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Oscillatory Singularities in Bianchi Models with Magnetic Fields

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Abstract. An idea which has been around in general relativity for more than 40 years is that in the approach to a big bang singularity solutions of the Einstein equations can be approximated by the Kasner map, which describes a succession of Kasner epochs. This is already a highly non-trivial statement in the spatially homogeneous case. There the Einstein equations reduce to ordinary differential equations and it becomes a statement that the solutions of the Einstein equations can be approximated by heteroclinic chains of the corresponding dynamical system. For a long time, progress on proving a statement of this kind rigorously was very slow but recently there has been new progress in this area, particularly in the case of the vacuum Einstein equations. In this paper we generalize some of these results to cases where the Einstein equations are coupled to matter fields, focussing on the example of a dynamical system arising from the Einstein–Maxwell equations with symmetry of Bianchi type VI_0. It turns out that this requires new techniques since certain eigenvalues are in a less favourable configuration than in the vacuum case. The difficulties which arise in that case are overcome by using the fact that the dynamical system of interest is of geometrical origin and thus has useful invariant manifolds.

1. Introduction

The fundamental equations of general relativity are the Einstein equations, possibly coupled to other equations describing the dynamics of the matter which generates the gravitational field. With a suitable choice of physical units the equations are

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}. \tag{1.1}$$

The unknowns in these equations are the spacetime metric $g_{\alpha\beta}$ and the matter fields. $R_{\alpha\beta}$ is the Ricci tensor of the Lorentzian metric $g_{\alpha\beta}$ and R its trace.

$T_{\alpha\beta}$ is the energy-momentum tensor. In this paper we are mainly concerned with the Einstein vacuum equations, where $T_{\alpha\beta} = 0$, and the Einstein–Maxwell equations. In the latter case, the source of the gravitational field is an electromagnetic field $F_{\alpha\beta}$ and the energy-momentum tensor is given by

$$T_{\alpha\beta} \;=\; F_\alpha{}^\gamma F_{\beta\gamma} - \frac{1}{4}(F^{\gamma\delta}F_{\gamma\delta})g_{\alpha\beta}. \tag{1.2}$$

The electromagnetic field tensor is antisymmetric ($F_{\alpha\beta} = -F_{\beta\alpha}$) and satisfies the source-free Maxwell equations

$$\nabla^\alpha F_{\alpha\beta} \;=\; 0, \qquad \nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} \;=\; 0. \tag{1.3}$$

It is well known that solutions of the Einstein equations generally develop singularities. In particular, there are solutions relevant to cosmology in which the singularity corresponds to the big bang. Belinskii, Khalatnikov and Lifshitz (hereafter abbreviated to BKL) developed a heuristic picture of the singularities in cosmological solutions of the Einstein equations. In this context they introduced a map of the circle to itself which we refer to as the Kasner map. It will be defined precisely below. They suggested that it provides a model for oscillations of the geometry in the approach to the singularity. For the original work see [2] and [3]. A modern discussion of these ideas can be found in [4]. An important idea in the BKL work is that spatially inhomogeneous solutions of the Einstein equations can be approximated by spatially homogeneous solutions near the singularity. Since from a mathematical point of view the dynamics of spatially homogeneous solutions is still far from being understood, it is natural at the present time to concentrate on understanding classes of spatially homogeneous solutions. This is the strategy we adopt in what follows.

A long-standing question in mathematical cosmology is to relate the Kasner map to the dynamics of actual solutions of the Einstein equations, possibly with matter. An important recent advance in this field is the paper [7] where a relation of this kind was established in a special case. These results concern solutions of the vacuum Einstein equations of Bianchi types VIII and IX. They complement earlier results of Ringström [10,11] by providing a more detailed description of the dynamics of the approach to the singularity in certain cases. The work of Ringström on vacuum spacetimes was preceded by results of Weaver [15] on solutions of the Einstein–Maxwell equations of Bianchi type VI_0 using a dynamical system introduced in [6]. In what follows we extend the results of [7] to this case of the Einstein–Maxwell equations. There is other recent work on this question in the vacuum case [1,8], but these papers use very different techniques from those which we will apply to the Einstein–Maxwell case and for this reason they will not be discussed further here.

In the next section the necessary background and the fundamental equations needed in the paper are introduced. The most important similarities and differences between the models with magnetic fields considered in what follows and the vacuum models which had previously been analysed are explained. The third section contains the main theorem and an exposition of the strategy of its proof. The central result is the existence of unstable manifolds of codimension one for some heteroclinic chains. To prove this it is necessary to obtain estimates for a solution during its passages close to the Kasner circle and for its behaviour between passages. This is done in Sects. 4 and 5, respectively. A central idea of the paper and one which is a major step beyond what was achieved in the vacuum case is the use of a specially constructed Riemannian metric to measure the distance between the heteroclinic chains and the approximating smooth solutions. The last section discusses future extensions of this research and interesting open problems.

2. The Basic Set-Up

Spatially homogeneous spacetimes are those solutions of the Einstein-matter equations where there is an action of a Lie group G by isometries of $g_{\alpha\beta}$ with three-dimensional spacelike orbits which leaves the matter fields invariant. The cases where the isotropy group is discrete can be classified according to the Lie algebra of G. It is common in general relativity to use the terminology due to Bianchi, who introduced types I to IX. It is also common to distinguish between two subsets of these types known as Class A and Class B. In what follows we will only be concerned with Class A models. More information on this subject can be found in [13] or [9].

The analyses of vacuum spacetimes mentioned above are based on the well-known Wainwright–Hsu system [14]. This is a system of ordinary differential equations for five variables $(\Sigma_+, \Sigma_-, N_1, N_2, N_3)$ which are subject to one constraint. It includes all the Bianchi models of Class A (i.e. types I, II, VI_0, VII_0,VIII and IX). The system is defined on a smooth hypersurface in $\mathbb{R}^5$. An analogous system for Bianchi spacetimes of type VI_0 with a magnetic field was introduced in [6]. It is also defined on a smooth hypersurface in $\mathbb{R}^5$ and it includes solutions of types I and II with a magnetic field. The variables are called $\Sigma_+, \Sigma_-, N_+, N_-, H$. The first two variables can be identified with the variables of the same name in the vacuum case since they have the same geometrical meaning in both cases. The variables N_+ and N_- correspond in a similar way to certain linear combinations of N_2 and N_3. More specifically, $N_+ = \frac{3}{2}(N_2 + N_3)$ and $N_- = \frac{\sqrt{3}}{2}(N_2 - N_3)$. The variable H corresponds to the magnetic field.

The dynamical system is

$$\begin{aligned}
\Sigma_+' &= -2N_-^2(1+\Sigma_+) + \frac{3}{2}H^2(2-\Sigma_+),\\
\Sigma_-' &= -\left(2N_-^2 + \frac{3}{2}H^2\right)\Sigma_- - 2N_+N_-,\\
N_+' &= \left(2\Sigma_+(1+\Sigma_+) + 2\Sigma_-^2 + \frac{3}{2}H^2\right)N_+ + 6\Sigma_-N_-,\\
N_-' &= \left(2\Sigma_+(1+\Sigma_+) + 2\Sigma_-^2 + \frac{3}{2}H^2\right)N_- + 2\Sigma_-N_+,\\
H' &= -(\Sigma_+(2-\Sigma_+) - \Sigma_-^2 + N_-^2)H.
\end{aligned} \tag{2.1}$$

The prime denotes a derivative with respect to a time variable τ which tends to $-\infty$ as the singularity is approached. These equations are taken from [15]. They arise as a special case of the equations for models with a magnetic field and a perfect fluid derived in [6] by setting the fluid density to zero. (Here a magnetic field means an electromagnetic field satisfying the condition that $F_{\alpha\beta}n^{\beta} = 0$, where n^{α} is the unit normal vector to the group orbits.) Solutions are considered which satisfy the condition

$$\Sigma_+^2 + \Sigma_-^2 + N_-^2 + \frac{3}{2}H^2 \;=\; 1. \tag{2.2}$$

This condition follows from the Einstein equations and is preserved by the evolution equations for $(\Sigma_+, \Sigma_-, N_+, N_-, H)$ just defined. The inequalities $N_- > 0$, $N_+^2 < 3N_-^2$ and $H > 0$ are assumed. These are also preserved by the evolution and define the region which corresponds to Bianchi type VI_0 solutions with non-zero magnetic field. Setting $H = 0$ while maintaining the other two inequalities gives a representation of the vacuum solutions of Bianchi type VI_0. Setting $N_+ = \pm\sqrt{3}N_-$ gives two different representations of solutions of type II with a magnetic field. Setting other combinations to zero leads to vacuum solutions of type II, solutions of type I with a magnetic field, and vacuum solutions of type I (the Kasner solutions). Note the invariant subspaces $\{N_2 = 0\}$ and $\{N_3 = 0\}$ which also appear in the Bianchi system with perfect fluid. In fact the invariant subspaces $\{N_2 = 0\}$, $\{N_3 = 0\}$, $\{H = 0\}$ will play a crucial role in our analysis, see (3.4).

The circle defined by $\Sigma_+^2 + \Sigma_-^2 = 1$ consists of stationary points. Each one of them corresponds to a Kasner solution and so this set is called the Kasner circle. There are three families of heteroclinic orbits between points on the Kasner circle whose projections to the (Σ_+, Σ_-)-plane are straight lines. Two of these families correspond to vacuum solutions of Bianchi type II and occur in both the vacuum case and the case with magnetic field. In the vacuum case there is a third family related to these two by symmetries of the system. In the case where a magnetic field is included, the two sets of Bianchi type II vacuum solutions are complemented by a family of Bianchi type I solutions with magnetic field. The projections of the latter to the (Σ_+, Σ_-)-plane are identical to those of the third family of Bianchi type II solutions in the vacuum case. There is thus a natural correspondence between heteroclinic chains consisting of Bianchi type II solutions in the vacuum case and heteroclinic chains in the case with a magnetic field which include orbits corresponding to both solutions of the vacuum Einstein equations of Bianchi type II and solutions of the Einstein–Maxwell equations of Bianchi type I. In the vacuum case there is a heteroclinic cycle consisting of three orbits and it is the central example considered in [7]. The projections of the orbits making up this cycle to the (Σ_+, Σ_-)-plane are related by rotations by multiples of $\frac{2\pi}{3}$. By what has already been said, there is a corresponding heteroclinic cycle in the system of [6]. See also Fig. 1. The Kasner solutions can be written in the explicit form

$$-\,dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2. \tag{2.3}$$

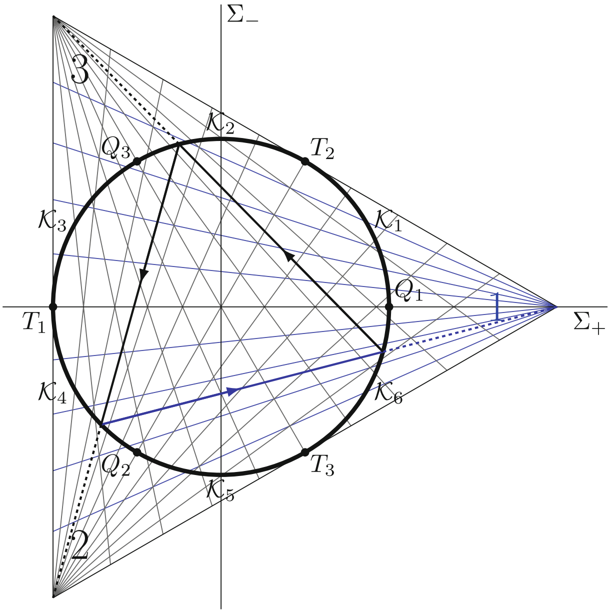

FIGURE 1. Kasner circle and period 3 cycle; Bianchi type II vacuum families to corners 2,3 (*in black*), Bianchi type I family with magnetic field to corner 1 (*in blue*); Kasner intervals $\mathcal{K}_i$ bounded by Taub points T_i of tangencies and antipodal points Q_i (colour figure online)

With a suitable choice of ordering the Kasner exponents p_1, p_2, p_3 are related to the variables Σ_+ and Σ_- by

$$\begin{aligned} p_1 &= \frac{1}{3}(1 - 2\Sigma_+), \\ p_2 &= \frac{1}{3}(1 + \Sigma_+ + \sqrt{3}\Sigma_-), \\ p_3 &= \frac{1}{3}(1 + \Sigma_+ - \sqrt{3}\Sigma_-). \end{aligned} \tag{2.4}$$

The eigenvalues of the linearisation of the Wainwright–Hsu system at a Kasner solution are $(6p_1, 6p_2, 6p_3)$ ([14], p. 1425). For the system describing solutions of the Einstein–Maxwell equations of Bianchi type VI_0 introduced in [6] the eigenvalues are $(3p_1, 6p_2, 6p_3)$ ([6], (4.1)). The Taub points of the Kasner circle are defined by the condition that the Kasner exponents are $(1, 0, 0)$ or a permutation thereof. All points of the Kasner circle other than the Taub points have a one-dimensional stable manifold. For $p_1 < 0$ the stable manifold is defined by a solution with non-vanishing magnetic field and the corresponding eigenvalue is $3p_1 < 0$. For $p_2 < 0$ or $p_3 < 0$ the stable manifold is defined by a vacuum solution.

In order to have an overview of the relative sizes of the different eigenvalues it is useful to introduce the Kasner parameter $u \in [1, \infty]$ which is defined implicitly by the relation (cf. [4])

$$p_1 p_2 p_3 = \frac{-u^2(1+u)^2}{(1+u+u^2)^3}. \tag{2.5}$$

Then the Kasner exponents arranged in ascending order are given by

$$\begin{aligned} \tilde{p}_1 &= \frac{-u}{1+u+u^2}, \\ \tilde{p}_2 &= \frac{1+u}{1+u+u^2}, \\ \tilde{p}_3 &= \frac{u(1+u)}{1+u+u^2}. \end{aligned} \tag{2.6}$$

Note that $\tilde{p}_2 \geq -\tilde{p}_1$ with equality only when $u = \infty$. On the other hand $\frac{\tilde{p}_2}{2} \leq -\tilde{p}_1$ and $\frac{\tilde{p}_3}{2} \geq -\tilde{p}_1$. The Kasner map is defined by $u \mapsto u - 1$ for $u \geq 2$ and $u \mapsto (u-1)^{-1}$ for $u \leq 2$. The central example in [7] is a set of heteroclinic orbits which form a cycle of order three. In fact the value of u corresponding to that example is invariant under the Kasner map. It solves the equation $u^2 - u - 1 = 0$ and is the golden ratio $\frac{1}{2}(1+\sqrt{5})$. The values of the Kasner exponents at the vertex of this cycle where $p_1 < p_2 < p_3$ are

$$\begin{aligned} p_1 &= \frac{1}{4}(1-\sqrt{5}), \\ p_2 &= \frac{1}{2}, \\ p_3 &= \frac{1}{4}(1+\sqrt{5}). \end{aligned} \tag{2.7}$$

For each value of the Kasner parameter u in the interval $(1, \infty)$ there are six points on the Kasner circle where u takes that value. Removing the Taub points T_i and their antipodal points Q_i from the Kasner circle leaves a union of six intervals K_i, $1 \leq i \leq 6$. They will be numbered as follows: Let K_1 be the region where $p_1 < 0$ and $p_2 > p_3$. Then number the others consecutively while moving anticlockwise along the Kasner circle, see Fig. 1.

In order to assess the stability of a heteroclinic cycle it is important to examine the eigenvalues of the linearisation of the system at the vertices. In both the vacuum case and the case with magnetic field there is one negative eigenvalue $-\mu_1$ and two positive eigenvalues μ_2, μ_3. Without loss of generality the labelling can be chosen so that $\mu_2 \leq \mu_3$. Then from what has been stated above it can be seen that in the vacuum case the inequalities $\mu_1 < \mu_2 < \mu_3$ hold at any point of the Kasner circle except T_i and Q_i. Call this the first linearisation condition. That this is true is one of the most important hypotheses of the main theorem of [7]. On the other hand this condition can fail in the case with magnetic field. It fails precisely when the Kasner exponent p_1 is intermediate in size between p_2 and p_3, i.e. when the point of the Kasner circle lies in the one of the sets K_2 and K_5. For then the eigenvalue $3p_1$ is smaller in magnitude than that of the negative eigenvalue, while the other positive eigenvalue is not. Then we have the situation that $\mu_2 < \mu_1 < \mu_3$. The situation that the eigenvalue $3p_1$ does not correspond to the eigendirection tangent to the heteroclinic orbit incoming towards the past is covered by the theorems in this paper. Call this the second linearisation condition. What is common to the first and second linearisation conditions is that the eigenvalue corresponding to the heteroclinic orbit incoming towards the past is larger in modulus than that corresponding to the heteroclinic orbit outgoing towards the past. In the example of the 3-cycle at least one of the two linearisation conditions just introduced holds at each of the vertices. See Fig. 1, the second eigenvalue condition holds in the intervals $\mathcal{K}_2$ and $\mathcal{K}_5$, whereas the first eigenvalue condition holds in the remaining intervals.

The linearisation conditions are not in themselves enough to make the theorems in this paper work. Additional geometrical information is required. This is the existence of a certain invariant manifold. It is tangent to the space spanned by the vectors tangent to the stable manifold and the centre manifold and the eigenvector corresponding to the largest eigenvalue. For a general dynamical system there is no reason why a manifold of this kind should exist. In the example of a Bianchi model of type VI_0 with magnetic field a manifold of this kind is defined by the vacuum solutions of type VI_0 or the solutions of type II with magnetic field.

Note that multiplying the last equation in (2.1) by H shows that (2.1) can be rewritten as a smooth dynamical system in the variables Σ_+, Σ_-, N_+, N_-, H^2. Call this the transformed system. From a physical point of view this corresponds to replacing the one independent component of the Maxwell field in this situation by the one independent component of the energy-momentum tensor as a basic variable. In the transformed system the problematic eigenvalue $3p_1$ is replaced by the value $6p_1$ familiar from the vacuum case. Thus in the context of the latter system the stability of the heteroclinic cycle can be analysed using the techniques of [7]. We have nevertheless chosen to present our discussion in terms of the original system (2.1) for the following reasons. The proofs of Sects. 3, 4 and 5 are formulated in a general setting which is intended to be applicable to a variety of Einstein-matter systems. It is to be expected that for most homogeneous spacetimes with matter the possibility of simplification by a clever change of variables will not be available. Usually, the matter fields and their equations of motion contain more information than can be encoded in the energy-momentum tensor alone. The example of the system (2.1) is convenient for illustrating how the new techniques developed in what follows can be applied to a dynamical system arising in general relativity. The role of the invariant manifolds highlights the special properties of this type of system among more general dynamical systems containing heteroclinic chains.

3. Main Result and Sketch of Proof

We shall prove the following result on the dynamics of the Bianchi model of type VI_0 (2.1, 2.2) with magnetic field:

Theorem 3.1. *The period 3 heteroclinic cycle given by* (2.7) *possesses a local codimension-one unstable manifold. In other words, system* (2.1, 2.2) *admits a codimension-one manifold, locally close to the heteroclinic cycle, of initial conditions whose backward trajectories converge to the heteroclinic cycle. The manifold is locally Lipschitz continuous in the open complement of the boundaries $N_1 = 0$, $N_2 = 0$, $H = 0$. It is Lipschitz continuous in every closed cone intersecting these boundaries only in the heteroclinic cycle.*

As pointed out at the end of the last section, passing to the transformed system would allow this result to be proved using the techniques of [7]. This can even be used to show that the invariant manifold is globally Lipschitz. Here the theorem will be proved using the original system since this illustrates the use of new techniques of wider applicability. The proof will only use certain properties of the particular structure of the Bianchi system (2.1) and can be sketched as follows:

Step 1: local passage, Sect. 4. In a neighbourhood of the equilibria of the heteroclinic cycle, i.e. close to the Kasner circle, the Bianchi system (2.1, 2.2) with reversed time direction can be smoothly transformed to a vector field

$$\dot{x} = f(x), \quad x = (x_{\mathrm{u}}, x_{\mathrm{ss}}, x_{\mathrm{s}}, x_{\mathrm{c}}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R},$$

that satisfies the following properties:

Conditions 3.2. **(loc-i)** *There is a straight line of equilibria,*

$$f(0, 0, 0, x_{\mathrm{c}}) \equiv 0. \tag{3.1}$$

(loc-ii). *The heteroclinic orbits of the original system correspond to the x_{ss}- and x_{u}-axes.*

(loc-iii). *The linearisation at the origin has the almost diagonal form*

$$Df(0) = \begin{pmatrix} \mu_{\mathrm{u}} & & & & & \\ & -\mu_{\mathrm{ss}} & & & & \\ & & -\mu_{\mathrm{s1}} & & & \\ & & & \ddots & & \\ & & & & -\mu_{\mathrm{sN}} & \\ * & * & * & \cdots & * & 0 \end{pmatrix}, \tag{3.2}$$

with $\mu_{\mathrm{u}}, \mu_{\mathrm{ss}}, \mu_{\mathrm{s1}}, \ldots, \mu_{\mathrm{sN}} > 0$.

(loc-iv). *The eigenvalue corresponding to the incoming direction is stronger than the eigenvalue corresponding to the outgoing direction,*

$$\mu_{\mathrm{u}}/\mu_{\mathrm{ss}} < 1. \tag{3.3}$$

(loc-v). *The codimension-one subspaces*

$$\{x_{\mathrm{u}} = 0\}, \quad \{x_{\mathrm{ss}} = 0\}, \quad \{x_{\mathrm{s1}} = 0\}, \ldots, \{x_{\mathrm{sN}} = 0\} \tag{3.4}$$

are invariant under the flow.

Note that, for the model (2.1, 2.2), we have $N = 1$. However, the generalization $N > 1$ is needed for applications discussed at the end of this section and in Sect. 6.

The last two properties are the most crucial for the proof. The invariance of the codimension-one subspaces is non-generic for systems admitting (loc-i)–(loc-iv) and is a very strong constraint on the system.

Under the above assumptions the local passage map, $\Psi^{\text{loc}} : \Sigma^{\text{in}} \to \Sigma^{\text{out}}$, from an in-section $\Sigma^{\text{in}} = \{x_{\text{ss}} = \varepsilon\}$ to an out-section $\Sigma^{\text{out}} = \{x_{\text{u}} = \varepsilon\}$, for sufficiently small ε, is a Lipschitz-continuous map with arbitrarily little change of the x_{c}-component and arbitrarily strong contraction transverse to the x_{c}-component. Unfortunately, this only holds true with respect to a non-Euclidean metric and represents one of the main difficulties of our investigation. Indeed, for $\mu_{\text{s1}} < \mu_{\text{u}}$ even a linear vector field f never gives rise to a Lipschitz-continuous map Ψ^{loc} with respect to the Euclidean metric. The case $\mu_{\text{u}} < \mu_{\text{ss}}, \mu_{\text{s1}}, \ldots \mu_{\text{sN}}$, on the other hand, does yield a Lipschitz-continuous map Ψ^{loc} and has been treated in [7].

Step 2: global excursion, Sect. 5. Close to the heteroclinic chain, by smooth dependence on initial conditions, the trajectories follow the heteroclinic orbit from the out-section of a local passage to the in-section of the next local passage. This map, $\Psi_k^{\text{glob}} : \Sigma_k^{\text{out}} \to \Sigma_{k+1}^{\text{in}}$, is a uniformly bounded diffeomorphism. In particular, any deformation imposed by Ψ^{glob} in directions transverse to x_{c} will turn out to be dominated by the strong contraction of the local passage map Ψ^{loc}. In x_{c}-direction, however, we gain an expansion given by the Kasner map. Thus, the global excursion $\Psi_k^{\text{glob}} : \Sigma_k^{\text{out}} \to \Sigma_{k+1}^{\text{in}}$ given by the Bianchi system (2.1, 2.2) with reversed time direction satisfies

Conditions 3.3. **(glob-i).** Ψ_k^{glob} *maps the origin of* Σ_k^{out} *to the origin of* Σ_{k+1}^{in} *and (local neighbourhoods of* 0 *of) the invariant subspaces* $\{x_{\text{ss}} = 0\}$, $\{x_{\text{s1}} = 0\}, \ldots, \{x_{\text{sN}} = 0\}$ *onto* $\{x_{\text{u}} = 0\}$, $\{x_{\text{s1}} = 0\}, \ldots, \{x_{\text{sN}} = 0\}$ *(in arbitrary order).*

(glob-ii). Ψ_k^{glob} *is a* $\mathcal{C}^2$ *Diffeomorphism. The bounds* $\|D\Psi_k^{\text{glob}}\|$, $\|D^2\Psi_k^{\text{glob}}\|$, $\|D(\Psi_k^{\text{glob}})^{-1}\|$, $\|D^2(\Psi_k^{\text{glob}})^{-1}\| < M$ *are independent of* k.

(glob-iii). *It uniformly expands in* x_{c}*-direction at the boundary. In other words,* $\Psi_k^{\text{glob}} : \{x_{\text{ss}}^{\text{out}} = 0, x_{\text{s}}^{\text{out}} = 0\} \to \{x_{\text{u}}^{\text{in}} = 0, x_{\text{s}}^{\text{in}} = 0\}$ *is Lipschitz continuous, and its inverse* $(\Psi_k^{\text{glob}}|_{\{x_{\text{ss}}^{\text{out}}=0, x_{\text{s}}^{\text{out}}=0\}})^{-1}$ *has Lipschitz constant less than* $L < 1$*, independent of* k.

Step 3: graph transform, Sect. 5. Combining local passage and global excursion yields maps from each in-section to the next, $\Psi = \Psi_k^{\text{glob}} \circ \Psi_k^{\text{loc}} : \Sigma_k^{\text{in}} \to \Sigma_{k+1}^{\text{in}}$, with uniform cone conditions. A standard graph-transform technique now yields the claimed invariant manifold as a fixed point in the space of Lipschitz-continuous graphs $x_{\text{c}}^k = x_{\text{c}}^k(x_{\text{u}}^k, x_{\text{s}}^k)$ in Σ_k^{in}. For completeness of presentation, we will give the necessary arguments in Sect. 5.

In fact, steps 1–3 prove a much more general theorem than 3.1, that is

Theorem 3.4. *Let a $\mathcal{C}^4$ vector field and a chain of heteroclinic orbits $h^k(t)$,*

$$\lim_{t\to\infty} h^{k-1}(t) = p^k = \lim_{t\to-\infty} h^k(t), \qquad k \in \mathbb{N},$$

be given. Assume that locally near p^k assumptions (loc-i)–(loc-v) hold and that along each h^k assumptions (glob-i)–(glob-iii) hold, with constants α, L, M independent of k.

Then there exists a local codimension-one stable manifold to the heteroclinic chain, i.e. a codimension-one manifold of initial conditions following the heteroclinic chain and converging to it.

The heteroclinic chain itself is contained in the boundary of this manifold. The manifold is locally Lipschitz continuous in the open complement of the invariant subspaces (loc-v). The manifold is uniformly Lipschitz continuous in every closed cone intersecting the invariant subspaces only in the heteroclinic chain itself.

This theorem covers not only the period 3 cycle of the Bianchi VI_0 system with magnetic field. In fact, it applies to every heteroclinic chain in the Bianchi VI_0 system with magnetic field that does not accumulate at any Taub point (required for uniformity of bounds) and such that the chain does not contain heteroclinic orbits of the magnetic family to points in the domain $K_2 \cup K_5$. See also Fig. 1. It also applies to every heteroclinic chain of the Bianchi A (VIII and IX) system without magnetic field but with ideal fluid as investigated in [7], as long as it does not accumulate at any Taub point. The proof given here completes the arguments sketched in the discussion section of [7] and relaxes the constraint on the matter model required there. All matter models that yield positive eigenvalues of the linearisation at the Kasner circle in the non-vacuum direction are included, due to the relaxed eigenvalue condition. In particular, as discussed in more detail in Sect. 6, it includes perfect fluids with equations of state which could not be treated by the methods of [7] and for which no method of reduction to that case is known. It is possible to remove the need to exclude the subset $K_2 \cup K_5$ of the Kasner circle using the modified system. This follows from results sketched in [7] and proved in detail in Sect. 5.

4. Local Passage Near a Line of Equilibria

In this section we study the passage of trajectories under a general flow near a line of equilibria with eigenvalue constraint (3.3) and invariant subspaces (3.4) consistent with the Kasner circle in the Bianchi VI_0 system with magnetic field. We will collect estimates on expansion and contraction rates to establish Lipschitz properties of the local map between sections to a reference orbit given by the passage near the line of equilibria, see Theorem 4.8 at the end of this section. Compared with [7] (Section 3), we assume the relaxed eigenvalue condition (3.3) without any constraint on $\mu_{\mathrm{s}1}, \ldots, \mu_{\mathrm{sN}}$. This requires the use of a non-Euclidean metric (4.21, 4.22).

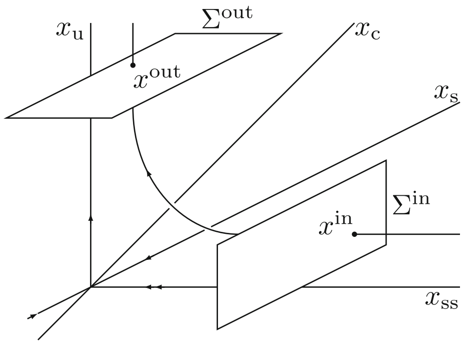

FIGURE 2. Local passage $\Psi^{loc} : \Sigma^{in} \to \Sigma^{out}$

Consider a $\mathcal{C}^k$ vector field, $k \geq 4$,

$$\dot{x} = f(x), \quad x = (x_u, x_{ss}, x_s, x_c) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}, \tag{4.1}$$

that satisfies Conditions 3.2 in a neighbourhood of the origin. Due to the invariant subspaces (3.4), the form of the linearisation (3.2) holds locally all along the line of equilibria,

$$Df(0,0,0,x_c) = \begin{pmatrix} \mu_u(x_c) & & & & & \\ & -\mu_{ss}(x_c) & & & & \\ & & -\mu_{s1(x_c)} & & & \\ & & & \ddots & & \\ & & & & -\mu_{sN}(x_c) & \\ * & * & * & \cdots & * & 0 \end{pmatrix}. \tag{4.2}$$

The stable and unstable manifolds as well as the strong stable foliation of the stable manifold are $\mathcal{C}^k$ and can be flattened, see e.g. [12] (Theorem 5.8). By a $\mathcal{C}^k$ change of coordinates the stable/strong stable/unstable manifolds to the equilibria locally coincide with the respective eigenspaces. In particular, and in addition to (3.4), the following stable and unstable fibres become invariant:

$$\begin{aligned} W^u(x_c) &= \{x_{ss} = 0, x_s = 0, x_c \text{ fixed}\}, \\ W^s(x_c) &= \{x_u = 0, x_c \text{ fixed}\}. \end{aligned} \tag{4.3}$$

Note that in the Bianchi system, $W^u(x_c)$ coincides with the outgoing heteroclinic orbit attached to the equilibrium $(0,0,0,x_c)$.

Due to (4.3), the linearisation becomes diagonal,

$$Df(0,0,0,x_c) = \begin{pmatrix} \mu_u(x_c) & & & & & \\ & -\mu_{ss}(x_c) & & & & \\ & & -\mu_{s1(x_c)} & & & \\ & & & \ddots & & \\ & & & & -\mu_{sN}(x_c) & \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.4}$$

Our aim is to study a local map from an in-section $\Sigma^{\mathrm{in}} = \{x_{\mathrm{ss}} = \varepsilon\}$ to an out-section $\Sigma^{\mathrm{out}} = \{x_{\mathrm{u}} = \varepsilon\}$ for $x_{\mathrm{s}}, x_{\mathrm{c}} \approx 0$, see Fig. 2. This corresponds to the passage near the Kasner circle in the Bianchi system in backwards time direction. (We reversed the time direction to obtain a well-defined local map.)

We rescale the system to

$$\dot{x} = A(x_{\mathrm{c}})x + \varepsilon g(x), \tag{4.5}$$

with ε arbitrarily fixed and g at least quadratic in $(x_{\mathrm{u}}, x_{\mathrm{ss}}, x_{\mathrm{s}})$. Due to the invariant subspaces (3.4) and (4.3), the vector field takes the form

$$\begin{aligned}
\dot{x}_{\mathrm{u}} &= \mu_{\mathrm{u}}(x_{\mathrm{c}})x_{\mathrm{u}} + \varepsilon g_{\mathrm{u}}(x)x_{\mathrm{u}},\\
\dot{x}_{\mathrm{ss}} &= -\mu_{\mathrm{ss}}(x_{\mathrm{c}})x_{\mathrm{ss}} + \varepsilon g_{\mathrm{ss}}(x)x_{\mathrm{ss}},\\
\dot{x}_{\mathrm{s}\ell} &= -\mu_{\mathrm{s}\ell}(x_{\mathrm{c}})x_{\mathrm{s}\ell} + \varepsilon g_{\mathrm{s}\ell}(x)x_{\mathrm{s}\ell}, \qquad \ell = 1, \ldots N,\\
\dot{x}_{\mathrm{c}} &= \varepsilon\left(g_{\mathrm{css}}(x)x_{\mathrm{ss}} + \sum_{\ell=1}^{N} g_{\mathrm{cs}\ell}(x)x_{\mathrm{s}\ell}\right)x_{\mathrm{u}},
\end{aligned} \tag{4.6}$$

with $\mathcal{C}^{k-1}$-functions g_{u}, $g_{\mathrm{ss}}, g_{\mathrm{s}\ell}$, vanishing along the line of equilibria, and $\mathcal{C}^{k-2}$-functions $g_{\mathrm{css}}, g_{\mathrm{cs}\ell}$. In particular,

$$\begin{aligned}
|g_{\mathrm{u}}(x)|, |g_{\mathrm{ss}}(x)|, |g_{\mathrm{s}1}(x)|, \ldots, |g_{\mathrm{sN}}(x)| &< C\max(|x_{\mathrm{u}}|, |x_{\mathrm{ss}}|, |x_{\mathrm{s}1}|, \ldots, |x_{\mathrm{sN}}|),\\
|g_{\mathrm{css}}(x)|, |g_{\mathrm{cs}1}(x)|, \ldots, |g_{\mathrm{csN}}(x)| &< C,
\end{aligned} \tag{4.7}$$

for some constant $C > 0$ independent of ε and $x \in \mathcal{U}$, where $\mathcal{U}$ is some local neighbourhood of the origin. Similarly, all derivatives of $g_{\mathrm{u}}, g_{\mathrm{ss}}, g_{\mathrm{s}\ell}, g_{\mathrm{css}}, g_{\mathrm{cs}\ell}$ are bounded by C for $x \in \mathcal{U}$. We choose

$$\mathcal{U} = (-2, 2)^{N+3}. \tag{4.8}$$

All further estimates will use this rescaled system (4.5) with flattened invariant manifolds (4.3) in the local neighbourhood $\mathcal{U}$. They will be valid for all $\varepsilon < \varepsilon_0$ and suitably chosen ε_0. In the original system (4.1), ε_0 bounds the size of the neighbourhood of the origin in which this local analysis is valid.

Proposition 4.1. *Let*

$$\mu_{\mathrm{u}} := \mu_{\mathrm{u}}(0), \quad -\mu_{\mathrm{ss}} := -\mu_{\mathrm{ss}}(0), \quad -\mu_{\mathrm{s}\ell} := -\mu_{\mathrm{s}\ell}(0), \qquad \ell = 1, \ldots, N,$$

be the eigenvalues of (4.4) *at the origin. Then for all* $0 < \alpha < 1$ *there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *in* (4.5) *and* $x \in \mathcal{U}$

$$\alpha \le \frac{\mu_{\mathrm{u}}(x_{\mathrm{c}})}{\mu_u}, \frac{\mu_{\mathrm{ss}}(x_{\mathrm{c}})}{\mu_{\mathrm{ss}}}, \frac{\mu_{\mathrm{s}1}(x_{\mathrm{c}})}{\mu_{\mathrm{s}1}}, \ldots, \frac{\mu_{\mathrm{sN}}(x_{\mathrm{c}})}{\mu_{\mathrm{sN}}} \le \alpha^{-1}. \tag{4.9}$$

Proof. Due to the invariant subspaces (3.4, 4.3), the linearisation of the system at equilibria close to the origin remains diagonal, and the eigenvalues depend differentiably on x_{c}, For the rescaled system (4.5) with small ε_0 this provides bounds in $\mathcal{U}$: Indeed, there exists a constant $C > 0$ independent of ε_0, ε, such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}x_{\mathrm{c}}}\mu_{\mathrm{u}}(x_{\mathrm{c}})\right|, \left|\frac{\mathrm{d}}{\mathrm{d}x_{\mathrm{c}}}\mu_{\mathrm{ss}}(x_{\mathrm{c}})\right|, \left|\frac{\mathrm{d}}{\mathrm{d}x_{\mathrm{c}}}\mu_{\mathrm{s}\ell}(x_{\mathrm{c}})\right| < \varepsilon C. \tag{4.10}$$

□

The scalar function $\theta(x) := \mu_{\mathrm{u}}\left(\mu_{\mathrm{u}}(x_{\mathrm{c}}) + \varepsilon g_{\mathrm{u}}(x)\right)^{-1}$ is, therefore, $\mathcal{C}^{k-1}$ and close to 1. The vector field

$$x' = \theta(x) f(x) = \frac{\mu_{\mathrm{u}}}{\mu_{\mathrm{u}}(x_{\mathrm{c}}) + \varepsilon g_{\mathrm{u}}(x)} f(x)$$

has the same trajectories as the original vector field and all previous considerations remain valid. Thus we can assume, without loss of generality, that $\theta(x) \equiv 1$ in $\mathcal{U}$, i.e.

$$\mu_{\mathrm{u}}(x_{\mathrm{c}}) \equiv \mu_{\mathrm{u}}, \qquad g_{\mathrm{u}}(x) \equiv 0. \tag{4.11}$$

At this step we have made use of the fact that the origin possesses exactly one unstable eigenvalue. The vector field to consider then has the form

$$\begin{aligned}
\dot{x}_{\mathrm{u}} &= \mu_{\mathrm{u}} x_{\mathrm{u}} \\
\dot{x}_{\mathrm{ss}} &= -\mu_{\mathrm{ss}}(x_{\mathrm{c}}) x_{\mathrm{ss}} + \varepsilon g_{\mathrm{ss}}(x) x_{\mathrm{ss}}, \\
\dot{x}_{\mathrm{s}\ell} &= -\mu_{\mathrm{s}\ell}(x_{\mathrm{c}}) x_{\mathrm{s}\ell} + \varepsilon g_{\mathrm{s}\ell}(x) x_{\mathrm{s}\ell}, \qquad \ell = 1, \ldots N, \\
\dot{x}_{\mathrm{c}} &= \varepsilon \left(g_{\mathrm{css}}(x) x_{\mathrm{ss}} + \sum_{\ell=1}^{N} g_{\mathrm{cs}\ell}(x) x_{\mathrm{s}\ell} \right) x_{\mathrm{u}}.
\end{aligned} \tag{4.12}$$

Lemma 4.2. *For all $0 < \alpha < 1$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $x(0) \in \mathcal{U}$ and $t \geq 0$, as long as $x(t)$ remains in $\mathcal{U}$ under the flow to the vector field (4.12), we can estimate*

$$x_{\mathrm{u}}(t) = \exp(\mu_{\mathrm{u}} t)\, x_{\mathrm{u}}(0), \tag{4.13}$$

$$x_{\mathrm{ss}}(t) \in \left[\exp\left(-\frac{1}{\alpha} \mu_{\mathrm{ss}} t \right), \exp\left(-\alpha \mu_{\mathrm{ss}} t \right) \right] x_{\mathrm{ss}}(0), \tag{4.14}$$

$$x_{\mathrm{s}\ell}(t) \in \left[\exp\left(-\frac{1}{\alpha} \mu_{\mathrm{s}\ell} t \right), \exp\left(-\alpha \mu_{\mathrm{s}\ell} t \right) \right] x_{\mathrm{s}\ell}(0), \qquad \ell = 1, \ldots, N, \tag{4.15}$$

$$|x_{\mathrm{c}}(t) - x_{\mathrm{c}}(0)| \leq \frac{2\varepsilon C}{\alpha} \left(\frac{1}{\mu_{\mathrm{ss}} - \mu_{\mathrm{u}}} |x_{\mathrm{u}}(0)| + \sum_{\ell=1}^{N} \frac{1}{\mu_{\mathrm{s}\ell}} |x_{\mathrm{s}\ell}(0)| \right). \tag{4.16}$$

Here, C is the uniform (in x and ε) bound from (4.7).

Proof. The unstable component (4.13) is given directly by the vector field. The estimates of the stable components (4.14, 4.15) follow from the vector field and the uniform bounds (4.9, 4.7). Indeed for arbitrary $0 < \tilde{\alpha} < 1$ and $\varepsilon < \varepsilon_0$ small enough, we have

$$\dot{x}_{\mathrm{ss}} \in \left[-\frac{1}{\tilde{\alpha}} \mu_{\mathrm{ss}} - \varepsilon C, -\tilde{\alpha} \mu_{\mathrm{ss}} + \varepsilon C \right] x_{\mathrm{ss}}.$$

Thus for arbitrary $0 < \alpha < 1$ we find suitable $\alpha < \tilde{\alpha} < 1$ and ε_0 small enough such that

$$\dot{x}_{\mathrm{ss}} \in \left[-\frac{1}{\alpha} \mu_{\mathrm{ss}}, -\alpha \mu_{\mathrm{ss}} \right] x_{\mathrm{ss}}.$$

Integration yields the claim. Bounds on $x_{s\ell}$ are obtained analogously. The centre component (4.16) is then estimated by plugging (4.13, 4.14, 4.15) into the vector field (4.12) and integrating

$$\begin{aligned}
&|x_c(t) - x_c(0)| \\
&\quad = \varepsilon \left| \int_0^t g_{css}(x(s)) x_{ss}(s) x_u(s) + \sum_{\ell=1}^{N} g_{cs\ell}(x(s)) x_{s\ell}(s) x_u(s)\, \mathrm{d}s \right| \\
&\quad \leq \varepsilon C \int_0^t \exp(\mu_u - \tilde{\alpha}\mu_{ss}) |x_{ss}(0) x_u(0)| + \sum_{\ell=1}^{N} \exp(-\tilde{\alpha}\mu_{s\ell}) |x_{s\ell}(0) x_u(s)|\, \mathrm{d}s \\
&\quad \leq 2\varepsilon C \int_0^t \exp(\mu_u - \tilde{\alpha}\mu_{ss}) |x_u(0)| + \sum_{\ell=1}^{N} \exp(-\tilde{\alpha}\mu_{s\ell}) |x_{s\ell}(0)|\, \mathrm{d}s \\
&\quad \leq 2\varepsilon C \int_0^t \exp(-\alpha(\mu_{ss} - \mu_u)) |x_u(0)| + \sum_{\ell=1}^{N} \exp(-\alpha\mu_{s\ell}) |x_{s\ell}(0)|\, \mathrm{d}s
\end{aligned}$$

The last inequality needs a slight adjustment of $\tilde{\alpha} \rightsquigarrow \alpha$ and uses the eigenvalue condition (3.3). Indeed, for all $0 < \alpha < 1$, we find a suitable $0 < \tilde{\alpha} < 1$ with $0 < \alpha(\mu_{ss} - \mu_u) < \tilde{\alpha}\mu_{ss} - \mu_u$. □

The local map

$$(x_u^{in}, x_s^{in}, x_c^{in}) \longmapsto (x_{ss}^{out}, x_s^{out}, x_c^{out}) = \Psi^{loc}(x_u^{in}, x_s^{in}, x_c^{in}) \tag{4.17}$$

is given by the first intersection of the solution of (4.12) to the initial value $(x_u^{in}, x_{ss}^{in} = 1, x_s^{in}, x_c^{in})$ with the out-section $\{x_u = 1\}$. See Fig. 2. The local map Ψ^{loc} is well defined on the in-section

$$\Sigma^{in} = \{ (x_u^{in}, x_{ss}^{in}, x_s^{in}, x_c^{in}) \,|\, x_{ss}^{in} = 1,\ 0 < x_u^{in} < 1,\ \|x_s^{in}\| < 1,\ |x_c^{in}| < 1 \}, \tag{4.18}$$

see Lemma 4.3 below. The singular points in the intersection of the stable manifold of the equilibrium line with the in-section are mapped to the respective points in the intersection of the unstable manifold of the equilibrium line with the out-section:

$$\Psi^{loc}(x_u^{in} = 0, x_s^{in}, x_c^{in}) = (x_{ss}^{out}, x_s^{out}, x_c^{out}) := (0, 0, x_c^{in}). \tag{4.19}$$

Note that there is no drift in x_c at the boundary due to the invariant fibres (4.3).

Lemma 4.3. *There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $x(0) = x^{in}$ in the in-section Σ^{in}, see (4.18), the trajectory $x(t)$ under the flow to the vector field (4.12) remains in $\mathcal{U}$ as long as $|x_u| \leq 1$, i.e. all along the passage defining the local map Ψ^{loc}. The passage time t^{loc} is given by*

$$t^{loc} = \frac{1}{\mu_u} \ln \frac{1}{|x_u^{in}|}. \tag{4.20}$$

Proof. We choose ε_0 smaller than $\frac{1}{2(N+1)C} \alpha \min\{\mu_{ss} - \mu_u, \mu_{s1}, \ldots, \mu_{sN}\})$, see Lemma 4.2. Then trajectories starting in Σ^{in} cannot leave $\mathcal{U}$ unless x_u becomes larger than 1, see (4.14, 4.15, 4.16). Furthermore, (4.13) ensures that x_u must grow beyond 1. Thus every trajectory starting in Σ^{in} intersects the out-section $\Sigma^{\mathrm{out}} = \{x_u = 1\}$ before leaving $\mathcal{U}$. Setting $x_{\mathrm{u}}(t^{\mathrm{loc}}) = 1$ in (4.13) determines the passage time t^{loc}. □

Corollary 4.4. *The local map Ψ^{loc} (4.17, 4.19), i.e. the local passage on the closed in-section $\overline{\Sigma^{\mathrm{in}}}$ including the singular boundary $\{x_{\mathrm{u}}^{\mathrm{in}} = 0\}$, is continuous. For all $0 < \alpha < 1$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the following estimates hold:*

$$
\begin{aligned}
|x_{\mathrm{c}}^{\mathrm{out}} - x_{\mathrm{c}}^{\mathrm{in}}| &\leq \varepsilon C \left(|x_{\mathrm{u}}^{\mathrm{in}}| + |x_{\mathrm{s}1}^{\mathrm{in}}| + \cdots + |x_{\mathrm{sN}}^{\mathrm{in}}|\right),\\
|x_{\mathrm{ss}}^{\mathrm{out}}| &\leq |x_{\mathrm{u}}^{\mathrm{in}}|^{\alpha\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-1}|x_{\mathrm{u}}^{\mathrm{in}}|,\\
|x_{\mathrm{s}\ell}^{\mathrm{out}}| &\leq |x_{\mathrm{u}}^{\mathrm{in}}|^{\alpha\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}}|x_{\mathrm{s}\ell}^{\mathrm{in}}|, \qquad \ell = 1, \ldots, N,
\end{aligned}
$$

with C independent of ε and x^{in}.

Thus the drift along the line of equilibria is arbitrarily small and the distance from the orbit to the union of the stable and unstable manifolds shrinks arbitrarily fast, close to the critical orbit.

Proof. The estimates follow directly from Lemma 4.2 applied to the local passage time given by Lemma 4.3. They also establish continuity of the local map Ψ^{loc} at the singular boundary $\{x_{\mathrm{u}}^{\mathrm{in}} = 0\}$. Note $0 < x_{\mathrm{u}}^{\mathrm{in}} < 1$ on the in-section. Note further that $\alpha\mu_{\mathrm{ss}}/\mu_{\mathrm{u}} - 1 > 0$ for α chosen close enough to 1. □

Unfortunately, there is no hope to obtain Lipschitz estimates for the local map Ψ^{loc} with respect to the standard metric. Even for $g = 0$, the linear vector field (4.5) yields a non-Lipschitz local passage, for $\mu_{\mathrm{s}\ell} < \mu_{\mathrm{u}}$.

To obtain Lipschitz bounds for the local map Ψ^{loc}, we have to introduce a non-Euclidean metric on the in- and out-sections. We define the Riemannian metrics

$$
\mathrm{d}s_*^2 \;=\; \frac{\|x_{\mathrm{u,s}}\|_2^2}{x_{\mathrm{u}}^2}\,\mathrm{d}x_{\mathrm{u}}^2 + \sum_{\ell=1}^{N} \frac{\|x_{\mathrm{u,s}}\|_2^2}{x_{\mathrm{s}\ell}^2}\,\mathrm{d}x_{\mathrm{s}\ell}^2 + \mathrm{d}x_{\mathrm{c}}^2 \tag{4.21}
$$

on the in-section Σ^{in} and

$$
\mathrm{d}s_*^2 \;=\; \frac{\|x_{\mathrm{ss,s}}\|_2^2}{x_{\mathrm{ss}}^2}\,\mathrm{d}x_{\mathrm{ss}}^2 + \sum_{\ell=1}^{N} \frac{\|x_{\mathrm{ss,s}}\|_2^2}{x_{\mathrm{s}\ell}^2}\,\mathrm{d}x_{\mathrm{s}\ell}^2 + \mathrm{d}x_{\mathrm{c}}^2 \tag{4.22}
$$

on the out-section Σ^{out}. We denoted the Euclidean norms

$$
\|x_{\mathrm{u,s}}\|_2^2 = \|(x_{\mathrm{u}}, x_{\mathrm{s}})\|_2^2 = x_{\mathrm{u}}^2 + \sum_{\ell=1}^{N} x_{\mathrm{s}\ell}^2,
$$

$$
\|x_{\mathrm{ss,s}}\|_2^2 = \|(x_{\mathrm{ss}}, x_{\mathrm{s}})\|_2^2 = x_{\mathrm{ss}}^2 + \sum_{\ell=1}^{N} x_{\mathrm{s}\ell}^2.
$$

The distance in Σ^{in}, Σ^{out} is then given by the length of the shortest connecting paths and denoted by dist_*. On fibres $\{x_{\mathrm{c}}\,\text{fixed}\}$ we define the metric analogously.

Let us discuss the new metric in the cone $x_{\mathrm{u,s}} = (x_{\mathrm{u}}, x_{\mathrm{s}}) \in [0,\infty)^{N+1}$ in Σ^{in}, ignoring the x_{c}-direction that remains unchanged. The metric becomes singular along the invariant boundaries $\{x_{\mathrm{u}} = 0\}, \{x_{\mathrm{s}\ell} = 0\}$, see (3.4). Inside the open cone $(0,\infty)^{N+1}$, the new metric $\mathrm{d}s_*$ is locally equivalent to the Euclidean metric $\mathrm{d}s$,

$$\mathrm{d}s^2 \le \mathrm{d}s_*^2 \le (N+1)\frac{\max\{|x_{\mathrm{u}}|, |x_{\mathrm{s1}}|, \ldots, |x_{\mathrm{sN}}|\}}{\min\{|x_{\mathrm{u}}|, |x_{\mathrm{s1}}|, \ldots, |x_{\mathrm{sN}}|\}}\,\mathrm{d}s^2, \tag{4.23}$$

and thus induces the same topology. The origin can be included. In fact the distance of any point to the origin is bounded by

$$\|x_{\mathrm{u,s}}\|_2 \le \mathrm{dist}_*(0, x_{\mathrm{u,s}}) \le (N+1)^{3/2}\|x_{\mathrm{u,s}}\|_2. \tag{4.24}$$

(The upper bound can easily be obtained by connecting the origin to $x_{\mathrm{u,s}}$ with a piecewise linear path along the space diagonals with respect to suitable coordinate directions.) Every curve in the open cone hitting the boundary away from the origin has infinite length.

In particular, the new metric is uniformly equivalent to the Euclidean metric in any closed cone that has finite, nonzero angle to the boundaries, i.e.

$$\{\,(x_{\mathrm{u,s}}, x_{\mathrm{c}}) \,|\, \|x_{\mathrm{u,s}}\|_2 \ge c\max\{|x_{\mathrm{u}}|, |x_{\mathrm{s1}}|, \ldots, |x_{\mathrm{sN}}|\}\,\}, \qquad c > 1.$$

Thus Lipschitz estimates with respect to the new metric carry over to the Euclidean metric.

We denote the in- and out-sections without the singular boundaries but with the origin by

$$\begin{aligned} \Sigma_*^{\mathrm{in}} &:= \Sigma^{\mathrm{in}} \cap ((0,\infty)^{N+1} \cup \{0\}) \times \mathbb{R}) \\ &= \{\,(x_{\mathrm{u,s}}, x_{\mathrm{c}}) \in \Sigma^{\mathrm{in}} \,|\, x_{\mathrm{u,s}} = 0 \text{ or } x_{\mathrm{u}}x_{\mathrm{s1}}\cdots x_{\mathrm{sN}} \ne 0\,\}, \\ \Sigma_*^{\mathrm{out}} &:= \Sigma^{\mathrm{out}} \cap ((0,\infty)^{N+1} \cup \{0\}) \times \mathbb{R}) \\ &= \{\,(x_{\mathrm{ss,s}}, x_{\mathrm{c}}) \in \Sigma^{\mathrm{out}} \,|\, x_{\mathrm{ss,s}} = 0 \text{ or } x_{\mathrm{ss}}x_{\mathrm{s1}}\cdots x_{\mathrm{sN}} \ne 0\,\}. \end{aligned} \tag{4.25}$$

Corollary 4.4 yields Lipschitz continuity of the local passage Ψ^{loc} at the origin with respect to the Euclidean metric, and by (4.24) also with respect to the new metrics (4.21, 4.22). To obtain Lipschitz estimates away from the invariant boundaries with respect to the new metric, we consider the linearisation of the vector field (4.12) along a trajectory $x(t)$ from the in- to the out-section to obtain bounds on $D\Psi^{\mathrm{loc}}$.

We start with a tangent vector $\delta^{\mathrm{in}} = (\delta_{\mathrm{u}}^{\mathrm{in}}, \delta_{\mathrm{ss}}^{\mathrm{in}} = 0, \delta_{\mathrm{s}}^{\mathrm{in}}, \delta_{\mathrm{c}}^{\mathrm{in}})$ of unit length with respect to the metric (4.21) at a point $x^{\mathrm{in}} = (x_{\mathrm{u,s}}^{\mathrm{in}}, x_{\mathrm{c}}^{\mathrm{in}}) \in \Sigma^{\mathrm{in}}$,

$$1 = \|\delta^{\mathrm{in}}\|_*^2 = \frac{\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^2}{|x_{\mathrm{u}}^{\mathrm{in}}|^2}|\delta_{\mathrm{u}}^{\mathrm{in}}|^2 + \sum_{\ell=1}^{N}\frac{\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^2}{|x_{\mathrm{s}\ell}^{\mathrm{in}}|^2}|\delta_{\mathrm{s}\ell}^{\mathrm{in}}|^2 + |\delta_{\mathrm{c}}^{\mathrm{in}}|^2. \tag{4.26}$$

First, we project δ^{in} along the vector field f into the hyperplane $\{\delta_{\mathrm{u}} = 0\}$, as this remains invariant under the linearised flow and corresponds to the out-section. The projected vector

$$\delta(0) = \delta^{\mathrm{in}} - \frac{\delta_{\mathrm{u}}^{\mathrm{in}}}{\mu_{\mathrm{u}}x_{\mathrm{u}}^{\mathrm{in}}}f(x^{\mathrm{in}}) \tag{4.27}$$

thus represents our initial condition to the linearised flow

$$\begin{aligned}
\dot\delta_{\mathrm{ss}} &= \left(-\mu_{\mathrm{ss}}(x_{\mathrm{c}}) + \varepsilon g_{\mathrm{ss}}(x) + \varepsilon(\partial_{x_{\mathrm{ss}}} g_{\mathrm{ss}}(x))x_{\mathrm{ss}}\right)\delta_{\mathrm{ss}}\\
&\quad + \varepsilon\sum_{\ell=1}^{N}(\partial_{x_{\mathrm{s}\ell}} g_{\mathrm{ss}}(x))x_{\mathrm{ss}}\delta_{\mathrm{s}\ell} + \left(-\mu'_{\mathrm{ss}}(x_{\mathrm{c}}) + \varepsilon(\partial_{x_{\mathrm{c}}} g_{\mathrm{ss}}(x))\right)x_{\mathrm{ss}}\delta_{\mathrm{c}},\\
\dot\delta_{\mathrm{s}\ell} &= \left(-\mu_{\mathrm{s}\ell}(x_{\mathrm{c}}) + \varepsilon g_{\mathrm{s}\ell}(x) + \varepsilon(\partial_{x_{\mathrm{s}\ell}} g_{\mathrm{s}\ell}(x))x_{\mathrm{s}\ell}\right)\delta_{\mathrm{s}\ell}\\
&\quad + \varepsilon(\partial_{x_{\mathrm{ss}}} g_{\mathrm{s}\ell}(x))x_{\mathrm{s}\ell}\delta_{\mathrm{ss}} + \varepsilon\sum_{\tilde\ell\neq\ell}(\partial_{x_{\mathrm{s}\tilde\ell}} g_{\mathrm{s}\ell}(x))x_{\mathrm{s}\ell}\delta_{\mathrm{s}\tilde\ell}\\
&\quad + \left(-\mu'_{\mathrm{s}\ell}(x_{\mathrm{c}}) + \varepsilon(\partial_{x_{\mathrm{c}}} g_{\mathrm{s}\ell}(x))\right)x_{\mathrm{s}\ell}\delta_{\mathrm{c}},\\
\dot\delta_{\mathrm{c}} &= \varepsilon\left(\partial_{x_{\mathrm{c}}} g_{\mathrm{css}}(x)x_{\mathrm{ss}}x_{\mathrm{u}} + \sum_{\ell=1}^{N}\partial_{x_{\mathrm{c}}} g_{\mathrm{cs}\ell}(x)x_{\mathrm{s}\ell}x_{\mathrm{u}}\right)\delta_{\mathrm{c}}\\
&\quad + \varepsilon\left(g_{\mathrm{css}}(x)x_{\mathrm{u}} + \partial_{x_{\mathrm{ss}}} g_{\mathrm{css}}(x)x_{\mathrm{ss}}x_{\mathrm{u}} + \sum_{\ell=1}^{N}\partial_{x_{\mathrm{ss}}} g_{\mathrm{cs}\ell}(x)x_{\mathrm{s}\ell}x_{\mathrm{u}}\right)\delta_{\mathrm{ss}}\\
&\quad + \varepsilon\sum_{\ell=1}^{N}\left(\partial_{x_{\mathrm{s}\ell}} g_{\mathrm{css}}(x)x_{\mathrm{ss}}x_{\mathrm{u}} + g_{\mathrm{cs}\ell}(x)x_{\mathrm{u}} + \sum_{\tilde\ell=1}^{N}\partial_{x_{\mathrm{s}\ell}} g_{\mathrm{cs}\tilde\ell}(x)x_{\mathrm{s}\tilde\ell}x_{\mathrm{u}}\right)\delta_{\mathrm{s}\ell}.
\end{aligned}\tag{4.28}$$

Here we already dropped the u-component.

Lemma 4.5. *Let a* **unit** *tangent vector δ^{in} to $x^{\mathrm{in}} = (x^{\mathrm{in}}_{\mathrm{u,s}}, x^{\mathrm{in}}_{\mathrm{c}}) \in \Sigma^{\mathrm{in}}$ with respect to the metric (4.21) be given. The projection $\delta(0)$ of δ^{in} along the vector field (4.12) into the plane $\{\delta_{\mathrm{u}} = 0\}$ is estimated by*

$$\begin{aligned}
|\delta_{\mathrm{ss}}(0)| &\le C\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1},\\
|\delta_{\mathrm{s}\ell}(0)| &\le C\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}|x^{\mathrm{in}}_{\mathrm{s}\ell}|,\\
|\delta_{\mathrm{c}}(0) - \delta^{\mathrm{in}}_{\mathrm{c}}| &\le \varepsilon C,
\end{aligned}$$

with a constant C independent of x^{in}, δ^{in}, and $\varepsilon < \varepsilon_0$, provided ε_0 is chosen small enough.

Proof. Apply (4.27) to (4.26) and use the bounds (4.7) on the nonlinear terms of the vector field (4.12).

Indeed, we find

$$\begin{aligned}
\delta(0) &= \delta^{\mathrm{in}} - \frac{\delta^{\mathrm{in}}_{\mathrm{u}}}{\mu_{\mathrm{u}} x^{\mathrm{in}}_{\mathrm{u}}} f(x^{\mathrm{in}})\\
&= \begin{pmatrix} \delta^{\mathrm{in}}_{\mathrm{u}}\\ \delta^{\mathrm{in}}_{\mathrm{ss}} = 0\\ \delta^{\mathrm{in}}_{\mathrm{s}\ell}\\ \delta^{\mathrm{in}}_{\mathrm{c}} \end{pmatrix} - \frac{\delta^{\mathrm{in}}_{\mathrm{u}}}{\mu_{\mathrm{u}} x^{\mathrm{in}}_{\mathrm{u}}} \begin{pmatrix} \mu_{\mathrm{u}} x^{\mathrm{in}}_{\mathrm{u}}\\ \left(-\mu_{\mathrm{ss}}(x^{\mathrm{in}}_{\mathrm{c}}) + \varepsilon g_{\mathrm{ss}}(x^{\mathrm{in}})\right) 1\\ \left(-\mu_{\mathrm{s}\ell}(x^{\mathrm{in}}_{\mathrm{c}}) + \varepsilon g_{\mathrm{s}\ell}(x^{\mathrm{in}})\right) x^{\mathrm{in}}_{\mathrm{s}\ell}\\ \varepsilon\left(g_{\mathrm{css}}(x^{\mathrm{in}})\, 1 + \sum_{\ell=1}^{N}(x^{\mathrm{in}})x^{\mathrm{in}}_{\mathrm{s}\ell}\right) x^{\mathrm{in}}_{\mathrm{u}} \end{pmatrix}.
\end{aligned}$$

Immediately, we have $\delta_{\mathrm{u}}^{\mathrm{in}}(0) = 0$. For the other components we again use the uniform bounds (4.7) on the nonlinearity g, the bounds (4.9) on the eigenvalues and the bounds (4.26) on the components of δ^{in}. We obtain for arbitrary $0 < \alpha < 1$, if $\varepsilon < \varepsilon_0$ is chosen small enough:

$$|\delta_{\mathrm{ss}}(0)| \leq \frac{|\delta_{\mathrm{u}}^{\mathrm{in}}|}{\mu_{\mathrm{u}}|x_{\mathrm{u}}^{\mathrm{in}}|}\frac{1}{\alpha}\mu_{\mathrm{ss}} \leq \frac{\mu_{\mathrm{ss}}}{\alpha\mu_{\mathrm{u}}}\frac{1}{\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2},$$

for the component transverse to the in-section,

$$|\delta_{\mathrm{s}\ell}(0)| \leq |\delta_{\mathrm{s}\ell}^{\mathrm{in}}| + \frac{|\delta_{\mathrm{u}}^{\mathrm{in}}|}{\mu_{\mathrm{u}}|x_{\mathrm{u}}^{\mathrm{in}}|}\frac{1}{\alpha}\mu_{\mathrm{s}\ell}|x_{\mathrm{s}\ell}^{\mathrm{in}}| \leq \left(1 + \frac{\mu_{\mathrm{s}\ell}}{\alpha\mu_{\mathrm{u}}}\right)\frac{|x_{\mathrm{s}\ell}^{\mathrm{in}}|}{\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2},$$

for each of the remaining N stable components, and

$$|\delta_{\mathrm{c}}(0) - \delta_{\mathrm{c}}^{\mathrm{in}}| \leq \frac{|\delta_{\mathrm{u}}^{\mathrm{in}}|}{\mu_{\mathrm{u}}|x_{\mathrm{u}}^{\mathrm{in}}|}\varepsilon C(N+1)|x_{\mathrm{u}}^{\mathrm{in}}| \leq \varepsilon\frac{N+1}{\mu_{\mathrm{u}}}C\frac{|x_{\mathrm{u}}^{\mathrm{in}}|}{\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2} \leq \varepsilon\frac{N+1}{\mu_{\mathrm{u}}}C,$$

for the centre component. An obvious choice of a new constant C yields all claimed estimates. □

Lemma 4.6. *For all $0 < \alpha < 1$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the linearised flow (4.28) can be estimated:*

$$|\delta_{\mathrm{ss}}|' \leq -\alpha\mu_{\mathrm{ss}}|\delta_{\mathrm{ss}}| + \varepsilon C|x_{\mathrm{ss}}|\left(|\delta_{\mathrm{c}}| + \sum_{\ell=1}^{N}|\delta_{\mathrm{s}\ell}|\right),$$

$$|\delta_{\mathrm{s}\ell}|' \leq -\alpha\mu_{\mathrm{s}\ell}|\delta_{\mathrm{s}\ell}| + \varepsilon C|x_{\mathrm{s}\ell}|\left(|\delta_{\mathrm{c}}| + |\delta_{\mathrm{ss}}| + \sum_{\tilde{\ell}\neq\ell}|\delta_{\mathrm{s}\tilde{\ell}}|\right),$$

$$|\dot{\delta}_{\mathrm{c}}| \leq \varepsilon C\left(|x_{\mathrm{ss}}| + \sum_{\ell=1}^{N}|x_{\mathrm{s}\ell}|\right)|x_{\mathrm{u}}||\delta_{\mathrm{c}}| + \varepsilon C|x_{\mathrm{u}}|\left(|\delta_{\mathrm{ss}}| + \sum_{\ell=1}^{N}|\delta_{\mathrm{s}\ell}|\right).$$

Here C is a constant independent of x^{in}, δ^{in} and $\varepsilon < \varepsilon_0$.

Proof. Use the bounds (4.7) on the nonlinear terms of the vector field (4.12) and the bounds (4.10) on the derivatives of the eigenvalues. Note that $x \in \mathcal{U} = [-2,2]^{N+3}$. This immediately yields the claimed estimates. □

Lemma 4.7. *For all $0 < \alpha < 1$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the following statement holds: Let a trajectory $x(t)$ of local passage, $x(0) = x^{\mathrm{in}} \in \Sigma_*^{\mathrm{in}}$, $x_{\mathrm{u,s}}^{\mathrm{in}} \neq 0$ be given. Let a unit tangent vector δ^{in} to $x^{\mathrm{in}} \in \Sigma^{\mathrm{in}}$ with respect to the metric (4.21) and its projection $\delta(0)$ be given. Then the evolution of δ under the linearised flow is estimated by*

$$\begin{aligned}|\delta_{\mathrm{ss}}(t)| &\leq \exp(-\alpha^2\mu_{\mathrm{ss}}t)C\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^{-1},\\ |\delta_{\mathrm{s}\ell}(t)| &\leq \exp(-\alpha^2\mu_{\mathrm{s}\ell}t)C\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^{-1}|x_{\mathrm{s}\ell}^{\mathrm{in}}|,\\ |\delta_{\mathrm{c}}(t) - \delta_{\mathrm{c}}^{\mathrm{in}}| &\leq \varepsilon C,\end{aligned}$$

all along the local passage, $t \in [0, t^{\mathrm{loc}}]$. The constant C is independent of x^{in}, δ^{in}, t, and $\varepsilon < \varepsilon_0$.

Proof. Assume

$$|\delta_{\mathrm{ss}}(\tau)| \leq 2C\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}, \qquad |\delta_{\mathrm{s}\ell}(\tau)| \leq 2C, \qquad |\delta_{\mathrm{c}}(\tau)| \leq 2, \tag{4.29}$$

on $\tau \in [0, t]$. This assumption holds for small t due the estimates of the initial values in Lemma 4.5. The constant C is taken from the lemma.

Then the estimates of lemmata 4.6 and 4.2 yield

$$\begin{aligned}
|\delta_{\mathrm{ss}}(\tau)|' &\leq -\alpha\mu_{\mathrm{ss}}|\delta_{\mathrm{ss}}(\tau)| + \varepsilon C\exp(-\alpha\mu_{\mathrm{ss}}\tau)\,(2+2NC)\\
&\leq -\alpha\mu_{\mathrm{ss}}|\delta_{\mathrm{ss}}(\tau)| + \varepsilon\tilde{C}\exp(-\alpha\mu_{\mathrm{ss}}\tau),\\
|\delta_{\mathrm{s}\ell}(\tau)|' &\leq -\alpha\mu_{\mathrm{s}\ell}|\delta_{\mathrm{s}\ell}(\tau)| + \varepsilon C\exp(-\alpha\mu_{\mathrm{s}\ell}\tau)|x^{\mathrm{in}}_{\mathrm{s}\ell}|\left(2+2C\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}+2(N-1)C\right)\\
&\leq -\alpha\mu_{\mathrm{s}\ell}|\delta_{\mathrm{s}\ell}(\tau)| + \varepsilon\tilde{C}\exp(-\alpha\mu_{\mathrm{s}\ell}\tau)\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}|x^{\mathrm{in}}_{\mathrm{s}\ell}|.
\end{aligned}$$

The last inequality uses $\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2 < \sqrt{N}$. The new constant is bounded by $\tilde{C} \leq (2+2NC)\sqrt{N}C$.

Then we can integrate[1] the above estimates to obtain

$$\begin{aligned}
|\delta_{\mathrm{ss}}(t)| &\leq \exp(-\alpha\mu_{\mathrm{ss}}t)(|\delta_{\mathrm{ss}}(0)| + \varepsilon\tilde{C}t)\\
|\delta_{\mathrm{s}\ell}(t)| &\leq \exp(-\alpha\mu_{\mathrm{s}\ell}t)(|\delta_{\mathrm{s}\ell}(0)| + \varepsilon\tilde{C}t\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}|x^{\mathrm{in}}_{\mathrm{s}\ell}|)
\end{aligned}$$

For small enough ε_0, this yields

$$\begin{aligned}
|\delta_{\mathrm{ss}}(t)| &\leq \exp(-\alpha^2\mu_{\mathrm{ss}}t)(|\delta_{\mathrm{ss}}(0)|+1)\\
&\leq \exp(-\alpha^2\mu_{\mathrm{ss}}t)(C\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}+1)\\
&\leq \exp(-\alpha^2\mu_{\mathrm{ss}}t)(C+1)\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1},\\
|\delta_{\mathrm{s}\ell}(t)| &\leq \exp(-\alpha^2\mu_{\mathrm{s}\ell}t)(|\delta_{\mathrm{s}\ell}(0)| + \|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}|x^{\mathrm{in}}_{\mathrm{s}\ell}|)\\
&\leq \exp(-\alpha^2\mu_{\mathrm{s}\ell}t)(C+1)\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}|x^{\mathrm{in}}_{\mathrm{s}\ell}|
\end{aligned}$$

In particular, assumptions (4.29) and the first two claims hold as long as $|\delta_{\mathrm{c}}(\tau)| \leq 2$, if the original constant was chosen larger than 1.

We use the new estimates of $\delta_{\mathrm{ss}}, \delta_{\mathrm{s}\ell}$, the assumption on δ_{c} and the bound on the trajectory given by Lemma 4.2 to estimate the centre component:

$$\begin{aligned}
|\dot{\delta}_{\mathrm{c}}(\tau)| &\leq \varepsilon C\left(|x_{\mathrm{ss}}| + \sum_{\ell=1}^{N}|x_{\mathrm{s}\ell}|\right)|x_{\mathrm{u}}||\delta_{\mathrm{c}}| + \varepsilon C|x_{\mathrm{u}}|\left(|\delta_{\mathrm{ss}}| + \sum_{\ell=1}^{N}|\delta_{\mathrm{s}\ell}|\right)\\
&\leq 2\varepsilon C\left(|x_{\mathrm{ss}}| + \sum_{\ell=1}^{N}|x_{\mathrm{s}\ell}|\right)|x_{\mathrm{u}}|\\
&\quad + \varepsilon C(C+1)\|x^{\mathrm{in}}_{\mathrm{u,s}}\|_2^{-1}|x_{\mathrm{u}}|\left(\exp(-\alpha^2\mu_{\mathrm{ss}}\tau) + \sum_{\ell=1}^{N}\exp(-\alpha^2\mu_{\mathrm{s}\ell}\tau)|x^{\mathrm{in}}_{\mathrm{s}\ell}|\right)
\end{aligned}$$

[1] This can also be seen as a Gronwall estimate for $\exp(\alpha\mu_{\mathrm{ss}}t)\delta_{\mathrm{ss}}(t)$ and $\exp(\alpha\mu_{\mathrm{s}\ell}t)\delta_{\mathrm{s}\ell}(t)$.

$$
\begin{aligned}
&\leq 2\varepsilon C\left(\exp(-(\alpha\mu_{\mathrm{ss}}-\mu_{\mathrm{u}})\tau)|x_{\mathrm{u}}^{\mathrm{in}}|+\sum_{\ell=1}^{N}\exp(-\alpha\mu_{\mathrm{s}\ell}\tau)|x_{\mathrm{s}\ell}^{\mathrm{in}}|\right)\\
&\quad+\varepsilon C(C+1)\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^{-1}\Bigg(\exp(-(\alpha^2\mu_{\mathrm{ss}}-\mu_{\mathrm{u}})\tau)|x_{\mathrm{u}}^{\mathrm{in}}|\\
&\qquad\qquad+\sum_{\ell=1}^{N}\exp(-\alpha^2\mu_{\mathrm{s}\ell}\tau)|x_{\mathrm{s}\ell}^{\mathrm{in}}|\Bigg)\\
&\leq \varepsilon\tilde{C}\left(\exp(-(\alpha^2\mu_{\mathrm{ss}}-\mu_{\mathrm{u}})\tau)+\sum_{\ell=1}^{N}\exp(-\alpha^2\mu_{\mathrm{s}\ell}\tau)\right),
\end{aligned}
$$

with new constant $\tilde{C}<C(C+3)$. Thus $|\dot{\delta}_{\mathrm{c}}(\tau)|$ decays exponentially for α close enough to 1. Integration yields

$$\int_0^t|\dot{\delta}_{\mathrm{c}}(\tau)|\leq\varepsilon\hat{C},$$

with $\hat{C}<\tilde{C}((\alpha^2\mu_{\mathrm{ss}}-\mu_{\mathrm{u}})^{-1}+\sum_{\ell=1}^{N}(\alpha^2\mu_{\mathrm{s}\ell})^{-1})$. If ε_0 is chosen small enough, this shows that the assumption (4.29) indeed holds all along the passage and the claimed estimates are valid. □

Theorem 4.8 (local Lipschitz map). *The local passage $\Psi^{\mathrm{loc}}:\Sigma_*^{\mathrm{in}}\to\Sigma_*^{\mathrm{out}}$ is Lipschitz continuous with respect to the metrics (4.21, 4.22).*

There exist $\beta>0$, $\varepsilon_0>0$ and $C>0$ such that for all $\varepsilon<\varepsilon_0$ the following estimates hold for all $x^{\mathrm{in}},\tilde{x}^{\mathrm{in}}$ with $0\leq\tilde{x}_{\mathrm{u}}^{\mathrm{in}}\leq x_{\mathrm{u}}^{\mathrm{in}}$:

$$
\begin{aligned}
\operatorname{dist}_*(\tilde{x}_{\mathrm{ss,s}}^{\mathrm{out}},x_{\mathrm{ss,s}}^{\mathrm{out}})&\leq|x_{\mathrm{u}}^{\mathrm{in}}|^{\beta}C\operatorname{dist}_*(\tilde{x}^{\mathrm{in}},x^{\mathrm{in}}),\\
|(\tilde{x}_{\mathrm{c}}^{\mathrm{out}}-x_{\mathrm{c}}^{\mathrm{out}})-(\tilde{x}_{\mathrm{c}}^{\mathrm{in}}-x_{\mathrm{c}}^{\mathrm{in}})|&\leq\varepsilon C\operatorname{dist}_*(\tilde{x}^{\mathrm{in}},x^{\mathrm{in}}).
\end{aligned}
$$

The domain Σ_*^{in}, as defined in (4.25), is given by the local section without the invariant singular boundaries but including the line $(0,x_{\mathrm{c}})$ representing the cap of heteroclinic orbits.

The drift in the centre direction can be made arbitrarily small by choosing a sufficiently small local neighbourhood. The contraction in the transverse directions is arbitrarily strong by restricting the in-section to the part close to the primary object, i.e. the stable manifold of the origin.

Proof. This is a corollary of Lemma 4.7 by applying the passage time (4.20) and the metric (4.22). Extension to the line $(0,x_{\mathrm{c}})$ is given by Corollary 4.4.

Indeed, in the out-section, the estimates of Lemma 4.7, read

$$
\begin{aligned}
|\delta_{\mathrm{ss}}^{\mathrm{out}}|&=|\delta_{\mathrm{ss}}(t^{\mathrm{loc}})|\leq|x_{\mathrm{u}}^{\mathrm{in}}|^{\alpha^2\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}}C\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^{-1},\\
|\delta_{\mathrm{s}\ell}^{\mathrm{out}}|&=|\delta_{\mathrm{s}\ell}(t^{\mathrm{loc}})|\leq|x_{\mathrm{u}}^{\mathrm{in}}|^{\alpha^2\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}}C\|x_{\mathrm{u,s}}^{\mathrm{in}}\|_2^{-1}|x_{\mathrm{s}\ell}^{\mathrm{in}}|,\\
|\delta_{\mathrm{c}}^{\mathrm{out}}-\delta_{\mathrm{c}}^{\mathrm{in}}|&=|\delta_{\mathrm{c}}(t^{\mathrm{loc}})-\delta_{\mathrm{c}}^{\mathrm{in}}|\leq\varepsilon C.
\end{aligned}
$$

With respect to the modified metric (4.22) we find using the estimates of Lemma 4.2:

$$\begin{aligned}
&\left(\frac{\|x^{\mathrm{out}}_{\mathrm{ss,s}}\|_2}{|x^{\mathrm{out}}_{\mathrm{ss}}|}|\delta^{\mathrm{out}}_{\mathrm{ss}}|\right)^2\\
&\quad\leq \left(1+\sum_{\ell=1}^{N}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\alpha\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}-\mu_{\mathrm{ss}}/\alpha\mu_{\mathrm{u}})}|x^{\mathrm{in}}_{\mathrm{s}\ell}|^2\right)|\delta^{\mathrm{out}}_{\mathrm{ss}}|^2\\
&\quad\leq C\frac{|x^{\mathrm{in}}_{\mathrm{u}}|^{2\xi}+\sum_{\ell=1}^{N}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\xi+\alpha\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}-\mu_{\mathrm{ss}}/\alpha\mu_{\mathrm{u}})}|x^{\mathrm{in}}_{\mathrm{s}\ell}|^2}{|x^{\mathrm{in}}_{\mathrm{u}}|^2+\sum_{\ell=1}^{N}|x^{\mathrm{in}}_{\mathrm{s}\ell}|^2}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\alpha^2\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-\xi)}\\
&\quad\leq |x^{\mathrm{in}}_{\mathrm{u}}|^{2\beta}C.
\end{aligned}$$

In the second inequality, we introduced a parameter ξ. The last inequality then needs

$$\begin{aligned}
&0<\alpha^2\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-\xi=:\beta,\\
&1\leq\xi,\\
&0\leq\xi+\alpha\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}-\mu_{\mathrm{ss}}/\alpha\mu_{\mathrm{u}},\quad \ell=1,\ldots,N.
\end{aligned}$$

For α close to 1, a suitable ξ exists. In fact we can obtain arbitrary

$$0\,<\,\beta\,<\,\min\left\{\frac{\mu_{\mathrm{ss}}}{\mu_{\mathrm{u}}}-1,\ \frac{\mu_{\mathrm{s}\ell}}{\mu_{\mathrm{u}}}\ (\ell=1,\ldots,N)\right\}.$$

Similarly we find for $1\leq\ell\leq N$,

$$\begin{aligned}
&\left(\frac{\|x^{\mathrm{out}}_{\mathrm{ss,s}}\|_2}{|x^{\mathrm{out}}_{\mathrm{s}\ell}|}|\delta^{\mathrm{out}}_{\mathrm{s}\ell}|\right)^2\\
&\quad\leq\left(|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\alpha\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-\mu_{\mathrm{s}\ell}/\alpha\mu_{\mathrm{u}})}+\sum_{\tilde{\ell}=1}^{N}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\alpha\mu_{\mathrm{s}\tilde{\ell}}/\mu_{\mathrm{u}}-\mu_{\mathrm{s}\ell}/\alpha\mu_{\mathrm{u}})}|x^{\mathrm{in}}_{\mathrm{s}\tilde{\ell}}|^2\right)\frac{|\delta^{\mathrm{out}}_{\mathrm{s}\ell}|^2}{|x^{\mathrm{in}}_{\mathrm{s}\ell}|^2}\\
&\quad\leq C\frac{\begin{pmatrix}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\xi+\alpha\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-\mu_{\mathrm{s}\ell}/\alpha\mu_{\mathrm{u}})}\\ \quad+\sum_{\tilde{\ell}=1}^{N}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\xi+\alpha\mu_{\mathrm{s}\tilde{\ell}}/\mu_{\mathrm{u}}-\mu_{\mathrm{s}\ell}/\alpha\mu_{\mathrm{u}})}|x^{\mathrm{in}}_{\mathrm{s}\tilde{\ell}}|^2\end{pmatrix}}{|x^{\mathrm{in}}_{\mathrm{u}}|^2+\sum_{\tilde{\ell}=1}^{N}|x^{\mathrm{in}}_{\mathrm{s}\tilde{\ell}}|^2}|x^{\mathrm{in}}_{\mathrm{u}}|^{2(\alpha^2\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}-\xi)}\\
&\quad\leq|x^{\mathrm{in}}_{\mathrm{u}}|^{2\tilde{\beta}}C.
\end{aligned}$$

This time we need for the last inequality

$$\begin{aligned}
&0<\alpha^2\mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}-\xi=:\tilde{\beta},\\
&1\leq\xi+\alpha\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-\mu_{\mathrm{s}\ell}/\alpha\mu_{\mathrm{u}},\\
&0\leq\xi+\alpha\mu_{\mathrm{s}\tilde{\ell}}/\mu_{\mathrm{u}}-\mu_{\mathrm{s}\ell}/\alpha\mu_{\mathrm{u}},\quad\tilde{\ell}=1,\ldots,N.
\end{aligned}$$

Again, for α close to 1, a suitable ξ exists. In fact, we can again obtain arbitrary

$$0\,<\,\tilde{\beta}\,<\,\min\{\,\mu_{\mathrm{ss}}/\mu_{\mathrm{u}}-1,\ \mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}\,\}.$$

Now, take a geodesic curve in Σ^{in} which connects $\tilde{x}^{\mathrm{in}}$, x^{in} and defines $\mathrm{dist}_*(\tilde{x}^{\mathrm{in}},x^{\mathrm{in}})$. The image of this curve under the passage Ψ^{loc} provides an upper bound on $\mathrm{dist}_*(\tilde{x}^{\mathrm{out}},x^{\mathrm{out}})$. In both sections the x_{c}-component can be separated. Therefore, the above estimates on the evolution of the tangent vectors immediately yield the claims of the theorem. □

Remark 4.9. In Theorem 4.8, the constant C only depends on the $\mathcal{C}^1$ bounds on the nonlinear part of the vector field and the derivatives of the eigenvalues of the linearisation along the line of equilibria. The exponent β only depends on the spectral gaps. In fact, it can be taken arbitrarily in the interval

$$0 < \beta < \min\{\, \mu_{\mathrm{ss}}/\mu_{\mathrm{u}} - 1,\, \mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}} \,\},$$

by choosing ε_0 small enough.

The last remark provides uniform Lipschitz estimates for the local passages near the Kasner circle in Bianchi models, provided they keep a uniform distance from the Taub points at which the spectral gap shrinks to zero.

5. Return Map and Graph Transform

In this section, we define a global excursion map for trajectories near a primary heteroclinic orbit to the Kasner circle, that is from the out-section of a local passage to the in-section of another local passage, both local passages as discussed in the previous section. Combining local passage and global excursion we obtain a return map from one in-section to the next,

$$\Psi_k \;:=\; \Psi_k^{\mathrm{glob}} \circ \Psi_k^{\mathrm{loc}} \;:\; \Sigma_k^{\mathrm{in}} \longrightarrow \Sigma_{k+1}^{\mathrm{in}}, \tag{5.1}$$

see Fig. 3. The given heteroclinic orbit corresponds to a fixed origin of this map.

We prove uniform Lipschitz- and cone properties of the return map, independently of the given heteroclinic orbit, as long as the orbit keeps a uniform distance from the Taub points. In fact, we prove uniform Lipschitz- and cone properties of the return map under the conditions 3.3 on the global excursion.

This yields a sequence of return maps, with uniform estimates, to every sequence of heteroclinic orbits to the Kasner circle that does not accumulate to any Taub point and satisfies the local conditions 3.2 at every equilibrium.

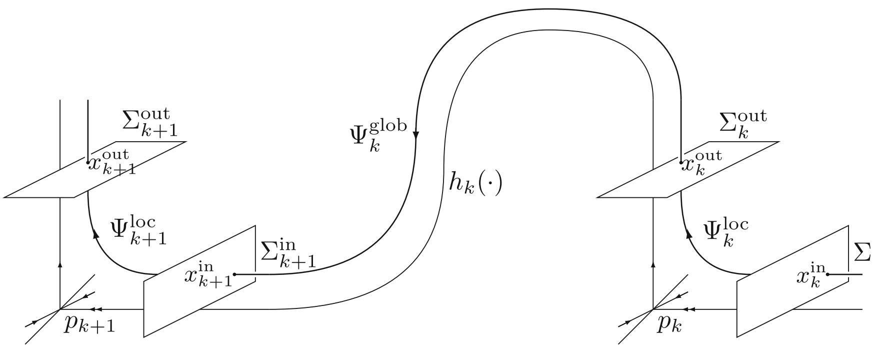

FIGURE 3. The return map $\Psi_k := \Psi_k^{\mathrm{glob}} \circ \Psi_k^{\mathrm{loc}}$

Due to their cone properties, the return maps induce a contracting map on a suitable space of sequences of Lipschitz curves. The fixed point provided by the contraction mapping theorem then yields the stable manifold of the heteroclinic sequence as claimed in Theorems 3.1, 3.4.

Take a sequence p^k, $k \in \mathbb{N}$ of equilibria on the Kasner circle, not accumulating at any Taub point and connected by heteroclinic orbits $h^k(t)$, $\lim_{t\to\infty} h^{k-1}(t) = p^k = \lim_{t\to-\infty} h^k(t)$, as in Theorem 3.4. Assume that the local conditions 3.2 hold uniformly at all p^k, in particular this includes uniform bounds $\sup_{k\in\mathbb{N}} \mu_{\mathrm{u}}(p^k)/\mu_{\mathrm{ss}}(p^k) < 1$ and $\inf_{k\in\mathbb{N}} \mu_{\mathrm{s}\ell}(p^k) > 0$. In the Bianchi VI_0 system (2.1) with magnetic field, these conditions are satisfied exactly for a chain of heteroclinic orbits not accumulating at Taub points and not containing heteroclinic orbits of the magnetic family to equilibria in the intervals $\mathcal{K}_2$, $\mathcal{K}_5$, see Fig. 1. In particular, the conditions hold for the period 3 cycle.

The previous section then applies to all p^k and the coefficients ε_0, β, C of the local estimates of Theorem 4.8 can be taken independent of k, see Remark 4.9.

Note the order of fixing the rescaling parameters: First ε_0 resp. ε is fixed small enough to yield our estimates of the local passages Ψ_k^{loc} with small Lipschitz constants, in particular $\varepsilon C \ll 1$ in Theorem 4.8. This amounts to a choice of the sections $\Sigma^{\mathrm{in}}(p_k)$ and $\Sigma^{\mathrm{out}}(p_k)$ in the original (unscaled) coordinates and also fixes the global excursion maps Ψ_k^{glob}.

Due to the non-Euclidean metric used in Theorem 4.8, we have to restrict our local passage map to $\Psi_k^{\mathrm{loc}} : \Sigma_k^{\mathrm{in}} \to \Sigma_k^{\mathrm{out}}$ by

$$\begin{aligned} \Sigma_k^{\mathrm{in}} &= \Sigma_*^{\mathrm{in}}(p_k) \\ &= \{ (x_{\mathrm{u,s}}, x_{\mathrm{c}}) \in \Sigma^{\mathrm{in}}(p_k) \,|\, x_{\mathrm{u,s}} = 0 \text{ or } x_{\mathrm{u}} x_{\mathrm{s1}} \cdots x_{\mathrm{sN}} \neq 0 \}, \\ \Sigma_k^{\mathrm{out}} &= \Sigma_*^{\mathrm{out}}(p_k) \\ &= \{ (x_{\mathrm{ss,s}}, x_{\mathrm{c}}) \in \Sigma^{\mathrm{out}}(p_k) \,|\, x_{\mathrm{ss,s}} = 0 \text{ or } x_{\mathrm{ss}} x_{\mathrm{s1}} \cdots x_{\mathrm{sN}} \neq 0 \}, \end{aligned} \tag{5.2}$$

see (4.25). Then a sufficiently small upper bound for $x_{\mathrm{u}}^{\mathrm{in}}$ is chosen, i.e. Ψ_k^{loc} are restricted to smaller sections

$$\begin{aligned} \tilde{\Sigma}_k^{\mathrm{in}} &= \{ x \in \Sigma_k^{\mathrm{in}} \,|\, 0 \le x_{\mathrm{u}} \le \delta,\, 0 \le x_{\mathrm{s}\ell} \le \delta,\, |x_{\mathrm{c}}| \le \delta \} \\ &= ((0,\delta]^{N+1} \cup \{0\}) \times [-\delta, \delta]. \end{aligned} \tag{5.3}$$

This makes the contraction of the local passage as strong as we like without changing Ψ_k^{loc}, Ψ_k^{glob}. It also ensures that trajectories of interest stay close to the Kasner caps of heteroclinic orbits, and therefore the global excursions Ψ_k^{glob} on the domain of interest are as close to the Kasner map as we like. It also ensures that all non-singular trajectories in these domains indeed return to the following in-sections $\Sigma_{k+1}^{\mathrm{in}}$.

The global conditions 3.3 hold accordingly: the invariant subspaces, (glob-i), are those of the Bianchi system; the uniform bound, (glob-ii), is fixed by choice of a uniform size ε_0 of all local neighbourhoods; expansion, (glob-iii), is given by the Kasner map. Uniform expansion again needs a uniform distance from the Taub points.

The following lemma relates the global excursions to the new metric used for the local estimates.

Lemma 5.1. *Let the global conditions 3.3 be satisfied for the sequence Ψ_k^{glob} : $\Sigma_k^{\text{out}} \to \Sigma_{k+1}^{\text{in}}$ of global excursions. Then condition $(glob - ii)$ also holds with respect to the new metrics (4.21, 4.22).*

Proof. Due to the invariant subspaces, (glob-iii), the linearisation $D\Psi_k^{\text{glob}}$ are diagonal at the origin and close to diagonal in the neighbourhoods of interest. Moreover, the transverse components of Ψ_k^{glob} have the form

$$[\Psi_k^{\text{glob}}(x)]_* = [\tilde{\Psi}_k^{\text{glob}}(x)]_* x_*, \qquad * = \text{ss}, \text{s1}, \ldots, \text{sN},$$

with smooth $\tilde{\Psi}_k^{\text{glob}}$. Now note the definitions (4.21, 4.22) of the new metric. The bounds (glob-ii), on first and second derivatives of Ψ_k^{glob}, yield uniform bounds on $\tilde{\Psi}_k^{\text{glob}}$ and their first derivatives. Thus, the ratio of the coefficients of the metric at an arbitrary $x \in \Sigma_k^{\text{out}}$ to the coefficients at $\Psi_k^{\text{glob}}(x) \in \Sigma_{k+1}^{\text{in}}$ is between $\|(\tilde{\Psi}_k^{\text{glob}})^{-1}\|^{-1}\|\tilde{\Psi}_k^{\text{glob}}\|^{-1}$ and $\|(\tilde{\Psi}_k^{\text{glob}})^{-1}\|\|\tilde{\Psi}_k^{\text{glob}}\|$, that is between M^{-2} and M^2, for a uniform constant M. This immediately yields new uniform bounds on the derivatives of Ψ^{glob} with respect to the new metric. □

Now we can proceed along the lines of [7] (Sect. 4) to establish the existence of stable manifolds by a graph-transform approach.

Lemma 5.2. *Assume Conditions 3.2 on the local passages and Conditions 3.3 on the global excursions.*

Then the return maps (5.1) are Lipschitz continuous with respect to the metric (4.21). Furthermore, there exist $\varepsilon > 0$, $\delta > 0$, $0 < \sigma < 1$, $K_{u,s} > 1$, and $K_\text{c} > (1-\sigma^2)^{-1} > 1$, such that the following cone conditions hold for

$$\Psi_k \;=\; \Psi_k^{\text{glob}} \circ \Psi_k^{\text{loc}} \;:\; \tilde{\Sigma}_k^{\text{in}} \to \Sigma_{k+1}^{\text{in}}.$$

Here Σ_k^{in} are the in-sections (5.2) corresponding to the choice of ε, and $\tilde{\Sigma}_k^{\text{in}}$ are suitable subsets of the form (5.3).

The cones are defined for $x \in \tilde{\Sigma}^{\text{in}}$ (omitting the index k) as

$$\begin{aligned} C_x^{\text{c}} &= \{\tilde{x} \in \tilde{\Sigma}^{\text{in}} \,|\, \text{dist}_*(\tilde{x}_{\text{u,s}}, x_{\text{u,s}}) \le \sigma|\tilde{x}_\text{c} - x_\text{c}|\}, \\ C_x^{\text{u,s}} &= \{\tilde{x} \in \tilde{\Sigma}^{\text{in}} \,|\, |\tilde{x}_\text{c} - x_\text{c}| \le \sigma \text{dist}_*(\tilde{x}_{\text{u,s}} - x_{\text{u,s}})\}. \end{aligned} \tag{5.4}$$

The cone conditions are

(i) *Invariance: $\Psi(C_x^\text{c}) \cap \tilde{\Sigma}^{\text{in}} \subset (\text{int}\, C_{\Psi x}^\text{c}) \cup \{\Psi x\}$ and $\Psi^{-1}(C_{\Psi x}^{\text{u,s}}) \cap \tilde{\Sigma}^{\text{in}} \subset (\text{int}\, C_x^{\text{u,s}}) \cup \{x\}$;*

(ii) *Contraction & Expansion: For all $\tilde{x} \in C_x^\text{c}$, in the centre cone, we have expansion in the centre direction: $|(\Psi\tilde{x})_c - (\Psi x)_c| \ge K_\text{c}|\tilde{x}_c - x_c|$, and for all $\Psi\tilde{x} \in C_{\Psi x}^{\text{u,s}}$, in the transverse cone, we have contraction in the transverse directions: $\text{dist}_*(\tilde{x}_{u,s}, x_{u,s}) \ge K_{u,s}\text{dist}_*((\Psi\tilde{x})_{u,s}, (\Psi x)_{u,s})$.*

They hold for all, $x, \tilde{x}, \Psi x, \Psi\tilde{x} \in \tilde{\Sigma}^{\text{in}}$. See also Fig. 4.

The coefficients σ, δ only depend on ε_0 and the uniform expansion (glob-iii), that is the distance to the Taub point in the Bianchi system.

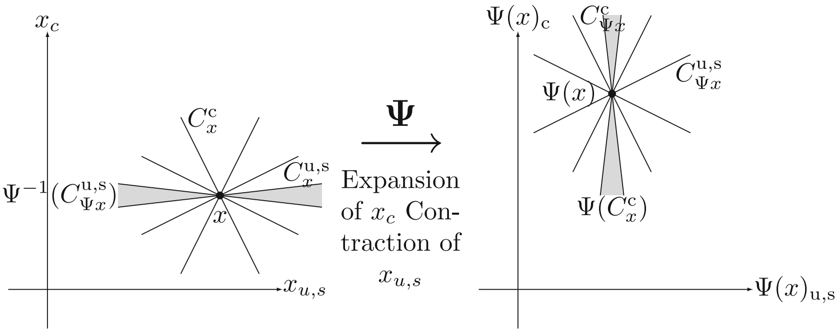

FIGURE 4. Cone properties of the return map Ψ

Proof. Lipschitz continuity of the return map Ψ_k follows directly from Lipschitz continuity of the local passage Ψ_k^{loc}, see Theorem 4.8, as the global excursion Ψ_k^{glob} is smooth. To simplify notation, we drop the index k from now on. All estimates will be uniform in k.

The cone conditions require the expansion in x_{c}-direction given by (glob-iii), corresponding to the expansion along the Kasner circle induced by the Kasner map of the Bianchi system. In fact, (glob-iii) states that we have

$$\Psi^{\mathrm{glob}}(x_{\mathrm{ss}}^{\mathrm{out}}=0, x_{\mathrm{s}}^{\mathrm{out}}=0, x_{\mathrm{c}}^{\mathrm{out}}) \;=\; (x_{\mathrm{u}}^{\mathrm{in}}=0, x_{\mathrm{s}}^{\mathrm{in}}=0, \Phi(x_{\mathrm{c}}^{\mathrm{out}})), \tag{5.5}$$

with Lipschitz continuous Φ, Φ^{-1}. The Lipschitz constant of Φ^{-1} is less than $L<1$ independent of k. Note again the invariant boundaries, (loc-v), (glob-i). Therefore we can write, as in the proof of Lemma 5.1,

$$\Psi^{\mathrm{glob}}(x_{\mathrm{ss}}, x_{\mathrm{s}}, x_{\mathrm{c}}) \;=\; (0,0,\Phi(x_{\mathrm{c}})) + \tilde{\Psi}^{\mathrm{glob}}(x_{\mathrm{ss}}, x_{\mathrm{s}}, x_{\mathrm{c}}) x_{\mathrm{ss,s}},$$

with a smooth matrix $\tilde{\Psi}^{\mathrm{glob}}(x_{\mathrm{ss}}, x_{\mathrm{s}}, x_{\mathrm{c}})$ and vector $x_{\mathrm{ss,s}} = (x_{\mathrm{ss}}, x_{\mathrm{s}})$.

Consider now two points $\tilde{x}, x \in \Sigma^{\mathrm{out}}$. Choose geodesic paths γ_1 from 0 to $x_{\mathrm{ss,s}}$ and γ_2 from $x_{\mathrm{ss,s}}$ to $\tilde{x}_{\mathrm{ss,s}}$, both with respect to the new metric (4.22). Then

$$\begin{aligned}
&\mathrm{dist}_*(\Psi^{\mathrm{glob}}(\tilde{x}), \Psi^{\mathrm{glob}}(x)) \le |\Phi(\tilde{x}_{\mathrm{c}}) - \Phi(x_{\mathrm{c}})| \\
&\qquad + \int_0^{\mathrm{dist}_*(0, x_{\mathrm{ss,s}})} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{dist}_*(\Psi^{\mathrm{glob}}(\gamma_1(s), \tilde{x}_{\mathrm{c}}), \Psi^{\mathrm{glob}}(\gamma_1(s), x_{\mathrm{c}}))\,\mathrm{d}s \\
&\qquad + \int_0^{\mathrm{dist}_*(\tilde{x}_{\mathrm{ss,s}}, x_{\mathrm{ss,s}})} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{dist}_*(\Psi^{\mathrm{glob}}(\gamma_2(s), \tilde{x}_{\mathrm{c}}), \Psi^{\mathrm{glob}}(x))\,\mathrm{d}s \\
&\quad \le |\Phi(\tilde{x}_{\mathrm{c}}) - \Phi(x_{\mathrm{c}})| \\
&\qquad + \int_0^{\mathrm{dist}_*(0, x_{\mathrm{ss,s}})} \frac{\mathrm{d}}{\mathrm{d}s} \int_{x_{\mathrm{c}}}^{\tilde{x}_{\mathrm{c}}} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{dist}_*(\Psi^{\mathrm{glob}}(\gamma_1(s), t), \Psi^{\mathrm{glob}}(\gamma_1(s), x_{\mathrm{c}}))\,\mathrm{d}t\,\mathrm{d}s \\
&\qquad + \int_0^{\mathrm{dist}_*(\tilde{x}_{\mathrm{ss,s}}, x_{\mathrm{ss,s}})} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{dist}_*(\Psi^{\mathrm{glob}}(\gamma_2(s), \tilde{x}_{\mathrm{c}}), \Psi^{\mathrm{glob}}(x))\,\mathrm{d}s,
\end{aligned}$$

and we obtain the following Lipschitz estimate:

$$\begin{aligned}
&\operatorname{dist}_*(\Psi^{\mathrm{glob}}(\tilde{x}) - (0,0,\Phi(\tilde{x}_{\mathrm{c}})), \Psi^{\mathrm{glob}}(x) - (0,0,\Phi(x_{\mathrm{c}}))) \\
&\quad \le \tilde{C}^{\mathrm{glob}} \left(\operatorname{dist}_*(0, x_{\mathrm{ss,s}})|\tilde{x}_{\mathrm{c}} - x_{\mathrm{c}}| + \operatorname{dist}_*(\tilde{x}_{\mathrm{ss,s}}, x_{\mathrm{ss,s}})\right) \\
&\quad \le C^{\mathrm{glob}} \left(\|x_{\mathrm{ss,s}}\||\tilde{x}_{\mathrm{c}} - x_{\mathrm{c}}| + \operatorname{dist}_*(\tilde{x}_{\mathrm{ss,s}}, x_{\mathrm{ss,s}})\right)
\end{aligned}$$

with C^{glob} only depending on the uniform bounds on $\|D\Psi^{\mathrm{glob}}\|, \|D^2\Psi^{\mathrm{glob}}\|$ with respect to the new metric provided by Lemma 5.1. The last inequality used the trivial upper bound (4.24) on the distance from the origin in the new metric.

Using $\Psi^{\mathrm{loc}}(\tilde{x})$ and $\Psi^{\mathrm{loc}}(x)$ instead of $\tilde{x}$ and x we get a similar estimate for the return map $\Psi(x) = \Psi^{\mathrm{glob}}(\Psi^{\mathrm{loc}}(x))$:

$$\begin{aligned}
&\operatorname{dist}_*(\Psi(\tilde{x}) - (0,0,\Phi(\Psi^{\mathrm{loc}}(\tilde{x})_{\mathrm{c}})), \Psi(x) - (0,0,\Phi(\Psi^{\mathrm{loc}}(x)_{\mathrm{c}}))) \\
&\quad \le C^{\mathrm{glob}}\big(\|\Psi^{\mathrm{loc}}(x)_{\mathrm{ss,s}}\||\Psi^{\mathrm{loc}}(\tilde{x})_{\mathrm{c}} - \Psi^{\mathrm{loc}}(x)_{\mathrm{c}}| \\
&\qquad\qquad + \operatorname{dist}_*(\Psi^{\mathrm{loc}}(\tilde{x})_{\mathrm{ss,s}}, \Psi^{\mathrm{loc}}(x)_{\mathrm{ss,s}})\big) \\
&\quad \le C^{\mathrm{glob}}\left(|x_{\mathrm{u}}|^{\beta}\|x\|(1+\varepsilon C)\operatorname{dist}_*(\tilde{x}, x) + |x_{\mathrm{u}}|^{\beta} C \operatorname{dist}_*(\tilde{x}, x)\right) \\
&\quad \le C^{\mathrm{return}}|x_{\mathrm{u}}|^{\beta}\operatorname{dist}_*(\tilde{x}, x).
\end{aligned} \tag{5.6}$$

The second last inequality uses the estimates of the local passage of Corollary 4.4 and Theorem 4.8 for the choice (w.l.o.g.) $0 \le \tilde{x}_{\mathrm{u}} \le x_{\mathrm{u}}$. Note that the estimates of Theorem 4.8 are used in the form

$$\begin{aligned}
|\Psi^{\mathrm{loc}}(\tilde{x})_{\mathrm{c}} - \Psi^{\mathrm{loc}}(x)_{\mathrm{c}}| &\le \varepsilon C \operatorname{dist}_*(\tilde{x}, x) + |\tilde{x}_{\mathrm{c}} - x_{\mathrm{c}}| \\
&\le (\varepsilon C + 1)\operatorname{dist}_*(\tilde{x}, x), \\
\operatorname{dist}_*(\Psi^{\mathrm{loc}}(\tilde{x})_{\mathrm{ss,s}}, \Psi^{\mathrm{loc}}(x)_{\mathrm{ss,s}}) &\le |x_{\mathrm{u}}|^{\beta} C \operatorname{dist}_*(\tilde{x}, x).
\end{aligned}$$

The constant C^{return} is uniform in x, $\tilde{x}$ in the in-section and the omitted number k of the section along the heteroclinic chain. Because $0 < \beta < \min\{\mu_{\mathrm{ss}}/\mu_{\mathrm{u}} - 1, \mu_{\mathrm{s}\ell}/\mu_{\mathrm{u}}\}$, we have an arbitrarily strong contraction for $x_{\mathrm{u}} < \delta$, if we choose δ small enough.

The map Φ given by (5.5), i.e. the Kasner map in the original Bianchi system, is expanding, see condition (glob-iii):

$$|\Phi(a) - \Phi(b)| \ \ge\ L^{-1}|a - b|,$$

for some uniform constant $L < 1$.

Now choose K_{c} with $1 < K_{\mathrm{c}} < L^{-1}$, and σ with $0 < \sigma < 1$ such that $K_{\mathrm{c}}(1-\sigma^2) > 1$. (The last relation is needed to obtain a contraction in Theorem 5.3.)

Consider the cone in centre direction with opening $\vartheta > 0$, i.e. assume $\operatorname{dist}_*(\tilde{x}_{\mathrm{u,s}}, x_{\mathrm{u,s}}) \le \vartheta|\tilde{x}_{\mathrm{c}} - x_{\mathrm{c}}|$. Then (5.6) using the local Lipschitz estimate of

Theorem 4.8 yields

$$\begin{aligned}
&|(\Psi(\tilde{x})-\Psi(x))_c| \\
&\quad \geq |\Phi(\Psi^{\rm loc}(\tilde{x})_{\rm c})-\Phi(\Psi^{\rm loc}(x)_{\rm c})| - C^{\rm return}|x_{\rm u}|^{\beta}{\rm dist}_*(\tilde{x},x) \\
&\quad \geq L^{-1}|\Psi^{\rm loc}(\tilde{x})_{\rm c}-\Psi^{\rm loc}(x)_{\rm c}| - C^{\rm return}|x_{\rm u}|^{\beta}{\rm dist}_*(\tilde{x},x) \\
&\quad \geq L^{-1}|\tilde{x}_{\rm c}-x_{\rm c}| - L^{-1}\varepsilon C{\rm dist}_*(\tilde{x},x) - C^{\rm return}|x_{\rm u}|^{\beta}{\rm dist}_*(\tilde{x},x) \\
&\quad \geq \left(L^{-1}-\left(L^{-1}\varepsilon C + C^{\rm return}|x_{\rm u}|^{\beta}\right)(1+\vartheta)\right)|\tilde{x}_{\rm c}-x_{\rm c}|. \qquad (5.7)
\end{aligned}$$

For ε and δ chosen small enough, using $|x_{\rm u}|\leq\delta$, we can achieve

$$K_{\rm c} \,<\, L^{-1}-\left(L^{-1}\varepsilon C + C^{\rm return}|x_{\rm u}|^{\beta}\right)(1+1/\sigma),$$

yielding the expansion not only in the cone $C_x^{\rm c}$, with $\vartheta=\sigma<1$, but also outside the cone $C_x^{\rm u,s}$, with $\vartheta=1/\sigma$.

Furthermore, using again (5.6), we see the invariance of the cones. Indeed, assume again ${\rm dist}_*(\tilde{x}_{\rm u,s},x_{\rm u,s})\leq\vartheta|\tilde{x}_{\rm c}-x_{\rm c}|$; then we have

$$\begin{aligned}
{\rm dist}_*(\Psi(\tilde{x})_{\rm u,s},\Psi(x)_{\rm u,s}) &\leq C^{\rm return}|x_u|^{\beta}{\rm dist}_*(\tilde{x},x) \\
&\leq C^{\rm return}|x_u|^{\beta}(1+\vartheta)|\tilde{x}_c-x_c| \\
&\leq C^{\rm return}|x_u|^{\beta}(1+\vartheta)K_{\rm c}^{-1}|\Psi(\tilde{x})_c-\Psi(x)_c|.
\end{aligned}$$

The last inequality uses the expansion in $x_{\rm c}$; thus it is valid for $\vartheta\leq 1/\sigma$. We choose δ small enough such that $C^{\rm return}|x_u|^{\beta}K_{\rm c}^{-1}<\sigma/(1+\sigma)$. Due to the monotone increase of $\vartheta/(1+\vartheta)$ we also have $C^{\rm return}|x_u|^{\beta}K_{\rm c}^{-1}<\vartheta/(1+\vartheta)$ for all $\vartheta\geq\sigma$. Thus we obtain the cone invariance

$${\rm dist}_*(\Psi(\tilde{x})_{\rm u,s},\Psi(x)_{\rm u,s}) \,\leq\, \vartheta|\Psi(\tilde{x})_{\rm c}-\Psi(x)_{\rm c}| \qquad (5.8)$$

for all $\sigma\leq\vartheta\leq 1/\sigma$.

The choice $\vartheta=\sigma$ yields (forward) invariance of the cone $C_x^{\rm c}$, and the choice $\vartheta=1/\sigma$ yields (backward) invariance of the cone $C_{\Psi x}^{\rm u,s}$. Note that the cone invariances are in fact strict as claimed in the lemma. The above estimates are strict inequalities for $x\neq\tilde{x}$.

Now consider the cone in transverse direction, that is $\Psi(\tilde{x})\in C_{\Psi x}^{\rm u,s}$, which amounts to $|\Psi(\tilde{x})_{\rm c}-\Psi(x)_{\rm c}|\leq\sigma{\rm dist}_*(\Psi(\tilde{x})_{\rm u,s},\Psi(x)_{\rm u,s})$. We have already established invariance. Thus $|\tilde{x}_{\rm c}-x_{\rm c}|\leq\sigma\|\tilde{x}_{\rm u,s}-x_{\rm u,s}\|$ and estimate (5.6) yields

$$\begin{aligned}
{\rm dist}_*(\Psi(\tilde{x})_{\rm u,s},\Psi(x)_{\rm u,s}) &\leq C^{\rm return}|x_u|^{\beta}{\rm dist}_*(\tilde{x},x) \\
&\leq C^{\rm return}|x_u|^{\beta}(1+\sigma){\rm dist}_*(\tilde{x}_{\rm u,s},x_{\rm u,s}).
\end{aligned}$$

This is the claimed contraction, $K_{\rm u,s}^{-1}=C^{\rm return}\delta^{\beta}(1+\sigma)<1$, for δ small enough. □

Theorem 5.3. *Assume Conditions 3.2 on the local passages and Conditions 3.3 on the global excursions.*

The (local) stable set of the origin under the sequence of return maps Ψ_k is given by

$$\mathcal{W}_k^{\rm loc}=\{(x_{\rm u}^k,x_{\rm s\ell}^k,x_{\rm c}^k)\,|\,x_{\rm c}^k=x_{\rm c}^k(x_{\rm u}^k,x_{\rm s\ell}^k)\}.$$

The functions x_{c}^k are Lipschitz continuous with respect to the metric (4.21)*. Furthermore, $x_{\mathrm{c}}^k(0) = 0$ and $\Psi(\mathcal{W}_k^{\mathrm{loc}}) \subset \mathcal{W}_{k+1}^{\mathrm{loc}}$.*

Proof. The idea of the proof is to define a graph transformation on the space of sequences of Lipschitz-continuous graphs $\{x_{\mathrm{u,s}} \mapsto x_{\mathrm{c}} = \zeta_k(x_{\mathrm{u,s}}) \,|\, k \in \mathbb{N}\}$ by the inverse return maps Ψ_k^{-1}. The uniform cone invariance provided by the previous lemma will ensure that the Lipschitz property of the graphs is preserved. Due to the expansion/contraction conditions of the previous lemma, the graph transformation turns out to be a contraction on the space of sequences of Lipschitz-continuous graphs. The fixed point of this contraction then yields the claim.

To make this idea precise, consider the Banach space of Lipschitz-continuous functions

$$X = \{\zeta \,:\, (0,\delta]^{1+N} \cup \{0\} \to [-\delta,\delta],\, x_{\mathrm{u,s}} = (x_{\mathrm{u}}, x_{\mathrm{s}}) \mapsto x_{\mathrm{c}} = \zeta(x_{\mathrm{u,s}}) \\ \text{such that } \mathrm{Lip}(\zeta) \le \sigma \text{ and } \zeta(0) = 0\}$$

with sup-norm. The parameters $\delta, \sigma < 1$ correspond to those of Lemma 5.2. Lipschitz continuity is considered with respect to the metric dist_* given by (4.21). Consider also the space of sequences

$$X^{\mathbb{N}} = \{\, (\zeta_k)_{k\in\mathbb{N}} \,|\, \zeta_k \in X \,\}$$

with sup-norm.

Define maps $G_k : X \to X$ as $\mathrm{graph}(G_k\zeta_{k+1}) := \Psi_k^{-1}\mathrm{graph}(\zeta_{k+1})$, i.e. as the transformations of the graphs of the functions in X. More precisely,

$$G_k\zeta\left(\left(\Psi_k^{-1}(x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}}))\right)_{\mathrm{u,s}}\right) := \left(\Psi_k^{-1}(x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}}))\right)_c, \quad \text{for } x_{\mathrm{u,s}} \neq 0,$$
$$G\zeta(0) := 0.$$

The first equation implicitly assumes that $(x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}}))$ has a pre-image under Ψ and that it lies in the domain. The second equation just gives the pre-image of the origin under Ψ. Note the restriction to non-negative $x_{\mathrm{u}}, x_{\mathrm{s1}}, \dots, x_{\mathrm{sN}}$ consistent with the invariant boundaries (loc-v), (glob-i).

We will prove the following claims, uniformly in the index k, (which is dropped from now on to simplify notation):

(i) domain of definition: for all $\zeta \in X$ and $x_{\mathrm{u,s}} \in (0,\delta]^{N+1}$ there exists $\tilde{x}_{\mathrm{u,s}} \in (0,\delta]^{N+1}$, such that $\left(\Psi^{-1}(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}}))\right)_{\mathrm{u,s}} = x_{\mathrm{u,s}}$.

(ii) well-definedness: for all $\zeta \in X$ and $x_{\mathrm{u,s}}, \tilde{x}_{\mathrm{u,s}} \in (0,\delta]^{N+1}$ the following holds: If $\left(\Psi^{-1}(x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}}))\right)_{\mathrm{u,s}} = \left(\Psi^{-1}(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}}))\right)_{\mathrm{u,s}} \in (0,\delta]^{N+1}$, then already $x_{\mathrm{u,s}} = \tilde{x}_{\mathrm{u,s}}$.

Conditions (i) and (ii) yield a well-defined function $G\zeta$ with $(G\zeta)(0) = 0$ for every $\zeta \in X$.

(iii) Lipschitz property: for all $\zeta \in X$ the function $G\zeta$ is again Lipschitz continuous with Lipschitz constant $\mathrm{Lip}(G\zeta) \le \sigma$. Note that the Lipschitz property is again considered with respect to the metric dist_*.

(iv) contraction: The exists a constant $0 < \kappa < 1$ such that for all $\zeta, \tilde{\zeta} \in X$ the estimate $\|G\tilde{\zeta} - G\zeta\|_{\sup} \le \kappa\|\tilde{\zeta} - \zeta\|_{\sup}$ holds.

Conditions (i)–(iii) prove that the graph transformation G indeed maps Lipschitz continuous functions in X to Lipschitz continuous functions in X, with respect to the metric dist_*. Condition (iv) provides a contraction. Uniformity of bounds yield a contraction on the space $X^{\mathbb{N}}$ of sequences. If all four conditions hold, then by contraction-mapping theorem there is a unique fixed point, i.e. a sequence of Lipschitz continuous function $\zeta_k^* \in X$ with $G_k \zeta_{k+1}^* = \zeta_k^*$.

Its graphs form a forward invariant set under Ψ composed of local manifold. It is also the stable set of the origin due to the cone conditions of Lemma 5.2. This yields the claim of the theorem. Therefore it remains to prove (i)–(iv):

(i) Let $\zeta \in X$ and $x_{\mathrm{u,s}} \in (0,\delta]^{N+1}$ be given. The straight line $\{x_{\mathrm{u,s}}\} \times [-\delta,\delta]$ is contained in the cone $C^{\mathrm{c}}_{(x_{\mathrm{u,s}},0)}$. We use Lemma 5.2: $\Psi(x_{\mathrm{u,s}},0) \in \tilde{\Sigma}^{\mathrm{in}}$ by invariance and contraction of the cone $C_0^{\mathrm{u,s}}$. Thus, by invariance and Expansion of $C^{\mathrm{c}}_{(x_{\mathrm{u,s}},0)}$, the image of the straight line $\{x_{\mathrm{u,s}}\} \times [-\delta,\delta]$ under Ψ contains a curve in $C^{\mathrm{c}}_{\Psi(x_{\mathrm{u,s}},0)}$ connecting the extremal planes $\{x_{\mathrm{c}} = \pm\delta\}$. By the intermediate value theorem this curve must intersect the graph of ζ.

(ii) Let $\zeta \in X$ and $x_{\mathrm{u,s}}, \tilde{x}_{\mathrm{u,s}} \in (0,\delta]^{N+1}$ be given with

$$\left(\Psi^{-1}(x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}}))\right)_{\mathrm{u,s}} = \left(\Psi^{-1}(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}}))\right)_{\mathrm{u,s}} \in (0,\delta]^{N+1}.$$

Then $\Psi^{-1}(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}})) \in C^{\mathrm{c}}_{\Psi^{-1}(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$, and by cone invariance we obtain $(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}})) \in C^{\mathrm{c}}_{(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$. The Lipschitz bound on $\zeta \in X$ on the other hand implies $(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}})) \in C^{\mathrm{u,s}}_{(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$; thus $(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}})) = (x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}}))$.

(iii) Again, the Lipschitz bound on $\zeta \in X$ translates to $(\tilde{x}_{\mathrm{u,s}}, \zeta(\tilde{x}_{\mathrm{u,s}})) \in C^{\mathrm{u,s}}_{(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$ for all $x, \tilde{x}$. Cone invariance and Lemma 5.2, immediately yield the Lipschitz bound on $G\zeta$.

(iv) The origin is fixed by construction; thus we only have to estimate the distance of the nonsingular part. Let $\zeta, \tilde{\zeta} \in X$ and $x_{\mathrm{u,s}}, \tilde{x}_{\mathrm{u,s}} \in (0,\delta]^{N+1}$ be given with $(\Psi^{-1}(x_{\mathrm{u,s}}, \zeta(x_{\mathrm{u,s}})))_{\mathrm{u,s}} = (\Psi^{-1}(\tilde{x}_{\mathrm{u,s}}, \tilde{\zeta}(\tilde{x}_{\mathrm{u,s}})))_{\mathrm{u,s}} \in (0,\delta]^{N+1}$.

Again, this implies $\Psi^{-1}(\tilde{x}_{\mathrm{u,s}}, \tilde{\zeta}(\tilde{x}_{\mathrm{u,s}})) \in C^{\mathrm{c}}_{\Psi^{-1}(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$, and by cone invariance we have $(\tilde{x}_{\mathrm{u,s}}, \tilde{\zeta}(\tilde{x}_{\mathrm{u,s}})) \in C^{\mathrm{c}}_{(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$. Thus we can estimate

$$\begin{aligned}|\tilde{\zeta}(\tilde{x}_{\mathrm{u,s}}) - \zeta(x_{\mathrm{u,s}})| &\le \|\tilde{\zeta} - \zeta\|_{\sup} + \sigma \mathrm{dist}_*(\tilde{x}_{\mathrm{u,s}}, x_{\mathrm{u,s}}) \\ &\le \|\tilde{\zeta} - \zeta\|_{\sup} + \sigma^2 |\tilde{\zeta}(\tilde{x}_{\mathrm{u,s}}) - \zeta(x_{\mathrm{u,s}})|\end{aligned}$$

The first inequality uses the Lipschitz bound on $\zeta \in X$, whereas the second one uses the aforementioned cone $C^{\mathrm{c}}_{(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$. We obtain

$$|\tilde{\zeta}(\tilde{x}_{\mathrm{u,s}}) - \zeta(x_{\mathrm{u,s}})| \le \frac{1}{1-\sigma^2} \|\tilde{\zeta} - \zeta\|_{\sup}.$$

On the other hand, the expansion of $C^{\mathrm{c}}_{\Psi^{-1}(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))}$ under Ψ yields

$$\begin{aligned}
&\left|(G\tilde{\zeta} - G\zeta)\left(\left(\Psi^{-1}(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))\right)_{\mathrm{u,s}}\right)\right| \\
&\quad = \left|\left(\Psi^{-1}(x_{\mathrm{u,s}},\zeta(x_{\mathrm{u,s}}))\right)_{\mathrm{c}} - \left(\Psi^{-1}(\tilde{x}_{\mathrm{u,s}},\tilde{\zeta}(\tilde{x}_{\mathrm{u,s}}))\right)_{\mathrm{c}}\right| \\
&\quad \leq \frac{1}{K_{\mathrm{c}}}\left|\zeta(x_{\mathrm{u,s}}) - \tilde{\zeta}(\tilde{x}_{\mathrm{u,s}})\right| \\
&\quad \leq \frac{1}{K_{\mathrm{c}}(1-\sigma^2)}\|\tilde{\zeta} - \zeta\|_{\mathrm{sup}}.
\end{aligned}$$

Lemma 5.2 provides constants K_{c}, σ with $K_{\mathrm{c}}(1-\sigma^2) > 1$. Therefore, the last estimates yield the claimed contraction, $\kappa = 1/(K_{\mathrm{c}}(1-\sigma^2))$, and this finishes the proof. □

With Theorem 5.3 we have finally proved the main Theorems 3.1, 3.4 as formulated in Sect. 3.

6. Discussion and Outlook

Unfortunately, the metric used to obtain the contraction in the proof of the main theorem of this paper is singular on the invariant subspaces. The proof gives no information on the way in which the manifolds constructed approach these boundaries. The only exception is the heteroclinic cycle itself. The cap of heteroclinic orbits corresponds to the line $\{0, x_{\mathrm{c}}\}$ and the new metric is regular there. In fact, the manifold is nicely attached to the given heteroclinic chain. Moreover, in the proof of Theorem 5.3 we could restrict to very small neighbourhoods of the primary heteroclinic chain, i.e. $\delta \to 0$. Then we can choose arbitrarily small Lipschitz bounds on the functions considered, i.e. $\sigma \to 0$. Thus the manifolds constructed are tangent to the fibre $\{x_{\mathrm{c}} = \text{constant}\}$ at the heteroclinic chain.

For completeness the following subtlety should be mentioned. The set of points in the domain of definition of the dynamical system corresponding to Bianchi type IX vacuum solutions or Bianchi type VI_0 solutions with magnetic field is an open subset bounded by invariant manifolds and it lies on only one side of these manifolds. The fact that the global excursion map has its image on the correct side of these manifolds is not mentioned in the analytical treatment above. Nevertheless, it follows immediately from the nature of the underlying geometrical problem.

Up to this point vacuum models of type IX were replaced by Einstein–Maxwell models of type VI_0 and one non-vanishing magnetic field component. Now some generalizations will be mentioned. In [7] some results were obtained for type IX solutions with perfect fluids having a linear equation of state $p = (\gamma - 1)\rho$. Restrictions had to be imposed on the value of γ. The techniques developed in this paper allow these results to be generalized to cases where these restrictions are relaxed. The four-dimensional dynamical system is replaced by a five-dimensional one and at each Kasner point there is an additional eigenvalue $3(2-\gamma)$. This situation can be treated for all $\gamma < 2$. In particular, the method applies for all values of γ in the physical range $[1, 2]$ except for the case $\gamma = 2$ where the dynamics is known to be very different. Bianchi type VI_0 solutions with a perfect fluid and a magnetic field can be treated in a very similar way. The additional eigenvalue arising from the fluid is the same as in the case without magnetic field [6].

It is possible to formulate the Bianchi type II models with a magnetic field as a five-dimensional dynamical system [5]. In this approach the magnetic field has only one non-zero component in the frame used but the metric has a non-zero off-diagonal component in that frame. The eigenvalues of the linearisation about a Kasner solution are given by

$$3p_1, \quad 6p_2, \quad 3(p_3 - p_1). \tag{6.1}$$

An important qualitative difference to the model of type VI_0 is that for some regions of the Kasner circle the stable manifold of the Kasner solution is two-dimensional. Thus in general the results of this paper do not apply to heteroclinic chains for the Bianchi type II model with magnetic field. In a similar way, it is possible to formulate the type I models with a magnetic field as a five-dimensional dynamical system with only one component of the magnetic field being non-zero [5]. In this case all the off-diagonal metric components are non-zero in general. Again it happens that the stable manifold can be two-dimensional. Note that it has been shown in [6] that it is not possible to have solutions of the Einstein–Maxwell equations of Bianchi type VIII or IX with a non-vanishing pure magnetic field.

Up to now there is no generalization of the results of [7] to oscillatory models of Bianchi class B. In fact it would be very interesting to have such results for Bianchi type $\mathrm{VI}_{-\frac{1}{9}}$ where oscillatory solutions are expected to exist. One obstacle is the existence of stable manifolds of dimension greater than one as in the examples with magnetic field above. Another is that invariant manifolds of the type which played such an important role in the proofs of this paper do not appear to exist for models of Bianchi class B.

In the case of Bianchi type IX vacuum models it has been proved that the α-limit set of each solution belongs to the union of points of type I and type II [11]. Interestingly, it is not known if the corresponding statement holds for the superficially similar type VIII. This contrasts with the fact that the results for type IX in [7] extend almost without change to type VIII. It is easy to formulate an analogue of the result of [11] for solutions of type VI_0 with magnetic field and it would be interesting to investigate whether it holds, especially since this might throw some new light on the unsolved Bianchi VIII problem.

To sum up, it is clear that the above complex of problems represents a promising opportunity to learn about the related questions of the BKL conjecture, the dynamics of Bianchi models near the initial singularity and the stability of heteroclinic cycles in more general dynamical systems.

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