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Twisted Equivariant Matter

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Abstract. We show how general principles of symmetry in quantum mechanics lead to twisted notions of a group representation. This framework generalizes both the classical threefold way of real/complex/ quaternionic representations as well as a corresponding tenfold way which has appeared in condensed matter and nuclear physics. We establish a foundation for discussing continuous families of quantum systems. Having done so, topological phases of quantum systems can be defined as deformation classes of continuous families of gapped Hamiltonians. For free particles, there is an additional algebraic structure on the deformation classes leading naturally to notions of *twisted* equivariant K-theory. In systems with a lattice of translational symmetries, we show that there is a *canonical* twisting of the equivariant K-theory of the Brillouin torus. We give precise mathematical definitions of two invariants of the topological phases which have played an important role in the study of topological insulators. Twisted equivariant K-theory provides a finer classification of topological insulators than has been previously available.

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0. Introduction

Increasingly sophisticated ideas from homotopy theory are being used to elucidate issues in quantum field theory and string theory. While relatively elementary topological notions were applied long ago to nonrelativistic condensed matter systems [42,60], it is only in the past few years that more modern topological invariants have been appearing. They are used to classify phases of noninteracting systems of electrons [18–21,28–30,32,33,35–41,43,48–51,53, 54,56,57,59,62,65]. This paper is particularly motivated and inspired by the beautiful results of Kitaev and of Ludwig et al. [20,37,56,57,53]. We show that fundamental principles of quantum mechanics—with a Wignerian emphasis on symmetry—suggest definitions of *QM symmetry classes* and *topological phases* of general quantum mechanical systems which lead to *twistings of K-theory* and *twisted K-theory*. Some of our arguments are quite general and apply in particular to interacting systems.

Our starting point is Wigner's theorem concerning symmetries of a quantum mechanical system. The pure states form the projective space $\mathbb{P}\mathcal{H}$ of a complex Hilbert space $\mathcal{H}$, and quantum symmetries are the *projective* transformations which preserve transition probabilities. Wigner's theorem asserts that a quantum symmetry has a real *linear* lift which is either unitary or antiunitary. Therefore, any group G of quantum symmetries comes equipped with a homomorphism $\phi\colon G \to \{\pm 1\}$ which encodes unitarity vs. antiunitarity, together with a group extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G^{\tau} \longrightarrow G \longrightarrow 1 \tag{0.1}$$

whose kernel $\mathbb{T} = \{e^{i\theta} : \theta \in \mathbb{R}\}$ is the group of scalar unitary transformations of $\mathcal{H}$, which act trivially on the space $\mathbb{P}\mathcal{H}$ of pure quantum states. We say (0.1) is a *ϕ-twisted extension* (by $\mathbb{T}$) since complex scalars commute with unitary transformations but complex conjugate when commuted past antiunitary transformations. The action of G^{τ} on $\mathcal{H}$ is a "twisted" version of a complex linear representation of G; the twisting is encoded by the abstract *QM symmetry class* (G, ϕ, τ). If we artificially ask that (0.1) be pulled back via ϕ, then there is a trichotomy of QM symmetry classes for fixed G, which roughly corresponds to the Frobenius–Schur trichotomy of complex/real/quaternionic representations; under a further splitting hypothesis $\mathcal{H}$ has a real or quaternionic structure if ϕ is onto.

Typically an abstract group G of quantum symmetries also acts on spacetime. In particular, there is a homomorphism $t\colon G \to \{\pm 1\}$ which encodes whether a symmetry preserves or reverses time-orientation. If in addition we fix a one-parameter group of time translations, so a one-parameter group of unitary operators generated by a self-adjoint Hamiltonian H, then a simple argument (Lemma 3.3) shows that the operator corresponding to a symmetry $g \in G$ commutes or anticommutes with H according to the product $\phi(g)t(g)$. In particular, if H is bounded below and unbounded above, then $\phi = t$. That restriction on H does not necessarily hold in finite quantum systems ($\dim \mathcal{H} < \infty$), nor in noninteracting single electron systems in the Dirac–Nambu picture (Sect. 4). Therefore, we study *extended QM symmetry classes* (G, ϕ, τ, c) in which $c := \phi t\colon G \to \{\pm 1\}$ is allowed to be nontrivial. If we now artificially restrict the ϕ-twisted extension to be a pullback via $(t, c)\colon G \to \{\pm 1\} \times \{\pm 1\}$, we recover a decachotomy ubiquitous in the condensed matter literature; it reduces to the previous trichotomy for $c \equiv 1$. Under a further splitting hypothesis the Hilbert space $\mathcal{H}$ is a module for one of the ten Morita classes of real and complex Clifford algebras. The refinement of a QM symmetry class (G, ϕ, τ) to an extended QM symmetry class (G, ϕ, τ, c) only has meaning if we fix an evolution of the system.

In Sect. 4, we discuss free fermion systems with finitely many degrees of freedom. Here we recover a result of Altland–Heinzner–Huckleberry–Zirnbauer [7,34] relating spaces of free fermion Hamiltonians compatible with a given symmetry and classical symmetric spaces of compact type. We also define the Dirac–Nambu Hilbert space, which motivates some of our later definitions.

We investigate deformation classes of quantum mechanical systems with fixed extended QM symmetry type (G, ϕ, τ, c). The first task is foundational: define a continuous family of quantum mechanical systems. Our general discussion of this point in Appendix D may have broader interest. A system is said to be *gapped* if the Hamiltonian is invertible, and two gapped systems are said to be in the same *topological phase* if there is a continuous path leading from one to the other. The set $\mathcal{TP}(G, \phi, \tau, c)$ of topological phases has an algebraic structure given by amalgamation of quantum systems. In standard quantum mechanics, the Hilbert space of the amalgam of quantum systems with Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is the *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$. However, in the Dirac–Nambu picture Hilbert spaces combine using *direct sum* $\mathcal{H}_1 \oplus \mathcal{H}_2$ instead. This nonstandard amalgamation is the only specific feature we abstract from noninteracting single fermion systems. With this algebraic structure $\mathcal{TP}(G, \phi, \tau, c)$ is a commutative monoid,[1] and it is natural to pass to an abelian group $\mathcal{RTP}(G, \phi, \tau, c)$ of *reduced topological phases*. Two systems in the same topological phase necessarily have the same reduced topological phase, but not vice versa, so $\mathcal{RTP}$ is a cruder invariant than $\mathcal{TP}$. On the other hand, it is more easily computable and is strong enough to distinguish interestingly different systems. We illustrate this by proving in Sect. 11 that two physically important topological invariants

[1] A *monoid* is a set with an associative composition law and a unit for composition. In the commutative case, we write the composition as '+' and write '0' for the unit. The commutative monoid is an *abelian group* if additive inverses exist.

of certain band insulators—the orbital magnetoelectric polarizability and the Kane–Mele invariant—factor through $\mathcal{RTP}$, which leads to a proof that they are equal.

The passage from the commutative monoid $\mathcal{TP}$ to the group $\mathcal{RTP}$ is reminiscent of K-theory, and indeed our main results identify $\mathcal{RTP}$ with a topological K-theory group. For that we need several additional hypotheses. First, we assume that there is a gap in the spectrum of the Hamiltonian H, which we shift to be at zero energy. Then after a homotopy, we may assume $H^2 = 1$, i.e., that H is the grading operator of a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{H}$. We also make certain finiteness and compactness hypotheses on the Hilbert space, Hamiltonian, and symmetry group. These have discrete consequences—for example, representations of compact Lie groups are rigid whereas representations of noncompact Lie groups often have continuous deformations—which are part of the proofs that we obtain K-theory groups. As a warmup in Sect. 8, we compute $\mathcal{RTP}$ for gapped systems with finite dimensional Hilbert space. While the proofs here are mainly a matter of unwinding definitions, this case illustrates important issues which arise in more intricate higher dimensional systems.

After these extensive preliminaries, we turn to quantum mechanical systems symmetric under a spacetime crystallographic group G with nontrivial magnetic point group $\widehat{P}$. Thus, G is a group extension

$$1 \longrightarrow \Pi \longrightarrow G \longrightarrow \widehat{P} \longrightarrow 1 \tag{0.2}$$

whose kernel is a full lattice Π of translations. The operators corresponding to elements of Π commute, so can be simultaneously diagonalized, and this Fourier transform (using *Bloch sums*) realizes the Hilbert space as L^2 sections of an infinite rank vector bundle $\mathcal{E} \to X_\Pi$ over the (Brillouin) torus X_Π of characters of Π. The group $\widehat{P}$ acts on X_Π, but if (0.2) is not split that action does not generally lift to $\mathcal{E}$. Rather, the extension (0.2) determines a central extension of the *groupoid* $X_\Pi/\!/\widehat{P}$, and it is this central extension which lifts. This central extension is known in a different language in the condensed matter literature: when describing the action of group elements of the point group on Bloch eigenfunctions, there are some phase ambiguities that are generally dealt with in an ad hoc fashion (see, e.g., [9]). These phases are encoded in the central extension of the groupoid. In general, a central extension of a groupoid is a special kind of *twisting* of its K-theory. This canonical twisting determined by a group extension was discovered in [26,27]. More generally, we consider an abstract symmetry group equipped with a homomorphism to (0.2), for example, in systems with internal symmetry, and it may come equipped with a nontrivial extended QM symmetry type unconnected with the extension (0.2). In that case, the QM symmetry type and canonical twisting from (0.2) combine, and $\mathcal{E} \to X_\Pi$ is a twisted bundle for the combined twisting. The Fourier transform based on Π brings us from twisted representations—a direct consequence of Wigner's theorem and spacetime symmetry—to twisted equivariant vector bundles that represent twisted K-theory classes:

$$\begin{aligned} \text{group} &\longrightarrow \text{groupoid} \\ \text{extended QM symmetry class} &\longrightarrow \text{twisting of } K\text{-theory} \\ \text{twisted representation} &\longrightarrow \text{twisted vector bundle} \\ \text{classes of twisted virtual representations} &\longrightarrow \text{twisted } K\text{-theory group.} \end{aligned}$$

The hypothesis of a gap in the Hamiltonian, specified by a Fermi level, induces a decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Due to Fermi–Dirac statistics, the subbundles have an interpretation as unfilled bands and filled bands, also known as conduction bands and valence bands, respectively. We assume that $\mathcal{E}^-$ has finite rank—there are only finitely many valence bands—and consider separately the finite and infinite rank possibilities for $\mathcal{E}^+$. Our main results (Theorems 10.15 and 10.21) identify the group of reduced topological phases with a twisted K-theory group. The nonequivariant specialization, which ignores the magnetic point group $\widehat{P}$, is well-known and expressed in terms of untwisted K-theory [37]; the result for the full symmetry group is new. (A special case is addressed in [62].) In case $\mathcal{E}^+$ has infinite rank we pull off an Eilenberg swindle to prove it does not carry any topological information.

Our results about topological phases are expressed in terms of monoids and groups of equivalence classes. These are sets of path components of topological spaces—or homotopy types—which classify families of quantum systems. The arguments we give can be adapted to identify classifying spaces of gapped quantum systems in terms of classifying spaces for K-theory.

A brief outline of the paper is as follows. We begin in Sect. 1 with an exposition of Wigner's theorem and QM symmetry classes. In Sect. 2, we review symmetries of an affine spacetime with Galilean structure. We take a pre-Cartesian non-formulaic approach to affine, Euclidean, and projective geometry in these introductory sections; it brings out the symmetries more clearly and is designed to convey the simplicity of the geometric ideas. In Sect. 3, we define extended QM symmetry classes and topological phases. Free fermions are the subject of Sect. 4, where we consider them in the context of algebraic quantum mechanics. We give a quick proof of the main theorem in [34]. The definitions of topological phases and reduced topological phases are the subject of Sect. 5. Classification results, including the decachotomy of special extended QM symmetry classes, are proved in Sect. 6. Section 7 gives a rapid introduction to special twistings of K-theory and twisted K-theory, beginning with twisted versions of complex representations of a group. There we also introduce groupoids, which simultaneously generalize groups and spaces. At this point we have enough to discuss the classification of systems with finite dimensional state spaces in Sect. 8. The canonical twisting of equivariant K-theory from a group extension is worked out in Sect. 9. The main theorems computing the group of phases of band insulators are stated and proved in Sect. 10. Section 11 is a treatment of the Kane–Mele invariant and the orbital magnetoelectric polarizability, which is a Chern–Simons invariant. We report on some computations. There are several appendices of background and supplementary material. Appendix A is a rapid compendium of definitions related to group extensions. As is well-known, the ten special extended QM symmetry classes correspond to the ten Morita classes of real and complex Clifford algebras. We prove a precise result in Appendix B, which we use at the end of Sect. 8 and Sect. 10 to recast special symmetries as a degree shift in K-theory. A beautiful paper of Dyson [17] also contains a tenfold way; we give a modern treatment in Appendix C. In Appendix D, we give a careful definition of a *continuous* family of quantum mechanical systems with fixed QM symmetry type. We also give a geometric treatment of Bloch sums and the Berry connection. Appendix E contains a lemma which ensures that the definition of twisted equivariant K-theory for the special twistings we encounter agrees with the general definition. An extended example, the 3-dimensional diamond structure, is treated in Appendix F.

While we do not discuss topological superconductors in detail, our considerations apply to free fermions with Bogoliubov-de Gennes Hamiltonians. We have nothing to say in this paper about the topology of electron systems with no gap in the spectrum. We stress that our discussion of $\mathcal{TP}$ as a commutative monoid and hence $\mathcal{RTP}$ as an abelian group is limited to noninteracting systems. The generalization to interacting systems is an interesting open problem; see [20, 21] for interesting recent results.

1. Quantum Symmetries and Twisted Extensions

1.1. Wigner's Theorem

The state space of a quantum system is the projective space $\mathbb{P}\mathcal{H}$ of a complex separable Hilbert space $\mathcal{H}$. In other words, the state of a quantum system is a line of vectors. (Sometimes the term 'ray' is used in place of 'line', and one may require that the state be normalized to have unit norm.) If $\ell, \ell' \in \mathbb{P}\mathcal{H}$ are states, then the *transition probability* from ℓ to ℓ' is defined as $|\langle\psi, \psi'\rangle|^2$, where $\psi, \psi' \in \mathcal{H}$ are unit norm vectors contained in the lines ℓ, ℓ', respectively, and $\langle -, -\rangle$ is the inner product on $\mathcal{H}$. The transition probability is a symmetric function

$$p\colon \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \longrightarrow [0, 1]. \tag{1.1}$$

A *projective quantum symmetry* of $\mathbb{P}\mathcal{H}$ is an invertible map $\mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ which preserves the function p. A basic theorem of Wigner [66], [64, Sect. 2.A] asserts that any projective quantum symmetry lifts to either a unitary or antiunitary[2] transformation $\mathcal{H} \to \mathcal{H}$, which we call a *linear quantum symmetry*. Recall that both unitary and antiunitary transformations preserve lengths and angles, but a unitary map is complex linear whereas an antiunitary map is complex antilinear.

Remark 1.2. We have defined projective quantum symmetries in terms of the transition probability structure on projective space, the symmetric function (1.1). The projective space $\mathbb{P}\mathcal{H}$ has a more conventional symmetric function which is induced from the inner product on $\mathcal{H}$: the distance function d of

[2] A real linear map $S\colon \mathcal{H} \to \mathcal{H}$ is *antiunitary* if it is antilinear ($S\lambda = \bar{\lambda}S,\ \lambda \in \mathbb{C}$) and if $\langle S\xi_1, S\xi_2\rangle = \overline{\langle \xi_1, \xi_2\rangle}$ for all $\xi_1, \xi_2 \in \mathcal{H}$.

the Fubini–Study metric. It is not difficult to compute that $p = \cos^2(d/2)$, from which the group of projective quantum symmetries is precisely the isometry group of projective space. Proofs of Wigner's theorem based on this identification are given in [23].

We recast Wigner's theorem in the following terms. Let $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ denote the group of projective quantum symmetries of $\mathbb{P}\mathcal{H}$ and $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ the group of linear quantum symmetries, that is, the group of all unitary and antiunitary transformations of $\mathcal{H}$. It is a group since the composition of two antiunitary transformations is unitary. In fact, $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ fits into a group extension (see Definition A.1)

$$1 \longrightarrow U(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H}) \xrightarrow{\phi_{\mathcal{H}}} \{\pm 1\} \longrightarrow 1 \tag{1.3}$$

with kernel of the group $U(\mathcal{H})$ of unitary operators: for $S \in \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ we have $\phi_{\mathcal{H}}(S) = 1$ if S is unitary and $\phi_{\mathcal{H}}(S) = -1$ if S is antiunitary. Wigner's theorem asserts that there is a group extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H}) \longrightarrow \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \longrightarrow 1 \tag{1.4}$$

in which the kernel is the group $\mathbb{T}$ of scalings of $\mathcal{H}$ by complex numbers of norm one.[3] In other words, every projective quantum symmetry lifts to a linear quantum symmetry which is unique up to composition with scalar multiplication by a unit norm complex number. Note that $\mathbb{T}$ is not central—for $\lambda \in \mathbb{T}$ and $S \in \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ we have

$$S\lambda = \begin{cases} \lambda S, & \phi_{\mathcal{H}}(S) = 1; \\ \bar{\lambda} S, & \phi_{\mathcal{H}}(S) = -1. \end{cases} \tag{1.5}$$

Hence (1.4) is not a central extension, but rather a slight twisting of a central extension. The twisting is expressed by the homomorphism $\phi_{\mathcal{H}}$ in (1.3), which descends to a homomorphism (with the same name)

$$\phi_{\mathcal{H}} \colon \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) \longrightarrow \{\pm 1\} \tag{1.6}$$

that encodes whether a quantum symmetry lifts to be unitary or antiunitary.

In the uniform (norm) topology, $U(\mathcal{H})$ is connected—in fact, contractible if $\mathcal{H}$ is infinite dimensional—and $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ has two components. It is more natural for us to topologize bounded operators with the compact-open topology, as we discuss in Appendix D. In that topology,[4] it is also true that $U(\mathcal{H})$ is contractible and $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ has two components, as we prove in Proposition D.3.

[3] We assume $\dim \mathcal{H} > 1$.

[4] We topologize these groups as subspaces of invertible operators, but invertible operators have a topology finer than the subspace topology inherited from the strong topology; see Definition D.1.

1.2. QM symmetry Classes and Twisted Representations

We abstract the structure which emerges from lifting projective quantum symmetries to linear quantum symmetries.

Definition 1.7. Let G be a topological group and $\phi\colon G \to \{\pm 1\}$ a continuous homomorphism. A *ϕ-twisted extension* (by $\mathbb{T}$) of G is a group extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{1.8}$$

which for all $S \in G^\tau$ and $\lambda \in \mathbb{T}$ satisfies (1.5) with $\phi_{\mathcal{H}}$ replaced by ϕ. We denote the extension as '$G^\tau \to G$', or simply 'τ'.

If $\phi(g) = 1$ for all $g \in G$, then (1.8) is a central extension, i.e., the kernel $\mathbb{T}$ commutes with every element of G^τ. For any ϕ, there is a trivial ϕ-twisted extension, the semidirect product $G \ltimes_\phi \mathbb{T}$ (see Definition A.9), where G acts on $\mathbb{T}$ by complex conjugation via ϕ. We will meet nontrivial twisted extensions below.

Now suppose G is any topological group of quantum symmetries of a quantum system whose state space is $\mathbb{P}\mathcal{H}$. In other words, G acts on $\mathbb{P}\mathcal{H}$ through a continuous homomorphism

$$\rho\colon G \longrightarrow \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}). \tag{1.9}$$

Set $\phi = \phi_{\mathcal{H}} \circ \rho\colon G \to \{\pm 1\}$ to be the composition of ρ with (1.6). There is a pullback ϕ-twisted extension of G (see Definition A.12) which fits into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & G^\tau & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \rho^\tau} & & \downarrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H}) & \longrightarrow & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) & \longrightarrow & 1 \end{array} \tag{1.10}$$

Simply put, the group G^τ consists of all linear quantum symmetries which lift the projective quantum symmetries in G. Note that ϕ encodes which operators in G^τ act linearly and which act antilinearly. Also, $\mathbb{T} \subset G^\tau$ acts on $\mathcal{H}$ by scalar multiplication. We formalize these properties.

Definition 1.11. Let G^τ be a ϕ-twisted extension of G, as in Definition 1.7. Then a homomorphism $\rho^\tau : G^\tau \to \mathrm{Aut}_{\mathbb{R}}(E)$ to the group of real automorphisms of a complex vector space E is a (ϕ, τ)-*twisted* representation of G if:

(i) $\rho^\tau(g)$ is complex linear if $\phi(g) = +1$, $\rho^\tau(g)$ is complex antilinear if $\phi(g) = -1$; and
(ii) $\mathbb{T} \subset G^\tau$ acts by complex scalar multiplication.

Note that if G—and hence also G^τ—is connected, then G acts by complex linear transformations.

Remark 1.12. A homomorphism ρ^τ which satisfies (i) is called a 'co-representation' by Wigner [66].

Next we abstract the structure (1.10) induced on a group of projective quantum symmetries. The definition parallels the standard notions of concrete group actions and abstract symmetry groups.

Definition 1.13. (i) Let $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ be a group of projective quantum symmetries. The *QM symmetry type* of (G, ρ) is the isomorphism class of the triple (G, ϕ, τ) consisting of G, the induced homomorphism $\phi\colon G \to \{\pm 1\}$, and the induced ϕ-twisted extension G^τ in (1.10).

(ii) A *QM symmetry class*[5] is an isomorphism class of triples (G, ϕ, τ), where G is a topological group, $\phi\colon G \to \{\pm 1\}$ is a continuous homomorphism, and G^τ is a ϕ-twisted extension as in Definition 1.7.

We often call a triple (G, ϕ, τ) a QM symmetry class, though strictly speaking the QM symmetry class is the equivalence class containing the triple. Also, we say the QM symmetry class is *based on* G. Recall from Definition A.3 that two ϕ-twisted extensions G^{τ_1}, G^{τ_2} of G are isomorphic if there is an isomorphism of groups $\varphi\colon G^{\tau_1} \to G^{\tau_2}$ which fits into the commutative diagram

$$\begin{array}{ccccccccc} & & & & G^{\tau_1} & & & & \\ & & & \nearrow & & \searrow & & & \\ 1 & \longrightarrow & \mathbb{T} & & \downarrow{\scriptstyle\varphi} & & G & \longrightarrow & 1 \\ & & & \searrow & & \nearrow & & & \\ & & & & G^{\tau_2} & & & & \end{array} \tag{1.14}$$

Two projective actions of G have the same QM symmetry type if both actions induce the same unitary vs. antiunitary dichotomy on elements of G and if the linear and antilinear lifts glue together into groups which are not only abstractly isomorphic, but are isomorphic in a way that matches the underlying projective symmetries. Symmetry *type* is an invariant of a group of projective quantum symmetries. A QM symmetry *class* is an abstract group of possible quantum symmetries, up to isomorphism. The classification of QM symmetry classes for fixed G can be expressed in terms of group cohomology, but for topological and Lie groups one must use continuous or smooth cohomology rather than the cohomology of discrete groups.

There are special QM symmetry classes we term *ϕ-standard*.

Definition 1.15. Let G be a Lie group equipped with a homomorphism $\phi\colon G \to \{\pm 1\}$. A QM symmetry class (G, ϕ, τ) is *ϕ-standard* if G^τ is pulled back from a twisted extension of $\phi(G) \subset \{\pm 1\}$.

Clearly a ϕ-standard QM symmetry class is determined by a subgroup $A \subset \{\pm 1\}$ and a QM symmetry class based on A with $\phi\colon A \hookrightarrow \{\pm 1\}$ the inclusion map. There are 2 possible subgroups and 3 possibilities in all.

Proposition 1.16 (threefold way).

(i) *There is a unique ϕ-standard QM symmetry class based on the trivial subgroup $A = 1$.*

[5] Altland–Zirnbauer [7,34] use the term 'symmetry classes' in a different way, referring to the classification of Hamiltonians with given symmetry. We make some comments on that problem in Sect. 4.

(ii) *There are two ϕ-standard QM symmetry classes based on $A = \{\pm 1\}$. We denote them generically by A^τ. Let $T \in A^\tau$ be a lift of $-1 \in \{\pm 1\}$; then the two QM symmetry classes are distinguished by $T^2 = \pm 1$.*

Remark 1.17. In (ii), the two groups A^τ may be identified with the two forms $\mathrm{Pin}_{\pm 2}$ of the pin group.

Remark 1.18. If ϕ is surjective there may not be an element $\widehat{T} \in G^\tau$ mapping to $\overline{T}$ with $\widehat{T}^2 = \pm 1$. A stronger assumption than ϕ-standard is that $G^\tau \cong A^\tau \times G_0$, where G_0 is the kernel of ϕ. If so, the operator $\widehat{T} = (T, 1)$ in (ii), where T is a real or quaternionic structure on $\mathcal{H}$ preserved by G_0.

Remark 1.19. We prove the more general Proposition 6.4 below so do not write out a proof of the simple Proposition 1.16 here. But we do observe that the key steps are contained in parts (i) and (ii) of Lemma 6.17.

1.3. Symmetries and Spacetime

In this section, we have discussed abstract quantum systems and their symmetries. Typically the symmetry group of a quantum system has a homomorphism to symmetries of spacetime, and so in the next section we review those symmetries in the Galilean setting. In Sect. 3, we resume the discussion of quantum symmetries and extend the notion of a QM symmetry class to account for one particular aspect of the spacetime symmetry: preservation or reversal of the arrow of time.

A nonrelativistic quantum system carries an action of (a subgroup of) the Galilean group, which as we review in Sect. 2 is a double cover of the connected component of symmetries of a Galilean spacetime. By (1.10) this determines a central extension of the Galilean group. There is in fact a nontrivial central extension, and it encodes the total mass of the system.[6] By contrast a relativistic quantum system carries an action of the Poincaré group—a double cover of the connected component of symmetries of a Minkowski spacetime—and the induced central extension (1.10) is split (see Definition A.5), as is every central extension of the Poincaré group. In other words, for G the Poincaré group, any homomorphism $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ has a lift $\tilde{\rho}\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$:

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & \overset{\tilde\rho}{\swarrow} & \downarrow{\scriptstyle\rho} & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H}) & \longrightarrow & \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H}) & \longrightarrow & 1 \end{array} \tag{1.20}$$

Nonrelativistic and relativistic quantum systems often have larger symmetry groups G equipped with a homomorphism to Galilean or Poincaré group. For example, G may include parity-reversal and/or time-reversal symmetries. There may also be other symmetries which commute with the Galilean or Poincaré symmetries. These are known as global symmetries, or sometimes as internal symmetries.

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[6] The mass central extension of the Galilean group can be constructed as a limit of the Poincaré group in which the speed of light tends to infinity; see [13, Sect. 1.4] for example. We describe the Lie algebra of the mass central extension in the proof of Lemma 6.25.

2. Symmetries of Galilean Spacetime

2.1. Affine Spaces: Global Parallelism

Just as Felix Klein's *Erlangen program* defines geometry as the study of properties invariant under a given symmetry group, so too models of physical systems are often built around symmetry principles tied to physical spacetime. In this spirit, nonrelativistic physics is the study of systems invariant under the Galilean group. It is useful to *define* the Galilean group as the group of symmetries of a *Galilean structure* Γ on spacetime, and one task in this subsection is to define that structure precisely. (In fact, we reserve the term 'Galilean group' for a double cover of the identity component of the group of symmetries of Γ.[7]) The symmetry group in a condensed matter system is often the subgroup that preserves a further *crystal structure*, as we explain. We linger a bit over the elementary material here to introduce ideas and definitions that we will use freely in more advanced contexts later.

Space E and spacetime M are real finite dimensional *affine* spaces. An affine space carries a simply transitive action of a real *vector* space V of translations: given two points p_0, p_1 in affine space there is a unique vector ξ such that p_1 is the point reached by starting at p_0 and displacing by ξ. We write simply $p_1 = p_0 + \xi$. The action of V on E encodes the *global parallelism* of affine space, its characteristic property. It is geometrically natural to distinguish points and displacement vectors. Displacements form a vector space: we can add displacements. However, it does not make sense to add geometric points in a space: in a flat model of Earth what would be the sum of New York and Boston? (Note, however, that weighted averages of points in affine space are well-defined: there is a definite point 2/3 of the way from New York to Boston.) Also, note that there is a distinguished zero displacement vector, whereas to distinguish a point in the idealized affine space of the world would be chauvinistic. Less prosaically, translations act as symmetries of affine space but not as symmetries of a vector space, since a symmetry of a vector space must preserve the zero vector. We denote the vector space of displacements of space E as V and the vector space of displacements of spacetime M as W. It is sometimes helpful to regard elements of V, W as global vector fields on E, M, respectively: affine spaces also have a global *infinitesimal* parallelism.

The symmetries of V are called *linear* transformations and the symmetries of E are called *affine* transformations. If we choose a basis of V, then we can identify the symmetry group $\mathrm{Aut}(V)$ as the group of invertible real square matrices of size $d = \dim V = \dim E$. An affine transformation $f\colon E \to E$ is a map which preserves the action of displacements. More precisely, if p_0, p_1, q_0, q_1 are points in E such that the displacements $p_1 - p_0$ and $q_1 - q_0$ are equal, then the displacements $f(p_1) - f(p_0)$ and $f(q_1) - f(q_0)$ are also equal. It follows that f induces a well-defined *linear* map $\dot{f}$ on displacements defined by

[7] One could imagine using the term without the spin double cover, or to include other components, but in fact our usage conforms to that adapted for the term 'Poincaré group' [12, Glossary] .

$\dot{f}(\xi) = f(p_1) - f(p_0)$ whenever $p_1 = p_0 + \xi$ for $\xi \in V$. The map $\dot{f}$ is the differential of f and it is constant on E. (The constancy of the derivative characterizes affine maps, and is defined using the global parallelism.) The affine maps f for which $\dot{f}$ is the identity map on displacements are the translations $f(p) = p + \xi_0$ by a fixed displacement vector ξ_0. This situation is summarized by the group extension

$$1 \longrightarrow V \longrightarrow \mathrm{Aff}(E) \longrightarrow \mathrm{Aut}(V) \longrightarrow 1. \tag{2.1}$$

Here 1 denotes the trivial group consisting only of an identity element; V is the abelian group of displacements under the vector addition; $\mathrm{Aff}(E)$ is the group of affine transformations of E; and $\mathrm{Aut}(V)$ the group of linear transformations of V. The map $V \to \mathrm{Aff}(E)$ is the inclusion of the translation subgroup and $\mathrm{Aff}(E) \to \mathrm{Aut}(V)$ is the derivative homomorphism $f \mapsto \dot{f}$ described above. Notice that a point $p \in E$ determines a splitting of (2.1): the image of $\mathrm{Aut}(V)$ in $\mathrm{Aff}(E)$ under the splitting is the subgroup of affine transformations which fix p.

2.2. Euclidean and Galilean Structures

A geometric structure on the vector space V induces a translation-invariant geometric structure on the affine space E. For example, the space E in a nonrelativistic system carries a Euclidean metric. It is specified by a positive definite inner (dot) product on V, which then determines a translation-invariant metric on E. The *Euclidean group* $\mathrm{Euc}(E)$ is the group of isometries of E with its Euclidean metric. Since the metric is translation-invariant, $\mathrm{Euc}(E)$ is a subgroup of $\mathrm{Aff}(E)$ and it is a group extension

$$1 \longrightarrow V \longrightarrow \mathrm{Euc}(E) \longrightarrow O(V) \longrightarrow 1 \tag{2.2}$$

where $O(V)$ is the orthogonal group of linear transformations which preserve the inner product. Relative to an orthonormal basis on V the orthogonal group $O(V)$ is the group of $d \times d$ orthogonal matrices. Recall that $O(V)$, hence $\mathrm{Euc}(E)$, has two components according as a symmetry preserves or reverses orientation. In terms of matrices, the components are distinguished by the determinant, which takes values ± 1. The Euclidean group contains the classical rigid motions: translations, rotations, and reflections.

Definition 2.3. A *Galilean structure* Γ on a spacetime M with translation group W is:

(i) a distinguished subspace $V \subset W$ of codimension one with a positive definite inner product;
(ii) a positive definite inner product on the quotient line W/V.

The pair (M, Γ) is called a *Galilean spacetime*.

The subspace V determines a translation-invariant codimension one foliation $\mathcal{S}$ by affine subspaces, and this gives a global notion of *simultaneity*: spacetime points on the same leaf occur at the same time. There is no distinguished complementary foliation defining worldlines of spatial points at rest—Galilean observers can undergo boosts. The metric on W/V sets a scale for time; the clock for any observer moves at a set rate. There is no set *arrow of time*, which would be an orientation of W/V, i.e., a positive and negative "side" of each simultaneity slice (leaf of $\mathcal{S}$).

Definition 2.4. Let (M,Γ) be a Galilean spacetime. Then $\mathrm{Aut}(M,\Gamma) \subset \mathrm{Aff}(M)$ is the group of affine transformations of M which preserve the Galilean structure Γ.

The symmetry group $\mathrm{Aut}(M,\Gamma)$ is a group extension

$$1 \longrightarrow W \longrightarrow \mathrm{Aut}(M,\Gamma) \longrightarrow \mathrm{Aut}(W,\Gamma) \longrightarrow 1 \tag{2.5}$$

where $\mathrm{Aut}(W,\Gamma)$ is the space of linear transformations on W which preserve the structures in Definition 2.3: the subspace V, its metric, and the metric on the quotient W/V. As above, let $d = \dim V$; then $\dim W = d+1$. Choose a basis of W so that the first basis vector projects onto a unit vector in W/V and the last d basis vectors form an orthonormal basis of V. Then elements of $\mathrm{Aut}(W,\Gamma)$ are represented by invertible $(d+1)\times(d+1)$ matrices of the triangular form

$$\begin{pmatrix} \pm 1 & 0 \\ * & A \end{pmatrix} \tag{2.6}$$

where A is a $d\times d$ orthogonal matrix. If $d>0$ the group $\mathrm{Aut}(W,\Gamma)$, hence also $\mathrm{Aut}(M,\Gamma)$, has four components according to $\det A = \pm 1$ and the sign of the upper left entry of (2.6). More invariantly, an affine transformation which preserves Γ also preserves the foliation $\mathcal{S}$ of simultaneous events, and it might or might not reverse the orientation of each leaf of $\mathcal{S}$ and might or might not reverse the orientation of the normal bundle to $\mathcal{S}$: the arrow of time. In other words, there are surjective (onto) homomorphisms

$$t,p\colon\ \mathrm{Aut}(M,\Gamma) \longrightarrow \{\pm 1\} \tag{2.7}$$

defined on transformations $f\colon M \to M$ such that $t(f) = \pm 1$ according to f preserves/reverses the arrow of time, and $p(f) = \pm 1$ according to f preserves/reverses the orientation of space. The value of t on the matrix in (2.6) is the upper left corner; the value of p is $\det A$. Any transformation f with $t(f) = -1$ is called *time-reversing*. For later purposes, we single out the following subgroup with two components.

Definition 2.8. $\mathrm{Aut}^+(M,\Gamma) = p^{-1}(+1) \subset \mathrm{Aut}(M,\Gamma)$.

In other words, $\mathrm{Aut}^+(M,\Gamma)$ is the subgroup of $\mathrm{Aut}(M,\Gamma)$ consisting of transformations which are orientation-preserving on each leaf of $\mathcal{S}$. It contains time-reversing transformations.

The fundamental group of $\mathrm{Aut}(M,\Gamma)$ is isomorphic to that of the special orthogonal group $SO(d)$ of space. The latter has a nontrivial double cover,[8] the spin group $\mathrm{Spin}(d)$, which is simply connected if $d \ge 3$. Note that $\mathrm{Spin}(0)$ and $\mathrm{Spin}(1)$ are each cyclic of order 2; $\mathrm{Spin}(2)$ is isomorphic to the circle group, so is connected but not simply connected.

[8] There are also double cover groups of the two-component orthogonal group $O(d)$, and these "pin groups" [3] can occur in systems with parity-reversing symmetries.

Definition 2.9. The *Galilean group* of (M, Γ) is the spin double cover of the identity component of $\mathrm{Aut}(M, \Gamma)$.

Some authors might use 'Galilean group' to refer to $\mathrm{Aut}(M, \Gamma)$ or some other group of symmetries of spacetime; as remarked earlier our choice is analogous to the definition of 'Poincaré group' in [12, Glossary]. We can realize the Galilean group as the group of symmetries of a geometric structure. Namely, endow the vector space V with an orientation and spin structure and also orient the quotient W/V. This induces a translation-invariant orientation and spin structure on the simultaneity foliation $\mathcal{S}$ as well as an orientation on the normal bundle to $\mathcal{S}$, an arrow of time. The Galilean group is the group of symmetries of M which preserve these structures. The Galilean group is connected for $d \geq 2$ and has two components for $d = 0, 1$.

2.3. Tying Abstract Symmetry Groups to Spacetime

Definition 2.10. Let (M, Γ) be a Galilean spacetime. A *Galilean symmetry group* is a triple (G, γ, j) consisting of a Lie group G, a homomorphism

$$\gamma : G \longrightarrow \mathrm{Aut}(M, \Gamma), \tag{2.11}$$

and a splitting

$$j : W \cap \gamma(G) \longrightarrow G \tag{2.12}$$

of γ over the group of spacetime translations in the image of γ. The image of j is required to be a normal subgroup of G.

Remark 2.13. The homomorphism γ need not be surjective or injective. The surjectivity is violated if, for example, G is the group of symmetries of a system which has only a proper subgroup of Galilean symmetries, as with the crystals below. If there is extra internal structure on spacetime—say extrinsic "spins" at each spacetime point or each point of a crystal in spacetime—then $\ker \gamma$ may consist of internal symmetries which fix the points of spacetime. Any such internal structure should have a global parallelism extending that of the affine space M, which is why we hypothesize the splitting (2.12) over spacetime translations. On the other hand, internal structures need not be homogeneous with respect to rotations, reflections, and boosts, which is why we only hypothesize the splitting over translations. The quotient of G by the image of j maps to the group $\mathrm{Aut}(W, \Gamma)$ of linear symmetries. We imagine these acting on some model internal structure in the linear space W, and it is the existence of this quotient which justifies the normality assumption on the image of j.

Remark 2.14. There is a similar definition for Minkowski spacetimes.

In some situations, a time direction is distinguished, and this breaks the boost symmetries.

Definition 2.15. Let (M, Γ) be a Galilean spacetime. A *time direction* is a splitting of the exact sequence

$$0 \longrightarrow V \longrightarrow W \longrightarrow W/V \longrightarrow 0 \tag{2.16}$$

of spacetime translations.

Let $U \subset W$ be the image of the splitting $W/V \hookrightarrow W$. Then U defines a distinguished translation-invariant set of affine lines in M (each with tangent U) which are transverse to the leaves of the simultaneity foliation $\mathcal{S}$. They are worldlines of distinguished observers. The inner product on W/V induces one on U, and so two distinguished unit-speed motions on each distinguished worldline. These are distinguished clocks in spacetime, though without a time-orientation the clocks can run either forward or backward (and no observer can distinguish forward from backward).

2.4. Crystals and Crystallographic Groups

We define subgroups of $\mathrm{Aut}(M, \Gamma)$ of interest in condensed matter physics, namely, those which preserve a *crystal* C in spacetime.

Definition 2.17. Let (M, Γ) be a Galilean spacetime.

(i) A *crystal* C is a subset of M such that (i) the subgroup $\Pi \subset V$ of spatial translations which preserve C is a lattice of full rank, and (ii) there is a line $U \subset W$ of spacetime translations complementary to V such that translations in U preserve C.

(ii) Let $G(C) \subset \mathrm{Aut}(M, \Gamma)$ be the subgroup of transformations which preserve a crystal C. The quotient $G(C)/U$ is the *spacetime crystallographic group* associated to the crystal C.

The subspace U is a time direction in the sense of Definition 2.15, so $G(C)$ contains no Galilean boosts. Note that the matrices (2.6) which preserve the direct sum decomposition $W = U \oplus V$ are block diagonal—the boost component $*$ must be the zero vector. The spacetime crystallographic group is an extension

$$1 \longrightarrow \Pi \longrightarrow G(C)/U \longrightarrow \widehat{P} \longrightarrow 1 \tag{2.18}$$

whose translation subgroup consists of the lattice Π of discrete spatial translations; the quotient is called the *magnetic point group* $\widehat{P} \subset \mathrm{Aut}(W, \Gamma)$. The crystallographic group is called *symmorphic* if (2.18) splits and *nonsymmorphic* otherwise. The homomorphism $t\colon \mathrm{Aut}(W, \Gamma) \to \{\pm 1\}$ in (2.7), which maps a linear Galilean transformation onto its action on W/V—which is either the identity (time-preserving) or minus the identity (time-reversing)—fits the magnetic point group into a group extension

$$1 \longrightarrow P \longrightarrow \widehat{P} \xrightarrow{\;t\;} \{\pm 1\} \longrightarrow 1 \tag{2.19}$$

whose kernel P is the *point group*. Both P and $\widehat{P}$ are finite groups. The extension (2.19) is naturally split by the involution ϵ of $W = U \oplus V$ which is -1 on U and $+1$ on V. Conjugation by ϵ reverses time translations and fixes spatial translations.

Example 2.20. Let $\mathbb{M}^3 = \mathbb{R} \times \mathbb{E}^2$ be the Cartesian product of the standard time line and standard Euclidean 2-space with the evident Galilean structure. Define C to be the Cartesian product of $\mathbb{R}$ with $C_0 \subset \mathbb{E}^2$, where

$$C_0 = \{(x^1, x^2) : x^1, x^2 \in \mathbb{Z}\} \cup \{(x^1 + \delta, x^2 + 1/2) : x^1, x^2 \in \mathbb{Z}\} \tag{2.21}$$

for $0 < \delta < 1/2$. Let $G \subset \mathrm{Euc}(\mathbb{E}^2)$ be the group of Euclidean symmetries which preserve C_0. It is the subgroup of the crystallographic group $G(C)/U$ of time-preserving transformations of modulo time translations. Then G is a group extension

$$1 \longrightarrow \Pi \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{2.22}$$

Here $\Pi \subset \mathbb{R}^2$ is the full lattice of translations by vectors (n^1, n^2) with integer components $n^1, n^2 \in \mathbb{Z}$. The quotient group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is generated by reflections g_1'', g_2'' in the two axes of the vector space $\mathbb{R}^2$. Then g_1'' lifts to the *glide reflection*

$$g_1 \colon (x^1, x^2) \longmapsto (-x^1 + \delta, x^2 + 1/2) \tag{2.23}$$

which interchanges the two sets in (2.21), whereas g_2'' lifts to the reflection

$$g_2 \colon (x^1, x^2) \longmapsto (x^1, -x^2). \tag{2.24}$$

Note that $(g_1)^2$ is translation by $(0, 1)$ whereas $(g_2)^2$ is the identity. Any other lift of g_1'' is τg_1 for some $\tau \in \Pi$, and $(\tau g_1)^2$ is again a nonzero translation. This proves that (2.22) is not split, and so $G(C)/U$ is nonsymmorphic.

For later use, we note that the commutator $g_2^{-1} g_1^{-1} g_2 g_1$ is translation by $(0, 1)$.

Finally, some condensed matter systems are described by specifying a subset of $M \times V$ rather than a subset of M. Intuitively, this is a set of points with spatial tangent vectors attached to each point.[9]

Definition 2.25. Let (M, Γ) be a Galilean spacetime. A *spin crystal* C is a subset of $M \times V$ such that (i) the subgroup $\Pi \subset V$ of spatial translations whose elements preserve C is a full lattice, and (ii) there is a line $U \subset W$ of spacetime translations complementary to V which preserve C.

Since we can take the tangent vectors to be zero, Definition 2.25 generalizes Definition 2.17. As before, the symmetry group $G(C)$ contains the group U of time translations, the quotient $G(C)/U$ is an extension (2.18), and we define the magnetic point group to be the quotient $\widehat{P} = G(C)/(U \times \Pi)$. But for spin crystals the homomorphism $t \colon \widehat{P} \to \{\pm 1\}$ need not be surjective—there might not be time-reversal symmetries—and even if t is surjective the extension (2.19) might not split, as the following example illustrates.

[9] More precisely, we should allow either V or $\bigwedge^{d-1} V^*$ depending on whether we attach vectors or axial vectors. Physically, these represent localized displacements (as in ferroelectrics) or current loops and atomic spins (as in ferromagnetics). One could of course consider more general order parameters.

Example 2.26. We use the standard Galilean spacetime $\mathbb{M}^3 = \mathbb{R} \times \mathbb{E}^2$ of Example 2.20. Fix $0 < \delta_1 \neq \delta_2 < 1/2$ and a nonzero vector $\xi \in V = \mathbb{R}^2$, and set $\mathbb{Z}^2 = \{(x^1, x^2) : x^1, x^2 \in \mathbb{Z}\} \subset \mathbb{E}^2$. Let

$$\begin{aligned} C_0 = &\big((\delta_1, \delta_2) + \mathbb{Z}^2\big) \times \{\xi\} \,\cup\, \big((-\delta_2, \delta_1) + \mathbb{Z}^2\big) \times \{-\xi\} \\ &\cup \big((-\delta_1, -\delta_2) + \mathbb{Z}^2\big) \times \{\xi\} \,\cup\, \big((\delta_2, -\delta_1) + \mathbb{Z}^2\big) \times \{-\xi\} \end{aligned} \tag{2.27}$$

which is a subset of $\mathbb{E}^2 \times \mathbb{R}^2$. The spin crystal is $C = \mathbb{R} \times C_0 \subset \mathbb{M}^3 \times \mathbb{R}^2$. Then $\widehat{P}$ is cyclic of order 4 with generator the linear transformation of $W = \mathbb{R} \oplus \mathbb{R}^2$ which is -1 on the first summand and a $\pi/2$ rotation on the second. The homomorphism $t\colon \widehat{P} \cong \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a nonsplit surjection.

Remark 2.28. As a crystal breaks symmetries which mix time and space, it can equally be embedded in a relativistic (Minkowski) spacetime and its symmetry group G embedded in the symmetry group of relativistic quantum mechanics.

3. Extended QM Symmetry Classes

3.1. Quantum Symmetries and Time-Reversal

Resuming the discussion in Sect. 1, suppose G is a Galilean symmetry group (Definition 2.10) on a Galilean spacetime (M, Γ), and let G act as projective quantum symmetries on a Hilbert space $\mathcal{H}$. Let $\rho\colon G \to \mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ denote the action of G, and let (G, ϕ, τ) be the induced QM symmetry type (Definition 1.13). Recall that the homomorphism

$$\phi\colon G \longrightarrow \{\pm 1\} \tag{3.1}$$

tracks which symmetries are implemented linearly and which antilinearly. Composing the homomorphism (2.11) with (2.7) we obtain a homomorphism

$$t\colon G \longrightarrow \{\pm 1\} \tag{3.2}$$

which tracks time-reversal: $t(g) = 1$ if a symmetry g preserves the arrow of time and $t(g) = -1$ if g reverses the arrow of time. A priori the homomorphisms ϕ, t are independent, but there is a standard argument which is often invoked to identify them. We review it now.

A one-parameter subgroup $\mathbb{R} \to G^\tau$ induces via ρ^τ in (1.10) a one-parameter group $s \mapsto e^{is\mathcal{O}/\hbar}$ of unitary operators on $\mathcal{H}$, where $\mathcal{O}$ is a self-adjoint operator. (The parameter s typically does not have dimensions of time.) In particular, suppose (M, Γ) has a distinguished time direction (Definition 2.15); the Galilean symmetry group G includes the distinguished time translation subgroup $U \subset G$, which comes with a fixed isomorphism $\mathbb{R} \cong U$; and assume that $G^\tau \to G$ is split over $U \subset G$. Then time translation gives a one-parameter group $t \mapsto e^{-itH/\hbar}$ of unitary transformations whose self-adjoint generator H is the *Hamiltonian* of the distinguished time evolution.[10] We assume oriented

[10] The sign convention [14] is that the self-adjoint Hamiltonian H corresponds to *minus* infinitesimal time translation.

time directions in the sequel, and so have systems with a well-defined Hamiltonian.

Lemma 3.3. (i) *For all* $g \in G^\tau$ *we have*

$$\rho^\tau(g)H = \phi(g)t(g)H\rho^\tau(g). \tag{3.4}$$

(ii) *If* H *is bounded below and unbounded above, then* $\phi = t$.

Proof. The distinguished time translations commute or anticommute with $g \in G^\tau$, according to $t(g)$, and so

$$\rho^\tau(g)\mathrm{e}^{-itH/\hbar} = \mathrm{e}^{-t(g)itH/\hbar}\rho^\tau(g). \tag{3.5}$$

Therefore, $\rho^\tau(g)iH = t(g)iH\rho^\tau(g)$, from which (3.4) follows. The conclusion in (ii) is immediate since if $\phi(g)t(g) = -1$, then $\rho^\tau(g)$ flips the spectrum of H. □

Because of this lemma it is usually assumed that time-reversing symmetries are antiunitary and time-preserving symmetries are unitary. However, the hypotheses do not apply to some Dirac Hamiltonians in condensed matter systems, and so we allow ϕ and t to be independent.

Define

$$c = \phi t \colon G \longrightarrow \{\pm 1\}. \tag{3.6}$$

This homomorphism tracks whether a symmetry commutes or anticommutes with H. A symmetry which commutes with H is called *Hamiltonian-preserving*; a symmetry which anticommutes with H is called *Hamiltonian-reversing*. Of course, any two of t, c, ϕ determine the third.

3.2. Extended QM Symmetry Classes and Gapped Systems

We now augment Definition 1.13(ii) of a QM symmetry class and Definition 1.11 of a twisted representation to include the homomorphism (3.6) which tracks whether symmetries commute or anticommute with the Hamiltonian.

Definition 3.7. (i) An *extended QM symmetry class* is an isomorphism class of quadruples (G, ϕ, τ, c) where G is a Lie group, $\phi\colon G \to \{\pm 1\}$ is a continuous homomorphism, G^τ is a ϕ-twisted extension, and $c\colon G \to \{\pm 1\}$ is a continuous homomorphism.

(ii) A homomorphism $\rho^\tau : G^\tau \to \mathrm{Aut}_{\mathbb{R}}(E)$ to real automorphisms of a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space E is a (ϕ, τ, c)-*twisted representation of* G if it satisfies the conditions of Definition 1.11 and in addition $\rho^\tau(g)$ is even if $c(g) = 1$ and odd if $c(g) = -1$.

As before, we often call a quartet (G, ϕ, τ, c) an extended QM symmetry class, though strictly speaking the QM symmetry class only remembers the isomorphism class of the ϕ-twisted extension $G^\tau \to G$. Lemma 3.3(i) specializes to the condition in Definition 3.7(ii) if we take H to be the grading operator on E.

We are now ready to abstract some basic properties of gapped free fermion systems.

Definition 3.8. A *gapped system with extended QM symmetry class* (G, ϕ, τ, c) is a triple $(\mathcal{H}, H, \rho^\tau)$ consisting of a complex separable $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H}$, a self-adjoint operator H acting on $\mathcal{H}$, and a (ϕ, τ, c)-twisted representation $\rho^\tau: G^\tau \to \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ such that

(i) the commutation relation (3.4) holds:

$$\rho^\tau(g)H = c(g)H\rho^\tau(g); \tag{3.9}$$

(ii) the Hamiltonian H is invertible with bounded inverse.

If H is an unbounded self-adjoint operator, then (ii) means that 0 is in its resolvent set, i.e., not in its spectrum.

Remark 3.10. It is probably better to include the Hamiltonian as part of G by assuming that G is a Galilean symmetry group with a distinguished oriented time direction $U \subset G$, which then comes with a fixed isomorphism $\mathbb{R} \cong U$, and further assuming that $G^\tau \to G$ is split over $U \subset G$. Then the Hamiltonian H is computed from the one-parameter group $e^{-itH/\hbar}$ which is part of the representation ρ^τ. Instead, we exclude time translations from our symmetry groups, as in Definition 2.17(ii) of crystallographic groups, and so keep the Hamiltonian distinct from the symmetry group. Nonetheless, for the definitions in Appendix D it is useful to incorporate the Hamiltonian into the symmetry, and we do so by constructing a QM symmetry class $(\widetilde{G}, \tilde{\phi}, \tilde{\tau})$ from an extended QM symmetry class as follows. Let $\widetilde{G} = G \ltimes_c \mathbb{R}$, where G acts on $\mathbb{R}$ via the homomorphism $c: G \to \{\pm 1\}$. (See Definition A.9 for semidirect products.) Similarly, set $\widetilde{G}^\tau = G^\tau \ltimes \mathbb{R}$, where G^τ acts via the composition $G^\tau \to G \xrightarrow{c} \{\pm 1\}$. Finally, define $\tilde{\phi}: \widetilde{G}^\tau \to G^\tau \to G \xrightarrow{\phi} \{\pm 1\}$. Then a gapped system with extended QM symmetry class (G, ϕ, τ, c) is precisely a $(\tilde{\phi}, \tilde{\tau})$-twisted representation of $\widetilde{G}$ such that the self-adjoint generator H of the unitary one-parameter group $\mathbb{R} \subset \widetilde{G}^\tau \to U(\mathcal{H})$ does not contain 0 in its spectrum.

Remark 3.11. We will add a finiteness hypothesis on the Hilbert space $\mathcal{H}$ for particular classes of gapped systems, depending on the dimension of the system and the symmetry group. These will be spelled out in Sects. 8 and 10. The representation ρ^τ obeys conditions which are specified in Definition 3.8. The second hypothesis in Definition 3.8, that the Hamiltonian be invertible, is specific to "gapped" systems of free fermions. We do *not* assume any boundedness of the Hamiltonian, such as appear in Lemma 3.3(ii). The assumption that 0 is not in the spectrum is a bit arbitrary. We can shift the Hamiltonian by a constant without changing the physics; our assumption is really that there is a spectral gap. Often one number, the *Fermi energy level*, is singled out in that gap. We take that Fermi level to be 0. We have nothing to say in this paper about systems without a gap.

4. Free Fermions

4.1. Symmetries in Algebraic Quantum Mechanics

Quantum mechanics is usually described, as in Sect. 1, in terms of a Hilbert space of states. Here we focus instead on the algebra of observables[11] in an abstract form. For a recent exposition of algebraic quantum mechanics, see [22, Sect. 2].

Let $\mathcal{A}$ be a *complex topological*[12] *$*$-algebra*. Thus, $\mathcal{A}$ is a complex topological vector space equipped with a continuous antilinear involution $*\colon \mathcal{A} \to \mathcal{A}$ which satisfies $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. An element $a \in \mathcal{A}$ is *real* (*self-adjoint*) if $a^* = a$. A self-adjoint element is an abstract observable; a *Hamiltonian* is a distinguished observable which is used to generate motion.

Remark 4.1. The abstract system described by $\mathcal{A}$ can be made concrete by the choice of an *$\mathcal{A}$-module*, a complex separable Hilbert space $\mathcal{H}$ and a continuous $*$-homomorphism $\mathcal{A} \to \mathrm{End}(\mathcal{H})$ into bounded[13] operators on $\mathcal{H}$. A *state* on $\mathcal{A}$ is a positive linear functional $\mathcal{A} \to \mathbb{C}$, and at least for C^*-algebras a state determines an $\mathcal{A}$-module via the GNS construction.

For systems with fermions, it is natural to assume that the operator algebra $\mathcal{A}$ is $\mathbb{Z}/2\mathbb{Z}$-graded. In that case, the Koszul sign rule demands that the $*$ involution satisfy

$$(ab)^* = (-1)^{|a|\,|b|} b^* a^* \tag{4.2}$$

for homogeneous elements $a, b \in \mathcal{A}$. Here $|a| \in \{0, 1\}$ is the parity of a.

Definition 4.3. (i) Let G be a topological group with $\mathbb{Z}/2\mathbb{Z}$-grading $\phi\colon G \to \{\pm 1\}$, and let $\mathcal{A}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded topological $*$-algebra. An *action* of (G, ϕ) on $\mathcal{A}$ is a homomorphism

$$\alpha\colon G \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{A}) \tag{4.4}$$

such that for all $g \in G$ the real linear map $\alpha(g)$ is complex linear or antilinear according as $\phi(g) = +1$ or -1 and $\alpha(g)$ preserves the $*$-structure:

$$\alpha(g)(a^*) = \big[\alpha(g)(a)\big]^* \tag{4.5}$$

for all $a \in \mathcal{A}$.

(ii) Let $H \in \mathcal{A}$ be a Hamiltonian. The action α is a *symmetry* of $(\mathcal{A}, H)$ if there exists a homomorphism $c\colon G \to \{\pm 1\}$ such that

$$\alpha(g)(H) = c(g)H, \quad g \in G. \tag{4.6}$$

The homomorphism c is the operator algebra analog of the homomorphism c in (3.9).

[11] The second author objects to this standard terminology since the observables, which are self-adjoint, do not form an associative algebra (though they do form a Jordan algebra).
[12] In the examples, we consider $\mathcal{A}$ is finite dimensional, so there is a unique topology compatible with the vector space structure.
[13] One can also allow unbounded operators. In our examples, $\mathcal{A}$ and $\mathcal{H}$ are finite dimensional and all operators are bounded.

Remark 4.7. If $\mathcal{H}$ is an *irreducible* $\mathcal{A}$-module preserved by the automorphism $\alpha(g)$ of $\mathcal{A}$, then that automorphism is realized on $\mathcal{H}$ by a line of complex linear or antilinear automorphisms. This leads to a ϕ-twisted extension of G and the (easy) analog of Wigner's theorem in this algebraic context. As before, the $\mathbb{Z}/2\mathbb{Z}$-grading ϕ also encodes the complex linearity vs. antilinearity of the real linear operators on $\mathcal{H}$.

4.2. Free Fermions

Let[14] $\mathcal{M}$ be a finite dimensional real vector space with positive definite inner product b. It encodes a system of finitely many fermions for which the operator algebra is

$$\mathcal{A} = \mathrm{Cl}(\mathcal{M}, b) \otimes \mathbb{C}. \tag{4.8}$$

Recall that the *Clifford algebra* $\mathrm{Cl}(\mathcal{M}, b)$ is the free unital real algebra generated by $\mathcal{M}$ subject to the canonical (anti)commutation relations

$$\xi_1\xi_2 + \xi_2\xi_1 = 2b(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \mathcal{M}. \tag{4.9}$$

(See Appendix B for a review of Clifford algebras.) Define the $*$-structure on (4.8) by[15]

$$\xi^* = \sqrt{-1}\,\xi, \quad \xi \in \mathcal{M}. \tag{4.10}$$

Then, for example, from (4.2) we deduce

$$(\xi_1\xi_2)^* = -\xi_2^*\xi_1^* = \xi_2\xi_1 \quad \text{for all } \xi_1, \xi_2 \in \mathcal{M}. \tag{4.11}$$

Let $O(\mathcal{M}, b)$ denote the orthogonal group of $(\mathcal{M}, b)$ and $\mathfrak{o}(\mathcal{M}, b)$ its Lie algebra. There is an embedding $\mathfrak{o}(\mathcal{M}, b) \otimes \mathbb{C} \subset \mathcal{A}$.

Definition 4.12. A *free fermion Hamiltonian*[16] is a self-adjoint element of $\mathfrak{o}(\mathcal{M}, b) \otimes \mathbb{C} \subset \mathcal{A}$.

Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathcal{M}$. An element of $\mathfrak{o}(\mathcal{M}, b) \otimes \mathbb{C} \subset \mathcal{A}$ has the form $H^{ij}e_ie_j$ for some $H^{ij} \in \mathbb{C}$ which satisfy $H^{ij} = -H^{ji}$. The self-adjointness implies $H^{ij} = \sqrt{-1}A^{ij}$ for $A^{ij} \in \mathbb{R}$, and necessarily $A^{ij} = -A^{ji}$.

Construction 4.13. Let $(\mathcal{M}, b)$ be a finite dimensional real inner product space and (G, ϕ) a $\mathbb{Z}/2\mathbb{Z}$-graded topological group. An action of G on the free fermion system based on $(\mathcal{M}, b)$ is determined by a continuous homomorphism $\beta\colon G \to O(\mathcal{M}, b)$. Extend the induced homomorphism $G \to \mathrm{Aut}_{\mathbb{R}}\bigl(\mathrm{Cl}(\mathcal{M}, b)\bigr)$ to a homomorphism (4.4) into real automorphisms of $\mathcal{A} = \mathrm{Cl}(\mathcal{M}, b) \otimes \mathbb{C}$ with $\alpha(g),\ g \in G$, complex linear or antilinear according as $\phi(g) = +1$ or -1. Recall

[14] The notation '$\mathcal{M}$' derives from 'mode space'.

[15] From the point of view of canonical quantization it is more natural [12, p. 679] to work with $\hat{\xi} = (\sqrt{-1}\,\hbar)^{1/2}\xi$, in which case the $*$-operation becomes complex conjugation.

[16] More fundamentally, a free fermion Hamiltonian is an even element in the second (quadratic) filtered subspace of the Clifford algebra, so has the form $H^{ij}e_ie_j + c$ for $H^{ij} \in \mathbb{C}$ arbitrary and $c \in \mathbb{C}$. Self-adjointness then leads to the same definition we give in the text, except we omit the constant c (which must be real).

that an orthogonal transformation $P \in O(\mathcal{M}, b)$ acts as an automorphism of the real Clifford algebra $\mathrm{Cl}(\mathcal{M}, b)$ by

$$P \cdot (\xi_1 \cdots \xi_k) = P\xi_1 \cdots P\xi_k, \quad \xi_i \in \mathcal{M}. \tag{4.14}$$

Let $\xi_1, \ldots, \xi_k \in \mathcal{M},\ x, y \in \mathbb{R}$. Then $g \in G$ acts on $\mathcal{A}$ by

$$\begin{aligned} &\alpha(g)\big(\xi_1 \cdots \xi_k \otimes (x + \sqrt{-1}\, y)\big) \\ &\quad = \begin{cases} \beta(g)\xi_1 \cdots \beta(g)\xi_k \otimes (x + \sqrt{-1}\, y), & \phi(g) = 1; \\ \beta(g)\xi_1 \cdots \beta(g)\xi_k \otimes (\sqrt{-1})^k (x - \sqrt{-1}\, y), & \phi(g) = -1. \end{cases} \end{aligned} \tag{4.15}$$

The extra factor of $(\sqrt{-1})^k$ is explained by footnote 15, in which $\hat{\xi}$ belongs to the subspace of $\mathcal{A}$ fixed by $*$. The action (4.15) satisfies (4.5).

The extension to real automorphisms of $\mathcal{A}$ provides the data of Definition 4.3.

4.3. Free Fermion Hamiltonians and Classical Symmetric Spaces

We reprove a theorem of Altland–Heinzner–Huckleberry–Zirnbauer [7,34].

Theorem 4.16. *Suppose the $\mathbb{Z}/2\mathbb{Z}$-graded group (G, ϕ) acts on the free fermion system based on the inner product space $(\mathcal{M}, b)$, as in Construction 4.13. Fix $c \colon G \to \{\pm 1\}$. Let $\mathfrak{p}$ denote the real vector space of free fermion Hamiltonians which are invariant under the action α, as in (4.6). Then $\exp(\mathfrak{p})$ is a classical symmetric space of compact type.*

Recall that a free fermion Hamiltonian has the form $H = \sqrt{-1}\, A^{ij} e_i e_j$ for $A^{ij} \in \mathbb{R}$ satisfying $A^{ij} = -A^{ji}$. Set $t = \phi c$. Then from (4.6) we deduce

$$\alpha(g)(A^{ij} e_i e_j) = t(g)(A^{ij} e_i e_j), \quad g \in G. \tag{4.17}$$

Note $A^{ij} e_i e_j \in \mathrm{Cl}(\mathcal{M}, b)$ lies in the *real* Clifford algebra. Identify $A^{ij} e_i e_j$ with an element $A \in \mathfrak{o}(\mathcal{M}, b)$. From the definition (4.14) of α we deduce

$$\mathfrak{p} = \{A \in \mathfrak{o}(\mathcal{M}, b) : \mathrm{Ad}_{\beta(g)}(A) = t(g)A \text{ for all } g \in G\}. \tag{4.18}$$

We may have $\mathfrak{p} = 0$, in which case $\exp(\mathfrak{p})$ is a single point.

Remark 4.19. Note that $\exp(\mathfrak{p}) = \{e^{tA} : t \in \mathbb{R},\ A \in \mathfrak{p}\} = \{e^{-\sqrt{-1}\, tH} : t \in \mathbb{R},\ \sqrt{-1}\, H \in \mathfrak{p}\}$, which may be construed as the space of unitary evolutions of the free fermion Hamiltonians. Here we exponentiate elements of $\mathfrak{o}(\mathcal{M}, b)$ to real orthogonal transformations of $\mathcal{M}$. We have not chosen an $\mathcal{A}$-module—here a fermionic Fock space—which would be the usual arena for unitary evolution of the free fermion.

Proof. Define

$$\mathfrak{k} = \{A \in \mathfrak{o}(\mathcal{M}, b) : \mathrm{Ad}_{\beta(g)}(A) = A \quad \text{for all } g \in G\}. \tag{4.20}$$

Use the embedding

$$\begin{aligned} \mathfrak{k} \oplus \mathfrak{p} &\longrightarrow \mathfrak{o}(\mathcal{M}, b) \oplus \mathfrak{o}(\mathcal{M}, b) \\ (A_1, A_2) &\longmapsto (A_1 + A_2, A_1 - A_2) \end{aligned} \tag{4.21}$$

to induce a Lie algebra structure on $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, and then an easy check shows

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{4.22}$$

Since the Killing form on $\mathfrak{o}(\mathcal{M}, b) \oplus \mathfrak{o}(\mathcal{M}, b)$ is negative definite, the same is true for $\mathfrak{h}$ and $\mathfrak{k}$, whence both are Lie algebras of compact type. It follows that $\exp(\mathfrak{p})$ is a compact symmetric space.

To see that $\mathfrak{h}$ and $\mathfrak{k}$ are classical Lie algebras—that is, matrix algebras over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$—we use an extended form of Theorem C.1 in Appendix C. That theorem asserts that the commutant of an *irreducible* real representation is a division algebra D over $\mathbb{R}$, so is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. An *isotypical* representation of G has the form $U \otimes V$, where G acts irreducibly on the real vector space V and trivially on the real vector space U. Then the commutant is $\mathrm{End}(U) \otimes D$, which is a matrix algebra over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Finally, the commutant of an arbitrary representation of G is a sum of matrix algebras over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Now $\mathfrak{k}$ is by definition (4.20) the commutant of a representation of G on $\mathcal{M}$, whence is classical. To prove that $\mathfrak{h}$ is classical, consider the representation of G on $\mathcal{M} \oplus \mathcal{M}$ given by

$$\begin{aligned} g &\longmapsto \begin{pmatrix} \beta(g) & 0 \\ 0 & \beta(g) \end{pmatrix}, \quad t(g) = +1, \\ g &\longmapsto \begin{pmatrix} 0 & \beta(g) \\ \beta(g) & 0 \end{pmatrix}, \quad t(g) = -1. \end{aligned} \tag{4.23}$$

The intersection of the commutant with the subalgebra of block diagonal matrices is $\mathfrak{h}$, embedded in $\mathrm{End}(\mathcal{M}) \oplus \mathrm{End}(\mathcal{M})$ by (4.21), so $\mathfrak{h}$ is also classical. □

Remark 4.24. There are ten sequences of symmetric spaces based on simple groups, and all of them can be realized by choosing G to be an appropriate Pin group using the construction of [44, Sect. 24]. See [45] for further details.

Remark 4.25. There is a parallel discussion for free bosons in which the bilinear form b is symplectic and the Clifford algebra is replaced by the Heisenberg-Weyl algebra. See [45] for further details.

4.4. The Dirac–Nambu Space

We continue with the free fermion system based on $(\mathcal{M}, b)$ with symmetry group (G, ϕ). Define the *Dirac–Nambu space*

$$\mathcal{H}_{\mathrm{DN}} := \mathcal{M} \otimes \mathbb{C} \tag{4.26}$$

While $\mathcal{H}_{\mathrm{DN}}$ is a complex vector space, we do not (yet) endow it with a hermitian metric. Define

$$\rho\colon G \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}_{\mathrm{DN}}) \tag{4.27}$$

by letting $\rho(g)$ be the complex linear or antilinear extension of $\beta(g)$ according as $\phi(g) = +1$ or -1. In symbols,

$$\begin{aligned} \rho(g)\big(\xi \otimes (x + \sqrt{-1}\, y)\big) = \beta(g)(\xi) \otimes (x + \phi(g)\sqrt{-1}\, y), \\ g \in G, \quad \xi \in \mathcal{M}, \quad x, y \in \mathbb{R}. \end{aligned} \tag{4.28}$$

We next choose a complex structure I on $\mathcal{M}$, which in geometric quantization plays the role of a complex polarization. Recall that I is a real isometry $I\colon \mathcal{M} \to \mathcal{M}$ which satisfies $I^2 = -\mathrm{id}_{\mathcal{M}}$. We require a compatibility condition with the symmetry group G, namely, $I \in \mathfrak{p}$. This means in particular that each $g \in G$ either preserves or reverses I. Such a complex structure may or may not exist. Assuming it does, extend I to a complex linear endomorphism of $\mathcal{H}_{\mathrm{DN}}$. Then $\sqrt{-1}\,I\colon \mathcal{H}_{\mathrm{DN}} \to \mathcal{H}_{\mathrm{DN}}$ squares to the identity, so induces a decomposition

$$\mathcal{H}_{\mathrm{DN}} \cong V \oplus \overline{V} \tag{4.29}$$

into its -1 and $+1$-eigenspaces. Here V denotes the complex vector space $\mathcal{M}$ with complex structure I. Since we assume $I \in \mathfrak{p}$, the map $\beta(g) \in O(\mathcal{M}, b)$ commutes or anticommutes with the complex structure I according as $t(g) = +1$ or -1. Then since

$$\rho(g)\sqrt{-1}\,I = c(g)\sqrt{-1}\,I\rho(g), \tag{4.30}$$

the automorphism $\rho(g)$ in (4.28) preserves or reverses the decomposition (4.29) according as $c(g) = +1$ or -1. The complex structure determines a hermitian structure on V:

$$\langle \xi_1, \xi_2\rangle := b(\xi_1, \xi_2) + \sqrt{-1}\,b(I\xi_1, \xi_2), \quad \xi_1, \xi_2 \in V. \tag{4.31}$$

As usual, the hermitian metric induces a linear isomorphism $\overline{V} \cong V^*$, a hermitian metric on that space, and so too a hermitian metric on $\mathcal{H}_{\mathrm{DN}}$.

We extend Hamiltonians to act linearly and self-adjointly on $\mathcal{H}_{\mathrm{DN}}$. We do *not* ask that a Hamiltonian be compatible with the complex structure, lest we rule out interesting examples such as Bogoliubov-de Gennes Hamiltonians. Nonetheless, for an *invertible* Hamiltonian $H = \sqrt{-1}\,A^{ij}e_ie_j \in \mathcal{A}$ invariant under the action (4.15) of (G, ϕ), there is a natural associated complex structure I_A. For this case, the associated $A \in \mathfrak{o}(\mathcal{M}, b)$ is also invertible, and we use the spectral calculus to rescale the eigenvalues and define $I_A = A/|A|$. Since $A \in \mathfrak{p}$, it follows that $I_A \in \mathfrak{p}$.

Remark 4.32. The Dirac–Nambu space is *not* an $\mathcal{A}$-module in the sense of Remark 4.1, so does not fit the standard paradigm of a realization of an abstract quantum system. Rather, given the complex structure I one usually forms the *fermionic Fock space* $\mathcal{H}_{\mathrm{Fock}} = \bigwedge^{\bullet} V$, which *is* an $\mathcal{A}$-module. The degree in the exterior algebra is the particle number, so $V \subset \mathcal{H}_{\mathrm{Fock}}$ may be identified with the 1-particle subspace: lines in V represent 1-particle states. The action of G on $\mathcal{A}$ is implemented as a *projective* action of G on $\mathcal{H}_{\mathrm{Fock}}$. Namely, the entire orthogonal group $O(\mathcal{M}, b)$ acts projectively on $\mathcal{H}_{\mathrm{Fock}}$ as the (s)pin representation, and there is a pullback central extension of G along $\beta\colon G \to O(\mathcal{M}, b)$. Contrast this projective action with the nonprojective action (4.27) of G on the Dirac–Nambu space $\mathcal{H}_{\mathrm{DN}}$. The symmetries in G are very special among orthogonal symmetries as they either preserve or reverse the complex structure I.

Despite the considerations in Remark 4.32, we use the Dirac–Nambu space $\mathcal{H}_{\mathrm{DN}}$ as the Hilbert space of a free fermion system. In the old Dirac picture of particles and holes, lines in $\overline{V} \subset \mathcal{H}_{\mathrm{DN}}$ represent "1-hole" states. Thus, we regard $\mathcal{H}_{\mathrm{DN}}$ as implementing this particle–hole picture, special to free fermions. From this point of view elements $g \in G$ with $c(g) = +1$ are *particle–hole preserving* whereas elements $g \in G$ with $c(g) = -1$ are *particle–hole reversing*.

5. Topological Phases

Our interest is in the topological properties of gapped systems, so we need to specify a notion of topological equivalence.

Definition 5.1. Gapped systems with extended QM symmetry class (G, ϕ, τ, c) are in the same *topological phase* if they are homotopic. We denote the set of equivalence classes by $\mathcal{TP}(G, \phi, \tau, c)$.

This definition requires explanation and perhaps motivation. First, the explanations. An isomorphism of gapped systems $(\mathcal{H}_0, H_0, \rho_0^\tau)$ and $(\mathcal{H}_1, H_1, \rho_1^\tau)$ is an isomorphism $\mathcal{H}_0 \to \mathcal{H}_1$ which intertwines the Hamiltonians H_i and the representations $\rho_i^\tau, i = 0, 1$. Next, two systems are homotopic if there exists a continuously varying family of gapped systems parametrized by the 1-simplex $\Delta^1 = [0, 1]$ whose restriction to $\{0\}$ is isomorphic to $(\mathcal{H}_0, H_0, \rho_0^\tau)$ and whose restriction to $\{1\}$ is isomorphic to $(\mathcal{H}_1, H_1, \rho_1^\tau)$. In Appendix D, we give a precise definition of a continuous family of quantum systems; in particular Definition D.12 defines a continuous family of gapped systems with given extended QM symmetry class. Isomorphism and homotopy are completely standard equivalence relations in many contexts in mathematics and, increasingly, in physics. The key is to define the notion of a *continuous* deformation—a slightly technical, but nonetheless crucial, task undertaken in Appendix D.

The set $\mathcal{TP}(G, \phi, \tau, c)$ has an algebraic structure given by amalgamation of quantum systems. Usually in quantum mechanics if $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces of two systems, then the Hilbert space of the composite quantum system is the (graded) *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$. This applies to the Fock spaces of free electron systems. Since the tensor product of Fock spaces corresponds to the *direct sum* of *Dirac–Nambu* Hilbert spaces (4.26), and we have in mind the Dirac–Nambu picture, we use direct sum to combine systems. Direct sum is compatible with the notion of topological equivalence in Definition 5.1, and so $\mathcal{TP}(G, \phi, \tau, c)$ is a commutative *monoid*, that is, a set with an associative, commutative composition law with an identity element. The composition is written additively and the identity element is represented by the zero Hilbert space.

Remark 5.2. We emphasize that in this paper we do not use any detailed structure of free systems of fermions. This use of direct sum rather than tensor product to combine quantum systems is the concrete manifestation of the Dirac–Nambu picture; in a general system only tensor products exist.

Definition 5.1 is quite general and seems to be what the physics demands. But it may be difficult to compute, and so we define a weaker invariant which is easier to compute and which is in addition an abelian group. We obtain it from the commutative monoid $\mathcal{TP}(G, \phi, \tau, c)$ by group completion or by imposing a further "topological triviality" relation: see Definitions 8.5, 10.14, and 10.20. We consider each of these in turn.

5.1. Group Completion

Construction 5.3 (*Grothendieck group completion*). Let M be a commutative monoid. Its *group completion* is the quotient A of $M \times M$ by the equivalence relation

$$(m_1', m_1'') \sim (m_2', m_2'') \quad \text{iff} \quad m_1' + m_2'' + m = m_1'' + m_2' + m \tag{5.4}$$

for some $m \in M$. Then A is an abelian group. There is a homomorphism $M \to A$ given by $m \mapsto (m, 0)$. The group completion A of M satisfies a universal property, which characterizes it up to unique isomorphism: if B is an abelian group and $\theta\colon M \to B$ a homomorphism of commutative monoids in the sense that $\theta(0) = 0$ and $\theta(m_1 + m_2) = \theta(m_1) + \theta(m_2)$ for all $m_1, m_2 \in M$, then there is a unique homomorphism of abelian groups $\bar{\theta}\colon A \to B$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \theta\ } & B \\ & \searrow & \nearrow_{\bar{\theta}} \\ & A & \end{array} \tag{5.5}$$

commutes.

Intuitively, (m', m'') represents $m' - m''$.

Example 5.6. There are two elementary, but illustrative, examples. First, the group completion of the commutative monoid $M = \mathbb{Z}^{\geq 0}$ of nonnegative integers is the abelian group of all integers. Note in this case that $\mathbb{Z}^{\geq 0}$ is also a semi-ring—there is an associative multiplication with identity which distributes over addition—and so the group completion $\mathbb{Z}$ is a ring. Next, suppose G is a compact Lie group. Let $\mathrm{Rep}(G)$ denote the set of equivalence classes of *finite dimensional* complex representations of G. Weyl's "unitary trick" implies that every representation is completely reducible—a direct sum of irreducible representations—so $\mathrm{Rep}(G)$ can also be defined as the free commutative monoid generated by isomorphism classes of irreducible complex representations of G. The group completion of $\mathrm{Rep}(G)$ is the *representation ring* K_G of G.

Example 5.7. For $G = SU_2$, there is an irreducible representation for each nonnegative integer, which labels its dimension, so as a commutative monoid $\mathrm{Rep}(G)$ is the free commutative monoid on $\mathbb{Z}^{\geq 0}$ whose elements are finite linear combinations of symbols $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ with positive integer coefficients. The structure of the *ring* K_G is much simpler: it is the polynomial ring $\mathbb{Z}[t]$ on a single generator $t = \mathbf{2}$, which represents the 2-dimensional irreducible representation. By Clebsch–Gordan, other irreducible representations are polynomials in t. For example, $\mathbf{3} = t^2 - 1$. The representation ring of a *connected, simply connected* compact Lie group is always a polynomial ring, but for a general compact Lie group there are relations. For example, the representation ring of the cyclic group of order 2 is isomorphic to $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$.

Remark 5.8. The multiplication in the representation ring K_G does not play a role in our considerations.

Remark 5.9. We illustrate why a finiteness hypothesis is crucial to obtain a nontrivial abelian group after completing. Resuming Example 5.6 consider the trivial group $G = \{1\}$, so $\mathrm{Rep}(G) \cong \mathbb{Z}^{\geq 0}$ is the set of isomorphism classes of finite dimensional vector spaces and the group completion is the integers. If we relax the finite dimensionality, so allow infinite dimensional vector spaces, then $\mathbb{Z}^{\geq 0}$ is replaced by the commutative monoid $\mathbb{Z}^{\geq 0} \cup \{\infty\}$. In this monoid, $n + \infty = \infty$ for any nonnegative integer n. The group completion of $\mathbb{Z}^{\geq 0} \cup \{\infty\}$ is the trivial abelian group. This is because for any pair of nonnegative integers (n', n''), we have

$$(n', n'') \sim (\infty, \infty) \sim (0, 0) \tag{5.10}$$

according to the equivalence relation (5.4) which defines the group completion. The maneuver (5.10) is called the *Eilenberg swindle*. We use a more sophisticated variation in the proof of Theorem 10.21.

5.2. Quotienting by Topological Triviality

There is another construction of the representation ring which illustrates the idea of "topological triviality". We start with the set $\mathrm{Rep}_s(G)$ of $\mathbb{Z}/2\mathbb{Z}$*-graded* finite dimensional complex representations up to isomorphism. (The 's' stands for 'super', which is a synonym for '$\mathbb{Z}/2\mathbb{Z}$-graded'.) A $\mathbb{Z}/2\mathbb{Z}$-graded representation is a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space $W = W^0 \oplus W^1$ with a homomorphism $\rho\colon G \to \mathrm{Aut}(W)$ to automorphisms which separately preserve W^0 and W^1. So it is the *direct sum* of two representations, but is to be thought of as the *formal difference* of the representations. The role of the grading is to encode the formal minus sign. Direct sum of super representations endows $\mathrm{Rep}_s(G)$ with the structure of a commutative monoid. Then define the submonoid $\mathrm{Triv}_s(G)$ to be equivalence classes of $\mathbb{Z}/2\mathbb{Z}$-graded representations $\rho\colon G \to \mathrm{Aut}(W)$ for which there exists an odd automorphism $P\colon W \to W$ such that

$$P\rho(g) = \rho(g)P, \quad g \in G. \tag{5.11}$$

Lemma 5.12. *The quotient* $\mathrm{Rep}_s(G)/\mathrm{Triv}_s(G)$ *is an abelian group.*

The additive inverse of $W^0 \oplus W^1$ is the parity-reversed representation $W^1 \oplus W^0$, which we define more precisely in the proof.

Proof. Let Π denote the complex line $\mathbb{C}$ in odd degree and $\pi \in \Pi$ the (odd) element $1 \in \mathbb{C}$. There is a canonical isomorphism $\Pi \otimes \Pi \cong \mathbb{C}$ under which $\pi \otimes \pi = 1$. If W is a finite dimensional representation of G, we define its parity-reversal to be the tensor product $\Pi \otimes W$, with G acting trivially on Π. Then $W \oplus (\Pi \otimes W)$ lies in $\mathrm{Triv}_s(G)$: for $x \in W \oplus (\Pi \otimes W)$ set $P(x) = \pi \otimes x$. Thus, $\Pi \otimes W$ is an inverse to W in $\mathrm{Rep}_s(G)/\mathrm{Triv}_s(G)$. □

The map $W^0 \oplus W^1 \mapsto (W^0, W^1)$ from $\mathbb{Z}/2\mathbb{Z}$-graded representations to the group completion of the monoid of equivalence classes of ungraded representations is an isomorphism, after modding out by $\mathrm{Triv}_s(G)$. In other words, there is an isomorphism

$$\mathrm{Rep}_s(G)/\,\mathrm{Triv}_s(G) \cong K_G. \tag{5.13}$$

5.3. Reduced Topological Phases

We call the abelian group obtained from the commutative monoid in Definition 5.1 by these processes—group completion or quotienting by "topological triviality"—the group of *reduced topological phases* $\mathcal{RTP}(G, \phi, \tau, c)$. The choice of process depends on the particular situation and will be specified in each case. Our main results identify $\mathcal{RTP}(G, \phi, \tau, c)$ with a topological K-theory group in three different situations: see Theorems 8.6, 10.15, and 10.21.

One immediate observation points the way to K-theory.

Lemma 5.14. *In every topological equivalence class of gapped systems, there is a Hamiltonian H with $H^2 = 1$.*

Proof. Let $(\mathcal{H}, H_0, \rho^\tau)$ be a gapped system. Define the one-parameter family of functions

$$\begin{aligned} f_t\colon \mathbb{R}\backslash\{0\} &\longrightarrow \mathbb{R}\backslash\{0\} \\ \lambda &\longmapsto (1-t)\lambda + t\lambda/|\lambda| \end{aligned} \tag{5.15}$$

where $t \in [0, 1]$. The spectral theorem gives a continuous one-parameter family of Hamiltonians $H_t = f_t(H_0)$ and $H = H_1$ squares to the identity. □

H is sometimes called the *spectral flattening* of H_0. It induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{H}$ whose homogeneous subspaces are the ± 1-eigenspaces of H. The $\mathbb{Z}/2\mathbb{Z}$-grading is a hallmark of K-theory, but to obtain nontrivial results we must impose finiteness conditions. We will specify these in Sects. 8 and 10; see Hypotheses 8.2, 8.3, and 10.9.

6. Special Extended QM symmetry Classes

6.1. A Restricted Set of Extended QM symmetry Classes

Let $\mathcal{C} = \{\pm 1\} \times \{\pm 1\}$ and

$$\mathcal{C} = \{\pm 1\} \times \{\pm 1\} \xrightarrow{\phi_{\mathcal{C}}} \{\pm 1\} \tag{6.1}$$

be multiplication. The element $\overline{T} = (-1, 1)$ represents time-reversal and $\overline{C} = (1, -1)$ represents Hamiltonian-reversal.

Definition 6.2. Let G be a Lie group equipped with a homomorphism $\psi = (t, c)\colon G \to \mathcal{C}$. An extended QM symmetry class (G, ϕ, τ, c) is *ψ-standard* if $\phi = \phi_{\mathcal{C}} \circ \psi = tc$ and G^τ is pulled back from a $\phi_{\mathcal{C}}$-twisted extension of $\psi(G) \subset \mathcal{C}$.

Thus, $\psi(g) = \big(t(g), c(g)\big)$ tells whether $g \in G$ is implemented as time-reversing and/or Hamiltonian-reversing, and g is implemented unitarily or antiunitarily according to $\phi(g)$: it is antiunitary if it is either time-reversing or Hamiltonian-reversing but not both, and it is unitary if it is neither or both. Furthermore, in a (ϕ, τ, c)-twisted representation [Definition 3.7(ii)] of G the subgroup of G^τ lying over $\ker \psi \subset G$ is split—there is no nontrivial central extension—but the hypothesis that the extension is pulled back is stronger.

Remark 6.3. The notion of a ψ-standard extended QM symmetry class is artificial[17] and only given here to reproduce the tenfold way which permeates the literature on topological classes of free fermion systems. While the restriction $\phi = \phi_{\mathcal{C}} \circ \psi = tc$ is physically reasonable, we do not see a rationale to require that the twisted central extension be a pullback. So in general there are more extended QM symmetry classes which are not pullbacks. We give an example below in Remark 6.5.

Returning now to free fermion systems, it is clear from Definition 6.2 that a ψ-standard extended QM symmetry class is determined by a subgroup $A \subset \mathcal{C}$ and an extended QM symmetry class based on A and the restriction of $\phi_{\mathcal{C}}$ to A. There are 5 subgroups and a total of 10 possibilities.

Proposition 6.4 (tenfold way).

(i) *There is a unique ψ-standard extended QM symmetry class based on the trivial subgroup $A = 1$.*

(ii) *There is a unique ψ-standard extended QM symmetry class based on the diagonal subgroup $A = \{\pm 1\} \subset \{\pm 1\} \times \{\pm 1\}$.*

(iii) *There are two ψ-standard extended QM symmetry classes based on $A = \{\pm 1\} \times 1$. Let $T \in A^\tau$ be a lift of $\overline{T}$; then the two QM symmetry classes are distinguished by $T^2 = \pm 1$.*

(iv) *Similarly, there are two QM symmetry classes based on $A = 1 \times \{\pm 1\}$, distinguished by $C^2 = \pm 1$, where $C \in A^\tau$ is a lift of $\overline{C}$.*

(v) *There are four ψ-standard extended QM symmetry classes based on $A = \mathcal{C}$, distinguished by $T^2 = \pm 1$ and $C^2 = \pm 1$.*

The presence or non-presence of $\overline{T}, \overline{C}, \overline{TC}$ and the squares of T, C are the standard invariants of the tenfold way. We term the pair (A, A^τ) of the subgroup and its central extension a 'CT type'. Propositions 1.16 and 6.4 have quick and easy proofs via group cohomology computations, once the problem of group extensions is translated to a cohomology question.[18] Instead we adopt a more direct, though unnecessarily lengthy, approach.

[17] There is a physical viewpoint which makes the assumption of ϕ-standard and ψ-standard symmetries somewhat more natural. In systems with low symmetry—such as systems with disorder—it might be that the *only* symmetries present are a time-reversal symmetry and—in the case of free fermions—a Hamiltonian-reversing symmetry.

[18] Untwisted central extensions of A up to isomorphism form an abelian group which is identified with the second group cohomology $H^2(A; \mathbb{T})$ with A acting trivially on the coefficients. More generally, a homomorphism $\phi_{\mathcal{C}} \colon A \to \{\pm 1\}$ determines an action of A on $\mathbb{T}$ (if $\phi_{\mathcal{C}}(a) = -1$, then a acts by $\lambda \mapsto \lambda^{-1}$), and $\phi_{\mathcal{C}}$-twisted extensions up to isomorphism are identified with $H^2(A; \widetilde{\mathbb{T}})$ with twisted coefficients. These cohomology groups are easy to compute for the small abelian groups A in Proposition 6.4.

Remark 6.5. As we have said, we find the ψ-standard and ϕ-standard conditions—which are generally assumed in the literature—to be artificial. It is certainly easy to give physically reasonable examples where they fail. For a non-ϕ-standard example, consider $G = \{\pm 1\} \times \{\pm 1\}$ with $\phi\colon G \to \{\pm 1\}$ defined as the product. Then we know from Proposition 6.4(v) that there are four nonisomorphic ϕ-twisted extensions G^τ of G, but there can be only two nontrivial pullbacks of ϕ-twisted extensions of $\{\pm 1\}$ with $\phi = \mathrm{id}$. Concretely, if in G^τ we have $T^2 = C^2 = 1$ or $T^2 = C^2 = -1$, then the extension is a pullback and hence is ϕ-standard, but if $T^2 = 1, C^2 = -1$ or $T^2 = -1, C^2 = +1$, then the extension is not a pullback. Similarly, one can make non-ψ-standard extended QM symmetry classes by considering $G = \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\}$ with $\psi\colon G \to \mathcal{C}$ given by multiplying the first two and the last two factors of G. There are eight ϕ-twisted extensions of G (defined by the signs of the squares of the lifts of the three generators of G) but there are only four pullbacks of extensions of $\mathcal{C}$.

6.2. Twisted Central Extensions and Semidirect Products

As a preliminary, we develop some facts about twisted extensions of semidirect products, despite the fact that to prove Propositions 1.16 and 6.4 we only need to consider direct products; we will encounter semidirect products later. Here we show how to build up complicated ϕ-twisted extensions out of simpler extensions.

Let G', G'' be topological groups and $\alpha\colon G'' \to \mathrm{Aut}(G')$ a continuous homomorphism. Recall the semidirect product $G = G'' \ltimes_\alpha G'$ defined in Definition A.9. Now suppose

$$1 \longrightarrow \mathbb{T} \longrightarrow H' \xrightarrow{\pi'} G' \longrightarrow 1 \tag{6.6}$$

is a *central* extension, $\phi\colon G'' \to \{\pm 1\}$ is a homomorphism and

$$1 \longrightarrow \mathbb{T} \longrightarrow H'' \xrightarrow{\pi''} G'' \longrightarrow 1 \tag{6.7}$$

a ϕ -twisted extension, and

$$\tilde{\alpha}\colon G'' \longrightarrow \mathrm{Aut}(H') \tag{6.8}$$

is a continuous homomorphism such that for $g'' \in G''$ and $\lambda \in \mathbb{T} \subset H'$,

$$\tilde{\alpha}(g'')(\lambda) = \begin{cases} \lambda, & \phi(g'') = +1; \\ \bar{\lambda}, & \phi(g'') = -1; \end{cases} \tag{6.9}$$

and for $g'' \in G''$, the diagram

$$\begin{array}{ccc} H' & \xrightarrow{\pi'} & G' \\ {\scriptstyle \tilde{\alpha}(g'')}\downarrow & & \downarrow{\scriptstyle \alpha(g'')} \\ H' & \xrightarrow{\pi'} & G' \end{array} \tag{6.10}$$

commutes. Let $\phi_G\colon G \to \{\pm 1\}$ be the composition $G \to G'' \xrightarrow{\phi} \{\pm 1\}$.

Construction 6.11. The data (6.6), (6.7), (6.8) determine a ϕ_G-twisted extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\tau \xrightarrow{\ \pi\ } G \longrightarrow 1 \tag{6.12}$$

which restricts on $G' \subset G$ to (6.6) and on $G'' \subset G$ to (6.7).

Proof. We use a variation of the semidirect product construction (A.10). Namely, define the set underlying G^τ as the quotient of $H'' \times H'$ by the diagonal embedding of $\mathbb{T}$: identify $(h'', h') \sim (\lambda h'', \lambda h')$ for all $h'' \in H'', h' \in H'$, $\lambda \in \mathbb{T}$. Multiplication is defined so that $H'' \subset G^\tau$ and $H' \subset G^\tau$ are subgroups, and

$$h'' \cdot h' = \tilde{\alpha}(\pi'' h'')(h') \cdot h'', \quad h'' \in H'', \quad h' \in H'. \tag{6.13}$$

A routine check shows that (6.13) is well-defined and descends to the quotient by the diagonal $\mathbb{T}$. □

Lemma 6.14. *Let $G = G'' \ltimes_\alpha G'$ be a semidirect product. Suppose $\phi_G \colon G \to \{\pm 1\}$ is a homomorphism which is pulled back from $\phi \colon G'' \to \{\pm 1\}$. Then any ϕ_G-twisted extension* (1.8) *is obtained via Construction 6.11.*

Proof. Given a ϕ_G-twisted extension (1.8), define the central extension (6.6) of G' as the restriction of (1.8) over $G' \subset G$ and the ϕ -twisted extension (6.7) of G'' as the restriction of (1.8) over $G'' \subset G$. For $h'' \in H'' \subset G^\tau$ and $h' \in H' \subset G^\tau$, define

$$\tilde{\alpha}(h'')(h') = (h'')h'(h'')^{-1}. \tag{6.15}$$

Straightforward checks show: (i) $\tilde{\alpha}$ depends only on $\pi(h'')$; (ii) $\tilde{\alpha}(h'')$ is an automorphism of H' which satisfies (6.9) and (6.10); and (iii) $\tilde{\alpha} \colon H'' \to \mathrm{Aut}(H')$ is a homomorphism. □

Lemma 6.16. *Let $\phi_G \colon G \to \{\pm 1\}$ be a homomorphism which is pulled back from $\phi \colon G'' \to \{\pm 1\}$ and fix a central extension* (6.6) *of $G' \subset G$. Then isomorphism classes of ϕ_G-twisted extensions of G which restrict on G' to* (6.6) *are in 1:1 correspondence with pairs $(\tilde{\alpha}, \epsilon'')$, where $\tilde{\alpha}$ is a continuous homomorphism as in* (6.8) *and ϵ'' is the isomorphism class of a ϕ-twisted extension as in* (6.7).

An isomorphism $\varphi \colon G^{\tau_1} \to G^{\tau_2}$ of ϕ_G-twisted extensions—as in (A.4)—is here required to be the identity map on the common subgroup H' of G^{τ_1} and G^{τ_2}.

Proof. To see that $\tilde{\alpha}$ is an invariant, apply an isomorphism $\varphi \colon G^{\tau_1} \to G^{\tau_2}$ to (6.15). Restrict φ over G'' to see that the isomorphism class of the restriction of G^τ over G'' is an invariant. Conversely, by Construction 6.11 every pair $(\tilde{\alpha}, \epsilon'')$ gives a ϕ_G-twisted extension. □

6.3. Proof of the Classification

Propositions 1.16 and 6.4 follow immediately from the next result.

Lemma 6.17. (i) *Every central extension*

$$1 \longrightarrow \mathbb{T} \longrightarrow H \longrightarrow \{\pm 1\} \longrightarrow 1 \tag{6.18}$$

is split, that is, is isomorphic to the trivial central extension.

(ii) *Let* $\phi\colon \{\pm 1\} \to \{\pm 1\}$ *be the identity map. Then there are two isomorphism classes of* ϕ*-twisted extensions* (6.18) *distinguished by whether the lift of* $-1 \in \{\pm 1\}$ *to* H *squares to* $+1$ *or* -1.

(iii) *Let* $G'' = G' = \{\pm 1\}$ *with generators* g'', g' *and let* $\phi\colon G'' \times G' \to \{\pm 1\}$ *be projection onto the first factor. Then there are four isomorphism classes of* ϕ*-twisted extensions*

$$1 \longrightarrow \mathbb{T} \longrightarrow H \longrightarrow G'' \times G' \longrightarrow 1. \tag{6.19}$$

Choose a lift h' *of* g' *so that* $(h')^2 = +1$, *and let* h'' *be any lift of* g''. *Then the four isomorphism classes of extensions are distinguished by the independent possibilities* $(h'')^2 = \pm 1$ *and* $(h'h'')^2 = \pm 1$.

Proof. For (i), if h is any lift of $-1 \in \{\pm 1\}$, then $h^2 = \mu \in \mathbb{T}$ for some μ, so replacing h by λh for $\lambda^2 = \mu^{-1}$ we have $(\lambda h)^2 = +1$. This splits (6.18).

For (ii), if h is any lift of $-1 \in \{\pm 1\}$, and $h^2 = \mu \in \mathbb{T}$, then $(\lambda h)^2 = \lambda\bar{\lambda}h^2 = \mu$ for all $\lambda \in \mathbb{T}$. Since $h = (h^2)h(h^2)^{-1} = \mu h \mu^{-1} = \mu^2 h$, we conclude $\mu^2 = +1$ and so $h^2 = \pm 1$.

For (iii), observe first that the restriction $H' \subset H$ of the extension to $G' \subset G' \times G''$ is split, by (i), and since ϕ is trivial on G' we have $H' \cong \{\pm 1\} \times \mathbb{T}$. Now apply Lemma 6.16. By (ii) there are two possibilities for the isomorphism class ϵ'' of the extension over G''. There are also two homomorphisms $\tilde{\alpha}\colon G'' \to \mathrm{Aut}(H')$ which satisfy (6.9) and (6.10): namely, $\beta = \tilde{\alpha}(-1) \in \mathrm{Aut}(H')$ is complex conjugation on $\mathbb{T} \subset H'$ and either $\beta(h') = h'$ or $\beta(h') = -h'$, where $-h'$ is the product of $-1 \in \mathbb{T} \subset H'$ and $h' \in H'$. □

6.4. More QM symmetry Classes

As a further illustration of these techniques, we prove that certain twisted extensions, though not ψ-standard (Definition 6.2), are nonetheless uniquely determined by simple data. In the following examples, we incorporate the mass central extension of the Galilean group, which is physically relevant, but which does not fit into the tenfold scheme of Proposition 6.4. Let (M, Γ) be a Galilean spacetime. Set

$$\begin{aligned} G_1 &= \text{identity component of } \mathrm{Aut}(M, \Gamma) \\ G_2 &= G_1 \times \{\pm 1\} \\ G_3 &= \mathrm{Aut}^+(M, \Gamma) \\ G_4 &= G_3 \times \{\pm 1\} \end{aligned} \tag{6.20}$$

Here G_3 is the group of parity-preserving symmetries of (M, Γ); see Definition 2.8. Notice that G_1 is *not* the Galilean group; the Galilean group is a double cover of G_1. Each of these groups is naturally equipped with a homomorphism t to $\{\pm 1\}$ which tracks time-reversal: it is trivial for G_1, G_2 and surjective for G_3, G_4. Let the homomorphism c to $\{\pm 1\}$ to be trivial on G_1, G_3 and either trivial or projection onto the second factor on G_2, G_4. (These two cases correspond to the two possibilities contemplated in Lemma 6.25 below.) Thus, for each G_i we have a homomorphism $\psi = (t, c)\colon G_i \to \mathfrak{C}$. Recall that there is a universal mass central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G_1^\tau \longrightarrow G_1 \longrightarrow 1 \tag{6.21}$$

of the identity component of $\mathrm{Aut}(M, \Gamma)$.[19]

Proposition 6.22. *Let (G, ϕ, τ) be a QM symmetry class with $G = G_i$ for some $i = 1, 2, 3, 4$ which satisfies (i) $\phi = \phi_{\mathcal{C}} \circ \psi$ and (ii) G^τ restricts to the mass central extension (6.21) over $G_1 \subset G$. Then the QM symmetry class is determined by a twisted extension of the quotient G/G_1.*

Remark 6.23. The hypothesis (ii) is a bit unnatural. For example, we should surely replace G_1 by its Galilean double cover, in which case there are more possibilities if $d = 0, 1$. Also, we could have a larger group of symmetries which commutes with the geometric spacetime symmetries, and that group may have nontrivial extensions.

As each of the groups $G = G_1, G_2, G_3, G_4$ is a semidirect product of G_1 and a finite group isomorphic to G/G_1, and the extension on the subgroup G_1 has been fixed, we can apply Lemma 6.14 to conclude that Proposition 6.22 follows if we prove that the homomorphism $\tilde{\alpha}$ in (6.8) is unique. Recall from (6.20) that G_1 is the component of the identity of symmetries of a Galilean spacetime (M, Γ). Let

$$1 \longrightarrow \mathbb{T} \longrightarrow G_1^{\tau_1} \xrightarrow{\pi_1} G_1 \longrightarrow 1 \tag{6.24}$$

be the mass central extension.

Lemma 6.25. (i) *The constant map with value $\mathrm{id}_{G_1^{\tau_1}}$ is the unique homomorphism*

$$\tilde{\alpha}\colon \{\pm 1\} \longrightarrow \mathrm{Aut}(G_1^{\tau_1}) \tag{6.26}$$

such that $\tilde{\alpha}(-1)(\lambda) = \lambda$ for $\lambda \in \mathbb{T}$ and $\tilde{\alpha}(-1)$ covers the identity automorphism of G_1.

(ii) *There is a unique homomorphism (6.26) such that $\tilde{\alpha}(-1)(\lambda) = \bar{\lambda}$ for $\lambda \in \mathbb{T}$ and $\tilde{\alpha}(-1)$ covers the automorphism of G_1 obtained by conjugation by a time-reversal symmetry.*

Proof. Set $\beta = \tilde{\alpha}(-1)$. Then for $h \in G_1^{\tau_1}$ with $\pi_1 h = g$, we have $\beta(h) = \mu(g)h$ for some character $\mu\colon G_1 \to \mathbb{T}$. Since G_1 is connected, μ is determined by its infinitesimal character $\dot{\mu}\colon \mathfrak{g}_1 \to \mathbb{R}$. Following the notation in Sect. 2 we write the Lie algebra $\mathfrak{g}_1$ of G_1 as an extension

$$0 \longrightarrow \mathfrak{o}(V) \ltimes (V_{\text{boost}} \oplus V_{\text{trans}}) \longrightarrow \mathfrak{g}_1 \longrightarrow W/V \longrightarrow 0, \tag{6.27}$$

where the orthogonal algebra $\mathfrak{o}(V)$ acts diagonally by the standard representation on $V_{\text{boost}} \oplus V_{\text{trans}}$ (infinitesimal boosts plus infinitesimal spatial translations).[20] Let $d = \dim V$. If $d \geq 3$, then $\mathfrak{o}(V)$ is semisimple and commutators in $\mathfrak{g}_1' = \mathfrak{o}(V) \ltimes (V_{\text{boost}} \oplus V_{\text{trans}})$ span $\mathfrak{g}_1'$, whence the infinitesimal character $\dot{\mu}$ vanishes on $\mathfrak{g}_1'$. Therefore, $\dot{\mu}$ factors through a linear map $W/V \to \mathbb{R}$. Exponentiating (6.27) we obtain a presentation of G_1 as a group extension whose

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[19] It may be more natural to take the kernel of (6.21) to be the universal cover $\mathbb{R}$ of $\mathbb{T}$, but an action $G_1 \to \mathrm{Aut}_{\text{qtm}}(\mathbb{P}\mathcal{H})$ induces a central extension by $\mathbb{T}$, not $\mathbb{R}$.

[20] The quotient W/V could be written W/V_{trans} in the current notation.

kernel is the normal subgroup of G_1 consisting of automorphisms of (M, Γ) which preserve the leaves of the simultaneity foliation $\mathcal{S}$. From the Lie algebra argument we conclude that μ factors through the quotient group, which is isomorphic to the translation group $\mathbb{R}$. But $\beta \circ \beta = \mathrm{id}$ and only the trivial character of $\mathbb{R}$ squares to the trivial character. This proves $\beta = \mathrm{id}_{G_1^{\tau_1}}$. Now the subgroup $V \oplus V$ of $\mathfrak{g}_1$ is an ideal with quotient isomorphic to $\mathfrak{o}(V) \oplus W/V$, and if $d = 2$ the latter is abelian. The group G_1 then also has characters pulled back from those on the group $SO(V) \cong SO(2)$. There are no nontrivial characters which square to the identity character, so once more $\beta = \mathrm{id}_{G_1^{\tau_1}}$. For $d = 1$, the group G_1 is the group of translations and boosts, and any character factors through the quotient by the normal subgroup $\exp(V_{\text{boost}} \oplus V_{\text{trans}})$ of spatial translations and boosts (similar to the situation for $d \geq 3$). That quotient is isomorphic to $\mathbb{R}$, so as before we conclude $\beta = \mathrm{id}_{G_1^{\tau_1}}$. Finally, for $d = 0$ the group G_1 consists only of time translations, and again $\beta = \mathrm{id}_{G_1^{\tau_1}}$. This proves (i).

For (ii) we first observe that since $G_1^{\tau_1}$ is connected, the automorphism $\beta = \tilde{\alpha}(-1)\colon G_1^{\tau_1} \to G_1^{\tau_1}$ is determined by its differential $\dot{\beta}\colon \mathfrak{g}_1^{\tau_1} \to \mathfrak{g}_1^{\tau_1}$ on the Lie algebra. Now a time-reversal symmetry determines a splitting $W = U \oplus V$, where U is the (-1)-eigenspace of the induced linear action on W. Then

$$\mathfrak{g}_1^{\tau_1} \cong \mathbb{R} \cdot c \oplus \mathfrak{o}(V) \ltimes (V_{\text{boost}} \oplus V_{\text{trans}}) \oplus U, \tag{6.28}$$

where c is central; there is a nontrivial commutator

$$[\xi, \eta] = \langle \xi, \eta \rangle\, c, \quad \xi \in V_{\text{boost}}, \quad \eta \in V_{\text{trans}}, \tag{6.29}$$

between an infinitesimal boost and an infinitesimal spatial translation—the angle brackets denote the Euclidean inner product on V; and there is a nonzero commutator between a (unit)[21] infinitesimal time translation $\tau \in U$ and an infinitesimal boost $\xi \in V_{\text{boost}}$: the bracket $[\tau, \xi]$ is $\xi \in V_{\text{trans}}$. The infinitesimal time-reversal is the automorphism of $\mathfrak{g}_1$ which is the identity on $\mathfrak{o}(V)$ and V_{trans}, and is minus the identity on V_{boost} and U. The unique lift to an automorphism $\dot{\beta}$ of $\mathfrak{g}_1^{\tau_1}$ sends $c \mapsto -c$. We must check that $\dot{\beta}$ exponentiates to an automorphism $\beta\colon G_1^{\tau_1} \to G_1^{\tau_1}$ of order 2. First, $\dot{\beta}$ exponentiates to an automorphism of the simply connected cover of $G_1^{\tau_1}$ which fixes the central element in the spin double cover of $SO(V)$ and inverts the kernel of the infinite cyclic cover $\mathbb{R} \to \mathbb{T}$ of the centers, whence it drops to an automorphism β of $G_1^{\tau_1}$. The differential of $\beta \circ \beta$ is $\dot{\beta} \circ \dot{\beta}$, which is the identity on $\mathfrak{g}_1^{\tau_1}$, whence $\beta \circ \beta$ is the identity on $G_1^{\tau_1}$. □

7. *K*-Theory and Some Twistings

There are several approaches to topological K-theory. It was introduced in the late 1950s by Atiyah and Hirzebruch [4] to formulate a topological analog of Grothendieck's Riemann–Roch theorem for sheaves. The basic object is a

[21] To write the formulas we orient U, and this arrow of time determines a sign for the action of boosts. Reversing the orientations does not change the Lie algebra.

complex vector bundle over a topological space X. Somewhat later Atiyah and Segal developed an equivariant version [55] for a compact Lie group G acting on X. Our main interest is in twistings of K-theory and twisted K-theory. There are many approaches and so many references, for example [6,8,15,26, 46,61]; the study by Gawędzki [31] includes the twists related to complex conjugation which we encounter here. The twistings we encounter are very special, and we give an appropriately tailored exposition. These twistings are a generalization of extended QM symmetry classes of groups [Definition 3.7(i)] to *groupoids*. Therefore, we begin with the case of groups and define the twisted K-theory, which is a twisted form of the representation ring (Example 5.6). The construction in Lemma 5.12 is relevant here. We then recast extended QM symmetry classes in terms of line bundles and generalize to groupoids. Twisted K-theory in this situation may still be defined in terms of finite rank vector bundles, which we prove in Appendix E. (For more general twistings, one must use bundles of infinite rank equipped with Fredholm operators [26].)

7.1. Twisted Representation Rings

Let G be a compact Lie group. Recall the two constructions of the representation ring K_G in Sect. 5. First, if $\operatorname{Rep}(G)$ is the set of isomorphism classes of finite dimensional complex representations of G, which is a commutative monoid under direct sum, then its group completion is K_G. Second, let $\operatorname{Rep}_s(G)$ be the commutative monoid of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded complex representations of G. Define the submonoid $\operatorname{Triv}_s(G)$ of $\mathbb{Z}/2\mathbb{Z}$-graded representations E which admit an odd automorphism $P\colon E \to E$ which satisfies (5.11). Then $K_G \cong \operatorname{Rep}_s(G)/\operatorname{Triv}_s(G)$; see (5.13).

As we explained in Sect. 1, quantum mechanics enhances the notion of an ordinary complex representation of a group in two ways: (i) there are both linear and antilinear transformations, and (ii) a twisted extension of the symmetry group acts on the vector space. In Sect. 3, we encountered a third enhancement: (iii) the representation space is $\mathbb{Z}/2\mathbb{Z}$-graded and there are both even and odd automorphisms. We formalize the resulting "twisted" version of K_G as follows.

Definition 7.1. Let G be a compact Lie group and (G, ϕ, τ, c) an extended QM symmetry class based on G [Definition 3.7(i)].

(i) Let $\operatorname{Rep}_s^{(\phi,\tau,c)}(G)$ denote the monoid of isomorphism classes of finite dimensional (ϕ, τ, c)-twisted representations [Definition 3.7(ii)].

(ii) Let the submonoid $\operatorname{Triv}_s^{(\phi,\tau,c)}(G)$ consist of (ϕ, τ, c)-twisted representations $\rho^\tau\colon G^\tau \to \operatorname{Aut}_{\mathbb{R}}(E)$ such that there exists an odd $\mathbb{C}$-linear automorphism $P\colon E \to E$ which satisfies $P\rho^\tau(g) = c(g)\rho^\tau(g)P$ for all $g \in G^\tau$.

(iii) Define ${}^{\phi}K_G^{\tau,c} = \operatorname{Rep}_s^{(\phi,\tau,c)}(G)/\operatorname{Triv}_s^{(\phi,\tau,c)}(G)$.

The proof of Lemma 5.12 carries over *verbatim* to prove the following.

Lemma 7.2. *The commutative monoid* ${}^{\phi}K_G^{\tau,c}$ *is a group.*

Remark 7.3. We always use the "Koszul sign rule" for tensor products of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Informally, it asserts that whenever commuting two homogeneous elements x, y one picks up a minus sign if both x and y are odd. See [16, Chapter 1] for the formal implementation as the symmetry in the symmetric monoidal category of super vector spaces, as well as a discussion of parity-reversal.

Remark 7.4. It would be reasonable to call an extended QM symmetry class (G, ϕ, τ, c) a 'ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension': the homomorphism c encodes the $\mathbb{Z}/2\mathbb{Z}$-grading and τ is the extension. This point of view is developed in the next subsection; see especially Remark 7.17.

Remark 7.5. We emphasize that ${}^{\phi}K_G^{\tau,c}$ is an abelian group which does not have a ring structure in general. Rather, it is a module over ${}^{\phi}K_G$, the Grothendieck ring of representations of G with the same homomorphism ϕ but trivial ϕ-twisted extension τ and trivial homomorphism c. Note that the tensor product of linear maps is linear and the tensor product of antilinear maps is antilinear, whereas the tensor product of odd maps is even and the tensor product of scalar multiplication by λ_1 and scalar multiplication by λ_2 is scalar multiplication by $\lambda_1\lambda_2$. So the tensor product of two representations with the same ϕ gives another representation with the same ϕ, whereas the homomorphism c and the central extension τ change under tensor product. This distinguished role of ϕ is reflected in the notation.

7.2. Extensions as Line Bundles

It is convenient (and standard) to recast the ϕ-twisted extension as a hermitian line bundle $L^{\tau} \to G$ with a "composition law". Namely, the extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G^{\tau} \longrightarrow G \longrightarrow 1 \tag{7.6}$$

is a principal $\mathbb{T}$-bundle over G, and L^{τ} is the associated hermitian line bundle $L^{\tau} = (G^{\tau} \times \mathbb{C})/\mathbb{T}$, where $\lambda \in \mathbb{T}$ acts on the right as $(\tilde{g}, z) \cdot \lambda = (\tilde{g}\lambda, \lambda^{-1}z)$ for $\tilde{g} \in G^{\tau}$ and $z \in \mathbb{C}$. The hermitian structure is induced from the standard hermitian structure on $\mathbb{C}$, which is preserved by multiplication by $\lambda \in \mathbb{T}$. Assume first that $\phi \equiv 1$ so that (7.6) is a central extension. We leave the reader to use the group law in G^{τ} to construct isometries

$$\lambda_{g_2,g_1} : L^{\tau}_{g_2} \otimes L^{\tau}_{g_1} \longrightarrow L^{\tau}_{g_2g_1}, \tag{7.7}$$

where L^{τ}_g is the fiber of $L^{\tau} \to G$ over $g \in G$, and to use the associativity property of G^{τ} to prove that the diagram

$$\begin{array}{ccc} L^{\tau}_{g_3} \otimes L^{\tau}_{g_2} \otimes L^{\tau}_{g_1} & \xrightarrow{\lambda_{g_3,g_2} \otimes \mathrm{id}} & L^{\tau}_{g_3g_2} \otimes L^{\tau}_{g_1} \\ {\scriptstyle \mathrm{id} \otimes \lambda_{g_2,g_1}} \downarrow & & \downarrow {\scriptstyle \lambda_{g_3g_2,g_1}} \\ L^{\tau}_{g_3} \otimes L^{\tau}_{g_2g_1} & \xrightarrow{\lambda_{g_3,g_2g_1}} & L^{\tau}_{g_3g_2g_1} \end{array} \tag{7.8}$$

commutes. This diagram can be turned into a cocycle condition; see Remark 7.12 below.

For a general ϕ, we need to bring in complex conjugation. Recall that if W is a complex vector space, then the *complex conjugate vector space* $\overline{W}$ has the same underlying real vector space as W but the complex multiplication is conjugated. Thus, $\ell \in W$ is the same element $\bar{\ell} \in \overline{W}$ in the set $W = \overline{W}$, and if $\lambda \in \mathbb{C}$ then $\bar{\lambda} \cdot \bar{\ell} = \overline{\lambda \cdot \ell}$. (The first scalar multiplication is in the complex vector space $\overline{W}$, the second in W.) For $\phi \in \{\pm 1\}$, define

$$ {}^{\phi}W = \begin{cases} W, & \phi = +1; \\ \overline{W}, & \phi = -1. \end{cases} \tag{7.9} $$

An antilinear map $W' \to W$ between complex vector spaces is equivalently a linear map $\overline{W'} \to W$. For arbitrary $\phi\colon G \to \{\pm 1\}$, replace (7.7) with a $\mathbb{C}$-linear isometry

$$ \lambda_{g_2,g_1}\colon {}^{\phi(g_1)}L^{\tau}_{g_2} \otimes L^{\tau}_{g_1} \longrightarrow L^{\tau}_{g_2 g_1}, \tag{7.10} $$

and replace (7.8) with

$$ \begin{CD} {}^{\phi(g_2g_1)}L^{\tau}_{g_3} \otimes {}^{\phi(g_1)}L^{\tau}_{g_2} \otimes L^{\tau}_{g_1} @>{\mathrm{id}\otimes\lambda_{g_2,g_1}}>> {}^{\phi(g_2g_1)}L^{\tau}_{g_3} \otimes L^{\tau}_{g_2g_1} \\ @V{\lambda_{g_3,g_2}\otimes\mathrm{id}}VV @VV{\lambda_{g_3,g_2g_1}}V \\ {}^{\phi(g_1)}L^{\tau}_{g_3g_2} \otimes L^{\tau}_{g_1} @>{\lambda_{g_3,g_2,g_1}}>> L^{\tau}_{g_3g_2g_1} \end{CD} \tag{7.11} $$

which can be interpreted as a twisted cocycle condition.

Remark 7.12. To see a cocycle α explicitly, choose a unit norm vector $\ell_g \in L^{\tau}_g$ for each $g \in G$. Then for $g_1, g_2 \in G$ define $\alpha(g_2, g_1) \in \mathbb{T}$ by

$$ \lambda_{g_2,g_1}({}^{\phi(g_1)}\ell_{g_2}, \ell_{g_1}) = \alpha(g_2, g_1)\ell_{g_2g_1}, \tag{7.13} $$

where, as in (7.9), the left superscript ϕ controls complex conjugation. The commutative diagram (7.11) translates into the twisted cocycle condition

$$ \alpha(g_2, g_1)\alpha(g_3, g_2g_1) = {}^{\phi(g_1)}\alpha(g_3, g_2)\alpha(g_3g_2, g_1), \quad g_1, g_2, g_3 \in G. \tag{7.14} $$

A different choice $\ell_g \mapsto b(g)\ell_g$, $b(g) \in \mathbb{T}$, shifts α by a twisted coboundary.

In preparation for twisted K-theory, let us note that a (ϕ, τ, c)-twisted representation of G is for each $g \in G$ a linear map

$$ \rho^{\tau}(g)\colon {}^{\phi(g)}(L^{\tau}_g \otimes W) \longrightarrow W \tag{7.15} $$

such that the diagram

$$ \begin{CD} {}^{\phi(g_2g_1)}\big({}^{\phi(g_1)}L^{\tau}_{g_2} \otimes L^{\tau}_{g_1} \otimes W\big) @>{\lambda_{g_2,g_1}\otimes\mathrm{id}}>> {}^{\phi(g_2g_1)}\big(L^{\tau}_{g_2g_1} \otimes W\big) \\ @V{\mathrm{id}\otimes\rho^{\tau}(g_1)}VV @VV{\rho^{\tau}(g_2g_1)}V \\ {}^{\phi(g_2)}(L^{\tau}_{g_2} \otimes W) @>{\rho^{\tau}(g_2)}>> W \end{CD} \tag{7.16} $$

commutes and the linear map $\rho^{\tau}(g)$ in (7.15) is even or odd according to $c(g)$.

Remark 7.17. It is convenient—and for the addition of twistings necessary—to combine the homomorphism $c\colon G \to \{\pm 1\}$ and the twisted extension τ into a $\mathbb{Z}/2\mathbb{Z}$*-graded* line bundle $L^{\tau} \to G$: the line L_g is even if $c(g) = +1$ and odd if $c(g) = -1$. The Koszul sign rule (Remark 7.3) must be used to permute tensor products of $\mathbb{Z}/2\mathbb{Z}$-graded lines.

7.3. Groupoids, Twistings, and Twisted Vector Bundles

We now generalize from groups to *groupoids*. The elements of a group may be pictured as arrows which begin and end at the same (abstract) point, and the group law is composition of arrows. A groupoid has a similar picture, but now there are many possible starting and ending points for the arrows and composition is limited to a sequence of arrows with the ending point of one arrow equal to the starting point of the next. Thus, a groupoid consists of a set $\mathcal{G}_0$ of points and $\mathcal{G}_1$ of arrows. The arrow $(x_0 \xrightarrow{\gamma} x_1)$ has source x_0 and target x_1. There is a partially defined associative composition law on arrows: for $\gamma_1, \gamma_2 \in \mathcal{G}_1$ the composition $\gamma_2 \circ \gamma_1 \in \mathcal{G}_1$ is defined only if the target for γ_1 equals the source for γ_2. There is an identity arrow for each state. Each arrow is invertible—two-sided inverse arrows exist. As we discuss below, $\mathcal{G}_0, \mathcal{G}_1$ may be topological spaces, or even manifolds, in which case we impose continuity (smoothness) on the structural maps which define the groupoid structure. The structural maps (except for composition) appear in the diagram

$$\mathcal{G}_0 \underset{p_1}{\overset{p_0}{\leftleftarrows}} \dashrightarrow \mathcal{G}_1 \tag{7.18}$$

The dashed map assigns the identity morphism to each object, the map p_1 takes each arrow to its source (=domain), and the map p_0 takes each arrow to its target (=codomain). See [26, Appendix] for a formal definition and development of groupoids and the associated K-theory.

We give several examples to help the reader warm up to this notion.

Example 7.19. As mentioned above, a group G may be viewed as the groupoid $\mathcal{G}$ with $\mathcal{G}_0 = \text{pt}$ and $\mathcal{G}_1 = G$. There is a unique object and any two arrows are composable.

Example 7.20. A set X is a groupoid $\mathcal{G}$ with only identity arrows, that is, $\mathcal{G}_0 = \mathcal{G}_1 = X$.

Example 7.21. Let X be a set and G a discrete group acting on X. We construct the quotient groupoid $\mathcal{G} = X/\!/G$ with $\mathcal{G}_0 = X$ and $\mathcal{G}_1 = X \times G$. The arrow $\gamma = (x, g) \in X \times G$ has domain x and codomain $g \cdot x$, where $g \cdot x$ is the result of acting g on x. Composition of arrows is defined by the action. For each x, the arrows which map x to itself form a group, called the stabilizer group of x; it is a subgroup of G. The orbit of an element in X also has a natural interpretation in terms of the groupoid $\mathcal{G}$. We use a topological version of $X/\!/G$ in which X is a nice topological space (in fact, a smooth torus of some dimension) and G is a compact Lie group. Then the K-theory of $X/\!/G$, as defined below, is identical to the G-equivariant K-theory of X.

A homomorphism of groupoids $\varphi\colon \mathcal{G}' \to \mathcal{G}$ is a pair of maps $\varphi_0\colon (\mathcal{G}')_0 \to \mathcal{G}_0$ and $\varphi_1\colon (\mathcal{G}')_1 \to \mathcal{G}_1$ which preserves compositions. For example, for any $\mathcal{G}$ there is an inclusion of the groupoid $\mathcal{G}'$ of identity arrows, which is a set or space as in Example 7.20: $(\mathcal{G}')_0 = (\mathcal{G}')_1 = \mathcal{G}_0$.

Next we generalize representations of groups to groupoids. We begin with discrete groupoids. A complex representation of a group G, viewed in terms of the associated groupoid $\mathcal{G}$ defined in Example 7.19, assigns a complex vector space W to the unique point in $\mathcal{G}_0$ and a complex linear map to each arrow in $\mathcal{G}_1$ such that the composite of two arrows maps to the composite of the linear maps. The generalization to groupoids is straightforward.

Definition 7.22. Let $\mathcal{G}$ be a groupoid. A *complex vector bundle over* $\mathcal{G}$ assigns a complex vector space W_x to each object $x \in \mathcal{G}_0$ and a complex linear map $\rho(\gamma)\colon W_x \to W_{x'}$ to each arrow $\gamma \in \mathcal{G}_1$ with domain x and codomain x'. The assignment ρ is a homomorphism in the sense that $\rho(\gamma_2 \circ \gamma_1) = \rho(\gamma_2) \circ \rho(\gamma_1)$ whenever $\gamma_2 \circ \gamma_1$ is defined. If $\mathcal{G}$ is a topological groupoid, then the vector spaces are required to fit together into a vector bundle $W \to \mathcal{G}_0$ and the linear maps $\rho(\gamma)$ to form a continuous map $p_1^*W \to p_0^*W$ over $\mathcal{G}_1$.

Recall that p_0, p_1 are defined in (7.18). If $\mathcal{G} = X/\!/G$ is the quotient of a set by a group action, then a vector bundle over $\mathcal{G}$ is a G-equivariant vector bundle $W \to X$.

We now introduce certain extensions of a groupoid, which are special twistings of its K-theory.

Definition 7.23. Let $\mathcal{G}$ be a groupoid.

(i) A *central extension* of $\mathcal{G}$ is a hermitian line bundle $L^\tau \to \mathcal{G}_1$ together with maps $\lambda_{\gamma_2,\gamma_1}$ as in (7.7) which satisfy the associativity (cocycle) constraint (7.8).

(ii) Let $\phi\colon \mathcal{G}_1 \to \{\pm 1\}$ be a homomorphism. A *ϕ-twisted extension* of $\mathcal{G}$ is a hermitian line bundle $L^\tau \to \mathcal{G}_1$ together with maps $\lambda_{\gamma_2,\gamma_1}$ as in (7.10) which satisfy the associativity (cocycle) constraint (7.11).

(iii) A *ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension* of $\mathcal{G}$ is a triple $\nu = (\phi, \tau, c)$ consisting of homomorphisms $\phi, c\colon \mathcal{G}_1 \to \{\pm 1\}$ and a ϕ-twisted extension τ.

(iv) If $\nu = (\phi, \tau, c)$ is a ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension, then a *ν-twisted vector bundle* W over $\mathcal{G}$ assigns a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space W_x to each object $x \in \mathcal{G}_0$ and a *linear* map

$$\rho^\tau(\gamma)\colon {}^{\phi(\gamma)}(L^\tau_\gamma \otimes W_{x_0}) \longrightarrow W_{x_1} \tag{7.24}$$

to each arrow $(x_0 \xrightarrow{\gamma} x_1) \in \mathcal{G}_1$. The map $\rho^\tau(\gamma)$ is even or odd according to $c(\gamma)$. If $\mathcal{G}$ is a topological groupoid, then we demand that W be a vector bundle and (7.24) define a continuous map ${}^{\phi(\gamma)}(L^\tau \otimes p_1^*W) \to p_0^*W$ over $\mathcal{G}_1$. An analog of the commutative diagram (7.16) expresses the compatibility between λ and ρ.

In (i), $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{G}_1$ are composable arrows, and in (7.7), (7.8), (7.10), (7.11), and (7.16) we replace 'g' with 'γ'. Note that central extensions and ϕ-twisted extensions are special cases of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions.

Remark 7.25. We emphasize that $W \to \mathcal{G}_0$ is an ordinary $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle. The twisting is in the action of $\mathcal{G}_1$, which is the same twisting as occurs in Definition 7.1. The only new idea is that of multiple objects, as parametrized by $\mathcal{G}_0$.

Remark 7.26. Let $\mathcal{G} = X//G$ be the quotient groupoid of the action of a group G on a space X. Suppose $\phi, c\colon G \to \{\pm 1\}$ are homomorphisms. Then there is an associated ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension $\nu(\phi, c)$ of $\mathcal{G}$. Namely, $\mathcal{G}_1 = X \times G$ and we define homomorphisms $\mathcal{G}_1 \to \{\pm 1\}$ by composition with projection $X \times G \to G$. The central extension $L^\tau \to \mathcal{G}_1$ is taken to be trivial. We use this ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension in Theorem 9.42 below.

Definition 7.27. Let $\mathcal{G}$ be a groupoid and $\nu = (\phi, \tau, c)$ and $\nu' = (\phi', \tau', c')$ ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions. Then if $\phi = \phi'$ and $c = c'$, an *isomorphism of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions* $\nu \to \nu'$ is an isomorphism $L^\tau \to L^{\tau'}$ of line bundles over $\mathcal{G}_1$ which is compatible with the structure maps (7.10).

Isomorphism of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions is an equivalence relation, and so there is a set of equivalence classes of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions.

Remark 7.28. We can then ask for a classification of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions up to isomorphism. If the groupoid $\mathcal{G}$ is sufficiently nice, for example, the groupoid associated to the action of a compact Lie group G on a nice topological space X as in Definition 7.33 below, then the set of isomorphism classes of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions is noncanonically isomorphic to

$$H^1_G(X; \mathbb{Z}/2\mathbb{Z})_{\mathrm{rel}} \times H^3_G(X; \mathbb{Z}_\phi)_{\mathrm{rel}}, \tag{7.29}$$

where

$$H^\bullet_G(X; \mathbb{Z}_\phi)_{\mathrm{rel}} = \ker\bigl(H^\bullet_G(X; \mathbb{Z}_\phi) \longrightarrow H^\bullet(X; \mathbb{Z})\bigr). \tag{7.30}$$

The subscript 'G' denotes equivariant cohomology, and '$\mathbb{Z}_\phi$' denotes the local system (twisted coefficients) defined by ϕ. See [26, Sect. 2.2.1] for more details and a proof (in case ϕ is trivial). If G is a finite group and X is compact, then (7.29) is a finite set.

Remark 7.31. There are various operations on ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions. If $\varphi\colon \mathcal{G}' \to \mathcal{G}$ is a homomorphism of groupoids, and $\nu = (\phi, \tau, c)$ is a ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension of $\mathcal{G}$, then there is a pullback ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension $\varphi^*\nu$ of $\mathcal{G}'$. Also, if $\mathcal{G}$ is a groupoid and $\nu_1 = (\phi, \tau_1, c_1), \nu_2 = (\phi, \tau_2, c_2)$ are ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions of $\mathcal{G}$ with the same homomorphism ϕ, then there is a new ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension $\nu = \nu_1 + \nu_2$ of $\mathcal{G}$, also with the same ϕ. To define it combine c_i, τ_i into a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle $L^{\tau_i} \to \mathcal{G}_1$ and set $L = L^{\tau_1} \otimes L^{\tau_2}$. The Koszul sign rule is used to define (7.10). The addition gives an abelian group law on (7.29) which is not, in general, the product of the natural abelian group laws on the factors.

Remark 7.32. Let $\varphi\colon \mathcal{G}' \to \mathcal{G}$ be the inclusion of the identity arrows $(\mathcal{G}')_0 = (\mathcal{G}')_1 = \mathcal{G}_0$. Then for any ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν, the pullback $\varphi^*\nu$ is a trivial twisting of $\mathcal{G}'$. In other words, the twistings in this paper are nonequivariantly trivial, hence simpler than general twistings of K-theory.

7.4. Twisted K-Theory

The following is an extension of Definition 7.1 to the special groupoids and special twistings (Remark 7.32) we encounter in this paper. It is justified in Appendix E.

Definition 7.33. Let X be a nice[22] compact topological space with a continuous action of a *finite* group G. Let $\mathcal{G} = X//G$ be the quotient groupoid and $\nu = (\phi, \tau, c)$ a ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension of $\mathcal{G}$.

(i) Let $s\mathrm{Vect}^{\nu}_{G}(X)$ denote the monoid of isomorphism classes of finite rank ν-twisted $\mathbb{Z}/2\mathbb{Z}$-graded G-equivariant vector bundles $W \to X$.

(ii) Let the submonoid $\mathrm{Triv}^{\nu}_{G}(X)$ consist of finite rank ν-twisted $\mathbb{Z}/2\mathbb{Z}$-graded G-equivariant vector bundles $W \to X$ such that there exists an odd automorphism $P \colon W \to W$.

(iii) Define the *twisted K-theory group* $K^{\nu}_{G}(X) = {}^{\phi}K^{\tau,c}_{G}(X) = \mathrm{Rep}^{(\phi,\tau,c)}_{s}(G)/\mathrm{Triv}^{(\phi,\tau,c)}_{s}(G)$.

Recall that a G-equivariant vector bundle over X, twisted or not, is an ordinary vector bundle over $X = \mathcal{G}_0$.

Remark 7.34. In case $G = 1$ is the trivial group, then Definition 7.33 reduces to the original definition of the topological K-theory group $K^0(X)$. Similarly, if $\phi \equiv c \equiv 1$ and τ is the trivial extension, then it reduces to the original definition of $K^0_G(X)$. See [1] for a nice, detailed exposition.

Remark 7.35. Suppose X is a space with involution $\sigma \colon X \to X$, which we regard as a space X with $G = \mathbb{Z}/2\mathbb{Z}$ action. Define $\phi \colon \mathcal{G}_1 \to \{\pm 1\}$ on the quotient groupoid $\mathcal{G} = X//G$ to be 1 on all identity arrows (as it must be) and -1 on all arrows which encode the σ action. Let c, τ be trivial. Then the twisted K-theory group ${}^{\phi}K_G(X) \cong KR^0(X)$ is Atiyah's (untwisted) KR-group; see [2]. More generally, if $G = G'' \times \mathbb{Z}/2\mathbb{Z}$ and ϕ is projection onto the second factor, then ${}^{\phi}K_G(X) \cong KR^0_{G''}(X)$ is an equivariant KR-theory group.

Remark 7.36. If $c \equiv 1$, so that ν is a ϕ-twisted extension, then the twisted K-theory is isomorphic to the group completion of the monoid $\mathrm{Vect}^{\nu}_{G}(X)$ of ungraded finite rank ν-twisted G-equivariant vector bundles $W \to X$.

Remark 7.37. K-theory is defined for a more general class of groupoids (*local quotient groupoids*) in [26, Appendix]. The general definition uses families of Fredholm operators. It is a *theo*, proved in Appendix E, that for the special twistings in Definition 7.23, namely, ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions of a global quotient by a finite group G, each twisted K-theory class has a representative which is a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle of finite rank. It may fail if the group G is a compact Lie group. For example, if $G = \mathbb{T}$ acts trivially on $X = \mathbb{T}$ there are nontrivial central extensions of infinite order, and only the zero twisted K-theory class has a finite rank representative.

Finally, we remark on the role of Clifford algebras in K-theory [3]. We review some basics about Clifford algebras in Appendix B. If R is a Clifford

[22] Locally contractible and completely regular.

algebra[23] then we can use R as an additional twisting of representations and of vector bundles in the following sense. Namely, we ask that a representation, or more generally the fibers of a vector bundle, be modules for the algebra R, and that the equivariance maps (7.15) commute with the R-module structure. This commutation must be understood in the graded sense, that is, using the Koszul sign rule. If $R = \mathrm{Cl}_n$ for some $n \in \mathbb{Z}$, this has the effect of shifting the degree in K-theory by n. Details may be found in many sources, for example [15,25], [26, Appendix].

8. Gapped Systems with a Finite Dimensional State Space

In this section, we take up systems of free fermions with a finite dimensional state space. A similar problem is treated in [5,37]. As quantum mechanical systems they have a time evolution, but we do not assume any symmetries from space. In other words, the underlying Galilean spacetime (M, Γ) is 1-dimensional. So $\mathrm{Aut}(M, \Gamma)$ is an extension

$$1 \longrightarrow U \longrightarrow \mathrm{Aut}(M, \Gamma) \longrightarrow \{\pm 1\} \longrightarrow 0 \tag{8.1}$$

where U is the line of time translations. The extension is split by choosing an origin of time, and a splitting maps $-1 \in \{\pm 1\}$ into the time-reversal which fixes the origin. Let (G, ϕ, τ, c) be an extended QM symmetry class, as in Definition 3.7. A Galilean symmetry group includes a homomorphism (2.11) to $\mathrm{Aut}(M, \Gamma)$, but as in Remark 3.10 we do not include time translations in G. Also, the homomorphism t to the quotient $\{\pm 1\}$ in (8.1) is $t = \phi c$, so is already included in the data of the extended QM symmetry class. We make the following "finiteness" hypothesis.

Hypothesis 8.2. The symmetry group G of a 0-dimensional gapped system is a compact Lie group.

In fact, it is usually assumed to be a finite group. Recall from Definition 3.8 that a gapped system $(\mathcal{H}, H, \rho^\tau)$ consists of a Hilbert space $\mathcal{H}$, a Hamiltonian H, and a (ϕ, τ, c)-twisted representation ρ^τ of G on $\mathcal{H}$. By Lemma 5.14 we may assume $H^2 = 1$, after a homotopy, so H is a grading operator on $\mathcal{H}$. The finiteness condition we work with is the following.

Hypothesis 8.3. In a finite dimensional gapped system (FDGS), the Hilbert space $\mathcal{H}$ is finite dimensional.

Proposition 8.4. *Assume Hypotheses 8.2 and 8.3. Then the commutative monoid $\mathcal{TP}_Z(G, \phi, \tau, c)$ of topological equivalence classes of FDGS with extended QM symmetry class (G, ϕ, τ, c) is isomorphic to $\mathrm{Rep}_s^{(\phi,\tau,c)}(G)$,*

The monoid $\mathcal{TP}_Z(G, \phi, \tau, c)$ is defined in Definition 5.1 and the monoid $\mathrm{Rep}_s^{(\phi,\tau,c)}(G)$ is defined in Definition 7.1(i).

[23] Or, more generally, a $\mathbb{Z}/2\mathbb{Z}$-graded central simple algebra [63].

Proof. First, since the space of positive definite hermitian metrics on $\mathcal{H}$ is contractible, and since we mod out by homotopy, we can ignore the metric on $\mathcal{H}$. Next, finite dimensional representations of compact Lie groups are discrete—they have no continuous deformations—so two homotopic representations are in fact isomorphic. $\square$

We impose a "topological triviality" relation on $\mathcal{TP}_Z(G,\phi,\tau,c)$ to define the abelian group of reduced topological phases $\mathcal{RTP}_Z(G,\phi,\tau,c)$; see the text preceding Lemma 5.14.

Definition 8.5. A FDGS $(\mathcal{H},H,\rho^\tau)$ with extended QM symmetry class (G,ϕ,τ,c) is *topologically trivial* if there exists an odd automorphism $P\colon \mathcal{H}\to\mathcal{H}$ such that $P\rho^\tau(g)=c(g)\rho^\tau(g)P$ for all $g\in G^\tau$.

Theorem 8.6. *There is an isomorphism* $\mathcal{RTP}_Z(G,\phi,\tau,c)\cong {}^{\phi}K_G^{\tau,c}$.

The twisted K-theory group in the theorem is given in Definition 7.1. The proof is immediate from the definitions.

We can identify the twisted virtual representation ring with a more standard K-theory group if we make an additional strong assumption. It is not, as far as we know, justified on physical grounds. In fact, it is stronger than the ψ-standard assumption in Sect. 6. We include it here to make contact with the literature, and in particular with Proposition 6.4. Consider $\psi=(t,c)\colon G\to \mathcal{C}=\{\pm1\}\times\{\pm1\}$ and let $A\subset\mathcal{C}$ be its image. Set $G_0=\ker\psi$.

Hypothesis 8.7. (i) The group G is a direct product $G\cong A\times G_0$ and under this isomorphism ψ is projection onto A.
(ii) The restriction $G_0^\tau\to G_0$ of $G^\tau\to G$ splits and we fix a splitting.
Therefore, there is fixed an isomorphism $G^\tau\cong A^\tau\times G_0$.

There are four nontrivial subgroups of $\mathcal{C}$; compare Proposition 6.4. Elements $\overline{T},\overline{C},\overline{S}=\overline{TC}$ may or may not be in A, depending on the case. They have special lifts T,C,S to the extension A^τ, hence to G^τ, as follows immediately from Proposition 6.4.

Lemma 8.8. (i) *If A is the diagonal subgroup of $\mathcal{C}$, then $\overline{S}$ has a lift $S\in A^\tau$ with $S^2=1$ and S central.*
(ii) *If $A=\{\pm1\}\times 1\subset\mathcal{C}$, then $\overline{T}$ has a lift $T\in A^\tau$ with $T^2=\pm1$.*
(iii) *If $A=1\times\{\pm1\}\subset\mathcal{C}$, then $\overline{C}$ has a lift $C\in A^\tau$ with $C^2=\pm1$.*
(iv) *If $A=\mathcal{C}$, then $\overline{T},\overline{C}$ have lifts $T,C\in A^\tau$ with $TC=CT, T^2=\pm1, C^2=\pm1$.*

Corollary 8.9. *If Hypothesis 8.7 holds, then we have the following table for* ${}^{\phi}K_G^{\tau,c}$:

A	1	diag	$\{\pm1\}\times1$	$\mathcal{C}$	$1\times\{\pm1\}$	$\mathcal{C}$	$\{\pm1\}\times1$	$\mathcal{C}$	$1\times\{\pm1\}$	$\mathcal{C}$
T^2			$+1$	$+1$		-1	-1	-1		$+1$
C^2				-1	-1	-1		$+1$	$+1$	$+1$
${}^{\phi}K_G^{t,c}$	$K_{G_0}^0$	$K_{G_0}^{-1}$	$KO_{G_0}^0$	$KO_{G_0}^{-1}$	$KO_{G_0}^{-2}$	$KO_{G_0}^{-3}$	$KO_{G_0}^{-4}$	$KO_{G_0}^{-5}$	$KO_{G_0}^{-6}$	$KO_{G_0}^{-7}$

Proof. An action of G on $\mathcal{H}$ of the type that occurs in Theorem 8.6 is equivalent to a linear representation of G_0 on $\mathcal{H}$ together with a *commuting* action of whichever elements $T, C, S = TC$ are present. Combining with the interpretation in terms of Clifford algebras developed in Appendix B, specifically Proposition B.4, and recalling from the end of Sect. 7 that Clifford modules represent K-theory in shifted degrees, we arrive at the table. □

9. Twistings from Group Extensions

9.1. Warmup: Direct Products

Let $G = G'' \times G'$ be a direct product group.[24] Then an irreducible complex representation of G is a tensor product $R'' \otimes R'$ of irreducible representations R'' of G'' and R' of G'. Hence any representation E of G decomposes as

$$\begin{aligned} E &\cong \bigoplus_{R'',R'} \mathrm{Hom}_G(R'' \otimes R', E) \otimes R'' \otimes R' \\ &\cong \bigoplus_{R'} \mathrm{Hom}_{G'}(R', E) \otimes R', \end{aligned} \tag{9.1}$$

where R'', R' run over distinguished representatives of the isomorphism classes of irreducible representations of G'', G', respectively. Recall that 'Hom_G' denotes the vector space of *intertwiners*—G-equivariant maps—between two representations of G. Thus, $\dim \mathrm{Hom}_{G'}(R', E)$ is the multiplicity of the irreducible representation R' in E. Decomposing under $G'' \subset G$ we deduce that the multiplicity space in the second line of (9.1) is

$$\mathrm{Hom}_{G'}(R', E) \cong \bigoplus_{R''} \mathrm{Hom}_G(R'' \otimes R', E) \otimes R''. \tag{9.2}$$

Let X be a set (or space) of irreducible representations of G'. We label points of X by the chosen distinguished irreducible representations R'. The multiplicity spaces form a family of complex vector spaces $\mathcal{E} \to X$ defined by

$$\mathcal{E}_{R'} = \mathrm{Hom}_{G'}(R', E), \quad R' \in X. \tag{9.3}$$

In general $\mathcal{E} \to X$ is a *sheaf*, not a vector bundle, since the family of vector spaces may not be locally trivial. The group G'' acts trivially on X, and from (9.2) we see that G'' acts on each fiber of $\mathcal{E}$: in each summand it acts trivially on $\mathrm{Hom}_G(R'' \otimes R', E)$ and irreducibly on R''. Furthermore, this construction induces an equivalence between finite dimensional representations of G and finitely supported G''-equivariant families of complex vector spaces parametrized by X. The inverse construction is straightforward: if $\mathcal{E} \to X$ has finite support, then

$$E = \bigoplus_{R' \in X} \mathcal{E}_{R'} \otimes R' \tag{9.4}$$

[24] For our motivational purposes, the reader may restrict to the simplest case in which G'', G' are finite groups, but with additional care the discussion applies to infinite discrete groups and Lie groups as well.

is a finite dimensional representation of G. If G is compact, then X is discrete and this construction induces an isomorphism of K-groups $K_G \cong K_{G''}(X)_{\mathrm{cpt}}$, where 'cpt' denotes compact (hence finite, since X is discrete) support.

Example 9.5. Consider $G' = \mathbb{Z}$ the free abelian group on one generator. Its irreducible complex representations are all one dimensional and parametrized by the circle group $X = \mathbb{T}$ of phases $\lambda = e^{i\theta}$. The underlying vector space of each representation is $\mathbb{C}$ and in the representation labeled λ the integer $n \in \mathbb{Z}$ acts on $\mathbb{C}$ as multiplication by λ^n. Let G'' be the trivial group. For the irreducible representation E, labeled by λ the vector space $\mathcal{E}_\mu,\ \mu \in \mathbb{T}$ is canonically $\mathbb{C}$ if $\mu = \lambda$ and is zero otherwise, so we obtain a sheaf which is not a vector bundle. On the other hand, if E is the vector space of complex-valued (L^2) functions on $\mathbb{Z}$, and $\mathbb{Z}$ acts by translation, then this construction reduces to the Fourier transform: each $\mathcal{E}_\lambda$ is canonically $\mathbb{C}$, and the construction identifies E as the space of complex-valued (L^2) functions on $\mathbb{T}$.

9.2. Group Extensions and Twistings

Let

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\ \pi\ } G'' \longrightarrow 1 \tag{9.6}$$

be a group extension. As before, let X be the space of isomorphism classes of irreducible complex representations of G'. Then there is a *right* action of G'' on X as follows. Let $\rho_V : G' \to \mathrm{Aut}(V)$ be an irreducible representation of G' and $g'' \in G''$. Choose $g \in G$ with $\pi(g) = g''$. Then if $[V] \in X$ denotes the isomorphism class of V, define $[V] \cdot g''$ to be the isomorphism class of the composition

$$\rho_{V^g} : G' \xrightarrow{\ \alpha(g)\ } G' \xrightarrow{\ \rho_V\ } \mathrm{Aut}(V), \tag{9.7}$$

where $\alpha(g)(g') = gg'g^{-1}$ for all $g' \in G'$. One checks that the isomorphism class of (9.7) depends only on $g'' \in G''$, not on the lift $g \in G$. Also, note that $(V^{g_1})^{g_2} = V^{g_1 g_2}$.

The following theorem is proved in case G is compact in [26, Examples 1.12–13] and in more general form in [27, Sect. 5]. We give an exposition here. Our main application is an extension in which the quotient G'' is a finite group and the kernel G' is isomorphic to a lattice $\mathbb{Z}^d$ for some $d \in \mathbb{Z}^{\geq 0}$. Since $\mathbb{Z}^d$ is abelian, its complex irreducible representations are one-dimensional. Hence, in this case, X is diffeomorphic to the d-dimensional torus of characters $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{T}) \simeq \mathbb{T}^d$. Recall Definition 7.23.

Theorem 9.8. *Let* (9.6) *be an extension of Lie groups and X the space of isomorphism classes of irreducible representations of G'. Assume G'' is compact and G' is either compact or a lattice.*

(i) *There is an induced central extension ν of the groupoid $X//G''$ such that a representation E of G induces a ν-twisted G''-equivariant sheaf over X. (Infinite dimensional representations are assumed unitary.)*
(ii) *If G is compact, there is an isomorphism $K_G \to K^{\nu}_{G''}(X)_{\mathrm{cpt}}$, where* 'cpt' *denotes 'compact support'.*

(iii) *If* G' *is abelian, then a splitting of* (9.6) *trivializes the central extension* ν*.*

A trivialization is defined in Definition 7.27 (with $\phi = \phi', c = c'$, and τ all trivial).

Remark 9.9. The ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν depends on some choices (of representative irreducible representations of G' in each isomorphism class), but we can track how ν changes under a change of these choices, and so we say ν is defined up to canonical isomorphism. In case G' is abelian, as it is in our application to gapped topological insulators in Sect. 10, there are canonical choices so no ambiguity in the definition of ν.

The following simple examples with finite groups illustrate the theorem.

Example 9.10. The cyclic group of order 6 is an extension

$$1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{9.11}$$

which is split: $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Each complex irreducible representation of the abelian group $\mathbb{Z}/3\mathbb{Z}$ is one-dimensional, whence the space X of isomorphism classes is the space of characters $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{T})$, which consists of 3 points. Let $\omega = e^{2\pi i/3}$. Label a character $\lambda\colon \mathbb{Z}/3\mathbb{Z} \to \mathbb{T}$ by $\lambda(1)$, so that $X = \{1, \omega, \omega^2\}$. The nonzero element of $\mathbb{Z}/2\mathbb{Z}$ acts on X by the trivial involution $\sigma = \mathrm{id}_X$. The induced central extension ν of $X/\!/(\mathbb{Z}/2\mathbb{Z})$ is canonically trivial.

Example 9.12. The automorphism group Σ_3 of a set of 3 elements is a split extension

$$1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \Sigma_3 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{9.13}$$

Now $\sigma\colon X \to X$ is the nontrivial involution which exchanges $\omega \leftrightarrow \omega^2$ and fixes 1. The induced central extension ν is trivializable and K_{Σ_3} is isomorphic to (untwisted) $K_{\mathbb{Z}/2\mathbb{Z}}(X)$.

Example 9.14. The 8-element quaternion group Q is a nonsplit extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Q \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{9.15}$$

The quotient $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts trivially on the space $X = \{1, -1\}$ of characters of $\mathbb{Z}/2\mathbb{Z}$. Let ν be the induced central extension of $X/\!/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. The restriction of ν to $\{1\} \subset X$ is trivial while the restriction of ν to $\{-1\} \subset X$ is represented by the nontrivial central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow (Q \times \mathbb{T})/\{\pm 1\} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{9.16}$$

This is a special case of Proposition 9.31 below. Moreover, the fact that (9.16) is nonsplit follows from Lemma 9.38 below.

Proof of Theorem 9.8. Assume first that X is discrete, so G'' acts on X through the discrete quotient $\pi_0 G''$. (The assumption that X is discrete remains in force until the last paragraph of the proof.) Choose a distinguished irreducible representation $\rho_V\colon G' \to \mathrm{Aut}(V)$ in each equivalence class $[V] \in X$. We construct a central extension [recall Definition 7.23(i)] of the groupoid $X/\!/G$ and then descend it to a central extension of $X/\!/G''$, which we define to be ν. Recall from (9.7) that for $[V] \in X$ and $g \in G$ we define a new representation V^g on the vector space V. Its equivalence class, denoted $[V] \cdot \pi(g) \in X$, is represented by one of our chosen representatives $\rho_W \colon G' \to \mathrm{Aut}(W)$. By Schur's lemma

$$\widetilde{L}_{[V],g} := \mathrm{Hom}_{G'}(W, V^g) \tag{9.17}$$

is a line: the space of intertwiners between isomorphic irreducible complex representations is one-dimensional. Composition of intertwiners defines an isomorphism

$$\begin{aligned} \widetilde{L}_{[V],g_1} \otimes \widetilde{L}_{[V]\cdot\pi(g_1),g_2} &\longrightarrow \widetilde{L}_{[V],g_1g_2}, \quad g_1, g_2 \in G \\ f_1 \otimes f_2 &\longmapsto f_1 \circ f_2 \end{aligned} \tag{9.18}$$

and the associative law [groupoid analog of (7.8)] is satisfied since composition of functions is associative. Now if $g' \in G'$ then $[V] \cdot \pi(g') = [V]$ and there is a canonical nonzero element of (9.17), namely, the map $\rho_V(g')$. So $\widetilde{L}_{[V],g'}$ is canonically trivial for $g' \in G'$, and taking $g_1 \in G'$ in (9.18) we see that the line $\widetilde{L}_{[V],g}$ depends up to canonical isomorphism only on $\pi(g) \in G''$. This is the descent mentioned above. More precisely, for $g'' \in G''$ define $L_{[V],g''}$ as the space of sections s of the map

$$\bigcup_{g \in \pi^{-1}(g'')} \widetilde{L}_{[V],g} \longrightarrow \pi^{-1}(g'') \tag{9.19}$$

such that $\rho_V(g') \otimes s(g)$ and $s(g'g)$ correspond under (9.18) for all $g \in \pi^{-1}(g''), g' \in G'$. In other words, $s(g) \in \widetilde{L}_{[V],g}$ and the set of s which satisfy the equivariance condition is a 1-dimensional vector space since s is determined by its value at any $g \in \pi^{-1}(g'')$. Then (9.18) induces isomorphisms

$$L_{[V],g_1''} \otimes L_{[V]\cdot g_1'',g_2''} \longrightarrow L_{[V],g_1''g_2''}, \quad g_1'', g_2'' \in G'', \tag{9.20}$$

and so a central extension ν of $X/\!/G''$.

Let $\rho_E \colon G \to \mathrm{Aut}(E)$ be a representation and as in (9.3) define a vector bundle $\mathcal{E} \to X$ by

$$\mathcal{E}_{[V]} = \mathrm{Hom}_{G'}(V, E). \tag{9.21}$$

(We are still assuming X is discrete, so any parametrized family of vector spaces $\mathcal{E} \to X$ is a vector bundle, possibly with infinite rank at some points.) If $g \in G$ and $[W] = [V] \cdot \pi(g)$ in X, then there is a map

$$\begin{aligned} \mathcal{E}_{[V]} \otimes \widetilde{L}_{[V],g} &\longrightarrow \mathcal{E}_{[W]} \\ \varphi \otimes f &\longmapsto \rho_E(g)^{-1} \circ \varphi \circ f \end{aligned} \tag{9.22}$$

which intertwines the G'-actions and satisfies associativity in that the diagram

$$\begin{array}{ccc} \mathcal{E}_{[V]} \otimes \widetilde{L}_{[V],g_1} \otimes \widetilde{L}_{[V]\cdot\pi(g_1),g_2} & \xrightarrow{(9.20)} & \mathcal{E}_{[V]} \otimes \widetilde{L}_{[V],g_1g_2} \\ {\scriptstyle (9.22)}\downarrow & & \downarrow{\scriptstyle (9.22)} \\ \mathcal{E}_{[V]\cdot\pi(g_1)} \otimes \widetilde{L}_{[V]\cdot\pi(g_1),g_2} & \xrightarrow{(9.22)} & \mathcal{E}_{[V]\cdot\pi(g_1g_2)} \end{array} \tag{9.23}$$

commutes. Apply (9.23) to $g_1 \in G'$ to conclude that under the descent described around (9.19) the action (9.22) descends to an action

$$\mathcal{E}_{[V]} \otimes L_{[V],g} \longrightarrow \mathcal{E}_{[W]} \tag{9.24}$$

Therefore, we obtain a ν-twisted vector bundle $\mathcal{E} \to X//G''$.

Assume now that G is compact, so every irreducible representation V is finite dimensional and rigid: the compactness of $G' \subset G$ implies that X is discrete. If $\mathcal{E} \to X$ is a ν-twisted vector bundle with compact—hence finite—support, then (9.4) defines a finite dimensional representation $\rho_E\colon G \to \mathrm{Aut}(E)$ on

$$E = \bigoplus_{[V]\in X} \mathcal{E}_{[V]} \otimes V \tag{9.25}$$

as follows. Fix $g \in G$ and $[V] \in X$, set $g'' = \pi(g)$, and suppose $[V] \cdot g'' \in X$ is represented by the distinguished representation $\rho_W\colon G' \to \mathrm{Aut}(W)$. Define ρ_E to map $\mathcal{E}_{[V]} \otimes V$ into $\mathcal{E}_{[W]} \otimes W$ by the composition

$$\mathcal{E}_{[V]}\otimes V \xrightarrow{\mathrm{id}\otimes\eta\otimes\mathrm{id}} \mathcal{E}_{[V]} \otimes \widetilde{L}_{[V],g} \otimes \mathrm{Hom}_{G'}(V^g, W)\otimes V \xrightarrow{\alpha\otimes\mathrm{ev}} \mathcal{E}_{[W]}\otimes W. \tag{9.26}$$

Here, recalling (9.17), η is the canonical isomorphism

$$\mathbb{C} \longrightarrow \mathrm{Hom}_{G'}(W, V^g) \otimes \mathrm{Hom}_{G'}(V^g, W) \tag{9.27}$$

defined using the fact that $\mathrm{Hom}_{G'}(W, V^g)$ and $\mathrm{Hom}_{G'}(V^g, W)$ are dual lines. Also, ev is the natural evaluation

$$\mathrm{Hom}_{G'}(V^g, W) \otimes V \longrightarrow W \tag{9.28}$$

and α is the structure map which defined $\mathcal{E} \to X$ as a ν-twisted bundle. The functors $E \mapsto \mathcal{E}$ and $\mathcal{E} \mapsto E$ are inverse equivalences, from which the isomorphism $K_G \cong K^{\nu}_{G''}(X)_c$ follows immediately. This completes the proof of (ii).

Assume G' is abelian, but not necessarily compact. Then every irreducible representation is 1-dimensional and given by a character $G' \to \mathbb{C}^{\times}$. We choose the representative vector space in each isomorphism class of irreducibles to be the trivial line $\mathbb{C}$. Then $\widetilde{L}_{[V],g} = \mathrm{Hom}_{G'}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ in (9.17) is canonically trivial and the composition (9.18) is the multiplication $\mathbb{C}\otimes\mathbb{C} \to \mathbb{C}$. If $j\colon G'' \to G$ is a splitting of (9.6), then we define $L_{[V],g''} = \widetilde{L}_{[V],j(g'')} = \mathbb{C}$. The cocycle map (9.20) is again multiplication $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, so under these identifications ν is the trivial central extension. This proves (iii).

It remains to prove (i) when G' is a lattice (finitely generated free abelian group) and X the Pontrjagin dual abelian group of unitary characters $\mathrm{Hom}(G',\mathbb{T})$. Then X is naturally topologized as a smooth torus. A unitary character $\rho_V\colon G'\to\mathbb{T}\subset\mathrm{Aut}(\mathbb{C})$ acts on $V=\mathbb{C}$ by multiplication. So we can take the lines $\widetilde{L}_{[V],g}$ in (9.17) to be the trivial line $\mathbb{C}$, as in the previous paragraph. However, the descent described in (9.19) may be nontrivial and so $L_{[V],g''}$ is not naturally trivialized. If $\rho_E\colon G\to\mathrm{Aut}(E)$ is a unitary representation on a Hilbert space E, then the spectral theorem gives a self-adjoint projection-valued measure μ_E on X. If $U\subset X$ is an open set, define $\mathcal{S}_U=\mathrm{image}\,\mu_E(U)\subset E$; it is a closed subspace of E. The proof of (i) is completed by the following lemma.

Lemma 9.29. *The assignment $U\mapsto\mathcal{S}_U$ is a sheaf on X.*

Proof. If $U'\subset U$ then $\mathcal{S}_{U'}\subset\mathcal{S}_U$. Define the restriction map $\mathcal{S}_U\to\mathcal{S}_{U'}$ to be orthogonal projection. This is obviously a presheaf. To verify the sheaf property suppose $U_1,U_2\subset X$ and $e_1\in\mathcal{S}_{U_1}$, $e_2\in\mathcal{S}_{U_2}$ have equal orthogonal projections $\bar{e}$ in $\mathcal{S}_{U_1\cap U_2}$. Then $e_1+e_2-\bar{e}\in\mathcal{S}_{U_1\cup U_2}$ is the unique vector which projects orthogonally onto $e_1\in\mathcal{S}_{U_1}$ and $e_2\in\mathcal{S}_{U_2}$. ☐

9.3. A Nontrivial Example

Assume G' is abelian and take X to be the space of unitary characters $\lambda\colon G'\to\mathbb{T}$. Fix $\lambda\in X$ and let $G''(\lambda)\subset G''$ be the subgroup which stabilizes λ. It may be regarded as a subgroupoid

$$\{\lambda\}/\!/G''(\lambda)\hookrightarrow X/\!/G''. \tag{9.30}$$

Proposition 9.31. *The restriction of the central extension ν in Theorem 9.8 to $\{\lambda\}/\!/G''(\lambda)$ is the central extension*

$$1\longrightarrow\mathbb{T}\longrightarrow H\longrightarrow G''(\lambda)\longrightarrow 1 \tag{9.32}$$

defined as the associated extension (Definition A.15) via $\lambda\colon G'\to\mathbb{T}$ of the pullback of (9.6) by the inclusion $G''(\lambda)\hookrightarrow G''$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T} & \longrightarrow & H & \longrightarrow & G''(\lambda) & \longrightarrow & 1\\
 & & \uparrow{\scriptstyle\lambda} & & \uparrow & & \| & & \\
1 & \longrightarrow & G' & \longrightarrow & G(\lambda) & \longrightarrow & G''(\lambda) & \longrightarrow & 1\\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 1
\end{array} \tag{9.33}$$

Proof. As in the proof of Theorem 9.8, we take the lines $\widetilde{L}_{[\lambda],g}$ ($\lambda\in X$, $g\in G$) to be trivial, and so (9.18) is multiplication. Assume $g''\in G''(\lambda)$. Then the set of equivariant sections of (9.19) is identified with the set of complex-valued functions $f\colon\pi^{-1}(g'')\to\mathbb{C}$ such that $f(g'g)=\lambda(g')f(g)$ for all $g\in\pi^{-1}(g'')\,,g'\in G'$. This is precisely the line at g'' associated to the central extension (9.32). ☐

Example 9.34. We resume Example 2.20, which concerns the nonsplit group extension (2.22). We show that the associated central extension ν of Theorem 9.8 is not trivializable. The characters of Π form the Pontrjagin dual torus X. We identify it with the standard torus $\mathbb{T}^2$ by letting $(\lambda_1, \lambda_2) \in \mathbb{T}^2$ act as the character

$$\lambda\colon (n^1, n^2) \longmapsto \lambda_1^{n^1}\lambda_2^{n^2}, \quad n^1, n^2 \in \mathbb{Z}. \tag{9.35}$$

The actions of the reflections which generate the quotient $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are

$$\begin{aligned} g_1''\colon (\lambda_1, \lambda_2) &\longmapsto (\lambda_1^{-1}, \lambda_2) \\ g_2''\colon (\lambda_1, \lambda_2) &\longmapsto (\lambda_1, \lambda_2^{-1}) \end{aligned} \tag{9.36}$$

The character $\lambda = (-1, -1)$ is fixed by both g_1'' and g_2''. By Proposition 9.31 the restriction of ν to $\{\lambda\}/\!/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ is the associated extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \longrightarrow & G & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \widetilde{G} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \end{array} \tag{9.37}$$

In G the commutator of lifts of g_1'', g_2'' is $(0, 1) \in \Pi$, as remarked at the end of Example 2.20. Therefore, the commutator of lifts of g_1'', g_2'' to $\widetilde{G}$ is $\lambda(0, 1) = -1$. The following lemma shows that the central extension $1 \to \mathbb{T} \to \widetilde{G} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is *not* split. It follows that ν is not trivializable.

Lemma 9.38. *Let*

$$1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{9.39}$$

be a central extension, and $g_1, g_2 \in \widetilde{G}$ lifts of the generators of the quotient. Then $g_2^{-1}g_1^{-1}g_2g_1 = \pm 1$ and (9.39) is split if and only if the sign is $+$.

The commutator is independent of the lifts since the extension is central.

Proof. By Lemma 6.17(i) the extension (9.39) is split over each factor of the quotient, so we can choose lifts g_1, g_2 such that $g_1^2 = g_2^2 = 1$. Then if $g_1g_2 = \mu g_2g_1$ for some $\mu \in \mathbb{T}$,

$$g_2 = g_1^2g_2 = \mu g_1g_2g_1 = \mu^2g_2g_1^2 = \mu^2g_2, \tag{9.40}$$

from which $\mu^2 = \pm 1$, as claimed. If $\mu = +1$ then g_1, g_2 generate a splitting of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in $\widetilde{G}$. Conversely, a splitting produces lifts with $g_1^2 = g_2^2 = 1$ and $g_1g_2 = g_2g_1$, so the sign in the commutator is $+$. ☐

9.4. A Generalization

For the application in the next section, we need to extend Theorem 9.8 to certain extended QM symmetry classes based on G. Namely, we assume given a homomorphism $\phi\colon G'' \to \{\pm 1\}$, which we pull back to G using $\pi\colon G \to G''$ in (9.6), and a $(\phi \circ \pi)$-twisted extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{9.41}$$

Also, let $c\colon G'' \to \{\pm 1\}$ be a homomorphism, which is pulled back to G using π. Overload the symbols 'ϕ' and 'c' by using them to denote the composition with π as well. We are interested in (ϕ, τ, c)-twisted representations of G, as in Definition 3.7(ii), so sketch here the modifications in the previous discussion necessary to account for ϕ, τ, and c.

First, let $1 \to \mathbb{T} \to (G')^\tau \to G' \to 1$ be the restriction of τ to G'. Notice that this is a *central* extension (untwisted). Let X^τ denote the space of isomorphism classes of τ-twisted irreducible representations of G', i.e., complex linear irreducible representations of $(G')^\tau$ on which $\mathbb{T} \subset (G')^\tau$ acts by scalar multiplication. We define an action of G'' on X^τ. Let $g'' \in G''$ and suppose first that $\phi(g'') = 1$. Let $[V] \in X^\tau$ be represented by $\rho_V\colon (G')^\tau \to \mathrm{Aut}(V)$ and choose $g \in G$ such that $\pi(g) = g''$. Choose a lift $\tilde{g} \in G^\tau$ of $g \in G$ and observe that $\alpha(g)\tilde{g}' = \tilde{g}\tilde{g}'\tilde{g}^{-1}$, $\tilde{g}' \in (G')^\tau$, is well-defined, independent of the lift $\tilde{g}$, since any two lifts differ by an element of the center $\mathbb{T}$. Then replace G' by $(G')^\tau$ in (9.7) to define $[V] \cdot g'' \in X^\tau$. If instead $\phi(g'') = -1$ modify the last step and take $[V] \cdot g''$ to be the isomorphism class of the complex conjugate representation $\overline{V^g}$ to (9.7).

Theorem 9.42. *Let (9.6) be an extension of Lie groups, $\phi, c\colon G'' \to \{\pm 1\}$ homomorphisms, and (9.41) a $(\phi \circ \pi)$-twisted extension. Let X^τ be the space of isomorphism classes of irreducible τ-twisted representations of G'. Assume G'' is compact and G' is either compact or a lattice.*

(i) *There is an induced ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν of the groupoid $X^\tau/\!/G''$ such that a $(\phi \circ \pi, \tau, c \circ \pi)$-twisted representation E of G induces a ν-twisted G''-equivariant sheaf over X^τ. (Infinite dimensional representations are assumed unitary.)*

(ii) *If G is compact, there is an isomorphism ${}^\phi K_G^{\tau, c} \to K^\nu_{G''}(X^\tau)_{\mathrm{cpt}}$.*

(iii) *If $(G')^\tau$ is abelian, then a splitting of the group extension*

$$1 \longrightarrow (G')^\tau \longrightarrow G^\tau \xrightarrow{\pi} G'' \longrightarrow 1 \tag{9.43}$$

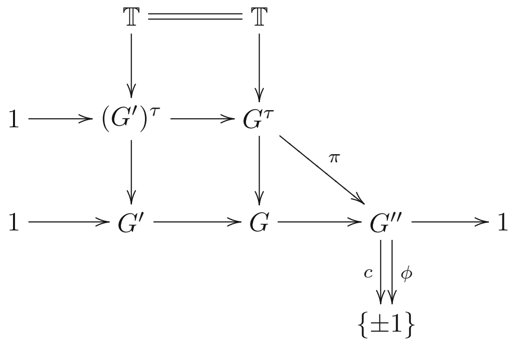
induces an isomorphism of ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions $\nu(\phi, c) \xrightarrow{\cong} \nu$, where $\nu(\phi, c)$ is the ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension in Remark 7.26.

Proof. The proof follows that of Theorem 9.8 with a few modifications. Throughout replace G' by $(G')^\tau$ and X by X^τ. If $g \in G$ and $(\phi \circ \pi)(g) = -1$, then replace (9.17) with

$$\widetilde{L}_{[V],g} := \mathrm{Hom}_{(G')^\tau}(\overline{W}, V^g), \tag{9.44}$$

where V^g is defined as before by (9.7). The descent to lines $L_{[V],g''}$ follows the argument around (9.19). The data of the ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν consist of the lines $L_{[V],g''}$ and the homomorphisms $(X^\tau/\!/G'')_1 \to \{\pm 1\}$ obtained by composing ϕ, c with the projection $(X^\tau/\!/G'')_1 = X^\tau \times G'' \to G''$.

For (iii) it helps organize the data in the diagram



(9.45)

A splitting of (9.43), which is a right inverse of π, induces a trivialization of the lines $L_{[V],g''}$ if G' is abelian, as in the proof of Theorem 9.8(iii). ☐

10. Gapped Topological Insulators

10.1. Periodic Systems of Electrons

We have in mind the following sort of quantum system. Let E be a Euclidean space and let $\overline{C} \subset E$ be the sites of a periodic system of atoms.[25] Assume there is a full lattice Π of translations of E which preserves $\overline{C}$. A typical Hilbert space in this situation is $\mathcal{H} = L^2(E; W)$, the space of L^2 functions on E with values in a finite dimensional complex vector space W. It is the Hilbert space of a single electron in the lattice. If we incorporate electron spin, then W is the 2-dimensional representation of $SU(2)$. Let X_Π be the abelian group of characters $\lambda\colon \Pi \to \mathbb{T}$, the *Brillouin torus*. Fourier transform writes a function $f\colon E \to W$ in terms of quasi-periodic functions f_λ, which satisfy

$$f_\lambda(x+\xi) = \lambda(\xi) f_\lambda(x), \quad \xi \in \Pi. \tag{10.1}$$

In condensed matter physics (10.1) is called the *Bloch wave condition*, and one usually writes $\lambda(\xi) = e^{ik\cdot\xi}$ where k (not uniquely fixed by λ) is the *Bloch momentum*; see Proposition D.17 for further discussion. Note that quasi-periodic functions are *not* in $L^2(E; W)$. If λ is the trivial character $\lambda \equiv 1$, then f_λ is periodic; for general λ, the function f_λ shifts by a phase when the argument is translated by an element of the lattice. Periodic functions on E are equivalently functions on the torus E/Π—it now makes sense to demand that they be L^2—and quasi-periodic functions which satisfy (10.1) are sections of a complex vector bundle[26] $\mathcal{L}_\lambda \otimes W \to E/\Pi$. Proposition D.17 is a precise formulation of the Fourier transform, or Bloch sum, which expresses an L^2 function

[25] A crystal (Definition 2.17) in spacetime consists of the trajectory in spacetime of a subset of space, on which the only restriction is invariance under a full lattice, whereas in this heuristic introduction we find it more convenient to freeze time and also assume that $\overline{C}$ is a discrete set of points.

[26] The line bundle $\mathcal{L}_\lambda \to E/\Pi$ is constructed in the proof of Proposition D.17.

on E as an L^2 section of a *Hilbert bundle* $\mathcal{E} \to X_\Pi$ whose fiber at $\lambda \in X_\Pi$ is the infinite dimensional Hilbert space $\mathcal{E}_\lambda = L^2(E/\Pi; \mathcal{L}_\lambda \otimes W)$:

$$\mathcal{H} = L^2(E; W) \cong L^2(X_\Pi; \mathcal{E}). \tag{10.2}$$

For a noninteracting single electron, the Hamiltonian H is typically a sum

$$H = \frac{\hbar^2}{2m}\triangle + u, \tag{10.3}$$

where m is the mass of the electron, $\triangle$ is the Laplace operator on Euclidean space, and $u = \sum_{\bar{c}\in\bar{C}} u_{\bar{c}}$ is a sum of local potential energy functions, thought of as localized near each site. We assume that H is invariant under Π, and so under the isomorphism (10.2) it corresponds to a family of Hamiltonians $\{H_\lambda\}$ parametrized by $\lambda \in X_\Pi$, which we prove in Proposition D.30 vary continuously in λ. In fact, H_λ is a Laplace operator on $L^2(E/\Pi; \mathcal{L}_\lambda)$—an elliptic operator on a compact manifold—so has discrete spectrum. We assume that the full Hamiltonian H has a gap in its spectrum. Often one fixes a particular *Fermi level* in the gap. We normalize the Fermi level to be at zero energy. In other words, H is invertible with bounded inverse. It follows that each H_λ is also invertible, and so there is a decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ of the Hilbert bundle as a sum of "valence bands" $\mathcal{E}^-$ and "conduction bands" $\mathcal{E}^+$ (see Proposition D.13). This is the moment we restrict to insulators. We further assume that there is a *finite* set of valence bands—$\mathcal{E}^-$ has finite rank—and therefore an *infinite* set of conduction bands—$\mathcal{E}^+$ has infinite rank. Because of this asymmetry, there are no Hamiltonian-reversing symmetries in this system (Lemma 3.3). More generally, we assume that the symmetry group G of the system is a Lie group containing Π as a normal subgroup such that $G'' = G/\Pi$ is compact. Then the quotient G'' acts on X_Π—the action factors through a finite group—and the action lifts to $\mathcal{E}$, but the lifted action is twisted, as explained in Sect. 9. There are further possible ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions of the groupoid $X_\Pi/\!/G''$ as G may have a nontrivial QM symmetry type which governs its representation on the Hilbert space $\mathcal{H}$.

Remark 10.4. Our considerations also apply to more realistic Hamiltonians than (10.3). In particular, in three dimensions, we could take W to be the 2-dimensional spin representation and then include the physically important spin-orbit term $\frac{1}{2m^2c^2}\vec{S}\cdot(\nabla u \times p)$, where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ is the spin operator and we have used unexplained notation instantly recognizable from the condensed matter literature, see e.g., [9]. More generally, the entire series of relativistic corrections can be included by taking W to be the four-dimensional Dirac representation and using the Dirac Hamiltonian $c\vec{\alpha}\cdot\vec{p} + \beta mc^2 + u$, where

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10.5}$$

Another important physical example—a spin system—is an analog with a discrete field supported on the sites in $\overline{C}$. For this purpose, we assume that the Π-action on $\overline{C}$ has only finitely many orbits. Let $W_{\bar{c}}$ be a finite dimensional complex vector space attached to each site $\bar{c} \in \overline{C}$, and assume the action of G on $\overline{C}$ is lifted to the vector bundle $W \to \overline{C}$. The quantum Hilbert space $\mathcal{H} = L^2(\overline{C}; W)$ is the space of L^2 sections of $W \to \overline{C}$. Now the Fourier transform produces a *finite rank* hermitian vector bundle $\mathcal{E} \to X_\Pi$, and Fourier transform produces an isomorphism $\mathcal{H} = L^2(\overline{C}; W) \cong L^2(X_\Pi; \mathcal{E})$, as in (10.2). So the Hamiltonians $\{H_\lambda\}$ are self-adjoint operators on finite dimensional vector spaces $\mathcal{E}_\lambda$.

We remark that sometimes a finite rank bundle $\mathcal{E} \to X_\Pi$ is also used to focus on a subbundle of the infinite rank bundle in the first example (10.3). This is done to model a finite number of bands. (If eigenvalues of H_λ do not cross as λ varies, then each band is a line bundle whose fiber at $\lambda \in X_\Pi$ is an eigenspace of H_λ, but of course the eigenvalues may cross and so $\mathcal{E}$ is not generally a sum of line bundles.)

With these examples in mind we craft Definition 10.7 and Hypothesis 10.9 below. We consider both finite rank (Type F) and infinite rank (Type I) Hilbert bundles $\mathcal{E} \to X_\Pi$.

10.2. Formal Setup

Let (M, Γ) be a Galilean spacetime with a crystal C (Definition 2.17) or, more generally, a spin crystal C (Definition 2.25). The symmetry group $G(C)$ contains a normal subgroup U of time translations; the quotient $G(C)/U$ is a spacetime crystallographic group, which is a group extension (2.18). The kernel Π is a full lattice of spatial translations. The quotient $\widehat{P}$ is a finite group, the *magnetic point group*, which is an extension (2.19) of the point group P of orthogonal spatial transformations by time-reversal symmetries. More generally, we take G to be a Galilean symmetry group in the sense of Definition 2.10, modified to mod out by time translation, so equipped with a homomorphism $\gamma\colon G \to \mathrm{Aut}(M, \Gamma)/U$ which is split over the intersection of $\gamma(G)$ with the spatial translation subgroup $V \subset \mathrm{Aut}(M, \Gamma)$. Because symmetries must preserve the crystal, we have $\gamma(G) \subset G(C)/U$. Assume $\Pi \subset \gamma(G)$, and then the splitting (2.12) gives an inclusion $\Pi \subset G$ as a normal subgroup. Summarizing, the symmetry group G fits into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \longrightarrow & G & \xrightarrow{\pi} & G'' & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\bar\gamma} & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & G(C)/U & \longrightarrow & \widehat{P} & \longrightarrow & 1 \end{array} \tag{10.6}$$

of group extensions. As the Hamiltonian H accounts for time translations U, we do not include U as a subgroup of G; see Remark 3.10.

Definition 10.7. A *band insulator* consists of the following data:

(i) a Galilean spacetime (M, Γ) with a spin crystal $C \subset M$;
(ii) a Galilean symmetry group G as described in the previous paragraph;
(iii) an extended QM symmetry class (G'', ϕ, τ, c); and
(iv) a gapped system $(\mathcal{H}, H, \rho^\tau)$ with extended QM symmetry type (G, ϕ, τ, c), where (G, ϕ, τ, c) is the pullback of (G'', ϕ, τ, c) along $G \xrightarrow{\pi} G''$.

Notice that the extension $G^\tau \to G$ splits over Π, and we regard Π as a subgroup of G^τ. Let $X_\Pi = \mathrm{Hom}(\Pi, \mathbb{T})$ be the compact abelian Lie group of characters of Π. Restrict $\rho^\tau \colon G^\tau \to \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ to $\Pi \subset G^\tau$, and note that $\rho^\tau(\Pi)$ is an abelian group of operators on $\mathcal{H}$ which commute with the Hamiltonian H, a self-adjoint operator on $\mathcal{H}$. The spectral theorem simultaneously diagonalizes the operators in $\rho^\tau(\Pi)$. This is encoded in a projection-valued measure $\mu_{\mathcal{H}}$ on X_Π whose value on a Borel subset $U \subset X_\Pi$ is orthogonal projection $\mu_{\mathcal{H}}(U)$ onto—very roughly speaking—the subspace $\mu_{\mathcal{H}}(U)(\mathcal{H}) \subset \mathcal{H}$ of vectors which transform under a character of Π which lies in U. Of course, this description only works for the discrete part of the spectrum, and as we assume below that the spectrum is continuous, vectors in $\mu_{\mathcal{H}}(U)(\mathcal{H})$ are formally a smearing of non-existent eigenfunctions. In case $\mathcal{H} = L^2(E; W)$, described in the introductory subsection above, quasi-periodic functions (10.1) are not L^2, so must be smeared to obtain vectors in $\mathcal{H}$. The assignment $U \mapsto \mu_{\mathcal{H}}(U)(\mathcal{H})$ is a *sheaf* $\mathcal{S}_{\mathcal{H}}$ of Hilbert spaces on X_Π; see Lemma 9.29.

Remark 10.8. We elaborate on the remarks in Example 9.5. If $\mathcal{H}$ is an irreducible representation of Π—that is, a particular character $\lambda \colon \Pi \to \mathbb{T}$ acting on a one-dimensional Hilbert space $\mathcal{H}$—then $\mathcal{S}_{\mathcal{H}}$ is the skyscraper sheaf at $\lambda \in X_\Pi$. If $\mathcal{H}$ is an irreducible representation of G^τ, then $\mathcal{H}$ is supported on an orbit of the G''-action on X_Π. This picture is familiar in the Wigner–Bargmann–Mackey theory of representations of the Poincaré group, for example, where the sheaf constructed from the translations in Minkowski spacetime is supported on an orbit of the Lorentz group on the characters of the translation group (mass shell). Hypothesis 10.9(ii) below imposes a regularity which is very far from irreducibility.

To define the monoid $\mathcal{TP}(G, \phi, \tau, c)$ of topological phases, we need to impose finiteness conditions on the symmetry group G and on the representation. Recall that the *Haar measure* on the compact abelian Lie group X_Π is the unique translation-invariant measure with prescribed total volume.

Hypothesis 10.9. (i) G'' is a compact Lie group.

(ii) The spectral measure $\mu_{\mathcal{H}}$ satisfies the following regularity property with respect to Haar measure: there is a G^τ-equivariant Hilbert bundle $\mathcal{E} \to X_\Pi$ and a G^τ-equivariant isomorphism

$$\mathcal{H} \xrightarrow{\cong} L^2(X_\Pi, \mathcal{E}) \tag{10.10}$$

under which Π acts by multiplication on each fiber; then $\mathcal{S}_{\mathcal{H}}$ is the sheaf of sections of the Hilbert bundle $\mathcal{E} \to X_\Pi$.

(iii) The Hamiltonian H induces a *continuous* family of self-adjoint operators $\{H_\lambda\}$ on the fibers of $\mathcal{E} \to X_\Pi$.

(iv) The invertible Hamiltonian H induces a decomposition $\mathcal{E} \cong \mathcal{E}^+ \oplus \mathcal{E}^-$ such that $L^2(X_\Pi, \mathcal{E}^+)$ is the subspace of $\mathcal{H}$ on which H is positive and $L^2(X_\Pi, \mathcal{E}^-)$ is the subspace of $\mathcal{H}$ on which H is negative. We assume $\mathcal{E}^-$ has *finite* rank.

(v) We make two distinct hypotheses on $\mathcal{E}^+$:

$$\text{Type F: } \mathcal{E}^+ \text{ has finite rank}$$
$$\text{Type I: } \mathcal{E}^+ \text{ has infinite rank}$$

Statement (i) requires no comment. In (ii), the action of $\xi \in \Pi$ on $\psi \in L^2(X_\Pi, \mathcal{E})$ is $(\xi \cdot \psi)(\lambda) = \lambda(\xi)\psi(\lambda)$ for $\lambda \in X_\Pi$. The G^τ-action on $\mathcal{E} \to X_\Pi$ descends to a twisted G''-action—better said, $\mathcal{E}$ is a twisted bundle for a ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension [Definition 7.23(iii)] of the groupoid $X_\Pi //G''$—*and the ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension is precisely the ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν of Theorem 9.42(i)*. Because H commutes with the action of Π, it preserves the fibers of $\mathcal{E} \to X_\Pi$; the import of (iii) is the continuity of the family of operators $\{H_\lambda\}$ in the sense of Definition D.10. The decomposition $\mathcal{E} \cong \mathcal{E}^+ \oplus \mathcal{E}^-$ then follows from Proposition D.13. The import of (iv) is the finite rank condition on $\mathcal{E}^-$. Finally, notice that in Type I we necessarily have $c \equiv 1$, since an odd symmetry would be an isomorphism between the infinite rank bundle $\mathcal{E}^+$ and the finite rank bundle $\mathcal{E}^-$.

In Appendix D, we prove that the periodic system of electrons discussed in the introductory subsection satisfies these hypotheses.

We can now compute the commutative monoid[27] $\mathcal{TP}(G, \phi, \tau, c)$ of topological phases of band insulators. We will obtain an explicit answer in Type F and only a partial answer in Type I; in both cases we will find a simple answer after passing to an abelian group $\mathcal{RTP}(G, \phi, \tau, c)$. Recall that Definition 5.1 of topological phase relies on the definition of continuous families of gapped systems. The discussion in Appendix D, especially Definition D.12, provides the necessary background.

10.3. Topological Phases: Type F

We show that the isomorphism classes of $\mathcal{E}^+, \mathcal{E}^-$ determine the topological phase. As in Definition 7.33, let $s\mathrm{Vect}^\nu_{G''}(X_\Pi)$ denote the commutative monoid of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded finite rank ν-twisted G''-equivariant vector bundles over X_Π.

Theorem 10.11. *For band insulators of Type F, the map*

$$\begin{aligned} \mathcal{TP}_F(G, \phi, \tau, c) &\longrightarrow s\mathrm{Vect}^\nu_{G''}(X_\Pi) \\ (\mathcal{H}, H, \rho^\tau) &\longmapsto \mathcal{E}^+ \oplus \mathcal{E}^- \end{aligned} \tag{10.12}$$

is an isomorphism of commutative monoids.

Proof. To begin we prove that (10.12) is well-defined. First, after a homotopy we may assume that H is the grading operator; see Lemma 5.14. In any case, $\mathcal{E}^+ \oplus \mathcal{E}^-$ is defined using the positive and negative spectral projections, which only depend on the grading operator associated to H. A homotopy of Type F band insulators is a Hilbert bundle $\mathcal{H} \to \Delta^1 \times X_\Pi$ together with a

[27] For Type I, the set $\mathcal{TP}_I(G, \phi, \tau, c)$ is not a monoid since there is no zero element—$\mathcal{E}^+$ has infinite rank and so the zero Hilbert space $\mathcal{H}$ is not allowed. We fix this by formally adjoining a zero element to $\mathcal{TP}_I(G, \phi, \tau, c)$.

continuous family of Hamiltonians H_t and representations $\rho_t^\tau, t \in \Delta^1$. (See Definition D.12.) The (Bloch) Hamiltonians $H_{t,\lambda}$, $(t, \lambda) \in \Delta^1 \times X_\Pi$ on the fibers of $\mathcal{E} \to \Delta^1 \times X_\Pi$ form a continuous family, by Hypothesis 10.9(iii), and the positive and negative spectral projections are obviously continuous since $\mathcal{E}$ has finite rank. Now because G'' and X_Π are compact, isomorphism classes of ν-twisted finite rank G''-equivariant super vector bundles over X_Π are rigid—they have no continuous deformations—and so bundles which are homotopic are in fact isomorphic. It follows that (10.12) is well-defined.

To prove that (10.12) is surjective, suppose given a ν-twisted bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \to X_\Pi$. Define

$$\mathcal{H} = L^2(X_\Pi; \mathcal{E}) = L^2(X_\Pi; \mathcal{E}^+) \oplus L^2(X_\Pi; \mathcal{E}^-) \tag{10.13}$$

and let H be the grading operator. The representation ρ^τ is constructed using the map (9.26) (which simplifies since Π is abelian).

To prove that (10.12) is injective, suppose $(\mathcal{H}_i, H_i, \rho_i^\tau)$, $i = 0, 1$ are band insulators which map to isomorphic ν-twisted bundles $\mathcal{E}_i = \mathcal{E}_i^+ \oplus \mathcal{E}_i^-$. Then an isomorphism $\theta\colon \mathcal{E}_0 \to \mathcal{E}_1$ induces an isomorphism $\mathcal{H}_0 \to \mathcal{H}_1$, since $\mathcal{H}_i \cong L^2(X_\Pi, \mathcal{E}_i)$. The grading operators correspond, since θ is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded bundles, and so too do the representations ρ_i^τ, since θ is an isomorphism of (twisted) equivariant bundles. □

To pass to an abelian group we impose that $\mathcal{E}^+ \oplus \mathcal{E}^-$ be the difference $\mathcal{E}^+ - \mathcal{E}^-$. This is convenient and leads to a computable K-theory group. Although it is used in the condensed matter literature, it is not clear to us that this subtraction is well-motivated physically.

Definition 10.14. A band insulator $(\mathcal{H}, H, \rho^\tau)$ of Type F is *topologically trivial* if there exists an odd automorphism P of the associated ν-twisted equivariant bundle $\mathcal{E}^+ \oplus \mathcal{E}^-$. Define $\mathcal{RTP}_F(G, \phi, \tau, c)$ to be the quotient of $\mathcal{TP}_F(G, \phi, \tau, c)$ by the submonoid of isomorphism classes of topologically trivial band insulators.

Definition 7.33 immediately implies the following.

Theorem 10.15. *For band insulators of Type F, there is an isomorphism*

$$\begin{aligned} \mathcal{RTP}_F(G, \phi, \tau, c) &\longrightarrow K^\nu_{G''}(X_\Pi) \\ (\mathcal{H}, H, \rho^\tau) &\longmapsto \mathcal{E}^+ \oplus \mathcal{E}^- \end{aligned} \tag{10.16}$$

10.4. Topological Phases: Type I

Let $(\mathcal{H}, H, \rho^\tau)$ be a band insulator of Type I. As observed after Hypothesis 10.9 because $\mathcal{E}^+$ has infinite rank and $\mathcal{E}^-$ has finite rank there are no odd symmetries which interchange them, whence $c \equiv 1$. Thus, $\mathcal{E}^+$ and $\mathcal{E}^-$ are independent, and we will show that they are separately invariants of $(\mathcal{H}, H, \rho^\tau)$ up to homotopy. The infinite rank bundle $\mathcal{E}^+ \to X_\Pi$ is trivializable as a nonequivariant bundle, although it may be nontrivial as an equivariant bundle, as we will see in Example 10.23 below. We do not know a standard commutative monoid in which to locate the isomorphism class of $\mathcal{E}^+$. The group completion of this monoid is trivial, by an Eilenberg swindle. For that reason, we only track the finite rank bundle $\mathcal{E}^-$.

As a preliminary, we prove that certain universal ν-twisted Hilbert bundles over X_Π exist. The theorem we need is a small modification of [26, Sect. 3.3], to which we refer for background and details. A ν-twisted G''-equivariant Hilbert bundle $\mathcal{F} \to X_\Pi$ is *universal* if for any ν-twisted G''-equivariant Hilbert bundle $\mathcal{E} \to X_\Pi$ there exists an embedding $\mathcal{E} \to \mathcal{F}$. By [26, Lemma A.24] a universal bundle $\mathcal{F}$ is *absorbing* in the sense that $\mathcal{F} \oplus \mathcal{E} \cong \mathcal{F}$ for any ν-twisted G''-equivariant Hilbert bundle $\mathcal{E}$. This last property shows that $\mathcal{F}$ has infinite rank. We remark that the universality property can be made stronger, so that it holds on open subsets as well, but we do not need this here.

Lemma 10.17. *There exists a universal, hence also absorbing, ν-twisted G''-equivariant Hilbert bundle $\mathcal{F} \to X_\Pi$.*

We remark that it is not a priori obvious that *any* nonzero ν-twisted G''-equivariant Hilbert bundle over X_Π exists. Lemma 10.17 holds for any ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν.

Proof. Let $\mathcal{G} = X_\Pi // G''$. Write $\nu = (\phi, \sigma)$, where σ is a ϕ-twisted extension of $\mathcal{G}$. The homomorphism ϕ on the arrows of the groupoid $\mathcal{G}$ determines a double cover $\widetilde{\mathcal{G}} \to \mathcal{G}$, and the ϕ-twisted extension σ of $\mathcal{G}$ pulls back to an untwisted central extension $\tilde{\nu}$ of $\widetilde{\mathcal{G}}$. We describe it explicitly (for general groupoids). Namely, $\widetilde{\mathcal{G}}^0 = \mathcal{G}_0 \times \mathbb{Z}/2\mathbb{Z}$ and $\widetilde{\mathcal{G}}^1 = \mathcal{G}_1 \times \mathbb{Z}/2\mathbb{Z}$: an arrow $(x \xrightarrow{\gamma} x') \in \mathcal{G}_1$ lifts to two arrows $\big((x,\epsilon) \xrightarrow{(\gamma,\epsilon)} (x',\epsilon')\big)$ which satisfy $\epsilon + \phi(\gamma) = \epsilon'$. The ϕ-twisted extension $L^\sigma \to \mathcal{G}_1$ lifts to an untwisted central extension $L^{\tilde{\sigma}} \to \widetilde{\mathcal{G}}^1$ with $L^{\tilde{\sigma}}_{(\gamma,0)} = L^\sigma_\gamma$ and $L^{\tilde{\sigma}}_{(\gamma,1)} = \overline{L^\sigma_\gamma}$. We leave the reader to check that the structure maps (7.10) for σ induce structure maps (7.7) for $\tilde{\sigma}$. By [26, Lemma 3.12] there exists a universal $\tilde{\nu}$-twisted Hilbert bundle $\mathcal{F}' \to \widetilde{\mathcal{G}}$. Let $\sigma \colon \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$ be the involution defined by $(x,\epsilon) \mapsto (x, \epsilon+1)$ and $(\gamma,\epsilon) \mapsto (\gamma,\epsilon+1)$. Set $\mathcal{F} = \mathcal{F}' \oplus \overline{\sigma^*\mathcal{F}'}$, a $\tilde{\sigma}$-twisted Hilbert bundle over $\widetilde{\mathcal{G}}$. For each $(x,\epsilon) \in \widetilde{\mathcal{G}}$, adjoin a new arrow $\big((x,\epsilon) \xrightarrow{\delta_{(x,\epsilon)}} (x,\epsilon+1)\big)$. Let $\mathcal{G}'$ denote the groupoid $\widetilde{\mathcal{G}}$ with the arrow $\delta_{(x,\epsilon)}$ adjoined. Define $\phi' \colon (\mathcal{G}')_1 \to \{\pm 1\}$ to be 0 on arrows $(\gamma,\epsilon) \in \widetilde{\mathcal{G}}$ and 1 on arrows $\delta_{(x,\epsilon)}$. Attach the trivial line to $\delta_{(x,\epsilon)}$; these trivial lines together with the lines $L^{\tilde{\sigma}}$ form a ϕ'-twisted extension σ' of $\mathcal{G}'$. The groupoid $\mathcal{G}'$ with twisting (ϕ', σ') is locally equivalent to the groupoid $\mathcal{G}$ with twisting $\nu = (\phi, \sigma)$. The Hilbert bundle $\mathcal{F} \to \widetilde{\mathcal{G}}$ extends to a (ϕ',σ')-twisted Hilbert bundle over $\mathcal{G}'$, and an easy argument proves that it is universal. $\square$

Let $\mathrm{Vect}^\nu_{G''}(X_\Pi)$ be the commutative monoid of equivalence classes of finite rank ν-twisted ungraded G''-equivariant vector bundles over X_Π.

Theorem 10.18. *For band insulators of Type I, the map*

$$\begin{aligned} \mathcal{TP}_I(G,\phi,\tau) &\longrightarrow \mathrm{Vect}^\nu_{G''}(X_\Pi) \\ (\mathcal{H}, H, \rho^\tau) &\longmapsto \mathcal{E}^- \end{aligned} \tag{10.19}$$

is a surjective homomorphism of commutative monoids. If G'' is trivial, then (10.19) is an isomorphism.

Proof. With one amplification, the argument in the Proof of Theorem 10.11 proves that (10.19) is well-defined. Namely, since $\mathcal{E}^+$ has infinite rank, we need to use Proposition D.13 to prove that the positive and negative spectral projections vary continuously. For the surjectivity, given $\mathcal{E}^- \to X_\Pi$ of finite rank, consider $\mathcal{F} \oplus \mathcal{E}^- \to X_\Pi$, where $\mathcal{F}$ is the universal bundle guaranteed by Lemma 10.17. (Here we only use the existence of an infinite rank bundle, not its universality.) Set $\mathcal{H} = L^2(X_\Pi; \mathcal{F}) \oplus L^2(X_\Pi; \mathcal{E}^-)$ and let H be the grading operator. Construct a representation ρ^τ of G^τ using (9.26). Then $\mathcal{E}^-$ is the image of $(\mathcal{H}, H, \rho^\tau)$ under (10.19). For the last statement, we observe that an infinite rank (nonequivariant) Hilbert bundle is trivializable; see the remark preceding Definition D.8. □

Our passage to an abelian group for Type I is more natural than that for Type F.

Definition 10.20. For band insulators of Type I, we define $\mathcal{RTP}_I(G, \phi, \tau)$ to be the group completion of the commutative monoid $\mathcal{TP}_I(G, \phi, \tau)$.

Theorem 10.21. *For band insulators of Type I, there is an isomorphism*

$$\begin{aligned} \mathcal{RTP}_I(G,\phi,\tau) &\longrightarrow K^{\nu}_{G''}(X_\Pi) \\ (\mathcal{H}, H, \rho^\tau) &\longmapsto \mathcal{E}^- \end{aligned} \tag{10.22}$$

Proof. The group completion of $\mathcal{TP}_I(G, \phi, \tau)$ is the direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2$ of the group completion $\mathcal{A}_1$ of the commutative monoid of infinite rank ν-twisted G''-equivariant Hilbert bundles $\mathcal{E}^+ \to X_\Pi$ and the group completion $\mathcal{A}_2$ of the commutative monoid of finite rank ν-twisted G''-equivariant Hilbert bundles $\mathcal{E}^- \to X_\Pi$. The latter is the twisted equivariant K-theory group $K^{\nu}_{G''}(X_\Pi)$, by Remark 7.36. We claim that $\mathcal{A}_1 = 0$. Let $\mathcal{F} \to X_\Pi$ be a universal ν-twisted G''-equivariant Hilbert bundle. We use it to pull off an Eilenberg swindle (Remark 5.9): given any $\mathcal{E}^+ \in \mathcal{A}_1$ the absorption property of $\mathcal{F}$ implies $\mathcal{F} + \mathcal{E}^+ = \mathcal{F}$ in $\mathcal{A}_1$, and since $\mathcal{A}_1$ is a group we can cancel $\mathcal{F}$ to obtain $\mathcal{E}^+ = 0$. □

Example 10.23. The following simple example shows that the commutative monoid built from bundles $\mathcal{E}^+ \to X_\Pi$ may be nontrivial if $G'' \neq \{1\}$. Let space be 1-dimensional and suppose $G = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ is the split extension $1 \to \mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ by -1. This is a possible symmetry group of a crystal in one spatial dimension, say the subset $\overline{C} = \mathbb{Z} \subset \mathbb{E}^1$ of the Euclidean line. Suppose ϕ, τ are trivial. Now $\mathbb{Z}/2\mathbb{Z}$ acts on the circle $X = \mathbb{T}$ of characters of $\mathbb{Z}$ by $\lambda \mapsto \bar{\lambda}$, which has fixed points at $\lambda = \pm 1$. Any $\mathbb{Z}/2\mathbb{Z}$-equivariant vector bundle $\mathcal{E}^+ \to \mathbb{T}$ restricts at the two fixed points to representations of the stabilizer group $\mathbb{Z}/2\mathbb{Z}$. Thus, at each point we obtain $n^+, n^- \in \mathbb{Z} \cup \{\infty\}$ which tell the dimensions of the subspaces on which $\mathbb{Z}/2\mathbb{Z}$ acts trivially and by the sign representation, respectively. The monoid of infinite rank bundles is then the set of quartets $(n_1^+, n_1^-, n_{-1}^+, n_{-1}^-) \in (\mathbb{Z} \cup \{\infty\})^{\times 4}$ such that the sums $n_1^+ + n_1^-$ and $n_{-1}^+ + n_{-1}^-$ are infinite. Note that equivariant line bundles which realize all four ways of ordering two 0s and two 1s are easily constructed, and then taking finite direct sums we can realize any such quartet. The easy

Eilenberg swindle which proves that the group completion of this monoid of quartets is trivial is based on (5.10). On the other hand, realistic physical examples seem to have $n_1^+ = n_1^- = n_{-1}^+ = n_{-1}^- = \infty$. For example, if $\mathcal{H} = L^2(\mathbb{E}^1)$, then the fiber of $\mathcal{E}$ at $\lambda = 1$ is the Hilbert space $L^2(\mathbb{E}^1/\mathbb{Z})$ of periodic functions. The Hamiltonian induces a self-adjoint operator on periodic functions, and by assumption it commutes with reflection on the circle $\mathbb{E}^1/\mathbb{Z}$, which is the action of the generator of $\mathbb{Z}/2\mathbb{Z} \subset G$. Each eigenspace of the reflection action is infinite dimensional, and since we assume the negative eigenspaces of the Hamiltonian are finite dimensional, it follows that $n_1^+ = n_1^- = \infty$. A similar argument applies to anti-periodic functions ($\lambda = -1$).

10.5. Simplifying Assumptions; More Familiar *K*-Theory Groups

As at the end of Sect. 8, Theorems 10.15 and 10.21 simplify if we assume certain splittings. This is not a physically valid hypothesis in general, as far as we know, but it does hold in many examples. Specifically, we assume Hypothesis 8.7, modified to account for the form (10.6) of G and, in the case of Type I, for the fact that $c \equiv 1$. Let $\psi = (t,c)\colon G'' \to \mathcal{C} = \{\pm 1\} \times \{\pm 1\}$ and $G_0'' = \ker\psi$. Set $G_0 = \pi^{-1}(G_0'')$, where $\pi\colon G \to G''$ is the projection.

Hypothesis 10.24. Let $A \subset \mathcal{C}$ be the image of ψ.

(i) The group G'' is a direct product $G'' \cong A \times G_0''$ and under this isomorphism ψ is projection onto A. Furthermore, (10.6) splits over $G_0'' \subset G''$ and so $G \cong A \times G_0$.

(ii) The restriction $G_0^\tau \to G_0$ of $G^\tau \to G$ splits and we fix a splitting.

Therefore, there is fixed an isomorphism $G^\tau \cong A^\tau \times G_0$.

Elements of $A^\tau \subset G^\tau$ commute with Π. Thus, if $a \in A^\tau$ satisfies $\phi(a) = -1$—that is, $\rho^\tau(a)$ acts antilinearly in any τ-compatible representation ρ^τ—then $\rho^\tau(a)$ transforms the character $\lambda \in X_\Pi$ to the inverse character $\lambda^{-1} \in X_\Pi$. Let $\sigma\colon X_\Pi \to X_\Pi$ denote the involution $\lambda \mapsto \lambda^{-1}$. Note the group G_0'' acts on X_Π through the homomorphism $\bar\gamma\colon G_0'' \to \widehat{P}$ to the magnetic point group $\widehat{P}$ in (10.6), and the magnetic point group acts on the lattice Π through the extension (2.18).

Corollary 10.25. *If Hypothesis 10.24 holds, then we have the following table for the twisted equivariant K-theory group in Theorem 10.15 for band insulators of Type F, where ν_0 is defined in (10.26) below:*

A	*1*	diag	$\{\pm 1\} \times 1$	$\mathcal{C}$	$1 \times \{\pm 1\}$
T^2			$+1$	$+1$	
C^2				-1	-1
$K^\nu_{G_0''}(X_\Pi)$	$K^{\nu_0}_{G_0''}(X_\Pi)$	$K^{\nu_0-1}_{G_0''}(X_\Pi)$	$KR^{\nu_0}_{G_0''}(X_\Pi)$	$KR^{\nu_0-1}_{G_0''}(X_\Pi)$	$KR^{\nu_0-2}_{G_0''}(X_\Pi)$

A	$\mathcal{C}$	$\{\pm 1\} \times 1$	$\mathcal{C}$	$1 \times \{\pm 1\}$	$\mathcal{C}$
T^2	-1	-1	-1		$+1$
C^2	-1		$+1$	$+1$	$+1$
$K^\nu_{G_0''}(X_\Pi)$	$KR^{\nu_0-3}_{G_0''}(X_\Pi)$	$KR^{\nu_0-4}_{G_0''}(X_\Pi)$	$KR^{\nu_0-5}_{G_0''}(X_\Pi)$	$KR^{\nu_0-6}_{G_0''}(X_\Pi)$	$KR^{\nu_0-7}_{G_0''}(X_\Pi)$

An element of $KR^q(X_\Pi)$ is represented by a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle $E \to X_\Pi$, a lift $\tilde{\sigma}$ of the involution σ to an antilinear involution of E, and an action of the Clifford algebra Cl_q on the fibers of E which (graded) commutes with $\tilde{\sigma}$. In the equivariant case $KR^q_{G''_0}(X_\Pi)$, there is an additional linear action of G''_0 on E which covers the action on X_Π and commutes with $\tilde{\sigma}$ and the Clifford algebra action. A refined version of Corollary 10.25 is an equivalence between the category of bundles $E \to X_\Pi$ of this type and the category of ν-twisted G''-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded bundles $\mathcal{E} \to X_\Pi$, where the chart tells the precise correspondence. The proof is modeled on that of Corollary 8.9—including the Morita equivalences therein—but with real structures replaced by $\tilde{\sigma}$. Note that both elements $\overline{T}, \overline{C}$ of A, if present, act on X_Π via the involution σ. This is the familiar fact in band structure theory that time reversal and "particle–hole conjugation" map $k \mapsto -k$, where k is the Bloch momentum.

Proof. Given a ν-twisted G''-equivariant super vector bundle $\mathcal{E}^+ \oplus \mathcal{E}^- \to X_\Pi$ we need to rewrite it as a $\mathbb{Z}/2\mathbb{Z}$-graded twisted G''_0-bundle and identify the new ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension. The ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension ν_0 in each entry arises from the extension

$$1 \longrightarrow \Pi \longrightarrow G_0 \longrightarrow G''_0 \longrightarrow 1, \tag{10.26}$$

as in Theorem 9.8(i), and since G_0 commutes with A^τ it simply adds to the twisting we now derive from the action of A^τ. The first two entries—the complex case—are straightforward: use S in the second entry to define the action of a complex Clifford algebra $\mathrm{Cl}^{\mathbb{C}}_{-1}$. The three columns with $T^2 = +1$ are also straightforward: set $\tilde{\sigma} = T$ and, if it is present, use C as a single Clifford generator. In the two columns in which C is present and T is absent, set $E = \mathcal{E} \oplus \sigma^*\overline{\mathcal{E}}$. There is an evident antilinear lift $\tilde{\sigma}$ of σ to E, and now we let the Clifford generators be $\tilde{\sigma}C, \tilde{\sigma}iC$. If $T^2 = -1$ and $\overline{C} \in A$ then we again set $E = \mathcal{E} \oplus \sigma^*\overline{\mathcal{E}}$ and use $\tilde{\sigma}C, \tilde{\sigma}iC, iTC$ as Clifford generators. In the remaining case in which $T^2 = -1$ and $\overline{C} \notin A$, set $E = (\mathcal{E} \oplus \Pi\mathcal{E}) \oplus (\sigma^*\overline{\mathcal{E}} \oplus \Pi\sigma^*\overline{\mathcal{E}})$, where Π is parity-reversal (see Lemma 7.2). The map $\tilde{\sigma}$ is the evident one, and the action of Cl_{-4} is modeled after (B.3):

$$\begin{gathered} e_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \\ e_3 = \begin{pmatrix} 0 & 0 & 0 & T \\ 0 & 0 & T & 0 \\ 0 & T & 0 & 0 \\ T & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & iT \\ 0 & 0 & iT & 0 \\ 0 & iT & 0 & 0 \\ iT & 0 & 0 & 0 \end{pmatrix} \end{gathered} \tag{10.27}$$

Conversely, for the inverse equivalence we begin with E and produce $\mathcal{E}$. In the complex case, which comprises the first two columns of the table, $\mathcal{E} = E$. For the columns with KR-theory in degrees $q = 0, \pm 1$ (recall we use the Morita equivalence which identifies degree -7 and degree $+1$), set $\mathcal{E} = E$ and let $T = \tilde{\sigma}$; if $q = \pm 1$, then set $C = e_1$. For degrees $q = \pm 2$, let $\mathcal{E}$ be the

$(+i)$-eigenspace of e_1e_2 and set $C = \tilde{\sigma}e_1$. For degrees $q = \pm 3$, let $\mathcal{E}$ be the $(+i)$-eigenspace of $\mp e_1e_2$ and set $C = \tilde{\sigma}e_1$ and $T = -\tilde{\sigma}e_2e_3$. For degree $q = -4$, let $\mathcal{E}$ be the simultaneous (-1)-eigenspace of $e_1e_2e_3e_4$ and $(+i)$-eigenspace of e_1e_2. Then $T = e_1e_3$ is even and squares to -1. $\square$

A subset of these arguments suffices for Type I. In that case, we work with the bundle $\mathcal{E}^- \to X_\Pi$.

Corollary 10.28. *If Hypothesis 10.24 holds, then we have the following table for the twisted equivariant K-theory group in Theorem 10.21 for band insulators of Type I:*

A	*1*	$\{\pm 1\} \times 1$	$\{\pm 1\} \times 1$
T^2	$+1$	$+1$	-1
$K^{\nu}_{G_0''}(X_\Pi)$	$K^{\nu_0}_{G_0''}(X_\Pi)$	$KR^{\nu_0}_{G_0''}(X_\Pi)$	$KR^{\nu_0-4}_{G_0''}(X_\Pi)$

11. Chern–Simons and Kane–Mele Invariants

To illustrate that the passage from $\mathcal{TP}$ to $\mathcal{RTP}$ retains information of physical interest, we consider two topological invariants with direct physical significance. These examples also illustrate our discussion in Sect. 10 in familiar examples from the contemporary condensed matter literature.

Both invariants involve the same basic data (Definition 10.7), which we now specify. The Galilean spacetime $\mathbb{M}^{d+1} = \mathbb{R} \times \mathbb{E}^d$ is the product of time and standard d-dimensional Euclidean space. We do not explicitly specify a crystal $C \subset \mathbb{M}^{d+1}$ as we only need a symmetry group G, which may be a subgroup of the complete symmetry group of a given crystal. The lattice subgroup of G is the standard $\Pi = \mathbb{Z}^d \subset \mathbb{R}^d$ acting by translations. These are systems of Type I, so $c \equiv 1$. We assume in addition either a time-reversal, parity-reversal symmetry, or both. In the time-reversal case,

$$G = \mathbb{Z}/2\mathbb{Z} \times \Pi \tag{11.1}$$

is a direct product with $\phi = t\colon \mathbb{Z}/2\mathbb{Z} \to \{\pm 1\}$ nontrivial on the generator $\overline{T}$ of $\mathbb{Z}/2\mathbb{Z}$. There is a nontrivial ϕ-twisted extension $G^\tau \to G$ given as the product of the nontrivial ϕ-twisted extension of $\mathbb{Z}/2\mathbb{Z}$ in Lemma 6.17(ii) with the group Π: if $T \in G^\tau$ projects to $\overline{T} \in \mathbb{Z}/2\mathbb{Z}$, then $T^2 = -\mathrm{id}$ (because we are describing spin 1/2 electrons). Let X_Π be the d-dimensional torus of characters $\lambda\colon \Pi \to \mathbb{T}$. Then $\overline{T}$ acts on X_Π by the involution $\sigma\colon \lambda \mapsto \bar{\lambda}$, according to the text preceding Theorem 9.42. In the parity-reversing case, the group G is the semidirect product

$$G = \mathbb{Z}/2\mathbb{Z} \ltimes \Pi \tag{11.2}$$

in which the generator $\overline{P} \in \mathbb{Z}/2\mathbb{Z}$ acts on $\Pi = \mathbb{Z}^d$ by $\xi \mapsto -\xi$. The homomorphism $\phi = t$ is trivial. The generator $\overline{P} \in \mathbb{Z}/2\mathbb{Z}$ acts on X_Π by the involution $\sigma\colon \lambda \mapsto \bar{\lambda}$. The Hilbert space is a space of vector-valued functions on $\mathbb{E}^d$, as in the introductory exposition in Sect. 10, and under Hypothesis 10.9 we obtain a finite rank complex vector bundle $\mathcal{E}^- \to X_\Pi$. The extra twist discussed in

Sect. 9 does not enter as in both (11.1) and (11.2) the projection $G \to \mathbb{Z}/2\mathbb{Z}$ is split; see Theorem 9.42(iii). So the nontrivial $\mathbb{Z}/2\mathbb{Z}$ action on X_Π lifts to a twisted action on $\mathcal{E}^-$. In the time-reversing case, it is a projective action (the square of the lift is $-\,\mathrm{id}$) and in addition is antilinear; in the parity-reversing case, it is an honest linear $\mathbb{Z}/2\mathbb{Z}$ action. If both a time-reversal and parity-reversal symmetry are present, then

$$G = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ltimes \Pi. \tag{11.3}$$

In this case, we have both a linear and antilinear lift of σ; their product is a quaternionic structure on $\mathcal{E}^-$. Let $G'' = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ be the extended point group in (11.1), (11.2), or (11.3).

We now go beyond Hypothesis 10.9 and use the fact that the Hilbert space consists of L^2 functions on $\mathbb{E}^d$ and that the bundle $\mathcal{E}^- \to X_\Pi$ is *smooth*; see the paragraph following Remark D.28. Then Proposition D.24 implies that there is a covariant derivative operator $\nabla^{\mathcal{E}^-}$ on $\mathcal{E}^- \to X_\Pi$, obtained by compression. It is commonly known as the *Berry connection*. Furthermore, it is invariant under the lifted, possibly twisted, action of G''. This follows from Remark D.28 and the fact that the Hamiltonian is assumed G-invariant, whence spectral projection $\mathcal{E} \to \mathcal{E}^-$ commutes with G. Contrary to the impression one gets from the condensed matter literature, we do not need the specific Berry covariant derivative $\nabla^{\mathcal{E}^-}$ to construct the two topological invariants discussed in this section.

Lemma 11.4. *The set of G''-invariant covariant derivatives on $\mathcal{E}^- \to X_\Pi$ is a nonempty affine space.*

Proof. The set of all covariant derivatives on $\mathcal{E}^- \to X_\Pi$ is an affine space over $\Omega^1(X_\Pi; \operatorname{End}\mathcal{E}^-)$. Fix a covariant derivative ∇ and average over G'' to obtain a G''-invariant covariant derivative. The difference of any two is a 1-form invariant under G'', so an element of the vector space $\Omega^1(X_\Pi; \operatorname{End}\mathcal{E}^-)^{G''}$. □

Note that the G''-action may be twisted—it acts via the ϕ-twisted extension $(G'')^\tau$ of G''—and if there are time-reversal symmetries ($\phi \not\equiv 1$) then there are antilinear transformations $\varphi\colon \mathcal{E}^- \to \mathcal{E}^-$. The covariant derivative ∇ is φ-invariant if $\varphi^*\nabla = \overline{\nabla}$, where $\overline{\nabla}$ is the covariant derivative on the complex conjugate bundle. In this case, $\Omega^1(X_\Pi; \operatorname{End}\mathcal{E}^-)^{G''}$ is a *real* vector space.

Corollary 10.28 gives a simple expression for the twisted equivariant K-theory group. For the time-reversing case (11.1), we obtain the KR-group

$$\mathcal{RTP}_I(G,\phi,\tau) \cong KR^{-4}(X_\Pi). \tag{11.5}$$

For the parity-reversing case (11.2), we have the untwisted equivariant K-theory group

$$\mathcal{RTP}_I(G,\phi,\tau) \cong K^0_{\mathbb{Z}/2\mathbb{Z}}(X_\Pi). \tag{11.6}$$

If both symmetries are present (11.3) we have

$$\mathcal{RTP}_I(G,\phi,\tau) \cong KR^{-4}_{\mathbb{Z}/2\mathbb{Z}}(X_\Pi) \cong KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(X_\Pi). \tag{11.7}$$

Recall that in this case elements are represented by complex bundles $\mathcal{E}^- \to X_\Pi$ with two commuting lifts of the involution σ, one linear and one antilinear, which accounts for the KR-theory description. The product of the lifts is a quaternionic structure on $\mathcal{E}^-$. In the KO-theory description of the group, we use this quaternionic structure and the commuting linear lift of σ.

11.1. Computations

In this subsection, we report on the computation of some relevant K-theory groups. Some of these were done in collaboration with Aaron Royer, and we hope to write a separate paper in the near future in which we make more computations and also provide details for the ones quoted here.

Consider the d-dimensional torus $(S^1)^{\times d} = S^1 \times \cdots \times S^1$ as an equivariant space for the group $(\mathbb{Z}/2\mathbb{Z})^{\times d} = \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$, where each $\mathbb{Z}/2\mathbb{Z}$ factor acts on the corresponding S^1 factor by reflection. The diagonal $\mathbb{Z}/2\mathbb{Z} \subset (\mathbb{Z}/2\mathbb{Z})^{\times d}$ is the involution $\sigma\colon \lambda \mapsto \bar{\lambda}$ which appears in the previous subsection. Mike Freedman brought the following theorem to our attention.

Theorem 11.8. *The torus $(S^1)^{\times d}$ is equivariantly stably homotopy equivalent to a wedge of spheres.*

In the cases $d = 2, 3$ we have

$$S^1 \times S^1 \underset{\text{stable}}{\sim} S^1 \vee S^1 \vee S^2 \tag{11.9}$$

$$S^1 \times S^1 \times S^1 \underset{\text{stable}}{\sim} S^1 \vee S^1 \vee S^1 \vee S^2 \vee S^2 \vee S^2 \vee S^3. \tag{11.10}$$

In the first splitting, the symmetry group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on the first S^1 summand via reflection after projection to the first $\mathbb{Z}/2\mathbb{Z}$ factor, on the second S^1 summand via reflection after projection to the second $\mathbb{Z}/2\mathbb{Z}$ factor, and on the $S^2 = I \times I \;/\; \partial(I \times I)$ summand by the reflection action on each $I = [-1/2, 1/2]$ factor. The actions in higher dimensions are similar, and easiest to see by writing $S^k = I^{\times k} \;/\; \partial(I^{\times k})$. The stable splitting in Theorem 11.8 is clearly also equivariant for any subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\times d}$.

This stable splitting is the key step in the computations we report here. An alternative derivation, which provides an independent check, is via the Kunneth spectral sequence for $RO(G)$-graded equivariant cohomology theories. We also use the twisted Thom isomorphism theorem, which we illustrate with a few specific cases. First, for any compact Lie group G and G-space X we have

$$\begin{aligned} KO^q_{G\times\mathbb{Z}/2\mathbb{Z}}(X \times \mathbb{R})_{\text{cv}} &\cong KO^q_G(X) \\ KO^q_{G\times\mathbb{Z}/2\mathbb{Z}}(X \times \mathbb{R}^2)_{\text{cv}} &\cong K^{q-2}_G(X) \\ KO^q_{G\times\mathbb{Z}/2\mathbb{Z}}(X \times \mathbb{R}^3)_{\text{cv}} &\cong KO^{q-4}_G(X) \end{aligned} \tag{11.11}$$

where $q \in \mathbb{Z}$, the group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^k$ by $\xi \mapsto -\xi$, and 'cv' denotes compact support on $\{x\} \times \mathbb{R}^k$ for all $x \in X$. There may be additional support conditions in the X-direction on both sides of (11.11). The second twisted Thom isomorphism is, for any space X,

$$KR^q(X \times \mathbb{R}^k)_{\text{cv}} \cong KR^{q+k}(X) \cong KO^{q+k}(X). \tag{11.12}$$

Here the involution which defines KR acts trivially on X and by $\xi \mapsto -\xi$ on $\mathbb{R}^k$. The compact support is in the $\mathbb{R}^k$-direction.

Recall that for any *equivariant* cohomology theory h, the *reduced cohomology* $\widetilde{h}(X)$ of a pointed space (X, x_0) is the kernel of the restriction map $h(X) \to h(\{x_0\})$ to the basepoint. In the equivariant case, the basepoint must be a fixed point of the group action.

A typical picture of the torus $(S^1)^{\times d}$ is as the quotient of $I^{\times d}$ by identifying opposite faces of the boundary, and so there is a collapse map

$$q\colon (S^1)^{\times d} \longrightarrow S^d \tag{11.13}$$

in which every point of the boundary $\partial(I^{\times d})$ maps to the basepoint of S^d. Stably, q corresponds to collapse of all spheres except S^d in the stable splitting of Theorem 11.8.

Theorem 11.14. (i) $KR^{-4}(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(ii) $KR^{-4}(S^1 \times S^1 \times S^1) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\times 4}$.

(iii) *The image of* $\widetilde{KR}^{-4}(S^3) \xrightarrow{q^*} \widetilde{KR}^{-4}(S^1 \times S^1 \times S^1)$ *is cyclic of order two.*

(iv) *The image of* $\widetilde{KO}^{-4}_{\mathbb{Z}/2\mathbb{Z}}(S^3) \xrightarrow{q^*} \widetilde{KO}^{-4}_{\mathbb{Z}/2\mathbb{Z}}(S^1 \times S^1 \times S^1)$ *is infinite cyclic.*

(v) *The natural map* $\widetilde{KO}^{-4}_{\mathbb{Z}/2\mathbb{Z}}(S^3) \to \widetilde{KR}^{-4}(S^3)$ *is reduction modulo two.*

The involutions in (i)–(v) are all simultaneous reflection on each S^1 factor of the torus; on $S^3 \subset \mathbb{E}^4$ centered at the origin, the involution is the composition of three orthogonal reflections in hyperplanes through the origin (with two fixed points). The latter may also be described as the one-point compactification of the three-dimensional sign representation of $\mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/2\mathbb{Z}$ factor in (i) is due to the S^2 in the stable splitting (11.9); the four $\mathbb{Z}/2\mathbb{Z}$ factors in (ii) are due to the S^2's and S^3 in the stable splitting (11.10). We will see in the next subsections that the orbital magnetoelectric polarizability and Kane–Mele invariant are detected on the subgroup of $KR^{-4}\big((S^1)^{\times 3}\big)$ identified in (iii). This is the so-called "strong" subgroup; the three $\mathbb{Z}/2\mathbb{Z}$ factors due to the S^2 summands in (11.10) are the "weak" subgroups. Assertions (iv) and (v) imply that in systems with parity-reversal and time-reversal, as in (11.3), the aforementioned $\mathbb{Z}/2\mathbb{Z}$-invariants have lifts to integer invariants.

Finally, we note that there are three projections

$$S^1 \times S^1 \times S^1 \longrightarrow S^1 \times S^1, \tag{11.15}$$

and it follows from the computations that these induce injections $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\times 4}$ on KR^{-4}. The three "weak" $\mathbb{Z}/2\mathbb{Z}$'s are in the image.

11.2. Orbital Magnetoelectric Polarizability

In this subsection $d = 3$. We use the notation established in the first subsection. Fix an orientation of X_Π; the choice will be irrelevant to the topological invariant (11.18) defined below. A finite rank complex vector bundle $\mathcal{E}^- \to X_\Pi$ with covariant derivative $\nabla^{\mathcal{E}^-}$ has a *Chern–Simons invariant* [24]

$$\theta(\nabla^{\mathcal{E}^-}) \in \mathbb{R}/2\pi\mathbb{Z} \tag{11.16}$$

In [18,19,47], a topological contribution to the orbital magnetoelectric polarizability is shown to be the Chern–Simons invariant of the Berry connection. It is a topological invariant in the presence of a time-reversal or parity-reversal symmetry.

Lemma 11.17. *Let* ∇ *be a* G''*-invariant covariant derivative on* $\mathcal{E}^- \to X_\Pi$*. Then either* $\theta(\nabla^{\mathcal{E}^-}) = 0$ *or* $\theta(\nabla^{\mathcal{E}^-}) = \pi$*. Furthermore,* $\theta(\nabla)$ *is independent of* ∇ *and the orientation of* X_Π*, so is a topological invariant* $\theta(\mathcal{E}^-)$ *of the bundle* $\mathcal{E}^- \to X_\Pi$*.*

Proof. In both the parity-reversing and time-reversing cases, the action of the generator of $\mathbb{Z}/2\mathbb{Z}$ reverses the orientation of X_Π. In the parity-reversing case, the lifted action preserves ∇, and since the Chern–Simons invariant changes sign when the orientation is reversed we have $\theta(\nabla) = -\theta(\nabla)$. The same argument works in the time-reversal case, but as the lifted action on $\mathcal{E}^- \to X_\Pi$ is antilinear, we must also observe that the Chern–Simons invariant of the complex conjugate bundle equals that of the original bundle.[28] The second statement is a consequence of Lemma 11.4: the continuous discrete function θ on the connected space of invariant connections is constant. □

It now follows from Theorem 10.18 that θ is an invariant of the topological phase of the system:

$$\theta\colon \mathcal{TP}_I(G,\phi,\tau) \longrightarrow \{0,\pi\}. \tag{11.18}$$

Proposition 11.19. *θ factors through a homomorphism $\bar\theta$ of abelian groups:*

$$\begin{array}{ccc} \mathcal{TP}_I(G,\phi,\tau) & \xrightarrow{\quad\theta\quad} & \{0,\pi\} \\ & \searrow \qquad\qquad \nearrow_{\bar\theta} & \\ & \mathcal{RTP}_I(G,\phi,\tau) & \end{array} \tag{11.20}$$

Proof. This follows directly from Definition 10.20 and the universal property (5.5) of the group completion, once we prove that θ is a homomorphism. But this is immediate: if $E = E_1 \oplus E_2$ is a direct sum, then we use a direct sum $\nabla_1 \oplus \nabla_2$ of invariant covariant derivatives to compute the Chern–Simons invariant, which satisfies $\theta(\nabla_1 \oplus \nabla_2) = \theta(\nabla_1) + \theta(\nabla_2)$. □

In case the system admits a parity-reversing symmetry, so the symmetry group is (11.3), there is an easy formula for $\theta(\mathcal{E}^-)$. As observed earlier, in this case $\mathcal{E}^-$ has a quaternionic structure. Let $P \to X_\Pi$ be the associated principal Sp_N-bundle, where the complex rank of $\mathcal{E}^- \to X_\Pi$ is $2N$, and endow it with a connection invariant under time-reversal (Lemma 11.4). Now any Sp_N-bundle over a 3-manifold is trivializable, so let $s\colon X_\Pi \to P$ be a section. Define $f\colon X_\Pi \to \mathrm{Sp}_N$ by $\hat\sigma^* s = s \cdot f$, where $\hat\sigma\colon P \to P$ is the lift of σ defined by the action of time-reversal on $\mathcal{E}^-$.

[28] This follows since $c_2(\overline{\mathcal{E}^-}) = c_2(\mathcal{E}^-)$ and this Chern–Simons invariant is a secondary invariant of the second Chern class c_2.

Proposition 11.21. $\theta(\mathcal{E}^-) = (\deg f)\pi \pmod{2\pi}$.

Here $\deg f$ is the induced map $f_*\colon H_3(X_\Pi) \to H_3(\mathrm{Sp}_N)$; observe that $H_3(X_\Pi) \cong H_3(\mathrm{Sp}_N) \cong \mathbb{Z}$.

Proof. Fix an orientation of X_Π and a generator of $H^3(\mathrm{Sp}_N;\mathbb{Z})$. Let $\alpha \in \Omega^3(P)$ be the Chern–Simons form of the connection on P. We apply [24, (1.28)] to the map $\hat{\sigma}$ to conclude that

$$\sigma^* s^* \alpha = s^*\alpha + \omega + d\eta, \tag{11.22}$$

where $\omega \in \Omega^3(X_\Pi)$ integrates to the degree of f and $\eta \in \Omega^2(X_\Pi)$. Integrate (11.22) over X_Π and use Stokes' theorem and the fact that σ is orientation-reversing. Since $\theta(\mathcal{E}^-) = 2\pi \int_{X_\Pi} s^*\alpha$ is the Chern–Simons invariant, the proposition follows. □

The group $\mathcal{RTP}_I(G,\phi,\tau)$ is expressed as a K-theory group in (11.5)–(11.7), and the appropriate K-theory group is computed in Theorem 11.14. Consider the time-reversal case (11.1) for which the K-theory group is computed in (11.5) and Theorem 11.14(ii) to be

$$\mathcal{RTP}_I(G,\phi,\tau) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\times 4}. \tag{11.23}$$

Proposition 11.24. *The invariant $\bar{\theta}$ vanishes on the three "weak" $\mathbb{Z}/2\mathbb{Z}$'s in the image of the projections* (11.15) *and is the identity map on the strong $\mathbb{Z}/2\mathbb{Z}$ which is the image of the collapse map* (11.13).

Proof. If the bundle $\mathcal{E}^- \to S^1 \times S^1 \times S^1$ is pulled back from a bundle over $S^1 \times S^1$, then we can choose a connection which is also pulled back. It follows easily that the Chern–Simons invariant vanishes (essentially because $S^1 \times S^1$ is 2-dimensional and it is computed as the integral of a 3-form).

Let $S^3 = \mathbb{R}^3 \cup \{\infty\}$ denote the 3-dimensional sphere with the involution $x \in \mathbb{R}$ maps to $-x$. Fix an equivariant map $q\colon X_\Pi \cong S^1 \times S^1 \times S^1 \to S^3$ as in (11.13). Let $U \subset \mathbb{H}$ denote the unit quaternions; U is diffeomorphic to a 3-sphere. There exists a continuous function $f\colon S^3 \to U$ such that

$$f(\infty) = 1, \quad f(0) = -1, \quad f(-x) = \overline{f(x)} \quad \text{for all } x \in \mathbb{R}^3 \cup \{\infty\}, \tag{11.25}$$

where the bar denotes conjugation in the quaternions: namely, f is the identity map $S^3 \to S^3$, after identifying $U \approx S^3$ and lining up the involutions. Now let $E \to S^3$ be the trivial bundle with fiber $\mathbb{H}$, and let $\mathbb{H}$ act by right quaternion multiplication on each fiber. This is the quaternionic structure on E. Define ϵ as the trivial lift of the involution $x \mapsto -x$ composed with left multiplication by f. Right multiplication by $j \in \mathbb{H}$ and ϵ generate the action of $G'' = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in (11.3). It follows that the pullback $q^*E \to X_\Pi$ has both parity-reversing and time-reversing symmetry. Now since the map q is a diffeomorphism off of a set of measure zero, we can compute the Chern–Simons invariant on S^3. Apply[29] Proposition 11.21. Since E is trivial, the associated

[29] Although Proposition 11.21 is stated and proved for the particular 3-manifold X_Π, the argument applies to any orientable 3-manifold with involution.

principal Sp_1-bundle has a canonical trivializing section s, and by definition $\hat{\sigma}^* s = s \cdot f$ for the function f in (11.25). Since f has degree 1, it follows that $\theta(q^*E)$ is nonzero. □

11.3. The Kane–Mele Invariant

This invariant was introduced in [40] and further studied in many papers, including [30]. There are many equivalent definitions.

We begin with some preliminaries on quaternionic vector spaces. Let W be a complex vector space and $J\colon W \to W$ a complex antilinear operator with $J^2 = \pm\,\mathrm{id}_W$. Then J is either a *real* structure $(+)$ or a *quaternionic* structure $(-)$. If $(W_i, J_i), i = 1, \dots, N$, is a finite set of real or quaternionic vector spaces, then the tensor product $W_1 \otimes \cdots \otimes W_N$ carries an antilinear operator $J_1 \otimes \cdots \otimes J_N$ which is a real or quaternionic structure according to the parity of the number of quaternionic structures in $\{J_i\}$. This applies to symmetric and antisymmetric tensor products as well. All tensor products are taken over $\mathbb{C}$. If (W, J) is quaternionic and finite dimensional, then $\dim_{\mathbb{C}} W = 2\ell$ is even, and the *determinant line* $\mathrm{Det}\, W := \bigwedge^{2\ell} W$ inherits a *real* structure $\det J := \bigwedge^{2\ell} J$. Thus, $\mathrm{Det}\, W$ is a complex line[30] with a distinguished real line $\mathrm{Det}_J\, W$ contained as a real subspace. We claim that the real line $\mathrm{Det}_J\, W$ has a canonical *orientation*—a choice of component of $\mathrm{Det}_J\, W \setminus \{0\}$. For if $e_1, \dots, e_\ell$ is a quaternionic basis of (W, J), then

$$(e_1 \wedge Je_1) \wedge (e_2 \wedge Je_2) \wedge \cdots \wedge (e_\ell \wedge Je_\ell) \in \mathrm{Det}_J\, W \subset \mathrm{Det}\, W \qquad (11.26)$$

is nonzero, so orients $\mathrm{Det}_J\, W$. The collection of quaternionic bases is connected, and (11.26) varies continuously, so the orientation is well-defined, independent of the basis. Alternatively, if we endow W with a hermitian metric $\langle -, - \rangle$ with respect to which J is antiunitary, then J determines a 2-form $\omega_J \in \bigwedge^2 W^*$ defined by

$$\omega_J(\xi_1, \xi_2) = \langle J\xi_1, \xi_2 \rangle, \quad \xi_1, \xi_2 \in W. \qquad (11.27)$$

The *pfaffian*

$$\mathrm{pfaff}\, \omega_J = \frac{\omega_J^\ell}{\ell!} = \frac{\omega_J \wedge \cdots \wedge \omega_J}{\ell!} \qquad (11.28)$$

is a nonzero element of $\mathrm{Det}_J\, W^*$. Since the space of hermitian metrics is connected, the induced orientation is independent of $\langle -, - \rangle$.

Assume given the band structure at the beginning of Sect. 10 for the time-reversal case (11.1). The complex finite rank vector bundle $\mathcal{E}^- \to X_\Pi$ has an antilinear lift $\tilde{\sigma}$ of the involution $\sigma\colon X_\Pi \to X_\Pi$ which complex conjugates characters. The involution σ has 2^d fixed points. Let $F \subset X_\Pi$ denote the fixed point set. At each fixed point $\lambda \in F$ the lift $\tilde{\sigma}$ of σ to the fiber $\mathcal{E}^-_\lambda$ is a quaternionic structure. Let $\mathrm{Det}\, \mathcal{E}^- \to X_\Pi$ be the (complex) determinant line bundle, obtained by replacing each fiber of $\mathcal{E}^-$ by its determinant line. Then $\tilde{\sigma}$ induces a lift $\det \tilde{\sigma}$ of σ to $\mathrm{Det}\, \mathcal{E}^-$, and $(\det \tilde{\sigma})^2 = \mathrm{id}_{\mathrm{Det}\, \mathcal{E}^-}$. At a fixed

[30] A *line* is a 1-dimensional vector space.

point $\lambda \in F$ the antilinear map $\det\tilde{\sigma}_\lambda$ is the real structure discussed in the previous paragraph—its fixed points comprise the real line $\mathrm{Det}_{\tilde{\sigma}}\,\mathcal{E}^-_\lambda \subset \mathrm{Det}\,\mathcal{E}^-_\lambda$.

Lemma 11.29. *Suppose $L \to X_\Pi$ is a complex line bundle with an antilinear lift α of σ whose square is id_L. Then $L \to X_\Pi$ admits a nowhere vanishing α-invariant section.*

Proof. We first prove that there is a nowhere vanishing section, equivalently that $L \to X_\Pi$ is nonequivariantly trivializable, equivalently that $c_1(L) \in H^2(X_\Pi;\mathbb{Z})$ vanishes. Now any class in $H^2(X_\Pi;\mathbb{Z})$ is determined by its value on the fundamental classes $[T]$ of 2-dimensional subtori $T \subset X_\Pi$. Since $L \cong \sigma^*\overline{L}$, we have

$$\langle c_1(L),[T]\rangle = \langle c_1(\sigma^*\overline{L}),[T]\rangle = \langle c_1(\overline{L}),\sigma_*[T]\rangle = -\langle c_1(L),[T]\rangle, \quad (11.30)$$

and so the pairing $\langle c_1(L),[T]\rangle \in \mathbb{Z}$ vanishes.

Let $s'\colon X_\Pi \to L$ be a nowhere vanishing section of $L \to X_\Pi$. Define $f\colon X_\Pi \to \mathbb{C}^\times$ by $\alpha^*s' = fs'$, where $\mathbb{C}^\times \subset \mathbb{C}$ is the multiplicative group of nonzero complex numbers; then $\sigma^*\bar{f}\cdot f = 1$. It follows that the winding number of f around any 1-dimensional subtorus $S \subset X_\Pi$ vanishes, whence f has a square root—a function $g\colon X_\Pi \to \mathbb{C}^\times$ such that $g^2 = f$. Then $(\sigma^*\bar{g}\cdot g)^2 = 1$, and since at a fixed point $\lambda \in F$ we have $|f(\lambda)| = 1$, it follows that $|g(\lambda)| = 1$ and so $\sigma^*\bar{g}\cdot g = 1$. The nowhere vanishing section $s = gs'$ is invariant: $\alpha^*s = s$. □

Apply Lemma 11.29 to $\mathrm{Det}\,\mathcal{E}^- \to X_\Pi$ to conclude that there exists a nowhere vanishing $\det\tilde{\sigma}$-invariant section s. At a fixed point $\lambda \in F$, we have $s(\lambda) \in \mathrm{Det}_{\tilde{\sigma}}\,\mathcal{E}^-_\lambda$, where recall $\mathrm{Det}_{\tilde{\sigma}}\,\mathcal{E}^-_\lambda \subset \mathrm{Det}\,\mathcal{E}^-_\lambda$ is a distinguished oriented real line. Define $\delta_\lambda(s) = \pm 1$ according as $s(\lambda)$ is in $(+)$ or not in $(-)$ the half-line of $\mathrm{Det}_{\tilde{\sigma}}\,\mathcal{E}^-_\lambda$ defined by the orientation.

Lemma 11.31. *$\prod_{\lambda\in F}\delta_\lambda(s)$ is independent of s if $d = \dim X_\Pi$ satisfies $d \geq 2$.*

Proof. Any other nowhere vanishing $\det\tilde{\sigma}$-invariant section has the form hs for $h\colon X_\Pi \to \mathbb{C}^\times$ with $\sigma^*\bar{h} = h$. Write $X_\Pi = S' \times S''$ for $S', S'' \subset X_\Pi$ subtori of dimensions $1, d-1$, respectively. Then $F = F' \times F''$ is the Cartesian product of the fixed point sets of σ restricted to S', S''. Let $m \in \mathbb{Z}$ be the winding number of $h\big|_{S'}$. Now at each fixed point h is a nonzero real number, so $\operatorname{sign} h = \pm 1$ is well-defined. If $F' = \{1,\lambda'\}$ and $\lambda'' \in F''$ it follows that $\operatorname{sign} h(1,\lambda'')\cdot\operatorname{sign} h(\lambda',\lambda'') = (-1)^m$. Since $d \geq 2$—equivalently, $F'' \neq \emptyset$—we conclude $\prod_{\lambda\in F}\operatorname{sign} h(\lambda) = 1$, as desired. □

Definition 11.32. The *Kane–Mele invariant* of $\mathcal{E}^- \to X_\Pi$ is

$$\kappa(\mathcal{E}^-) = \prod_{\lambda\in F}\delta_\lambda(s) \;\in \{\pm 1\}, \quad (11.33)$$

where s is any nowhere vanishing $\det\tilde{\sigma}$-invariant section of $\mathrm{Det}\,\mathcal{E}^- \to X_\Pi$.

Analogous to Proposition 11.19, and with almost the same proof, we have the following.

Proposition 11.34. *κ factors through a homomorphism $\bar{\kappa}$ of abelian groups;*

$$\begin{array}{ccc} \mathcal{TP}_I(G,\phi,\tau) & \xrightarrow{\quad\kappa\quad} & \{\pm 1\} \\ & \searrow \qquad \nearrow_{\bar{\kappa}} & \\ & \mathcal{RTP}_I(G,\phi,\tau) & \end{array} \tag{11.35}$$

Proof. If $E = E_1 \oplus E_2$ is a direct sum, then $\operatorname{Det} E \cong \operatorname{Det} E_1 \otimes \operatorname{Det} E_2$ and we use the tensor product of nowhere vanishing invariant sections to verify that $\kappa(E) = \kappa(E_1)\kappa(E_2)$ is a homomorphism. □

Remark 11.36. Recall from (11.5) that $\mathcal{RTP}(G,\phi,\tau) \cong KR^{-4}(X_\Pi)$. We observe that there is a Kane–Mele invariant defined on each subtorus $T \subset X_\Pi$ of dimension at least two.

In case the system admits a parity-reversing symmetry, so the symmetry group is (11.3), then there is a well-known easy formula for $\kappa(\mathcal{E}^-)$. The parity-reversing symmetry also acts on X_Π by the involution σ, and its lift ϵ to $\mathcal{E}^- \to X_\Pi$ is complex linear, squares to $\mathrm{id}_{\mathcal{E}^-}$, and commutes with $\tilde{\sigma}$. The composition $j = \epsilon \circ \tilde{\sigma}$ then covers id_{X_Π}, is complex antilinear, and $j^2 = -\mathrm{id}_{\mathcal{E}^-}$. Thus, j is a *global* quaternionic structure on $\mathcal{E}^- \to X_\Pi$. Now at a fixed point $\lambda \in F$ there are two quaternionic structures—$\tilde{\sigma}_\lambda$ and j_λ—and $\tilde{\sigma}_\lambda = \epsilon_\lambda \circ j_\lambda$, where $\epsilon_\lambda \in \operatorname{End} \mathcal{E}^-_\lambda$ squares to $\mathrm{id}_{\mathcal{E}^-_\lambda}$ and commutes with j_λ. Therefore, there is a quaternionic basis $e_1, \dots, e_\ell$ of $(\mathcal{E}^-_\lambda, j_\lambda)$ such that $\epsilon_\lambda(e_i) = \pm e_i$ for each i. Let $\Delta_\lambda = \pm 1$ be the product over the basis of the signs.

Proposition 11.37. $\kappa(\mathcal{E}^-) = \prod_{\lambda \in F} \Delta_\lambda$.

Proof. Fix a hermitian metric on $\mathcal{E}^- \to X_\Pi$ which is invariant under j, ϵ, and hence also $\tilde{\sigma}$. Then as in (11.28) the pfaffian $s = \operatorname{pfaff} \omega_j$ is a nowhere vanishing $\det j$-invariant section of $\mathcal{E}^- \to X_\Pi$. To compute κ we compare the canonical orientations of $\operatorname{Det}_j \mathcal{E}^- = \operatorname{Det}_{\tilde{\sigma}} \mathcal{E}^-$ at each $\lambda \in F$. That the ratio is Δ_λ follows from the formula

$$(e_1 \wedge \tilde{\sigma}_\lambda e_1) \wedge \cdots \wedge (e_\ell \wedge \tilde{\sigma}_\lambda e_\ell) = \Delta_\lambda\, (e_1 \wedge j e_1) \wedge \cdots \wedge (e_\ell \wedge j e_\ell), \tag{11.38}$$

where as above $e_1, \dots, e_\ell$ is a quaternionic basis of $(\mathcal{E}^-_\lambda, j_\lambda)$. □

Proposition 11.39. *The invariant $\bar{\kappa}$ in Proposition 11.34 vanishes on the three "weak" $\mathbb{Z}/2\mathbb{Z}$'s in the image of the projections (11.15) and is the identity map on the strong $\mathbb{Z}/2\mathbb{Z}$ which is the image of the collapse map (11.13).*

Proof. If the bundle $\mathcal{E}^- \to S^1 \times S^1 \times S^1$ is pulled back from a bundle over $S^1 \times S^1$, then we can choose the section s to also be a pullback. Since pairs of fixed points in the 3-torus are mapped to a single fixed point of the 2-torus, we see that each sign in (11.33) occurs with even multiplicity, whence $\kappa(\mathcal{E}^-) = +1$.

Now consider the bundle pulled back from the trivial bundle $E \to S^3$ via $q\colon S^1 \times S^1 \times S^1 \to S^3$, as in the paragraph containing (11.25). We describe q

more explicitly. Choose an isomorphism $X_\Pi \cong \mathbb{R}^3/\mathbb{Z}^3$ and map the fundamental domain

$$R = \{x = (x^1, \dots, x^3) \in \mathbb{R}^3 : -1/2 \leq x^i \leq 1/2\} \tag{11.40}$$

to $\mathbb{R}^3 \cup \{\infty\}$ by $\pi\colon x \mapsto x/\epsilon(x)$, where $\epsilon(x)$ is the Euclidean distance from x to ∂R. Then q maps $2^3 - 1$ points of $F \subset X_\Pi \approx S^1 \times S^1 \times S^1$ to $\infty \in S^3$ and the remaining fixed point to $0 \in S^3$. Since $q^*E \to X_\Pi$ has both time-reversal and parity-reversal symmetry, we can apply Proposition 11.37. It is easy to see $\Delta_\lambda = +1$ at $2^3 - 1$ points of F and $\Delta_\lambda = -1$ at the remaining fixed point, whence $\kappa(q^*E) = -1$, as desired. □

Proposition 11.24 and Proposition 11.39 together with the computation Theorem 11.14(ii) and the remark around (11.15) immediately imply the following result of Wang-Qi-Zhang.

Corollary 11.41 ([67]). *The orbital magnetoelectric polarizability and 3-dimensional Kane–Mele invariant are equal: $\theta = \kappa$.*

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Appendix A. Group Extensions

Group extensions appear throughout the main body of this paper, and for the reader's convenience we summarize some basic definitions in this appendix. We do not comment further on topology, but say once and for all that for topological groups all group homomorphisms are assumed continuous, and for Lie groups all group homomorphisms are assumed smooth.

Definition A.1. A *group extension* is a sequence of group homomorphisms

$$1 \longrightarrow G' \xrightarrow{\;i\;} G \xrightarrow{\;\pi\;} G'' \longrightarrow 1 \tag{A.2}$$

such that the composition of any two consecutive arrows is the identity and, more strongly, the kernel of any homomorphism equals the image of the preceding homomorphism. We call G' the *kernel* and G'' the *quotient*.

The equality of kernel and image is called *exactness*, and it has consequences at the three "interior nodes" of (A.2). At G' it implies that i is injective (1:1). We use the inclusion i to identify G' with its image, which is a subgroup of G. At G the exactness implies that π factors through an injective map of the quotient group G/G' into G''. At G'' the exactness implies that π is surjective (onto), hence the quotient map $G/G' \to G''$ an isomorphism. We use it to identify G'' as this quotient.

As with any algebraic structure, there is a notion of homomorphism.

Definition A.3. Let G', G'' be groups and G^{τ_1}, G^{τ_2} group extensions with kernel G' and quotient G''. A *homomorphism of group extensions* G^{τ_1}, G^{τ_2} is a group homomorphism $\varphi\colon G^{\tau_1} \to G^{\tau_2}$ which fits into the commutative diagram

$$\begin{array}{ccccccc} & & & G^{\tau_1} & & & \\ & & \nearrow & & \searrow & & \\ 1 \longrightarrow & G' & & \downarrow \varphi & & G'' & \longrightarrow 1 \\ & & \searrow & & \nearrow & & \\ & & & G^{\tau_2} & & & \end{array} \tag{A.4}$$

As usual, φ is an *isomorphism* if there exists a homomorphism $\psi\colon G^{\tau_2} \to G^{\tau_1}$ of group extensions so that the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps, which is simply equivalent to the condition that φ be an isomorphism of groups..

There is a notion of a trivialization of a group extension.

Definition A.5. A *splitting* of the group extension (A.2) is a homomorphism $j\colon G'' \to G$ such that $\pi \circ j = \mathrm{id}_{G''}$.

Not every group extension is split: for example, the cyclic group of order 4 is a nonsplit extension of the cyclic group of order 2 by the cyclic group of order 2:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{A.6}$$

For example, the group extension (1.3) is split whereas (1.4) is not split. A splitting $j\colon G'' \to G$ of (A.2) determines a homomorphism

$$\alpha\colon G'' \longrightarrow \mathrm{Aut}(G'), \tag{A.7}$$

which we also call an action of G'' on G'. Namely, the action of $g'' \in G''$ is

$$\alpha(g'')(g') = gg'g^{-1}, \quad g' \in G', \tag{A.8}$$

where $g = j(g'') \in G$. (We remark that if G' is *abelian*, then (A.7) is defined without a splitting: let g be any element of G such that $\pi(g) = g''$.) There is a converse construction.

Definition A.9. Let G', G'' be groups and $\alpha\colon G'' \to \mathrm{Aut}(G')$ a homomorphism. The *semidirect product* $G = G'' \ltimes_\alpha G'$ is defined to be the Cartesian product set $G'' \times G'$ with multiplication defined so that $G'' \times 1$ and $1 \times G'$ are subgroups, and

$$g'' \cdot g' = \alpha(g'')(g') \cdot g'', \quad g'' \in G'', \quad g' \in G'. \tag{A.10}$$

A semidirect product sits in a split extension

$$1 \longrightarrow G' \xrightarrow{\ i\ } G \underset{\pi}{\overset{j}{\rightleftarrows}} G'' \longrightarrow 1 \tag{A.11}$$

and conversely every split extension is isomorphic to a semidirect product. We say that a group extension is *trivializable* if it is isomorphic to a split extension; a choice of splitting is a trivialization.

Group extensions can be pulled back and, in very special situations, pushed out.

Definition A.12. Suppose (A.2) is a group extension and $\rho'' \colon \widetilde{G}'' \to G''$ a group homomorphism. Then there is a *pullback* group extension with kernel G' and quotient $\widetilde{G}''$ which fits into the commutative diagram

$$\begin{CD} 1 @>>> G' @>>> \widetilde{G} @>\tilde{\pi}>> \widetilde{G}'' @>>> 1 \\ @. @| @VV\rho V @VV\rho'' V @. \\ 1 @>>> G' @>>> G @>\pi>> G'' @>>> 1 \end{CD} \tag{A.13}$$

It is defined by setting

$$\widetilde{G} = \bigl\{ (g, \tilde{g}'') \in G \times \widetilde{G}'' : \pi(g) = \rho''(\tilde{g}'') \bigr\}. \tag{A.14}$$

Note that $\widetilde{G}$ is a subgroup of the direct product group $G \times \widetilde{G}''$. The homomorphisms ρ and $\tilde{\pi}$ are defined by restricting the projections onto the factors of $G \times \widetilde{G}''$, respectively.

Definition A.15. Suppose (A.2) is a group extension and $\rho' \colon G' \to \widetilde{G}'$ a group homomorphism with $\widetilde{G}'$ abelian. Assume further that the action of G'' on G' fixes ρ'. Then there is an *associated group extension* with kernel $\widetilde{G}'$ and quotient G'' which fits into the commutative diagram

$$\begin{CD} 1 @>>> G' @>i>> G @>>> G'' @>>> 1 \\ @. @VV\rho' V @VV\rho V @| @. \\ 1 @>>> \widetilde{G}' @>\tilde{i}>> \widetilde{G} @>>> G'' @>>> 1 \end{CD} \tag{A.16}$$

It is defined by setting

$$\widetilde{G} = G \times \widetilde{G}' \,/\!\sim, \quad (g, \tilde{g}') \sim \bigl(i(g')g, \rho'(g')\tilde{g}' \bigr) \tag{A.17}$$

for all $g \in G, \tilde{g}' \in \widetilde{G}', g' \in G'$.

If we view the top row of (A.16) as a principal G'-bundle, then this is the associated bundle construction; the hypotheses allow us to descend the product group structure on $G \times \widetilde{G}'$ to $\widetilde{G}$.

Appendix B. Clifford Algebras and the Tenfold Way

The classic reference is [3]. We begin with a summary of some basics. Recall that for each integer $n \in \mathbb{Z}$ the standard Clifford algebra Cl_n is the real associative algebra with unit generated by $|n|$ elements $e_1, e_2, \dots, e_{|n|}$ with $e_i e_j + e_j e_i = 0$ if $i \neq j$, and $e_i^2 = \mathrm{sign}(n)$ for all i. The Clifford algebra Cl_0 is the field $\mathbb{R}$. The Clifford algebra is naturally $\mathbb{Z}/2\mathbb{Z}$-graded: the generators e_i

are odd. The corresponding *complex* superalgebras are denoted $\mathrm{Cl}_n^{\mathbb{C}}$. For n, even the complex Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$ is isomorphic to the superalgebra of endomorphisms of a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space $S = S^0 \oplus S^1$ with $\dim S^0 = \dim S^1$. As usual, after choosing a basis we write elements of $\mathrm{End}\, S$ as square matrices, which relative to the decomposition $S = S^0 \oplus S^1$ have a 2×2 block form. The even elements of $\mathrm{End}\, S$ are block diagonal matrices; the odd elements are block off-diagonal matrices. Over the reals the Clifford algebras Cl_n are matrix superalgebras if n is divisible by 8. It happens that as *ungraded* algebras some other Clifford algebras are ungraded matrix algebras. For example, as an ungraded algebra Cl_2 is the algebra of 2×2 real matrices. This algebra is generated by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.1}$$

which satisfy the Clifford relations. However, the matrix which represents e_1 is diagonal hence even, not odd. Thus the isomorphism (B.1) between Cl_2 and 2×2 matrices does not preserve the $\mathbb{Z}/2\mathbb{Z}$-grading. On the other hand, the mixed signature Clifford algebra with two anticommuting generators e_+, e_- whose squares are $e_+^2 = +1, e_-^2 = -1$ is isomorphic to the superalgebra of real 2×2 matrices and the isomorphism preserves the grading:

$$e_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{B.2}$$

If $S = S' \amalg S''$ is a finite set of generators, written as a disjoint union, then the Clifford algebra with generating set S is a graded tensor product $\mathrm{Cl}(S) \cong \mathrm{Cl}(S') \otimes \mathrm{Cl}(S'')$.

Our interest is in representations of Clifford algebras, or better *Clifford modules*. A left Clifford module is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V with a Clifford algebra acting consistently with the grading. Clifford modules for a fixed Clifford algebra R are the objects of a category R-Mod, and superalgebras R_0 and R_1 are *Morita equivalent* if R_0-Mod and R_1-Mod are equivalent categories. This holds if $R_1 \otimes \mathrm{End}(S_1) \cong R_0 \otimes \mathrm{End}(S_0)$ for some super vector spaces S_0, S_1. Here, as always, we use the graded tensor product defined with the Koszul sign rule (Remark 7.3). The equivalence maps an R_0-module V_0 to the R_1-module $V_0 \otimes S$. In the other direction, an R_1-module V_1 maps to the R_0-module $\mathrm{Hom}_{\mathrm{End}(S)}(S, V_1)$. The Morita class of a complex Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$ is determined by $n \pmod 2$; the Morita class of a real Clifford algebra Cl_n by $n \pmod 8$. Finally, there is an equivalence between quaternionic $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces and Cl_{-4}-modules; in other words, Cl_{-4} is Morita equivalent to the (even) quaternion algebra $\mathbb{H}$. Let W be a quaternionic super vector space, i.e., a real $\mathbb{Z}/2\mathbb{Z}$-graded vector space with an even action of the quaternions i, j, k and a grading operator σ (with $\sigma^2 = 1$). The associated Cl_{-4}-module is the real $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = W \oplus W$ with grading $\left(\begin{smallmatrix} \sigma & 0 \\ 0 & -\sigma \end{smallmatrix}\right)$ and Clifford action

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \tag{B.3}$$

Conversely, given a Cl_{-4}-module V, the (-1)-eigenspace of $\gamma = e_1e_2e_3e_4$ is a quaternionic vector space with e_1e_2, e_1e_3, e_1e_4 acting as unit quaternions i, j, k.

We now consider the universal group $\mathcal{C} = \{\pm 1\} \times \{\pm 1\}$ which tracks time-reversal and Hamiltonian-reversal. It is equipped with homomorphisms $c, \phi_{\mathcal{C}} \colon \mathcal{C} \to \{\pm 1\}$, where c is projection onto the second factor, and $\phi_{\mathcal{C}}$ is multiplication $\{\pm 1\} \times \{\pm 1\} \to \{\pm 1\}$. As in Sect. 6, we define a *CT type* as a pair (A, A^τ) consisting of a subgroup A of $\mathcal{C}$ and a ϕ-twisted extension $1 \to \mathbb{T} \to A^\tau \to A \to 1$, where ϕ is the restriction of $\phi_{\mathcal{C}}$ to A. There are ten isomorphism classes of CT types (Proposition 6.4). These may be labeled by the special lifts T, C, S of the non-identity elements $\overline{T} = (-1, 1), \overline{C} = (1, -1), \overline{S} = (-1, -1)$ of $\mathcal{C}$, as in Lemma 8.8. Recall from Definition 3.8(ii) that a (ϕ, τ, c)-twisted representation of A is a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space W with an action of A^τ for which ϕ tracks linearity/antilinearity and c tracks evenness/oddness.

Proposition B.4. *Each CT type (A, A^τ) is associated to a real or complex Clifford algebra R, as indicated in the following table, such that the category of (ϕ, τ, c)-twisted representations of A is equivalent to the category of real or complex R-modules.*

A	1	diag	$\{\pm1\}\times 1$	$\mathcal{C}$	$1\times\{\pm1\}$	$\mathcal{C}$	$\{\pm1\}\times 1$	$\mathcal{C}$	$1\times\{\pm1\}$	$\mathcal{C}$
T^2			$+1$	$+1$		-1	-1	-1		$+1$
C^2				-1	-1	-1		$+1$	$+1$	$+1$
R	$\mathrm{Cl}_0^{\mathbb{C}}$	$\mathrm{Cl}_{-1}^{\mathbb{C}}$	Cl_0	Cl_{-1}	Cl_{-2}	Cl_{-3}	Cl_{-4}	Cl_{-5}	Cl_{-6}	Cl_{-7}

The Clifford algebra is only determined up to Morita equivalence, which we use to identify $\mathrm{Cl}_{-5}, \mathrm{Cl}_{-6}, \mathrm{Cl}_{-7}$ with $\mathrm{Cl}_3, \mathrm{Cl}_2, \mathrm{Cl}_1$, respectively. A closely related discussion and table appear in [20, Sect. 5] and in [5].

Proof. We first indicate the equivalence which maps a (ϕ, τ, c)-twisted representation W to a Clifford module. Recall that the vector space underlying a (ϕ, τ, c)-twisted representation of A is *complex*. Also, the representation of A^τ is determined by the action of the special lifts T, C, S, not all of which may be present and some of which may be redundant. The argument proceeds on a case-by-case basis.

For $A = 1$, the representation is a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space, which is a $\mathrm{Cl}_0^{\mathbb{C}}$-module. For $A = \text{diag}$, there is in addition an odd operator S with $S^2 = 1$, and iS generates the complex Clifford algebra $\mathrm{Cl}_{-1}^{\mathbb{C}}$. (One could equally well use S to generate the Morita equivalent $\mathrm{Cl}_{+1}^{\mathbb{C}}$.)

If $\overline{C} \in A$ and $\overline{T} \notin A$, then C and iC are odd and generate a Clifford algebra on the real super vector space $W_{\mathbb{R}}$ underlying W. If $C^2 = (iC)^2 = -1$ this is Cl_{-2}, whence the fifth column of the table. If $C^2 = (iC)^2 = +1$ the Clifford algebra is Cl_{+2}, ergo the penultimate column.

If $\overline{T} \in A$, then the lift $T \in A^\tau$ acts as an even antilinear transformation with square ± 1 on the complex super vector space W. If $T^2 = +1$ it is a real structure on W, and if $T^2 = -1$ it is a quaternionic structure. This, together with the equivalence above between quaternionic super vector spaces and Cl_{-4}-modules, gives the columns with $R = \mathrm{Cl}_0, \mathrm{Cl}_{-4}$. In the real case $T^2 = +1$, if in addition $\overline{C} \in A$, then C is odd and commutes with T, so if $C^2 = -1$ it generates a real Clifford algebra Cl_{-1}, and this gives the equivalence in the fourth column of the table. If $C^2 = +1$, then C generates the Clifford algebra Cl_{+1}, and this gives the last column. Finally, if $T^2 = -1$ and $\overline{C} \in A$, then C, iC, iTC are odd endomorphisms of $W_{\mathbb{R}}$ and generate a $\mathrm{Cl}_{\pm 3}$-action, according to $C^2 = \pm 1$. This completes the table.

The inverse equivalence takes a Clifford module V to a (ϕ, τ, c)-twisted representation W of the corresponding CT group A as follows. By Morita equivalence we may assume that V is a Clifford module for $\mathrm{Cl}_n^{\mathbb{C}}$ with $n = 0, 1$ or for Cl_n with $|n| \le 4$. In the complex cases, we take $W = V$; for $n = 1$ the action of e_1 generates the representation of $A = \mathrm{diag}$. If V is a real Cl_n-module with $n = 0, \pm 1$, then let $W = V \otimes \mathbb{C}$ be the complexification of V and $T = \mathrm{id}_V \otimes \kappa$, where κ is complex conjugation; for $n = \pm 1$ set $C = e_1 \otimes \kappa$. For V a real $\mathrm{Cl}_{\pm 2}$-module, set $W = V$ with complex structure $e_1 e_2$ and $C = e_1$. For V a real $\mathrm{Cl}_{\pm 3}$-module, set $W = V$ with complex structure $\mp e_1 e_2$, but now let $C = e_1$ and $T = -e_2 e_3$. Finally, motivated by (B.3), if V is a real Cl_{-4}-module, let W be the (-1)-eigenspace of $e_1 e_2 e_3 e_4$ with complex structure $e_1 e_2$. Then $T = e_1 e_3$ is even, preserves W, and squares to -1. □

Appendix C. Dyson's Tenfold Way

Dyson [17] studied *irreducible* corepresentations (Definition C.3 below) of a group G equipped with a homomorphism $\phi\colon G \to \{\pm 1\}$ and classified them into ten types. We re-derive his classification by studying commutants, as did Dyson, with the difference that we take into account a $\mathbb{Z}/2\mathbb{Z}$-grading and use the Koszul sign rule. The argument then reduces to a $\mathbb{Z}/2\mathbb{Z}$-graded version of Schur's lemma. We make contact with Wall's graded Brauer groups [63], though he too ignored the Koszul sign rule in his definition of the center. We follow instead the treatment by Deligne [11, Sect. 3]. The arguments apply to vector spaces over very general fields, but the reader should keep in mind the real numbers, which is the case of interest in this paper. All algebras are unital and associative.

Recall that an associative unital algebra D over a field k is a *division algebra* if every nonzero element of D is invertible.

Theorem C.1 (Schur). *Let k be a field, G a group, and $\rho\colon G \to \mathrm{Aut}_k(V)$ an irreducible representation of G on a vector space V over k. Then the commutant*

$$D = \{a \in \mathrm{End}_k(V) : a\rho(g) = \rho(g)a \text{ for all } g \in G\} \tag{C.2}$$

is a division algebra.

Proof. Let $a \in D$ and consider $\ker(a) \subset V$. Since a commutes with each $\rho(g)$ it follows that $\ker(a)$ is G-invariant. But ρ is irreducible, whence $\ker(a) = V$ or $\ker(a) = 0$, i.e., $a = 0$ or a is injective. Similarly, the image of a is either 0 or V, so $a = 0$ or a is surjective. Therefore, $a = 0$ or a is invertible. □

According to a classical theorem of Frobenius, there are three division algebras over $k = \mathbb{R}$ up to isomorphism: namely, the field of real numbers $D = \mathbb{R}$, the field of complex numbers $D = \mathbb{C}$, and the noncommutative algebra of quaternions $D = \mathbb{H}$. Thus, there is a corresponding trichotomy of irreducible real representations.

From now on suppose $1 \neq -1$ in the field k, i.e., k does not have characteristic 2.

Definition C.3. Let G be a group and $\phi\colon G \to \{\pm 1\}$ a homomorphism. Let V be a vector space over k and $I \in \mathrm{End}_k(V)$ an automorphism with $I^2 \in k^\times$ a nonzero scalar transformation.

(i) Let $\mathrm{Aut}_k(V, I)$ be the $\mathbb{Z}/2\mathbb{Z}$-graded group of automorphisms of V which either commute or anticommute with I.

(ii) A *corepresentation* of G on V is a homomorphism $\rho\colon G \to \mathrm{Aut}_k(V, I)$ of $\mathbb{Z}/2\mathbb{Z}$-graded groups, which then satisfies

$$\rho(g)I = \phi(g)I\rho(g) \quad \text{for all } g \in G. \tag{C.4}$$

(iii) The corepresentation ρ is *irreducible* if there is no proper subspace of V which is invariant under I and $\rho(g)$ for all $g \in G$.

The pair (G, ϕ) is called a *$\mathbb{Z}/2\mathbb{Z}$-graded group*. The $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Aut}_k(V, I)$ tracks whether an automorphism commutes or anticommutes with I. The algebra $\mathrm{End}_k(V)$ is $\mathbb{Z}/2\mathbb{Z}$-graded by the involution $a \mapsto IaI^{-1}$, which squares to the identity map: even elements commute with I and odd elements anticommute. If $k = \mathbb{R}$ and I is a complex structure on V, then Definition C.3 reduces to Definition 1.11 (with τ trivial). In this case, even elements of $\mathrm{End}_{\mathbb{R}}(V)$ are complex linear and odd elements are complex antilinear. Note that each $\rho(g)$ commutes or anticommutes with I. The sign in (C.4) is then the usual Koszul sign. In (iii), recall that a *proper* subspace of V is neither 0 nor V.

Definition C.5. A super algebra D over k is a *super division algebra* if every nonzero homogeneous element is invertible.

It is not true that every non-homogeneous element is invertible.

Theorem C.6 (super Schur). *Let (G, ϕ) be a $\mathbb{Z}/2\mathbb{Z}$-graded group and $\rho\colon G \to \mathrm{Aut}_k(V, I)$ an irreducible corepresentation on (V, I). Then the graded commutant*

$$\begin{aligned} D = \{a^0 + a^1 \in \mathrm{End}_k(V) : \quad & a^0\rho(g) = \rho(g)a^0 \\ & a^1\rho(g) = \phi(g)\rho(g)a^1 \quad \textit{for all } g \in G\} \end{aligned} \tag{C.7}$$

is a super division algebra over k.

In (C.7), the general element of $\mathrm{End}_k(V)$ is written as a sum $a^0 + a^1$ of even and odd elements: a^0 commutes with I and a^1 anticommmutes with I.

Proof. As in the classical case—Theorem C.1—the kernel and image of a homogeneous element $a \in D$ are I-invariant and G-invariant subspaces of V, so by irreducibility are either V or 0. □

We can classify a corepresentation by the isomorphism class of its graded commutant. For $k = \mathbb{R}$, there are 10 possibilities [63], and they are exhibited in the first line of the table below. Note that 9 of them are Clifford algebras—the real and complex Clifford algebras of smallest dimension—but the 10th ($\mathbb{H}$) is not. Also, the entries $\mathbb{C}, \mathbb{R}, \mathbb{H}$ are purely even, and hence are the usual (ungraded) division algebras over the reals.

Remark C.8. The set of isomorphism classes of (super) division algebras over k may be identified with the set of (super) simple algebras over k up to Morita equivalence (see Appendix B). If we fix the (graded) center then they form an abelian group—the *Brauer group*—under tensor product. For $k = \mathbb{R}$, the (graded) center is either $\mathbb{R}$ or $\mathbb{C}$. In the graded case there are 8 isomorphism classes of super division algebras with center $\mathbb{R}$ and 2 isomorphism classes with center $\mathbb{C}$. The Brauer groups are cyclic, and each Morita equivalence class of super central simple algebras is represented by a Clifford algebra.

Example C.9. Let $G = \mathbb{Z}/2\mathbb{Z}$ with the nontrivial grading ϕ. Let $V = \mathbb{R}^2$ with its usual complex structure I, which we use to identify $\mathbb{R}^2 = \mathbb{C}$. Define a ϕ-twisted representation ρ by letting $\rho(g)z = \bar{z}$, where $g \in G$ is the non-zero element and $z \in \mathbb{C}$. Let $D = D^0 \oplus D^1$ be the graded commutant. Then D^0 consists of real scalar endomorphisms $z \mapsto rz$ ($r \in \mathbb{R}$) and D^1 consists of endomorphisms $z \mapsto si\bar{z}$ ($s \in \mathbb{R}$). So $D \cong \mathrm{Cl}_1$. There are many more examples in [17, Sect. V].

The CT types studied in Propositions 6.4 and B.4 correspond to super division algebras over $\mathbb{R}$ as follows. A CT type is a pair (A, A^τ) consisting of a subgroup $A \subset \{\pm1\} \times \{\pm1\}$ and a twisted extension $\mathbb{T} \to A^\tau \to A$. Recall that A may or may not contain the generators $\overline{T}, \overline{C}$ of $\{\pm1\} \times \{\pm1\}$, which have special lifts $T, C \in A^\tau$ which satisfy $T^2 = \pm1$, $C^2 = \pm1$, $TC = CT$. Given a CT type (A, A^τ), the construction beginning in (7.6) gives a hermitian line bundle $L^\tau \to A$ such that $\widetilde{D} = \oplus_{a\in A} L^\tau_a$ is a super algebra which contains A^τ. Note that the multiplication is not complex linear if the homomorphism $\phi\colon A \to \{\pm1\}$ is nontrivial. Also, the homomorphism $c\colon A \to \{\pm1\}$ is used to grade $\widetilde{D}$. Define a super algebra D as follows. If $\overline{T} \notin A$ set $D = \widetilde{D}$; it is a *complex* division algebra. If $\overline{T} \in A$ and $T^2 = -1$ let $D = \widetilde{D}_{\mathbb{R}}$ be the underlying real super algebra. If $\overline{T} \in A$ and $T^2 = +1$, then conjugation by T is a real structure on $\widetilde{D}$, and we let D be the real points of the subalgebra of $\widetilde{D}$ generated by C, if $\overline{C} \in A$; otherwise D consists only of real scalars.

We summarize some of the various tenfold ways in a compact table:

sDivAlg	$\mathbb{C}$	$\mathrm{Cl}_1^{\mathbb{C}}$	$\mathbb{R}$	Cl_{-1}	Cl_{-2}	Cl_{-3}	$\mathbb{H}$	Cl_3	Cl_2	Cl_1
CT	1	diag	$+0$	$+-$	$0-$	$--$	-0	$-+$	$0+$	$++$
Dyson	$\mathbb{CC}_1$	$\mathbb{CC}_2$	$\mathbb{RC}$	$\mathbb{RH}$	$\mathbb{CH}$	$\mathbb{HH}$	$\mathbb{HC}$	$\mathbb{HR}$	$\mathbb{CR}$	$\mathbb{RR}$

The first row enumerates the ten super division algebras which arise from Theorem C.6 with $k = \mathbb{R}$. As mentioned above, all but $\mathbb{H}$ are Clifford algebras. Note that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are purely even and are the classical division algebras over $\mathbb{R}$. The second row enumerates the 10 CT types in a compact notation: the first two entries indicate the subgroup $A \subset \{\pm 1\} \times \{\pm 1\}$ and the last 8 tell the status of $T, C \in A^\tau$: '0' if not present, '+' if it squares to $+1$, and '−' if it squares to -1. The columns line up with those of the table in Proposition B.4, which gives the same data in a more elaborate form. The final row of the present table gives Dyson's labels for the ten types. For the reader's convenience, we present a quick resumé of the first half of [17] where these labels are defined.

Dyson's analysis is based on the classical Theorem C.1 and the Wedderburn theorem, which asserts that a finite dimensional simple algebra over a field k is a matrix algebra over a division algebra. For $k = \mathbb{R}$, by the Frobenius theorem, the algebra must be a matrix algebra over one of the division algebras $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and we call this division algebra its *Wedderburn type*. Let (G, ϕ) be a $\mathbb{Z}/2\mathbb{Z}$-graded group as in Definition C.3. Dyson assumes that the homomorphism ϕ is surjective, so the kernel $G_0 \subset G$ of ϕ is an index 2 subgroup. Let V be an irreducible co-representation, so a complex vector space on which elements of G_0 act complex linearly and elements of $G \setminus G_0$ act complex antilinearly. Let[31] A be the subalgebra of $\mathrm{End}_{\mathbb{R}}(V)$ generated by $\rho(g_0)$, $g_0 \in G_0$ and D the subalgebra of $\mathrm{End}_{\mathbb{R}}(V)$ generated by $\rho(g)$, $g \in G$ and the complex structure I. If the co-representation is irreducible, then D is a simple algebra, so by the Wedderburn theorem is a matrix algebra over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. The algebra A is not simple, since the real representation of G_0 on $V_{\mathbb{R}}$ may be reducible, but Dyson proves that the irreducible summands are all of the same Wedderburn type. He further shows that the Wedderburn types of D and A are independent in that all 9 possibilities can occur. The Dyson type is the conjunction of the Wedderburn types of D and A. Thus, for example, for Dyson type $\mathbb{HC}$ the algebra D is quaternionic and the algebra A is complex. The type $\mathbb{CC}$ splits into two subtypes according as the complex representation of G_0 on V splits into the sum of two nonisomorphic or isomorphic irreducible representations. In his analysis, Dyson studies the ungraded commutator algebras of A and D, denoted X and Z, and they have the same Wedderburn types as A and D, respectively. The graded commutant of the algebra generated by $\rho(g), g \in G$, which is the super division algebra we associate with the representation ρ, lies between Dyson's algebras Z and X and uniquely characterizes the Dyson type. We highly recommend the beautiful paper [17] to the reader.

Appendix D. Continuous Families of Quantum Systems

For a finite dimensional complex vector space V, there is only one reasonable topology on the algebra $\mathrm{End}\, V$ of linear maps $V \to V$. The invertible operators $\mathrm{Aut}\, V \subset \mathrm{End}\, V$ inherit a topology in which inversion and multiplication are continuous: $\mathrm{Aut}\, V$ is a topological group. The situation is quite different when V is an infinite dimensional topological vector space, as then there are several possible topologies on the vector space of continuous linear operators. In this appendix, we discuss the compact-open topology for V a Hilbert space.

[31] In this paragraph, 'A' and 'D' are as used in [17], not as used in the previous text.

This is the crucial ingredient in our definition of a continuous family of quantum systems, and so too in the definition of homotopy invariants of quantum systems. The main references for our discussion are [6, Appendices] and [52]; we follow [6] particularly closely in the next subsection.

D.1 Topologies on Mapping Spaces

Quite generally, if Y is a topological space and Z a metric space, then there are several natural topologies on the set $\mathrm{Map}(Y, Z)$ of continuous maps $f : Y \to Z$. In the *topology of pointwise convergence*, or equivalently *point-open topology*, a sequence $f_n \to f$ iff $f_n(y) \to f(y)$ for all $y \in Y$. In the finer[32] *topology of uniform convergence on compact sets*, or equivalently *compact-open topology*,[33] a sequence $f_n \to f$ iff for each compact subset $C \subset Y$, the convergence $f_n(y) \to f(y)$ is uniform for $y \in C$. If X is a topological space, and $F : X \times Y \to Z$ a continuous map, then $x \mapsto (y \mapsto F(x, y))$ is a continuous map from X to $\mathrm{Map}(Y, Z)$ with the compact-open topology.

If $Y = Z = \mathcal{H}$ is an infinite dimensional separable complex Hilbert space, and $\mathrm{End}(\mathcal{H}) \subset \mathrm{Map}(\mathcal{H}, \mathcal{H})$ the set of continuous, hence bounded, linear maps $\mathcal{H} \to \mathcal{H}$, then the topology of pointwise convergence is called the *strong operator topology*. The compact-open topology is finer, but not so different: as shown in [6, Appendix 1] the compact sets in the two topologies coincide. It follows that if X is a *compactly generated*[34] topological space the continuous maps $X \to \mathrm{End}(\mathcal{H})$ are the same in both topologies. We write $\mathrm{End}(\mathcal{H})_{\mathrm{c.o.}}$ for the space of bounded linear operators with the compact-open topology. There is a finer *norm topology* on $\mathrm{End}(\mathcal{H})$ which is often used; then $f_n \to f$ iff $f_n(x) \to f(x)$ uniformly for x in the unit sphere of $\mathcal{H}$. In other words, $f_n \to f$ in the metric topology of the operator norm: $\|f - f_n\| \to 0$.

We choose to work with the compact-open topology on $\mathrm{End}(\mathcal{H})$ for several reasons:

(i) If H is an *unbounded* self-adjoint operator on $\mathcal{H}$, then the one-parameter group $t \mapsto e^{-itH/\hbar}$ is continuous in the compact-open topology but not in the norm topology.
(ii) A fiber bundle of Hilbert spaces (Definition D.6) has transition functions which are continuous in the compact-open topology.
(iii) The projection-valued measure associated to a self-adjoint operator is countably additive in the compact-open topology [52, Sects. VII.3,VIII.3].

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[32] A topology τ on a set X is a collection of subsets of X. A topology τ' is finer, or stronger, than a topology τ if $\tau \subset \tau'$.

[33] The compact-open topology is defined for Z a general topological space, not necessarily a metric space.

[34] A topological space X is compactly generated if a subset $A \subset X$ is closed if and only if $A \cap C$ is closed for all compact subsets $C \subset X$. If X is compactly generated, then one can test continuity of maps with domain X on compact subsets of X.Topological spaces whose topology can be defined by a metric are called *metrizable*, and they are compactly generated. See [15, Sect. 6.1] for a summary of Steenrod's classic paper [58] on compactly generated spaces.

(iv) While the norm topology is too strong for many purposes, the *weak operator topology* is too weak to control the spectrum.[35]

We warn that the compact-open topology is not sufficiently strong for problems involving index theory: the space of Fredholm operators is contractible in the compact-open topology [6, Theorem A2.1].

There are other difficulties with the compact-open topology [6, Appendix 1], assuming $\mathcal{H}$ is infinite dimensional, which stem from the fact that $f \mapsto f^{-1}$ is *not* a continuous map $\mathrm{Aut}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{H})$ in the induced topology on the subset $\mathrm{Aut}(\mathcal{H}) \subset \mathrm{End}(\mathcal{H})$ of invertible operators.

Definition D.1. The topology on the set $\mathrm{Aut}(\mathcal{H})$ induced from

$$\begin{aligned} \mathrm{Aut}(\mathcal{H}) &\longrightarrow \mathrm{End}(\mathcal{H})_{\text{c.o.}} \times \mathrm{End}(\mathcal{H})_{\text{c.o.}} \\ f &\longmapsto (f, f^{-1}) \end{aligned} \tag{D.2}$$

is called the *compact-open topology* and is denoted $\mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$. The compact-open topology on the unitary group $U(\mathcal{H})$ is the topology induced by the inclusion $U(\mathcal{H}) \subset \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$.

Even though inversion $\mathrm{Aut}(\mathcal{H})_{\text{c.o.}} \to \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ is continuous, by design using Definition D.1, multiplication $\mathrm{Aut}(\mathcal{H})_{\text{c.o.}} \times \mathrm{Aut}(\mathcal{H})_{\text{c.o.}} \to \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ is *not* continuous, if $\mathcal{H}$ is infinite dimensional, whence $\mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ is not a topological group. But if X is compactly generated, then the set of continuous maps $\mathrm{Map}\big(X, \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}\big)$ is a topological group in the compact-open topology. The spaces $\mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ and $U(\mathcal{H})_{\text{c.o.}}$ are contractible [6, Theorem A2.1].

Proposition D.3. *The subgroup* $\mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H}) \subset \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ *has two contractible components in the compact-open topology.*

Proof. Each $S \subset \mathrm{Aut}_{\mathrm{qtm}}(\mathcal{H})$ is real linear and $S(i\psi) = \pm i\psi$ for all $\psi \in \mathcal{H}$ and one choice of sign. The sign is unchanged under limits $S_n \to S$ in the point-open, hence compact-open, topology. □

D.2 Hilbert Bundles and Continuous Families of Quantum Systems

We begin by defining a representation of a group as a continuous *action* on a vector space.

Definition D.4. Let G be a topological group and $\mathcal{H}$ a Hilbert space. A *representation* of G on $\mathcal{H}$ is a continuous linear action $\alpha\colon G \times \mathcal{H} \to \mathcal{H}$. The representation is *unitary* if the linear operator $\alpha(g, -)$ is unitary for all $g \in G$.

The (left) action property is

$$\alpha\big(g_1, \alpha(g_2, \psi)\big) = \alpha(g_1 g_2, \psi), \quad g_1, g_2 \in G, \quad \psi \in \mathcal{H}. \tag{D.5}$$

The continuity of α implies that the induced map $\rho\colon G \to \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ is continuous. Conversely, a continuous map $\rho\colon G \to \mathrm{Aut}(\mathcal{H})_{\text{c.o.}}$ is a representation

[35] For example, if $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$, and f_n orthogonal projection onto the line spanned by $e_1 + e_n$, then $\{f_n\}$ converges weakly to 1/2 times orthogonal projection onto the span of e_1. This exhibits an operator with spectrum $\{0, 1/2\}$ as a weak limit of operators with spectrum $\{0, 1\}$.

if the topology of G is compactly generated.[36] We have already remarked that one-parameter groups ($G = \mathbb{R}$) of unitary operators generated by an unbounded self-adjoint operator are continuous in the sense of Definition D.4.

The compact-open topology is also natural for fiber bundles.

Definition D.6. Let $X, \mathcal{E}$ be topological spaces and $\pi\colon \mathcal{E} \to X$ a continuous map. Suppose each fiber $\mathcal{E}_x = \pi^{-1}(x), x \in X$, is endowed with the structure of a complex Hilbert space. Then π is a *Hilbert bundle* if it is locally trivial, i.e., if about each $x \in X$ there exists an open set $U \subset X$, a Hilbert space $\mathcal{H}$, and a homeomorphism φ which fits into the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \varphi\ } & U \times \mathcal{H} \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U & \end{array} \tag{D.7}$$

and preserves the Hilbert space structure on each fiber.

Here π_1 is projection onto the first factor. The homeomorphism φ is a *local trivialization*, and the transition function between two local trivializations is a continuous map into $U(\mathcal{H})_{\text{c.o.}}$. Conversely, if X is compactly generated a fiber bundle can be constructed from coherent continuous transition functions into $U(\mathcal{H})_{\text{c.o.}}$. A Hilbert bundle with infinite dimensional fibers is trivializable, but may not come with a natural trivialization (e.g., Proposition D.17).

Definition D.8. Let G be a topological group and X a topological space. Then a *continuous family of unitary representations of* G *parametrized by* X is a Hilbert bundle $\pi\colon \mathcal{E} \to X$ and a continuous action $\alpha\colon G \times \mathcal{E} \to \mathcal{E}$ which preserves fibers: the diagram

$$\begin{array}{ccc} G \times \mathcal{E} & \xrightarrow{\ \alpha\ } & \mathcal{E} \\ & {\scriptstyle \pi \circ \pi_2} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & X & \end{array} \tag{D.9}$$

commutes.

If $\mathcal{E} = X \times \mathcal{H} \to X$ is the trivial bundle with fiber a fixed Hilbert space $\mathcal{H}$, then α defines a continuous map $G \times X \to U(\mathcal{H})_{\text{c.o.}}$. Conversely, if G and X are compactly generated, then a continuous map $G \times X \to U(\mathcal{H})_{\text{c.o.}}$ which is a homomorphism for each $x \in X$ determines a continuous family of representations.

A specialization of Definition D.8 applies to a family of Hamiltonians parametrized by X. We use the equivalence between self-adjoint operators and one-parameter groups of unitary operators given by Stone's theorem [52, Sect. VIII.4].

[36] We remind that metrizable spaces are compactly generated, so this holds for Lie groups G.

Definition D.10. A family of (possibly unbounded) self-adjoint operators $\{H_x\}, x \in X$, acting on a Hilbert bundle $\mathcal{E} \to X$ is *continuous* iff the associated family $t \mapsto e^{-itH_x/\hbar}$ of unitary one-parameter groups is a continuous family of unitary representations of $\mathbb{R}$ on $\mathcal{E} \to X$.

Recall that $\lambda \in \mathbb{C}$ is in the *resolvent set* of a self-adjoint operator H if $\lambda - H\colon \mathrm{domain}(H) \to \mathcal{H}$ is bijective with bounded inverse $R_\lambda(H)$, the *resolvent*. Then $\lambda \mapsto R_\lambda(H)$ is a holomorphic function whose domain is the resolvent set—the complement of the spectrum—so includes $\mathbb{C} \setminus \mathbb{R}$ since H is self-adjoint, and whose codomain is the space of bounded self-adjoint operators. A family $\{H_x\}_{x\in X}$ of self-adjoint operators on a fixed Hilbert space $\mathcal{H}$ has a *common core* $D \subset \mathcal{H}$ if $D \subset \mathcal{H}$ is dense, $D \subset \mathrm{domain}(H_x)$ for all $x \in X$, and the closure of the restriction of H_x to D is H_x.

Proposition D.11 ([52, Sect. VIII.7]). *Let X be a compactly generated topological space, $\mathcal{E} \to X$ a Hilbert bundle, and $\{H_x\}_{x\in X}$ a family of self-adjoint operators acting on the fibers of $\mathcal{E} \to X$.*

(i) *$\{H_x\}$ is a continuous family iff the associated resolvent function $(\lambda, x) \mapsto R_\lambda(H_x)$ is continuous in the compact-open topology for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $x \in X$.*
(ii) *Suppose $\mathcal{E} \to X$ is the trivial bundle with fiber $\mathcal{H}$ and the operators H_x, $x \in X$ have a common core D. Then if for each convergent sequence $x_n \to x$ in X and $\psi \in D$ we have $H_{x_n}\psi \to H_x\psi$, then $\{H_x\}$ is a continuous family.*

We remind that we can substitute the strong topology for the compact-open topology.

Next we give a precise version of Definition 5.1. Recall from Remark 3.10 that an extended QM symmetry class (G, ϕ, τ, c) has a canonically associated QM symmetry class $(\widetilde{G}, \tilde{\phi}, \tilde{\tau})$ which includes time translations.

Definition D.12. Let X be a topological space. A *continuous family of gapped systems with extended QM symmetry class* (G, ϕ, τ, c) is a continuous family of $(\tilde{\phi}, \tilde{\tau})$-twisted representations of $\widetilde{G}$ parametrized by X such that for all $x \in X$ the Hamiltonian H_x does not have 0 in its spectrum.

In other words, there is a Hilbert bundle $\pi\colon \mathcal{E} \to X$ and a continuous action $\widetilde{G}^\tau \times \mathcal{E} \to \mathcal{E}$ which on each fiber of π is a $(\tilde{\phi}, \tilde{\tau})$-twisted representation of $\widetilde{G}$. The Hamiltonian H_x is the self-adjoint generator of the one-parameter group $\mathbb{R} \subset \widetilde{G}^\tau \to U(\mathcal{E}_x)_{\text{c.o.}}$.

In Sect. 10, we use the positive and negative spectral projections associated to Hamiltonians which do not contain 0 in their spectrum.

Proposition D.13. *Let X be a compactly generated topological space, $\mathcal{E} \to X$ a Hilbert bundle, and $\{H_x\}$ a continuous family of self-adjoint operators acting on $\mathcal{E}$. Assume 0 is not in the spectrum of H_x for all $x \in X$. Then*

(i) *the positive spectral projections $\{\chi^+(H_x)\}$ and negative spectral projections $\{\chi^-(H_x)\}$ form continuous families of bounded self-adjoint operators; and*

(ii) *if the negative spectral projections have finite rank, then the images of* χ^+ *and* χ^- *are sub-Hilbert bundles* $\mathcal{E}^+$ *and* $\mathcal{E}^-$*.*

A corollary of the proof of (ii) is that if $\{E_x \subset \mathcal{E}_x\}_{x\in X}$ is the image of a family of finite rank spectral projections $\chi^{[a,b]}$ with neither a nor b in the spectrum of each H_x, then $E \to X$ is a locally trivial vector bundle.

Proof. The statements are local and by the local triviality of $\mathcal{E} \to X$ we may assume H_x acts on a constant Hilbert space $\mathcal{H}$. Let $\Gamma\colon \mathbb{R} \to \mathbb{C}$ be a smooth embedding such that $\Gamma(0) = 0$ and $\Gamma(-t) = t + i, \Gamma(t) = t - i$ for $t > 1$. Then if H is self-adjoint and 0 is not in the spectrum, the spectral calculus implies

$$\chi^+(H) = \int_\Gamma d\lambda\, R_\lambda(H), \tag{D.14}$$

where the integral is defined as $\int_\Gamma d\lambda = \lim_{M\to\infty} \int_{-M}^{M} dt$. Let $H_n \to H$ be a convergent sequence in the sense of Definition D.10, so that by Proposition D.11 we have $R_{\Gamma(t)}(H_n) \to R_{\Gamma(t)}(H)$ in the strong topology for all $t \neq 0$. For any vector ψ, the norm $\|R_{\Gamma(t)}(H)\psi\|$ is uniformly bounded, since it is bounded by $\|\psi\|$ for $|t| > 1$. The Lebesgue dominated convergence theorem now implies $\chi^+(H_n) \to \chi^+(H)$ in the strong topology. This proves (i).

For (ii) we must prove that the finite dimensional vector spaces $\{\mathcal{E}_x^- = \chi^-(H_x)(\mathcal{H})\}$ form a locally trivial family of subspaces of $\mathcal{H}$. Fix $x_0 \in X$ and let $\pi_x\colon \mathcal{E}_{x_0}^- \to \mathcal{E}_x^-$ be orthogonal projection, i.e., the restriction of $\chi^-(H_x)$ to $\mathcal{E}_{x_0}^-$. We claim that for x in an open neighborhood of x_0 the map π_x is an isomorphism, hence a local trivialization after applying Gram-Schmidt to make it an isometry. The claim follows since by strong continuity the matrix of inner products $\big(\langle\chi^-(H_x)e_i, \chi^-(H_x)e_j\rangle\big)_{i,j=1\dots N}$ is invertible for x sufficiently close to x_0, where $\{e_i\}_{i=1}^N$ is an orthonormal basis of $\mathcal{E}_{x_0}^-$. □

D.3 Fourier Transform (Bloch Sums) and the Berry Connection

Next, we prove that Hypothesis 10.9(ii) is satisfied in the standard situation described at the beginning of Sect. 10. We need the following preliminary.

Lemma D.15. *Let* X, Y *be locally compact Hausdorff topological spaces with Borel measures, and assume that* Y *is compact. Let* $\mathcal{L} \to X \times Y$ *be a hermitian line bundle. Then the Hilbert spaces* $\mathcal{E}_x = L^2\big(Y; \mathcal{L}\big|_{\{x\}\times Y}\big), x \in X$*, are the fibers of a Hilbert bundle* $\mathcal{E} \to X$ *and there exists an isomorphism of Hilbert spaces* $L^2(X \times Y; \mathcal{L}) \cong L^2(X; \mathcal{E})$*.*

Proof. About each $x \in X$ is an open neighborhood $U \subset X$ together with a continuous isomorphism $\phi\colon \mathcal{L}\big|_{U\times Y} \to U \times \mathcal{L}\big|_{\{x\}\times Y}$. Then ϕ induces a local trivialization $\varphi\colon \mathcal{E}\big|_U \to U \times \mathcal{E}_x$ which we use to topologize $\mathcal{E}$. To prove the resulting transition functions are continuous, use the fact that if $h\colon U \times Y \to \mathbb{C}^\times$ is continuous and $\mathcal{K} \to Y$ a hermitian line bundle, then

$$x \mapsto \text{multiplication by } h(x, -) \text{ on } L^2(Y; \mathcal{K}) \tag{D.16}$$

is continuous in the compact-open topology. □

Let E be a Euclidean space of dimension d and Π a full lattice of translations of E. The Pontrjagin dual X_Π is the torus of unitary characters $\lambda\colon \Pi \to \mathbb{T}$. The affine torus E/Π is a torsor over the dual torus to X_Π. The lattice Π acts on the Hilbert space $L^2(E;\mathbb{C})$ of L^2 functions with respect to Lebesgue measure, so determines a sheaf $\mathcal{S}$ of Hilbert spaces, by Lemma 9.29.

Proposition D.17. *$\mathcal{S}$ is the sheaf of sections of a Hilbert bundle $\mathcal{E} \to X_\Pi$.*

Proof. Let $\mathcal{L} \to X_\Pi \times E/\Pi$ be the "Poincaré line bundle" whose sections are functions $\hat{f}\colon X_\Pi \times E \to \mathbb{C}$ which satisfy the quasi-periodicity condition

$$\hat{f}(\lambda, x+\xi) = \lambda(\xi)\hat{f}(\lambda, x), \quad \xi \in \Pi. \tag{D.18}$$

Fourier transform defines an isomorphism of Hilbert spaces

$$L^2(E;\mathbb{C}) \longrightarrow L^2(X_\Pi \times E/\Pi;\mathcal{L}) \tag{D.19}$$

which takes $f \in L^2(E;\mathbb{C})$ to

$$\hat{f}(\lambda, x) = \sum_{\xi' \in \Pi} \lambda(\xi')^{-1} f(x+\xi'); \tag{D.20}$$

the inverse maps $\hat{f} \in L^2(X_\Pi \times E/\Pi;\mathcal{L})$ to

$$f(x) = \int_{X_\Pi} d\lambda\, \hat{f}(\lambda, x), \tag{D.21}$$

where $d\lambda$ is Haar measure with unit total volume. Lemma D.15 identifies the codomain of (D.19) with the space of L^2 sections of the Hilbert bundle $\mathcal{E} \to X_\Pi$ whose fiber at $\lambda \in X_\Pi$ is

$$\mathcal{E}_\lambda = L^2\big(E/\Pi;\mathcal{L}_\lambda\big), \tag{D.22}$$

where $\mathcal{L}_\lambda = \mathcal{L}\big|_{\{\lambda\}\times E/\Pi}$. □

Remark D.23. In the condensed matter physics literature, (D.18) is known as the *Bloch wave condition* and (D.20) is known as a *Bloch sum*.

The Hilbert bundle $\mathcal{E} \to X_\Pi$ is in fact smooth and can be built from locally constant transition functions. Let V denote the vector space of translations of E, so $\Pi \subset V$, and let $\Pi^* \subset V^*$ be the dual lattice of linear functionals $k\colon V \to \mathbb{R}$ such that $\langle k, \xi\rangle \in \mathbb{Z}$ for all $\xi \in \Pi$. Here the bracket is the pairing between V^* and V. We identify $H_1(X_\Pi;\mathbb{Z}) \cong \Pi^*$.

Proposition D.24. *For each choice of origin $x_0 \in E$, there is a flat covariant derivative $\nabla^{\mathcal{E}}$ on $\mathcal{E} \to X_\Pi$ whose holonomy around a loop with homology class $k \in \Pi^*$ is multiplication by the periodic function*

$$h_{\pi^*}(x) = e^{2\pi i\langle k, x-x_0\rangle}, \quad x \in E. \tag{D.25}$$

Proof. Let $\hat{f}_0\colon E \to \mathbb{C}$ be quasi-periodic for fixed λ_0, as in (D.18), and let $\{\lambda_t\colon 0 \le t \le 1\}$, be a smooth path in X_Π. Identify $X_\Pi = V^*/\Pi^*$ and choose a lift $k_0 \in V^*$ of λ_0; then there is a unique smooth lift $\{k_t\} \subset V^*$ of the path $\{\lambda_t\} \subset X_\Pi$. Define the parallel transport of $\hat{f}_0$ along $\{\lambda_t\}$ to be the path of functions

$$\hat{f}_t(x) = e^{2\pi i \langle k_t - k_0,\, x - x_0\rangle} \hat{f}_0(x), \quad x \in E, \quad 0 \le t \le 1. \tag{D.26}$$

It is independent of the choice of k_0 and defines the parallel transport of a flat covariant derivative with the stated holonomy. □

Remark D.27. The dependence on the origin x_0 is easily worked out from (D.25) and (D.26) and is the familiar indeterminacy of the covariant derivative on the Poincaré line bundle: we can tensor with a flat covariant derivative pulled back from X_Π.

Remark D.28. If Π is extended to a larger symmetry group G, as in (10.6), then the quotient G'' acts on X_Π and with a possible central extension the groupoid $X_\Pi /\!/ G''$ lifts to $\mathcal{E} \to X_\Pi$. We claim that this lifted action commutes with the covariant derivative $\nabla^{\mathcal{E}}$. This follows simply by observing that it preserves the parallel transport (D.26), since the action of G'' preserves the pairing $\langle -,-\rangle\colon V^* \otimes V \to \mathbb{R}$.

In band theory, one encounters smooth finite rank subbundles $F \subset \mathcal{E}$. Let $F \xrightarrow{i} \mathcal{E} \xrightarrow{p} F$ denote the inclusion and orthogonal projection. Then the compression $p \circ \nabla^{\mathcal{E}} \circ i$ is a covariant derivative on $F \to X_\Pi$ called the *Berry connection*. If the "Bloch Hamiltonians" on the fibers of $\mathcal{E} \to X_\Pi$ are *elliptic* differential operators, and if $F \to X_\Pi$ is obtained by finite rank spectral projection, then $F \to X_\Pi$ is a smooth vector bundle. (That it is a *continuous* vector bundle follows from the proof of Proposition D.13.) The proof, which we omit, uses elliptic regularity.

Finally, we prove that differential operators such as (10.3)—elliptic or not—lead to continuous families of Bloch Hamiltonians. Let W be a finite dimensional hermitian vector space, M a smooth manifold, and $\{H_m\}_{m\in M}$ a smooth family of self-adjoint differential operators of fixed (finite) order $k \in \mathbb{Z}^{\ge 0}$ acting on functions $E \to W$. Explicitly, let $x^1, \dots, x^n$ be coordinates on E. Then we can write

$$H_m = \sum_{q=0}^{K} (\sqrt{-1})^q\, a_{i_1,\dots,i_q}(m; x^1, \dots, x^n)\, \frac{\partial^q}{\partial x^{i_1}, \dots, \partial x^{i_q}}, \quad m \in M, \tag{D.29}$$

where $a_{i_1,\dots,i_q}\colon M \times E \to \mathrm{End}(W)$ is smooth and each $a_{i_1,\dots,i_q}(m;x)$ is hermitian. There is an implicit sum over the symmetric multi-index $i_1, \dots, i_q$. Assume H_m is invariant under translation by Π, i.e., $a_{i_1,\dots,i_q}(m; x+\xi) = a_{i_1,\dots,i_q}(m;x)$ for all $m \in M$, $x \in E$, $\xi \in \Pi$. Then for each $\lambda \in X_\Pi$ there is an induced self-adjoint differential operator $H_{m,\lambda}$ on W-valued quasi-periodic functions (D.18), which are sections of the vector bundle $\mathcal{L}_\lambda \otimes W \to E/\Pi$. Let $\mathcal{E} \to M \times X_\Pi$ be the Hilbert bundle with fiber $\mathcal{E}_{m,\lambda} = L^2(E/\Pi; \mathcal{L}_\lambda \otimes W)$.

Proposition D.30. *The family of self-adjoint operators $\{H_{m,\lambda}\}$ is continuous.*

Proof. Relative to the local trivialization (D.26) we view $H_{m,\lambda}$ as acting on

$$D = \{e^{2\pi i\langle k_t - k_0, x - x_0\rangle}\hat{f}_0(x) : \hat{f}_0 \colon E \to W \text{ is smooth with quasi-periodicity } \lambda_0\}. \tag{D.31}$$

Now apply Proposition D.11(ii). □

Appendix E. Twisted K-Theory on Orbifolds

In this appendix, we prove the theorem alluded to in Remark 7.37. It insures that Definition 7.33 reproduces standard twisted K-theory for global quotients $X/\!/G$, where X is a nice compact space and G a finite group.

Untwisted K-theory, equivariant or not, is defined in terms of finite rank vector bundles [1]. It was realized early on (Atiyah-Jänich) that Fredholm operators also provide a model for K-theory, a crucial observation for the Atiyah-Singer index theorem, for example. By contrast, twisted K-theory classes cannot in general be represented by finite rank bundles. For example, if the twisting has infinite order in the abelian group of twistings, then only the zero class has a finite rank representative. Hence models for twisted K-theory are built using Fredholm operators [6,26]. For the application to gapped topological insulators (Sect. 10), specifically for Theorem 10.15 and Theorem 10.21, it is crucial that every twisted equivariant K-theory class over the Brillouin torus have a finite rank representative. This is what we sketch here. We freely use [26, Appendix].

The starting point is [26, Proposition A.37], which asserts that if $\mathcal{G}$ is a compact groupoid which is *locally* of the form $Y/\!/H$ for Y a compact space and H a compact Lie group, then $\mathcal{G}$ admits a universal Hilbert bundle which is a sum of finite rank bundles if and only if the groupoid $\mathcal{G}$ is locally equivalent to a *global* quotient $Y/\!/H$. The standard argument [1, Appendix] that a family of (untwisted) Fredholm operators over a compact space is represented by a finite rank $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle can be carried out on the groupoid $\mathcal{G}$ to prove that if there is a universal Hilbert bundle which is a sum of finite rank bundles, then every K-theory class on $\mathcal{G}$ is represented by a finite rank $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle. For twisted K-theory, we use [26, Lemma 3.11] to reduce to the untwisted case on the centrally extended groupoid. Below we comment on the extra $\mathbb{Z}/2\mathbb{Z}$-twists.

The only new point, then, is the following. Let X be a nice topological space with a continuous action of a finite group G. Set $\mathcal{G} = X/\!/G$. Suppose $L^\tau \to \mathcal{G}_1$ is a central extension, as in Definition 7.23(i). The elements of unit norm $\mathcal{G}_1^\tau \subset L^\tau$ are the morphisms of a groupoid $\mathcal{G}^\tau$ with $\mathcal{G}_0^\tau = \mathcal{G}_0 = X$.

Lemma E.1. *The groupoid $\mathcal{G}^\tau$ is locally equivalent to a global quotient $Y/\!/H$ of a compact space Y by a compact Lie group H.*

Proof. The action map $p\colon \mathcal{G}_1 = X \times G \to X$ has finite fibers, so the pushforward $p_*L^\tau \to X$ is a rank N hermitian vector bundle, where N is the cardinality of G. The fiber at $x \in X$ is

$$\oplus L^{\tau}_{(p \xrightarrow{g} x)} \tag{E.2}$$

where the sum is over the finite set of arrows $(z \xrightarrow{g} x)$ with target x. The bundle p_*L^τ is equivariant for $\mathcal{G}_1^\tau$: the $\mathbb{T}$-torsor of unit norm elements in $L^{\tau}_{(x \xrightarrow{f} y)}$ acts via the multiplication

$$L^{\tau}_{(x \xrightarrow{f} y)} \otimes L^{\tau}_{(z \xrightarrow{g} x)} \longrightarrow L^{\tau}_{(z \xrightarrow{fg} y)} \tag{E.3}$$

in the central extension.

Let $P \to X$ be the U_N-bundle of frames associated to $p_*L^\tau \to X$. It is also $\mathcal{G}^\tau$-equivariant, and furthermore there is a unique arrow in $\mathcal{G}^\tau$ between any two points of P: elements in the central extension $\mathcal{G}_1^\tau$ which cover non-identity elements in the stabilizer $G_x \subset G$ at $x \in X$ permute the lines (2.2) nontrivially. So the groupoid with objects P and arrows $\mathcal{G}^\tau$ is equivalent to a space Y, and the groupoid with objects X and arrows $\mathcal{G}^\tau$ is locally equivalent to the global quotient $Y//U_N$. □

Lemma E.1 leads to a universal twisted Hilbert bundle over $\mathcal{G}$ which is a sum of finite rank bundles. The argument given covers central extensions of $\mathcal{G}$. For ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extensions, as in Definition 7.23(iii), we first apply the argument of Lemma 10.17 to account for a nontrivial homomorphism $\phi\colon \mathcal{G}_1 \to \{\pm 1\}$. A similar argument applies to nontrivial $c\colon \mathcal{G}_1 \to \{\pm 1\}$: replace the complex conjugate Hilbert bundle by the parity-reversed Hilbert bundle.

Appendix F. Diamonds and Dust

In this appendix, we give an illustration of the canonical ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension of Theorem 9.8 for the three-dimensional diamond structure, a common structure in materials science found in many materials, for example, the carbon column of the periodic table: diamond, silicon, germanium, and grey tin. We will describe the pullback of the ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension to the subvarieties of the Brillouin torus X_Π with nontrivial stabilizer group. This appendix is included partially to assist some readers in translating between the language in the body of the text and more standard terminology in condensed matter physics.

In the diamond structure, the lattice $\Pi \subset V = \mathbb{R}^3$ is the face-centered-cubic (fcc) lattice. This may also be viewed as the lattice $D_3 \cong A_3$, the root lattice of the Lie algebra $so(6) \cong su(4)$. Concretely,

$$D_3 = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 \equiv 0 \pmod 2\} \subset \mathbb{R}^3. \tag{F.1}$$

The "reciprocal lattice" lattice is, up to scale, (see below) the weight lattice D_3^*. It is also known as the body-centered-cubic (bcc) lattice. Concretely it is

$$D_3^* = D_3 \cup (D_3 + s) \cup (D_3 + v) \cup (D_3 + s'), \tag{F.2}$$

where

$$\begin{aligned} s &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\ v &= (0, 0, 1) \\ s' &= \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \end{aligned} \tag{F.3}$$

are spinor, vector, and conjugate spinor weights. The diamond crystal is

$$\bar{C} = D_3 \cup (D_3 + s) \subset \mathbb{E}^3. \tag{F.4}$$

Even if we choose a lattice point as an origin (thus identifying the affine space $\mathbb{E}^3$, which contains the crystal $\bar{C}$, with the vector space $V \cong \mathbb{R}^3$, which contains the lattice Π), $\bar{C}$ is not a lattice since $2s \cong v$ in the discriminant group $D_3^*/D_3 \cong \mathbb{Z}/4\mathbb{Z}$. It is useful to regard $2\bar{C}$ as the set of points $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with $x_i \equiv x_j \pmod 2$ and $\sum_i x_i \equiv 0, 1 \pmod 4$.

In nature, the fcc lattice D_3 comes scaled by $a/2$ where the fcc lattice is constructed from a cubic lattice of side length a and we add lattice points at the centers of the sides of the cubes. In condensed matter, the "reciprocal lattice" which is in the (Fourier) dual space V^* is then scaled by $2/a$. The condensed matter convention is to take $\sum n^j k_j \in 2\pi\mathbb{Z}$ where $(n^1, n^2, n^3) \in \Pi$ is in the space lattice and $k = (k_1, k_2, k_3)$ is in the reciprocal lattice. Therefore, in comparing to the condensed matter literature we should scale D_3^* by $\frac{2}{a} \times 2\pi$. For simplicity, we will work with $a = 2$, and we define our reciprocal vectors without the factor of 2π. Thus, for us the Brillouin torus X_Π is $\mathbb{R}^3/D_3^*$. A standard fundamental domain is the Wigner–Seitz domain whose closure is given by points in $\mathbb{R}^3$ closest to the lattice point 0 (in the lattice D_3^*). In mathematics, this is known as the Voronoi cell. The notation shown here is standard notation going back to the classic paper analyzing the irreducible representations of the space group in [10].

Another useful way to think about the Brillouin torus is as the quotient of points $(k_1, k_2, k_3) \in \mathbb{R}^3$ by the equivalence relation $k_i \sim k_i + 1$ *and* $k \sim k + s$. Thus, the Brillouin torus is a quotient of the standard torus $\mathbb{R}^3/\mathbb{Z}^3$ by a free $\mathbb{Z}/2\mathbb{Z}$-action. In this example, $\phi = t = c = 1$. The extension (2.18) for the crystallographic group of the fcc lattice splits, but that for the diamond structure is not split, i.e., it is nonsymmorphic. In both cases, we have the sequence:

$$1 \to D_3 \to G(\bar{C}) \to \mathbb{Z}/2\mathbb{Z} \times O_h \to 1 \tag{F.5}$$

where $G(\bar{C}) := G(C)/U$, the extended point group is $\mathbb{Z}/2\mathbb{Z} \times O_h$ and the point group is $P = O_h$, the cubic group. The latter can be regarded as $S_4 \times \mathbb{Z}/2\mathbb{Z}$, where S_4 is the Weyl group of $so(6) = su(4)$ and the $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^3$ by inversion $x \to -x$. For our purposes, a more useful characterization is $O_h \cong (\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$ where S_3 acts on (x_1, x_2, x_3) by permutation and the $(\mathbb{Z}/2\mathbb{Z})^3$ subgroup is generated by $\epsilon_1, \epsilon_2, \epsilon_3$, where ϵ_i flips the sign of x_i, leaving the other coordinates fixed.

It is now convenient to introduce the Seitz notation for elements of the Euclidean group (the group $Euc(E)$ of Sect. 2): Again we choose an origin to identify $\mathbb{E}^d \cong \mathbb{R}^d$ and then define $\{R|v\}$ by

$$\{R|v\}x := Rx + v, \tag{F.6}$$

where Rx means $R^i{}_j x^j, R \in O(d)$.

To see that the space group of the diamond structure is nonsymmorphic (i.e., (F.5) does not split) we note that 24 elements of O_h preserve $(\sum_i 2x_i)$ mod 4, and hence can be lifted $R \to \{R|0\}$ in the space group of the diamond crystal, but the remaining 24 elements shift $(\sum_i 2x_i) \to [(\sum_i 2x_i) - 1] \mod 4$ and hence require a shift by an element of $s + D_3$ to preserve the crystal $\bar{C}$. We choose to lift such elements R to the screw-displacement $\{R|s\}$.

Let us now describe the groupoid $X_\Pi//O_h$. The action of the point group O_h on the Brillouin torus is defined by

$$(\{R|v\} \cdot \lambda)(\xi) := \lambda\left(\{R|v\}\{1|\xi\}\{R|v\}^{-1}\right) = \lambda(R\xi) \tag{F.7}$$

where $\lambda \in X_\Pi$ is a character of Π and $\xi \in \Pi$. It is expressed (nonuniquely!) in terms of more standard k-vectors $k = (k_1, k_2, k_3) \in V^* \cong \mathbb{R}^3$ by $\lambda(\xi) = \exp[2\pi i \sum_{j=1}^3 k_j \xi^j]$. The O_h action on X_Π has nontrivial stabilizer groups at special points, circles, and (two-dimensional) tori. These subvarieties with enhanced symmetry form orbits under the O_h action. The nontrivial submanifolds, together with their stabilizer groups may be summarized by the following table:

Typical character k	Stabilizer group $G''(\lambda)$
(y, y, y')	$\mathbb{Z}/2\mathbb{Z}$
$(y, y', 0)$	$\mathbb{Z}/2\mathbb{Z}$
$(y, 0, 0)$	D_4
$(y, 0, 1/2)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$(y, y, 0)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
(y, y, y)	S_3
$(1/4, 1/4, 1/4)$	$\mathbb{Z}/2\mathbb{Z} \times S_3$
$(1/2, 0, 0)$	$\mathbb{Z}/2\mathbb{Z} \times D_4$
$(0, 0, 0)$	O_h

The notation here is as follows: y denotes a generic real number and we identify $y \sim y + 1$. A prime indicates that the number y' is different from y modulo 1. For each example of fixed point loci shown, there are others obtained by the action of the point group on X_Π. For example, the 2-torus obtained by the closure of points of type (y, y, y') can be mapped to a 2-torus of points $(\bar{y}, y', y)$. Similarly, the point $(1/2, 0, 0)$ sits in an orbit of three points, the other two being represented by $(0, 1/2, 0)$ and $(0, 0, 1/2)$. These three points are known as X-points in the notation of [10]. The other special points are the four L-points: $(1/4, 1/4, 1/4)$ and an orbit of $(-1/4, 1/4, 1/4)$ under the point group, together with the trivial character $k = 0$. The trivial character is usually denoted as the Γ-point in the condensed matter literature.

We can detect if the canonical ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension is nontrivial by pulling back to these special loci and determining whether the central extension of (9.32) is trivializable or not. A case-by-case analysis reveals that the nontrivial central extensions occur only on the three circles with generic points $(y, 0, 1/2), (0, y, 1/2)$, and $(0, 1/2, y)$. For example, the point $(y, 0, 1/2)$, with y generic has stabilizer group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by ϵ_2 and ϵ_3. Their lifts to $G(\bar{C})$ are $\{\epsilon_2|s\}$ and $\{\epsilon_3|s\}$, whose group commutator is

$$[\{\epsilon_2|s\}, \{\epsilon_3|s\}] = \{1|\xi\} \tag{F.8}$$

with $\xi = (0, -1, 1)$, and for this lattice vector $\lambda(\xi) = -1$. So, by the same reasoning as in Example 9.34 the extension $H(\lambda)$ of $G''(\lambda)$ is nontrivial. At the enhanced symmetry points X, which lie on these three circles the stabilizer group is $\mathbb{Z}/2\mathbb{Z} \times D_4$ and the central extension remains nontrivial. For example, for the point $(1/2, 0, 0)$, the commutator function (that is, the commutator of the lift of elements to $H(\lambda)$), which can be shown to characterize the central extension up to isomorphism, is nontrivial for the pairs $s(\epsilon_1, \epsilon_2) = s(\epsilon_1, \epsilon_3) = -1$, (and trivial on other pairs).

There are also examples of materials in nature with nontrivial time-reversal symmetries leading to nontrivial extensions of the Brillouin groupoid, such as those described in Theorem 9.42. An example is probably provided by manganese dioxide in its rutile structure.[37] The space group (number 136) is nonsymmorphic, with a point group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times D_4$. Each manganese atom is surrounded by a regular octahedron of oxygen atoms. There are unpaired d-electrons on the manganese atoms which provide local magnetic moments and these moments can be staggered so that time reversal must be accompanied by a nontrivial element of the space group. Both sequences (2.18) and (2.19) are nonsplit. We leave a detailed analysis of the ϕ-twisted $\mathbb{Z}/2\mathbb{Z}$-graded extension and equivariant K-theory for another occasion.

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