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Inverse Scattering Theory for Discrete Schrödinger Operators on the Hexagonal Lattice

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Abstract. We consider the spectral theory and inverse scattering problem for discrete Schrödinger operators on the hexagonal lattice. We give a procedure for reconstructing finitely supported potentials from the scattering matrices for all energies. The same procedure is applicable for the inverse scattering problem on the triangle lattice.

1. Introduction

Let us begin with recalling the study of inverse scattering problems for Schrödinger operators on $\mathbb{R}^n$. For $n = 1$, it was solved by Gel'fand and Levitan [9], Marčhenko [24] in the 1950s. For $n = 3$, Faddeev [7] established the uniqueness of the potential with given scattering matrices using high-energy Born approximation soon after that. In the 1970s, Faddeev [8] and Newton [26] investigated an analogue of Gel'fand–Levitan theory for the multidimensional case by making use of Faddeev's Green operator. A breakthrough in this direction, called $\overline{\partial}$-approach, was brought in the late 1980s by Beals and Coifman [2], Nachman and Ablowitz [1], Khenkin and Novikov [16], Weder [31]. See, e.g. a survey article of Isozaki [12].

In discrete settings, it was pointed out early that the theory of Gel'fand–Levitan–Marčhenko is applicable for the discrete Schrödinger operator on $\mathbb{Z}$ (e.g. Case and Kac [4]). On $\mathbb{Z}^n$, $n \geq 2$, Isozaki and Korotyaev [13] studied the inverse scattering problem for the discrete Schrödinger operator with a finitely supported potential, recently. They used a kind of complex Born approximation and showed a procedure for reconstructing the potential.

In this paper, we consider the inverse scattering problem for the discrete Schrödinger operator on the hexagonal lattice and derive a reconstruction procedure for the potential from the scattering matrices of all energies. The hexagonal lattice is a sort of two-dimensional lattice, which covers the plane by

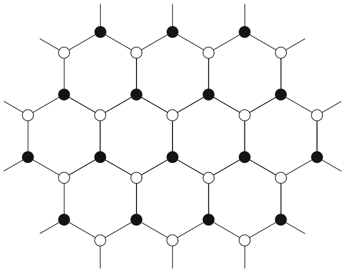

Figure 1. The hexagonal lattice as graph

equilateral hexagons with honeycomb structure. We regard it as graph. Before stating our main theorem, let us briefly recall the graph theory.

We denote by $G = (V(G), E(G))$ the graph that consists of a vertex set $V(G)$, whose cardinality is at most countable, and an edge set $E(G)$, each element of which connects a pair of vertices. Let v, $u \in V(G)$, and $e \in E(G)$. We denote by $v \overset{e}{\sim} u$, or simply $v \sim u$, when v is adjacent to u by e; by $N_v = \{u \in V(G); v \sim u\}$ the set of vertices which are adjacent to v. We also denote by $\deg(v) = \sharp N_v$ the degree of v. We assume that the graph G is connected, which implies that $\deg(v) > 0$ for any $v \in V(G)$, locally finite, that is, $\deg(v) < \infty$ for any $v \in V(G)$, and simple, that is, there are neither self-loops nor multiple edges. Here, a self-loop is an edge which joins a vertex to itself, and multiple edges are two or more edges which join the same two vertices. The discrete Laplacian Δ_d on G is defined as

$$(\Delta_d \hat{f})(v) = \frac{1}{\deg(v)} \sum_{u \in N_v} \hat{f}(u) - \hat{f}(v)$$

for the function $\hat{f}$ on $V(G)$. It is well known that $-\Delta_d$ is a bounded, self-adjoint operator on

$$l^2(G) = \left\{ \hat{f} : V(G) \to \mathbb{C}; \|\hat{f}\|^2_{l^2(G)} = \sum_{v \in V(G)} |\hat{f}(v)|^2 \deg(v) < \infty \right\},$$

and $\sigma(-\Delta_d) \subset [0, 2]$. See, e.g. Chung [6].

Let G be the hexagonal lattice. Figure 1 illustrates it, where the vertices are represented as big black and white dots and the edges as segments between two of them. For simplicity's sake, we adopt as a free Hamiltonian $\hat{H}_0$ the operator which is unitarily equivalent to $3(\Delta_d + 1)$ on G. See Sect. 2 for the details. The discrete Schrödinger operator $\hat{H}$ is defined as $\hat{H}_0$ plus the potential $\hat{q}$, which is a multiplication operator by a real-valued function: $(\hat{q}\hat{f})(v) = \hat{q}(v)\hat{f}(v)$. Our main result is

Theorem 1.1. *Assume that $\hat{q}(v) = 0$ except for a finite number of $v \in V(G)$. Then from the scattering amplitude $A(\lambda)$ for all $\lambda \in \sigma_{ess}(\hat{H})$, one can uniquely compute $(\hat{q}(v))_{v \in V(G)}$.*

Note that, since our potential $\hat{q}$ is a finite-dimensional operator, the existence and completeness of wave operators are guaranteed by the finite rank and trace class perturbation theory, which were developed from the late 1950s to the early 1960s by Kato [14], Rosenblum [29], Kuroda [20,21], and Birman [3]; see also historical notes in [28].

The hexagonal lattice can be viewed as a discrete model of graphene, which is a two-dimensional, single-layered carbon sheet with honeycomb structure. Graphene is one of the most interesting subjects due to the peculiar behavior of electrons and very actively studied recently in physics, where tight binding models for the Schrödinger operator are employed as standard tool, which give interesting band structures in momentum space. Moreover, the tight binding models for single-layered graphene can be easily extended to more complicated structures, such as bilayer graphene and graphene nanoribbons (Neto et al. [5]). Another approach would be a quantum graph model (Kuchment and Post [19]); unfortunately, there are no physical papers on graphene which are using it as far as the author knows at this point.

As for the inverse problem, there are few results on graphene media. We mention Korotyaev and Kutsenko [17], in which some inverse spectral problem for graphene nanoribbons in external electric fields is considered.

Let us consider the tight binding Hamiltonian on the hexagonal lattice using Fourier transform. Then we have the characteristic energy bands, which are similar to those for particles described by the massless Dirac equation, at Dirac points. See, e.g. [5]. In fact, by expanding the Hamiltonian with respect to momentum around Dirac points, we have the Dirac operator on $\mathbb{R}^2$ approximately (Semenoff [30], González et al. [10]). Therefore, in a sense, we are considering a discrete version of the inverse scattering problem for Dirac operators in two dimensions. It is not known so much about the inverse scattering theory for Dirac operators, relative to that for Schrödinger operators. We refer to Isozaki [11].

The rest of this paper is organized as follows: In Sect. 2, we review spectral properties of the free Hamiltonian on the hexagonal lattice, and define a conjugate operator to derive Mourre estimates. As a consequence, we show the absolute continuity of the discrete Schrödinger operator on the hexagonal lattice. In Sect. 3, we construct a spectral representation and then obtain a representation of the S-matrix. Our main result, the reconstruction procedure for the potential, is shown in Sect. 6, and the key lemmas for it, analytic continuation and estimates of the resolvent, are proved in Sects. 4 and 5, respectively. Section 7 remarks that our reconstruction procedure also works on the triangle lattice in almost the same way as on the hexagonal lattice and the two-dimensional square lattice.

2. Discrete Schrödinger Operators on the Hexagonal Lattice

2.1. Preliminaries

As far as the discrete Laplacian is concerned, we only have to take care of the adjacent relations between the vertices, which enables us to illustrate the

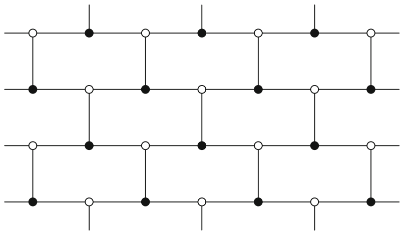

FIGURE 2. The hexagonal lattice can be deformed as above, where the *black* and *white dots* are placed on $\mathbb{Z}^2$ and correspond to those in Fig. 1, respectively

hexagonal lattice G as Fig. 2, where the vertices are placed on $\mathbb{Z}^2$. Therefore, we can regard that

$$V(G) = \mathbb{Z}^2,$$

$$\begin{aligned} E(G) =& \{[m,n]; m,n \in \mathbb{Z}^2, n_1 = m_1 + 1, m_2 = n_2\} \\ & \cup \{[m,n]; m,n \in \mathbb{Z}^2, m_1 = n_1, m_2 \in 2\mathbb{Z}+1, n_2 = m_2 + 1\} \\ & \cup \{[m,n]; m,n \in \mathbb{Z}^2, m_1 = n_1, m_2 \in 2\mathbb{Z}, n_2 = m_2 + 1\}, \end{aligned}$$

where $m = (m_1, m_2)$ and $n = (n_1, n_2)$, and the segment between m and n is denoted by $[m, n]$. Also, noticing that the degree of the vertices on the hexagonal lattice is always three, we can regard $l^2(G)$ as just the l^2-space on $\mathbb{Z}^2$ equipped with the norm $\|\hat{g}\|^2_{l^2(G)} = 3\sum_{n\in\mathbb{Z}^2} |\hat{g}(n)|^2$.

We next introduce the hexagonal lattice structure on $\mathbb{Z}^2$, which is different from the standard lattice one, in the following manner. We split $\mathbb{Z}^2$ into two parts: $\mathbb{Z}^2 = \mathbb{Z}^2_e \cup \mathbb{Z}^2_o$, where

$$\mathbb{Z}^2_e = \{(n_1, n_2); n_1 + n_2 \in 2\mathbb{Z}\},\ \mathbb{Z}^2_o = \{(n_1, n_2); n_1 + n_2 - 1 \in 2\mathbb{Z}\}.$$

In Fig. 2, $\mathbb{Z}^2_e$ and $\mathbb{Z}^2_o$ are represented as the black dots and the white ones, respectively, each of which has a lattice structure with basis $\vec{e}_1 = (1, 1)$ and $\vec{e}_2 = (-1, 1)$. Therefore, there exist canonical isomorphisms:

$$\mathbb{Z}^2_e \ni (n_1, n_2) \longmapsto (m_1, m_2) \in \mathbb{Z}^2,\ n_1 = m_1 - m_2,\ n_2 = m_1 + m_2, \tag{2.1}$$

$$\mathbb{Z}^2_o \ni (n_1, n_2) \longmapsto (m_1, m_2) \in \mathbb{Z}^2,\ n_1 = m_1 - m_2,\ n_2 = 1 + m_1 + m_2. \tag{2.2}$$

Noticing that $\mathbb{Z}^2_e$ and $\mathbb{Z}^2_o$ are the minimal periodic lattice structures in the hexagonal lattice, we need to use a system of difference operators to analyze the discrete Laplacian on the hexagonal lattice.

2.2. Discrete Laplacian

There is a natural unitary mapping $\mathcal{J} : l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2) \to l^2(G)$; more precisely,

$$\mathcal{J} : (\hat{f}_1, \hat{f}_2) \longmapsto \hat{g},\ \sqrt{3}\hat{g}(n) = \begin{cases} \hat{f}_1(m), & \text{if } n \in \mathbb{Z}^2_e, \\ \hat{f}_2(m), & \text{if } n \in \mathbb{Z}^2_o, \end{cases}$$

where $m = (m_1, m_2)$ is defined by (2.1) and (2.2), respectively. Note that $\sqrt{3}$ comes from the degree of the vertices, which contributes to the definition of the norm of $l^2(G)$.

The adjacent vertices of $(n_1, n_2) \in \mathbb{Z}^2_e$ are $(n_1 \pm 1, n_2)$, $(n_1, n_2 + 1)$, and those of $\mathbb{Z}^2_o$ are $(n_1 \pm 1, n_2)$, $(n_1, n_2 - 1)$. Therefore, $3(\Delta_d + 1)$ on G is written as

$$(3(\Delta_d + 1)\hat{g})(n) = \begin{cases} \hat{g}(n_1, n_2 + 1) + \hat{g}(n_1 - 1, n_2) + \hat{g}(n_1 + 1, n_2), & \text{if } n \in \mathbb{Z}^2_e, \\ \hat{g}(n_1, n_2 - 1) + \hat{g}(n_1 - 1, n_2) + \hat{g}(n_1 + 1, n_2), & \text{if } n \in \mathbb{Z}^2_o, \end{cases}$$

for $\hat{g} \in l^2(G)$. This implies that

$$(\mathcal{J}^* 3(\Delta_d + 1)\mathcal{J}\hat{f})_1(m) = \hat{f}_2(m_1, m_2) + \hat{f}_2(m_1 - 1, m_2) + \hat{f}_2(m_1, m_2 - 1),$$

$$(\mathcal{J}^* 3(\Delta_d + 1)\mathcal{J}\hat{f})_2(m) = \hat{f}_1(m_1, m_2) + \hat{f}_1(m_1 + 1, m_2) + \hat{f}_1(m_1, m_2 + 1),$$

for $\hat{f} = ((\hat{f}_1(m))_{m\in\mathbb{Z}^2}, (\hat{f}_2(m))_{m\in\mathbb{Z}^2}) \in l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2)$.

Let us define

$$l^2(\mathbb{Z}^2; \mathbb{C}^2) = \left\{ \hat{f} = \left(\begin{pmatrix} \hat{f}_1(m) \\ \hat{f}_2(m) \end{pmatrix} \right)_{m\in\mathbb{Z}^2} ; \|\hat{f}\|^2_{l^2(\mathbb{Z}^2;\mathbb{C}^2)} = \sum_{m\in\mathbb{Z}^2} (|\hat{f}_1(m)|^2 + |\hat{f}_2(m)|^2) < \infty \right\}.$$

There is a unitary mapping $\mathcal{I} : l^2(\mathbb{Z}^2; \mathbb{C}^2) \to l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2)$, under which we can naturally identify $l^2(\mathbb{Z}^2; \mathbb{C}^2)$ with $l^2(\mathbb{Z}^2) \oplus l^2(\mathbb{Z}^2)$; more precisely,

$$\mathcal{I} : \left(\begin{pmatrix} \hat{f}_1(m) \\ \hat{f}_2(m) \end{pmatrix} \right)_{m\in\mathbb{Z}^2} \longmapsto ((\hat{f}_1(m))_{m\in\mathbb{Z}^2}, (\hat{f}_2(m))_{m\in\mathbb{Z}^2}).$$

Let $\hat{H}_0 = \mathcal{I}^* \mathcal{J}^* 3(\Delta_d + 1)\mathcal{J}\mathcal{I}$. Then we can write

$$(\hat{H}_0 \hat{f})(m) = \begin{pmatrix} \hat{f}_2(m_1, m_2) + \hat{f}_2(m_1 - 1, m_2) + \hat{f}_2(m_1, m_2 - 1) \\ \hat{f}_1(m_1, m_2) + \hat{f}_1(m_1 + 1, m_2) + \hat{f}_1(m_1, m_2 + 1) \end{pmatrix},$$

for $\hat{f} = ((\hat{f}_1(m), \hat{f}_2(m)))_{m\in\mathbb{Z}^2} \in l^2(\mathbb{Z}^2; \mathbb{C}^2)$.

We put $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and

$$L^2(\mathbb{T}^2; \mathbb{C}^2) = \left\{ f = (f_1(\xi), f_2(\xi)); \|f\|^2_{L^2(\mathbb{T}^2;\mathbb{C}^2)} = \int_{\mathbb{T}^2} (|f_1(\xi)|^2 + |f_2(\xi)|^2) \mathrm{d}\xi < \infty \right\}.$$

We define a unitary operator $\mathcal{F} : l^2(\mathbb{Z}^2) \to L^2(\mathbb{T}^2)$ and its adjoint $\mathcal{F}^*$ by

$$(\mathcal{F}\hat{f})(\xi) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}^2} \hat{f}(n) \mathrm{e}^{in\xi}, \ (\mathcal{F}^* f)(n) = \frac{1}{2\pi} \int_{\mathbb{T}^2} f(\xi) \mathrm{e}^{-in\xi} \mathrm{d}\xi,$$

and extend them naturally on $l^2(\mathbb{Z}^2; \mathbb{C}^2)$ and $L^2(\mathbb{T}^2; \mathbb{C}^2)$. They are the usual Fourier series and Fourier coefficients for each component. We also put

$$H_0 = \mathcal{F}\hat{H}_0\mathcal{F}^*.$$

Then H_0 is a multiplication operator by a symmetric matrix on $L^2(\mathbb{T}^2;\mathbb{C}^2)$:

$$(H_0 f)(\xi) = H_0(\xi)f(\xi),\ H_0(\xi) = \begin{pmatrix} 0 & \alpha(\xi) \\ \overline{\alpha}(\xi) & 0 \end{pmatrix},$$

where

$$\alpha(\xi) = \alpha_1(\xi) = 1 + \mathrm{e}^{i\xi_1} + \mathrm{e}^{i\xi_2},\ \xi = (\xi_1, \xi_2) \in \mathbb{T}^2, \tag{2.3}$$

$$\overline{\alpha}(\xi) = \alpha_2(\xi) = \overline{\alpha(\xi)}. \tag{2.4}$$

Next, let us define the unitary operator $\mathcal{U}$ as

$$(\mathcal{U}f)(\xi) = U(\xi)f(\xi),\ U(\xi) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \alpha(\xi)/|\alpha(\xi)| \\ 1 & -\alpha(\xi)/|\alpha(\xi)| \end{pmatrix}$$

on $L^2(\mathbb{T}^2;\mathbb{C}^2)$, where $U(\xi)$ is a unitary matrix for each $\xi \in \mathbb{T}^2$. Its adjoint $\mathcal{U}^*$ is a multiplication operator by the adjoint matrix $U^*(\xi) = U(\xi)^*$. We put

$$\tilde{H}_0 = \mathcal{U} H_0 \mathcal{U}^*.$$

The conjugation by $U(\xi)$ diagonalizes $H_0(\xi)$, which means that $\tilde{H}_0$ is a multiplication operator by a diagonal matrix on $L^2(\mathbb{T}^2;\mathbb{C}^2)$:

$$(\tilde{H}_0 f)(\xi) = \tilde{H}_0(\xi)f(\xi),\ \tilde{H}_0(\xi) = U(\xi)H_0(\xi)U^*(\xi) = \begin{pmatrix} p(\xi) & 0 \\ 0 & -p(\xi) \end{pmatrix},$$

where

$$p(\xi) = |\alpha(\xi)| = \sqrt{3 + 2\cos\xi_1 + 2\cos\xi_2 + 2\cos(\xi_1 - \xi_2)}. \tag{2.5}$$

One computes that

$$p(\xi)^2 = 4\left(\cos\frac{\xi_1 - \xi_2}{2} + \frac{1}{2}\cos\frac{\xi_1 + \xi_2}{2}\right)^2 - \cos^2\frac{\xi_1 + \xi_2}{2} + 1. \tag{2.6}$$

We put

$$\xi_{0,1} = \left(\frac{2\pi}{3}, -\frac{2\pi}{3}\right),\ \xi_{0,2} = \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \in \mathbb{T}^2. \tag{2.7}$$

Then we have

Proposition 2.1. *$p(\xi)$ is continuous and*

$$0 \le p(\xi) \le 3$$

on $\mathbb{T}^2$. Moreover,

1. *$p(\xi) = 0$ if and only if $\xi = \xi_{0,1}$ or $\xi_{0,2}$,*
2. *$p(\xi) = 3$ if and only if $\xi = (0, 0)$,*
3. *$p(\xi)$ is real-analytic on $\mathbb{T}^2 \setminus \{\xi_{0,1}, \xi_{0,2}\}$.*

Proof. Clearly, $p(\xi)^2$ is real-analytic and

$$0 \le p(\xi)^2 = 3 + 2\cos\xi_1 + 2\cos\xi_2 + 2\cos(\xi_1 - \xi_2) \le 9$$

on $\mathbb{T}^2$, which proves the first statement.

By using (2.6), we have $p(\xi) = 0$ if and only if

$$\cos\frac{\xi_1 - \xi_2}{2} + \frac{1}{2}\cos\frac{\xi_1 + \xi_2}{2} = 0,\ \cos\frac{\xi_1 + \xi_2}{2} = \pm 1,$$

which means that the zeros of $p(\xi)$ are exactly $\xi_{0,j}$, $j \in \{1, 2\}$.

The next statement is obvious.

Noticing that $\xi_{0,1}$ and $\xi_{0,2}$ are all the zeros of $p(\xi)^2$, we have the last one. □

We denote by $\sigma(T)$ the spectrum of a self-adjoint operator T; by $\sigma_{ac}(T)$, $\sigma_{sc}(T)$, and $\sigma_{pp}(T)$ its absolutely continuous one, its singularly continuous one, and its pure point one, respectively. By using the above proposition, we have

Proposition 2.2. $\sigma(\tilde{H}_0) = \sigma_{ac}(\tilde{H}_0) = [-3, 3]$ *and* $\sigma_{pp}(\tilde{H}_0) = \sigma_{sc}(\tilde{H}_0) = \emptyset$.

Remark 2.3. In the graph theory, the hexagonal lattice is, in general, called an abelian covering graph of a finite graph. For more general treatments for the spectrum of the discrete Laplacian on such a graph, we refer to Kotani et al. [18].

2.3. Discrete Schrödinger Operators

The potential on $l^2(\mathbb{Z}^2; \mathbb{C}^2)$ is denoted by $\hat{q}$, which is a multiplication operator by real-valued, diagonal, 2×2 matrices:

$$\hat{q} = \sum_{n \in \mathbb{Z}^2} \hat{q}(n)\hat{P}(n),\ \hat{q}(n) = \begin{pmatrix} \hat{q}_1(n) & 0 \\ 0 & \hat{q}_2(n) \end{pmatrix},\ \hat{q}_j(n) \in \mathbb{R},\ j \in \{1, 2\}, \quad (2.8)$$

where $\hat{P}(n)$ is the projection onto the site $n \in \mathbb{Z}^2$ on $l^2(\mathbb{Z}^2; \mathbb{C}^2)$ written as

$$(\hat{P}(n)\hat{f})(m) = \delta_{nm}\hat{f}(m).$$

Here δ_{nm} is Kronecker's delta. Throughout the paper, we shall assume that

(A). $\hat{q}$ is finitely supported, that is, $\hat{q}(n) = 0$ except for a finite number of $n \in \mathbb{Z}^2$.

Note that some parts of our arguments are extended to more general decaying potentials.

We put

$$q = \mathcal{F}\hat{q}\mathcal{F}^*,\ \tilde{q} = \mathcal{U}q\mathcal{U}^*.$$

Then q and $\tilde{q}$ are written as

$$(qf)(\xi) = \frac{1}{2\pi}\begin{pmatrix} \displaystyle\int_{\mathbb{T}^2} q_1(\xi - \zeta) f_1(\zeta) \mathrm{d}\zeta \\ \displaystyle\int_{\mathbb{T}^2} q_2(\xi - \zeta) f_2(\zeta) \mathrm{d}\zeta \end{pmatrix}, \quad (\tilde{q}f)(\xi) = \begin{pmatrix} (\tilde{q}f)_1(\xi) \\ (\tilde{q}f)_2(\xi) \end{pmatrix},$$

for $f \in L^2(\mathbb{T}^2;\mathbb{C}^2)$, where

$$q_j(\xi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}^2} \hat{q}_j(n)\mathrm{e}^{in\xi}, \quad j \in \{1,2\},$$

$$(\tilde{q}f)_1(\xi) = \frac{1}{2(2\pi)}\int_{\mathbb{T}^2}(q_1(\xi-\zeta) + \frac{\alpha(\xi)}{p(\xi)}q_2(\xi-\zeta)\frac{\overline{\alpha}(\zeta)}{p(\zeta)})f_1(\zeta)\mathrm{d}\zeta$$
$$+\frac{1}{2(2\pi)}\int_{\mathbb{T}^2}(q_1(\xi-\zeta) - \frac{\alpha(\xi)}{p(\xi)}q_2(\xi-\zeta)\frac{\overline{\alpha}(\zeta)}{p(\zeta)})f_2(\zeta)\mathrm{d}\zeta,$$

$$(\tilde{q}f)_2(\xi) = \frac{1}{2(2\pi)}\int_{\mathbb{T}^2}(q_1(\xi-\zeta) - \frac{\alpha(\xi)}{p(\xi)}q_2(\xi-\zeta)\frac{\overline{\alpha}(\zeta)}{p(\zeta)})f_1(\zeta)\mathrm{d}\zeta$$
$$+\frac{1}{2(2\pi)}\int_{\mathbb{T}^2}(q_1(\xi-\zeta) + \frac{\alpha(\xi)}{p(\xi)}q_2(\xi-\zeta)\frac{\overline{\alpha}(\zeta)}{p(\zeta)})f_2(\zeta)\mathrm{d}\zeta.$$

The discrete Schrödinger operator is denoted by

$$\hat{H} = \hat{H}_0 + \hat{q}.$$

We also put

$$H = \mathcal{F}\hat{H}\mathcal{F}^* = H_0 + q, \ \tilde{H} = \mathcal{U}H\mathcal{U}^* = \tilde{H}_0 + \tilde{q}.$$

We denote by $\sigma_{ess}(T)$ the essential spectrum of a self-adjoint operator T. Then, by noticing that the potential is a compact perturbation, we have

Proposition 2.4. $\sigma_{ess}(\tilde{H}) = [-3,3]$.

2.4. Sobolev Spaces and Mourre Estimates

Let L_0 be the self-adjoint extension of the operator

$$\begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$$

on $C^\infty(\mathbb{T}^2;\mathbb{C}^2) = \{\mathbb{C}^2\text{-valued smooth function on } \mathbb{T}^2\}$, where $\Delta = (\frac{\partial^2}{\partial{\xi_1}^2} + \frac{\partial^2}{\partial{\xi_2}^2})$. We denote by $\mathcal{H}^s$, $s \in \mathbb{R}$, the domain of the operator $L_0^{s/2}$:

$$\mathcal{H}^s = \{f \in \mathcal{D}^{'}(\mathbb{T}^2;\mathbb{C}^2); \|f\|_s = \|(1+L_0)^{s/2}f\| < \infty\},$$

where $\mathcal{D}^{'}(\mathbb{T}^2;\mathbb{C}^2) = \{\mathbb{C}^2\text{-valued distribution on } \mathbb{T}^2\}$. Also, we denote

$$\hat{\mathcal{H}}^s = \mathcal{F}^*\mathcal{H}^s.$$

We put $\mathcal{H} = \mathcal{H}^0 = L^2(\mathbb{T}^2;\mathbb{C}^2)$ and $\hat{\mathcal{H}} = \hat{\mathcal{H}}^0 = l^2(\mathbb{Z}^2;\mathbb{C}^2)$. The next proposition, which characterizes $\mathcal{H}^s$, is obvious by Parseval's equation.

Proposition 2.5. *$f \in \mathcal{H}^s$, if and only if $\sum_{n\in\mathbb{Z}^2}(1+|n|^2)^s\|\hat{f}(n)\|^2_{\mathbb{C}^2} = \sum_{n\in\mathbb{Z}^2}(1+|n|^2)^s(|\hat{f}_1(n)|^2 + |\hat{f}_2(n)|^2) < \infty$, where $f = \mathcal{F}\hat{f}$.*

We derive Mourre estimates. At first, we perform formal calculations and then modify the conjugate operator by introducing a cut-off function to prove its self-adjointness. Let A be a first-order differential operator defined as

$$A = i[\tilde{H}_0, L_0] = i \begin{pmatrix} \nabla p \cdot \nabla + \nabla \cdot \nabla p & 0 \\ 0 & -(\nabla p \cdot \nabla + \nabla \cdot \nabla p) \end{pmatrix}.$$

By a straightforward calculation, the commutator of $\tilde{H}_0$ and A is

$$i[\tilde{H}_0, A] = 2|\nabla p(\xi)|^2 I_2,$$

where I_2 is the 2×2 identity matrix.

A simple calculation shows that

$$\nabla p(\xi) = \left(\frac{-\sin \xi_1 - \sin(\xi_1 - \xi_2)}{p(\xi)}, \frac{-\sin \xi_2 + \sin(\xi_1 - \xi_2)}{p(\xi)} \right). \tag{2.9}$$

If $\sin \xi_1 + \sin(\xi_1 - \xi_2) = 0$, then

$$\xi_1 - \xi_2 = \xi_1 + \pi \text{ or } -\xi_1 \pmod{2\pi}.$$

Also, if $\sin \xi_2 - \sin(\xi_1 - \xi_2) = 0$, then

$$\xi_1 - \xi_2 = \xi_2 \text{ or } -\xi_2 + \pi \pmod{2\pi}.$$

Note that we cannot define $\nabla p(\xi)$ at $\xi_{0,1}$ and $\xi_{0,2}$. Therefore, we have

$$\{\xi \in \mathbb{T}^2; \nabla p(\xi) = 0\} = \{(0,0), (0,-\pi), (-\pi,0), (-\pi,-\pi)\}.$$

From (2.6), $p(\xi) = 1$ if and only if $\cos \frac{\xi_1 - \xi_2}{2} = 0$ or $\cos \frac{\xi_1 - \xi_2}{2} = -\cos \frac{\xi_1 + \xi_2}{2}$. If $\cos \frac{\xi_1 - \xi_2}{2} = 0$, then

$$\frac{\xi_1 - \xi_2}{2} = \frac{\pi}{2} \pmod{\pi}.$$

Also, if $\cos \frac{\xi_1 - \xi_2}{2} = -\cos \frac{\xi_1 + \xi_2}{2}$, then

$$\frac{\xi_1 - \xi_2}{2} = \pm \frac{\xi_1 + \xi_2}{2} + \pi \pmod{2\pi}.$$

Therefore, we have

$$\begin{aligned} &\{\xi \in \mathbb{T}^2; p(\xi) = 1\} \\ &= \{(\xi_1, \xi_2) \in [-\pi, \pi)^2; \xi_1 = -\pi \text{ or } \xi_2 = -\pi \text{ or } \xi_2 = \xi_1 \pm \pi\}, \end{aligned}$$

which includes the set $\{\xi \in \mathbb{T}^2; \nabla p(\xi) = 0\}$ except the origin.

Let us define

$$\mathcal{M}_E = \{\xi \in \mathbb{T}^2; p(\xi) = E\}. \tag{2.10}$$

Then we have

Proposition 2.6. *For $E \in (0,3) \setminus \{1\}$, $\mathcal{M}_E$ is a real-analytic compact manifold.*

Proof. We have $\nabla p(\xi) \neq 0$ on $\mathcal{M}_E$ for $E \in (0,3) \setminus \{1\}$. Noticing Proposition 2.1, we have the claim. □

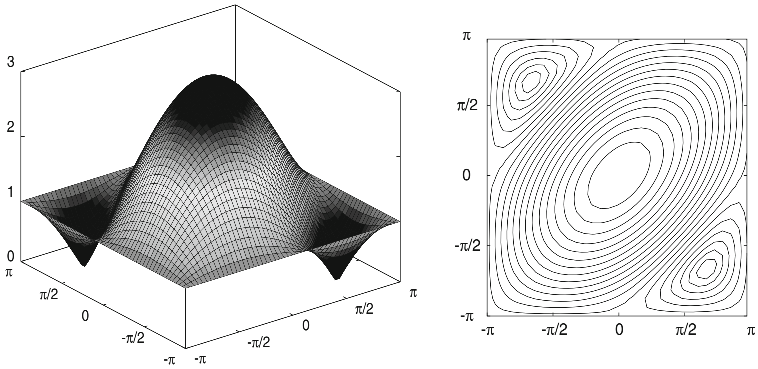

FIGURE 3. The surface graph and the contour lines of $p(\xi)$ on $[-\pi,\pi)^2$

Put $C_0(E) = 2\inf_{\xi\in\mathcal{M}_E} |\nabla p(\xi)|^2$. Note that $C_0(E) > 0$ for $E \in (0,3)\setminus\{1\}$. Then, for small $\varepsilon > 0$, there is $\delta > 0$ such that

$$|\nabla p(\xi)|^2 \geq C_0(E) - \varepsilon > 0$$

on $p^{-1}([E-\delta, E+\delta])$, which shall imply that

$$i[\tilde{H}_0, A] \geq C_0(E) - \varepsilon.$$

We have already shown that $p(\xi_{0,j}) = 0$, $j \in \{1,2\}$, which means that $\nabla p(\xi)$ has singularities at $\xi_{0,1}$ and $\xi_{0,2}$. To avoid them, we introduce a smooth cut-off function $\chi(\xi)$: for small $\mu > 0$,

$$\chi(\xi) = \chi_\mu(\xi) = \begin{cases} 0, & \text{if } \xi \in B_\mu(\xi_{0,1}) \cup B_\mu(\xi_{0,2}), \\ 1, & \text{if } \xi \notin B_{2\mu}(\xi_{0,1}) \cup B_{2\mu}(\xi_{0,2}), \end{cases} \tag{2.11}$$

where $B_\mu(\xi_0) = \{\xi \in \mathbb{T}^2; |\xi - \xi_0| < \mu\}$. Let us define

$$A_\chi = \chi A \chi.$$

Then, by Nelson's commutator theorem (Theorem X.37, Reed-Simon [27], where we take $N = L_0 + 1$), A_χ is essentially self-adjoint on $C^\infty(\mathbb{T}^2;\mathbb{C}^2)$.

Let us choose sufficiently small $\mu > 0$ depending on E and $\delta > 0$. Then we have the Mourre estimate for $\tilde{H}_0$:

$$f(\tilde{H}_0) i[\tilde{H}_0, A_\chi] f(\tilde{H}_0) \geq (C_0(E) - \varepsilon) f(\tilde{H}_0)^2 \tag{2.12}$$

for any real-valued $f \in C_0^\infty((E-\delta, E+\delta))$. Figure 3 helps us to understand what we have done so far in this section.

We can write $(i[\tilde{q}, A_\chi]f)(\xi)$ as a sum of the following terms: for some $a(\xi)$, $b(\xi)$, $c(\xi)$, $d(\xi) \in L^\infty(\mathbb{T}^2)$, and j, k, $l \in \{1,2\}$,

$$a(\xi)\int_{\mathbb{T}^2} q_j(\xi-\zeta) b(\zeta) f_k(\zeta) \mathrm{d}\zeta, \ \ c(\xi)\int_{\mathbb{T}^2} \frac{\partial q_j}{\partial \xi_l}(\xi-\zeta) \mathrm{d}(\zeta) f_k(\zeta) \mathrm{d}\zeta.$$

Our assumption (A) makes those operators compact. As a result, we have the Mourre estimate for $\tilde{H}$, that is, there is a compact operator $K = K_I$ depending on the interval $I = (E - \delta/2, E + \delta/2)$ such that

$$E_{\tilde{H}}(I)[\tilde{H}, iA_\chi]E_{\tilde{H}}(I) \geq (C_0(E) - \varepsilon)E_{\tilde{H}}(I) + K, \tag{2.13}$$

where $E_{\tilde{H}}(\cdot)$ is the spectral projection for $\tilde{H}$.

Let $\tilde{R}(z) = (\tilde{H} - z)^{-1}$. We denote by $\mathbf{B}(X, Y)$, or simply $\mathbf{B}(X)$ when $X = Y$, the set of all bounded operators from X to Y, where X and Y are Banach spaces. We also denote the domain $D((1+|A_\chi|)^s)$ equipped with graph norm by $\mathcal{H}_{A_\chi}$. Then, with the aid of (2.12) and (2.13), by the well-known Mourre theory (Mourre [25]), we have the following theorem:

Theorem 2.7. *Let $I_0 = (-3, 3) \setminus \{\pm 1, 0\}$. Then*

1. *The eigenvalues of $\tilde{H}$ are of finite multiplicities with possible accumulation points 0, ± 1, ± 3.*
2. *There is no singularly continuous spectrum:*

$$\mathcal{H} = \mathcal{H}_{pp}(\tilde{H}) \oplus \mathcal{H}_{ac}(\tilde{H}).$$

3. *Let $s > 1/2$ and $\lambda \in I_0$. Then there is a norm limit $\tilde{R}(\lambda \pm i0) := \lim_{\varepsilon \searrow 0} \tilde{R}(\lambda \pm i\varepsilon)$ in $\mathbf{B}(\mathcal{H}^s_{A_\chi}, \mathcal{H}^{-s}_{A_\chi})$, and $I_0 \ni \lambda \to \tilde{R}(\lambda \pm i0) \in \mathbf{B}(\mathcal{H}^s_{A_\chi}, \mathcal{H}^{-s}_{A_\chi})$ is norm continuous. Furthermore, we have*

$$\sup_{\lambda \in J} \|\tilde{R}(\lambda \pm i0)\|_{\mathbf{B}(\mathcal{H}^s_{A_\chi}, \mathcal{H}^{-s}_{A_\chi})} < \infty$$

for any compact interval J in $I_0 \setminus \sigma_{pp}(\tilde{H})$.

Corollary 2.8. *If $s > 1/2$,*

$$\sup_{\lambda \in J} \|\tilde{R}(\lambda \pm i0)\|_{\mathbf{B}(\mathcal{H}^s, \mathcal{H}^{-s})} < \infty. \tag{2.14}$$

Proof of the Corollary. We only have to note that the inclusions $\mathcal{H}^s \subset \mathcal{H}^s_{A_\chi}$ and $(\mathcal{H}^s_{A_\chi})^* \simeq \mathcal{H}^{-s}_{A_\chi} \subset \mathcal{H}^{-s} \simeq (\mathcal{H}^s)^*$ are continuous for $s > 0$. □

Conjugating by $\mathcal{U}$, we have the same statements for H; moreover, by $\mathcal{F}$, we also have those for $\hat{H}$, where we use $\hat{\mathcal{H}}^s$ instead of $\mathcal{H}^s$.

3. Eigenoperators and Scattering Matrix

3.1. Trace Operators

Let $\tilde{R}_0(z) = (\tilde{H}_0 - z)^{-1}$. Then we have

$$(\tilde{R}_0(z)f, g)_{L^2(\mathbb{Z}^2;\mathbb{C}^2)} = \int_{\mathbb{T}^2} \frac{f_1(\xi)\overline{g_1(\xi)}}{p(\xi) - z} \mathrm{d}\xi + \int_{\mathbb{T}^2} \frac{f_2(\xi)\overline{g_2(\xi)}}{-p(\xi) - z} \mathrm{d}\xi$$

for $f = (f_1, f_2), g = (g_1, g_2) \in C^\infty(\mathbb{T}^2; \mathbb{C}^2)$.

It has been already shown that $\mathcal{M}_\lambda$ is a real-analytic manifold for $\lambda \in (0,3) \setminus \{1\}$. Thus we can introduce local coordinates ω on $\mathcal{M}_\lambda$, which induce the measure

$$\mathrm{d}\xi_1 \mathrm{d}\xi_2 = \mathrm{d}\mathcal{M}_\lambda(\omega)\mathrm{d}\lambda = J(\lambda,\omega)\mathrm{d}\omega\mathrm{d}\lambda, \tag{3.1}$$

where $J(\lambda,\omega)$ is real-analytic with respect to λ and ω.

We note that the set $\{\xi \in \mathbb{T}^2; p(\xi) \in \{0,1,3\}\}$, which includes all the extreme points and the critical points of $p(\xi)$, has null Lebesgue measure. It enables us to write

$$(\tilde{R}_0(z)f,g)_{L^2(\mathbb{Z}^2;\mathbb{C}^2)} = \int_0^3 \frac{h_1(\rho)}{\rho - z}\mathrm{d}\rho + \int_{-3}^0 \frac{h_2(-\rho)}{\rho - z}\mathrm{d}\rho = \int_{-3}^3 \frac{h(\rho)}{\rho - z}\mathrm{d}\rho,$$

where

$$h_j(\rho) = \int_{\mathcal{M}_{|\rho|}} f_j(\xi(\rho,\omega))\overline{g_j(\xi(\rho,\omega))}\mathrm{d}\mathcal{M}_{|\rho|}(\omega), \quad j \in \{1,2\},$$

$$h(\rho) = \begin{cases} h_1(\rho), & \text{if } \rho \in (0,3) \setminus \{1\}, \\ h_2(-\rho), & \text{if } \rho \in (-3,0) \setminus \{-1\}. \end{cases}$$

Then we have

$$\begin{aligned} &(\tilde{R}_0(\lambda \pm i0)f,g)_{L^2(\mathbb{T}^2;\mathbb{Z}^2)} \\ &\quad = \pm i\pi h(\lambda) + \text{p.v.}\int_{\lambda-\delta}^{\lambda-\delta} \frac{h(\rho)-h(\lambda)}{\rho - \lambda}\mathrm{d}\rho + \int_{|\lambda-\rho|>\delta} \frac{h(\rho)}{\rho-\lambda}\mathrm{d}\rho \end{aligned}$$

for $\lambda \in (-3,3) \setminus \{\pm 1, 0\}$, which leads us to

$$\frac{1}{2\pi i}((\tilde{R}_0(\lambda + i0) - \tilde{R}_0(\lambda - i0))f,g)_{L^2(\mathbb{T}^2;\mathbb{C}^2)} = h(\lambda). \tag{3.2}$$

Let us define the trace operator $\tilde{\mathcal{F}}_0(\lambda)$ as

$$(\tilde{\mathcal{F}}_0(\lambda)f) = \begin{pmatrix} \chi_{(0,3)}(\lambda)f_1\big|_{\mathcal{M}_\lambda} \\ \chi_{(-3,0)}(\lambda)f_2\big|_{\mathcal{M}_\lambda} \end{pmatrix}$$

for $f \in C^\infty(\mathbb{T}^2;\mathbb{C}^2)$, where χ_I is a characteristic function of interval $I \subset \mathbb{R}$, that is, $\chi_I(\lambda) = 1$ if $\lambda \in I$, and $\chi_I(\lambda) = 0$ otherwise. From (3.2), we have

$$\frac{1}{2\pi i}((\tilde{R}_0(\lambda + i0) - \tilde{R}_0(\lambda - i0))f,g)_{L^2(\mathbb{T}^2;\mathbb{C}^2)} = (\tilde{\mathcal{F}}_0(\lambda)f, \tilde{\mathcal{F}}_0(\lambda)g)_{L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)}. \tag{3.3}$$

for $\lambda \in (-3,3) \setminus \{\pm 1, 0\}$; moreover, by Corollary 2.8 and (3.3),

$$\tilde{\mathcal{F}}_0(\lambda) \in \mathbf{B}(\mathcal{H}^s, L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)) \tag{3.4}$$

for $s > 1/2$.

By the definition, we have

$$\tilde{\mathcal{F}}_0(\lambda)(\tilde{H}_0 - \lambda) = 0. \tag{3.5}$$

We pass from $\tilde{H}_0$ to H_0. For $s \in \mathbb{R}$ and small $\varepsilon > 0$, let $\mathcal{H}^s_\varepsilon$ be the completion of $C^\infty_\varepsilon(\mathbb{T}^2;\mathbb{C}^2)$ in $\mathcal{H}^s$, where

$$C^\infty_\varepsilon(\mathbb{T}^2;\mathbb{C}^2) = \{f \in C^\infty(\mathbb{T}^2;\mathbb{C}^2); \mathrm{supp} f \subset \mathbb{T}^2 \setminus (B_\varepsilon(\xi_{0,1}) \cup B_\varepsilon(\xi_{0,2}))\}.$$

Note that $\tilde{R}_0(z)C^\infty_\varepsilon(\mathbb{T}^2;\mathbb{C}^2) \subset C^\infty_\varepsilon(\mathbb{T}^2;\mathbb{C}^2)$. Since $\mathcal{H}^s_\varepsilon$ avoids the singular points of $\alpha(\xi)/|\alpha(\xi)|$, we have

Lemma 3.1. *The operator $\mathcal{U}$ is defined as a homomorphism on $\mathcal{H}^s_\varepsilon$.*

For sufficiently small $\varepsilon > 0$, for example $0 < \varepsilon < |\lambda|/2$, we have $\tilde{R}_0(\lambda \pm i0) \in \mathbf{B}(L^2(B_\varepsilon(\xi_{0,1}) \cup B_\varepsilon(\xi_{0,2});\mathbb{C}^2))$; moreover, $\tilde{R}_0(\lambda + i0)f = \tilde{R}_0(\lambda - i0)f$ for $f \in L^2(B_\varepsilon(\xi_{0,1}) \cup B_\varepsilon(\xi_{0,2});\mathbb{C}^2)$. Therefore, we can extend $\tilde{\mathcal{F}}_0(\lambda)$ uniquely as a null operator on $L^2(B_\varepsilon(\xi_{0,1}) \cup B_\varepsilon(\xi_{0,2});\mathbb{C}^2)$.

Let $\mathcal{U}^\dagger_\lambda$ be a multiplication operator on $L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)$ by the unitary matrix

$$U^\dagger(\lambda,\omega) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ |\lambda|^{-1}\overline{\alpha}(\xi(|\lambda|,\omega)) & -|\lambda|^{-1}\overline{\alpha}(\xi(|\lambda|,\omega)) \end{pmatrix},$$

which means that $\mathcal{U}^\dagger_\lambda$ is a unitary operator. Then, from Lemma 3.1 and its subsequent descriptions, we can define the operator

$$\mathcal{F}_0(\lambda) = \mathcal{U}^\dagger_\lambda \tilde{\mathcal{F}}_0(\lambda)\mathcal{U},$$

which is written as

$$(\mathcal{F}_0(\lambda)f)_k(\omega) = \frac{1}{2}f_k(\xi(|\lambda|,\omega)) + \frac{1}{2}\lambda^{-1}\alpha_k(\xi(|\lambda|,\omega))f_l(\xi(|\lambda|,\omega))$$

for $f \in \mathcal{H}^s$, $s > 1/2$, where $(k,l) = (1,2)$ or $(2,1)$.

Lemma 3.2. *For $s > 1/2$, $\mathcal{F}_0(\lambda) \in \mathbf{B}(\mathcal{H}^s, L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2))$.*

Proof. Let $\chi = \chi_\varepsilon \in C^\infty(\mathbb{T}^2)$ be defined as (2.11). Then, $1 - \chi$ as a multiplication operator is continuous from $\mathcal{H}^s$ to $L^2(B_\varepsilon(\xi_{0,1}) \cup B_\varepsilon(\xi_{0,2});\mathbb{C}^2)$; so is χ from $\mathcal{H}^s$ to $\mathcal{H}^s_{\varepsilon/4}$. Therefore, using Lemma 3.1 and (3.4), and noticing that $\mathcal{U}^\dagger_\lambda$ is unitary, we have the lemma. □

Let $R_0(z) = (H_0 - z)^{-1}$. Passing through $\mathcal{U}^\dagger_\lambda$ and $\mathcal{U}$ in (3.3), we have

Lemma 3.3. *For $f, g \in \mathcal{H}^s$, $s > 1/2$,*

$$\frac{1}{2\pi i}((R_0(\lambda + i0) - R_0(\lambda - i0))f, g)_{L^2(\mathbb{T}^2;\mathbb{C}^2)} = (\mathcal{F}_0(\lambda)f, \mathcal{F}_0(\lambda)g)_{L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)}.$$

By Corollary 2.8, Lemma 3.3, and Stone's formula, we have

$$(f,g)_{L^2(\mathbb{T}^2;\mathbb{C}^2)} = \int_{-3}^{3} (\mathcal{F}_0(\lambda)f, \mathcal{F}_0(\lambda)g)_{L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)} \mathrm{d}\lambda,$$

for $f, g \in \mathcal{H}^s$, $s > 1/2$; moreover, from (3.5),

$$\mathcal{F}_0(\lambda)(H_0 - \lambda) = 0. \tag{3.6}$$

For $\lambda \in (-3,3) \setminus (\{\pm 1, 0\} \cup \sigma_{pp}(H))$, let us define the operator

$$\mathcal{F}^{(\pm)}(\lambda) = \mathcal{F}_0(\lambda)(I - qR(\lambda \pm i0)).$$

Then we have

Lemma 3.4. *For $s > 1/2$ and $\lambda \in (-3,3) \setminus (\{\pm 1, 0\} \cup \sigma_{pp}(H))$,*

$$\mathcal{F}^{(\pm)}(\lambda) \in \mathbf{B}(\mathcal{H}^s, L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2)).$$

Proof. Noticing that $q \in \mathbf{B}(\mathcal{H}^{-s}, \mathcal{H}^s)$, this is an immediate consequence of Corollary 2.8 and Lemma 3.2. □

By the definition of $\mathcal{F}^{(\pm)}(\lambda)$ and (3.6), we have

$$\mathcal{F}^{(\pm)}(\lambda)(H - \lambda) = 0. \tag{3.7}$$

The next lemma is proved by calculating $2\pi i \mathcal{F}^{(\pm)}(\lambda)^* \mathcal{F}^{(\pm)}(\lambda)$ in the same way as Lemma 3.3 of [13].

Lemma 3.5. *For $f, g \in \mathcal{H}^s$, $s > 1/2$,*

$$\begin{aligned}\frac{1}{2\pi i}&((R(\lambda + i0) - R(\lambda - i0))f, g)_{L^2(\mathbb{T}^2;\mathbb{C}^2)} \\ &= (\mathcal{F}^{(\pm)}(\lambda) f, \mathcal{F}^{(\pm)}(\lambda) g)_{L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)}.\end{aligned}$$

3.2. Eigenoperators and Spectral Representations

The adjoint operator $\mathcal{F}_0(\lambda)^*$ is defined by

$$(\mathcal{F}_0(\lambda) f, \phi)_{L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)} = (f, \mathcal{F}_0(\lambda)^* \phi)_{L^2(\mathbb{T}^2;\mathbb{C}^2)}$$

for $f \in \mathcal{H}^s$, $s > 1/2$, and $\phi \in L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2)$. By the definition, we have $\mathcal{F}_0(\lambda)^* \in \mathbf{B}(L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2), \mathcal{H}^{-s})$, which is written as

$$\begin{aligned}&(\mathcal{F}_0(\lambda)^* \phi)_k(\xi(|\rho|, \omega)) \\ &\quad = \frac{1}{2}\delta(p(\xi(|\rho|, \omega)) - |\lambda|)\phi_k(\omega) + \frac{1}{2}\frac{\alpha_k(\xi(|\rho|, \omega))}{\lambda}\delta(p(\xi(|\rho|, \omega)) - |\lambda|)\phi_l(\omega),\end{aligned} \tag{3.8}$$

for $\phi \in C^\infty(\mathcal{M}_{|\lambda|}; \mathbb{C}^2)$, where $\delta(\cdot)$ is Dirac's delta and $(k, l) = (1, 2)$ or $(2, 1)$.

By taking the adjoint of (3.6), we have

$$(H_0 - \lambda)\mathcal{F}_0(\lambda)^* = 0,$$

which means that $\mathcal{F}_0(\lambda)^*$ is the eigenoperator of H_0.

Let us define the operator $\mathcal{F}_0$ by

$$(\mathcal{F}_0 f)(\lambda, \omega) = (\mathcal{F}_0(\lambda) f)(\omega)$$

for $f \in \mathcal{H}^s$, $s > 1/2$. The proof of the next theorem can be done in the same way as that for Theorem 3.2 in [13].

Theorem 3.6. 1. *$\mathcal{F}_0$ is uniquely extended to a unitary operator from $L^2(\mathbb{T}^2; \mathbb{C}^2)$ to $L^2((-3, 3); L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2); \mathrm{d}\lambda)$.*

2. *$\mathcal{F}_0$ diagonalizes H_0:*

$$(\mathcal{F}_0 H_0 f)(\lambda) = \lambda(\mathcal{F}_0 f)(\lambda).$$

3. *For any compact interval* $I \subset (-3,3) \setminus \{\pm 1, 0\}$,

$$\int_I \mathcal{F}_0(\lambda)^* g(\lambda) \mathrm{d}\lambda \in L^2(\mathbb{T}^2; \mathbb{C}^2)$$

for $g \in L^2((-3,3); L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2); \mathrm{d}\lambda)$. *Moreover, the following inversion formula holds:*

$$f = s\text{-}\lim_{N\to\infty} \int_{I_N} \mathcal{F}_0(\lambda)^* (\mathcal{F}_0 f)(\lambda) \mathrm{d}\lambda$$

for $f \in L^2(\mathbb{T}^2; \mathbb{C}^2)$, *where* I_N *is a finite union of compact intervals in* $(-3,3) \setminus \{\pm 1, 0\}$ *such that* $I_N \to (-3,3)$ *as* $N \to \infty$.

Let us also define the operators $\mathcal{F}^{(\pm)}$ as

$$(\mathcal{F}^{(\pm)} f)(\lambda, \omega) = (\mathcal{F}^{(\pm)}(\lambda) f)(\omega)$$

for $f \in \mathcal{H}^s$, $s > 1/2$. The proof of the next theorem can be also done in the same way as that for Theorem 3.4 in [13].

Theorem 3.7. 1. $\mathcal{F}^{(\pm)}$ *are uniquely extended to partial isometries from* $L^2(\mathbb{T}^2; \mathbb{C}^2)$ *to* $L^2((-3,3); L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}); \mathrm{d}\lambda)$ *with the initial set* $\mathcal{H}_{ac}(H)$ *and the final set* $L^2((-3,3); L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2); \mathrm{d}\lambda)$. *Moreover,* $\mathcal{F}^{(\pm)}$ *diagonalize* H*:*

$$(\mathcal{F}^{(\pm)} H f)(\lambda) = \lambda (\mathcal{F}^{(\pm)} f)(\lambda).$$

2. *The following inversion formula holds:*

$$f = s\text{-}\lim_{N\to\infty} \int_{I_N} \mathcal{F}^{(\pm)}(\lambda)^* (\mathcal{F}^{(\pm)} f)(\lambda) \mathrm{d}\lambda$$

for $f \in \mathcal{H}_{ac}(H)$, *where* I_N *is a finite union of compact intervals in* $(-3,3) \setminus \big(\{\pm 1, 0\} \cup \sigma_{pp}(H)\big)$ *such that* $I_N \to (-3,3)$ *as* $N \to \infty$.

3. $\mathcal{F}^{(\pm)}(\lambda)^* \in \mathbf{B}(L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2), \mathcal{H}^{-s})$ *are eigenoperators for* H *in the following sense:*

$$(H - \lambda) \mathcal{F}^{(\pm)}(\lambda)^* \phi = 0$$

for $\phi \in L^2(\mathcal{M}_{|\lambda|}; \mathbb{C}^2)$.

Now, we turn to the spectral representation for $\hat{H}$ on the lattice. For $\omega \in \mathcal{M}_{|\lambda|}$, let us define the distribution $\delta_\omega(\cdot) \in \mathcal{D}'(\mathcal{M}_{|\lambda|})$ by

$$\langle \delta_\omega, \phi \rangle = \phi(\omega)$$

for $\phi \in C^\infty(\mathcal{M}_{|\lambda|})$. Note that $\mathcal{M}_{|\lambda|}$ is an analytic manifold for $\lambda \in (-3,3) \setminus \{\pm 1, 0\}$. Then, from (3.8), we have the matrix-valued distribution kernel

$\psi^{(0)}(\xi;\lambda,\theta)$ of $\mathcal{F}_0(\lambda)^*$, which is written as

$$\begin{aligned}&\psi^{(0)}(\xi;\lambda,\theta)\\&\quad=\psi^{(0)}(\xi(|\rho|,\omega);\lambda,\theta)\\&\quad=\frac{1}{2}\begin{pmatrix}1 & \lambda^{-1}\alpha(\xi(|\rho|,\omega))\\ \lambda^{-1}\overline{\alpha}(\xi(|\rho|,\omega)) & 1\end{pmatrix}\delta(p(\xi(|\rho|,\omega))-|\lambda|)\otimes\delta_\omega(\theta),\end{aligned}$$

where $\omega\in\mathcal{M}_{|\rho|}$ and $\theta\in\mathcal{M}_{|\lambda|}$.

In the lattice space, we consider the Fourier coefficients of $\psi^{(0)}(\xi;\lambda,\theta)$:

$$\hat{\psi}^{(0)}(\lambda,\theta)=(\hat{\psi}^{(0)}(n;\lambda,\theta))_{n\in\mathbb{Z}^2},\tag{3.9}$$

$$\begin{aligned}\hat{\psi}^{(0)}(n;\lambda,\theta)&=\frac{1}{2\pi}\int_{\mathbb{T}^2}\psi^{(0)}(\xi;\lambda,\theta)\mathrm{e}^{-in\xi}\mathrm{d}\xi\\&=\frac{1}{2}\frac{1}{2\pi}J(|\lambda|,\theta)\mathrm{e}^{in\xi(|\lambda|,\theta)}\begin{pmatrix}1 & \lambda^{-1}\alpha(\xi(|\lambda|,\theta))\\ \lambda^{-1}\overline{\alpha}(\xi(|\lambda|,\theta)) & 1\end{pmatrix},\end{aligned}\tag{3.10}$$

where $J(|\lambda|,\theta)$ is the density introduced in (3.1). We put

$$\hat{\mathcal{F}}_0=\mathcal{F}_0\mathcal{F}.$$

Then we have

$$\begin{aligned}&(\mathcal{F}_0(\lambda)\mathcal{F}\hat{f})_k(\theta)\\&\quad=\frac{1}{2}\frac{1}{2\pi}\sum_{n\in\mathbb{Z}^2}\mathrm{e}^{in\xi(|\lambda|,\theta)}\hat{f}_k(n)+\frac{1}{2}\frac{1}{2\pi}\frac{\alpha_k(\xi(|\lambda|,\theta))}{\lambda}\sum_{n\in\mathbb{Z}^2}\mathrm{e}^{in\xi(|\lambda|,\theta)}\hat{f}_l(n),\end{aligned}\tag{3.11}$$

for $\hat{f}\in\hat{\mathcal{H}}^s$, $s>1/2$, where $(k,l)=(1,2)$ or $(2,1)$. Also, we have

$$\hat{\mathcal{F}}_0^*\phi(\lambda)=(\hat{\mathcal{F}}_0^*\phi(n;\lambda))_{n\in\mathbb{Z}^2},$$

for $\phi\in L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)$, which is written as

$$\hat{\mathcal{F}}_0^*\phi(n;\lambda)=(\hat{\mathcal{F}}_0(\lambda)^*\phi)(n)=\int_{\mathcal{M}_{|\lambda|}}\hat{\psi}^{(0)}(n;\lambda,\theta)\phi(\theta)\mathrm{d}\mathcal{M}_{|\lambda|}(\theta),$$

$$\begin{aligned}(\hat{\mathcal{F}}_0(\lambda)^*\phi)_k(n)&=\frac{1}{2\pi}\int_{\mathcal{M}_{|\lambda|}}\frac{1}{2}\mathrm{e}^{-in\xi(|\lambda|,\theta)}\phi_k(\theta)\mathrm{d}\mathcal{M}_{|\lambda|}(\theta)\\&\quad+\frac{1}{2\pi}\int_{\mathcal{M}_{|\lambda|}}\frac{1}{2}\frac{\alpha_k(\xi(|\lambda|,\theta))}{\lambda}\mathrm{e}^{-in\xi(|\lambda|,\theta)}\phi_l(\theta)\mathrm{d}\mathcal{M}_{|\lambda|}(\theta),\end{aligned}\tag{3.12}$$

where $(k,l)=(1,2)$ or $(2,1)$.

We put

$$\hat{\mathcal{F}}^{(\pm)}=\mathcal{F}^{(\pm)}\mathcal{F}.$$

Passing it to the Fourier transform in Theorem 3.7, we have

Theorem 3.8. 1. *$\hat{\mathcal{F}}^{(\pm)}$ are uniquely extended to partial isometries from $l^2(\mathbb{Z}^2;\mathbb{C}^2)$ to $L^2((-3,3);L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2);\mathrm{d}\lambda)$ with the initial set $\mathcal{H}_{ac}(\hat{H})$ and the final set $L^2((-3,3);L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2);\mathrm{d}\lambda)$. Moreover, $\hat{\mathcal{F}}^{(\pm)}$ diagonalize $\hat{H}$:*

$$(\hat{\mathcal{F}}^{(\pm)}\hat{H}\hat{f})(\lambda) = \lambda(\hat{\mathcal{F}}^{(\pm)}\hat{f})(\lambda).$$

2. *The following inversion formula holds:*

$$\hat{f} = s\text{-}\lim_{N\to\infty}\int_{I_N}\hat{\mathcal{F}}^{(\pm)}(\lambda)^*(\hat{\mathcal{F}}^{(\pm)}\hat{f})(\lambda)\mathrm{d}\lambda$$

for $f \in \mathcal{H}_{ac}(\hat{H})$, where I_N is a finite union of compact intervals in $(-3,3)\setminus\big(\{\pm1,0\}\cup\sigma_{pp}(H)\big)$ such that $I_N \to (-3,3)$ as $N\to\infty$.

3. *$\hat{\mathcal{F}}^{(\pm)}(\lambda)^* \in \mathbf{B}(L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2),\hat{\mathcal{H}}^{-s})$ are eigenoperators for $\hat{H}$ in the following sense:*

$$(\hat{H}-\lambda)\hat{\mathcal{F}}^{(\pm)}(\lambda)^*\phi = 0$$

for $\phi\in L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)$.

3.3. Scattering Matrix

Since $\hat{q}$ is a finite rank perturbation, we can prove that the wave operators

$$W_\pm = \text{s-}\lim_{t\to\pm\infty}\mathrm{e}^{it\hat{H}}\mathrm{e}^{-it\hat{H}_0}$$

exist, and that they are complete (Theorem XI.8, Reed-Simon [28]). Then the scattering operator S is defined by

$$S = W_+^*W_-.$$

We put

$$\hat{S} = \hat{\mathcal{F}}_0 S\hat{\mathcal{F}}_0^*.$$

Since S commutes with $\hat{H}_0$ and $\hat{\mathcal{F}}_0$ diagonalizes $\hat{H}_0$, $\hat{S}$ is decomposable on $L^2((-3,3);L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2);\mathrm{d}\lambda) = \int_{-3}^{3}\oplus L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)\mathrm{d}\lambda$:

$$\hat{S} = \int_{-3}^{3}\oplus\hat{S}(\lambda)\mathrm{d}\lambda,$$

where $\hat{S}(\lambda)$ is the S-matrix, which is unitary on $L^2(\mathcal{M}_{|\lambda|};\mathbb{C}^2)$.

Let us define the scattering amplitude $A(\lambda)$ by

$$\hat{S}(\lambda) = I - 2\pi i A(\lambda).$$

Then, by using the abstract stationary scattering theory (Kato and Kuroda [15], Kuroda [22,23]), we have

$$A(\lambda) = \hat{\mathcal{F}}_0(\lambda)\hat{q}\hat{\mathcal{F}}_0(\lambda)^* - \hat{\mathcal{F}}_0(\lambda)\hat{q}\hat{R}(\lambda+i0)\hat{q}\hat{\mathcal{F}}_0(\lambda)^*.$$

Let $A(\lambda,\theta,\theta')$ be the integral kernel of $A(\lambda)$, which is written as

$$A(\lambda,\theta,\theta') = \begin{pmatrix} A_{11}(\lambda,\theta,\theta') & A_{12}(\lambda,\theta,\theta') \\ A_{21}(\lambda,\theta,\theta') & A_{22}(\lambda,\theta,\theta')\end{pmatrix}.$$

Then, using (2.8), (3.11) and (3.12), we have the following representations for $A_{jj}(\lambda, \theta, \theta')$, $j \in \{1, 2\}$:

$$\begin{aligned} &A_{11}(\lambda, \theta, \theta') \\ &\quad = \frac{1}{4}\frac{1}{(2\pi)^2} J(|\lambda|, \theta) J(|\lambda|, \theta') \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in(\xi(|\lambda|,\theta)-\xi(|\lambda|,\theta'))} \\ &\qquad \cdot \big(\hat{q}_1(n) + \frac{\alpha(\xi(|\lambda|, \theta))}{\lambda} \frac{\overline{\alpha}(\xi(|\lambda|, \theta'))}{\lambda} \hat{q}_2(n)\big) \\ &\qquad + \frac{1}{2}\frac{1}{2\pi} J(|\lambda|, \theta) \Big(\sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(|\lambda|,\theta)} \hat{q}_1(n) \big(\hat{R}(\lambda + i0)\hat{q}\hat{\psi}^{(0)}(\lambda, \theta')\big)_{11}(n) \\ &\qquad + \frac{\alpha(\xi(|\lambda|, \theta))}{\lambda} \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(|\lambda|,\theta)} \hat{q}_2(n) \big(\hat{R}(\lambda + i0)\hat{q}\hat{\psi}^{(0)}(\lambda, \theta')\big)_{21}(n) \Big), \end{aligned}$$

$$\begin{aligned} A_{22}(\lambda, \theta, \theta') &= \frac{1}{4}\frac{1}{(2\pi)^2} J(|\lambda|, \theta) J(|\lambda|, \theta') \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in(\xi(|\lambda|,\theta)-\xi(|\lambda|,\theta'))} \\ &\cdot \big(\frac{\overline{\alpha}(\xi(|\lambda|, \theta))}{\lambda} \frac{\alpha(\xi(|\lambda|, \theta'))}{\lambda} \hat{q}_1(n) + \hat{q}_2(n)\big) \\ &+ \frac{1}{2}\frac{1}{2\pi} J(|\lambda|, \theta) \left(\frac{\overline{\alpha}(\xi(|\lambda|, \theta))}{\lambda} \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(|\lambda|,\theta)} \hat{q}_1(n) \big(\hat{R}(\lambda+i0)\hat{q}\hat{\psi}^{(0)}(\lambda, \theta')\big)_{12}(n) \right. \\ &\left. + \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(|\lambda|,\theta)} \hat{q}_2(n) \big(\hat{R}(\lambda + i0)\hat{q}\hat{\psi}^{(0)}(\lambda, \theta')\big)_{22}(n) \right). \end{aligned}$$

We omit the formulas for $A_{ij}(\lambda, \theta, \theta')$, $(i, j) = (1, 2)$ or $(2, 1)$, since we do not use them later.

4. Analytic Continuation

A simple calculation shows that if $\xi \in \mathcal{M}_\lambda$,

$$\left(\cos\frac{\xi_1}{2}\cos\frac{\xi_2}{2} + \frac{1}{2}\sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2} \right)^2 - \left(\frac{1}{2}\sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2} \right)^2 = \frac{1}{8}(\lambda^2 - 1). \tag{4.1}$$

We let $\lambda = \sqrt{8k^2 + 1}$ so that k varies over $(-1, 0) \cup (0, 1)$. Then we have

$$\cos\frac{\xi_1}{2}\cos\frac{\xi_2}{2} + \frac{1}{2}\sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2} = \pm k\cosh\theta,\ \ \frac{1}{2}\sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2} = \pm k\sinh\theta, \tag{4.2}$$

where $\theta \in \mathbb{R}$. We parametrize $\xi \in \mathcal{M}_\lambda$ using (4.2), for which we need to solve (4.2) with respect to ξ_1 and ξ_2.

From (4.2), we have

$$\cos^2\frac{\xi_1}{2}\cos^2\frac{\xi_2}{2} = k^2 e^{-2\theta},\ \ \sin^2\frac{\xi_1}{2}\sin^2\frac{\xi_2}{2} = 4k^2\sinh^2\theta. \tag{4.3}$$

By solving (4.3) with respect to $\sin^2 \frac{\xi_j}{2}$ and $\cos^2 \frac{\xi_j}{2}$, $j \in \{1,2\}$, we put

$$\sin^2 \frac{\xi_1}{2} = f_{s,+,\theta}(k^2),\ \sin^2 \frac{\xi_2}{2} = f_{s,-,\theta}(k^2), \tag{4.4}$$

$$\cos^2 \frac{\xi_1}{2} = f_{c,-,\theta}(k^2),\ \cos^2 \frac{\xi_2}{2} = f_{c,+,\theta}(k^2), \tag{4.5}$$

where

$$f_{s,\pm,\theta}(x) = \frac{1}{2}\left\{1 - (2-e^{2\theta})x \pm \sqrt{\left(1-(2-e^{2\theta})\,x\right)^2 - 4^2 x \sinh^2\theta}\right\},$$

$$f_{c,\pm,\theta}(x) = \frac{1}{2}\left\{1 + (2-e^{2\theta})x \pm \sqrt{\left(1-(2-e^{2\theta})\,x\right)^2 - 4^2 x \sinh^2\theta}\right\}.$$

In the following, we consider the case that $\xi_1, \xi_2 \in (0,\pi)$ so that $\sin\frac{\xi_j}{2} > 0$ and $\cos\frac{\xi_j}{2} > 0$, $j \in \{1,2\}$ and define $\xi_1(k,\theta)$, $\xi_2(k,\theta)$ by the equations (4.4) and (4.5). We also put

$$a = a(\theta) = \sqrt{2-e^{2\theta}}, \quad b = b(\theta) = \frac{2\sinh\theta}{\sqrt{2-e^{2\theta}}}. \tag{4.6}$$

Then we have

Lemma 4.1. *Assume $0 < \theta < \frac{1}{2}\log 2$. Then $\xi_j(k,\theta)$ has the analytic continuations $\zeta_j(z,\theta)$, $j \in \{1,2\}$, with the following properties:*

1. *$\zeta_j(z,\theta)$, $j \in \{1,2\}$, are analytic with respect to $z \in \mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$.*
2. *If $k \in (-1,0) \cup (0,1)$, we have $\zeta_j(k+i0,\theta) = \xi_j(k,\theta)$, $j \in \{1,2\}$.*
3. *If $0 < \xi_j < \pi$, $j \in \{1,2\}$, there exist m_1, $m_2 \in \mathbb{Z}$ such that, as $N \to +\infty$,*

$$\Re\zeta_1(1+iN,\theta) = 2m_1\pi + \mathcal{O}(N^{-3}), \tag{4.7}$$

$$\Im\zeta_1(1+iN,\theta) = 2\log\left(b + \sqrt{b^2+1}\right) + \mathcal{O}(N^{-2}), \tag{4.8}$$

$$\Re\zeta_2(1+iN,\theta) = (2m_2+1)\pi + \mathcal{O}(N^{-1}), \tag{4.9}$$

$$\Im\zeta_2(1+iN,\theta) = 2\log N + \log(4a^2) + \mathcal{O}(N^{-2}). \tag{4.10}$$

Proof. We note that

$$0 \le f_{s,\pm,\theta}(x) \le 1,\ x \in [0,\infty),$$
$$f_{s,-,\theta}(0) = 0,\ f_{s,+,\theta}(0) = 1,$$
$$\lim_{x\to+\infty} f_{s,-,\theta}(x) = 1,\ \lim_{x\to+\infty} f_{s,+,\theta}(x) = 0.$$

A simple calculation shows that $f_{s,-,\theta}(x)$ (resp. $f_{s,+,\theta}(x)$) are increasing (resp. decreasing) monotonically.

We put

$$g(z) = \left(1 - \left(2-e^{2\theta}\right) z^2\right)^2 - 4^2 z^2 \sinh^2\theta.$$

Another simple calculation shows that all the zeros of $g(z)$ belong to $\mathbb{R}$. Then we have the analytic continuations of $f_{s,\pm,\theta}(z^2)$ with respect to $z \in \mathbb{C}_+$.

Furthermore, another simple calculation shows that $0 \le f_{s,\pm,\theta}(z^2) \le 1$ if and only if $z \in \mathbb{R}$. Therefore, we have the analytic continuations $\zeta_j(z,\theta)$ of $\xi_j(k,\theta)$, $j \in \{1,2\}$, $z \in \mathbb{C}_+$, which have the imaginary parts of fixed signs.

Let $|z| \to \infty$. Then, noticing that $\sin \frac{\xi_j}{2} > 0$ and $\cos \frac{\xi_j}{2} > 0$, since $0 < \xi_j < \pi$, $j \in \{1, 2\}$, we have

$$\sin \frac{\zeta_1}{2} = i \frac{2 \sinh \theta}{\sqrt{2 - e^{2\theta}}} \left\{ 1 + \frac{e^{-2\theta}}{2\,(2 - e^{2\theta})^2} z^{-2} + \mathcal{O}(|z|^{-4}) \right\}, \tag{4.11}$$

$$\sin \frac{\zeta_2}{2} = i \sqrt{2 - e^{2\theta}} z \left\{ 1 - \frac{e^{-2\theta}}{2\,(2 - e^{2\theta})^2} z^{-2} + \mathcal{O}(|z|^{-4}) \right\}, \tag{4.12}$$

$$\cos \frac{\zeta_1}{2} = \frac{e^{-\theta}}{\sqrt{2 - e^{2\theta}}} \left\{ 1 + 2 \frac{\sinh^2 \theta}{(2 - e^{2\theta})^2} z^{-2} + \mathcal{O}(|z|^{-4}) \right\}, \tag{4.13}$$

$$\cos \frac{\zeta_2}{2} = \sqrt{2 - e^{2\theta}} z \left\{ 1 - 2 \frac{\sinh^2 \theta}{(2 - e^{2\theta})^2} z^{-2} + \mathcal{O}(|z|^{-4}) \right\}. \tag{4.14}$$

We put $z = 1 + iN$. Then, from (4.11) and (4.14), we have

$$\sin \frac{\zeta_1}{2} = x_{s,1} + i y_{s,1}, \quad \cos \frac{\zeta_2}{2} = x_{c,2} + i y_{c,2}, \tag{4.15}$$

where

$$x_{s,1} = b(\theta) \left\{ \frac{e^{-2\theta}}{(2 - e^{2\theta})^2} N^{-3} + \mathcal{O}(N^{-5}) \right\}, \tag{4.16}$$

$$y_{s,1} = b(\theta) \left\{ 1 - \frac{e^{-2\theta}}{2\,(2 - e^{2\theta})^2} N^{-2} + \mathcal{O}(N^{-4}) \right\}, \tag{4.17}$$

$$x_{c,2} = a(\theta) \left\{ 1 - 2 \frac{\sinh^2 \theta}{(2 - e^{2\theta})^2} N^{-2} + \mathcal{O}(N^{-4}) \right\}, \tag{4.18}$$

$$y_{c,2} = a(\theta) \left\{ N + 2 \frac{\sinh^2 \theta}{(2 - e^{2\theta})^2} N^{-1} + \mathcal{O}(N^{-3}) \right\}, \tag{4.19}$$

and $a(\theta)$ and $b(\theta)$ is defined by (4.6), respectively. We also put $\zeta_j = \eta_j + i\kappa_j$, $j \in \{1, 2\}$. Then, from (4.15), we have

$$\sin \frac{\eta_1}{2} \cosh \frac{\kappa_1}{2} = x_{s,1}, \ \cos \frac{\eta_1}{2} \sinh \frac{\kappa_1}{2} = y_{s,1}, \tag{4.20}$$

$$\cos \frac{\eta_2}{2} \cosh \frac{\kappa_2}{2} = x_{c,2}, \ -\sin \frac{\eta_2}{2} \sinh \frac{\kappa_2}{2} = y_{c,2}. \tag{4.21}$$

For sufficiently large $N > 0$, from (4.16) and (4.18), we have $x_{s,1} > 0$ and $x_{c,2} > 0$, which implies, from (4.20) and (4.21), that $\sin \frac{\eta_1}{2} > 0$ and $\cos \frac{\eta_2}{2} > 0$, since $\cosh \frac{\kappa_j}{2} > 0$, $j \in \{1, 2\}$. We can now prove (4.7), (4.8), (4.9), and (4.10) in the same way as in [13] by carefully investigating the asymptotic behaviors as $N \to +\infty$. □

Remark 4.2. In the case that $\xi_1, \xi_2 \in (-\pi, 0)$, we have $\sin \frac{\xi_j}{2} < 0$ and $\cos \frac{\xi_j}{2} > 0$, $j \in \{1, 2\}$. Therefore, also in this case, we can prove the same lemma as Lemma 4.1, where we replace the third statement in Lemma 4.1 with the following one:

3′ If $-\pi < \xi_j < 0$, $j \in \{1, 2\}$, there exist $m_3, m_4 \in \mathbb{Z}$ such that, as $N \to +\infty$,

$$\Re\zeta_1(1 + iN, \theta) = 2m_3\pi + \mathcal{O}(N^{-3}), \tag{4.22}$$

$$\Im\zeta_1(1 + iN, \theta) = -2\log(b + \sqrt{b^2 + 1}) + \mathcal{O}(N^{-2}), \tag{4.23}$$

$$\Re\zeta_2(1 + iN, \theta) = (2m_4 + 1)\pi + \mathcal{O}(N^{-1}), \tag{4.24}$$

$$\Im\zeta_2(1 + iN, \theta) = -2\log N - \log(4a^2) + \mathcal{O}(N^{-2}). \tag{4.25}$$

5. Resolvent Estimates

A direct calculation shows that $R_0(z) = (H_0 - z)^{-1}$ is a multiplication operator by the matrix $R_0(z, \xi)$, which is written as

$$R_0(z, \xi) = \frac{1}{z^2 - r(\xi)} \begin{pmatrix} -z & -\alpha(\xi) \\ -\overline{\alpha}(\xi) & -z \end{pmatrix}, \; r(\xi) = |\alpha(\xi)|^2.$$

Let $\hat{r}_0(z) = (\hat{r}_0(z, n))_{n\in\mathbb{Z}^2}$ be the Fourier coefficients of $R_0(z, \xi)$:

$$\hat{r}_0(z, n) = \frac{1}{2\pi} \int_{\mathbb{T}^2} R_0(z, \xi)\mathrm{e}^{-in\xi}\mathrm{d}\xi = \begin{pmatrix} \hat{r}_{0,11}(z, n) & \hat{r}_{0,12}(z, n) \\ \hat{r}_{0,21}(z, n) & \hat{r}_{0,22}(z, n) \end{pmatrix}.$$

Then the resolvent $\hat{R}_0(z) = (\hat{H}_0 - z)^{-1}$ is a convolution operator by $\hat{r}_0(z)$:

$$(\hat{R}_0(z)\hat{f})(n) = \begin{pmatrix} \sum_{k=1,2} \sum_{m\in\mathbb{Z}^2} \hat{r}_{0,1k}(z, n - m)\hat{f}_k(m) \\ \sum_{k=1,2} \sum_{m\in\mathbb{Z}^2} \hat{r}_{0,2k}(z, n - m)\hat{f}_k(m) \end{pmatrix}. \tag{5.1}$$

Let $|z|$ be sufficiently large. Then we have

$$\hat{r}_{0,jj}(z, n) = \sum_{s=0}^{\infty} z^{-2s-1}\hat{r}_{0,s,jj}(n), \quad \hat{r}_{0,ij}(z, n) = \sum_{s=0}^{\infty} z^{-2s-2}\hat{r}_{0,s,ij}(n), \tag{5.2}$$

for $(i, j) = (1, 2)$ or $(2, 1)$, where

$$\hat{r}_{0,s,jj}(n) = \frac{-1}{2\pi} \int_{\mathbb{T}^2} r(\xi)^s \mathrm{e}^{-in\xi}\mathrm{d}\xi, \quad \hat{r}_{0,s,12}(n) = \frac{-1}{2\pi} \int_{\mathbb{T}^2} r(\xi)^s \alpha(\xi)\mathrm{e}^{-in\xi}\mathrm{d}\xi,$$

and $\hat{r}_{0,s,21}(n) = \overline{\hat{r}_{0,s,12}(-n)}$.

We define the following non-negative quantities: for $n = (n_1, n_2) \in \mathbb{Z}^2$,

$$d(n) = d_{11}(n) = d_{22}(n) = \begin{cases} |n_1| + |n_2|, & \text{if } n_1 \cdot n_2 \geq 0, \\ \max\{|n_1|, |n_2|\}, & \text{if } n_1 \cdot n_2 \leq 0, \end{cases} \tag{5.3}$$

$$d_{12}(n) = d_{21}(-n) = \begin{cases} |n_1| + |n_2| - 1, & \text{if } n_1 > 0, n_2 > 0, \\ \max\{|n_1| - 1, |n_2|\}, & \text{if } n_1 > 0, n_2 \leq 0, \\ |n_1| + |n_2|, & \text{if } n_1 \leq 0, n_2 \leq 0, \\ \max\{|n_1|, |n_2| - 1\}, & \text{if } n_1 \leq 0, n_2 > 0. \end{cases} \tag{5.4}$$

The contour lines of $d_{ij}(\cdot)$, $i, j \in \{1, 2\}$, are shown in Fig. 4. The quantity $d(\cdot)$ is used to estimate the support of the Fourier coefficients of $r(\xi)^s$ in

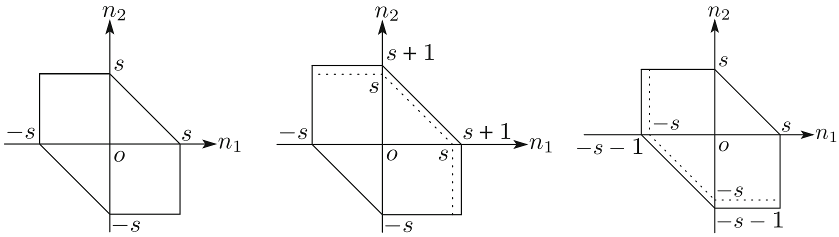

FIGURE 4. The shape of contour lines of $d_{ij}(n) = s$; $(i,j) = (1,1)$ or $(2,2)$, $(1,2)$, and $(2,1)$ from the *left* to the *right*

Lemma 5.7. Also, $d_{ij}(\cdot)$, $(i,j) = (1,2)$ or $(2,1)$, are used for $r(\xi)^s\alpha(\xi)$ or $r(\xi)^s\overline{\alpha}(\xi)$ in Lemmas 5.8 or 5.9, respectively. By making use of those Lemmas, we can derive the resolvent estimates at the end this section.

The following two lemmas are immediate consequences from the definitions (5.3) and (5.4).

Lemma 5.1. *For $n = (n_1, n_2) \in \mathbb{Z}^2$,*

$$d(n) \geq |n_2|,\ d_{12}(n) \geq \begin{cases} |n_2| - 1, & \text{if } n_2 > 0, \\ |n_2|, & \text{if } n_2 \leq 0, \end{cases}$$

$$d_{21}(n) \geq \begin{cases} |n_2|, & \text{if } n_2 \geq 0, \\ |n_2| - 1, & \text{if } n_2 < 0. \end{cases}$$

Lemma 5.2. *For $i, j \in \{1, 2\}$,*

$$d(n) - 1 \leq d_{ij}(n) \leq d(n).$$

We can easily show that $d(an) = |a|d(n)$ for $a \in \mathbb{Z}$; also, that $d(n) = 0$ if and only if $n = (0,0)$. Moreover, we have

Lemma 5.3. *$d(\cdot)$ is a norm on $\mathbb{Z}^2$. In particular, it satisfies the triangle inequality:*

$$d(m - n) \leq d(m - l) + d(l - n). \tag{5.5}$$

Proof. We only have to prove (5.5).

Assume that $m_1 - n_1 \geq 0$ and $m_2 - n_2 \geq 0$, where $m = (m_1, m_2), n = (n_1, n_2) \in \mathbb{Z}^2$. The left figure in Fig. 5 illustrates the possible positions of $l = (l_1, l_2) \in \mathbb{Z}^2$, which are numbered from (1) to (9). In each case, we can prove the triangle inequality (5.5) by a simple calculation.

Next, assume that $m_1 - n_1 \leq 0$ and $m_2 - n_2 \geq 0$. This time, we treat the cases from $(1)'$ to $(9)'$ illustrated in the right figure in Fig. 5, which correspond to the possible positions of l. We can also prove the triangle inequality (5.5) in the same way as above.

The case that $m_1 - n_1 \leq 0$ and $m_2 - n_2 \leq 0$ is reduced to the first one and the remaining one to the second. □

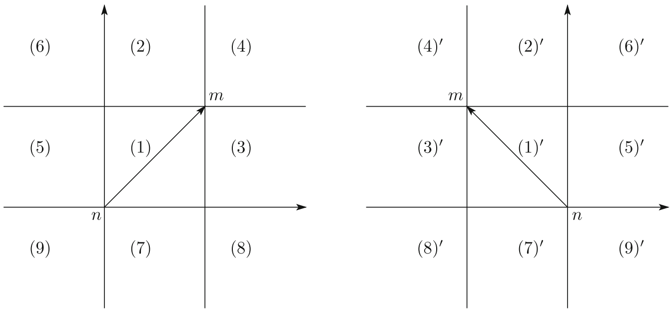

FIGURE 5. The *left* and *right figures* show the possible positions of $l \in \mathbb{Z}^2$

Remark 5.4. One can convince oneself of the above lemma if one bears the shape of the contour lines of $d_{ij}(\cdot)$ in mind and shifts them. One can also have insight into Lemmas 5.5 and 5.6 in the same manner.

We can prove the following two lemmas in the same way as shown above.

Lemma 5.5. *For i, $j \in \{1,2\}$,*

$$d_{ij}(m-n) \le d_{ii}(m-l) + d_{ij}(l-n),$$
$$d_{ij}(m-n) \le d_{ij}(m-l) + d_{jj}(l-n).$$

Lemma 5.6. *For i, $j \in \{1,2\}$,*

$$d_{ii}(m-n) \le d_{ij}(m-l) + d_{ji}(l-n) + 1.$$

Next, we estimate the support of the Fourier coefficients of $r(\xi)^s$.

Lemma 5.7. *Let $s \in \mathbb{Z}$, $s \ge 0$. Then the Fourier coefficients of $r(\xi)^s$ are supported on the set $\{n \in \mathbb{Z}^2; d(n) \le s\}$, that is, $\hat{r}_{0,s,jj}(n) = 0$, $j \in \{1,2\}$, when $d(n) > s$.*

Proof. The claim in the lemma is obvious if $s = 0$.

By the definition of $r(\xi)$, we have

$$r(\xi) = 3 + \mathrm{e}^{i\xi_1} + \mathrm{e}^{-i\xi_1} + \mathrm{e}^{i\xi_2} + \mathrm{e}^{-i\xi_2} + \mathrm{e}^{i(\xi_1-\xi_2)} + \mathrm{e}^{-i(\xi_1-\xi_2)}, \tag{5.6}$$

which means that $r(\xi)$ is a linear combination of $\mathrm{e}^{in_0\xi}$, $d(n_0) \le 1$. Assume that we have proved the claim for $s \le s_0$. Then $r(\xi)^{s_0}$ is a linear combination of $\mathrm{e}^{in\xi}$, $d(n) \le s_0$. Therefore, $r(\xi)^{s_0+1}$ is a linear combination of $\mathrm{e}^{in\xi}r(\xi)$, $d(n) \le s_0$, which means, from (5.6), that it is that of $\mathrm{e}^{im\xi}$, $m = n + n_0$, where $d(n) \le s_0$ and $d(n_0) \le 1$. By using the triangle inequality (5.5), we have

$$d(m) = d(n+n_0) \le d(n) + d(n_0) \le s_0 + 1,$$

which proves the lemma by the induction argument with respect to s. □

Similarly, we can estimate that of $r(\xi)^s\alpha(\xi)$.

Lemma 5.8. *Let $s \in \mathbb{Z}$, $s \ge 0$. Then the Fourier coefficients of $r(\xi)^s\alpha(\xi)$ are supported on the set $\{n \in \mathbb{Z}^2; d_{12}(n) \le s\}$, that is, $\hat{r}_{0,s,12}(n) = 0$, when $d_{12}(n) > s$.*

Proof. We have already proved that $r(\xi)^s$ is a linear combination of $\mathrm{e}^{in\xi}$, $d(n) \leq s$. From (2.3), $\alpha(\xi)$ is a sum of $\mathrm{e}^{in_0\xi}$, where $n_0 \in \{(0,0),(1,0),(0,1)\}$. Therefore, $r(\xi)^s\alpha(\xi)$ is a linear combination of $\mathrm{e}^{in\xi}\alpha(\xi)$, $d(n) \leq s$, which means that it is that of $\mathrm{e}^{im\xi}$, $m = n + n_0$, where $d(n) \leq s$ and $n_0 \in \{(0,0),(1,0),(0,1)\}$. By the definition (5.4), we have $d_{12}(m) \leq s$ (see also the shape of the contour line of $d_{12}(\cdot)$ in Fig. 4), which proves the lemma. □

Also, noticing that $\hat{r}_{0,s,21}(n) = \overline{\hat{r}_{0,s,12}(-n)}$ and $d_{12}(n) = d_{21}(-n)$, we have

Lemma 5.9. *Let $s \in \mathbb{Z}$, $s \geq 0$. Then the Fourier coefficients of $r(\xi)^s\overline{\alpha}(\xi)$ are supported on the set $\{n \in \mathbb{Z}^2; d_{21}(n) \leq s\}$, that is, $\hat{r}_{0,s,21}(n) = 0$, when $d_{21}(n) > s$.*

Now, let us investigate the asymptotic behavior of the resolvents as $|z| \to \infty$. We begin with $\hat{R}_0(z)$. We will use $\hat{P}(n)$ as the projection onto the site $n \in \mathbb{Z}^2$ on $l^2(\mathbb{Z}^2)$ as well.

Lemma 5.10. *If $\hat{f} \in l^2(\mathbb{Z}^2;\mathbb{C}^2)$ is finitely supported, then we have*

$$\hat{R}_0(z)\hat{f} = \begin{pmatrix} \sum_{s=0}^{\infty} z^{-2s-1}\hat{R}_{0,s,11}\hat{f}_1 + \sum_{s=0}^{\infty} z^{-2s-2}\hat{R}_{0,s,12}\hat{f}_2 \\ \sum_{s=0}^{\infty} z^{-2s-2}\hat{R}_{0,s,21}\hat{f}_1 + \sum_{s=0}^{\infty} z^{-2s-1}\hat{R}_{0,s,22}\hat{f}_2 \end{pmatrix}$$

for sufficiently large $|z|$, where $\hat{R}_{0,s,ij} \in \mathbf{B}(l^2(\mathbb{Z}^2))$ with the properties

$$\hat{P}(m)\hat{R}_{0,s,ij}(z)\hat{P}(n) = 0,$$

if $d_{ij}(m-n) > s$, $i,j \in \{1,2\}$.

Proof. From (5.1) and (5.2), we have

$$(\hat{R}_{0,s,ij}\hat{f}_j)(n) = \sum_{m\in\mathbb{Z}^2} \hat{r}_{0,s,ij}(n-m)\hat{f}_j(m).$$

Lemmas 5.7, 5.8, and 5.9 imply the above properties of $\hat{R}_{0,s,ij}$, $i,j \in \{1,2\}$, which proves the lemma. □

Next, we consider the asymptotic behavior of $\hat{R}(z)$ for large $|z|$.

Lemma 5.11. *For sufficiently large $|z|$,*

$$\hat{P}(m)\hat{R}(z)\hat{P}(n) = \begin{pmatrix} \mathcal{O}(\langle z\rangle^{-2d(m-n)-1}) & \mathcal{O}(\langle z\rangle^{-2d_{12}(m-n)-2}) \\ \mathcal{O}(\langle z\rangle^{-2d_{21}(m-n)-2}) & \mathcal{O}(\langle z\rangle^{-2d(m-n)-1}) \end{pmatrix}. \tag{5.7}$$

Proof. Let $p = d(m-n)$. Then, using the resolvent equation repeatedly, we have the following expansion:

$$\hat{R}(z) = \hat{R}_0(z) - \cdots + (-1)^{2p-1} \underbrace{\hat{R}_0(z)\hat{q}\hat{R}_0(z)\cdots\hat{q}\hat{R}_0(z)}_{2p\ \hat{R}_0(z)\text{'s and } 2p-1\ \hat{q}\text{'s}} + \mathcal{O}(\langle z\rangle^{-2p-1}). \tag{5.8}$$

By multiplying $\hat{P}(m)$ and $\hat{P}(n)$ from the left and the right of (5.8), respectively, we have

$$\hat{P}(m)\hat{R}_0(z)\hat{P}(n) = \begin{pmatrix} \sum_{s=0}^{\infty} z^{-2s-1}\hat{P}(m)\hat{R}_{0,s,11}\hat{P}(n) & \sum_{s=0}^{\infty} z^{-2s-2}\hat{P}(m)\hat{R}_{0,s,12}\hat{P}(n) \\ \sum_{s=0}^{\infty} z^{-2s-2}\hat{P}(m)\hat{R}_{0,s,21}\hat{P}(n) & \sum_{s=0}^{\infty} z^{-2s-1}\hat{P}(m)\hat{R}_{0,s,22}\hat{P}(n) \end{pmatrix}$$

as the first term of the right-hand side. Then, from Lemma 5.10, we have

$$\hat{P}(m)\hat{R}_0(z)\hat{P}(n)_{ii} = \sum_{s\geq d(m-n)} z^{-2s-1}\hat{P}(m)\hat{R}_{0,s,ii}\hat{P}(n) = \mathcal{O}(\langle z\rangle^{-2d(m-n)-1}),$$
$$\hat{P}(m)\hat{R}_0(z)\hat{P}(n)_{ij} = \sum_{s\geq d_{ij}(m-n)} z^{-2s-2}\hat{P}(m)\hat{R}_{0,s,ij}\hat{P}(n) = \mathcal{O}(\langle z\rangle^{-2d_{ij}(m-n)-2}),$$

for $(i,j) = (1,2)$ or $(2,1)$.

About the k-th term, $1 \leq k \leq p$, we use inductions. Such a term is written as a sum of

$$\hat{P}(m)\hat{R}_0(z)\hat{P}(l^{(1)})\hat{q}(l^{(1)})\hat{P}(l^{(1)})\hat{R}_0(z)\cdots\hat{P}(l^{(k)})\hat{q}(l^{(k)})\hat{P}(l^{(k)})\hat{R}_0(z)\hat{P}(n), \tag{5.9}$$

since $\hat{q} = \sum_{\sharp\{l\in\mathbb{Z}^2\}<\infty}\hat{P}(l)\hat{q}(l)\hat{P}(l)$.

Assume that (5.9) is estimated as (5.7) for $k = k_0 - 1$. Then, by the hypothesis of the induction, we have

$$\begin{aligned}&\hat{P}(m)\hat{R}_0(z)\hat{P}(l^{(1)})\hat{q}(l^{(1)})\hat{P}(l^{(1)})\hat{R}_0(z)\cdots\hat{q}(l^{(k_0-1)})\hat{P}(l^{(k_0-1)})\hat{R}_0(z)\hat{P}(l^{(k_0)})\\ &\quad\cdot\hat{q}(l^{(k_0)})\cdot\hat{P}(l^{(k_0)})\hat{R}_0(z)\hat{P}(n)\\ &= \begin{pmatrix} \mathcal{O}(\langle z\rangle^{-2d(m-l^{(k_0)})-1}) & \mathcal{O}(\langle z\rangle^{-2d_{12}(m-l^{(k_0)})-2}) \\ \mathcal{O}(\langle z\rangle^{-2d_{21}(m-l^{(k_0)})-2}) & \mathcal{O}(\langle z\rangle^{-2d(m-l^{(k_0)})-1}) \end{pmatrix}\cdot\begin{pmatrix} \hat{q}_1(l^{(k_0)}) & 0 \\ 0 & \hat{q}_2(l^{(k_0)}) \end{pmatrix}\\ &\quad\cdot\begin{pmatrix} \mathcal{O}(\langle z\rangle^{-2d(l^{(k_0)}-n)-1}) & \mathcal{O}(\langle z\rangle^{-2d_{12}(l^{(k_0)}-n)-2}) \\ \mathcal{O}(\langle z\rangle^{-2d_{21}(l^{(k_0)}-n)-2}) & \mathcal{O}(\langle z\rangle^{-2d(l^{(k_0)}-n)-1}) \end{pmatrix},\end{aligned}$$

which reads, from Lemmas 5.3, 5.5, and 5.6, that the upper left element is

$$\mathcal{O}(\langle z\rangle^{-2d(m-l^{(k_0)})-2d(l^{(k_0)}-n)-2}) + \mathcal{O}(\langle z\rangle^{-2d_{12}(m-l^{(k_0)})-2d_{21}(l^{(k_0)}-n)-4}) = \mathcal{O}(\langle z\rangle^{-2d(m-n)-2}),$$

and that the upper right one is

$$\mathcal{O}(\langle z\rangle^{-2d(m-l^{(k_0)})-2d_{12}(l^{(k_0)}-n)-3}) + \mathcal{O}(\langle z\rangle^{-2d_{12}(m-l^{(k_0)})-2d(l^{(k_0)}-n)-3}) = \mathcal{O}(\langle z\rangle^{-2d_{12}(m-n)-3}).$$

We also have the lower left and right elements as desired, which proves the lemma. □

Remark 5.12. A more detailed argument in the induction above gives us the estimate (5.7) for the non-diagonal $\hat{q}(n)$'s.

6. Proof of the Main Theorem

We will essentially follow the idea of [13]. Assume that $\lambda > 0$; for $\lambda < 0$, we can argue in the same way. Put

$$B(k,\theta,\theta') = 4(2\pi)^2 \big(J(k,\theta)J(k,\theta')\big)^{-1} A(k,\theta,\theta'),$$

where $A(k,\theta,\theta')$ is the integral kernel of the scattering amplitude defined in Sect. 3.3 and $\lambda = \sqrt{8k^2+1}$ in Sect. 4. Then, we have $B(k,\theta,\theta') = B_0(k,\theta,\theta') - B_1(k,\theta,\theta')$, where

$$\begin{aligned} B_0(k,\theta,\theta') &= 4(2\pi)^2 \big(J(k,\theta)J(k,\theta')\big)^{-1} \hat{\mathcal{F}}_0(\lambda)\hat{q}\hat{\mathcal{F}}_0(\lambda)^* \\ &= \begin{pmatrix} B_{0,11}(k,\theta,\theta') & B_{0,12}(k,\theta,\theta') \\ B_{0,21}(k,\theta,\theta') & B_{0,22}(k,\theta,\theta') \end{pmatrix}, \end{aligned}$$

$$B_{0,11}(k,\theta,\theta') = \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in(\xi(k,\theta)-\xi(k,\theta'))} \left(\hat{q}_1(n) + \frac{\alpha(\xi(k,\theta))}{\sqrt{8k^2+1}} \frac{\overline{\alpha}(\xi(k,\theta'))}{\sqrt{8k^2+1}} \hat{q}_2(n) \right),$$

$$B_{0,22}(k,\theta,\theta') = \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in(\xi(k,\theta)-\xi(k,\theta'))} \left(\frac{\overline{\alpha}(\xi(k,\theta))}{\sqrt{8k^2+1}} \frac{\alpha(\xi(k,\theta'))}{\sqrt{8k^2+1}} \hat{q}_1(n) + \hat{q}_2(n) \right)$$

($B_{0,12}$ and $B_{0,21}$ are omitted), and

$$\begin{aligned} B_1(k,\theta,\theta') &= 4(2\pi)^2 \big(J(k,\theta)J(k,\theta')\big)^{-1} \hat{\mathcal{F}}_0(\lambda)\hat{q}\hat{R}(\lambda+i0)\hat{q}\hat{\mathcal{F}}_0(\lambda)^* \\ &= \begin{pmatrix} B_{1,11}(k,\theta,\theta') & B_{1,12}(k,\theta,\theta') \\ B_{1,21}(k,\theta,\theta') & B_{1,22}(k,\theta,\theta') \end{pmatrix}, \\ B_{1,11}(k,\theta,\theta') &= \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(k,\theta)} \hat{q}_1(n) \big(\hat{R}(\lambda+i0)\hat{q}\hat{\varphi}^{(0)}(k,\theta')\big)_{11}(n) \\ &\quad + \frac{\alpha(\xi(k,\theta))}{\sqrt{8k^2+1}} \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(k,\theta)} \hat{q}_2(n) \big(\hat{R}(\lambda+i0)\hat{q}\hat{\varphi}^{(0)}(k,\theta')\big)_{21}(n), \\ B_{1,22}(k,\theta,\theta') &= \frac{\overline{\alpha}(\xi(k,\theta))}{\sqrt{8k^2+1}} \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(k,\theta)} \hat{q}_1(n) \big(\hat{R}(\lambda+i0)\hat{q}\hat{\varphi}^{(0)}(k,\theta')\big)_{12}(n) \\ &\quad + \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{in\xi(k,\theta)} \hat{q}_2(n) \big(\hat{R}(\lambda+i0)\hat{q}\hat{\varphi}^{(0)}(k,\theta')\big)_{22}(n) \end{aligned}$$

($B_{1,12}$ and $B_{1,21}$ are omitted). Here we put

$$\begin{aligned} \hat{\varphi}^{(0)}(k,\theta') &= 2(2\pi)J(k,\theta')^{-1}\hat{\psi}^{(0)}(k,\theta') = (\hat{\varphi}^{(0)}(n;k,\theta'))_{n\in\mathbb{Z}^2}, \\ \hat{\varphi}^{(0)}(n;k,\theta') &= 2(2\pi)J(k,\theta')^{-1}\hat{\psi}^{(0)}(n;k,\theta') = \mathrm{e}^{-in\xi(k,\theta')}\hat{\eta}^{(0)}(k,\theta'), \ n\in\mathbb{Z}^2, \end{aligned}$$

where $\hat{\psi}^{(0)}(k,\theta')$ and $\hat{\psi}^{(0)}(n;k,\theta')$ are defined by (3.9) and (3.10), respectively, and

$$\hat{\eta}^{(0)}(k,\theta') = \begin{pmatrix} 1 & \dfrac{\alpha(\xi(k,\theta'))}{\sqrt{8k^2+1}} \\ \dfrac{\overline{\alpha}(\xi(k,\theta'))}{\sqrt{8k^2+1}} & 1 \end{pmatrix}. \tag{6.1}$$

Let $\zeta_\pm(z,\theta) = (\zeta_{\pm,1}(z,\theta), \zeta_{\pm,2}(z,\theta))$ be the analytic continuations of $\xi(k,\theta) = (\xi_1(k,\theta), \xi_2(k,\theta))$ in Lemma 4.1. Then $B_0(k,\theta,\theta')$ and $B_1(k,\theta,\theta')$ have the analytic continuations $B_0(z,\theta,\theta')$, and $B_1(z,\theta,\theta')$ for $z \in \mathbb{C}_+$, respectively, which are defined with k replaced by z, $\xi(k,\theta)$ by $\zeta_+(z,\theta)$, and $\xi(k,\theta')$ by $\zeta_-(z,\theta')$.

Let us take $-\pi < \xi_j(k,\theta) < 0$ and $0 < \xi_j(k,\theta') < \pi$ for $j \in \{1,2\}$. From Lemma 4.1, we have

$$\mathrm{e}^{in\zeta_+(z,\theta)} \sim N^{2n_2} b_1(\theta)^{n_1} a_1(\theta)^{n_2}, \quad \mathrm{e}^{-in\zeta_-(z,\theta')} \sim N^{2n_2} b_1(\theta')^{n_1} a_1(\theta')^{n_2} \tag{6.2}$$

as $N \to \infty$, where $z = 1 + iN$ and

$$a_1(\theta) = (2a(\theta))^2 = 4(2 - e^{2\theta}), \quad b_1(\theta) = (b(\theta) + \sqrt{b(\theta)^2+1})^2 = \frac{e^{2\theta}}{2 - e^{2\theta}}. \tag{6.3}$$

We put

$$\alpha_{1,+}(z,\theta) = \frac{\alpha(\zeta_+(z,\theta))}{\sqrt{8z^2+1}}, \quad \alpha_{2,+}(z,\theta) = \frac{\overline{\alpha}(\zeta_+(z,\theta))}{\sqrt{8z^2+1}},$$

$$\alpha_{1,-}(z,\theta') = \frac{\alpha(\zeta_-(z,\theta'))}{\sqrt{8z^2+1}}, \quad \alpha_{2,-}(z,\theta') = \frac{\overline{\alpha}(\zeta_-(z,\theta'))}{\sqrt{8z^2+1}}.$$

Then, we have

$$\alpha_{1,+}(z,\theta) \sim a_2(\theta)N + \mathcal{O}(1), \tag{6.4}$$
$$\alpha_{2,+}(z,\theta) \sim b_2(\theta)N^{-1} + \mathcal{O}(N^{-2}), \tag{6.5}$$
$$\alpha_{1,-}(z,\theta') \sim b_2(\theta')N^{-1} + \mathcal{O}(N^{-2}), \tag{6.6}$$
$$\alpha_{2,-}(z,\theta') \sim a_2(\theta')N + \mathcal{O}(1), \tag{6.7}$$

where

$$a_2(\theta) = -\frac{a_1(\theta)}{2\sqrt{2}} i = -\sqrt{2}(2 - e^{2\theta}) i, \quad b_2(\theta) = -\frac{1 + b_1(\theta)^{-1}}{2\sqrt{2}} i = -\frac{e^{-2\theta}}{\sqrt{2}} i.$$

By our assumption (A), the potential $\hat{q}$ is finitely supported; more precisely, for some $M \geq 0$, $M \in \mathbb{Z}$,

$$|n|_{l^1} = |n_1| + |n_2| > M \Longrightarrow \hat{q}(n) = 0.$$

We put

$$n^{(M)} = (0, M) \in \mathbb{Z}^2.$$

Then, by using (6.2), (6.5), and (6.6), we have the following asymptotic expansion:

$$B_{0,22}(z,\theta,\theta') \sim N^{4M}(a_1(\theta)a_1(\theta'))^{4M} \hat{q}_2(n^{(M)}). \tag{6.8}$$

Let us investigate the asymptotic behavior of $B_{1,22}(z,\theta,\theta')$. By introducing Dirac's notation, we have

$$\hat{P}(m) = |\hat{p}_m\rangle\langle\hat{p}_m|,\ (\hat{p}_m)(n) = \begin{pmatrix}\delta_{mn}\\ \delta_{mn}\end{pmatrix} \in l^2(\mathbb{Z}^2;\mathbb{C}^2).$$

From (6.6) and (6.7), we note that

$$\hat{\eta}^{(0)}(z,\theta') = \begin{pmatrix}1 & \alpha_{1,-}(z,\theta')\\ \alpha_{2,-}(z,\theta') & 1\end{pmatrix} \sim \begin{pmatrix}1 & \mathcal{O}(N^{-1})\\ \mathcal{O}(N) & 1\end{pmatrix}. \tag{6.9}$$

From Lemma 5.11, using (6.9), we have

$$\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta') = \begin{pmatrix}\mathcal{O}_{11} & \mathcal{O}_{12}\\ \mathcal{O}_{21} & \mathcal{O}_{22}\end{pmatrix}, \tag{6.10}$$

where

$$\begin{aligned}
\mathcal{O}_{11} &= \mathcal{O}(\langle z\rangle^{-2d(n-m)-1})\hat{q}_1(m) + \mathcal{O}(\langle z\rangle^{-2d_{12}(n-m)-2})\hat{q}_2(m)\alpha_{2,-}(z,\theta'),\\
\mathcal{O}_{12} &= \mathcal{O}(\langle z\rangle^{-2d(n-m)-1})\hat{q}_1(m)\alpha_{1,-}(z,\theta') + \mathcal{O}(\langle z\rangle^{-2d_{12}(n-m)-2})\hat{q}_2(m),\\
\mathcal{O}_{21} &= \mathcal{O}(\langle z\rangle^{-2d_{21}(n-m)-2})\hat{q}_1(m) + \mathcal{O}(\langle z\rangle^{-2d(n-m)-1})\hat{q}_2(m)\alpha_{2,-}(z,\theta'),\\
\mathcal{O}_{22} &= \mathcal{O}(\langle z\rangle^{-2d_{21}(n-m)-2})\hat{q}_1(m)\alpha_{1,-}(z,\theta') + \mathcal{O}(\langle z\rangle^{-2d(n-m)-1})\hat{q}_2(m),
\end{aligned}$$

which reads, using (6.6) and (6.7), that

$$\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta') \sim \begin{pmatrix}\mathcal{O}(N^{-1}) & \mathcal{O}(N^{-2})\\ \mathcal{O}(1) & \mathcal{O}(N^{-1})\end{pmatrix}, \tag{6.11}$$

since $d_{ij}(n) \geq 0$. Then we have

$$\begin{aligned}
&B_{1,22}(z,\theta,\theta')\\
&\quad= \alpha_{2,+}(z,\theta)\sum_{n_2,m_2\leq M} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_1(n)\\
&\qquad\cdot\left(\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\right)_{12}\\
&\qquad+\sum_{n_2,m_2\leq M} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_2(n)\left(\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\right)_{22}\\
&\quad\sim \mathcal{O}(N^{4M-1}).
\end{aligned}$$

Therefore, noticing that $a_1(\cdot)$ in (6.8) is defined as (6.3), we can compute $\hat{q}_2(n^{(M)})$ from the asymptotic expansion of $B_{22}(z,\theta,\theta')$.

We turn to $B_{11}(z,\theta,\theta')$. By using (6.2), (6.4), (6.7), and (6.11), we have the following asymptotic expansion:

$$\begin{aligned}
B_{0,11}(z,\theta,\theta') \sim\ &\mathrm{e}^{in^{(M)}(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\alpha_{1,+}(z,\theta)\alpha_{2,-}(z,\theta')\hat{q}_2(n^{(M)})\\
&+N^{4M}(a_1(\theta)a_1(\theta'))^M\hat{q}_1(n^{(M)}) + \mathcal{O}(N^{4M-2}).
\end{aligned} \tag{6.12}$$

Note that the asymptotic behavior of the first term of the right-hand side is known, since $\hat{q}_2(n^{(M)})$ has already been computed.

Let us investigate the asymptotic behavior of $B_{1,11}(z,\theta,\theta')$. We put

$$\hat{q}_{(\le r,\le s)} = \sum_{n_2\le r} \hat{P}(n)E_1\hat{q}_1(n)\hat{P}(n) + \sum_{n_2\le s} \hat{P}(n)E_2\hat{q}_2(n)\hat{P}(n),$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \hat{q}_{(>r,>s)} = \hat{q} - \hat{q}_{(\le r,\le s)},$$

$$\hat{H}_{(>r,>s)} = \hat{H}_0 + \hat{q}_{(>r,>s)},\ \hat{R}_{(>r,>s)}(z) = (\hat{H}_{(>r,>s)} - z)^{-1}.$$

If $r = s$, we write $\hat{q}_{\le r} = \hat{q}_{(\le r,\le r)}$, $\hat{q}_{>r} = \hat{q}_{(>r,>r)}$, $\hat{H}_{>r} = \hat{H}_{(>r,>r)}$, and $\hat{R}_{>r}(z) = \hat{R}_{(>r,>r)}(z)$ for brevity.

From (6.11), we have

$$\big(\langle \hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11} \sim \mathcal{O}(N^{-1}),$$

which implies, using (6.2), the following asymptotic expansion of the fist term of $B_{1,11}(z,\theta,\theta')$:

$$\sum_{n_2,m_2\le M} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_1(n)\big(\langle \hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}$$
$$\sim \mathcal{O}(N^{4M-1}).$$

We put the second term of $B_{1,11}(z,\theta,\theta')$ as

$$I = \alpha_{1,+}(z,\theta) \sum_{n,m\in\mathbb{Z}^2} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_2(n)$$
$$\cdot \big(\langle \hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{21},$$

and split it into four parts: $I = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \sum_{n=n^{(M)}} \sum_{m=n^{(M)}},\quad I_2 = \sum_{n=n^{(M)}} \sum_{m_2\le M-1},$$
$$I_3 = \sum_{n_2\le M-1} \sum_{m=n^{(M)}},\quad I_4 = \sum_{n_2\le M-1} \sum_{m_2\le M-1}.$$

By using (6.2), (6.4), and (6.11), we have $I_j \sim \mathcal{O}(N^{4M-1})$ for $j = 2,3$; also, $I_4 \sim \mathcal{O}(N^{4M-3})$.

About I_1, using the resolvent equation

$$\hat{R}(z) = \hat{R}_{(>M,>M-1)}(z) - \hat{R}_{(>M,>M-1)}(z)\hat{q}_{(\le M,\le M-1)}\hat{R}(z), \tag{6.13}$$

we split it into four parts: $I_1 = I_{1,1} + I_{1,2,1} - I_{1,2,2,1} - I_{1,2,2,2}$, where

$$I_{1,1} = \alpha_{1,+}(z,\theta)\mathrm{e}^{in^{(M)}(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\hat{q}_2(n^{(M)})$$
$$\cdot\big(\langle \hat{P}(n^{(M)})|\hat{R}(z)|\hat{P}(n^{(M)})\rangle E_2\hat{q}_1(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21},$$

$$I_{1,2,1} = \alpha_{1,+}(z,\theta)\mathrm{e}^{in^{(M)}(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\hat{q}_2(n^{(M)})$$
$$\cdot\big(\langle \hat{P}(n^{(M)})|\hat{R}_{(>M,>M-1)}(z)|\hat{P}(n^{(M)}))\rangle E_2\hat{q}_2(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21},$$

$$\begin{aligned} I_{1,2,2,1} &= \alpha_{1,+}(z,\theta)\mathrm{e}^{in^{(M)}(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\hat{q}_2(n^{(M)}) \\ &\quad \cdot \big(\langle \hat{P}(n^{(M)})|\hat{R}_{(>M,>M-1)}(z)\hat{q}_{\le M-1}\hat{R}(z)|\hat{P}(n^{(M)})\rangle \\ &\qquad \cdot E_2\hat{q}_2(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21}, \\ I_{1,2,2,2} &= \alpha_{1,+}(z,\theta)\mathrm{e}^{in^{(M)}(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\hat{q}(n^{(M)}) \\ &\quad \cdot \big(\langle \hat{P}(n^{(M)})|\hat{R}_{(>M,>M-1)}(z)E_1\hat{q}_1(n^{(M)})\hat{P}(n^{(M)})\hat{R}(z)|\hat{P}(n^{(M)})\rangle \\ &\qquad \cdot E_2\hat{q}_2(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21}. \end{aligned}$$

We know $\hat{R}_{(>M,>M-1)}(z)$, since we have already computed $\hat{q}_2(n^{(M)})$, which implies that $I_{1,2,1}$ is a known term.

By using (6.10) and (6.9), we have

$$\big(\langle \hat{P}(n^{(M)})|\hat{R}(z)|\hat{P}(n^{(M)})\rangle E_1\hat{q}_1(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21} \sim \mathcal{O}(N^{-2}),$$

which implies, using (6.2) and (6.4), that $I_{1,1} \sim \mathcal{O}(N^{4M-1})$.

We can write $I_{1,2,2,1}$ as a sum of the following terms:

$$\begin{aligned} &\alpha_{1,+}(z,\theta)\mathrm{e}^{in^{(M)}(\zeta_+(z,\theta)-\zeta_-(z,\theta'))} \\ &\quad \cdot \big(\langle \hat{P}(n^{(M)})|\hat{R}_{(>M,>M-1)}(z)|\hat{P}(k)\rangle\hat{q}(k)\langle\hat{P}(k)|\hat{R}(z)|\hat{P}(n^{(M)})\rangle \\ &\quad \cdot E_2\hat{q}_2(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21}, \end{aligned}$$

where $k=(k_1,k_2)\in\mathbb{Z}^2, k_2\le M-1$. From Lemma 5.1, we have

$$d(n^{(M)}-k) = d(k-n^{(M)}) \ge 1,\ d_{21}(n^{(M)}-k) = d_{12}(k-n^{(M)}) \ge 1, \tag{6.14}$$

since $(n^{(M)}-k)_2 > 0$. Then, using (5.7), (2.8), (6.10), (6.7), and (6.14), we have

$$\begin{aligned} &\big(\langle \hat{P}(n^{(M)})|\hat{R}_{(>M,>M-1)}(z)|\hat{P}(k)\rangle\hat{q}(k)\langle\hat{P}(k)|\hat{R}(z)|\hat{P}(n^{(M)})\rangle \\ &\qquad \cdot E_2\hat{q}_2(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21} \\ &\quad = \mathcal{O}(\langle z\rangle^{-2d_{21}(n^{(M)}-k)-2})\hat{q}_1(k)\mathcal{O}(\langle z\rangle^{-2d_{12}(k-n^{(M)})-2})\hat{q}_2(n^{(M)})\alpha_{2,-}(z,\theta') \\ &\qquad + \mathcal{O}(\langle z\rangle^{-2d(n^{(M)}-k)-1})\hat{q}_2(k)\mathcal{O}(\langle z\rangle^{-2d(k-n^{(M)})-1})\hat{q}_2(n^{(M)})\alpha_{2,-}(z,\theta') \\ &\qquad \sim \mathcal{O}(N^{-5}), \end{aligned}$$

which implies, using (6.2) and (6.4) that $I_{1,2,2,1} \sim \mathcal{O}(N^{4M-4})$.

In the same way as shown above, we have

$$\begin{aligned} &\big(\langle \hat{P}(n^{(M)})|\hat{R}_{(>M,>M-1)}(z)|\hat{P}(n^{(M)})\rangle E_1\hat{q}_1(n^{(M)}) \\ &\qquad \cdot \langle \hat{P}(n^{(M)})|\hat{R}(z)|\hat{P}(n^{(M)})\rangle E_2\hat{q}_2(n^{(M)})\hat{\eta}^{(0)}(z,\theta')\big)_{21} \\ &\quad = \mathcal{O}(\langle z\rangle^{-2d_{21}((0,0))-2d_{12}((0,0))-4})\alpha_{2,-}(z,\theta') \sim \mathcal{O}(N^{-3}), \end{aligned}$$

which implies $I_{1,2,2,2} \sim \mathcal{O}(N^{-4M-2})$.

Therefore, by taking the known terms into consideration, we can compute $\hat{q}_1(n^{(M)})$ from the asymptotic expansion of $B_{11}(z,\theta,\theta')$.

Suppose that we have computed $\hat{q}(n)$ for $n_2 \geq p+1$. Then, using (6.2), (6.4), and (6.6), we have

$$B_{0,22}(z,\theta,\theta') - \sum_{n_2 \geq p+1} \mathrm{e}^{in(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\big(\alpha_{2,+}(z,\theta)\alpha_{1,-}(z,\theta')\hat{q}_1(n) + \hat{q}_2(n)\big)$$
$$\sim N^{4p}(a_1(\theta)a_1(\theta'))^p \sum_{n_2=p} (b_1(\theta)b_1(\theta'))^{n_1}\hat{q}_2(n). \tag{6.15}$$

Let us investigate the asymptotic behavior of $B_{1,22}(z,\theta,\theta') = \mathit{II} + \mathit{III}$, where

$$\mathit{II} = \alpha_{2,+}(z,\theta) \sum_{n,m\in\mathbb{Z}^2} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_1(n)$$
$$\cdot\big(\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{12},$$
$$\mathit{III} = \sum_{n,m\in\mathbb{Z}^2} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_2(n)\big(\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{22}.$$

Lemma 6.1. *We have $\mathit{II} \sim \mathcal{O}(N^{4p-1})$ up to a known term, which is written as*

$$\alpha_{2,+}(z,\theta) \sum_{n_2,m_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_1(n)$$
$$\cdot\,\big(\langle\hat{P}(n)|\hat{R}_{>p}(z)|\hat{P}(m)\rangle\hat{q}_{>p}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{12}, \tag{6.16}$$

and $\mathit{III} \sim \mathcal{O}(N^{4p-1})$ up to a known term, which is written as

$$\sum_{n_2,m_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_2(n)\big(\langle\hat{P}(n)|\hat{R}_{>p}(z)|\hat{P}(m)\rangle\hat{q}_{>p}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{22}. \tag{6.17}$$

Proof. We split II into four parts: $\mathit{II} = \mathit{II}_1 + \mathit{II}_2 + \mathit{II}_3 + \mathit{II}_4$, where

$$\mathit{II}_1 = \sum_{n_2\geq p+1}\sum_{m_2\geq p+1}, \qquad \mathit{II}_2 = \sum_{n_2\geq p+1}\sum_{m_2\leq p},$$
$$\mathit{II}_3 = \sum_{n_2\leq p}\sum_{m_2\geq p+1}, \qquad \mathit{II}_4 = \sum_{n_2\leq p}\sum_{m_2\leq p}.$$

Moreover, using the resolvent equation

$$\hat{R}(z) = \hat{R}_{>p}(z) - \hat{R}_{>p}(z)\hat{q}_{\leq p}\hat{R}(z), \tag{6.18}$$

we split II_1 into two parts: $\mathit{II}_1 = \mathit{II}_{1,1} - \mathit{II}_{1,2}$. Then, $\mathit{II}_{1,1}$ is exactly (6.16), which is a known term, since we have already computed $\hat{q}(n)$ for $n_2 > p$ by the assumption. We can prove that the other terms behave at most $\mathcal{O}(N^{4p-1})$ in a similar way to [13].

We deal with III in the same way as II, that is,

$$\mathit{III} = \mathit{III}_1 + \mathit{III}_2 + \mathit{III}_3 + \mathit{III}_4,$$

and using (6.18),

$$\mathit{III}_1 = \mathit{III}_{1,1} - \mathit{III}_{1,2},$$

where $III_{1,1}$ is exactly (6.17), and the other terms can be proved to be at most $\mathcal{O}(N^{4p-1})$ as shown above. □

Therefore, we can compute $\hat{q}_2(n)$ for $n_2 = p$ from (6.15), since the set $\{b_1(\theta)b_1(\theta') \in \mathbb{R}; \theta, \theta' \in (0, \frac{1}{2}\log 2)\}$ contains non empty open intervals, where $b_1(\cdot)$ is defined by (6.3).

Finally, we compute $\hat{q}_1(n)$ for $n_2 = p$. By using (6.2), (6.4), and (6.7), we have

$$\begin{aligned} B_{0,11}(z,\theta,\theta') &- \sum_{n_2 \geq p+1} \mathrm{e}^{in(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\big(\hat{q}_1(n) + \alpha_{1,+}(z,\theta)\alpha_{2,-}(z,\theta')\hat{q}_2(n)\big) \\ &\sim N^{4p}(a_1(\theta)a_1(\theta'))^p \sum_{n_2=p} (b_1(\theta)b_1(\theta'))^{n_1}\hat{q}_1(n) \\ &+ \sum_{n_2=p} \mathrm{e}^{in(\zeta_+(z,\theta)-\zeta_-(z,\theta'))}\alpha_{1,+}(z,\theta)\alpha_{2,-}(z,\theta')\hat{q}_2(n) + \mathcal{O}(N^{4p-2}). \end{aligned} \tag{6.19}$$

Note that the second term of the right-hand side in (6.19) is known, since we have already computed $\hat{q}_2(n)$ for $n_2 = p$.

Let us investigate the asymptotic behavior of $B_{1,11}(z,\theta,\theta') = IV + V$, where

$$IV = \sum_{n,m\in\mathbb{Z}^2} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_1(n)\big(\langle\hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle\hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11},$$

and $V = I$. By using the next lemma, we can compute $\hat{q}_1(n)$ for $n_2 = p$ from (6.19) as before.

Lemma 6.2. *We have $IV \sim \mathcal{O}(N^{4p-1})$ up to a known term, which is written as*

$$\begin{aligned} &\sum_{n_2,m_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_1(n) \\ &\quad\cdot \big(\langle\hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)|\hat{P}(m)\rangle\hat{q}_{(>p,>p-1)}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \end{aligned} \tag{6.20}$$

and $V \sim \mathcal{O}(N^{4p-1})$ up to a known term, which is written as

$$\begin{aligned} &\alpha_{1,+}(z,\theta) \sum_{n_2\geq p, m_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)-im\zeta_-(z,\theta')}\hat{q}_2(n) \\ &\quad\cdot \big(\langle\hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)|\hat{P}(m)\rangle\hat{q}_{(>p,>p-1)}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{21}. \end{aligned} \tag{6.21}$$

Proof. We split IV into four parts: $IV = IV_1 + IV_2 + IV_3 + IV_4$, where

$$IV_1 = \sum_{n_2\geq p+1}\sum_{m_2\geq p+1}, IV_2 = \sum_{n_2\geq p+1}\sum_{m_2\leq p},$$
$$IV_3 = \sum_{n_2\leq p}\sum_{m_2\geq p+1}, IV_4 = \sum_{n_2\leq p}\sum_{m_2\leq p}.$$

By using the resolvent equation

$$\hat{R}(z) = \hat{R}_{(>p,>p-1)}(z) - \hat{R}_{(>p,>p-1)}(z)\hat{q}_{(\leq p,\leq p-1)}\hat{R}(z), \tag{6.22}$$

we split IV_2 into five parts:$IV_2 = IV_{2,1}+IV_{2,2,1}+IV_{2,2,2,1}-IV_{2,2,2,2,1}-IV_{2,2,2,2,2}$, where

$$\begin{aligned}
IV_{2,1} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2\leq p-1} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \big(\langle \hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \\
IV_{2,2,1} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2=p} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \big(\langle \hat{P}(n)|\hat{R}(z)|\hat{P}(m)\rangle E_1\hat{q}_1(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \\
IV_{2,2,2,1} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2=p} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \big(\langle \hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)|\hat{P}(m)\rangle E_2\hat{q}_2(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \\
IV_{2,2,2,2,1} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2=p} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \big(\langle \hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)\hat{q}_{\leq p-1}\hat{R}(z)|\hat{P}(m)\rangle E_2\hat{q}_2(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \\
IV_{2,2,2,2,2} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2=p} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \sum_{k_2=p} \big(\langle \hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)E_1\hat{q}_1(k)\hat{P}(k)\hat{R}(z)|\hat{P}(m)\rangle \\
&\qquad\qquad E_2\hat{q}_2(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}.
\end{aligned}$$

Moreover, by using (6.22), we split IV_1 into three parts: $IV_1 = IV_{1,1}-IV_{1,2,1}-IV_{1,2,2}$, where

$$\begin{aligned}
IV_{1,1} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2\geq p+1} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \big(\langle \hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \\
IV_{1,2,1} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2\geq p+1} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \big(\langle \hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)\hat{q}_{\leq p-1}\hat{R}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}, \\
IV_{1,2,2} &= \sum_{n_2\geq p+1} \mathrm{e}^{in\zeta_+(z,\theta)}\hat{q}_1(n) \sum_{m_2\geq p+1} \mathrm{e}^{-im\zeta_-(z,\theta')} \\
&\quad\cdot \sum_{k_2=p} \big(\langle \hat{P}(n)|\hat{R}_{(>p,>p-1)}(z)E_1\hat{q}_1(k)\hat{P}(k)\hat{R}(z)|\hat{P}(m)\rangle \hat{q}(m)\hat{\eta}^{(0)}(z,\theta')\big)_{11}.
\end{aligned}$$

Then, $IV_{2,2,2,1}$ and $IV_{1,1}$ are known terms, since we have already computed $\hat{q}(n)$ for $n_2 \geq p+1$ and $\hat{q}_2(n)$ for $n_2 = p$, and the sum of them is exactly (6.20). We can prove that the other terms behave at most $\mathcal{O}(N^{4p-1})$ in a similar way to the case of I.

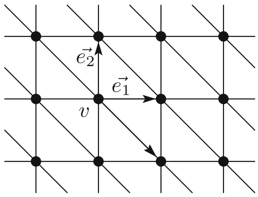

FIGURE 6. The triangle lattice

We deal with V in the same way as IV, that is,

$$V = V_1 + V_2 + V_3 + V_4,$$
$$V_1 = \sum_{n_2 \geq p} \sum_{m_2 \geq p+1}, V_2 = \sum_{n_2 \geq p} \sum_{m_2 \leq p}, V_3 = \sum_{n_2 \leq p-1} \sum_{m_2 \geq p+1}, V_4 = \sum_{n_2 \leq p-1} \sum_{m_2 \leq p},$$

and by using (6.22)

$$V_2 = V_{2,1} + V_{2,2,1} + V_{2,2,2,1} - V_{2,2,2,2,1} - V_{2,2,2,2,2},$$
$$V_1 = V_{1,1} - V_{1,2,1} - V_{1,2,2},$$

where $V_{2,2,2,1}$ and $V_{1,1}$ are known terms, the sum of which is exactly (6.21); the other terms can be proved to be at most $\mathcal{O}(N^{4p-1})$. □

Thus, we can compute all the $q(n)$'s inductively.

7. The Triangle Lattice

Let G be the triangle lattice. Then, in the same way as in Sect. 2, we have $\hat{H}_0 \simeq 6(\Delta_d + 1)$ on $l^2(\mathbb{Z}^2) \simeq l^2(G)$ as follows:

$$(\hat{H}_0 \hat{f})(n) = \hat{f}(n_1+1, n_2) + \hat{f}(n_1-1, n_2) + \hat{f}(n_1, n_2+1) + \hat{f}(n_1, n_2-1) + \hat{f}(n_1-1, n_2+1) + \hat{f}(n_1+1, n_2-1)$$

for $\hat{f} = (\hat{f}(n))_{n \in \mathbb{Z}^2}$. See Fig. 6.

In this case, $\mathcal{F}$ is just the Fourier series on $\mathbb{Z}^2$, which implies that $H_0 = \mathcal{F}\hat{H}_0\mathcal{F}^*$ is a multiplication operator by

$$p(\xi) = 2(\cos\xi_1 + \cos\xi_2 + \cos(\xi_1 - \xi_2)) \tag{7.1}$$

on $L^2(\mathbb{T}^2)$. Here, we note the similarity between (7.1) and the square of (2.5). Therefore, Mourre estimate is obtained with the conjugate operator $A = \nabla p \cdot \nabla + \nabla \cdot \nabla p$ on any closed interval $I \subset (-3, 6) \setminus \{0\}$. Note also that we no longer need a cut-off function which appears in the hexagonal lattice's case. By using limiting absorption principle, we have a spectral representation for $\hat{H} = \hat{H}_0 + \hat{q}$ and also a representation for the kernel of the scattering amplitude $A(\lambda; \theta, \theta')$, where θ and $\theta' \in \mathbb{R}$ are local coordinates of $\mathcal{M}_\lambda = \{\xi \in \mathbb{T}^2; p(\xi) = \lambda\}$, which satisfy

$$\cos^2\frac{\xi_1}{2}\cos^2\frac{\xi_2}{2} + \sin\frac{\xi_1}{2}\sin\frac{\xi_2}{2}\cos\frac{\xi_1}{2}\cos\frac{\xi_2}{2} = \frac{1}{8}(\lambda + 2).$$

We have the analytic continuations $\zeta_{\pm}(z,\theta) = (\zeta_{\pm,1}(z,\theta), \zeta_{\pm,2}(z,\theta))$ from $\xi(\lambda,\theta) = (\xi_1(\lambda,\theta), \xi_2(\lambda,\theta))$ in the same way as in Sect. 4 and also, the resolvent estimates as in [13], where we only have to replace $|\cdot|_{l^1}$ with $d(\cdot)$. Therefore, we can adopt the same strategy for our reconstruction procedure for the potential as in the hexagonal lattice.

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