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Existence of Wave Operators with Time-Dependent Modifiers for the Schrödinger Equations with Long-Range Potentials on Scattering Manifolds

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Abstract. We construct time-dependent wave operators for Schrödinger equations with long-range potentials on a manifold M with asymptotically conic structure. We use the two space scattering theory formalism, and a reference operator on a space of the form $\mathbb{R} \times \partial M$, where ∂M is the boundary of M at infinity. We construct exact solutions to the Hamilton–Jacobi equation on the reference system $\mathbb{R} \times \partial M$ and prove the existence of the modified wave operators.

1. Introduction

In this paper, we show the existence of wave operators for the Schödinger equations with long-range potentials on scattering manifolds, which have asymptotically conic structure at infinity (see Melrose [16] about scattering manifolds). We employ the formulation of Ito–Nakamura [13], which uses the two-space scattering framework of Kato [14]. Following Hörmander [10] and Dereziński and Gérard [4], we construct exact solutions to the Hamilton–Jacobi equation on the reference system and show the existence of the modified two-space wave operators using the stationary phase method.

Let M be an n-dimensional smooth non-compact manifold such that M is decomposed to $M_C \cup M_\infty$, where M_C is relatively compact, and M_∞ is diffeomorphic to $\mathbb{R}_+ \times \partial M$ with a compact manifold ∂M. We fix an identification map:

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$$\iota : M_\infty \longrightarrow \mathbb{R}_+ \times \partial M \ni (r, \theta).$$

We suppose $M_c \cap M_\infty \subset (0, 1/2) \times \partial M$ under this identification. We also suppose that ∂M is equipped with a measure $H(\theta)\mathrm{d}\theta$ where $H(\theta)$ is a smooth positive density.

Let $\{\phi_\lambda : U_\lambda \to \mathbb{R}^{n-1}\}, U_\lambda \subset \partial M$, be a local coordinate system of ∂M. We set $\{\tilde{\phi}_\lambda : \mathbb{R}_+ \times U_\lambda \to \mathbb{R} \times \mathbb{R}^{n-1}\}$ to be a local coordinate system of $M_\infty \cong \mathbb{R}_+ \times \partial M$, and we denote $(r, \theta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ to represent a point in M_∞. We suppose $G(x)$ is a smooth positive density on M such that

$$G(x)\mathrm{d}x = r^{n-1} H(\theta)\mathrm{d}r\mathrm{d}\theta \quad \text{on } \left(\frac{1}{2}, \infty\right) \times \partial M \subset M_\infty,$$

and we set

$$\mathscr{H} = L^2(M, G(x)\mathrm{d}x).$$

Let P_0 be a formally self-adjoint second-order elliptic operator on $\mathscr{H}$ of the form

$$P_0 = -\frac{1}{2} G^{-1}(\partial_r, \partial_\theta / r) G \begin{pmatrix} 1 + a_1 & a_2 \\ {}^t a_2 & a_3 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta / r \end{pmatrix} \quad \text{on } \tilde{M}_\infty = (1, \infty) \times \partial M$$

where a_1, a_2, and a_3 are real-valued smooth tensors.

Assumption 1. For any $l \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^{n-1}$ and $j = 1, 2, 3$, there is $C_{l,\alpha}$ such that

$$|\partial_r^l \partial_\theta^\alpha a_j(r, \theta)| \le C_{l,\alpha} r^{-\mu_j - l}$$

on $\tilde{M}_\infty$, where $\mu_j \ge 0$. Note that we use the coordinate system in M_∞ described above.

We will construct a time-dependent scattering theory for $P_0 + V$ on $\mathscr{H}$ where V is a potential.

Definition 2. Let $\mu_s > 0$. A finite rank differential operator V^S of the form $V^S = \sum_{l,\alpha} V_{l,\alpha}^S(r, \theta) \partial_r^l \partial_\theta^\alpha$ on M_∞ is said to be a short-range perturbation of μ_S type if for every l, α the coefficient $V_{l,\alpha}^S$ is a L_{loc}^2 tensor and satisfies

$$\int_{\mathbb{R}_+ \times U_\lambda} |V_{l,\alpha}^S(x)|^2 \langle r \rangle^{-M} G(x)\mathrm{d}x < \infty$$

for some M, and almost every $(\rho_0, \theta_0) \in \mathbb{R} \times \partial M$ has a neighborhood $\omega_{\rho_0, \theta_0}$ such that

$$\int_1^\infty \left(\int_{(\rho,\theta) \in \omega_{\rho_0,\theta_0}} |V_{l,\alpha}^S(t\rho, \theta)|^2 \mathrm{d}\rho H(\theta)\mathrm{d}\theta \right)^{1/2} t^{\mu_S |\alpha|} \mathrm{d}t < \infty.$$

Let $\mu_L > 0$. V^L is called a long-range smooth potential if V^L is a real-valued C^∞ function with support in $\tilde{M}_\infty$ and satisfies for any indices j, α,

$$|D_r^j D_\theta^\alpha V^L(r, \theta)| \le C_{j,\alpha} r^{-\mu_L - j}.$$

A differential operator V on M is called an admissible long-range perturbation of P_0 if V is of the form $V = V^S + V^L$ where V^S is a short-range perturbation of μ_S type and V^L is a long-range smooth potential and

$$\varepsilon = \mu_1 = \mu_2 = \mu_L > 0, \quad \mu_3 = 0, \quad \mu_S = 1 - \varepsilon.$$

Example 1. If $V^S = V^S(r, \theta)$ is a multiplication operator and $|V^S(r, \theta)| \leq Cr^{-1-\eta}$, $\eta > 0$, then V^S satisfies the aforementioned short-range condition.

If $V^S = \sum_{|\alpha|=1} V^S_\alpha \partial^\alpha_\theta$ and $|V^S_\alpha(r, \theta)| \leq Cr^{-1-\mu_S-\eta}, \eta > 0$, then V^S satisfies the aforementioned short-range condition. As the order of the derivative with respect to θ-variable increases, we need more rapid decay conditions on the coefficients.

Remark 2. If V^S is a smooth function, then $P_0 + V$ is essentially self-adjoint. More generally, if V^S is at most second-order differential operator with "small" smooth coefficients, then V^S is P_0-bounded with relative bound less than one, and $P_0 + V$ is essentially self-adjoint. In this paper, we assume that $P_0 + V$ is essentially self-adjoint on suitable domains (see Theorem 3) and do not investigate the conditions of self-adjointness.

Remark 3. If we assume ∂M is equipped with a positive (2, 0)-tensor $h = (h^{jk}(\theta))$, for some $\varepsilon > 0$,

$$|\partial^l_r \partial^\alpha_\theta (a_3(r, \theta) - h(\theta))| \leq C_{l,\alpha} r^{-\varepsilon - l},$$

and $V^S = 0$, then $P_0 + V$ has a self-adjoint extension H and corresponds (via a unitary equivalence) to the Laplacian on Riemannian manifolds with asymptotically conic structure. Since $\varepsilon > 0$, our model includes the scattering metric of long-range type described in [12]. Thus our results are generalizations of [13].

We prepare a reference system as follows:

$$M_f = \mathbb{R} \times \partial M, \quad \mathscr{H}_f = L^2(M_f, H(\theta)\mathrm{d}r\mathrm{d}\theta), \quad P_f = -\frac{1}{2}\frac{\partial^2}{\partial r^2} \quad \text{on } M_f$$

Note that P_f is essentially self-adjoint on $C_0^\infty(M_f)$, and we denote the unique self-adjoint extension by the same symbol. Let $j(r) \in C^\infty(\mathbb{R})$ be a real-valued function such that $j(r) = 1$ if $r \geq 1$ and $j(r) = 0$ if $r \leq 1/2$. We define the identification operator $J : \mathscr{H}_f \to \mathscr{H}$ by

$$(Ju)(r, \theta) = r^{-(n-1)/2} j(r) u(r, \theta) \quad \text{if} \quad (r, \theta) \in M_\infty$$

and $Ju(x) = 0$ if $x \notin M_\infty$, where $u \in \mathscr{H}_f$. We denote the Fourier transform with respect to r-variable by $\mathscr{F}$:

$$\mathscr{F}u(\rho, \theta) = \hat{u}(\rho) = \int_{-\infty}^{\infty} \mathrm{e}^{-ir\rho} u(r, \theta)\mathrm{d}r, \quad \text{for} \quad u \in C_0^\infty(M_f).$$

We decompose the reference Hilbert space $\mathscr{H}_f$ as $\mathscr{H}_f = \mathscr{H}_f^+ \oplus \mathscr{H}_f^-$, where $\mathscr{H}_f^\pm$ are defined by

$$\mathscr{H}_f^+ = \{u \in \mathscr{H} | \mathrm{supp}(\mathscr{F}u) \subset [0,\infty) \times \partial M\},$$
$$\mathscr{H}_f^- = \{u \in \mathscr{H} | \mathrm{supp}(\mathscr{F}u) \subset (-\infty,0] \times \partial M\}.$$

We use the following notation throughout the paper: For $x \in M$, we write

$$\langle x \rangle = \langle r \rangle = \begin{cases} 1 + rj(r) & \text{for } x \in M_\infty, \\ 1 & \text{for } x \in M_c. \end{cases}$$

We state our main theorem.

Theorem 3. *Suppose $V = V^L + V^S$ is an admissible long-range perturbation of P_0, V is symmetric on $J\mathscr{F}^{-1}C_0^\infty(M_f)$, and $P_0 + V$ has a self-adjoint extension H. Let $S(t,\rho,\theta)$ be a solution to the Hamilton–Jacobi equation which is constructed in Theorem 13. Then the modified wave operators*

$$\Omega_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH} J \mathrm{e}^{-iS(t,D_r,\theta)}$$

exist, and are partial isometries from $\mathscr{H}_f^\pm$ into $\mathscr{H}$ intertwining H and P_f:

$$\mathrm{e}^{isH}\Omega_\pm = \Omega_\pm \mathrm{e}^{isP_f}.$$

We refer Reed and Simon [20], Dereziński and Gérard [4], and Yafaev [22] for general concepts of wave operators and scattering theory for Schödinger equations. We here briefly review the history of wave operators. The concept of wave operator was introduced by Møller [17]. The existence of wave operators has long been studied (see Cook [2] and Kuroda [15]) for short-range potentials, which decay faster than the Coulomb potential. For the Coulomb potential, it was proved by Dollard [5,6] that the wave operators do not exist unless the definition is modified. Dollard introduced the concept of the modified wave operators $\text{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-iS(t,D_x)}$. Buslaev–Mateev [1] showed the existence of modified wave operators using stationary phase method and by employing an approximate solution to the Hamilton–Jacobi equation as a modifier function $S(t,\xi)$. Hörmander [10] constructed exact solutions to the Hamilton–Jacobi equation (see also [11] vol. IV).

The spectral properties of Laplace operators on a class of non-compact manifolds were studied by Froese et al. [8,9], and Donnelly [7] using the Mourre theory (see, the original paper Mourre [18], and Perry et al. [19]). In early 1990s, Melrose introduced a new framework of scattering theory on a class of Riemannian manifolds with metrics called scattering metrics (see [16] and references therein) and showed that the absolute scattering matrix, which is defined through the asymptotic expansion of generalized eigenfunctions, is a Fourier integral operator. Vasy [21] studied Laplace operators on such manifolds with long-range potentials of Coulomb-type decay ($|V(r,\theta)| \le Cr^{-1}$).

Ito and Nakamura [13] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds. They used the two-space scattering framework of Kato [14] with a simple reference operator $D_r^2/2$ on a space of the form $\mathbb{R} \times \partial M$, where ∂M is the boundary of the scattering manifold M.

We employ the formulation of Ito and Nakamura [13] and consider general long-range metric perturbations and potential perturbations. We assume that the scalar potential decay as $|V(r,\theta)| \le Cr^{-\varepsilon}, \varepsilon > 0$.

We make some remarks along with the outline of the proof. The time-dependent modifier function $S(t,\rho,\theta)$ is not uniquely determined. Our choice is a solution to the Hamilton–Jacobi equation on the reference manifold $\mathbb{R}\times\partial M$ with the long-range potential V^L:

$$h\left(\frac{\partial S}{\partial \rho}, \theta, \rho, -\frac{\partial S}{\partial \theta}\right) = \frac{\partial S}{\partial t},$$
$$h(r,\theta,\rho,\omega) = \frac{1}{2}\rho^2 + \frac{1}{2}a_1\rho^2 + \frac{1}{r}a_2^j\rho\omega_j + \frac{1}{r^2}a_3^{jk}\omega_j\omega_k + V^L, \tag{1}$$

for large t and for every ρ in any fixed compact set of $\mathbb{R}\backslash\{0\}$, where h is the corresponding classical Hamiltonian. We choose ρ and θ as variables of S because ρ and θ components of the classical trajectories have limits as t goes to infinity. The time-dependent modifier $\mathrm{e}^{-S(t,D_r,\theta)}$ is a Fourier multiplier in r-variable for each θ and we only need to consider the 1-dimensional Fourier transform with respect to r-variable. We construct solutions to the Hamilton–Jacobi equation mainly following Dereziński and Gérard [4].

In Sect. 2.1, we consider the boundary value problem for Newton equations on $\mathbb{R}\times\partial M$ with time-dependent slowly-decaying forces, which decay in time (Definition 4). In Theorem 5, we construct solutions and show several estimates. We use an integral equation and Banach's contraction mapping theorem (Proposition 7, refer Dereziński [3] and Sect. 1.5 of [4]). In the definition of slowly decaying forces (Definition 4) and the function spaces (Definition 6), we assume different decaying rates on different variables r, θ, ρ, and ω. These are efficiently used to show Proposition 7. We observe that the classical trajectories will stay in outgoing (incoming) regions as $t\to+\infty(-\infty)$.

In Sect. 2.2, we consider Newton equations with time-independent long-range forces which decay in space (Definition 8) in appropriate outgoing (incoming) regions. By inserting time-dependent cut-off functions, we introduce an effective time-dependent force and reduce the time-independent problem to the time-dependent one (Theorem 9). Our model (the Hamiltonian flow induced by the classical Hamiltonian) turns out to fit into this framework (Lemma 11). These tricks are also used in [4] for Hamiltonians with long-range potentials on Euclidean spaces.

Finally, in Sect. 2.3, in Theorem 13 we construct exact solutions to the Hamilton–Jacobi equation, using the classical trajectories with their dependence on initial data. Here we use the idea by Hörmander [10], see also Sect. 2.7 of [4]. We show that these solutions with their derivatives satisfy "good estimates", which are used to show the existence of the modifiers. Once we obtain a suitable modifier $S(t,\rho,\theta)$, we can show the existence of modified wave operators through stationary phase method (Sect. 3).

Using the Cook–Kuroda method (see Cook [2], and Kuroda [15]) and 1-dimensional Fourier transform, we deduce the proof of the main theorem to estimates of the integral (Proposition 14):

$$\int \left[h(r,\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta)) - h\left(\frac{\partial S}{\partial \rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta)\right)\right] \cdot \mathrm{e}^{ir\rho - iS(t,\rho,\theta)}\hat{u}(\rho,\theta)\mathrm{d}\rho.$$

In Sect. 3, we apply the stationary phase method (Lemma 16, we refer Hörmander [11, Sect. 7.7]). In the asymptotic expansion of the above integral, the terms in which h is not differentiated vanish since the equation $r = \partial S/\partial \rho$ holds at the stationary points. To show the uniformly boundedness of constants which appear in the asymptotic expansions of the integral, we construct diffeomorphisms in small neighborhoods of the stationary points which transform the phase function into quadratic forms there (Lemma 15). In the constructions of these diffeomorphisms, we use the estimates on the modifier function S.

In summary, we extended the time-dependent methods in [4] of quantum scattering in Euclidean spaces to the case of long-range scattering metric with long-range potential with the help of a simple reference system proposed in [13]. We expect that many of other methods in Euclidean scattering can also be applied to scattering on non-compact manifolds.

Notations. We use the following notation throughout the paper: Let $t \in \mathbb{R}$ and s be a parameter. We write $f(t,s) \in g(s)\mathrm{O}(\langle t\rangle^{-m})$ if $f(t,s) \le Cg(s)\langle t\rangle^{-m}$ uniformly for t and s. We write $f(t,s) \in \mathrm{O}(\langle t\rangle^{-\infty})$ if $f = 0$. We denote $f(t,s) \in g(s)\mathrm{o}(t^0)$ if $\lim_{t\to\infty} f(t,s)/g(s) = 0$.

2. Classical Mechanics

In this section, we study classical trajectories and solutions to the Hamilton–Jacobi equation.

2.1. Classical Trajectories with Slowly Decaying Time-Dependent Force

Let $(r,\theta,\rho,\omega) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1})$ and consider the Newton's equation:

$$(\dot{r},\dot{\theta},\dot{\rho},\dot{\omega})(t) = (\rho + F_r, F_\theta, F_\rho, F_\omega)(t,(r,\theta,\rho,\omega)(t)) \tag{2}$$

where

$$F = F(t,r,\theta,\rho,\omega) = (F_r,F_\theta,F_\rho,F_\omega)(t,r,\theta,\rho,\omega)$$

is a time-dependent force. Let $\varepsilon > 0$ and $\tilde{\varepsilon} = \frac{1}{2}\varepsilon$.

Definition 4. A time-dependent force F is said to be slowly decaying if F satisfies

$$\sup_{(r,\rho,\theta,\omega)\in T^*(\mathbb{R}\times\mathbb{R}^{n-1})} |\partial_r^l \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (F_*)(t)| \in \mathrm{O}(\langle t\rangle^{-n_*(l,\alpha,k,\beta)}), \quad * = \{r,\theta,\rho,\omega\} \tag{3}$$

where

$$\begin{aligned} &n_r(l,\alpha,k,\beta) = m(l,\alpha,k+1,\beta), \quad n_\theta(l,\alpha,k,\beta) = m(l,\alpha,k,\beta+e_i),\\ &n_\rho(l,\alpha,k,\beta) = m(l+1,\alpha,k,\beta), \quad n_\omega(l,\alpha,k,\beta) = m(l,\alpha+e_i,k,\beta),\\ &m(l,\alpha,0,0) = l+\varepsilon, \quad m(l,\alpha,1,0) = l+\varepsilon, \quad m(l,\alpha,2,0) = l+\varepsilon, \end{aligned} \tag{4}$$

$$m(l,\alpha,0,e_i) = l+1+\tilde{\varepsilon},\quad m(l,\alpha,1,e_i) = l+1+\varepsilon,\quad m(l,\alpha,0,e_i+e_j) = l+2,$$
$$m(l,\alpha,k,\beta) = +\infty,\qquad \text{if } k+|\beta|\geq 3,$$

$i,j = 1,\ldots,n-1$, and $e_i = (0,\ldots,1,0,\ldots,0) \in \mathbb{Z}_+^{n-1}$ is the canonical unit vector, i.e., every component of e_i is 0 except ith component.

In the next theorem, we show the unique existence of trajectories for the dynamics (2) where the boundary conditions are the initial position and the final momentum.

Theorem 5. *Assume that F is a time-dependent slowly decaying force in the sense of Definition 4. Then there exists T such that if $T \leq t_1 < t_2 \leq \infty$ (we allow t_2 to be ∞) and $(r_i,\theta_f,\rho_f,\omega_i) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1})$, there exists a unique trajectory*

$$[t_1,t_2] \ni s \mapsto (\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)$$

satisfying

$$\begin{aligned}
&\partial_s(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)\\
&\quad= (\rho+F_r,F_\theta,F_\rho,F_\omega)(s,(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)),\\
&\qquad(\tilde{r},\tilde{\omega})(t_1,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i) = (r_i,\omega_i),\\
&\qquad(\tilde{\theta},\tilde{\rho})(t_2,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i) = (\theta_f,\rho_f).
\end{aligned}$$

We set $\underline{r}(s),\underline{\theta}(s),\underline{\rho}(s),\underline{\omega}(s)$ by

$$\underline{r}(s) = \tilde{r}(s) - r_i - (s-t_1)\rho_f,\quad \underline{\theta}(s) = \tilde{\theta}(s) - \theta_f,$$
$$\underline{\rho}(s) = \tilde{\rho}(s) - \rho_f,\quad \underline{\omega}(s) = \tilde{\omega}(s) - \omega_i.$$

Moreover, the solution satisfies the following estimates uniformly for $T \leq t_1 \leq s \leq t_2 \leq \infty$, $(r_i,\theta_f,\rho_f,\omega_i) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1})$:

$$\begin{aligned}
&|\underline{r}(s)| \in \mathrm{o}(s^0)|s-t_1|,\quad |\underline{\theta}(s)| \in \mathrm{O}(s^{-\tilde{\varepsilon}}),\\
&|\underline{\rho}(s)| \in \mathrm{O}(s^{-\tilde{\varepsilon}}),\quad |\underline{\omega}(s)| \in \mathrm{o}(s^0)|s-t_1|^{1-\tilde{\varepsilon}},
\end{aligned}\tag{5}$$

$$\begin{aligned}
&\begin{pmatrix}
\partial_{r_i}\underline{r}(s) & \partial_{\theta_f}\underline{r}(s) & \partial_{\rho_f}\underline{r}(s) & \partial_{\omega_i}\underline{r}(s)\\
\partial_{r_i}\underline{\theta}(s) & \partial_{\theta_f}\underline{\theta}(s) & \partial_{\rho_f}\underline{\theta}(s) & \partial_{\omega_i}\underline{\theta}(s)\\
\partial_{r_i}\underline{\rho}(s) & \partial_{\theta_f}\underline{\rho}(s) & \partial_{\rho_f}\underline{\rho}(s) & \partial_{\omega_i}\underline{\rho}(s)\\
\partial_{r_i}\underline{\omega}(s) & \partial_{\theta_f}\underline{\omega}(s) & \partial_{\rho_f}\underline{\omega}(s) & \partial_{\omega_i}\underline{\omega}(s)
\end{pmatrix}\\
&\quad\in \begin{pmatrix}
\mathrm{o}(s^0)|s-t_1|\\
\mathrm{O}(1)\\
\mathrm{O}(s^{-\tilde{\varepsilon}})\\
\mathrm{o}(s^0)|s-t_1|^{1-\tilde{\varepsilon}}
\end{pmatrix} \otimes (t_1^{-1-\tilde{\varepsilon}},t_1^{-\tilde{\varepsilon}},t_1^{-\tilde{\varepsilon}},t_1^{-1}),
\end{aligned}\tag{6}$$

$$\partial_{r_i}^l\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta\begin{pmatrix}\underline{r}\\ \underline{\theta}\\ \underline{\rho}\\ \underline{\omega}\end{pmatrix} \in \begin{pmatrix}
\mathrm{o}(s^0)|s-t_1|\\
\mathrm{O}(1)\\
\mathrm{O}(s^{-\tilde{\varepsilon}})\\
\mathrm{o}(s^0)|s-t_1|^{1-\tilde{\varepsilon}}
\end{pmatrix}\cdot t_1^{-l-|\beta|}.\tag{7}$$

Here $\otimes$ is an outer product and (6) means, for example, $\partial_{r_i}\underline{r}(s) \in \mathrm{o}(s^0)|s-t_1|t_1^{-1-\tilde{\varepsilon}}$ and $\partial_{\theta_f}\underline{\theta}(s) \in \mathrm{O}(1)t_1^{-\tilde{\varepsilon}}$. Estimates such as $f_{T,t_1,t_2}(s) \in \mathrm{o}(s^0)$ mean

$\sup |f_{T,t_1,t_2}(s)| \to 0$ *as* $s \to \infty$ *where supremum is taken over* T, t_1, *and* t_2 *satisfying* $T \leq t_1 \leq s \leq t_2 \leq \infty$.

A straightforward computation shows that $(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s)$ satisfies the following integral equation:

$$(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s) = (P_r, P_\theta, P_\rho, P_\omega)(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s)$$
$$:= \begin{pmatrix} \int_{t_1}^{s} \big(\underline{\rho}(u) + F_r(u, r_i + (u-t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))\big)\mathrm{d}u \\ -\int_{s}^{t_2} \big(F_\theta(u, r_i + (u-t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))\big)\mathrm{d}u \\ -\int_{s}^{t_2} \big(F_\rho(u, r_i + (u-t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))\big)\mathrm{d}u \\ \int_{t_1}^{s} \big(F_\omega(u, r_i + (u-t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))\big)\mathrm{d}u \end{pmatrix} \tag{8}$$

where the map $P = (P_r, P_\theta, P_\rho, P_\omega)$ depends on the parameters $t_1, t_2, r_i, \theta_f, \rho_f, \omega_i$. We will apply the fixed point theorem to solve (8). We define the Banach space on which the map P is defined as follows:

Definition 6. For $m \geq 0$, we define

$$Z_T^m := \left\{ z \in C([T,\infty)) : \sup \frac{|z(t)|}{|t-T|^m} < \infty \right\},$$
$$Z_{T,\infty}^m := \left\{ z \in Z_T^m : \lim_{t\to\infty} \frac{|z(t)|}{|t-T|^m} = 0 \right\}.$$

For $m < 0$, we define

$$Z_T^m := \left\{ z \in C([T,\infty)) : \sup \frac{|z(t)|}{\langle t\rangle^m} < \infty \right\}.$$

We define

$$Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}} := \{(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega}) \in Z_{t_1,\infty}^1 \times Z_{t_1}^0 \times Z_{t_1}^{-\tilde{\varepsilon}} \times Z_{t_1,\infty}^{1-\tilde{\varepsilon}}\}.$$

Then we have the following proposition:

Proposition 7. *For large enough* $T > 0$, *the map* P *is a contraction map on* $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$ *for any* $T \leq t_1 \leq t_2 \leq \infty, (r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$. *Indeed, for some constant* c *which does not depend on* $t_1, t_2, (r_i, \theta_f, \rho_f, \omega_i)$ *but on* T, *we have*

$$\|\nabla_x P(x)\|_{B\left(Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}\right)} < c < 1. \tag{9}$$

Proof. We first note that P is well defined as a map of $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$ into itself. Indeed, for example, if $x = (\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega}) \in Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$,

$$\begin{aligned} &|P_r(x)(s)| \\ &\quad \leq \int_{t_1}^{s} |\big(\underline{\rho}(u) + F_r(u, r_i + (s-t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))\big)|\mathrm{d}u \\ &\quad \leq \int_{t_1}^{s} |C\langle u\rangle^{-\tilde{\varepsilon}} + C\langle u\rangle^{-\varepsilon}|\mathrm{d}u, \end{aligned}$$

which implies $P_r(x)(s) \in Z_{t_1,\infty}^1$. Others are similar to prove.

Now we check that P is a contraction on $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$. It suffices to show (9) for some constant c which does not depend on $t_1, t_2, (r_i, \theta_f, \rho_f, \omega_i)$ but on T. Let $v \in Z^1_{t_1,\infty}$. Then

$$
\begin{aligned}
|s-t_1|^{-1}(\nabla_r P_r(x)v)(s) &\leq |s-t_1|^{-1}\int_{t_1}^{s} \|\nabla_r F_r(u,\cdot)\|_\infty |u-t_1| \|v\|_{Z^1_{t_1,\infty}} \mathrm{d}u \\
&\leq \|v\|_{Z^1_{t_1,\infty}} \int_{t_1}^{s} |s-t_1|^{-1}|u-t_1|\langle u\rangle^{-1-\varepsilon}\mathrm{d}u.
\end{aligned}
$$

If we let $T \to \infty$, the right-hand side goes to zero uniformly for $T \leq t_1 \leq t_2 \leq \infty$. Moreover, the right-hand side goes to zero as $s \to \infty$. Hence taking T large enough, we may assure that

$$\|\nabla_r P_r\|_{B(Z^1_{t_1,\infty})} < c < 1$$

for some constant c for any $T \leq t_1 \leq t_2 \leq \infty$ and for any $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$. In a similar way, we can show that for some large enough T, (9) holds for any $T \leq t_1 \leq t_2 \leq \infty$ and for any $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1})$. ☐

Proof of Theorem 5. The fixed point theorem together with Proposition 7 implies that there exists a unique solution $(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s) \in Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$ for the integral equation (8) for each $T \leq t_1 < t_2 \leq \infty$ and $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$ if T is large enough. $(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s) \in Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$ directly means (5).

Let us now prove (6). We use the identity

$$
(I-\nabla_x P(x))\partial^\gamma(x) = h^\gamma = (h^\gamma_r, h^\gamma_\theta, h^\gamma_\rho, h^\gamma_\omega)
$$

$$
:= \begin{pmatrix} \int_{t_1}^{s}(\nabla F_r)(u,y)\partial^\gamma(y-x)\mathrm{d}u \\ -\int_{s}^{t_2}(\nabla F_\theta)(u,y)\partial^\gamma(y-x)\mathrm{d}u \\ -\int_{s}^{t_2}(\nabla F_\rho)(u,y)\partial^\gamma(y-x)\mathrm{d}u \\ \int_{t_1}^{s}(\nabla F_\omega)(u,y)\partial^\gamma(y-x)\mathrm{d}u \end{pmatrix} \tag{10}
$$

where $\partial^\gamma = \partial_{r_i}, \partial_{\theta_f}, \partial_{\rho_f}$, or ∂_{ω_i}, $x = (\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})$ is the solution of (8), and $y = (r_i + (u-t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))$. By a straight computation we have

$$(h^{\partial_{r_i}}, h^{\partial_{\theta_i}}, h^{\partial_{\rho_i}}, h^{\partial_{\omega_i}}) \in (\langle t_1\rangle^{-1-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-1}) Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}.$$

(9) implies that $I - \nabla_x P(x)$ is invertible on $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$. Using (10), we get

$$(\partial_{r_i}x, \partial_{\theta_f}x, \partial_{\rho_f}x, \partial_{\omega_i}x) \in Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}},$$

and

$$\|(\partial_{r_i}x, \partial_{\theta_f}x, \partial_{\rho_f}x, \partial_{\omega_i}x)\|_{Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}} \in \mathrm{O}(\langle t_1\rangle^{-1-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-1}),$$

which implies (6).

Now we prove (7) by an induction. Assume that $\partial^\gamma = \partial_{r_i}^l \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta$, $l + |\alpha| + k + |\beta| = n \geq 2$, and (7) is true for $l + |\alpha| + k + |\beta| \leq n - 1$. We use the identity

$$(I - \nabla_x P(x))\partial^\gamma(x) = h^\gamma = (h_r^\gamma, h_\theta^\gamma, h_\rho^\gamma, h_\omega^\gamma)$$

$$:= \begin{pmatrix} \int_{t_1}^s \sum_{q\geq 2} (\nabla^q F_r)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)\mathrm{d}u \\ -\int_s^{t_2} \sum_{q\geq 2} (\nabla^q F_\theta)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)\mathrm{d}u \\ -\int_s^{t_2} \sum_{q\geq 2} (\nabla^q F_\rho)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)\mathrm{d}u \\ \int_{t_1}^s \sum_{q\geq 2} (\nabla^q F_\omega)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)\mathrm{d}u \end{pmatrix} \tag{11}$$

where the sum is taken over $\gamma = \sum_{p=1}^q \gamma_p, q \geq 2$. The induction hypothesis with a straight computation shows that

$$(h^\gamma) \in (\langle t_1\rangle^{-l-|\beta|}) Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}.$$

Thus we have

$$\|\partial_\gamma x\|_{Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}} \in (\langle t_1\rangle^{-l-|\beta|}),$$

which implies (7). □

2.2. Classical Trajectories with Long-Range Time-Independent Force

We denote the outgoing region by $\Gamma_{R,U,J,Q}^{+,\tilde{\varepsilon}}$:

$$\Gamma_{R,U,J,Q}^{+,\tilde{\varepsilon}} := \{(r,\theta,\rho,\omega) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1}) : r > R, \theta \in U, \rho \in J, |\omega| \leq Qr^{1-\tilde{\varepsilon}}\}$$

for $R > 0, U \subset \mathbb{R}^{n-1}, J \subset \mathbb{R}, Q > 0$.

We now consider the dynamics with time-independent long-range forces.

Definition 8. A time-independent force F is said to be a long-range force if it satisfies

$$\sup_{(r,\theta,\rho,\omega)\in\Gamma_{R,U,J,Q}^{+,\tilde{\varepsilon}}} |\partial_r^l \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (F_r, F_\theta, F_\rho, F_\omega)(r,\theta,\rho,\omega)| \in \mathrm{O}(\langle R\rangle^{-n_{r,\theta,\rho,\omega}(l,\alpha,k,\beta)}) \tag{12}$$

for any $R > 0$, $U \Subset \mathbb{R}^{n-1}$, $J \Subset (0,\infty)$, $Q > 0$.

As in Theorem 5, we show the unique existence of trajectories for the dynamics where the boundary conditions are the initial position and the final momentum.

Theorem 9. *Assume that F is a time-independent long-range force in the sense of Definition 8. Then for any open $U \Subset \tilde{U} \Subset \mathbb{R}^{n-1}$, open $J \Subset \tilde{J} \Subset (0,\infty)$, and $Q > 0$, there exists $R > 0$ such that for any $t \geq 0$ and for any $(r_i, \theta_f, \rho_f, \omega_i) \in \Gamma_{R,U,J,Q}^{+,\tilde{\varepsilon}}$, there exists a unique trajectory*

$$[0,t] \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s,t,r_i,\theta_f,\rho_f,\omega_i)$$

satisfying

$$\partial_s(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t,r_i,\theta_f,\rho_f,\omega_i)=(\rho+F_r,F_\theta,F_\rho,F_\omega)((\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t,r_i,\theta_f,\rho_f,\omega_i)) \tag{13}$$

$$(\tilde{r},\tilde{\omega})(0,t,r_i,\theta_f,\rho_f,\omega_i)=(r_i,\omega_i),\quad (\tilde{\theta},\tilde{\rho})(t,t,r_i,\theta_f,\rho_f,\omega_i)=(\theta_f,\rho_f),$$

and the estimates

$$\begin{aligned} &|\underline{r}(s)|\in \mathrm{o}((s+\langle r_i\rangle)^0)|s|,\quad |\underline{\theta}(s)|\in \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}}),\\ &|\underline{\rho}(s)|\in \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}}),\quad |\underline{\omega}(s)|\in \mathrm{o}((s+\langle r_i\rangle)^0)|s|^{1-\tilde{\varepsilon}}, \end{aligned} \tag{14}$$

and

$$\tilde{\theta}(s,t,r_i,\theta_f,\rho_f,\omega_i)\in \tilde{U},\quad \tilde{\rho}(s,t,r_i,\theta_f,\rho_f,\omega_i)\in \tilde{J}$$

where

$$\underline{r}(s)=\tilde{r}(s)-r_i-s\rho_f,\quad \underline{\theta}(s)=\tilde{\theta}(s)-\theta_f$$
$$\underline{\rho}(s)=\tilde{\rho}(s)-\rho_f,\quad \underline{\omega}(s)=\tilde{\omega}(s)-\omega_i.$$

Moreover, the solution satisfies the following estimates uniformly for $0\le s\le t\le\infty$, $(r_i,\theta_f,\rho_f,\omega_i)\in\Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$*:*

$$\begin{pmatrix} \partial_{r_i}\underline{r}(s) & \partial_{\theta_f}\underline{r}(s) & \partial_{\rho_f}\underline{r}(s) & \partial_{\omega_i}\underline{r}(s)\\ \partial_{r_i}\underline{\theta}(s) & \partial_{\theta_f}\underline{\theta}(s) & \partial_{\rho_f}\underline{\theta}(s) & \partial_{\omega_i}\underline{\theta}(s)\\ \partial_{r_i}\underline{\rho}(s) & \partial_{\theta_f}\underline{\rho}(s) & \partial_{\rho_f}\underline{\rho}(s) & \partial_{\omega_i}\underline{\rho}(s)\\ \partial_{r_i}\underline{\omega}(s) & \partial_{\theta_f}\underline{\omega}(s) & \partial_{\rho_f}\underline{\omega}(s) & \partial_{\omega_i}\underline{\omega}(s) \end{pmatrix}$$

$$\in \begin{pmatrix} \mathrm{o}((s+\langle r_i\rangle)^0)|s|\\ \mathrm{O}(1)\\ \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}})\\ \mathrm{o}((s+\langle r_i\rangle)^0)|s|^{1-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_i\rangle^{-1-\tilde{\varepsilon}},\langle r_i\rangle^{-\tilde{\varepsilon}},\langle r_i\rangle^{-\tilde{\varepsilon}},\langle r_i\rangle^{-1}),$$

$$\partial_{r_i}^l\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta \begin{pmatrix} \underline{r}\\ \underline{\theta}\\ \underline{\rho}\\ \underline{\omega} \end{pmatrix} \in \begin{pmatrix} \mathrm{o}((s+\langle r_i\rangle)^0)|s|\\ \mathrm{O}(1)\\ \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}})\\ \mathrm{o}((s+\langle r_i\rangle)^0)|s|^{1-\tilde{\varepsilon}} \end{pmatrix} \cdot \langle r_i\rangle^{-l-|\beta|}.$$

Proof. There exists C_0 such that if $\rho\in J$ and $r,s>0$, then

$$|r+s\rho|\ge C_0(|s|+r). \tag{15}$$

We fix constants $\varepsilon_0,\tilde{Q},\varepsilon_1$ such that

$$0<\varepsilon_0<C_0,\quad \tilde{Q}\ge\frac{2Q}{C_0^{1-\tilde{\varepsilon}}},\quad 0<\varepsilon_1<\frac{1}{2}\tilde{Q}(C_0-\varepsilon_0)^{1-\tilde{\varepsilon}},$$

and introduce cut-off functions $I_r,I_\theta,I_\rho,I_\omega$ as follows: We take $I_r\in C^\infty(0,\infty)$ such that $I_r=1$ on a neighborhood of $\{r;r>C_0-\varepsilon_0\}$, $I_\theta\in C_0^\infty(\mathbb{R}^{n-1})$ such that $I_\theta=1$ on $\tilde{U}$, $I_\rho\in C_0^\infty(0,\infty)$ such that $I_\rho=1$ on $\tilde{J}$, and $I_\omega\in C_0^\infty(\mathbb{R}^{n-1})$ such that $I_\omega=1$ on a neighborhood of $\{\omega:|\omega|<\tilde{Q}\}$. Using these cut-off functions, we define the effective time-dependent force F_{e} by

$$F_{\mathrm{e}}(t,r,\theta,\rho,\omega)=I_r\left(\frac{r}{t}\right)I_\theta(\theta)I_\rho(\rho)I_\omega\left(\frac{\omega}{r^{1-\tilde{\varepsilon}}}\right)F(r,\theta,\rho,\omega).$$

It follows from (12) that $F_{\rm e}(t,r,\theta,\rho,\omega)$ is a slowly decaying force in the sense of Definition 4. Therefore, we can find T such that the boundary value problem considered in Theorem 5 possesses a unique solution for any $T \le t_1 \le t_2$ and any $r_i,\theta_f,\rho_f,\omega_i$. Let us denote this solution by

$$(\tilde r_e,\tilde\theta_e,\tilde\rho_e,\tilde\omega_e)(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i).$$

By enlarging T if needed, we can guarantee that

$$\begin{aligned}|\tilde r_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)-r_i-(s-t_1)\rho_f| &\le \varepsilon_0|s-t_1|, && (16)\\ |\tilde\theta_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)-\theta_f| &\le \mathrm{dist}(U,\tilde U^C)\\ |\tilde\rho_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)-\rho_f| &\le \mathrm{dist}(J,\tilde J^C)\\ |\tilde\omega_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i)-\omega_i| &\le \varepsilon_1|s-t_1|^{1-\tilde\varepsilon}. && (17)\end{aligned}$$

We claim that if $R=T(C_0-\varepsilon_0)/C_0$ and $(r_i,\theta_f,\rho_f,\omega_i)\in\Gamma^{+,\tilde\varepsilon}_{R,U,J,Q}$, then we can solve our boundary problem by setting

$$(\tilde r,\tilde\theta,\tilde\rho,\tilde\omega)(s,t,r_i,\theta_f,\rho_f,\omega_i):=(\tilde r_e,\tilde\theta_e,\tilde\rho_e,\tilde\omega_e)(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i) \quad (18)$$

where $r=|r_i|C_0/(C_0-\varepsilon)$. Indeed, from (15), (16), and (17) we see that

$$\begin{gathered}|\tilde r_e(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i)|\ge (C_0-\varepsilon_0)|s+r|,\\ \tilde\theta_e(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i)\in\tilde U,\\ \tilde\rho_e(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i)\in\tilde J,\end{gathered}$$

and

$$\begin{aligned}|\tilde\omega_e(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i)| &\le \varepsilon_1|s|^{1-\tilde\varepsilon}+\omega_i\le\varepsilon_1|s|^{1-\tilde\varepsilon}+Qr_i^{1-\tilde\varepsilon}\\ &\le \varepsilon_1|s|^{1-\tilde\varepsilon}+Q\big(\frac{C_0-\varepsilon_0}{C_0}\big)^{1-\tilde\varepsilon}r^{1-\tilde\varepsilon}\le\tilde Q(C_0-\varepsilon_0)^{1-\tilde\varepsilon}|s+r|^{1-\tilde\varepsilon}\\ &\le \tilde Q|\tilde r_e(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i)|^{1-\tilde\varepsilon}.\end{aligned}$$

Hence we have

$$\begin{aligned}&F_{\rm e}(r+s,(\tilde r_e,\tilde\theta_e,\tilde\rho_e,\tilde\omega_e)(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i))\\ &\quad=F((\tilde r_e,\tilde\theta_e,\tilde\rho_e,\tilde\omega_e)(r+s,r,r+t,r_i,\theta_f,\rho_f,\omega_i)).\end{aligned}$$

Therefore, the function (18) solves the boundary problem (13) with the initial time-independent force.

The estimates on $(\tilde r,\tilde\theta,\tilde\rho,\tilde\omega)(s,t,r_i,\theta_f,\rho_f,\omega_i)$ are obtained directly from those of Theorem 5 using the identity (18) and replacing s,t_1,t_2 there by $s+\langle r_i\rangle,\langle r_i\rangle,t+\langle r_i\rangle$.

Finally, the uniqueness of the solution follows from the fact that any solution of (13) with (14) is also a solution of the problem considered in Theorem 5 for the force $F_{\rm e}(t,r,\theta,\rho,\omega)$ if time t is large enough. □

Now we solve the dynamics with initial conditions.

Theorem 10. *Assume F is a time-independent long-range force in the sense of Definition 8. Then for any open $U\Subset\tilde U\Subset\mathbb{R}^{n-1}$, open $J\Subset\tilde J\Subset(0,\infty)$, and*

$Q > 0$, there exists $R > 0$ such that for any $(r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$, there exists a unique trajectory

$$[0, \infty) \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0)$$

satisfying

$$\partial_s(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0) = (\rho + F_r, F_\theta, F_\rho, F_\omega)((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0)),$$
$$(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(0, r_0, \theta_0, \rho_0, \omega_0) = (r_0, \theta_0, \rho_0, \omega_0).$$

Set

$$\underline{r}(s) = \tilde{r}(s) - r_0 - s\rho_0, \quad \underline{\theta}(s) = \tilde{\theta}(s) - \theta_0,$$
$$\underline{\rho}(s) = \tilde{\rho}(s) - \rho_0, \quad \underline{\omega}(s) = \tilde{\omega}(s) - \omega_0.$$

Moreover, the solution satisfies the following estimates uniformly for $0 \le s \le t \le \infty$, $(r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$:

$$\tilde{\theta}(s, r_0, \theta_0, \rho_0, \omega_0) \in \tilde{U}, \quad \tilde{\rho}(s, r_0, \theta_0, \rho_0, \omega_0) \in \tilde{J},$$

$$|\underline{r}(s)| \in \mathrm{o}((s + \langle r_0\rangle)^0)|s|, \qquad |\underline{\theta}(s)| \in \mathrm{O}((\langle r_0\rangle)^{-\tilde{\varepsilon}}),$$
$$|\underline{\rho}(s)| \in \mathrm{O}((\langle r_0\rangle)^{-\tilde{\varepsilon}}), \quad |\underline{\omega}(s)| \in \mathrm{o}((s + \langle r_0\rangle)^0)|s|^{1-\tilde{\varepsilon}},$$
$$\begin{pmatrix} \partial_{r_0}\underline{r}(s) & \partial_{\theta_0}\underline{r}(s) & \partial_{\rho_0}\underline{r}(s) & \partial_{\omega_0}\underline{r}(s) \\ \partial_{r_0}\underline{\theta}(s) & \partial_{\theta_0}\underline{\theta}(s) & \partial_{\rho_0}\underline{\theta}(s) & \partial_{\omega_0}\underline{\theta}(s) \\ \partial_{r_0}\underline{\rho}(s) & \partial_{\theta_0}\underline{\rho}(s) & \partial_{\rho_0}\underline{\rho}(s) & \partial_{\omega_0}\underline{\rho}(s) \\ \partial_{r_0}\underline{\omega}(s) & \partial_{\theta_0}\underline{\omega}(s) & \partial_{\rho_0}\underline{\omega}(s) & \partial_{\omega_0}\underline{\omega}(s) \end{pmatrix}$$
$$\in \begin{pmatrix} \mathrm{o}((s + \langle r_0\rangle)^0)|s| \\ \mathrm{O}(1) \\ \mathrm{O}((s + \langle r_0\rangle)^{-\tilde{\varepsilon}}) \\ \mathrm{o}((s + \langle r_0\rangle)^0)|s|^{1-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_0\rangle^{-1-\tilde{\varepsilon}}, \langle r_0\rangle^{-\tilde{\varepsilon}}, \langle r_0\rangle^{-\tilde{\varepsilon}}, \langle r_0\rangle^{-1}),$$

$$\partial^l_{r_0}\partial^\alpha_{\theta_0}\partial^k_{\rho_0}\partial^\beta_{\omega_0} \begin{pmatrix} \underline{r} \\ \underline{\theta} \\ \underline{\rho} \\ \underline{\omega} \end{pmatrix} \in \begin{pmatrix} \mathrm{o}((s + \langle r_0\rangle)^0)|s - t_1| \\ \mathrm{O}(1) \\ (\langle r_0\rangle)^{-\tilde{\varepsilon}} \\ \mathrm{o}((s + \langle r_0\rangle)^0)|s - t_1|^{1-\tilde{\varepsilon}} \end{pmatrix} \cdot \langle r_0\rangle^{-l-|\beta|}.$$

Proof. Let $(\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})$ be the solutions in Theorem 9 with $t = \infty$:

$$[0, \infty] \ni s \mapsto (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i),$$
$$\partial_s(\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i)$$
$$= (\rho + F_r, F_\theta, F_\rho, F_\omega)((\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i)),$$
$$(\bar{r}, \bar{\omega})(0, \infty, r_i, \theta_f, \rho_f, \omega_i) = (r_i, \omega_i), \quad (\bar{\theta}, \bar{\rho})(\infty, \infty, r_i, \theta_f, \rho_f, \omega_i) = (\theta_f, \rho_f).$$

Set

$$(r_0, \theta_0, \rho_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i) := (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(0, \infty, r_i, \theta_f, \rho_f, \omega_i).$$

It is clear that

$$(r_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i) = (r_i, \omega_i).$$

Theorem 9 assures the following estimates:

$$|\theta_0(r_i,\theta_f,\rho_f,\omega_i)-\theta_f| \in \mathrm{O}(\langle r_i\rangle^{-\tilde{\varepsilon}}), \quad |\rho_0(r_i,\theta_f,\rho_f,\omega_i)-\rho_f| \in \mathrm{O}(\langle r_i\rangle^{-\tilde{\varepsilon}}),$$

$$\begin{pmatrix} \partial_{r_i}(\theta_0-\theta_f) & \partial_{\theta_f}(\theta_0-\theta_f) & \partial_{\rho_f}(\theta_0-\theta_f) & \partial_{\omega_i}(\theta_0-\theta_f) \\ \partial_{r_i}(\rho_0-\rho_f) & \partial_{\theta_f}(\rho_0-\rho_f) & \partial_{\rho_f}(\rho_0-\rho_f) & \partial_{\omega_i}(\rho_0-\rho_f) \end{pmatrix}$$
$$\in \begin{pmatrix} \mathrm{O}(1) \\ (\langle r_i\rangle)^{-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_1\rangle^{-1-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-1}). \tag{19}$$

By taking R large enough, we can assure that the map $(r_0,\theta_0,\rho_0,\omega_0)(r_i,\theta_f,\rho_f,\omega_i)$ is injective. Let $(r_i,\theta_i,\rho_i,\omega_i)(r_0,\theta_0,\rho_0,\omega_0)$ be the inverse function. We will show that

$$(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,r_0,\theta_0,\rho_0,\omega_0) := (\bar{r},\bar{\theta},\bar{\rho},\bar{\omega})(s,\infty,(r_i,\theta_f,\rho_f,\omega_i)(r_0,\theta_0,\rho_0,\omega_0))$$

gives the desired function. (19) implies

$$\begin{pmatrix} \partial_{r_0}(\theta_f-\theta_0) & \partial_{\theta_0}(\theta_f-\theta_0) & \partial_{\rho_0}(\theta_f-\theta_0) & \partial_{\omega_0}(\theta_f-\theta_0) \\ \partial_{r_0}(\rho_f-\rho_0) & \partial_{\theta_0}(\rho_f-\rho_0) & \partial_{\rho_0}(\rho_f-\rho_0) & \partial_{\omega_0}(\rho_f-\rho_0) \end{pmatrix}$$
$$\in \begin{pmatrix} \mathrm{O}(1) \\ (\langle r_0\rangle)^{-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_1\rangle^{-1-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-1}). \tag{20}$$

Moreover, it is easy to see that

$$\partial_{r_0}^l \partial_{\theta_0}^\alpha \partial_{\rho_0}^k \partial_{\omega_0}^\beta \begin{pmatrix} \theta_f-\theta_0 \\ \rho_f-\rho_0 \end{pmatrix} \in \begin{pmatrix} \mathrm{O}(\langle r_0\rangle^{-l-|\beta|}) \\ \mathrm{O}(\langle r_0\rangle^{-l-|\beta|-\tilde{\varepsilon}}) \end{pmatrix}. \tag{21}$$

(20) and (21) shows the desired estimates. □

2.3. Solutions to the Hamilton–Jacobi Equation

We state a lemma which relates the hamiltonian h with the time-independent force F.

Lemma 11. *Let*

$$h(r,\theta,\rho,\omega) = \frac{1}{2}\rho^2 + \tilde{h}(r,\theta,\rho,\omega)$$
$$\tilde{h}(r,\theta,\rho,\omega) = \frac{1}{2}a_1(r,\theta)\rho^2 + \frac{1}{r}a_2^j(r,\theta)\rho\omega_j + \frac{1}{2r^2}a_3^{jk}(r,\theta)\omega_j\omega_k + V^L(r,\theta).$$

Assume

$$|\partial_r^l\partial_\theta^\alpha a_j(r,\theta)| \le C_{l,\alpha} r^{-\mu_j-l}, \quad |D_r^j D_\theta^\alpha V^L(r,\theta)| \le C_j r^{-\mu_L-j},$$

with

$$\mu_1 = \mu_2 = \mu_L = \varepsilon > 0, \quad \mu_3 = 0.$$

Then for any $U \Subset \mathbb{R}^{n-1}$, $J \Subset \mathbb{R}$, and $Q > 0$,

$$\sup_{(r,\theta,\rho,\omega)\in\Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}} |\partial_r^l\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\tilde{h})(r,\theta,\rho,\omega)| \in \mathrm{O}(\langle R\rangle^{-m(l,\alpha,k,\beta)}). \tag{22}$$

This immediately implies that setting

$$(F_r,F_\theta,F_\rho,F_\omega) = (\partial_\rho\tilde{h}, \partial_\omega\tilde{h}, -\partial_r\tilde{h}, -\partial_\theta\tilde{h}), \tag{23}$$

we have (12), i.e., h defines a long-range time-independent force via (23).

Combining Theorem 10 and Lemma 11, we obtain solutions to the Hamilton–Jacobi equation:

Theorem 12. *Let* $h, \tilde{h}$ *be as in Lemma* 11*. For any* $\tilde{U} \Subset U \Subset \mathbb{R}^{n-1}$, $\tilde{J} \Subset J \Subset (0, \infty)$, $C_{j,\alpha} > 0$, *there exists* $T > 0$ *satisfying the following conditions: if a smooth function* $\psi(\rho, \theta)$ *defined on* $J \times U$ *satisfies*

$$\left|\partial_\rho^j \partial_\theta^\alpha \left(\psi(\rho,\theta) - \frac{1}{2}s\rho^2\right)\right| \le C_{j,\alpha}\langle s\rangle^{1-\tilde{\varepsilon}} \tag{24}$$

for some $s > T$*, then there exists a unique function* $S(t, \rho, \theta)$ *defined on a region* $\Theta \subset (0, \infty) \times (0, \infty) \times \mathbb{R}^{n-1}$ *(which will be defined in the proof), with* $\Theta \supset (0, \infty) \times \tilde{J} \times \tilde{U}$*, which satisfies the Hamilton–Jacobi equation:*

$$(\partial_t S)(t, \rho, \theta) = h((\partial_\rho S)(t, \rho, \theta), \theta, \rho, -(\partial_\theta S)(t, \rho, \theta))$$

with the initial value

$$S(0, \rho, \theta) = \psi(\rho, \theta).$$

Moreover, the function S *satisfies the following estimates:*

$$\left|\partial_\rho^j \partial_\theta^\alpha \left(S(t,\rho,\theta) - \frac{1}{2}t\rho^2\right)\right| \le \tilde{C}_{j,\alpha}\langle t\rangle^{1-\varepsilon}. \tag{25}$$

Proof. We fix $\tilde{U} \Subset U \Subset \mathbb{R}^{n-1}$, $\tilde{J} \Subset J \Subset (0, \infty)$, and $C_{j,\alpha} > 0$. Theorem 10 implies that there is a sufficiently large $R > 0$ such that there exists a unique trajectory

$$[0, \infty) \ni t \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, r_0, \theta_0, \rho_0, \omega_0)$$

of the Hamilton equations with initial value problem:

$$\partial_t(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, r_0, \theta_0, \rho_0, \omega_0) = (\rho + F_r, F_\theta, F_\rho, F_\omega)((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, r_0, \theta_0, \rho_0, \omega_0)),$$
$$(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(0, r_0, \theta_0, \rho_0, \omega_0) = (r_0, \theta_0, \rho_0, \omega_0),$$

for any $(r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$. We take $T_0 > 0$ large enough such that for any $\psi(\rho, \theta)$ satisfying (24) with some $s > T_0$,

$$\{((\partial_\rho\psi)(\rho_0, \theta_0), \theta_0, \rho_0, -(\partial_\theta\psi)(\rho_0, \theta_0)) : (\theta_0, \rho_0) \in U \times J\} \subset \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}.$$

Fix such $\psi(\rho, \theta)$ we set

$$(r, \theta, \rho, \omega)(t; \rho_0, \theta_0) := (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(t, (\partial_\rho\psi)(\rho_0, \theta_0), \theta_0, \rho_0, -(\partial_\theta\psi)(\rho_0, \theta_0)).$$

We consider the map

$$(\rho_0, \theta_0) \mapsto (\rho, \theta)(t; \rho_0, \theta_0) \tag{26}$$

and its first derivatives. We set $\Theta := \{(t, (\rho, \theta)(t; \rho_0, \theta_0)) | (\rho_0, \theta_0) \in J \times U\}$. By a straight computation, we obtain

$$\left|\frac{\partial(\rho, \theta)(t, \rho_0, \theta_0)}{\partial(\rho_0, \theta_0)} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| \le C\langle s\rangle^{-\tilde{\varepsilon}},$$

where C depends on $C_{j,\alpha}$ but does not depend on the choice of ψ as long as ψ satisfies (24) for some $s > T_0$. Indeed, using $|\partial\psi(\rho, \theta)(\rho_0, \theta_0)| \ge C\langle s\rangle$,

$$
\begin{aligned}
&\partial_{\rho_0}\rho(t;\rho_0,\theta_0)\\
&\quad= \partial_{\rho_0}[\tilde\rho(t,(\partial_\rho\psi)(\rho_0,\theta_0),\theta_0,\rho_0,-(\partial_\theta\psi)(\rho_0,\theta_0))]\\
&\quad= (\partial_{r_0}\tilde\rho)(t)\cdot(\partial_\rho^2\psi)+(\partial_{\rho_0}\tilde\rho)(t)-(\partial_{\omega_0}\tilde\rho)(t)\cdot(\partial_\rho\partial_\theta\psi)\\
&\quad= \mathrm{O}(\langle s\rangle^{-1-\varepsilon}\cdot\langle s\rangle)+(1+\mathrm{O}(\langle s\rangle^{-\varepsilon}))+\mathrm{O}(\langle s\rangle^{-1-\tilde\varepsilon}\cdot\langle s\rangle^{1-\tilde\varepsilon})\\
&\quad= 1+\mathrm{O}(\langle s\rangle^{-\varepsilon}),\\
&\partial_{\theta_0}\rho(t;\rho_0,\theta_0)\\
&\quad= \partial_{\theta_0}[\tilde\rho(t,(\partial_\rho\psi)(\rho_0,\theta_0),\theta_0,\rho_0,-(\partial_\theta\psi)(\rho_0,\theta_0))]\\
&\quad= (\partial_{r_0}\tilde\rho)(t)\cdot(\partial_\rho\partial_\theta\psi)+(\partial_{\theta_0}\tilde\rho)(t)-(\partial_{\omega_0}\tilde\rho)(t)\cdot(\partial_\theta\partial_\theta\psi)\\
&\quad= \mathrm{O}(\langle s\rangle^{-1-\varepsilon}\cdot\langle s\rangle^{1-\tilde\varepsilon})+\mathrm{O}(\langle s\rangle^{-\varepsilon})+\mathrm{O}(\langle s\rangle^{-1-\tilde\varepsilon}\cdot\langle s\rangle^{1-\tilde\varepsilon})\\
&\quad= \mathrm{O}(\langle s\rangle^{-\varepsilon}),\\
&\partial_{\rho_0}\theta(t;\rho_0,\theta_0)\\
&\quad= \partial_{\rho_0}[\tilde\theta(t,(\partial_\rho\psi)(\rho_0,\theta_0),\theta_0,\rho_0,-(\partial_\theta\psi)(\rho_0,\theta_0))]\\
&\quad= (\partial_{r_0}\tilde\theta)(t)\cdot(\partial_\rho^2\psi)+(\partial_{\rho_0}\tilde\theta)(t)-(\partial_{\omega_0}\tilde\theta)(t)\cdot(\partial_\rho\partial_\theta\psi)\\
&\quad= \mathrm{O}(\langle s\rangle^{-1-\tilde\varepsilon}\cdot\langle s\rangle)+\mathrm{O}(\langle s\rangle^{-\tilde\varepsilon})+\mathrm{O}(\langle s\rangle^{-1}\cdot\langle s\rangle^{1-\tilde\varepsilon})\\
&\quad= \mathrm{O}(\langle s\rangle^{-\tilde\varepsilon}),\\
&\partial_{\theta_0}\theta(t;\rho_0,\theta_0)\\
&\quad= \partial_{\theta_0}[\tilde\theta(t,(\partial_\rho\psi)(\rho_0,\theta_0),\theta_0,\rho_0,-(\partial_\theta\psi)(\rho_0,\theta_0))]\\
&\quad= (\partial_{r_0}\tilde\theta)(t)\cdot(\partial_\rho\partial_\theta\psi)+(\partial_{\theta_0}\tilde\theta)(t)-(\partial_{\omega_0}\tilde\theta)(t)\cdot(\partial_\theta\partial_\theta\psi)\\
&\quad= \mathrm{O}(\langle s\rangle^{-1-\tilde\varepsilon}\cdot\langle s\rangle^{1-\tilde\varepsilon})+(1+\mathrm{O}(\langle s\rangle^{-\tilde\varepsilon}))+\mathrm{O}(\langle s\rangle^{-1}\cdot\langle s\rangle^{1-\tilde\varepsilon})\\
&\quad= 1+\mathrm{O}(\langle s\rangle^{-\tilde\varepsilon}).
\end{aligned}
$$

We fix a large enough $T>0$ so that for any $s>T$ we have

$$
\left|\frac{\partial(\rho,\theta)(t,\rho_0,\theta_0)}{\partial(\rho_0,\theta_0)}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|\ll 1.
$$

Now (26) becomes an injective map for every $t>0$. We denote its inverse by

$$
(\rho,\theta)\mapsto(\rho_0,\theta_0)(t;\rho,\theta).
$$

Let

$$
\begin{aligned}
Q(t;\rho_0,\theta_0) = \psi(\rho_0,\theta_0)+\int_0^t&[h((r,\theta,\rho,\omega)(u;\rho_0\omega_0)\\
&+\langle r(t;\rho_0,\theta_0),(\partial_u\rho)(u;\rho,\theta)\rangle-\langle\omega(t;\rho_0,\theta_0),(\partial_u\theta)(u;\rho,\theta)\rangle]\mathrm{d}u.
\end{aligned}
$$

Then the function

$$
S(t,\rho,\theta)=Q(t;(\rho_0,\theta_0)(t;\rho,\theta))
$$

defined on Θ is the desired solution to the Hamilton–Jacobi equation (see, for example, [4] Appendix A.3). Moreover,

$$
\begin{aligned}
(\partial_\rho S)(t,\rho,\theta)&=r(t;\rho_0(t,\rho,\theta),\theta_0(t,\rho,\theta)),\\
-(\partial_\theta S)(t,\rho,\theta)&=\omega(t;\rho_0(t,\rho,\theta),\theta_0(t,\rho,\theta)).
\end{aligned}
$$

The derivatives of $S(t,\rho,\theta)$

$$\partial_\rho^j \partial_\theta^\alpha \partial_t \left(S(t,\rho,\theta) - \frac{1}{2} t\rho^2 \right) = \partial_\rho^j \partial_\theta^\alpha \tilde{h}(\partial_\rho S(t,\rho,\theta), \theta, \rho, -\partial_\theta S(t,\rho,\theta))$$

is a summation of the terms of the type

$$(\partial_r^l \partial_\theta^\beta \partial_\rho^k \partial_\omega^\gamma \tilde{h})(\partial_\rho S(t,\rho,\theta), \theta, \rho, -\partial_\theta(t,\rho,\theta)) \\ \times \prod_{i=1}^{l} \partial_\rho^{k_i} \partial_\theta^{\beta_i} (\partial_\rho S)(t,\rho,\theta) \times \prod_{d=1}^{n-1} \prod_{j=1}^{\gamma_d} \partial_\rho^{k_{d,j}} \partial_\theta^{\beta_{d,j}} (-\partial_{\theta_d} S)(t,\rho,\theta)),$$

which belongs to $\mathrm{O}(\langle t\rangle^{-m(l,\beta,k,\gamma)+l+(1-\tilde{\varepsilon})|\gamma|}) \subset \mathrm{O}(\langle t\rangle^{-\varepsilon})$. This shows (25). □

Finally, we extend $S(t,\rho,\theta)$ to a globally defined function on $\mathbb{R} \times (0,\infty) \times \partial M$ which satisfies the same kind of estimates locally.

Theorem 13. *Let $h, \tilde{h}$ be as in Lemma 11 defined on $T^*\mathbb{R} \times T^*\partial M$. Then there exists a function $S(t,\rho,\theta)$ defined on $T^*\mathbb{R} \times T^*\partial M$ such that for every $J \Subset \mathbb{R}\backslash\{0\}$, there exists $T > 0$ such that the Hamilton–Jacobi equation:*

$$(\partial_t S)(t,\rho,\theta) = h((\partial_\rho S)(\rho,\theta), \theta, \rho, -(\partial_\theta S)(\rho,\theta)) \tag{27}$$

is satisfied for $t > |T|$, $\rho \in J$, and $\theta \in \partial M$. Moreover, the function S satisfies the following estimates:

$$\left| \partial_\rho^j \partial_\theta^\alpha \left(S(t,\rho,\theta) - \frac{1}{2} t\rho^2 \right) \right| \le \tilde{C}_{j,\alpha} \langle t\rangle^{1-\varepsilon}. \tag{28}$$

Proof. First note that since ∂M is compact and the Hamilton–Jacobi equation is defined in a coordinate invariant manner, we can extend U in Theorem 12 to ∂M. It is sufficient to consider the case $J \Subset (0,\infty)$ and $t > T$, since we can extend the function S in a C^∞-fashion.

Take a sequence of open sets in $(0,\infty)$ such that

$$J_0 \Subset J_1 \Subset J_2 \Subset J_3 \Subset \ldots, \quad \bigcup_{n=0}^{\infty} J_n = (0,\infty).$$

First we solve the Cauchy problem for the Hamilton–Jacobi equation with initial data

$$S(t,\rho,\theta) = \frac{1}{2} t\rho^2 \quad \text{when} \quad \rho \in J_1, t = T_1 > 0$$

for a large enough T_1 by Theorem 12 with U replaced by ∂M. We denote the solution by S_1. We can assume that S_1 is defined on $(T_1,\infty) \times J_0 \times \partial M$. S_1 also satisfies (25) for $\rho \in J_1$ and $t \ge T_1$.

Next we take $\chi_1 \in C_0^\infty(J_1)$ equal to 1 in a neighborhood of $\overline{J_0}$ (the closure of J_0). We solve the Cauchy Problem with initial data

$$S(t,\rho,\theta) = \chi_1 S_1 + (1-\chi_1)\frac{t}{2}\rho^2 \quad \text{when} \quad \rho \in J_2, t = T_2.$$

By taking $T_2 > T_1$ large enough, the right-hand side satisfies the conditions for T_2 in Theorem 12. So we can solve the Cauchy Problem for such T_2. We denote the solution by S_2.

Repeating this procedure, we obtain a sequence S_n of functions and a sequence $T_1 < T_2 < \dots$ such that S_n is defined on $J_n \times [T_n, \infty) \times \partial M$,

$$S_{n+1} = S_m \quad \text{for} \quad m \geq n+1 \quad \text{on} \quad J_n \times [T_m, \infty) \times \partial M,$$

and satisfies (25). Thus by extending in a C^∞ fashion, we can construct a C^∞ function S which satisfies (27), and (25) for large enough t and ρ in any fixed compact subset of $(0, \infty)$. □

3. Proof of Theorem 3

In this section, we give the proof of Theorem 3. First we give the outline of the proof.

Outline of the Proof. We consider the $t \to +\infty$ case. By the density argument, it is sufficient to show the existence of the norm limit

$$\lim_{t\to\infty} \mathrm{e}^{itH} J \mathrm{e}^{-iS(t,D_r,\theta)} u$$

for all $\hat{u} \in C_0^\infty((\mathbb{R}\backslash\{0\}) \times U_\lambda)$ for all λ. For such u, we have

$$\frac{1}{i}\mathrm{e}^{-itH}\frac{\partial}{\partial t}[\mathrm{e}^{itH} J \mathrm{e}^{-iS(t,D_r,\theta)} u] = \left[HJ - J\frac{\partial S}{\partial t}(t, D_r, \theta)\right] \mathrm{e}^{-iS(t,D_r,\theta)} u.$$

By the Cook–Kuroda method we only need to show that

$$\left\| \left[HJ - J\frac{\partial S}{\partial t}(t, D_r, \theta)\right] \mathrm{e}^{-iS(t,D_r,\theta)} u \right\|_{\mathscr{H}} \in L^1_t(1, \infty).$$

We decompose

$$\begin{aligned}&\left[HJ - J\frac{\partial S}{\partial t}(t, D_r, \theta)\right] \\ &\quad = [P_0 J - J\tilde{P}_0] + V_S J + [V^L J - J V^L] + J\left[\tilde{P}_0 + V^L(r) - \frac{\partial S}{\partial t}(t, D_r, \theta)\right].\end{aligned}$$

Here we set

$$\tilde{P}_0 = -\frac{1}{2} j(r) H(\theta)^{-1} (\partial_r, \partial_\theta / r) H(\theta) \begin{pmatrix} 1 + a_1 & a_2 \\ {}^t a_2 & a_3 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta / r \end{pmatrix} j(r),$$

which is a symmetric operator on $\mathscr{H}_f$. The first three terms are essentially short-range terms. It is easy to check

$$\|[P_0 J - J\tilde{P}_0] + V_S J + [V^L J - J V^L] \mathrm{e}^{-iS(t,D_r,\theta)} u\|_{\mathscr{H}} \in L^1_t(1, \infty). \tag{29}$$

We examine the last term:

$$\begin{aligned}&[\tilde{P}_0 + V^L(r) - (\partial_t S)(t, D_r, \theta)] \mathrm{e}^{-iS(t,D_r,\theta)} u \\ &\quad = h(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) \mathrm{e}^{-iS(t,D_r,\theta)} u \\ &\qquad - h((\partial_\rho S)(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)) \mathrm{e}^{-iS(t,D_r,\theta)} u\end{aligned}$$

$$+\Big(\frac{1}{r}a_2^j D_r + \frac{1}{2r^2}a_3^{jk}\frac{\partial S}{\partial \theta^k}(t, D_r, \theta)\Big)\mathrm{e}^{-iS(t,D_r,\theta))}(\partial_{\theta^j}u)$$
$$-\frac{1}{2r^2}a_3^{jk}\mathrm{e}^{-iS(t,D_r,\theta))}(\partial_{\theta^j}\partial_{\theta^k}u)$$
$$+ \text{[short range terms]}.$$

Here we use pseudo-differential operators under the right-quantization: for a function $f(r,\rho)$ of r and ρ, we set

$$(f(r,D_r)u)(r) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{ir\rho}f(r,\rho)\hat{u}(\rho)\mathrm{d}\rho$$

where $\hat{u}(\rho) = (\mathscr{F}u)(\rho)$ is the Fourier transform of u with respect to the r-variable. We apply the stationary phase method to the first two terms. Then the first terms which appear in the asymptotic expansion will vanish since the relation

$$(\partial_\rho S)(t,\rho,\theta) = r$$

gives the stationary point with respect to the ρ-variable. Therefore, we obtain

$$\|[\tilde{P}_0 + V^L(r) - (\partial_t S)(t, D_r, \theta)]\mathrm{e}^{-iS(t,D_r,\theta)}u\|_{\mathscr{H}_f} \in L_t^1(1,\infty). \tag{30}$$

We give a detailed proof of (29) and (30) in the remaining of this section. □

First we consider the long-range term (30). The next proposition is our key estimate.

Proposition 14. *Assume the assumptions of Theorem 3. Suppose u satisfies $\hat{u} \in C_0^\infty((\mathbb{R}\backslash\{0\}) \times U_\lambda)$ and $J \times U$ is a neighborhood of supp $\hat{u}$. Then we have*

$$\left|\left[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)) - \tilde{h}\left(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\right)\right] \cdot \mathrm{e}^{-iS(t,D_r,\theta)}u(r,\theta)\right| \le Ct^{-\frac{1}{2}-1-\varepsilon} \tag{31}$$

for $(\frac{r}{t},\theta) \in J \times U \Subset (0,\infty) \times \partial M$, and

$$\left|\left[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)) - \tilde{h}\left(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\right)\right] \cdot \mathrm{e}^{-iS(t,D_r,\theta)}u(r,\theta)\right| \le C_N(1+|r|+|t|)^{-N} \tag{32}$$

for any N and for $(\frac{r}{t},\theta) \notin J \times U$.

Proof of (30). We fix a neighborhood $J \times U$ of supp $\hat{u}$ which appears in Proposition 14. Then

$$\int_1^\infty \Big\|J\Big[\tilde{h}\Big(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\Big) - \tilde{h}\Big(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\Big)\Big] \times \mathrm{e}^{-iS(t,D_r,\theta)}u\Big\|_{\mathscr{H}}\mathrm{d}t$$

$$= \int_1^\infty \Big(\int_{\mathbb{R}_+ \times \partial M} \Big| j(r) \Big[\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta))$$

$$-\tilde{h}\left(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\right)\Big] \mathrm{e}^{-iS(t,D_r,\theta)} u(r,\theta)\Big|^2 \mathrm{d}r H(\theta) \mathrm{d}\theta\Big)^{\frac{1}{2}} \mathrm{d}t$$

$$\leq \int_1^\infty \Big(\int_{\frac{r}{t} \in J} |j(r)[\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta))$$

$$-\tilde{h}\left(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\right)]\mathrm{e}^{-iS(t,D_r,\theta)} u(r,\theta)|^2 \mathrm{d}r H(\theta) \mathrm{d}\theta\Big)^{\frac{1}{2}}$$

$$+\Big(\int_{\frac{r}{t} \notin J} |j(r)\Big[\tilde{h}(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta))$$

$$-\tilde{h}\Big(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\Big)\Big]\mathrm{e}^{-iS(t,D_r,\theta)} u(r,\theta)|^2 \mathrm{d}r H(\theta) \mathrm{d}\theta\Big)^{\frac{1}{2}} \mathrm{d}t.$$

By (31), the first term is finite:

$$\int_1^\infty \Big(\int_{\frac{r}{t} \in J} \Big| j(r) \Big[\tilde{h}\Big(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\Big)$$

$$-\tilde{h}\Big(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\Big)\Big]$$

$$\times \mathrm{e}^{-iS(t,D_r,\theta)} u(r,\theta)\Big|^2 \mathrm{d}r H(\theta) \mathrm{d}\theta\Big)^{\frac{1}{2}} \mathrm{d}t$$

$$\leq \int_1^\infty \Big(\int_{R \in J} |Ct^{-\frac{1}{2}-1-\varepsilon}|^2 t \mathrm{d}R\Big)^{\frac{1}{2}} \mathrm{d}t \leq C \int_1^\infty t^{-1-\varepsilon} \mathrm{d}t < \infty.$$

By (32), the second term is also finite:

$$\int_1^\infty \Big(\int_{\frac{r}{t} \notin J} \Big| j(r) \Big[\tilde{h}\Big(r, \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\Big)$$

$$-\tilde{h}\Big(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\Big)\Big]$$

$$\times \mathrm{e}^{-iS(t,D_r,\theta)} u(r,\theta)\Big|^2 \mathrm{d}r H(\theta) \mathrm{d}\theta\Big)^{\frac{1}{2}} \mathrm{d}t$$

$$\leq \int_1^\infty \Big(\int_{\frac{r}{t} \notin J} C(1+|r|+|t|)^{-N} \mathrm{d}r\Big)^{\frac{1}{2}} \mathrm{d}t < \infty$$

Therefore,

$$\Big\| J\Big[V^L(r, D_r, \theta) - V^L\Big(\frac{\partial W}{\partial \rho}(D_r, \theta, t), D_r, \theta\Big)\Big] \mathrm{e}^{-iW(D_r,\theta,t)} u\Big\|_{\mathscr{H}} \in L^1_t(1,\infty).$$

□

In order to prove Proposition 14, we prepare a lemma.

Lemma 15. *Let $S(t,\rho,\theta)$ satisfy the properties listed in Theorem 13. Set*

$$f_{r,\theta,t}(\rho) := \frac{1}{t}(r\rho - S(t,\rho,\theta)).$$

For ρ in any fixed compact subset of $\mathbb{R}\setminus\{0\}$ and for large enough $|t|$, there exists a function $\Xi_{\theta,t}(r)$ which gives the critical point of $f_{r,\theta,t}(\rho)$:

$$\partial_\rho f_{r,\theta,t}(\rho) = 0 \iff \rho = \Xi_{\theta,t}(r).$$

Set $\Omega_d := (-d,d)$. Then there exist $0 < \tilde{d} < d$ and a function $\phi_{r,\theta,t} \in C^\infty(\Omega_d;\mathbb{R})$ such that $\Omega_{2\tilde{d}} \Subset \phi_{r,\theta,t}(\Omega_d)$. Setting

$$\psi_{r,\theta,t}(y) := \Xi_{\theta,t}(r) + \phi_{r,\theta,t}(y),$$
$$(f_{r,\theta,t}\circ\psi_{r,\theta,t})(y) = f_{r,\theta,t}(\Xi_{\theta,t}(r)) + \langle A_{r,\theta,t}y,y\rangle/2,$$

where

$$A_{r,\theta,t} = (\partial_\rho^2 f_{r,\theta,t})(\Xi_{\theta,t}(r)),$$

we have

$$|\partial_y^k\psi_{r,\theta,t}(0)| \le Ct^{-\varepsilon} \quad (k\ge 2), \quad \partial_y\psi_{r,\theta,t}(0) = 1. \tag{33}$$

Proof. We only consider the $t>0$ and $\rho>0$ case. First we prove that $\Xi_{\theta,t}(r)$ is well defined. Compute

$$0 = \partial_\rho f_{r,\theta,t}(\rho) = \frac{1}{t}\Big[r - \frac{\partial S}{\partial\rho}(t,\rho,\theta)\Big]$$

We note that by (28),

$$\Big|\frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\rho,\theta) - 1\Big| \le Ct^{-\varepsilon}.$$

This implies that $\frac{1}{t}\frac{\partial S}{\partial\rho}$ is monotonously increasing with respect to ρ for large enough t. Thus there is a unique inverse function $\Xi_{\theta,t}(r)$ such that

$$(\partial_\rho f_{r,\theta,t})(\Xi_{\theta,t}(r)) = 0$$

for large enough t and $\frac{r}{t}\in J$, a fixed compact subset of $(0,\infty)$.

Now we construct $\phi_{r,\theta,t}$ and $\psi_{r,\theta,t}$. We set

$$A_{r,\theta,t} := f''_{r,\theta,t}(\Xi_{\theta,t}(r)) \ = -\frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\Xi_{\theta,t}(r),\theta).$$

Then (28) implies that

$$|A_{r,\theta,t}+1| \le Ct^{-\varepsilon}.$$

Hence we have $A_{r,\theta,t}\to -1$ uniformly for $r/t\in J$. If we set

$$g_{r,\theta,t}(\rho) := f_{r\theta,t}(\Xi_{\theta,t}(r)+\rho),$$

then

$$g'_{r,\theta,t}(0) := f'_{r\theta,t}(\Xi_{\theta,t}(r)) = 0, \quad g''_{r,\theta,t}(0) := f''_{r\theta,t}(\Xi_{\theta,t}(r)) = A_{r,\theta,t},$$
$$g_{r,\theta,t}(\rho) - g_{r,\theta,t}(0) = \langle B_{r,\theta,t}(\rho)\rho,\rho\rangle/2,$$

where

$$B_{r,\theta,t}(\rho) := 2\int_0^1 g_{r,\theta,t}(s\rho)(1-s)\mathrm{d}s, \quad B_{r,\theta,t}(0) = A_{r,\theta,t}$$

by Taylor's formula. Now we compute

$$\begin{aligned}
|B_{r,\theta,t}(\rho) - A_{r,\theta,t}| &= |B_{r,\theta,t}(\rho) - B_{r,\theta,t}(0)| \\
&= 2\Big|\int_0^1 (g''_{r,\theta,t}(s\rho) - g''_{r,\theta,t}(0))(1-s)\mathrm{d}s\Big| \\
&\le 2\sup_{0\le s\le 1} |g''_{r,\theta,t}(s\rho) - g''_{r,\theta,t}(0)| \\
&\le 2\sup_{0\le s\le 1} \Big|\frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\Xi_{\theta,t}(r)+s\rho,\theta) - \frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\Xi_{\theta,t}(r),\theta)\Big| \\
&\le Ct^{-\varepsilon} \to 0
\end{aligned}$$

as $t \to \infty$, uniformly for $\frac{r}{t}, \rho \in J$ by (28). Hence by taking t sufficiently large, we may assume $\big|\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}(\rho)} - 1\big| < 1/2$ is uniformly very small. For such t, $\frac{r}{t}$,and ρ, we set

$$X_{r,\theta,t}(\rho) := \sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}} \cdot \rho.$$

Then we have

$$g_{r,\theta,t}(\rho) - g_{r,\theta,t}(0) = \langle A_{r,\theta,t}X_{r,\theta,t}(\rho), X_{r,\theta,t}(\rho)\rangle/2.$$

Now we compute

$$\begin{aligned}
(\partial_\rho X_{t,\theta,t})(\rho) &= \left(\sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}}\right)' \cdot \rho + \sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}} \cdot 1 \\
&= \frac{1}{\sqrt{A_{r,\theta,t}}} 2\sqrt{B_{r,\theta,t}(\rho)} \cdot B'_{r,\theta,t}(\rho) \cdot \rho + \sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}} \cdot 1, \\
(\partial_\rho B_{r,\theta,t})(\rho) &= 2\int_0^1 g'''_{r,\theta,t}(s\rho)s(1-s)\mathrm{d}s, \\
|g'''_{r,\theta,t}(s\rho)| &= |-\frac{1}{t}(\partial_\rho^3 S)(t,\Xi_{\theta,t}(r)+s\rho,\theta)| \le Ct^{-\varepsilon}, \\
|\partial_\rho X_{r,\theta,t}(\rho) - 1| &\le Ct^{-\varepsilon}.
\end{aligned}$$

Hence for small enough $d_0 > 0$ and for $|\rho| \le d_0$, we have $|\partial_\rho X_{r,\theta,t}(\rho) - 1|$ arbitrary small for all large enough t, and $X_{r,\theta,t} : \Omega_{d_0} \to X_{r,\theta,t}(\Omega_{d_0})$ is a C^∞-diffeomorphism. We can pick $d > 0$ such that, $\Omega_d \subset X_{r,\theta,t}(\Omega_{d_0})$ for all r, θ, large enough t, $\frac{r}{t} \in J$. Let $\phi_{r,\theta,t}$ be the inverse map of $X_{r,\theta,t}$ with domain Ω_d.

Then we can also pick $\tilde{d} > 0$ such that $\Omega_{\tilde{d}} \subset \phi_{r,\theta,t}(\Omega_d)$ for all r, θ, large enough t, $\frac{r}{t} \in J$. We note that

$$g_{r,\theta,t} \circ \phi_{r,\theta,t}(y) - g_{r,\theta,t}(0) = \langle A_{r,\theta,t} X_{r,\theta,t} \circ \phi_{r,\theta,t}(y), X_{r,\theta,t} \circ \phi_{r,\theta,t}(y)\rangle/2. \\ = \langle A_{r,\theta,t} y, y\rangle/2.$$

We set

$$\psi_{r,\theta,t}(y) = \phi_{r,\theta,t}(y) + \Xi_{\theta,t}(r).$$

Then we have

$$f_{r,\theta,t} \circ \psi_{r,\theta,t}(y) = f_{r,\theta,t}(\Xi_{\theta,t}(r)) + \langle A_{r,\theta,t} y, y\rangle/2.$$

Last, we prove the estimates (33). For $k \geq 1$,

$$|\partial_\rho^k B_{r,\theta,t}(\rho)| = 2\left|\int_0^1 g_{r,\theta,t}^{2+k}(s\rho) s^k (1-s) \mathrm{d}s\right| \leq 2 \sup |g_{r,\theta,t}^{2+k}(s\rho)| \\ \leq 2 \sup \left|\frac{1}{t} \partial_\rho^{2+k} S(t, \Xi_{\theta,t}(r) + s\rho, \theta)\right| \leq Ct^{-\varepsilon}$$

by (28). We also have

$$|\partial_\rho^k \sqrt{B_{r,\theta,t}}(\rho)| \leq Ct^{-\varepsilon}.$$

Therefore,

$$|\partial_\rho^k X_{r,\theta,t}(\rho)| \leq Ct^{-\varepsilon} \quad (k \geq 2), \quad |\partial_\rho X_{r,\theta,t}(0) - 1| = Ct^{-\varepsilon},$$

and we have

$$|\partial_y^k \psi_{r,\theta,t}(y)| = |\partial_y^k \phi_{r,\theta,t}(y)| \leq Ct^{-\varepsilon} \quad (k \geq 2), \\ |\partial_y \psi_{r,\theta,t}(0) - 1| = |\partial_y \phi_{r,\theta,t}(0) - 1| \leq Ct^{-\varepsilon}.$$

Then we complete the proof of Lemma 15. □

We quote a lemma from Hörmander [11]:

Lemma 16. *Let A be a symmetric non-degenerate matrix with $Im\, A \geq 0$. Then we have for every integer $k > 0$ and integer $s > n/2$*

$$\int u(x) \mathrm{e}^{i\omega\langle Ax, x\rangle \mathrm{d}x} - (det(\omega A/2\pi i))^{-1/2} T_k(\omega)| \\ \leq C_k(\|A^{-1}\|/\omega)^{n/2+k} \sum_{|\alpha| \leq 2k+s} \|D^\alpha u\|_{L^2}, \quad u \in \mathscr{S}(\mathbb{R}^n), \\ T_k(\omega) = \sum_{j=0}^{k-1} (2i\omega)^{-j} \langle A^{-1} D, D\rangle^j u(0)/j!.$$

Proof. See Lemma 7.3.3. of Hörmander [11] volume I. □

Proof of Proposition 14. First we prove (31) for $\frac{r}{t} \in J$. We fix $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ if $|x| \le \frac{1}{2}$, and $\chi(x) = 0$ if $|x| \ge 1$. We split u into two terms depending on r, θ, and t:

$$\hat{u}^c_{r,\theta,t}(\rho,\theta) = \hat{u}(\rho,\theta)\chi\Big(\frac{\rho - \Xi_{\theta,t}(r)}{\tilde{d}}\Big),$$
$$\hat{u}^d_{r,\theta,t}(\rho,\theta) = \hat{u}(\rho,\theta)\Big[1 - \chi\Big(\frac{\rho - \Xi_{\theta,t}(r)}{\tilde{d}}\Big)\Big]$$

where we use notations defined in Lemma 15. The support of $\hat{u}^c_{r,\theta,t}$ is close to the critical point of $r\rho - S(t,\rho,\theta)$, while that of $\hat{u}^d_{r,\theta,t}$ is apart from it. Note that

$$\mathrm{supp}\hat{u}^c_{r,\theta,t} \subset \Xi_{\theta,t}(r) + \Omega_{\tilde{d}} \subset \mathrm{Ran}(\psi_{r,\theta,t}).$$

By a change of variables we have

$$\begin{aligned}
&\tilde{h}\Big(\frac{\partial S}{\partial \rho}(t, D_r, \theta), \theta, D_r, -\frac{\partial S}{\partial \theta}(t, D_r, \theta)\Big)\mathrm{e}^{-iS(t,D_r,\theta)} u^c_{r,\theta,t}(r,\theta) \\
&\quad = \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{h}\Big(\frac{\partial S}{\partial \rho}(t,\rho,\theta), \theta, \rho, -\frac{\partial S}{\partial \theta}(t,\rho,\theta)\Big)\mathrm{e}^{ir\rho - iS(t,\rho,\theta)}\hat{u}^c_{r,\theta,t}(\rho,\theta)\mathrm{d}\rho \\
&\quad = \frac{1}{2\pi}\int_{\Omega_d} \tilde{h}\Big(\frac{\partial S}{\partial \rho}(t,\psi_{r,\theta,t}(y),\theta), \theta, \psi_{r,\theta,t}(y), -\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta)\Big)\hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta) \\
&\qquad \cdot J_{r,\theta,t}(y)\mathrm{e}^{itf_{r,\theta,t}(\Xi_{\theta,t}(r))}\mathrm{e}^{it\langle A_{r,\theta,t}y,y\rangle/2}\mathrm{d}y
\end{aligned}$$

where $J_{r,\theta,t}(y) = |\psi'_{r,\theta,t}(y)|$ is the Jacobian. Since

$$\Big|D^j_\rho \tilde{h}\Big(\frac{\partial S}{\partial \rho}(t,\rho,\theta), \theta, \rho, -\frac{\partial S}{\partial \theta}(t,\rho,\theta)\Big)\Big| \le Ct^{-\varepsilon},$$

we have

$$\begin{gathered}
\Big|D^j_y \tilde{h}\Big(\frac{\partial S}{\partial \rho}(t,\psi_{r,\theta,t}(y),\theta), \theta, \psi_{r,\theta,t}(y), -\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta)\Big)\Big| \le Ct^{-\varepsilon}, \\
\Big|\tilde{h}(r,\theta,\psi_{r,\theta,t}(y), -\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta))\Big| \le Ct^{-\varepsilon}, \\
|D^j_y \hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)| \le C, \\
|D^j_y J_{r,\theta,t}(y)| \le C,
\end{gathered}$$

for $y \in \Omega_d$, $\frac{r}{t} \in J$. Now we apply the stationary phase method (Lemma 16) to the integral. In the asymptotic expansion of

$$
\begin{aligned}
&\Big[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)) - \tilde{h}\Big(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\Big)\Big] \\
&\quad\cdot \mathrm{e}^{-iS(t,D_r,\theta)} u^c_{r,\theta,t}(r,\theta) \\
&= \frac{1}{2\pi}\int\limits_{\Omega_d}\Big[\tilde{h}(r,\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,\psi_{r,\theta,t}(y),\theta)) \\
&\quad -\tilde{h}\Big(\frac{\partial S}{\partial\rho}(t,\psi_{r,\theta,t}(y),\theta),\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,\psi_{r,\theta,t}(y),\theta)\Big)\Big] \\
&\quad \cdot \hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)\cdot J_{r,\theta,t}(y)\mathrm{e}^{itf_{r,\theta,t}(\Xi_{\theta,t}(r))}\mathrm{e}^{it\langle A_{r,\theta,t}y,y\rangle/2}\mathrm{d}y,
\end{aligned}
$$

the terms in which $\tilde{h}$ is not differentiated will vanish since

$$
\frac{\partial S}{\partial\rho}(t,\psi_{r,\theta,t}(0),\theta) = r.
$$

Especially, in the first step of the asymptotic expansion, we need to estimate only the remainder terms. Therefore, we have

$$
\begin{aligned}
&\Big|\Big[\tilde{h}\Big(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\Big) - \tilde{h}\Big(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta)\Big)\Big] \\
&\quad\cdot \mathrm{e}^{-iS(t,D_r,\theta)} u^c_{r,\theta,t}(r,\theta)\Big| \\
&\le Ct^{-\frac{1}{2}-1}\sum_{|k|\le 3}\sup\Big\|D^k_y\Big[\tilde{h}(r,\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,\psi_{r,\theta,t}(y),\theta)) \\
&\quad -\tilde{h}\Big(\frac{\partial S}{\partial\rho}(t,\psi_{r,\theta,t}(y),\theta),\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,\psi_{r,\theta,t}(y),\theta)\Big)\Big] \\
&\quad \cdot \hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)\cdot J_{r,\theta,t}(y)\Big\|_{L^2} \\
&\le Ct^{-\frac{1}{2}-1-\varepsilon}.
\end{aligned}
$$

We now consider $u^d_{r,\theta,t}$ term.

$$
\begin{aligned}
&\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))\mathrm{e}^{-iS(t,D_r,\theta)}u^d_{r,\theta,t}(r,\theta) \\
&\quad = \frac{1}{2\pi}\Big(\int\limits_{-\infty}^{\Xi_{\theta,t}(r)-\frac{1}{2}\tilde{d}} + \int\limits_{\Xi_{\theta,t}(r)+\frac{1}{2}\tilde{d}}^{\infty}\Big) \\
&\qquad \tilde{h}\Big(\frac{\partial S}{\partial\rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial\theta}(t,\rho,\theta)\Big)\mathrm{e}^{ir\rho-iS(t,\rho,\theta)}\hat{u}^d_{r,\theta,t}(\rho,\theta)\mathrm{d}\rho
\end{aligned}
$$

We consider integration over $\ge \Xi_{\theta,t}(r)+\frac{1}{2}\tilde{d}$ only (the other part is similar to prove). (28) implies

$$
\begin{aligned}
&\partial_\rho f_{r,\theta,t}(\rho) \le -C < 0, \\
&|\partial^j_\rho f_{r,\theta,t}(\rho)| \le Ct^{-\varepsilon}, \quad j\ge 2
\end{aligned}
$$

in this region. Let $y \mapsto h_{r,\theta,t}(y)$ be the inverse of $\rho \mapsto f_{r,\theta,t}(\rho)$. Then

$$|\partial_y h_{r,\theta,t}(y)| \le C,$$
$$|\partial_y^j h_{r,\theta,t}(y)| \le Ct^{-\varepsilon}, \quad j \ge 2.$$

By a change of the variables we obtain

$$\begin{aligned}
&\Big| \int_{\Xi_{\theta,t}(r)+\frac{1}{2}\tilde{d}}^{\infty} \tilde{h}\Big(\frac{\partial S}{\partial \rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta)\Big)\mathrm{e}^{itf_{r,\theta,t}(\rho)}\hat{u}^d_{r,\theta,t}(\rho,\theta)\mathrm{d}\rho\Big| \\
&= \Big| \int \mathrm{e}^{ity}\tilde{h}\Big(\frac{\partial S}{\partial \rho}(t,h_{r,\theta,t}(y),\theta),\theta,h_{r,\theta,t}(y),-\frac{\partial S}{\partial \theta}(t,h_{r,\theta,t}(y),\theta)\Big)\hat{u}^d_{r,\theta,t}(h_{r,\theta,t}(y),\theta) \\
&\quad \cdot |h'_{r,\theta,t}(y)|\mathrm{d}y\Big| \\
&\le Ct^{-N}.
\end{aligned}$$

We can show the same kind of estimations for $\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))$ $\mathrm{e}^{-iS(t,D_r,\theta)}u^d_{r,\theta,t}(r,\theta)$. We have ended the proof of (31).

Now we show (32). (28) implies that there exists $\tilde{J}$ such that $\frac{1}{t}\frac{\partial S}{\partial \rho} \in \tilde{J} \Subset J$ for large enough t. Thus the absolute value of the derivative of $\rho \mapsto (r\rho - S(t,\rho,\theta))/(|r|+|t|)$ is bounded below for $\frac{r}{t} \notin J$, large enough t, and $\rho \in \mathrm{supp}\hat{u}$. Thus we can apply the non-stationary phase method and obtain (32). □

Proof of (29)*, partial isometry, and intertwining property.* First we consider the short-range terms. On $\tilde{M}_0$,

$$\begin{aligned}
P_0J - J\tilde{P}_0 + V^LJ - JV^L = O(r^{-\frac{n-1}{2}}r^{-1-\varepsilon})\partial_r\partial_\theta + O(r^{-\frac{n-1}{2}}r^{-2})\partial_\theta^2 \\
+ \sum_j O(r^{-\frac{n-1}{2}}r^{-1-\varepsilon})\partial_r^j.
\end{aligned}$$

These terms can be treated as a short-range perturbation of $(1-\varepsilon) = \mu_S$ type. Hence on $\tilde{M}_0$, $P_0J - J\tilde{P}_0 + V^LJ - JV^L + V^SJ$ is a finite sum of terms of the form $v_{j,\alpha}(r,\theta)r^{-\frac{n-1}{2}}D_r^j\partial_\theta^\alpha$ where $v_{j,\alpha}$ satisfy

$$\int_{\mathbb{R}_+\times U_\lambda} |v^S_{j,\alpha}(x)|^2\langle x\rangle^{-M}G(x)\mathrm{d}x < \infty,$$

$$\int_1^\infty \Big(\int_{(\rho,\theta)\in J\times U} |v^S_{j,\alpha}(t\rho,\theta)|^2\mathrm{d}\rho\mathrm{d}\theta\Big)^{1/2} t^{(1-\varepsilon)|\alpha|}\mathrm{d}t < \infty,$$

for some neighborhood $J \times U$ of almost every $(\rho_0,\theta_0) \in \mathbb{R} \times \partial M$. We assume supp $\hat{u} \subset J \times U$.

We consider the differential operators with respect to θ-variable. ∂_θ $\mathrm{e}^{-iS(t,D_r,\theta)}$ yields $(\partial_\theta S)(t,D_r,\theta)$ terms which increase as $t^{1-\varepsilon}$. Hence, as in the long-range case, the inequalities (28) implies

$$\begin{aligned}
|D_r^j\partial_\theta^\alpha \mathrm{e}^{-iS(t,D_r,\theta)}u(r,\theta)| &= \frac{1}{2\pi}\Big|\int_{\mathbb{R}} \partial_\theta^\alpha[\mathrm{e}^{i(r\rho - S(t,\rho,\theta))}\rho^j\hat{u}(\rho,\theta)]\mathrm{d}\rho\Big| \\
&\le Ct^{-\frac{1}{2}+|\alpha|(1-\varepsilon)} \qquad (34)
\end{aligned}$$

for $(\frac{r}{t},\theta) \in J \times U$, and

$$|\partial_r^j \partial_\theta^\alpha \mathrm{e}^{-iS(t,D_r,\theta)} u(r,\theta)| \le C_N(1+|r|+|t|)^{-N} \tag{35}$$

for any N for $(\frac{r}{t},\theta) \notin J \times U$. Thus we obtain for such $v_{j,\alpha}$

$$\|v_{j,\alpha}(r,\theta) r^{-\frac{n-1}{2}} D_r^j \partial_\theta^\alpha \mathrm{e}^{-iS(t,D_r,\theta)} u\|_{\mathscr{H}} \in L_t^1(1,\infty),$$

which proves

$$\|\big([P_0 J - J\tilde{P}_0] + V^S J + [V^L J - J V^L]\big)\mathrm{e}^{-iS(t,D_r,\theta)} u\|_{\mathscr{H}} \in L_t^1(1,\infty).$$

We have proved the existence of the modified wave operators.

(31), (32), (34), and (35) also show that $W_\pm$ are partial isometries from $\mathscr{H}_f$ into $\mathscr{H}$.

The intertwining property follows from

$$\operatorname*{s-lim}_{t\to\infty}(\mathrm{e}^{-iS(t+s,D_r,\theta)} - \mathrm{e}^{-iS(D_r,\theta,t)}\mathrm{e}^{-isP_f}) = 0 \tag{36}$$

which can be proved using (28) and the dominated convergence theorem. The proof of the theorem is complete. □

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