

(In)Finite Extent of Stationary Perfect Fluids in Newtonian Theory

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Abstract. For stationary, barotropic fluids in Newtonian gravity we give simple criteria on the equation of state and the “law of motion” which guarantee finite or infinite extent of the fluid region (providing a priori estimates for the corresponding stationary Newton–Euler system). Under more restrictive conditions, we can also exclude the presence of “hollow” configurations. Our main result, which does not assume axial symmetry, uses the virial theorem as the key ingredient and generalises a known result in the static case. In the axially symmetric case stronger results are obtained and examples are discussed.

1. Introduction

This work deals with a priori estimates of solutions of the stationary Newton–Euler system of equations. For definiteness the latter includes a barotropic equation of state (EOS) $\rho_g = \rho_g(p)$ relating the matter density and the pressure, and a “law of motion” (LOM) specifying the velocity $\vec{U}(x^i)$ or the centrifugal potential $\phi_c = \phi_c(x^k)$ as a function of position. This system can be used to model stars or galaxies, provided the perfect fluid is a viable model for the corresponding multi-particle system consisting of molecules or stars, respectively. The equations have been studied accordingly, from mathematical as well as from physical viewpoints, see e.g. [1–4]. Key problems are existence, uniqueness, (axial) symmetry, stability and parametrization of the solutions, where uniqueness may be understood modulo global parameters, like mass and angular momentum. Other useful parametrizations are the pressure or the density at the centre or at the axis, depending on the symmetry, or, as we shall see below, the sum of the gravitational potential ϕ_g and the centrifugal potential ϕ_c on the surface, called Φ_S .

In the present work we assume a connected fluid region and focus on investigating the problem if the Newton–Euler system admits (only) solutions

of finite or infinite extent (i.e., compact support of the density function). In fact this question can be regarded as a special case of the problem of obtaining a formula for Φ_S or a bound thereon, for which we give results as well. Normally finite extent of solutions is a prerequisite for the physical significance of a model. This always applies to models for single stars. However, for the polytropic EOS

$$p = \frac{1}{n+1} \rho_o^{-\frac{1}{n}} \rho_g^{\frac{n+1}{n}}, \quad \rho_o = \text{const.} > 0 \quad (1)$$

with index $n = 5$, all static configurations are infinitely extended. They are known as “Plummer’s model” for non-rotating globular clusters of stars (see e.g., [4]).

In the static case, systematic treatments of the finiteness problem, which do not rest on the assumption of spherical symmetry, can be found in [5–7]; for stronger results in spherical symmetry, cf. [8]. Formally, the task consists of determining if the surface value Φ_S of the gravitational potential ϕ_g agrees with its value at infinity, which we set to zero. Physically, the key ingredient is the continuum form of the virial theorem, and mathematically it is a modified Pohozaev–Rellich identity. Moreover, since $\rho_g \geq 0$, the maximum principle is available as well. In [5, 7] the functional

$$F(p) = \rho \int_0^p \frac{dp'}{\rho_g(p')} - 6p \quad (2)$$

was identified as the crucial quantity in the sense that $F \leq 0$ for all p guarantees finiteness of the solutions unless $F \equiv 0$, which characterises the polytropes of index 5. Moreover, $F \geq 0$ for all p but $F \not\equiv 0$ implies that there are no solutions with finite mass. If F changes sign, the analysis of the spherically symmetric Newton–Euler system is much more involved and uses dynamical system techniques (see e.g., [9]).

In the stationary case, results on finiteness of which we are aware are basically amendments to theorems on existence (cf. the classic paper [10], the recent account [11] and the references therein). All these results require axial symmetry and conditions on the EOS which are in many respects more restrictive than ours. In contrast, the present paper ignores the problem of existence, which means that we are interested in “a priori” estimates. On the other hand, some of our results do not require axial symmetry, which does not hold in general for stationary perfect fluids (cf. Sect. 3 below) and we use appropriately adapted conditions on the EOS and the LOM. Formally, we compare now the value of the “effective potential” $\phi = \phi_g + \phi_c$ at infinity (where we can again set it to zero unless it diverges) and at the surface (Φ_S), and the main technical tool is still the virial theorem. The quantity characterising rotation which arises in our analysis is

$$D = x^i \frac{\partial}{\partial x^i} \phi_c + \frac{1}{2} \phi_c, \quad (3)$$

where x^i denote Cartesian coordinates. Finiteness of the solution is guaranteed provided D and F have the same sign everywhere but do not both vanish identically. Under the additional requirement that $\Delta\phi_c \geq 0$ the case with D and F being non-negative can be excluded by the maximum principle, again unless both F and D vanish everywhere. This latter case leads to a particular LOM for the $n = 5$ polytrope, which will be analyzed separately.

For polytropes with index n which rotate according to a “power law” $\vec{U} \propto r^{-m} \partial/\partial\varphi$, where r is the distance from an axis and $m \in \mathbf{R}$, our conditions $F \leq 0$ and $D \leq 0$ enforcing finiteness read $n \leq 5$ and $m \geq 5/4$, respectively. In particular, a polytropic fluid with $n \leq 5$ whose layers rotate at or near their Kepler orbits ($m \simeq 3/2$) must be finite. This is somewhat counter-intuitive for the following reason: For such a fluid the gravitational attraction between its “layers” will be balanced locally only by the centrifugal force and not by pressure, whence the fluid should behave like “dust”. However, there is no obvious reason as to why Keplerian orbits of dust (-particles) should not extend to infinity. Of course this “paradox” must disappear upon properly taking into account the gravitational interaction between the dust particles or fluid layers. However, this reminds of heuristic arguments trying to estimate the amount of dark matter necessary to stabilise galaxies and to explain the observed rotation law $m \approx 1$. Recent analyses of the dark matter problem by different methods (see e.g., [12] and [13]) indicate that for this purpose much less dark matter is needed than previous approximations suggested. In resolving this issue Vlasov–Poisson theory should play a key role and also establish a connection to the phenomena in fluid mechanics described here (see e.g., [14]).

While we believe that our results are new, they are rather elementary, and the axially symmetric ones might well be contained in the vast astronomical, physical, and mathematical literature on rotating fluids since Euler’s and Newton’s time. On the other hand, our exposition is motivated by and partially adapted to the corresponding relativistic problem. In fact for static perfect fluids in general relativity the quantity $F(p)$ in (2) can be replaced by

$$G(p) = \rho \left[\exp \left(\int_0^p \frac{dp'}{\rho_g(p') + p} \right) - 1 \right] - 6p \quad (4)$$

to obtain analogous conclusions regarding finiteness and infiniteness as in the Newtonian case [5, 7]. However, since the known relativistic virial theorems [15, 16] are not suitable for the present purpose the methods are different, and generalizations to the stationary case are not straightforward. We intend to present relativistic analogues of some of the results given below elsewhere.

2. Assumptions and Basic Results

This section is divided into five subsections. In Sect. 2.1 we give some basic definitions and continue in Sect. 2.2 with writing the Newton–Euler system in a form suitable for our purposes. Section 2.3 contains a discussion of the EOS

$\rho_g = \rho_g(p)$ and the relationships $\rho_g(\phi)$ and $p(\phi)$. (The symbols are defined in the introduction). The space-time dependence of the gravitational variables ρ_g , ϕ_g and the rotational variables ρ_c , ϕ_c will be discussed in Sects. 2.4 and 2.5, respectively, and the latter section also contains the key Lemma 2.5.2 on (in)finiteness.

We denote Cartesian coordinates by either x^i ($i = 1, 2, 3$) or x, y and z . $R = \sqrt{x^2 + x^2 + z^2}$, ϑ and φ are spherical polar coordinates, and $r = \sqrt{x^2 + y^2}$ and φ are cylindrical polar coordinates. The volume element on \mathbf{R}^3 is denoted by $d\nu$.

2.1. Finiteness Versus Infiniteness

The vacuum and the fluid regions \mathcal{V} and \mathcal{F} are *by definition* 3-dimensional, open sets where $\rho_g = 0$ and $\rho_g \neq 0$, respectively. This means that all points $v \in \mathcal{V}$ and $f \in \mathcal{F}$ have open 3-neighbourhoods in \mathcal{V} and \mathcal{F} . We take \mathcal{F} to be connected but possibly with non-trivial topology; in particular, \mathcal{V} and $\mathcal{B} = \partial\mathcal{F}$ may be disconnected. The boundary \mathcal{B} is required to be a C^1 submanifold with \mathcal{F} lying only on one side of $\partial\mathcal{F}$. In other words, ρ_g only vanishes in vacuum and possibly at the boundary, but not on sets of dimension less than three “inside the region occupied by the fluid”. The reason for this requirement is to avoid trouble with integrating Euler’s equation (9). We also note that the boundary can be characterized by vanishing pressure, cf. Sect. 2.3.

The following definition introduces a shorthand for our main issue.

Definition 2.1. The fluid region \mathcal{F} is called finite if it stays within a compact subset of \mathbf{R}^3 , and infinite otherwise.

In particular, fluids which extend to infinity in at least one direction are called infinite. In the axially symmetric case dealt with in Sect. 3 we will distinguish between (in)finite extent in axial and equatorial directions.

2.2. The Newton–Euler System

We denote by \vec{U} the velocity of the fluid and by \mathcal{I} the integral

$$\mathcal{I} = \int_0^p \frac{dp'}{\rho_g(p')} \quad (5)$$

(the specific enthalpy), whose existence is assumed for finite p ; this is satisfied in particular for polytropes $\rho_g(p) \propto p^a$ when $a < 1$. Further restrictions on the EOS are discussed in Sect. 2.3. The stationary Newton–Euler system can be written as follows

$$\rho_g = \rho_g(p), \quad (6)$$

$$\Delta\phi_g = 4\pi\rho_g, \quad (7)$$

$$\nabla(\rho_g \vec{U}) = 0, \quad (8)$$

$$-(\vec{U} \cdot \nabla)\vec{U} = \nabla\phi_g + \frac{\nabla p}{\rho_g} = \nabla(\phi_g + \mathcal{I}). \quad (9)$$

In terms of the centrifugal potential ϕ_c defined up to a constant (which will be specified in Sect. 2.5) by

$$(\vec{U} \cdot \nabla) \vec{U} = \nabla \phi_c, \tag{10}$$

(9) yields the “Bernoulli” equation

$$\phi_g + \phi_c + \mathcal{I}(p) = \Phi_S, \tag{11}$$

where Φ_S is a constant. We also introduce a “centrifugal charge density”

$$\rho_c = \frac{1}{4\pi} \operatorname{div}[(\vec{U} \cdot \nabla) \vec{U}], \tag{12}$$

in terms of which Eq. (9) yields

$$\Delta \phi_c = 4\pi \rho_c. \tag{13}$$

Remark. 1. The existence theorems in the axially symmetric case show that the motion of the fluid has to be specified somehow in the Newton–Euler system. A simple way of doing so is to prescribe the velocity in terms of position $\vec{U} = \vec{U}(x^i)$ or to prescribe the centrifugal potential $\phi_c(x^i)$ which we do below (without restriction to axial symmetry). An alternative is to specify the angular momentum per unit mass (see e.g. [10]).

2. The above definitions of \vec{U} , ϕ_c and ρ_c , and the equations containing them, are understood to hold only in \mathcal{F} . It will however be convenient to extend these quantities, and some relations between them, to \mathcal{V} . In Sect. 2.5 we will discuss four alternative extensions which will be used in the following sections.
3. In the work focusing on existence, the aim is to specify only the EOS and the velocity or the specific angular momentum as functions in space, and to get information on the spatial behaviour of all variables. In this respect the present work has the same scope, although existence is not the issue here. However, except for crucial conditions on the EOS and the LOM, we will in the sequel also have to make differentiability and falloff requirements for the space-time dependence of our functions.

2.3. The Equation of State and the Effective Potential

The “effective potential” is defined by $\phi = \phi_g + \phi_c$; its gradient is called “effective gravity” in Sect. 3.2.1 of [3].

Our first Lemma serves mainly to list the assumptions on the equation of state required later. The proof is an easy consequence of the Bernoulli equation (11).

Lemma 2.3.1. *We assume that $\rho_g(p)$ is piecewise continuous, $0 \leq \rho_g(p) < \infty$, and that the integral \mathcal{I} exists for finite p . Then in \mathcal{F} the effective potential $\phi(p)$ is C^0 , piecewise C^1 , and strictly monotonic; the same applies to the inverse $p = p(\phi - \Phi_S) = p(\phi)$, and the density ρ_g is also a C^0 and piecewise C^1 function of ϕ that satisfies $\rho_g = dp(\phi)/d\phi$.*

Remark. The Lemma implies that the surface $p = 0$ is an equipotential surface of ϕ , and $\phi(p = 0)$ takes the value Φ_S . If we allowed for disconnected fluid

regions, a consistent definition of ϕ on \mathbf{R}^3 (cf. Sect. 2.5) would imply different constants Φ_S on each component in general.

The next Lemma (which is known, see e.g., [17]) contains a stronger assumption on the EOS in a neighbourhood of $p = 0$, which will also be made in Proposition 4.2.

Lemma 2.3.2. *In addition to the requirements of Lemma 2.3.1, we assume that $\rho_g(p)$ is C^0 in $[0, \delta)$ and C^1 in $(0, \delta)$ for some $\delta > 0$. Then $\lim_{p \rightarrow 0} p/\rho_g(p) = 0$.*

Proof. The result is obvious if $\rho_g(0) \neq 0$. If $\rho_g(0) = 0$ we first note that near $p = 0$ the inverse $p = p(\rho_g)$ exists, and $dp/d\rho_g \geq 0$. We can thus replace the assertion by $\lim_{\rho \rightarrow 0} p(\rho_g)/\rho_g = 0$ (dropping the subscript g on ρ_g when the latter is a sub- or superscript itself). We obtain

$$\begin{aligned} \infty &> \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^p \frac{dp'}{\rho_g(p')} = \lim_{\epsilon \rightarrow 0} \int_{\rho(\epsilon)}^{\rho_g(p)} \frac{dp'}{d\rho'_g} \frac{d\rho'_g}{\rho'_g} \\ &\geq \lim_{\epsilon \rightarrow 0} \left\{ \left[\inf_{[\rho_g(\epsilon), \rho_g(p)]} \frac{dp(\rho_g)}{d\rho_g} \right] \int_{\rho(\epsilon)}^{\rho(p)} \frac{d\rho'_g}{\rho'_g} \right\} \\ &\geq \lim_{\rho \rightarrow 0} \frac{dp(\rho_g)}{d\rho_g} \lim_{\epsilon \rightarrow 0} \int_{\rho(\epsilon)}^{\rho(p)} \frac{d\rho'_g}{\rho'_g} = \lim_{\rho \rightarrow 0} \frac{dp(\rho_g)}{d\rho_g} \cdot \infty. \end{aligned} \tag{14}$$

Hence $0 = \lim_{\rho \rightarrow 0} dp(\rho_g)/d\rho_g = \lim_{\rho \rightarrow 0} p(\rho_g)/\rho_g$ by de l'Hospital's rule. \square

2.4. The Gravitational Variables

We work in the weighted Sobolev spaces $W_\delta^{k,p}$ ($1 \leq p \in \mathbf{R}, \delta \in \mathbf{R}, k \in \mathbf{N}_0$), based on the weighted Lebesgue norms

$$\|u\|_{k,p,\delta} = \sum_0^k \|D^j u\|_{p,\delta-j}, \quad \|u\|_{p,\delta} = \left(\int_{\mathbf{R}^3} |u|^p \sigma^{-\delta p-3} d\nu \right)^{1/p} \tag{15}$$

for measurable functions $u \in L^p_{loc}(\mathbf{R}^3)$, where $\sigma = (1+R^2)^{1/2}$. This is Bartnik's index convention, cf. [19], see also remark 2 below.

We always require that the fluid has finite mass $m = \int_{\mathbf{R}^3} \rho_g d\nu$. The following Lemma is standard; we use [18, 19] for the inversion of the Laplacian in (7), and the strong maximum principle, Thm. 9.6 of [20].

Lemma 2.4. *Let $\rho_g \in W_{-3-\alpha}^{0,2}, 0 < \alpha < 1$. Then there is a unique solution $\phi_g \in W_{loc}^{2,2}$ of (7) with $\psi_g = \phi_g - m/\sigma \in W_{-1-\alpha}^{2,2}$. Moreover, $\phi_g \leq 0$.*

Remarks. 1. Weighted Sobolev spaces have the important property that a rather slow falloff of the density function in one or two directions is admitted as long as it is compensated by sufficiently fast falloff in the other direction(s). In particular, an axially symmetric disk of finite thickness with a measurable density function $\rho_g \in W^{0,2} = L^2$ and falloff

$\rho_g = O(1/r^{2+\epsilon}), \epsilon > 0$, satisfies the requirement of Lemma 2.4 and of the subsequent results.

2. With the conventions of Bartnik [19] used above, the index β for $f \in W_\beta^{k,p}$ is related to the growth of f at infinity; in particular $f = o(R^\beta)$ provided $kp > 3$ (throughout the paper, falloff conditions are always understood for large r or R). In fact for the function ψ_g introduced in Lemma 2.4 it follows that $\psi_g = o(R^{-1-\epsilon})$. However, getting the corresponding fall-off for the derivatives, namely $\nabla_i \psi_g = o(R^{-2-\epsilon})$ would require $p > 3$. While such first derivatives do occur in Theorem 4.2 below, less precise information on their falloff, which follows from $p = 2$, will suffice.

2.5. The Rotational Variables

As already mentioned in the introduction, the strategy of our finiteness argument is to compare the value of the effective potential $\phi = \phi_g + \phi_c$ at the surface with its value at infinity. On the other hand, integrating Euler’s equation (9) defines ϕ_c only in \mathcal{F} , and in fact only up to a constant.

We therefore extend now ϕ_c to \mathcal{V} . Below we consider four alternative definitions of the rotational variables valid on \mathbf{R}^3 , labelled A, $B_<$, $B_>$ and C. In extensions $B_<$, $B_>$ and C we follow standard practice and prescribe $\phi_c(x^i)$ a priori on \mathbf{R}^3 , irrespective of the fluid distribution. Of course we can also prescribe $\vec{U}(x^i)$ instead of $\phi_c(x^i)$, together with a constant in the resulting ϕ_c . On the other hand, extension A is motivated by the analogy between the Poisson equations (7) and (13), which suggests treating ρ_c as a “source” for ϕ_c .

While A, $B_<$ and $B_>$ are compatible with any symmetry, definition C refers only to a cylindrically symmetric velocity distribution $\phi_c(r)$. All definitions are in fact not only extensions from \mathcal{F} to \mathcal{V} , but involve also extra conditions on the falloff of the velocity at infinity if the fluid spreads there (cf. the remarks after Definition 2.5 for details).

Definition 2.5. For a solution of (6–11) with \mathcal{I} finite, ρ_g and ϕ_g as in Lemma 2.4, and $1 < q < \infty, 0 < \epsilon < 1$, we assume one of A, B or C:

- A. $\rho_c \in W_{-2-\epsilon}^{0,q}$ in $\mathcal{F}, \rho_c \equiv 0$ in \mathcal{V} and $\phi_c \rightarrow 0$ at infinity.
- B. $\phi_c(x^i) \in W^{2,q}$ given on \mathbf{R}^3 such that, for all radial unit vectors \vec{n} there is a unique limit

$$\lim_{R \rightarrow \infty} \phi_c(R \vec{n}) = \phi_\infty(\vec{n}) = \phi_\infty(\vartheta, \varphi) \tag{16}$$

and either

- $B_<$. $\sup_{\vartheta, \varphi} \phi_\infty(\vartheta, \varphi) = 0$, or
- $B_>$. $\inf_{\vartheta, \varphi} \phi_\infty(\vartheta, \varphi) = 0$.

- C. $\phi_c(r) \in C^1(\mathbf{R} \setminus \{0\})$ given (and possibly divergent at 0), and $\lim_{r \rightarrow \infty} \phi_c(r) = 0$.

Remarks. 1. While condition A is somewhat alien to the Newtonian case, it is mainly motivated by relativity. There the norm and twist potentials of the stationary Killing field satisfy elliptic equations, while on the other hand counterparts of the coordinate conditions B and C will hardly make good sense.

2. We note that cylindrically symmetric potentials $\phi_c(r)$ do not satisfy (16) unless $\phi_c(r) = \text{const.}$, since the limit $\lim_{R \rightarrow \infty} \phi_c(r) = \lim_{z \rightarrow \infty} \phi_c(r) = \phi_c(r)$ in the axis direction depends manifestly on r , while there should be a unique $\phi_\infty(\vartheta = 0, \phi)$. On the other hand, condition B is meaningful in particular for “almost spherical” velocity distributions.
3. Since ϕ_c was defined in (11) only up to a constant, condition B amounts to requiring (16), and that either $\sup_{\vartheta, \varphi} \phi_\infty(\vartheta, \varphi) < \infty$, or $\inf_{\vartheta, \varphi} \phi_\infty(\vartheta, \varphi) > -\infty$, while C includes the requirement that $\lim_{r \rightarrow \infty} \phi_c(r) > -\infty$. This latter requirement can in fact be removed, so that $\phi_c(r)$ is allowed to diverge both at the axis and at infinity. However, in order not to overload the subsequent definitions and results, this option will be considered only in the final remark of Sect. 3.2 and in examples.
4. The falloff conditions $B_<$ and $B_>$ and C are in some sense less restrictive than A. However in Theorem 4.2 the former requirements need to be explicitly supplemented by a falloff condition on the pressure which, on the other hand, follows from A automatically by virtue of the Bernoulli equation.
5. None of the definitions A, $B_<$, $B_>$ or C extends the validity of the Bernoulli equation from \mathcal{F} to the vacuum region \mathcal{V} in general. Such an extension could simply be afforded by setting $\phi_c = -\phi_g + \Phi_S$ in \mathcal{V} , but it seems to be of little use. Note in particular that such a ϕ_c would not necessarily be C^1 at the surface.
6. In principle, we could also consider axially symmetric velocity distributions of the form $\phi_c(r, z)$. However, the Poincaré–Wavre theorem, Lemma 3.1, then already implies cylindrical symmetry in \mathcal{F} , so an extension to \mathcal{V} with the same symmetry is the natural choice.

The following Lemma is analogous to Lemma 2.4 but, compared to the gravitational variables, the falloffs are slower here, and in contrast to the natural condition $\rho_g \geq 0$, the assumption $\rho_c \geq 0$ is highly restrictive.

Lemma 2.5.1. *If $\rho_c \in W_{-2-\alpha}^{0,2}$, $\alpha > 0$, then ϕ_c can be chosen such that $\phi_c \in W_{-\alpha}^{2,2}$. Moreover, if $\rho_c \geq 0$, then $\phi_c \leq 0$, and therefore also $\Phi_S \leq 0$.*

We now have the following easy to prove, but important Lemma.

Lemma 2.5.2. *Under the requirements and with the labelling of Definition 2.5 the following holds:*

- A. *If $\Phi_S \neq 0$ then the fluid is finite. Moreover, if $\rho_c \geq 0$ the fluid is finite if $\Phi_S < 0$ and infinite if $\Phi_S = 0$.*
- B. *If $\phi_\infty(\vartheta, \varphi)$ does not agree with Φ_S for some (ϑ, φ) , the fluid is finite “in the direction (ϑ, φ) ,” or more precisely, it does not intersect any 2-sphere \mathbf{S}_R^2 of sufficiently large radius R at the points $(R, \vartheta, \varphi) \in \mathbf{S}_R^2$. In particular, for extensions $B_<$, $B_>$ the fluid is finite if $\Phi_S > 0$, $\Phi_S < 0$ respectively.*
- C. *If $\Phi_S < 0$, the fluid is finite (in all directions), and if $\Phi_S > 0$ the fluid is finite in every direction except possibly in the axial one. Moreover, if $\Phi_S = 0$, and the fluid extends to infinity in the axis direction at some radius r_1 , it is static for all $r \geq r_1$.*

Proof. The proofs of parts A and B follow easily from the definitions, and the second part of case A from the maximum principle. The proof of C is postponed to Sect. 3.2. \square

Remark. We recall from a previous remark that a cylindrically symmetric $\phi_c(r)$ considered under C in Lemma 2.5.2 is not compatible with B, in particular not with (16). This accounts for differences in the conclusions and the proofs in the corresponding parts of Lemma 2.5.2.

3. Results with Symmetry

Assumptions and results on symmetry of barotropic perfect fluids require a careful discussion (cf. [21] where the relativistic case is included as well).

It is important to distinguish between the symmetry of the velocity field $\vec{U}(x^i)$ and its trajectories on the one hand, and the symmetry of the whole configuration on the other hand. In fact, a well-known example important in theory are non-axially symmetric fluid trajectories winding around an axially symmetric torus (cf. [1] in the Newtonian case and [22] in relativity). Fluids with non-axially symmetric configuration have been discussed; cf. [21] and Sect. 2.8.3 of [3]. On the other hand, known existence proofs of reasonable generality all assume axially symmetric velocities (cf. [11] and the references therein).

In the Newton–Euler system (6–9) the gravitational and velocity variables can enjoy different symmetries, as long as the coupling between the potentials via the Bernoulli equation (11) is respected. In this work we do not assume any symmetry of the matter variables ρ_g and ϕ_g . As to the rotational variables we first state in Sect. 3.1 the definitions of axial and cylindrical symmetry. Then we continue with a known result (the “Poincaré–Wavre theorem”) relating them, and give a simple application. In Sect. 3.2 we examine systematically the results arising from the integrals of the cylindrically symmetric Bernoulli equation via the limits $r \rightarrow \infty$ and $z \rightarrow \infty$. In particular, Sect. 3.2 also contains the proof of Lemma 2.5.2.C.

3.1. The Poincaré–Wavre Theorem

Definition 3.1. This refers to solutions to the Newton–Euler system (6–9):

1. A solution has axially symmetric velocity if the velocity \vec{U} is proportional to the axial Killing vector, and if $|\vec{U}|$ (and hence also ϕ_c) are rotation invariant.
2. An axially symmetric solution has cylindrically symmetric velocity if \vec{U} , (and hence also ϕ_c) are invariant under translations along an axis.

Lemma 3.1. *For a solution of Newton–Euler system (6–9) with axially symmetric centrifugal potential $\phi_c(r, z)$, the latter is in fact cylindrically symmetric, viz.*

$$\phi_c(r, z) = \phi_c(r), \quad \vec{U} = \omega(r) \frac{\partial}{\partial \varphi}, \quad (17)$$

and it holds

$$\phi_c(r) = - \int_{r_1}^r \omega^2(r')r' \, dr', \quad r > r_1 = \text{const.} \quad (18)$$

Proof. The proof is obvious from Euler's equation which reads

$$\frac{\partial \phi_c(r, z)}{\partial z} = 0, \quad \frac{\partial \phi_c(r, z)}{\partial r} = -\omega^2(r, z)r \quad (19)$$

under the stated assumptions. \square

Here is a simple application of the previous Lemma.

Proposition 3.1. *We assume a cylindrically symmetric velocity distribution and an EOS as in Lemma 2.3.1. If the fluid extends to infinity in the axis direction at two different radii $r_1 \geq 0$ and $r_2 > r_1$, the intermediate region $\mathcal{A} = \{(r, z) \text{ with } r_1 < r < r_2 \text{ and } -\infty \leq z \leq \infty\}$ must be static, i.e., $\phi_c = \text{const.}$ in \mathcal{A} .*

Proof. From the Bernoulli equation we obtain

$$\phi_c(r_1) = \lim_{z \rightarrow \infty} \phi_c(r_1) = \lim_{z \rightarrow \infty} (\Phi_S - \phi_g - \mathcal{I}) = \Phi_S, \quad (20)$$

and analogously $\phi_c(r_2) = \Phi_S$, which contradicts (18), unless $\omega \equiv 0$, and the solution is static. \square

Remark. This result implies in particular that solutions with axially symmetric velocity distribution which fill the whole space must be static and hence spherically symmetric.

3.2. Finiteness and Φ_S

In this section we discuss the finiteness question under the assumption that we know the sign or the vanishing of Φ_S ; in particular we will prove Lemma 2.5.2.C. On the other hand, in Sect. 4 the virial theorem will be employed to obtain the required information on Φ_S from the EOS and the LOM, independently of symmetry assumptions.

Finiteness in the radial direction, and the behaviour near the axis, can be obtained rather easily from a qualitative discussion of the form of the potential. The key features of the potentials to be kept in mind here are:

1. $\phi_g \leq 0$.
2. $\phi_c(r)$ behaves as in Definition 2.5.C and is monotonically decreasing with r from (18).
3. Inside the fluid region $\phi = \phi_g + \phi_c \leq \Phi_S$, and $\phi \nearrow \Phi_S$, as one approaches a locus of zero pressure (irrespective of its location in space).

Figure 1a–d show qualitatively possible forms of the potentials together with the respective fluid fillings (shaded), differing in their range with respect to infinity and the axis. For positive potentials ϕ as in Fig. 1a no solution exists, irrespective of the value of Φ_S , while the other figures show ranges for ϕ which are a priori admissible but still without guarantee of existence of a

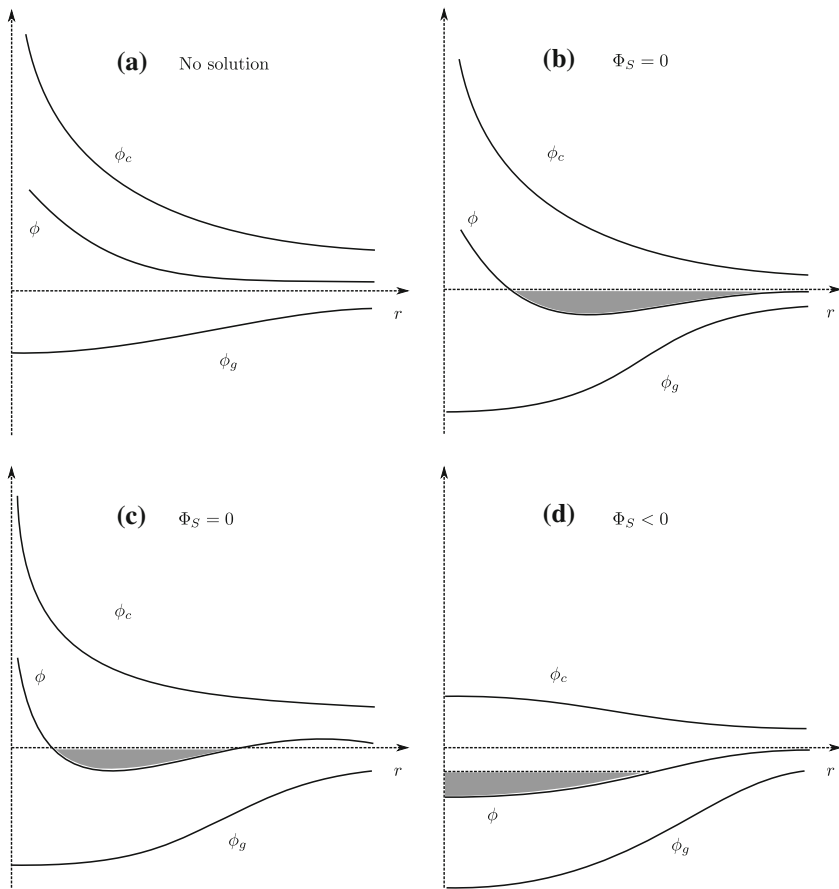


FIGURE 1. Different a priori possible forms of the potentials ϕ_g, ϕ_c and ϕ

solution. If the fluid is not axially symmetric, the curves are understood to be sections of higher-dimensional level surfaces of the potentials.

While the behaviour of the fluid in the axis direction as included in Lemma 2.5.2.C is not seen from the above diagrams, it is a simple consequence of the Bernoulli equation. The Lemma is now proven as follows.

Proof of Lemma 2.5.2.C. By contradiction: Assume that the fluid is infinite in the r direction. Then (11) and Definition 2.5.C yield

$$\Phi_S = \lim_{r \rightarrow \infty} (\phi + \mathcal{I}) = 0, \tag{21}$$

which contradicts the behaviour of Φ_S , as claimed for finiteness in the r direction. Here (21) in fact holds for the limit in any direction not parallel to the axis. The final assertion of the Lemma is obtained from (11), (18) and Definition 2.5.C. as follows: Assuming that the fluid extends to infinity in the z

direction at r we get

$$\Phi_S = \lim_{z \rightarrow \infty} (\phi + \mathcal{I}) = \phi_c(r) = \int_r^\infty \omega^2(r') r' dr', \quad (22)$$

and hence $\Phi_S = 0$ implies $\omega = 0$. \square

Remark. As remarked after Definition 2.5, the condition in part C that $\lim_{r \rightarrow \infty} \phi_c(r) > -\infty$ can be removed. In fact, if $\lim_{r \rightarrow \infty} \phi_c(r) = -\infty$, the statement of Lemma 2.5.2.C remains correct and the proof is trivially adapted.

4. General Results

In Sect. 4.1 we give a result which, under rather restrictive conditions, forbids shells or “hollow” bodies. We continue in Sect. 4.2 with our main finiteness theorem, which rests on the suitably adapted virial theorem, and we also obtain a more general bound on Φ_S .

4.1. A “no-shell” Result

Proposition 4.1. *We consider a solution of (6–13) and assume part A of Definition 2.5, with $\rho_c \geq 0$. Then there cannot be any vacuum region not connected to infinity, i.e., the fluid \mathcal{F} is not a “shell”, and it is not “hollow”.*

Proof. Assuming the contrary, \mathcal{F} has an inner and an outer boundary, the latter either separating \mathcal{F} from infinity or located at infinity, where ϕ takes on the value Φ_S . We first apply the strong maximum principle, Thm. 9.6 of [20], to (13) on the region \mathcal{R} consisting of \mathcal{F} and the “inner vacuum,” but not the vacuum component connected to infinity. This implies that $\phi = \Phi_S$ is the maximum taken at the boundary of \mathcal{R} (the outer boundary of \mathcal{F}). But this maximum is also taken on at interior points of \mathcal{R} , namely the inner boundary of \mathcal{F} , and hence the maximum principle implies that ϕ is constant on \mathcal{R} . The latter, however, is easily excluded from the assumptions. \square

Remark. Clearly, this result does not exclude toroidal rings of finite thickness, with or without central body.

4.2. The Finiteness Theorem

Our main result is now obtained by combining various Lemmas of the preceding sections. The quantities F and D are defined in the Introduction; the main conditions (23) and (24) are discussed in remarks after the following Theorem and in Sect. 5.

Theorem 4.2. *Assume we are given a solution of (6–11) such that the EOS (6) satisfies the requirements of Lemma 2.3.1, and that $\rho_g \in W_{-3-\epsilon}^{0,2}$, $\epsilon > 0$ as in Lemma 2.4. Moreover, we require that ϕ_c or ρ_c should satisfy one of A, B $_{<}$, B $_{>}$ or C in Definition 2.5 and that there holds in case*

A. one of

$$F(p) \geq 0 \quad \forall p \quad \text{and} \quad D(x) \geq 0 \quad \forall x, \tag{23}$$

$$F(p) \leq 0 \quad \forall p \quad \text{and} \quad D(x) \leq 0 \quad \forall x, \tag{24}$$

B_<. $p \in W_{-4-\epsilon}^{1,1}$ and (23),

B_>. $p \in W_{-4-\epsilon}^{1,1}$ and (24),

C. $p \in W_{-4-\epsilon}^{1,1}$ and (24).

Then either $\Phi_S \neq 0$ and the fluid is finite, or $F \equiv 0, D \equiv 0$ and $\Phi_S = 0$.

Corollary 4.2.1. *In case C above, assume that (23) holds instead of (24). Then the fluid is finite in all directions except possibly in the axial one.*

Corollary 4.2.2. *In case A above, assume that there hold (24) and $\rho_c \geq 0$. Then either $\Phi_S < 0$ and the fluid is finite, or $F \equiv 0, D \equiv 0$ and $\Phi_S = 0$, and the fluid is infinite.*

Proof. We use the following modified version of the Pohozaev–Rellich identity [24, 25]: Let $\vec{\xi} = x^i \frac{\partial}{\partial x^i}$ be a dilation, i.e., $\nabla_{(i} \xi_{j)} = \frac{1}{2}(\nabla_i \xi_j + \nabla_j \xi_i) = g_{ij}$, where g_{ij} denotes the components of the 3-metric, and ∇_i is the covariant derivative with respect to g_{ij} . Then a simple calculation (compare [5, 7]) shows that

$$\begin{aligned} \nabla_i \left[\left(\xi^j \nabla_j \phi_g + \frac{1}{2} \phi_g \right) \nabla^i \phi_g - \frac{1}{2} \xi^i \nabla_j \phi_g \nabla^j \phi_g + 4\pi p \xi^i \right] \\ = 2\pi [\rho_g(\phi_g - 2\xi^i \nabla_i \phi_c) + 6p]. \end{aligned} \tag{25}$$

We first note that $p \in W_{-4-\epsilon}^{1,q}$, which in case A follows from Euler’s equation (9). Next, by Gauss’s law, the left hand side can be written as a surface term over a ball \mathcal{S} of radius R . We now insert the forms of $\phi_g, \nabla \phi_g$, obtained in Lemma 2.4, and use the fact that ψ_g and $\nabla_i \psi_g$ defined in this Lemma “fall off faster” than the leading terms, to show that the surface integral vanishes as $R \rightarrow \infty$. This follows straightforwardly via Cauchy–Schwarz estimates and by applying the “trace theorem” (see e.g., Sect. 5.5 of [23]):

$$\|f\|_{\partial \mathcal{S}} \|L^2(\partial \mathcal{S})\| \leq C \|f\|_{W^{1,q}(\mathcal{S})} \tag{26}$$

(which holds for any function $f \in W^{1,q}(\mathcal{S}), q \geq 1$, and its extension $f|_{\partial \mathcal{S}}$ to $\partial \mathcal{S}$), to the functions $\psi_g, \nabla_i \psi_g$ and p . In the limit $R \rightarrow \infty$ we are left with

$$0 = 2\pi \int_{\mathbf{R}^3} [\rho_g(\phi_g - 2\xi^i \nabla_i \phi_c) + 6p] d\nu. \tag{27}$$

Inserting (11) into the above expression gives

$$m\Phi_S = \int_{\mathbf{R}^3} [F(x) + 2\rho_g D(x)] d\nu. \tag{28}$$

The theorem is now obvious from the requirements and from Lemma 2.5.2. □

- Remarks.* 1. Note that the conclusion does not require falloff conditions for $F(x)$, and only the mild falloff conditions on $\phi_c(x^j)$ or ρ_c from Definition 2.5. Even without such conditions, Eq. (28) and finiteness of $m\Phi_S$ imply that the integral on the right exists.
2. In the static case existence of finite, spherically symmetric solutions is known for polytropes with index $n \leq 5$, which corresponds to $F \leq 0$. On the other hand, if $F \geq 0$ solutions with finite mass do not exist except for the polytropes of index $n = 5$. This suggests that the “realistic” range for rotating fluids is given by conditions (24). In fact the requirements of existence results for axisymmetric rotating fluids of which we are aware ([11] and the references therein) fall in this range, whereas (23) might not allow for solutions at all.
3. If ρ_c is at least weakly differentiable, the second term on the right in (28) can be rewritten as

$$\int_{\mathbf{R}^3} \rho_g D(x) \, d\nu = \int_{\mathbf{R}^3} \phi_g \left[x^i \frac{\partial}{\partial x^i} \rho_c + \frac{5}{2} \rho_c \right] \, d\nu + \text{surface terms}, \quad (29)$$

and, since $\phi_g \leq 0$, the sign of the expression in the bracket determines (in)finite extent of the fluid region in the same manner as $D(x^i)$. The surface terms vanish in case A but have to be handled with care in cases $B_<$, $B_>$ and C.

4. To show the relation with the classical virial theorem (cf. Sect.1.4 of [15] or Sect.2.8.1 of [3]) we use (25) and definition (10) to obtain (30) below. To get (31) which is the sum of the potential energy $E_{\text{pot}} = 1/2 \int_{\mathbf{R}^3} \rho_g \phi_g \, d\nu$ the (bulk) kinetic energy $E_{\text{kin}} = 1/2 \int_{\mathbf{R}^3} \rho_g |U|^2 \, d\nu$ and the thermal energy (kinetic energy of the thermal motion) $E_{\text{therm}} = 3/2 \int_{\mathbf{R}^3} p \, d\nu$ one has to remove the second term in (30) by partial integration using the continuity equation (8), and assume that the velocity \vec{U} falls off suitably in order for the surface terms to vanish.

$$0 = 2\pi \int_{\mathbf{R}^3} \{ \rho_g [\phi_g - 2U^j \nabla_j (\xi^i U_i) + 2|U|^2] + 6p \} \, d\nu \quad (30)$$

$$= 4\pi (E_{\text{pot}} + 2E_{\text{kin}} + 2E_{\text{therm}}). \quad (31)$$

5. In the static case Φ_S is the gravitational potential at the surface, and it is related to the observed redshift. For a rotating, extended object, the redshift on the rim clearly depends on the velocity of the rotation as well, but does not seem to bear any obvious relation to Φ_S .
6. Equation (28) is a formula for Φ_S , but as such it requires a knowledge of $F(x)$, as opposed to the mere sign condition on $F(p)$ used before. An estimate for Φ_S in terms of $F(p)$ and $D(x)$ is the following.

Proposition 4.2. *Under the requirements of Lemma 2.3.2,*

$$|\Phi_S| \leq \sup_{\mathcal{F}} \left| \frac{F(p)}{\rho_g} \right| + 2 \sup_{\mathcal{F}} |D(x)|. \quad (32)$$

Proof. From (28),

$$\begin{aligned}
 m|\Phi_S| &\leq \int_{\mathbf{R}^3} \rho_g \sup \left| \frac{F(x)}{\rho_g} + 2D(x) \right| d\nu \\
 &\leq m \sup_{\mathcal{F}} \left| \frac{F(\rho_g)}{\rho_g} + 2D(x) \right| \leq m \sup_{\mathcal{F}} \left| \frac{F(p)}{\rho_g} \right| + 2m \sup_{\mathcal{F}} |D(x)|, \quad (33)
 \end{aligned}$$

and the supremum of $F(p)/\rho_g(p)$ exists, since from Lemma 2.3.2 $\lim_{p \rightarrow 0} F(p)/\rho_g(p) = \lim_{p \rightarrow 0} p/\rho_g(p) = 0$. \square

5. Examples

We examine here the limiting case of Theorem 4.2, and conclude with discussing “power law rotations.” Explicit examples are only available in the cylindrically symmetric case.

5.1. The Limiting Case

The limiting case of Theorem 4.2 is $F(p) \equiv 0, D \equiv 0$ and $\Phi_S \equiv 0$. The first condition yields the 1-parameter family of polytropic EOS (1) with index $n = 5$. In the static case, the resulting PDE

$$\Delta \phi_g = -4\pi \rho_o \phi_g^5 \tag{34}$$

has for each ρ_o the well-known (cf. e.g. [4]) family of solutions

$$\phi_g = -\frac{m}{\sqrt{\frac{4\pi}{3} \rho_o m^4 + R^2}} \tag{35}$$

parametrized by the mass m . All these solutions extend to infinity.

In the stationary case, if ρ_c is differentiable, it follows from $0 = D = x^i \partial_i \phi_c + \frac{1}{2} \phi_c$ that $x^i \partial_i \rho_c + \frac{5}{2} \rho_c = 0$. This yields that ϕ_c and ρ_c are homogeneous functions of degree $-1/2$ and $-5/2$, respectively. That is to say, these functions have the form

$$\phi_c = z^{-\frac{1}{2}} \sigma \left(\frac{x}{z}, \frac{y}{z} \right), \quad \rho_c = z^{-\frac{5}{2}} \tau \left(\frac{x}{z}, \frac{y}{z} \right), \quad z \neq 0 \tag{36}$$

for some arbitrary (but mutually related) functions $\sigma(x, y)$ and $\tau(x, y)$. To determine the solution, we have to solve

$$\Delta \phi = \Delta (\phi_g + \phi_c) = -4\pi \rho_o \phi^5 + 4\pi z^{-\frac{5}{2}} \tau \left(\frac{x}{z}, \frac{y}{z} \right) \tag{37}$$

in \mathcal{F} , and $\Delta \phi = 0$ in \mathcal{V} (if present). We remark that (37) is scale invariant under

$$\phi(x^i) \longrightarrow \sqrt{\frac{1}{\lambda}} \phi \left(\frac{x^i}{\lambda} \right), \quad \forall \lambda = \text{const.} > 0. \tag{38}$$

Thus the motion determined by $D \equiv 0$ could be called “scale invariantly rotating polytrope of index 5.”

Scale invariance is sometimes useful for getting information about (non-) existence of solutions, in particular in combination with the scaling behaviour of the energy functional. We do not go into details here.

Under restriction to axially symmetric rotation laws (about the z -axis) (36) becomes

$$\phi_c = z^{-\frac{1}{2}}\alpha\left(\frac{r}{z}\right), \quad \rho_c = z^{-\frac{5}{2}}\beta\left(\frac{r}{z}\right), \quad z \neq 0 \tag{39}$$

for some arbitrary (again related) functions $\alpha(r)$ and $\beta(r)$. However, by Lemma 3.1, the rotation law must be cylindrically symmetric. Choosing α and β in (39) appropriately, we have

$$\phi_c(r) = 2\frac{C^2}{\sqrt{r}}, \quad \rho_c(r) = \frac{C^2}{8\pi r^{\frac{5}{2}}}, \quad C = \text{const. in } \mathcal{F}. \tag{40}$$

We obtain the following behaviour for the “scale invariant rotation” $\omega_{si}(r)$, whose falloff interestingly lies between the Kepler angular velocity ω_{\odot} (actually Copernicus [26], as this is for circular orbits) and the observed galaxy rotation curves $\omega_{\textcircled{a}}$:

$$\omega_{\odot} = \frac{C}{r^{\frac{3}{2}}}, \quad \omega_{si} = \frac{C}{r^{\frac{5}{4}}}, \quad \omega_{\textcircled{a}} \sim \frac{C}{r}. \tag{41}$$

Regarding finiteness, we note that ω and ϕ_c are singular on the axis, whence Theorem 4.2 A is not applicable; in particular the maximum of ϕ is obviously taken at the axis. However, (40) implies that $\phi = \phi_c + \phi_g = 2C^2/\sqrt{r} - m/R + o(1/R^{1+\epsilon}) \geq 0$ for sufficiently large r . Hence if a solution exists, the form of ϕ must be as in Fig. 1c, while the form of Fig. 1b is excluded at least in the cylindrically symmetric case considered here. Thus the fluid is either finite (with a hole near the axis) or it is infinite in the z -direction (respecting Proposition 3.1). The former case would probably give a toroidal configuration.

We can now discuss the consequences of the above arguments for “power law rotations”.

5.2. Power Law Rotation

Proposition 5.2. *Assume (6–11) have a solution which satisfies the conditions of Theorem 4.2.C with the rotation law*

$$\phi_c(r) = \frac{C^2}{2(k-1)r^{2(k-1)}} \iff \omega(r) = \frac{C}{r^k}, \quad k, C \in \mathbf{R}. \tag{42}$$

Then depending on the value of k the fluid has the following properties regarding infinity and the axis:

1. $k < \frac{3}{2}$: It is finite in the radial direction.
2. $k > 1$: It has a “hole” near the axis.
3. $k \geq \frac{5}{4}$, $F \leq 0$ and $p \in W_{-4-\epsilon}^{1,1}$: It is either finite (with a hole near the axis), or infinite in the axis direction (respecting Proposition 3.1).

Moreover, if the solution is finite in the axis direction in case (3), the conclusion (finiteness) holds without the falloff condition on the pressure.

Proof. For $k \leq 1$ statement (1) follows from the fact that ϕ_c diverges at infinity. For $1 < k < 3/2$ the same conclusion can be inferred from the argument used in Sect. 5.1 to show that for the “scale invariantly” rotating polytrope

$\phi = \phi_c + \phi_g \geq 0$ for sufficiently large r . Conclusion (2) is obvious from the fact that ϕ_c diverges at the axis. To see (3) we calculate from Definition (3)

$$D(r) = \frac{C^2(5 - 4k)}{4(k - 1)r^{2(k-1)}}, \quad (43)$$

and use Theorem 4.2.C for the generic case, and the discussion of Sect. 5.1 for the limiting case with $\Phi_S = 0$. For the final statement, we note that (9) with Lemma 2.4 and $k \geq 5/4$ imply that $p = O(r^{-7/2})$. This is sufficient in order for the surface term in the integral of (25) to vanish as $R \rightarrow \infty$, provided the fluid is finite in the axis direction. \square

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