}}(\alpha)$ *and* $B_{\mathrm{V}}(\alpha)$ *as follows.*

For the automorphisms $R_s(s \in \{s_1, s_3, \sigma\})$,

$$R_s\left(L_{\mathrm{V}}(\alpha), B_{\mathrm{V}}(\alpha)\right) = \left(L_{\mathrm{V}}(s(\alpha)), B_{\mathrm{V}}(s(\alpha)) + f_s\right),$$

where

$$f_{s_1} = \kappa\alpha_1(\alpha_3 + t), \quad f_{s_3} = \kappa\alpha_1\alpha_3, \quad f_\sigma = -\kappa\alpha_1 t.$$

For the automorphism R^2_π,

$$\begin{aligned}
&R^2_\pi\left((x-q)L_{\mathrm{V}}(\alpha), B_{\mathrm{V}}(\alpha)\right)\\
&\quad = \left((q-x)L_{\mathrm{V}}(\pi^2(\alpha)), B_{\mathrm{V}}(\pi^2(\alpha)) - \kappa(\alpha_2 + \alpha_3 + \kappa)t\right).
\end{aligned}$$

For the automorphisms $T_r(r \in \{\sigma s_1 s_3 s_2, s_1 s_2 s_3 \pi^{-1}\})$,

$$\begin{aligned}
l_{T_r} T_r\left((x-q)L_{\mathrm{V}}(\alpha)\right) &= (x-q)L_{\mathrm{V}}\left(r(\alpha)\right),\\
T_r\left(B_{\mathrm{V}}(\alpha)\right) &= B_{\mathrm{V}}\left(r(\alpha)\right) + f_{T_r},
\end{aligned} \tag{2.8}$$

where

$$f_{T_{\sigma s_1 s_3 s_2}} = \kappa(\alpha_2 - \kappa)(\alpha_0 - t), \quad f_{T_{s_1 s_2 s_3 \pi^{-1}}} = 0,$$

and $l_{T_{\sigma s_1 s_3 s_2}}$, $l_{T_{s_1 s_2 s_3 \pi^{-1}}}$ *are some elements in* $\mathcal{K}$ *whose explicit forms are given in the proof.*

For the automorphism S,

$$
\begin{aligned}
&yp\left(R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2+\kappa)\left((x-q)L_{\mathrm{V}}(\alpha)\right)\right)D(\alpha_2)\\
&\quad=((x-q)yp+(\alpha_2+\kappa-\epsilon_2)y+(\epsilon_1-\alpha_2)p)\\
&\qquad\times\left(D(\alpha_2)-\frac{(\alpha_2+\epsilon_1)(\epsilon_1+\epsilon_2)}{(x-q)^2}\right)L_{\mathrm{V}}(\alpha_0,\tilde{\alpha_1},-\alpha_2-2\epsilon_1,\tilde{\alpha_3}),\\
&S\left(B_{\mathrm{V}}(\alpha)+f_S\right)=B_{\mathrm{V}}(\alpha_0,\tilde{\alpha_1},-\alpha_2-2\epsilon_1,\tilde{\alpha_3})\\
&\qquad\qquad-D(\alpha_2)^{-1}\frac{2(\alpha_2+\epsilon_1)\epsilon_1\epsilon_2}{(x-q)^2}L_{\mathrm{V}}(\alpha_0,\tilde{\alpha_1},-\alpha_2-2\epsilon_1,\tilde{\alpha_3}),
\end{aligned}
$$

where $\tilde{\alpha_i}=\alpha_i+\alpha_2+\epsilon_1$ $(i=1,3)$ and

$$f_S=\kappa(\alpha_2+\epsilon_1)(\alpha_1+\alpha_2+\alpha_3+\epsilon_1+t).$$

Proof. For the cases of the automorphisms T_r ($r\in\{\sigma s_1s_3s_2,s_1s_2s_3\pi^{-1}\}$ acting $L_{\mathrm{V}}(\alpha)$, we show that

$$
\begin{aligned}
&((x-q)p-\alpha_2-\kappa)\,yR^x_{s_2}(-\alpha_2-\kappa)R^x_{s_1}(\alpha_1)R^x_{s_3}(\alpha_3)\mathrm{Ad}\\
&\qquad\times\left(\exp\left(-\frac{xt}{\epsilon_1}\right)\right)((x-q)L_{\mathrm{V}}(\alpha))\\
&\quad=((x-q)y+\alpha_2+\kappa)\,pR^2_{s_2}(-\alpha_2-\kappa)R^q_{s_1}(\bar{\alpha_1})R^q_{s_3}(\bar{\alpha_3})\\
&\qquad\circ(t\mapsto-t)\mathrm{Ad}\left(\exp\left(-\frac{qt}{\epsilon_2}\right)\right)((x-q)L_{\mathrm{V}}(\sigma s_1s_3s_2(\alpha))\,,
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&AR^x_\pi\left((x-1)R^x_{s_3}(s_1s_2(\alpha_3))\left(yR^x_{s_2}(s_1(\alpha_2))R^x_{s_1}(\alpha_1)\left((x-q)L_{\mathrm{V}}(\alpha)\right)\right)\right)\\
&\quad=BR^q_\pi\left((q-1)R^q_{s_3}(s_1s_2(\alpha_3))\left(pR^q_{s_2}(s_1(\alpha_2))R^q_{s_1}(\alpha_1-\kappa)\right.\right.\\
&\qquad\left.\left.\times\left((x-q)L_{\mathrm{V}}(s_1s_2s_3\pi^{-1}(\alpha))\right)\right)\right),
\end{aligned}
\tag{2.10}
$$

which are the explicit forms of (2.8). Here A, B are elements in $\mathcal{K}$ such that

$$
\begin{aligned}
A&=a_{0,3}p^3-\frac{q-1}{x-1}yp^2+a_{0,2}p^2\\
&\quad+(\alpha_3-\epsilon_2+t(1-q)(1+x))yp+a_{1,0}y+a_{0,1}p+a_{0,0},\\
B&=y^3+b_{2,1}y^2p+b_{2,0}y^2+b_{1,1}yp+b_{1,0}y+b_{0,1}p+b_{0,0}.
\end{aligned}
$$

where $a_{i,j}$, $b_{i,j}$ are rational functions of x, q, t, α_i $(i=1,2,3)$, ϵ_1, ϵ_2. We omit the proofs of (2.9), (2.10), since they are similar to that of Theorem 2.5.

Proofs of the other cases follow from direct computations using Proposition 2.8. □

Actions involving $R^x_{s_0}(\alpha_0)$, $R^q_{s_0}(\alpha_0)$ on the quantum Lax operators can be obtained from Theorem 2.10, because of the relations

$$R_\sigma R^x_{s_0}(\alpha_2)R_\sigma=R^x_{s_2}(\alpha_2),\quad R_\sigma R^q_{s_0}(\alpha_2)R_\sigma=R^q_{s_2}(\alpha_2).$$

2.3. $\mathbf{P_{IV}}$ Case

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ defined by the generators x, y, q, p, t, d, α_i $(0\le i\le 2)$, ϵ_1,ϵ_2, and the commutation relations:

$$[y,x]=\epsilon_1,\quad[p,q]=\epsilon_2,\quad[d,t]=1,$$

and the other commutation relations are zero, and a relation $\alpha_0 + \alpha_1 + \alpha_2 = -\epsilon_1 + \epsilon_2$.

Let $H^x_{\mathrm{IV}}(\alpha)$ ($\alpha = (\alpha_0, \alpha_1, \alpha_2)$) be the Hamiltonian for the quantum fourth Painlevé equation defined by

$$H^x_{\mathrm{IV}}(\alpha) = yxy - xyx - txy - \alpha_2 x - \alpha_1 y.$$

Let $H^q_{\mathrm{IV}}(\alpha)$ be defined by replacing x, y, ϵ_1, ϵ_2 in $H^x_{\mathrm{IV}}(\alpha)$ with q, p, ϵ_2, ϵ_1, respectively.

Let us introduce the quantum Lax operators $L_{\mathrm{IV}}(\alpha)$ and $B_{\mathrm{IV}}(\alpha)$ for the fourth Painlevé equation defined by

$$\begin{aligned} L_{\mathrm{IV}}(\alpha) &= H^x_{\mathrm{IV}}(\alpha_0, \alpha_1, \alpha_2) - H^q_{\mathrm{IV}}(\alpha_0 + \kappa, \alpha_1, \alpha_2 + \kappa) - \frac{\kappa}{x-q}(xy - qp), \\ B_{\mathrm{IV}}(\alpha) &= \epsilon_2 H^x_{\mathrm{IV}}(\alpha_0, \alpha_1, \alpha_2) - \epsilon_1 H^q_{\mathrm{IV}}(\alpha_0 + \kappa, \alpha_1, \alpha_2 + \kappa) - \kappa\epsilon_1\epsilon_2 d. \end{aligned}$$

Let us recall the extended affine Weyl group $\widetilde{W}(A^{(1)}_2)$ symmetry of the quantum fourth Painlevé equation. Here, $\widetilde{W}(A^{(1)}_2) = W(A^{(1)}_2) \rtimes G$, where $W(A^{(1)}_2) = \langle s_0, s_1, s_2 \rangle$ is the affine Weyl group of type $A^{(1)}_2$ and $G = \langle \pi, \sigma \rangle$ is the automorphism group of the Dynkin diagram of type $A^{(1)}_2$.

Definition 2.11 (cf. [12]). Let the automorphisms s^q for $s \in \{s_0, s_1, s_2, \pi, \sigma\}$ on $\mathcal{K}$ be defined by the following table:

z	α_0	α_1	α_2	q	p	t	d
$s^q_0(z)$	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	$q + \frac{\alpha_0}{p-q-t}$	$p + \frac{\alpha_0}{p-q-t}$	t	$d + \frac{\alpha_0/\epsilon_2}{p-q-t}$
$s^q_1(z)$	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	q	$p - \frac{\alpha_1}{q}$	t	d
$s^q_2(z)$	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$q + \frac{\alpha_2}{p}$	p	t	d
$\pi^q(z)$	α_1	α_2	α_0	$-p$	$-p + q + t$	t	$d - p$
$\sigma^q(z)$	α_2	α_1	α_0	$\sqrt{-1}q$	$-\sqrt{-1}(p - q - t)$	$\sqrt{-1}t$	$\sqrt{-1}(-d + \frac{q}{\epsilon_2})$

Definition 2.12 (cf. [17]). Let the automorphisms $R^x_{s_i}(\alpha_i)$ ($i = 0, 1, 2$), R^x_π, R_σ on $\mathcal{K}$ be defined by

$$\begin{aligned} R^x_{s_0}(\alpha_0) &= \mathrm{Ad}\left(\exp\left(\left(\frac{x^2}{2} + xt\right)\frac{1}{\epsilon_1}\right)\right) \circ \mathcal{L}_x^{-1} \circ \mathrm{Ad}(x^{-\frac{\alpha_0}{\epsilon_1}}) \\ &\quad \circ \mathcal{L}_x \circ \mathrm{Ad}\left(\exp\left(\left(-\frac{x^2}{2} - xt\right)\frac{1}{\epsilon_1}\right)\right), \\ R^x_{s_1}(\alpha_1) &= \mathrm{Ad}(x^{-\frac{\alpha_1}{\epsilon_1}}), \quad R^x_{s_2}(\alpha_2) = \mathcal{L}_x^{-1} \circ \mathrm{Ad}(x^{-\frac{\alpha_2}{\epsilon_1}}) \circ \mathcal{L}_x, \\ R^x_\pi &= \mathcal{L}_x \circ \mathrm{Ad}\left(\exp\left(\left(-\frac{x^2}{2} - xt\right)\frac{1}{\epsilon_1}\right)\right), \\ R_\sigma &= \left(x \mapsto \sqrt{-1}x, q \mapsto \sqrt{-1}q, t \mapsto \sqrt{-1}t\right) \circ \mathrm{Ad}\left(\exp\left(\left(-\frac{x^2}{2} - xt\right)\frac{1}{\epsilon_1}\right)\right) \\ &\quad \circ \mathrm{Ad}\left(\exp\left(\left(-\frac{q^2}{2} - qt\right)\frac{1}{\epsilon_2}\right)\right). \end{aligned}$$

The automorphisms $R^q_{s_i}(\alpha_i)$ ($i = 0, 1, 2$), R^q_π are defined by replacing x, ϵ_1 in $R^x_{s_i}(\alpha_i)$, R^x_π with q, ϵ_2, respectively.

Proposition 2.13 [12,17]. *The automorphisms $R^x_{s_i}(\alpha_i)(i=0,1,2), R^x_\pi$, R_σ preserve the Hamiltonian $H^x_{\mathrm{IV}}(\alpha)$ in the following sense.*

$$\begin{aligned} R^x_{s_i}(\alpha_i)\left(H^x_{\mathrm{IV}}(\alpha)\right) &= H^x_{\mathrm{IV}}(s_i(\alpha)) + C_{s_i},\\ R^x_\pi\left(H^x_{\mathrm{IV}}(\alpha)\right) &= H^x_{\mathrm{IV}}(\pi^{-1}(\alpha)) + C_\pi,\\ R_\sigma\left(H^x_{\mathrm{IV}}(\alpha)\right) &= -\sqrt{-1}H^x_{\mathrm{IV}}(\sigma(\alpha)) + C_\sigma, \end{aligned}$$

where

$$\begin{aligned} &C_{s_0} = -\frac{\kappa\alpha_0}{y-x-t}, \quad C_{s_1} = -\alpha_1 t, \quad C_{s_2} = \alpha_2 t,\\ &C_\pi = -\alpha_1 t - \kappa y, \quad C_\sigma = -\sqrt{-1}((\alpha_1-\epsilon_1)t - \kappa x). \end{aligned}$$

By definition, the automorphisms $R^q_{s_i}(\alpha_i)$ $(i=0,1,2)$, R^q_π and R_σ act the Hamiltonian $H^q_{\mathrm{IV}}(\alpha)$ in the same way above.

Definition 2.14. Let the automorphisms $R_{s_1}, T_{\sigma s_1 s_2}, T_{s_1 s_2 \pi^{-1}}, S$ on $\mathcal{K}$ be defined by

$$\begin{aligned} R_{s_1} &= R^x_{s_1}(\alpha_1)R^q_{s_1}(\alpha_1),\\ T_{\sigma s_1 s_2} &= R^x_{s_2}(-\alpha_2-\kappa)R^x_{s_1}(\alpha_1)R_\sigma R^q_{s_1}(-\sigma s_1 s_2(\alpha_1))R^q_{s_2}(\alpha_2+\kappa),\\ T_{s_1 s_2 \pi^{-1}} &= R^x_\pi R^x_{s_2}(\alpha_1+\alpha_2)R^x_{s_1}(\alpha_1)R^q_{s_1}(-\alpha_1+\kappa)R^q_{s_2}(\alpha_0+\kappa)\left(R^q_\pi\right)^{-1},\\ S &= \mathrm{Ad}(D(\alpha_2)^{-1})R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2+\kappa). \end{aligned}$$

Theorem 2.15. *The automorphisms R_{s_1}, R_σ, $T_{\sigma s_1 s_2}$, $T_{s_1 s_2 \pi^{-1}}$ and S act the quantum Lax operators $L_{\mathrm{IV}}(\alpha)$ and $B_{\mathrm{IV}}(\alpha)$ as follows.*

For the automorphisms $R_s(s\in\{s_1,\sigma\})$,

$$R_s\left(L_{\mathrm{IV}}(\alpha), B_{\mathrm{IV}}(\alpha)\right) = c_s\left(L_{\mathrm{IV}}(s(\alpha)), B_{\mathrm{IV}}(s(\alpha)) + f_s\right),$$

where

$$c_{s_1} = 1, \quad c_\sigma = -\sqrt{-1}, \quad f_{s_1} = \kappa\alpha_1 t, \quad f_\sigma = -\kappa\alpha_1 t.$$

For the automorphism $T_r(r\in\{\sigma s_1 s_2, s_1 s_2\pi^{-1})\}$,

$$\begin{aligned} l_{T_r}T_r\left((x-q)L_{\mathrm{IV}}(\alpha)\right) &= (x-q)L_{\mathrm{IV}}\left(r(\alpha)\right),\\ T_r\left(B_{\mathrm{IV}}(\alpha)\right) &= -\sqrt{-1}B_{\mathrm{IV}}\left(r(\alpha)\right) + f_{T_r}, \end{aligned} \tag{2.11}$$

where

$$f_{T_{\sigma s_1 s_2}} = -\kappa(\alpha_2+\kappa)t, \quad f_{T_{s_1 s_2\pi^{-1}}} = 0,$$

and $l_{T_{\sigma s_1 s_2}}$, $l_{T_{s_1 s_2 \pi^{-1}}}$ are some elements in $\mathcal{K}$ whose explicit forms are given in the proof.

For the automorphism S,

$$\begin{aligned} &yp\left(R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2+\kappa)\left((x-q)L_{\mathrm{IV}}(\alpha)\right)\right)D(\alpha_2)\\ &\quad=((x-q)yp+(\alpha_2+\kappa-\epsilon_2)y+(\epsilon_1-\alpha_2)p)\\ &\qquad\times\left(D(\alpha_2)-\frac{(\alpha_2+\epsilon_1)(\epsilon_1+\epsilon_2)}{(x-q)^2}\right)L_{\mathrm{IV}}(\alpha_0,\tilde{\alpha_1},-\alpha_2-2\epsilon_1),\\ &S\left(B_{\mathrm{IV}}(\alpha)+f_S\right)=B_{\mathrm{IV}}(\alpha_0,\tilde{\alpha_1},-\alpha_2-2\epsilon_1)\\ &\qquad\qquad-D(\alpha_2)^{-1}\frac{2(\alpha_2+\epsilon_1)\epsilon_1\epsilon_2}{(x-q)^2}L_{\mathrm{IV}}(\alpha_0,\tilde{\alpha_1},-\alpha_2-2\epsilon_1),\end{aligned}$$

where $\tilde{\alpha_1}=\alpha_1+\alpha_2+\epsilon_1$ *and*

$$f_S=\kappa(\alpha_2+\epsilon_1)t.$$

Proof. For the cases of the automorphisms $T_r(r\in\{\sigma s_1s_2,s_1s_2\pi^{-1}\})$ acting $L_{\mathrm{IV}}(\alpha)$, we show that

$$\begin{aligned} &((x-q)p-\alpha_2-\kappa)\,yR^x_{s_2}(-\alpha_2-\kappa)R^x_{s_1}(\alpha_1)\mathrm{Ad}\left(\exp\left(\left(-\frac{x^2}{2}-xt\right)\frac{1}{\epsilon_1}\right)\right)\\ &\quad\times((x-q)L_{\mathrm{IV}}(\alpha))\\ &=((x-q)y+\alpha_2+\kappa)\,pR^2_{s_2}(-\alpha_2-\kappa)R^q_{s_1}(\sigma s_1s_2(\alpha_1))\\ &\quad\circ\left(x\mapsto\sqrt{-1}x,q\mapsto\sqrt{-1}q,t\mapsto\sqrt{-1}t\right)\mathrm{Ad}\left(\exp\left(\left(-\frac{q^2}{2}-qt\right)\frac{1}{\epsilon_2}\right)\right)\\ &\quad\times((x-q)L_{\mathrm{IV}}(\sigma s_1s_2(\alpha))\,, \end{aligned}\tag{2.12}$$

$$\begin{aligned} &AR^x_\pi\left(yR^x_{s_2}(s_1(\alpha_2))R^x_{s_1}(\alpha_1)\left((x-q)L_{\mathrm{IV}}(\alpha)\right)\right)\\ &=BR^q_\pi\left(pR^q_{s_2}(s_1(\alpha_2))R^q_{s_1}(\alpha_1-\kappa)\left((x-q)L_{\mathrm{IV}}(s_1s_2\pi^{-1}(\alpha))\right)\right), \end{aligned}\tag{2.13}$$

which is the explicit form of (2.11). Here A, B are elements in $\mathcal{K}$ such that

$$\begin{aligned} A&=a_{0,3}p^3-yp^2+a_{0,2}p^2+(q-x+t)yp\\ &\quad+(\alpha_1+\alpha_2+(q+t)x)y+a_{0,1}p+a_{0,0},\\ B&=b_{3,0}y^3+b_{2,1}y^2p+b_{2,0}y^2+b_{1,1}yp+b_{1,0}y+b_{0,1}p+b_{0,0}. \end{aligned}$$

where $a_{i,j}$, $b_{i,j}$ are rational functions of x, q, t, α_i $(i=1,2,3)$, ϵ_1, ϵ_2. We omit the proofs of (2.12), (2.13), since they are similar to that of Theorem 2.5.

Proofs of the other cases follow from direct computations using Proposition 2.13. □

Actions involving $R^x_{s_0}(\alpha_0)$, $R^q_{s_0}(\alpha_0)$ on the quantum Lax operators can be obtained from Theorem 2.15, because of the relations

$$(R_\sigma)^{-1}R^x_{s_0}(\alpha_2)R_\sigma=R^x_{s_2}(\alpha_2),\quad (R_\sigma)^{-1}R^q_{s_0}(\alpha_2)R_\sigma=R^q_{s_2}(\alpha_2).$$

2.4. $\mathbf{P_{III}}$ Case

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ defined by the generators x, y, q, p, t, d, α_i $(0\le i\le 2)$, ϵ_1,ϵ_2, and the commutation relations:

$$[y,x]=\epsilon_1,\quad [p,q]=\epsilon_2,\quad [d,t]=1,$$

and the other commutation relations are zero, and a relation $\alpha_0 + 2\alpha_1 + \alpha_2 = -\epsilon_1 + \epsilon_2$.

Let $H^x_{\mathrm{III}}(\alpha)$ ($\alpha = (\alpha_0, \alpha_1, \alpha_2)$) be the Hamiltonian for the quantum third Painlevé equation defined by

$$H^x_{\mathrm{III}}(\alpha) = xyxy - xyx + (\alpha_0 + \alpha_2 + \epsilon_1)xy - \alpha_2 x + ty.$$

Let $H^q_{\mathrm{III}}(\alpha)$ be defined by replacing x, y, ϵ_1, ϵ_2 in $H^x_{\mathrm{III}}(\alpha)$ with q, p, ϵ_2, ϵ_1, respectively.

Let us introduce the quantum Lax operators $L_{\mathrm{III}}(\alpha)$ and $B_{\mathrm{III}}(\alpha)$ for the third Painlevé equation defined by

$$L_{\mathrm{III}}(\alpha) = H^x_{\mathrm{III}}(\alpha_0, \alpha_1, \alpha_2) - H^q_{\mathrm{III}}(\alpha_0 + \kappa, \alpha_1, \alpha_2 + \kappa) - \frac{\kappa x q}{x - q}(y - p),$$

$$B_{\mathrm{III}}(\alpha) = \epsilon_2 H^x_{\mathrm{III}}(\alpha_0, \alpha_1, \alpha_2) - \epsilon_1 H^q_{\mathrm{III}}(\alpha_0 + \kappa, \alpha_1, \alpha_2 + \kappa) - \kappa\epsilon_1\epsilon_2 t d.$$

Let us recall the extended affine Weyl group $\widetilde{W}(C_2^{(1)})$ symmetry of the quantum third Painlevé equation. Here, $\widetilde{W}(C_2^{(1)}) = W(C_2^{(1)}) \rtimes G$, where $W(C_2^{(1)}) = \langle s_0, s_1, s_2 \rangle$ is the affine Weyl group of type $C_2^{(1)}$ and $G = \langle \sigma \rangle$ is the automorphism group of the Dynkin diagram of type $C_2^{(1)}$.

Definition 2.16 (cf. [7,13]). Let the automorphisms s^q for $s \in \{s_0, s_1, s_2, \sigma\}$ on $\mathcal{K}$ be defined by the following table:

z	α_0	α_1	α_2	q	p	z	d
$s_0^q(z)$	$-\alpha_0$	$\alpha_1 + \alpha_0$	α_2	$q + \frac{\alpha_0}{p-1}$	p	t	d
$s_1^q(z)$	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$\alpha_2 + 2\alpha_1$	q	$p - \frac{2\alpha_1}{q} + \frac{t}{q^2}$	$-t$	$-d + \frac{1}{\epsilon_2 q}$
$s_2^q(z)$	α_0	$\alpha_1 + \alpha_2$	$-\alpha_2$	$q + \frac{\alpha_2}{p}$	p	t	d
$\sigma^q(z)$	α_2	α_1	α_0	$-q$	$1 - p$	$-t$	$-d$

Definition 2.17 (cf. [17]). Let the automorphisms $R^x_{s_i}(\alpha_i)$ ($i = 0, 1, 2$), R_σ on $\mathcal{K}$ be defined by

$$R^x_{s_0}(\alpha_0) = \mathcal{L}_x^{-1} \circ \mathrm{Ad}\left((x-1)^{-\frac{\alpha_0}{\epsilon_1}}\right) \circ \mathcal{L}_x,$$

$$R^x_{s_1}(\alpha_1) = (t \mapsto -t) \circ \mathrm{Ad}\left(\exp\left(-\frac{t}{\epsilon_1 x}\right) x^{-\frac{2\alpha_1}{\epsilon_1}}\right),$$

$$R^x_{s_2}(\alpha_2) = \mathcal{L}_x^{-1} \circ \mathrm{Ad}\left(x^{-\frac{\alpha_2}{\epsilon_1}}\right) \circ \mathcal{L}_x,$$

$$R_\sigma = (x \mapsto -x, q \mapsto -q,\ t \mapsto -t) \circ \mathrm{Ad}\left(\exp\left(-\frac{x}{\epsilon_1}\right)\right) \circ \mathrm{Ad}\left(\exp\left(-\frac{q}{\epsilon_2}\right)\right).$$

The automorphisms $R^q_{s_i}(\alpha_i)$ ($i = 0, 1, 2$) are defined by replacing x, ϵ_1 in $R^x_{s_i}(\alpha_i)$, with q, ϵ_2, respectively.

Proposition 2.18 [17]. *The automorphisms $R^x_{s_i}(\alpha_i)(i = 0, 1, 2), R_\sigma$ preserve the Hamiltonian $H^x_{\mathrm{III}}(\alpha)$ in the following sense.*

$$R^x_{s_i}(\alpha_i)\left(H^x_{\mathrm{III}}(\alpha)\right) = H^x_{\mathrm{III}}(s_i(\alpha)) + C_{s_i},$$

$$R_\sigma\left(H^x_{\mathrm{III}}(\alpha)\right) = H^x_{\mathrm{III}}(\sigma(\alpha)) + C_\sigma,$$

where

$$C_{s_0} = -(\alpha_0 + \epsilon_1)(\alpha_2 + \epsilon_1), \quad C_{s_1} = 2\alpha_1\epsilon_2 - t - \frac{\kappa t}{x},$$
$$C_{s_2} = -\alpha_2(\alpha_0 + \epsilon_1), \quad C_\sigma = \kappa t.$$

By definition, the automorphisms $R^q_{s_i}(\alpha_i)$ $(i = 0, 1, 2)$ and R_σ act the Hamiltonian $H^q_{\mathrm{III}}(\alpha)$ in the same way above.

Definition 2.19. Let the automorphisms R_{s_1}, $T_{\sigma s_1 s_2}$, S on $\mathcal{K}$ be defined by

$$\begin{aligned} R_{s_1} &= R^x_{s_1}(\alpha_1)R^q_{s_1}(\alpha_1), \\ T_{\sigma s_1 s_2} &= R^x_{s_2}(-\alpha_2 - \kappa)R^x_{s_1}(\alpha_1)R_\sigma(t \mapsto t)R^q_{s_1}(-\sigma s_1 s_2(\alpha_1))R^q_{s_2}(\alpha_2 + \kappa), \\ S &= \mathrm{Ad}(D(\alpha_2)^{-1})R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2 + \kappa). \end{aligned}$$

Theorem 2.20. *The automorphisms R_{s_1}, R_σ, $T_{\sigma s_1 s_2}$ and S act the quantum Lax operators $L_{\mathrm{III}}(\alpha)$ and $B_{\mathrm{III}}(\alpha)$ as follows.*

For the automorphisms $R_s(s \in \{s_1, \sigma\})$,

$$R_s\left(L_{\mathrm{III}}(\alpha), B_{\mathrm{III}}(\alpha)\right) = \left(L_{\mathrm{III}}(s(\alpha)), B_{\mathrm{III}}(s(\alpha)) + f_s\right),$$

where

$$f_{s_1} = \kappa(t - 2\alpha_1(\epsilon_1 + \epsilon_2)), \quad f_\sigma = \kappa t.$$

For the automorphism $T_{\sigma s_1 s_2}$,

$$\begin{aligned} l_{T_{\sigma s_1 s_2}} T_{\sigma s_1 s_2}\left((x-q)L_{\mathrm{III}}(\alpha)\right) &= (x-q)L_{\mathrm{III}}\left(\sigma s_1 s_2(\alpha)\right), \\ T_{\sigma s_1 s_2}\left(B_{\mathrm{III}}(\alpha)\right) &= B_{\mathrm{III}}\left(\sigma s_1 s_2(\alpha)\right) + f_{T_{\sigma s_1 s_2}}, \end{aligned} \tag{2.14}$$

where

$$f_{T_{\sigma s_1 s_2}} = \kappa\alpha_0(\alpha_2 + 2\epsilon_1).$$

and $l_{T_{\sigma s_1 s_2}}$ is some element in $\mathcal{K}$ whose explicit form is given in the proof.

For the automorphism S,

$$\begin{aligned} & yp\left(R^x_{s_2}(\alpha_2)R^q_{s_2}(\alpha_2+\kappa)\left((x-q)L_{\mathrm{III}}(\alpha)\right)\right)D(\alpha_2) \\ &\quad = ((x-q)yp + (\alpha_2 + \kappa - \epsilon_2)y + (\epsilon_1 - \alpha_2)p) \\ &\qquad \times \left(D(\alpha_2) - \frac{(\alpha_2+\epsilon_1)(\epsilon_1+\epsilon_2)}{(x-q)^2}\right)L_{\mathrm{III}}(\alpha_0, \tilde{\alpha_1}, -\alpha_2 - 2\epsilon_1), \\ & S\left(B_{\mathrm{III}}(\alpha) + f_S\right) = B_{\mathrm{III}}(\alpha_0, \tilde{\alpha_1}, -\alpha_2 - 2\epsilon_1) \\ &\qquad\qquad - D(\alpha_2)^{-1}\frac{2(\alpha_2+\epsilon_1)\epsilon_1\epsilon_2}{(x-q)^2}L_{\mathrm{III}}(\alpha_0, \tilde{\alpha_1}, -\alpha_2 - 2\epsilon_1), \end{aligned}$$

where $\tilde{\alpha_1} = \alpha_1 + \alpha_2 + \epsilon_1$ and

$$f_S = -\kappa(\alpha_0 + \epsilon_1)(\alpha_2 + \epsilon_1).$$

Proof. For the cases of the automorphisms $T_{\sigma s_1 s_2}$ acting $L_{\mathrm{III}}(\alpha)$, we show that

$$
\begin{aligned}
&((x-q)p-\alpha_2-\kappa)\, yR^x_{s_2}(-\alpha_2-\kappa)R^x_{s_1}(\alpha_1)\\
&\mathrm{Ad}\left(\exp\left(\left(-\frac{t}{x}-x\right)\frac{1}{\epsilon_1}\right)x^{-\frac{2\alpha_1}{\epsilon_1}}\right)((x-q)L_{\mathrm{III}}(\alpha))\\
&\quad=-\left((x-q)y+\alpha_2+\kappa\right)pR^2_{s_2}(-\alpha_2-\kappa)R^q_{s_1}(\sigma s_1 s_2(\alpha_1))\\
&\qquad\circ(x\mapsto -x, q\mapsto -q)\,\mathrm{Ad}\left(\exp\left(\left(-\frac{t}{q}-q\right)\frac{1}{\epsilon_2}\right)q^{-\frac{2\sigma s_1 s_2(\alpha_1)}{\epsilon_2}}\right)\\
&\qquad((x-q)L_{\mathrm{III}}(\sigma s_1 s_2(\alpha))\,,
\end{aligned}
\tag{2.15}
$$

which is the explicit form of (2.14). We omit the proofs of (**??**), since they are similar to that of Theorem 2.5.

Proofs of the other cases follow from direct computations using Proposition 2.18. □

Actions involving $R^x_{s_0}(\alpha_0)$, $R^q_{s_0}(\alpha_0)$ on the quantum Lax operators can be obtained from Theorem 2.20, because of the relations

$$R_\sigma R^x_{s_0}(\alpha_2)R_\sigma = R^x_{s_2}(\alpha_2),\quad R_\sigma R^q_{s_0}(\alpha_2)R_\sigma = R^q_{s_2}(\alpha_2).$$

2.5. $\mathbf{P_{III}^{D_7}}$ Case

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ defined by the generators x, y, q, p, t, d, α_0, α_1, ϵ_1, ϵ_2, and the commutation relations:

$$[y,x]=\epsilon_1,\quad [p,q]=\epsilon_2,\quad [d,t]=1,$$

and the other commutation relations are zero, and a relation $\alpha_0+\alpha_1=-\epsilon_1+\epsilon_2$.

Let $H^{D_7,x}_{\mathrm{III}}(\alpha)$ $(\alpha=(\alpha_0,\alpha_1))$ be the Hamiltonian for the quantum third Painlevé equation of type D_7 defined by

$$H^{D_7,x}_{\mathrm{III}}(\alpha)=xyxy+(-\alpha_0+\epsilon_2)xy+ty+x.$$

Let $H^{D_7,q}_{\mathrm{III}}(\alpha)$ be defined by replacing x, y, ϵ_1, ϵ_2 in $H^{D_7,x}_{\mathrm{III}}(\alpha)$ with q, p, ϵ_2, ϵ_1, respectively.

Let us introduce the quantum Lax operators $L^{D_7}_{\mathrm{III}}(\alpha)$ and $B^{D_7}_{\mathrm{III}}(\alpha)$ for the third Painlevé equation of type D_7 defined by

$$
\begin{aligned}
L^{D_7}_{\mathrm{III}}(\alpha)&=H^{D_7,x}_{\mathrm{III}}(\alpha_0,\alpha_1)-H^{D_7,q}_{\mathrm{III}}(\alpha_0,\alpha_1+2\kappa)-\frac{\kappa xq}{x-q}(y-p),\\
B^{D_7}_{\mathrm{III}}(\alpha)&=\epsilon_2 H^{D_7,x}_{\mathrm{III}}(\alpha_0,\alpha_1)-\epsilon_1 H^{D_7,q}_{\mathrm{III}}(\alpha_0,\alpha_1+2\kappa)-\kappa\epsilon_1\epsilon_2 td.
\end{aligned}
$$

We introduce the extended affine Weyl group $\widetilde{W}(A^{(1)}_1)$ symmetry of the quantum third Painlevé equation of type D_7. Here, $\widetilde{W}(A^{(1)}_1)=W(A^{(1)}_1)\rtimes G$, where $W(A^{(1)}_1)=\langle s_0,s_1\rangle$ is the affine Weyl group of type $A^{(1)}_1$ and $G=\langle\pi\rangle$ is the automorphism group of the Dynkin diagram of type $A^{(1)}_1$.

Definition 2.21. Let the automorphisms s^q for $s\in\{s_0,s_1,\pi\}$ on $\mathcal{K}$ be defined by the following table:

z	α_0	α_1	q	p	t	d
$s_0^q(z)$	$-\alpha_0$	$\alpha_1+2\alpha_0$	q	$p-\frac{\alpha_0}{q}+\frac{t}{q^2}$	$-t$	$-d+\frac{1}{\epsilon_2 q}$
$s_1^q(z)$	$\alpha_0+2\alpha_1$	$-\alpha_1$	$-q-\frac{\alpha_1}{p}-\frac{1}{p^2}$	$-p$	$-t$	$-d$
$\pi^q(z)$	α_1	α_0	tp	$-\frac{q}{t}$	$-t$	$-d-\frac{qp}{\epsilon_2 t}$

Definition 2.22. Let the automorphisms $R^x_{s_i}(\alpha_i)$ $(i=0,1)$, R^x_π on $\mathcal{K}$ be defined by

$$\begin{aligned}R^x_{s_0}(\alpha_0) &= (t\mapsto -t)\circ \mathrm{Ad}\left(\exp\left(-\frac{t}{\epsilon_1 x}\right)x^{-\frac{\alpha_0}{\epsilon_1}}\right),\\ R^x_\pi &= \left(x\mapsto -\frac{x}{t}, t\mapsto -t\right)\circ \mathcal{L}_x,\\ R^x_{s_1}(\alpha_1) &= R^x_\pi\circ R^x_{s_0}(\alpha_1)\circ R^x_\pi.\end{aligned}$$

The automorphisms $R^q_{s_i}(\alpha_i)$ $(i=0,1)$ and R^q_π are defined by replacing x, ϵ_1 in $R^x_{s_i}(\alpha_i)$, R^x_π with q, ϵ_2, respectively.

Proposition 2.23. *The automorphisms $R^x_{s_0}(\alpha_0)$, R_π preserve the Hamiltonian $H^{D_7,x}_{\mathrm{III}}(\alpha)$ in the following sense.*

$$\begin{aligned}R^x_{s_0}(\alpha_0)\left(H^{D_7,x}_{\mathrm{III}}(\alpha)\right) &= H^{D_7,x}_{\mathrm{III}}(s_0(\alpha))+C_{s_0},\\ R_\pi\left(H^{D_7,x}_{\mathrm{III}}(\alpha)\right) &= H^{D_7,x}_{\mathrm{III}}(\pi(\alpha))+C_\pi,\end{aligned}$$

where

$$C_{s_0}=\epsilon_2\alpha_0-\frac{\kappa t}{x},\quad C_\pi=-\epsilon_1\alpha_1+\kappa xy.$$

By definition, the automorphisms $R^q_{s_0}(\alpha_0)$ and R_π act the Hamiltonian $H^{D_7,q}_{\mathrm{III}}(\alpha)$ in the same way above.

Definition 2.24. Let the automorphisms R_{s_0}, $T_{s_0\pi}$, S on $\mathcal{K}$ be defined by

$$\begin{aligned}R_{s_0} &= R^x_{s_0}(\alpha_0)R^q_{s_0}(\alpha_0),\\ T_{s_0\pi} &= R^q_{s_0}(-\alpha_0+\kappa)\left(R^q_\pi\right)^{-1}R^x_\pi R^x_{s_0}(\alpha_0),\\ S &= \mathrm{Ad}(D^{-1}_{\mathrm{III}})R^q_\pi R^q_{s_0}(\alpha_0),\end{aligned}$$

where

$$D_{\mathrm{III}}=\frac{x}{x-q}y+\frac{q}{q-x}p.$$

Theorem 2.25. *The automorphisms R_{s_0}, $T_{s_0\pi}$ and S act the quantum Lax operators $L^{D_7}_{\mathrm{III}}(\alpha)$ and $B^{D_7}_{\mathrm{III}}(\alpha)$ as follows.*

For the automorphisms R_{s_0},

$$R_{s_0}\left(L^{D_7}_{\mathrm{III}}(\alpha),B^{D_7}_{\mathrm{III}}(\alpha)\right)=\left(L^{D_7}_{\mathrm{III}}(s_0(\alpha)),B^{D_7}_{\mathrm{III}}(s_0(\alpha))-\kappa\alpha_0(\epsilon_1+\epsilon_2)\right).$$

For the automorphism $T_{s_0\pi}$,

$$\begin{aligned}l_{T_{s_0\pi}}T_{s_0\pi}\left((x-q)L^{D_7}_{\mathrm{III}}(\alpha)\right) &= (x-q)L^{D_7}_{\mathrm{III}}\left(s_0\pi(\alpha)\right),\\ T_{s_0\pi}\left(B^{D_7}_{\mathrm{III}}(\alpha)\right) &= B^{D_7}_{\mathrm{III}}\left(s_0\pi(\alpha)\right)+(\alpha_0-\epsilon_1)\kappa^2,\end{aligned}\qquad(2.16)$$

where $l_{T_{s_0\pi}}$ is some element in $\mathcal{K}$ whose explicit form is given in the proof.

For the automorphism S*,*

$$\begin{aligned}&\left(R^q_\pi R^q_{s_0}(\alpha_0)\left((x-q)L^{D_7}_{\mathrm{III}}(\alpha)\right)\right)D_{\mathrm{III}}\\&\quad=\left(\frac{1}{x-q}\left(x+\frac{t\epsilon_2}{q-x}-tp\right)\left(\frac{\epsilon_1 q+\epsilon_2 x}{x-q}+qp-xy\right)+\frac{\kappa x}{x-q}\right)\\&\qquad\times L^{D_7}_{\mathrm{III}}(\alpha_0+\epsilon_1,\alpha_1-\epsilon_1),\\&S\left(B^{D_7}_{\mathrm{III}}(\alpha)\right)=\left(\frac{x}{x-q}y+\frac{q}{q-x}p\right)B^{D_7}_{\mathrm{III}}(\alpha_0+\epsilon_1,\alpha_1-\epsilon_1)\\&\qquad\qquad-\epsilon_1\epsilon_2 D^{-1}_{\mathrm{III}}\frac{x+q}{(x-q)^2}L^{D_7}_{\mathrm{III}}(\alpha_0+\epsilon_1,\alpha_1-\epsilon_1).\end{aligned}$$

Proof. For the cases of the automorphisms $T_{s_0\pi}$ acting $L^{D_7}_{\mathrm{III}}(\alpha)$, we show that

$$\begin{aligned}&(-tp+x)\,R^x_\pi R^x_{s_0}(\alpha_0)\left((x-q)L^{D_7}_{\mathrm{III}}(\alpha)\right)\\&\quad=-(ty-q)\,R^q_\pi R^q_{s_0}(\alpha_0-\kappa)\left((x-q)L^{D_7}_{\mathrm{III}}(s_0\pi(\alpha)\right),\end{aligned}\tag{2.17}$$

which is the explicit form of (2.16). We omit the proofs of (2.17), since they are similar to that of Theorem 2.5.

Proofs of the other cases follow from direct computations using Proposition 2.23. □

Actions involving $R^x_{s_1}(\alpha_1)$, $R^q_{s_1}(\alpha_1)$ on the quantum Lax operators can be obtained from Theorem 2.25, because of the definitions of $R^x_{s_1}(\alpha_1)$, $R^q_{s_1}(\alpha_1)$.

2.6. $\mathbf{P_{II}}$ Case

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ defined by the generators x, y, q, p, t, d, α_0, α_1, ϵ_1, ϵ_2, and the commutation relations:

$$[y,x]=\epsilon_1,\quad [p,q]=\epsilon_2,\quad [d,t]=1,$$

and the other commutation relations are zero, and a relation $\alpha_0+\alpha_1=-\epsilon_1+\epsilon_2$.

Let $H^x_{\mathrm{II}}(\alpha)$ ($\alpha=(\alpha_0,\alpha_1)$) be the Hamiltonians for the quantum second Painlevé equation defined by

$$H^x_{\mathrm{II}}(\alpha)=\frac{y^2}{2}-xyx-\frac{t}{2}y-\alpha_1 x.$$

Let $H^q_{\mathrm{II}}(\alpha)$ be defined by replacing x, y, ϵ_1, ϵ_2 in $H^x_{\mathrm{II}}(\alpha)$ with q, p, ϵ_2, ϵ_1, respectively.

Let us introduce the quantum Lax operators $L_{\mathrm{II}}(\alpha)$ and $B_{\mathrm{II}}(\alpha)$ for the second Painlevé equation defined by

$$\begin{aligned}L_{\mathrm{II}}(\alpha)&=H^x_{\mathrm{II}}(\alpha_0,\alpha_1)-H^q_{\mathrm{II}}(\alpha_0+\kappa,\alpha_1+\kappa)-\frac{\kappa}{2(x-q)}(y-p),\\B_{\mathrm{II}}(\alpha)&=\epsilon_2 H^x_{\mathrm{II}}(\alpha_0,\alpha_1)-\epsilon_1 H^q_{\mathrm{II}}(\alpha_0+\kappa,\alpha_1+\kappa)-\kappa\epsilon_1\epsilon_2 d.\end{aligned}$$

Let us recall the extended affine Weyl group $\widetilde{W}(A^{(1)}_1)$ symmetry of the quantum second Painlevé equation. Here, $\widetilde{W}(A^{(1)}_1)=W(A^{(1)}_1)\rtimes G$, where

$W(A_1^{(1)}) = \langle s_0, s_1 \rangle$ is the affine Weyl group of type $A_1^{(1)}$ and $G = \langle \pi \rangle$ is the automorphism group of the Dynkin diagram of type $A_1^{(1)}$.

Definition 2.26 (cf. [12,17]). Let the automorphisms s^q for $s \in \{s_0, s_1, \pi\}$ on $\mathcal{K}$ be defined by the following table:

z	α_0	α_1	q	p	t	d
$s_0^q(z)$	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q + \frac{\alpha_0}{f}$	$p + 2q\frac{\alpha_0}{f} + 2\frac{\alpha_0}{f}q + 2\frac{\alpha_0^2}{f^2}$	t	$d + \frac{\alpha_0/\epsilon_2}{f}$
$s_1^q(z)$	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \frac{\alpha_1}{p}$	p	t	d
$\pi^q(z)$	α_1	α_0	$-q$	$-f$	t	$d - \frac{q}{\epsilon_2}$

where $f = p - 2q^2 - t$.

Definition 2.27 (cf. [17]). Let the automorphisms $R^x_{s_i}(\alpha_i)$ $(i = 0, 1)$, R_π on $\mathcal{K}$ be defined by

$$\begin{aligned} R^x_{s_1}(\alpha_1) &= \mathcal{L}_x^{-1} \circ \mathrm{Ad}(x^{-\frac{\alpha_1}{\epsilon_1}}) \circ \mathcal{L}_x, \\ R_\pi &= (x \mapsto -x, q \mapsto -q) \circ \mathrm{Ad}\left(\exp\left(\left(-\frac{2}{3}x^3 - xt\right)\frac{1}{\epsilon_1}\right)\right) \\ &\quad \circ \mathrm{Ad}\left(\exp\left(\left(-\frac{2}{3}q^3 - qt\right)\frac{1}{\epsilon_2}\right)\right), \\ R^x_{s_0}(\alpha_0) &= R_\pi \circ R^x_{s_1}(\alpha_0) \circ R^x_\pi. \end{aligned}$$

The automorphisms $R^q_{s_i}(\alpha_i)$ $(i = 0, 1, 2)$ are defined by replacing x, ϵ_1 in $R^x_{s_i}(\alpha_i)$, with q, ϵ_2, respectively.

Proposition 2.28 [17]. *The automorphisms $R^x_{s_i}(\alpha_i)(i = 0, 1), R_\pi$ preserve the Hamiltonian $H^x_{\mathrm{II}}(\alpha)$ in the following sense.*

$$\begin{aligned} R^x_{s_i}(\alpha_i)\left(H^x_{\mathrm{II}}(\alpha)\right) &= H^x_{\mathrm{II}}(s_i(\alpha)) + C_{s_i}, \\ R_\pi\left(H^x_{\mathrm{II}}(\alpha)\right) &= H^x_{\mathrm{II}}(\pi(\alpha)) + C_\pi, \end{aligned}$$

where

$$C_{s_0} = -\frac{\kappa\alpha_0}{f}, \quad C_{s_1} = 0, \quad C_\pi = -\kappa x.$$

By definition, the automorphisms $R^q_{s_i}(\alpha_i)$ $(i = 0, 1)$ and R_π act the Hamiltonian $H^q_{\mathrm{II}}(\alpha)$ in the same way above.

Definition 2.29. Let the automorphisms $T_{\pi s_1}$, S on $\mathcal{K}$ be defined by

$$\begin{aligned} T_{\pi s_1} &= R^x_{s_1}(-\alpha_1 - \kappa) R_\pi R^q_{s_1}(\alpha_1 + \kappa), \\ S &= \mathrm{Ad}(D(\alpha_1)^{-1}) R^x_{s_1}(\alpha_1) R^q_{s_1}(\alpha_1 + \kappa). \end{aligned}$$

Theorem 2.30. *The automorphisms R_π, $T_{\pi s_1}$ and S act the quantum Lax operators $L_{\mathrm{II}}(\alpha)$ and $B_{\mathrm{II}}(\alpha)$ as follows.*

For the automorphism R_π,

$$R_\pi\left(L_{\mathrm{II}}(\alpha), B_{\mathrm{II}}(\alpha)\right) = \left(L_{\mathrm{II}}(\pi(\alpha)), B_{\mathrm{II}}(\pi(\alpha))\right).$$

For the automorphism $T_{\pi s_1}$*,*

$$l_{T_{\pi s_1}} T_{\pi s_1}\left((x-q)L_{\rm II}(\alpha)\right) = (q-x)L_{\rm II}\left(\pi s_1(\alpha)\right), \qquad (2.18)$$
$$T_{\pi s_1}\left(B_{\rm II}(\alpha)\right) = B_{\rm II}\left(\pi s_1(\alpha)\right),$$

where $l_{T_{\pi s_1}}$ *is some element in* $\mathcal{K}$ *whose explicit form is given in the proof.*
For the automorphism S*,*

$$\begin{aligned} & yp\left(R^x_{s_1}(\alpha_1)R^q_{s_1}(\alpha_1+\kappa)\left((x-q)L_{\rm II}(\alpha)\right)\right)D(\alpha_1) \\ & \quad = \left((x-q)yp + (\alpha_1+\kappa-\epsilon_2)y + (\epsilon_1-\alpha_1)p\right) \\ & \qquad \times \left(D(\alpha_1) - \frac{(\alpha_1+\epsilon_1)(\epsilon_1+\epsilon_2)}{(x-q)^2}\right)L_{\rm II}(\alpha_1+\epsilon_1+\epsilon_2, -\alpha_1-2\epsilon_1), \\ & S\left(B_{\rm II}(\alpha)\right) = B_{\rm II}(\alpha_1+\epsilon_1+\epsilon_2, -\alpha_1-2\epsilon_1) \\ & \qquad\qquad - D(\alpha_1)^{-1}\frac{2(\alpha_1+\epsilon_1)\epsilon_1\epsilon_2}{(x-q)^2}L_{\rm II}(\alpha_1+\epsilon_1+\epsilon_2, -\alpha_1-2\epsilon_1). \end{aligned}$$

Proof. For the cases of the automorphisms $T_{\pi s_1}$ acting $L_{\rm II}(\alpha)$, we show that

$$\begin{aligned} & \left((x-q)p - \alpha_2 - \kappa\right) yR^x_{s_1}(-\alpha_1-\kappa){\rm Ad}\left(\exp\left(\left(-\frac{2}{3}x^3 - xt\right)\frac{1}{\epsilon_1}\right)\right) \\ & \times\left((x-q)L_{\rm II}(\alpha)\right) = -\left((x-q)y + \alpha_2 + \kappa\right)pR^q_{s_1}(-\alpha_1-\kappa) \\ & \circ\left(x\mapsto -x, q\mapsto -q\right){\rm Ad}\left(\exp\left(\left(-\frac{2}{3}q^3 - qt\right)\frac{1}{\epsilon_2}\right)\right)\left((x-q)L_{\rm II}(\pi s_1(\alpha)\right), \end{aligned} \qquad (2.19)$$

which is the explicit form of (**??**). We omit the proofs of (2.19), since they are similar to that of Theorem 2.5.

Proofs of the other cases follow from direct computations using Proposition 2.28. □

Actions involving $R^x_{s_0}(\alpha_0)$, $R^q_{s_0}(\alpha_0)$ on the quantum Lax operators can be obtained from Theorem 2.30, because of the definitions of $R^x_{s_0}(\alpha_0)$, $R^q_{s_0}(\alpha_0)$.

3. Derivation of the Quantum Lax Pair from CFT

In this section, we derive the quantum Lax operators $L_{\rm J}$ and $B_{\rm J}$ (${\rm J} = {\rm I}, \ldots {\rm VI}$) from Virasoro conformal field theory. Note that the quantum Lax operators $L_{\rm J}$ and $B_{\rm J}$ introduced in Sect. 2 are linear combinations of $L_{\rm J}$ and $B_{\rm J}$ in this section, up to gauge transformations, and the parameters α_i in Sect. 2 are also linear combinations of a_i in this section (see Remark 3.1).

The central charge c and conformal dimension (L_0-eigenvalue) h of the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$ are parameterized as [1]

$$c = 1 + 6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}, \quad h(\alpha) = \frac{\frac{\alpha}{2}(\epsilon_1+\epsilon_2-\frac{\alpha}{2})}{\epsilon_1\epsilon_2}. \qquad (3.1)$$

Following [?], we introduce the kth confluent operator $\Phi^{[k]}(z)$, depending on parameters $u_0, \ldots, u_k$ as

$$\Phi^{[k]}(z) = \exp\left\{u_0\varphi(z) + \frac{u_1}{1!}\varphi'(z) + \cdots + \frac{u_k}{k!}\varphi^{(k)}(z)\right\}, \tag{3.2}$$

where $\varphi(z)$ is a free boson such that $\varphi(z)\varphi(w) = \log(z-w) + \text{regular}$. $\Phi^{[0]}$ corresponds to the usual primary field. The OPEs of $\Phi^{[k]}$ with $J(z) = \varphi'(z)$ and $T(z) = \frac{1}{2}J(z)^2 + \rho J'(z)$ are

$$\begin{aligned}
J(z)\Phi^{[k]}(w) &= \sum_n J_n^{(w)}(z-w)^{-n-1}\Phi^{[k]}(w)\\
&= \left\{\frac{u_k}{(z-w)^{k+1}} + \frac{u_{k-1}}{(z-w)^k} + \cdots\right\}\Phi^{[k]}(w),\\
J(z)\Phi^{[k]}(\infty) &= \sum_n J_n^{(\infty)} z^{n-1}\Phi^{[k]}(\infty) = \{u_k z^{k-1} + u_{k-1}z^{k-2} + \cdots\}\Phi^{[k]}(\infty),\\
T(z)\Phi^{[k]}(w) &= \sum_n L_n^{(w)}(z-w)^{-n-2}\Phi^{[k]}(w)\\
&= \left\{\frac{u_k^2}{2(z-w)^{2k+2}} + \frac{u_k u_{k-1}}{(z-w)^{2k+1}} + \cdots\right\}\Phi^{[k]}(w),\\
T(z)\Phi^{[k]}(\infty) &= \sum_n L_n^{(\infty)} z^{n-2}\Phi^{[k]}(\infty)\\
&= \left\{\frac{u_k^2}{2}z^{2k-2} + u_k u_{k-1}z^{2k-3} + \cdots\right\}\Phi^{[k]}(\infty).
\end{aligned}$$

More explicitly, in case of $k=3$ for instance, we have

$$\begin{aligned}
T(z)\Phi^{[k]}_{u_0,\ldots,u_3}(\infty) = \bigg\{&\frac{u_3^2}{2}z^4 + u_3u_2z^3 + \left(\frac{u_2^2}{2} + u_3u_1\right)z^2\\
&+ (u_2u_1 + u_3u_0 + 2\rho u_3)z + \left(\frac{u_1^2}{2} + u_2u_0 + u_3\frac{\partial}{\partial u_1} + \rho u_2\right)\\
&+ \left(u_1u_0 + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2}\right)z^{-1} + \cdots\bigg\}\Phi^{[k]}(\infty).
\end{aligned} \tag{3.3}$$

3.1. $\mathbf{P_{VI}}$ Case

Let $\Psi^{CFT}_{\text{VI}}(q, x, t)$ be a correlation function on $\mathbb{P}^1$ defined as

$$\Psi^{CFT}_{\text{VI}} = \langle\mathcal{O}_{\text{VI}}\rangle, \quad \mathcal{O}_{\text{VI}} = \Phi_{h_0}(0)\Phi_{h_1}(1)\Phi_{h_t}(t)\Phi_{h_\infty}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x), \tag{3.4}$$

where Φ_{h_i} is the primary field of dimension $h_i = h(a_i)$, $(i = 0, 1, t, \infty, q, x)$. We put[2] $a_q = -\epsilon_1$ and $a_x = -\epsilon_2$, then we have the null field constraints

$$L^{(q)}_{-2}\Phi_{h_q}(q) = -\frac{\epsilon_2}{\epsilon_1}\frac{\partial^2}{\partial q^2}\Phi_{h_q}(q), \quad L^{(x)}_{-2}\Phi_{h_x}(x) = -\frac{\epsilon_1}{\epsilon_2}\frac{\partial^2}{\partial x^2}\Phi_{h_x}(x). \tag{3.5}$$

From the residue theorem $\sum_{\text{poles}}\text{Res}(\xi(z)\langle T(z)\mathcal{O}\rangle \mathrm{d}z) = 0$ for the vector field $\xi_{L_{\text{VI}}}(z)\frac{\partial}{\partial z} = \frac{z(z-1)(z-t)}{(z-q)(z-x)}\frac{\partial}{\partial z}$, we obtain a linear relation between

[2] We apply these specializations also for J = II, …, V cases below.

$\{L_0^{(i)}\}_{i=0,1,t,\infty,q,x}$, $L_{-1}^{(q)}, L_{-2}^{(q)}, L_{-1}^{(x)}$ and $L_{-2}^{(x)}$ which gives a differential equation for $\Psi_{\rm VI}^{CFT}$ of second order in q and x. Under the gauge transformation $\Psi_J^{CFT} = g_J\Psi_J$ where $g_J = (x-q)^{-\frac{1}{2}} f_J(x)^{\frac{1}{2\epsilon_1}} f_J(q)^{\frac{1}{2\epsilon_2}}$ with $f_{\rm VI}(z) = z^{a_0}(z-1)^{a_1}(z-t)^{a_t}$, we obtain the desired equation $L_{\rm VI}\Psi_{\rm VI} = 0$. Similarly, taking the vector field as $\xi_{B_{\rm VI}}(z) = \frac{z(z-1)}{z-q}$, we have the deformation equation $B_{\rm VI}\Psi_{\rm VI} = 0$. The final results are as follows

$$\begin{aligned}
L_{\rm VI} = &-(x-1)x(x-t)\Big\{\frac{a_t-\epsilon_2}{x-t}+\frac{a_0-\epsilon_2}{x}+\frac{a_1-\epsilon_2}{x-1}+\frac{\epsilon_2-\epsilon_1}{x-q}\Big\}\epsilon_1\partial_x\\
&+(q-1)q(q-t)\Big\{\frac{a_t-\epsilon_1}{q-t}+\frac{a_0-\epsilon_1}{q}+\frac{a_1-\epsilon_1}{q-1}+\frac{\epsilon_1-\epsilon_2}{q-x}\Big\}\epsilon_2\partial_q\\
&+C(q-x)-(x-1)x(x-t){\epsilon_1}^2{\partial_x}^2+(q-1)q(q-t){\epsilon_2}^2{\partial_q}^2,\\
B_{\rm VI} = &\,(q-1)q\Big\{-\frac{a_t}{q-t}+\frac{\epsilon_1-a_0}{q}+\frac{\epsilon_1-a_1}{q-1}+\frac{\epsilon_2}{q-x}\Big\}\epsilon_2\partial_q\\
&-\Big\{\frac{(a_0t+a_1t-a_0)\,a_t}{2(q-t)}+C\Big\}-\frac{(t-1)t}{q-t}\epsilon_1\epsilon_2\partial_t\\
&-\frac{(x-1)x}{q-x}\epsilon_1\epsilon_2\partial_x-(q-1)q{\epsilon_2}^2{\partial_q}^2.
\end{aligned}$$

where $C = (-a_t - a_\infty - a_0 - a_1 + 3\epsilon_1 + 3\epsilon_2)(-a_t + a_\infty - a_0 - a_1 + \epsilon_1 + \epsilon_2)/4$. We note that the deformation equation $B_{\rm VI}\Psi_{\rm VI} = 0$ is equivalent to the BPZ equation (5.17) in [3] associated to the field $\Phi_{h_q}(q)$ and $L_{\rm VI}\Psi_{\rm VI} = 0$ is a linear combination of the BPZ equations associated to the fields $\Phi_{h_q}(q)$ and $\Phi_{h_x}(x)$.

Remark 3.1. Denote by $\widehat{L}_{\rm VI}$ and $\widehat{B}_{\rm VI}$ the quantum Lax operators defined by (2.1) and (2.2), respectively. The quantum Lax operators $\widehat{L}_{\rm VI}$ and $\widehat{B}_{\rm VI}$ are expressed in terms of $L_{\rm VI}$ and $B_{\rm VI}$ from Virasoro conformal field theory as follows:

$$\begin{aligned}
\widehat{L}_{\rm VI} &= -\,L_{\rm VI},\\
\widehat{B}_{\rm VI} &= -\,\epsilon_2 L_{\rm VI} + (\epsilon_1-\epsilon_2)(q-t)B_{\rm VI} + b,
\end{aligned}$$

where

$$y=\epsilon_1\partial_x,\quad p=\epsilon_2\partial_q,\quad d=\partial_t,\quad \begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_3\\ \alpha_4\end{pmatrix}=\begin{pmatrix}-a_t\\ -a_\infty\\ -a_1\\ -a_0\end{pmatrix}+(\epsilon_1+\epsilon_2)\begin{pmatrix}1\\1\\1\\1\end{pmatrix},$$

and $b = (\epsilon_1 - \epsilon_2)\{(a_0t + a_1t - a_0)\,a_t/2 - Ct\}$ can be removed by some gauge transformation.

3.2. $\mathbf{P_V}$ Case

Operators: $\mathcal{O}_{\rm V} = \Phi_{h_0}(0)\Phi_{h_1}(1)\Phi_{\rm V}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$, where $\Phi_{\rm V} \in \{\Phi^{[1]}\}$ such as

$$\begin{aligned}
&T(z)\Phi_{\rm V}(\infty)\\
&\quad=\left\{\frac{-t^2}{4\epsilon_1\epsilon_2}+\frac{t(\epsilon_1+\epsilon_2+2a_2-a_0-a_1)}{2\epsilon_1\epsilon_2}z^{-1}+t\frac{\partial}{\partial t}z^{-2}+\cdots\right\}\Phi_{\rm V}(\infty). \qquad (3.6)
\end{aligned}$$

Vector fields: $\xi_{L_{\mathrm{V}}}(z) = \frac{z(z-1)}{(z-q)(z-x)}$, $\xi_{B_{\mathrm{V}}}(z) = \frac{z(z-1)}{z-q}$. Gauge factor: $f_{\mathrm{V}}(z) = z^{a_0}(z-1)^{a_1}e^{tz}$.

$$\begin{aligned}
L_V &= (x-1)x\Big\{\frac{a_0-\epsilon_2}{x} + \frac{a_1-\epsilon_2}{x-1} + \frac{\epsilon_2-\epsilon_1}{x-q} + t\Big\}\epsilon_1\partial_x \\
&\quad - (q-1)q\Big\{\frac{a_0-\epsilon_1}{q} + \frac{a_1-\epsilon_1}{q-1} + \frac{\epsilon_1-\epsilon_2}{q-x} + t\Big\}\epsilon_2\partial_q \\
&\quad - ta_2(q-x) + (x-1)x{\epsilon_1}^2{\partial_x}^2 - (q-1)q{\epsilon_2}^2{\partial_q}^2, \\
B_V &= (q-1)q\Big\{\frac{\epsilon_1-a_0}{q} + \frac{\epsilon_1-a_1}{q-1} + \frac{\epsilon_2}{q-x} - t\Big\}\epsilon_2\partial_q \\
&\quad - t\epsilon_1\epsilon_2\partial_t - \frac{(x-1)x}{q-x}\epsilon_1\epsilon_2\partial_x - (q-1)q{\epsilon_2}^2{\partial_q}^2 \\
&\quad + \Big\{\frac{1}{2}t\left(-a_1 - 2a_2q + 2a_2\right) \\
&\quad - \frac{1}{4}\left(-a_0 - a_1 + \epsilon_1 + \epsilon_2\right)\left(-a_0 - a_1 + 3\epsilon_1 + 3\epsilon_2\right)\Big\}.
\end{aligned}$$

3.3. $\mathbf{P_{IV}}$ Case

Operators: $\mathcal{O}_{\mathrm{IV}} = \Phi_{h_0}(0)\Phi_{\mathrm{IV}}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$, where $\Phi_{\mathrm{IV}} \in \{\Phi^{[2]}\}$ such as

$$\begin{aligned}
&T(z)\Phi_{\mathrm{IV}}(\infty) \\
&\quad = \left\{\frac{-1}{16\epsilon_1\epsilon_2}z^2 + \frac{-t}{4\epsilon_1\epsilon_2}z + \frac{-t^2+2a_1-a_0}{4\epsilon_1\epsilon_2} + \frac{1}{2}\partial_t z^{-1} + \cdots\right\}\Phi_{\mathrm{IV}}(\infty). \quad (3.7)
\end{aligned}$$

Vector fields: $\xi_{L_{\mathrm{IV}}}(z) = \frac{z}{(z-q)(z-x)}$, $\xi_{B_{\mathrm{IV}}}(z) = \frac{z}{z-q}$. Gauge factor: $f_{\mathrm{IV}}(z) = z^{a_0}e^{-tz-z^2/4}$.

$$\begin{aligned}
L_{\mathrm{IV}} &= x\Big\{\frac{a_0-\epsilon_2}{x} + \frac{\epsilon_2-\epsilon_1}{x-q} - t - \frac{x}{2}\Big\}\epsilon_1\partial_x \\
&\quad - q\Big\{\frac{a_0-\epsilon_1}{q} + \frac{\epsilon_1-\epsilon_2}{q-x} - \frac{q}{2} - t\Big\}\epsilon_2\partial_q + x{\epsilon_1}^2{\partial_x}^2 - q{\epsilon_2}^2{\partial_q}^2 + \frac{1}{2}a_1(q-x), \\
B_{\mathrm{IV}} &= q\Big\{\frac{\epsilon_1-a_0}{q} + \frac{\epsilon_2}{q-x} + \frac{q}{2} + t\Big\}\epsilon_2\partial_q + \frac{1}{2}\Big\{t(a_0-\epsilon_1-\epsilon_2) + a_1q\Big\} \\
&\quad - \frac{1}{2}\epsilon_1\epsilon_2\partial_t - \frac{x}{q-x}\epsilon_1\epsilon_2\partial_x - q{\epsilon_2}^2{\partial_q}^2.
\end{aligned}$$

3.4. $\mathbf{P_{III}}$ Case

Operators: $\mathcal{O}_{\mathrm{III}} = \Phi_{\mathrm{III}}(0)\Phi_{\mathrm{III}'}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$, where $\Phi_{\mathrm{III}}(0), \Phi_{\mathrm{III}'}(\infty) \in \{\Phi^{[1]}\}$ such as

$$\begin{aligned}
T(z)\Phi_{\mathrm{III}}(0) &= \left\{\frac{-t^2}{4\epsilon_1\epsilon_2}z^{-4} + \frac{t(2\epsilon_1+2\epsilon_2-a_0)}{2\epsilon_1\epsilon_2}z^{-3} + t\partial_t z^{-2} + \cdots\right\}\Phi_{\mathrm{III}}(0), \\
T(z)\Phi_{\mathrm{III}'}(\infty) &= \left\{\frac{-1}{4\epsilon_1\epsilon_2} + \frac{-\epsilon_1-\epsilon_2-2a_1+a_0}{2\epsilon_1\epsilon_2}z^{-1} + \cdots\right\}\Phi_{\mathrm{III}'}(\infty).
\end{aligned}$$

Vector fields: $\xi_{L_{\rm III}}(z) = \frac{z}{(z-q)(z-x)}$, $\xi_{B_{\rm III}}(z) = \frac{z}{z-q}$. Gauge factor: $f_{\rm III}(z) = z^{\alpha_0}e^{t/z+z}$.

$$\begin{aligned} L_{\rm III} &= -q^2\left\{\frac{a_0-2\epsilon_1}{q}+\frac{t}{q^2}+\frac{\epsilon_1-\epsilon_2}{q-x}-1\right\}\epsilon_2\partial_q \\ &\quad + x^2\left\{\frac{a_0-2\epsilon_2}{x}+\frac{\epsilon_2-\epsilon_1}{x-q}+\frac{t}{x^2}-1\right\}\epsilon_1\partial_x \\ &\quad - q^2{\epsilon_2}^2{\partial_q}^2 + x^2{\epsilon_1}^2{\partial_x}^2 + a_1(q-x), \\ B_{\rm III} &= -q^2\left\{\frac{a_0-\epsilon_1}{q}+\frac{t}{q^2}-\frac{\epsilon_2}{q-x}-1\right\}\epsilon_2\partial_q \\ &\quad + \left\{\frac{1}{4}a_0(-a_0+2\epsilon_1+2\epsilon_2)+a_1q+\frac{t}{2}\right\} \\ &\quad - q^2{\epsilon_2}^2{\partial_q}^2 - \frac{qx}{q-x}\epsilon_1\epsilon_2\partial_x - t\epsilon_1\epsilon_2\partial_t. \end{aligned}$$

3.5. $\mathbf{P_{II}}$ Case

Operators: $\mathcal{O}_{\rm II} = \Phi_{\rm II}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$, where $\Phi_{\rm II} \in \{\Phi^{[3]}\}$ such that[3]

$$\begin{aligned} &T(z)\Phi_{\rm II}(\infty) \\ &\quad = \left\{\frac{-1}{\epsilon_1\epsilon_2}z^4+\frac{-t}{16\epsilon_1\epsilon_2}z^2+\frac{-2a-\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2}z+2\partial_t+\frac{2a+\epsilon_1-\epsilon_2}{\epsilon_1\epsilon_2}tz^{-1}+\cdots\right\}\Phi_{\rm II}(\infty). \end{aligned} \tag{3.8}$$

Vector fields: $\xi_{L_{\rm II}}(z) = \frac{1}{2(z-q)(z-x)}$, $\xi_{B_{\rm II}}(z) = \frac{1}{z-q}$. Gauge factor: $f_{\rm II}(z) = e^{-tz-\frac{2}{3}z^3}$.

$$\begin{aligned} L_{\rm II} &= 2(a+\epsilon_1)(q-x) + \left\{2q^2+t+\frac{\epsilon_2-\epsilon_1}{q-x}\right\}\epsilon_2\partial_q \\ &\quad - \left\{2x^2+t+\frac{\epsilon_1-\epsilon_2}{x-q}\right\}\epsilon_1\partial_x + {\epsilon_1}^2{\partial_x}^2 - {\epsilon_2}^2{\partial_q}^2, \\ B_{\rm II} &= 2(a+\epsilon_1)q + \left\{2q^2+t+\frac{\epsilon_2}{q-x}\right\}\epsilon_2\partial_q - 2\epsilon_1\epsilon_2\partial_t - \frac{\epsilon_2}{q-x}\epsilon_1\partial_x - {\epsilon_2}^2{\partial_q}^2. \end{aligned}$$

3.6. $\mathbf{P_I}$ Case

Operators: $\mathcal{O}_{\rm I} = \Phi_{\rm I}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$, where $\Phi_{\rm I}$ is a degenerate case of $\Phi^{[3]}$ such that

$$T(z)\Phi_{\rm I}(\infty) = \left\{\frac{-4}{\epsilon_1\epsilon_2}z^3+\frac{-2t}{\epsilon_1\epsilon_2}z+2\partial_t+\cdots\right\}\Phi_{\rm I}(\infty). \tag{3.9}$$

Vector fields: $\xi_{L_{\rm I}}(z) = \frac{1}{2(z-q)(z-x)}$, $\xi_{B_{\rm I}}(z) = \frac{1}{z-q}$. Gauge factor: $f_{\rm I}(z) = 1$.

$$\begin{aligned} L_{\rm I} &= (4q^3+2qt-4x^3-2xt) - \frac{\epsilon_1-\epsilon_2}{q-x}\epsilon_1\partial_x + \frac{\epsilon_2-\epsilon_1}{x-q}\epsilon_2\partial_q + {\epsilon_1}^2{\partial_x}^2 - {\epsilon_2}^2{\partial_q}^2, \\ B_{\rm I} &= (4q^3+2qt) + \frac{1}{q-x}{\epsilon_2}^2\partial_q - \frac{1}{q-x}\epsilon_1\epsilon_2\partial_x - {\epsilon_2}^2{\partial_q}^2 - 2\epsilon_1\epsilon_2\partial_t. \end{aligned}$$

[3] We have set the additional parameter u_2=0 in the corresponding equation (3.3).

3.7. $\mathrm{P_{III}}$ (D_7) Case

Operators: $\mathcal{O}_{\rm III}^{(D_7)} = \Phi_{\rm III}^{(D_7)}(0)\Phi_{\rm III'}^{(D_7)}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$,

$$T(z)\Phi_{\rm III}^{(D_7)}(0) = \left\{\frac{-t^2}{4\epsilon_1\epsilon_2}z^{-4} + \frac{t(2\epsilon_1+\epsilon_2-a_0)}{2\epsilon_1\epsilon_2}z^{-3} + t\partial_t z^{-2} + \cdots\right\}\Phi_{\rm III}^{(D_7)}(0),$$

$$T(z)\Phi_{\rm III'}^{(D_7)}(\infty) = \left\{\frac{1}{\epsilon_1\epsilon_2}z^{-1} + \cdots\right\}\Phi_{\rm III'}^{(D_7)}(\infty).$$

Vector fields: $\xi_{L_{\rm III}^{(D_7)}}(z) = \frac{z^2}{(z-q)(z-x)}$, $\xi_{B_{\rm III}^{(D_7)}}(z) = \frac{z}{z-q}$. Gauge factor: $f_{\rm III}^{(D_7)}(z) = e^{-t/(2z)}z^{a_0}$.

$$\begin{aligned} L_{\rm III}^{(D_7)} &= \left\{q(2\epsilon_1-a_0) - \frac{q^2(\epsilon_1-\epsilon_2)}{q-x} - t\right\}\epsilon_2\partial_q \\ &\quad + \left\{x(a_0-2\epsilon_2) + \frac{x^2(\epsilon_1-\epsilon_2)}{q-x} + t\right\}\epsilon_1\partial_x \\ &\quad - q^2{\epsilon_2}^2{\partial_q}^2 - \frac{(q-x)(2qx+t\epsilon_2)}{2qx} + x^2{\epsilon_1}^2{\partial_x}^2, \\ B_{\rm III}^{(D_7)} &= \left\{-a_0q + \frac{q^2\epsilon_2}{q-x} + q\epsilon_1 - t\right\}\epsilon_2\partial_q \\ &\quad + \left\{\frac{1}{4}a_0(-a_0+2\epsilon_1+2\epsilon_2) + \frac{t\epsilon_2}{2q} - q\right\} \\ &\quad - q^2{\epsilon_2}^2{\partial_q}^2 - t\epsilon_1\epsilon_2\partial_t - \frac{qx}{q-x}\epsilon_1\epsilon_2\partial_x. \end{aligned}$$

3.8. $\mathrm{P_{III}}$ (D_8) Case

Operators: $\mathcal{O}_{\rm III}^{(D_8)} = \Phi_{\rm III}^{(D_8)}(0)\Phi_{\rm III'}^{(D_8)}(\infty)\Phi_{h_q}(q)\Phi_{h_x}(x)$,

$$T(z)\Phi_{\rm III}^{(D_8)}(0) = \left\{\frac{t}{\epsilon_1\epsilon_2}z^{-3} + t\partial_t z^{-2} + \cdots\right\}\Phi_{\rm III}^{(D_8)}(0),$$

$$T(z)\Phi_{\rm III'}^{(D_8)}(\infty) = \left\{\frac{1}{\epsilon_1\epsilon_2}z^{-1} + \cdots\right\}\Phi_{\rm III'}^{(D_8)}(\infty).$$

Vector fields: $\xi_{L_{\rm III}^{(D_8)}}(z) = \frac{z^2}{(z-q)(z-x)}$, $\xi_{B_{\rm III}^{(D_8)}}(z) = \frac{z}{z-q}$. Gauge factor: $f_{\rm III}^{(D_8)}(z) = z^{\epsilon_1+\epsilon_2}$.

$$\begin{aligned} L_{\rm III}^{(D_8)} &= -\frac{qx(\epsilon_1-\epsilon_2)}{q-x}\epsilon_2\partial_q - q^2{\epsilon_2}^2{\partial_q}^2 \\ &\quad + \frac{(q-x)(t-qx)}{qx} + x^2{\epsilon_1}^2{\partial_x}^2 + \frac{qx(\epsilon_1-\epsilon_2)}{q-x}\epsilon_1\partial_x, \\ B_{\rm III}^{(D_8)} &= -q^2{\epsilon_2}^2{\partial_q}^2 + \left\{\frac{(\epsilon_1+\epsilon_2)^2}{4} - q - \frac{t}{q}\right\} \\ &\quad - t\epsilon_1\epsilon_2\partial_t - \frac{qx}{q-x}\epsilon_1\epsilon_2\partial_x + \frac{qx}{q-x}\epsilon_2^2\partial_q. \end{aligned}$$

Remark 3.2. It is known that the classical limit of the Knizhnik–Zamolodchikov equations are the Schlesinger equations [5, 24]. Similarly, all the above operators L_J, B_J from Virasoro conformal field theory give the Lax pair for the classical Painlevé equations P_J (see Appendix A) under the limit $\epsilon_2 \to 0$ with $\epsilon_2 \partial_q \to p$, up to a gauge factor independent of z. See [11,27] for the more detail.

Remark 3.3. In a similar way, one can derive the Lax pair for quantum Garnier system of N-variables, by inserting N-primary fields $\Phi_{-\epsilon_1}(q_i)$ $(i = 1, \ldots, N)$.

Remark 3.4. The confluent/degeneration scheme of the Painlevé equation is summarized by the following diagram

$$\begin{array}{ccccccccc} \mathrm{P_{VI}}(1,1,1,1) & \to & \mathrm{P_V}(2,1,1) & \to & \mathrm{P_{III}}(2,2) & \to & \mathrm{P_{III}^{D_7}}(2,\frac{3}{2}) & \to & \mathrm{P_{III}^{D_8}}(\frac{3}{2},\frac{3}{2}) \\ & & & \searrow & & \searrow & & & \\ & & & & \mathrm{P_{IV}}(3,1) & \to & \mathrm{P_{II}}(4) & \to & \mathrm{P_I}(\frac{7}{2}) \end{array} \tag{3.10}$$

where the numbers $(i_1, i_2, \ldots)$ represent the 'Poincaré rank +1' of the singularities. The cases $\mathrm{P_{III}^{D_7}}(2, \frac{3}{2})$, $\mathrm{P_{III}^{D_8}}(\frac{3}{2}, \frac{3}{2})$ are degenerate case of $\mathrm{P_{III}}$ and studied systematically in [23]. In view of the 4d $\mathcal{N} = 2$ gauge theory, the series $(1,1,1,1) \to (2,1,1) \to (2,2) \to (2, \frac{3}{2}) \to (\frac{3}{2}, \frac{3}{2})$ correspond to the $SU(2)$ gauge theories with $N_f = 4, 3, 2, 1, 0$, and the series $(3,1) \to (4) \to (\frac{7}{2})$ corresponds to the AD theories [8].

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Appendix A. Classical Cases

A.1. Data for the Classical Painlevé Equations

We will summarize some relevant data for the classical case [19–23].

$$\begin{aligned} \mathrm{P_I}:\ H_{\mathrm{I}} &= \frac{p^2}{2} - 2q^3 - tq, \\ L_{\mathrm{I}} &= \Big\{ -4x^3 - 2tx - 2H_{\mathrm{I}} + \frac{p}{x-q} \Big\} - \frac{1}{x-q}\partial_x + {\partial_x}^2, \\ B_{\mathrm{I}} &= \partial_t - \frac{1}{2(x-q)}\partial_x + \frac{p}{2(x-q)}, \end{aligned}$$

$$\begin{aligned} \mathrm{P_{II}}:\ H_{\mathrm{II}} &= \frac{p^2}{2} - \Big\{ q^2 + \frac{t}{2} \Big\} p - a_1 q, \\ L_{\mathrm{II}} &= \Big\{ \frac{p}{x-q} - 2H_{\mathrm{II}} - 2a_1 x \Big\} - \Big\{ 2x^2 + t + \frac{1}{x-q} \Big\}\partial_x + {\partial_x}^2, \\ B_{\mathrm{II}} &= \partial_t - \frac{1}{2(x-q)}\partial_x + \frac{p}{2(x-q)}, \\ s_1 &= \{ a_1 \mapsto -a_1, q \mapsto q + a_1/p \}, \\ \pi &= \{ a_1 \mapsto 1 - a_1, q \mapsto -q, p \mapsto -p + 2q^2 + t \}. \end{aligned}$$

$$
\begin{aligned}
\mathrm{P_{III}}:\ H_{\mathrm{III}} &= \frac{1}{t}\Big\{p^2q^2-(q^2+a_1q-t)p-a_0q\Big\},\\
L_{\mathrm{III}} &= \Big\{-\frac{a_0}{x}+\frac{pq}{x(x-q)}-\frac{tH_{\mathrm{III}}}{x^2}\Big\}\\
&\quad+\Big\{\frac{1-a_1}{x}-\frac{1}{x-q}+\frac{t}{x^2}-1\Big\}\partial_x+{\partial_x}^2,\\
B_{\mathrm{III}} &= \partial_t-\frac{xq}{t(x-q)}\partial_x+\frac{pq^2}{t(x-q)},\\
s_0 &= \{a_0\mapsto -a_0, a_1\mapsto a_1+2a_0, q\mapsto q+a_0/p\},\\
s_1 &= \{a_0\mapsto 1+a_0+a_1, a_1\mapsto -2-a_1, p\mapsto p\\
&\quad -(a_1+1)/q+t/q^2, t\mapsto -t\},\\
s_2 &= \{a_1\mapsto -2a_0-a_1, q\mapsto q-(a_0+a_1)/(p-1)\}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}_{\mathrm{III}}^{(D_7)}:\ H_{\mathrm{III}}^{(D_7)} &= \frac{1}{t}(p^2q^2+q+pt+a_1pq),\\
L_{\mathrm{III}} &= \Big\{\frac{1-p}{x}+\frac{p}{x-q}-\frac{tH_{\mathrm{III}}^{(D_7)}}{x^2}\Big\}\\
&\quad+\Big\{\frac{a_1+1}{x}-\frac{1}{x-q}+\frac{t}{x^2}\Big\}\partial_x+{\partial_x}^2,\\
B_{\mathrm{III}} &= \partial_t-\frac{xq}{t(x-q)}\partial_x+\frac{pq^2}{t(x-q)},\\
s_0 &= \{a_1\mapsto 2-a_1, p\mapsto p-(1-a_1)/q+t/q^2, t\mapsto -t\},\\
s_1 &= \{a_1\mapsto -a_1, p\mapsto -p, q\mapsto -q-a_1/p-1/p^2, t\mapsto -t\},\\
\pi &= \{a_1\mapsto 1-a_1, q\mapsto tp, p\mapsto -q/t, t\mapsto -t\}.\\
\mathrm{P}_{\mathrm{III}}^{(D_8)}:H_{\mathrm{III}}^{(D_8)} &= \frac{1}{t}(p^2q^2+pq+q+\frac{t}{q}),\\
L_{\mathrm{III}}^{(D_8)} &= \Big\{\frac{1-p}{x}+\frac{p}{x-q}-\frac{tH_{\mathrm{III}}^{(D_8)}}{x^2}+\frac{t}{x^3}\Big\}+\Big\{\frac{2}{x}-\frac{1}{x-q}\Big\}\partial_x+{\partial_x}^2,\\
B_{\mathrm{III}}^{(D_8)} &= \partial_t-\frac{xq}{t(x-q)}\partial_x+\frac{pq^2}{t(x-q)},\\
\pi &= \{q\mapsto t/q, p\mapsto -q(2qp+1)/(2t)\}.\\
\mathrm{P_{IV}}:\ H_{\mathrm{IV}} &= qpf-a_1p-a_2q,\quad f=p-q-t,\\
L_{\mathrm{IV}} &= \Big\{-a_2-\frac{H_{\mathrm{IV}}}{x}+\frac{pq}{x(x-q)}\Big\}\\
&\quad+\Big\{\frac{1-a_1}{x}-t-x-\frac{1}{x-q}\Big\}\partial_x+{\partial_x}^2,\\
B_{\mathrm{IV}} &= \partial_t-\frac{x}{x-q}\partial_x+\frac{pq}{x-q},\\
s_0 &= \{p\mapsto p+(1-a_1-a_2)/f, q\mapsto q\\
&\quad +(1-a_1-a_2)/f, a_1\mapsto 1-a_2, a_2\mapsto 1-a_1\},
\end{aligned}
$$

$$s_1 = \{p \mapsto p - a_1/q, a_1 \mapsto -a_1, a_2 \mapsto a_1 + a_2\},$$
$$s_2 = \{q \mapsto q + a_2/p, a_1 \mapsto a_1 + a_2, a_2 \mapsto -a_2\},$$
$$\pi = \{p \mapsto -f, q \mapsto -p, a_1 \mapsto a_2, a_2 \mapsto 1 - a_1 - a_2\}.$$

$$\mathrm{P_V}: \ H_\mathrm{V} = \frac{1}{t}\Big\{(q-1)q(p+t)p + \{a_1 - (a_1+a_3)q\}p + a_2 q t\Big\},$$
$$L_\mathrm{V} = \Big\{\frac{p(q-1)q}{(x-1)x(x-q)} + \frac{a_2 t x - t H_\mathrm{V}}{(x-1)x}\Big\}$$
$$+\Big\{\frac{1-a_1}{x} + t + \frac{1-a_3}{x-1} - \frac{1}{x-q}\Big\}\partial_x + {\partial_x}^2,$$
$$B_\mathrm{V} = \partial_t - \frac{(x-1)x}{t(x-q)}\partial_x + \frac{p(q-1)q}{t(x-q)},$$
$$s_0 = \{a_1 \mapsto 1 - a_2 - a_3, a_3 \mapsto 1 - a_1 - a_2, q \mapsto q$$
$$+(1 - a_1 - a_2 - a_3)/(p+t)\},$$
$$s_1 = \{a_1 \mapsto -a_1, a_2 \mapsto a_1 + a_2, a_3 \mapsto a_3, p \mapsto p - a_1/q\},$$
$$s_2 = \{a_1 \mapsto a_1 + a_2, a_3 \mapsto a_2 + a_3, a_2 \mapsto -a_2, q \mapsto q + a_2/p\},$$
$$s_3 = \{a_3 \mapsto -a_3, a_2 \mapsto a_2 + a_3, p \mapsto p - a_3/(q-1)\},$$
$$\pi = \{a_1 \mapsto a_2, a_2 \mapsto a_3, a_3 \mapsto 1 - a_1 - a_2 - a_3,$$
$$q \mapsto -p/t, p \mapsto (q-1)t\}.$$

$$\mathrm{P_{VI}}: \ H_\mathrm{VI} = \frac{q(q-1)(q-t)}{t(t-1)}\Big\{p^2 - \big(\frac{a_0-1}{q-t} + \frac{a_3}{q-1} + \frac{a_4}{q}\big)p\Big\} + \frac{(q-t)a_2(a_1+a_2)}{t(t-1)},$$
$$L_\mathrm{VI} = \Big\{\frac{p(q-1)q}{(x-1)x(x-q)} + \frac{a_2(a_1+a_2)}{(x-1)x} - \frac{t(t-1)H_\mathrm{VI}}{(x-1)x(x-t)}\Big\}$$
$$+\Big\{\frac{1-a_0}{x-t} + \frac{1-a_3}{x-1} + \frac{1-a_4}{x} - \frac{1}{x-q}\Big\}\partial_x + {\partial_x}^2,$$
$$B_\mathrm{VI} = \partial_t - \frac{(t-q)(x-1)x}{(t-1)t(q-x)}\partial_x + \frac{p(q-1)q(q-t)}{(t-1)t(x-q)},$$
$$s_0 = \{a_0 \mapsto -a_0, a_2 \mapsto a_0 + a_2, p \mapsto p - a_0/(q-t)\},$$
$$s_1 = \{a_1 \mapsto -a_1, a_2 \mapsto a_1 + a_2\},$$
$$s_2 = \{a_0 \mapsto a_0 + a_2, a_1 \mapsto a_1 + a_2, a_2 \mapsto -a_2, a_3 \mapsto a_2 + a_3, a_4$$
$$\mapsto a_2 + a_4, q \mapsto q + a_2/p\},$$
$$s_3 = \{a_2 \mapsto a_2 + a_3, a_3 \mapsto -a_3, p \mapsto p - a_3/(q-1)\},$$
$$s_4 = \{a_2 \mapsto a_2 + a_4, a_4 \mapsto -a_4, p \mapsto p - a_4/q\},$$
$$\pi_1 = \Bigg\{a_0 \mapsto a_1, a_1 \mapsto a_0, a_3 \mapsto a_4, a_4 \mapsto a_3,$$
$$p \mapsto -\frac{(q-t)(p(q-t)+a_2)}{t(t-1)}, q \mapsto \frac{(q-1)t}{q-t}\Bigg\},$$

$$\pi_2 = \left\{a_0 \mapsto a_3, a_1 \mapsto a_4, a_3 \mapsto a_0, a_4 \mapsto a_1, p \mapsto -\frac{q(pq+a_2)}{t}, q \mapsto \frac{t}{q}\right\},$$

$$\pi_3 = \left\{a_0 \mapsto a_4, a_1 \mapsto a_3, a_3 \mapsto a_1, a_4 \mapsto a_0, \right.$$
$$\left. p \mapsto \frac{(q-1)(p(q-1)+a_2)}{t-1}, q \mapsto \frac{q-t}{q-1}\right\}.$$

A.2. Symmetry of the Classical Lax Operator

We will summarize some relevant data for the classical case [9,26].

Proposition A.1. *If* $L_{\rm J}y(x) = 0$ *then* $\ell w(L_{\rm J})\tilde{y} = 0$*, where*

$$\tilde{y} = (\partial_x)^{2-a_1} y,\ w = \pi s_1 \pi,\ \ell = \partial_x + \frac{1-a_1}{x-q} + \frac{1}{x-w(q)}, \quad (for\ {\rm J} = {\rm II})$$

$$\tilde{y} = (\partial_x)^{2-a_1} e^{\frac{1}{\partial_x}} y,\ w = \pi s_1 \pi s_1,\ \ell = \partial_x^2 - \left(\frac{1}{x-q} + \frac{a_1-1}{s_1(q)}\right)$$
$$\partial_x - \frac{p}{x-q} + \frac{p+1}{x+s_1(q)}, \quad (for\ {\rm J} = {\rm III}^{D_7})$$

$$\tilde{y} = (\partial_x)^{2-a_0} y,\ w = s_1 s_2 s_1 s_0,$$
$$\ell = \partial_x + \frac{2}{x} + \frac{1-a_0}{x-q} + \frac{1}{x-w(q)}, \quad (for\ {\rm J} = {\rm III})$$

$$\tilde{y} = (\partial_x)^{2-a_2} y,\ w = s_1 s_0 s_1,\ \ell = \partial_x + \frac{1}{x} + \frac{1-a_2}{x-q} + \frac{1}{x-w(q)},$$
$$(for\ {\rm J} = {\rm IV})$$

$$\tilde{y} = (\partial_x)^{2-a_2} y,\ w = s_3 s_0 s_1 s_0 s_3,$$
$$\ell = \partial_x + \frac{1}{x} + \frac{1}{x-1} + \frac{1-a_2}{x-q} + \frac{1}{x-w(q)}, \quad (for\ {\rm J} = {\rm V})$$

$$\tilde{y} = (\partial_x)^{2-a_2} y,\ w = s_4 s_3 s_1 s_0 s_2 s_4 s_3 s_1 s_0,$$
$$\ell = \partial_x + \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t} + \frac{1-a_2}{x-q} + \frac{1}{x-w(q)},$$
$$(for\ {\rm J} = {\rm VI}).$$

Proposition A.2. *If* $L_J y(x) = 0$ *then* $w(L_J)\tilde{y} = 0$*, where*

$$\tilde{y} = \{a_1 y + (x-q-a_1/p) y_x\}/(x-q),\ w = s_1 \pi s_1 \pi, \quad (for\ {\rm J} = {\rm II})$$
$$\tilde{y} = \{x y_x - qpy\}/(x-q),\ w = s_1 \pi, \quad (for\ {\rm J} = {\rm III}^{D_7})$$
$$\tilde{y} = \{a_0 y + (x-q-a_0/p) y_x\}/(x-q),\ w = s_1 s_2 s_1, \quad (for\ {\rm J} = {\rm III})$$
$$\tilde{y} = \{a_2 y + (x-q-a_2/p) y_x\}/(x-q),\ w = s_1 s_0 s_1 s_2, \quad (for\ {\rm J} = {\rm IV})$$
$$\tilde{y} = \{a_2 y + (x-q-a_2/p) y_x\}/(x-q),\ w = s_3 s_0 s_1 s_0 s_3 s_2, \quad (for\ {\rm J} = {\rm V})$$
$$\tilde{y} = \{a_2 y + (x-q-a_2/p) y_x\}/(x-q),\ w = s_4 s_3 s_1 s_0 s_2 s_4 s_3 s_1 s_0 s_2.$$
$$(for\ {\rm J} = {\rm VI})$$

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