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Characterization of the Anderson Metal–Insulator Transition for Non Ergodic Operators and Application

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Abstract. We study the Anderson metal–insulator transition for non ergodic random Schrödinger operators in both annealed and quenched regimes, based on a dynamical approach of localization, improving known results for ergodic operators into this more general setting. In the procedure, we reformulate the Bootstrap Multiscale Analysis of Germinet and Klein to fit the non ergodic setting. We obtain uniform Wegner Estimates needed to perform this adapted Multiscale Analysis in the case of Delone-Anderson type potentials, that is, Anderson potentials modeling aperiodic solids, where the impurities lie on a Delone set rather than a lattice, yielding a break of ergodicity. As an application we study the Landau operator with a Delone-Anderson potential and show the existence of a mobility edge between regions of dynamical localization and dynamical delocalization.

1. Introduction

Under the effect of a random perturbation, the spectrum of an ergodic Schrödinger operator is expected to undergo a transition where we can identify two distinct regimes: the insulator region, characterized by localized states and the metallic region, characterized by extended states. The passage from one to the other under a certain disorder regime is known as the Anderson metal–insulator transition. Although a precise spectral description of this phenomenon is still out of reach, this transition is better characterized in terms of its dynamical properties. Germinet and Klein tackled this problem in [17] by introducing a local transport exponent $\beta(E)$ to measure the spreading of a wave packet initially localized in space and in energy evolving under the effect of the random operator. This provides a proper dynamical characterization of

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the metal–insulator transition, and the mobility edge, i.e., the energy where the transition occurs, is shown to be a discontinuity point of $\beta(E)$.

Since ergodicity is a basic feature in the theory of random Schrödinger operators, Germinet and Klein's work was done in that framework. However, more real models may lack this fundamental property. Examples of this kind of systems are Schrödinger operators with Anderson-type potentials where the random variables are not i.i.d. or where impurities are located in aperiodic discrete sets. The first case (sparse models, decaying randomness, surfacic potentials) has been studied in [1,6,26,33], while the second case (Delone-Anderson type potentials) has been treated in [3]. In the deterministic case, Delone operators have been studied with a dynamical systems approach in [25,27–29] and [31].

We aim to study the Anderson metal–insulator characterization in a general non ergodic setting, with minimal requirements on the model to fit the dynamical characterization of localization/delocalization using the local transport exponent $\beta(E)$, extending the results of [17] to the non ergodic models mentioned above. The main tool in the study of the transport transition is the Multiscale Analysis (MSA), initially developed by Frölich and Spencer [12], it has been improved over the last three decades to its strongest version so far, the Bootstrap MSA by Germinet and Klein [15]. The Bootstrap MSA yields among other features strong dynamical localization in the Hilbert–Schmidt norm, and so it can be used to characterize the set of energies where the transport exponent is zero, that is associated to dynamically localized states [17], but since it was originally developed in the frame of ergodic operators it is not suitable when there is lack of ergodicity, so we adapt it to our model. What completes the dynamical characterization is the fact that, in the ergodic case, slow transport in average over the randomness, the so-called annealed regime, implies dynamical localization. This holds in our new setting and, moreover, this can be improved and it can be shown that it is enough to have slow transport with a good probability, that is, in a quenched regime, to obtain dynamical localization, so in both quenched and annealed regimes the metal–insulator transition can be characterized in an analog way. There are examples related to the Parabolic Anderson model where the behavior of the solution in both regimes differ from each other and this can depend on the density of the random variables [22].

We obtain uniform Wegner estimates needed for the adapted version of the Bootstrap MSA for operators with Delone-Anderson potentials, that is, Anderson potentials where the impurities are located on sites of an a priori aperiodic set, called a *Delone* set. It is known that a way to obtain Wegner estimate is to "lift" the spectrum by considering the random Hamiltonian as a negative perturbation of a periodic Hamiltonian whose spectrum starts above a certain energy above the bottom of the spectrum of the original free Hamiltonian (called fluctuation boundary). In this way the Wegner estimate is obtained "outside" the spectrum of the periodic operator, as in [2]. We stress the fact that this approach is not convenient in our case since we have no information on where the fluctuation boundary lies. On the other hand, [9] and [10] take a different approach by using a unique continuation property to prove Wegner estimates without a covering condition on the single-site potential, and not using fluctuation boundaries. The results in [9] rely strongly on the periodicity of the lattice and the use of Floquet theory, which, again, cannot be used in our model since our set of impurities is aperiodic. However, this was improved in [11] to obtain a positivity estimate for the Landau Hamiltonian that does not rely on Floquet theory, but on specific features of the spectral projectors inherent to the magnetic case. This makes it convenient for our setting and therefore we can extend these results to the non ergodic case. Furthermore, we can use the approach in [10] to obtain Wegner estimates outside the unperturbed spectrum in the case where the background Hamiltonian is periodic, and for all energies in the case where the Integrated Density of States (IDS) of the background operator is Hölder continuous. In the case of the free Laplacian, a uniform Wegner estimate can be obtained as well, at the bottom of the spectrum in an interval whose length depends only on the Delone set parameters and not in the disorder parameter λ. We refer to [21] where a *spatial averaging* method as in [7,13,14] is exploited to prove the required positivity estimate, and bypass the use of Floquet theory.

For the Landau operator, and as an application of the main results, we can show the existence of a metal–insulator transition, as expected from the ergodic case [19]. Since the lattice is a particular case of a Delone set, these results imply in particular those of the ergodic setting. By the lack of ergodicity we cannot make use of the Integrated Density of States to prove the existence of a non random spectrum for H_ω, nor use the characterization of the spectrum in terms of the spectra of periodic operators as done in [20] to locate the spectrum in the Landau band. Therefore, to show our results are not empty we need to prove that we can almost surely find spectrum near the band edges, which is done adapting an argument in [8, Appendix B] in a not necessarily perturbative regime of the disorder parameter λ. We stress that we consider a general Delone set and do not assume any geometric property, like repetitivity or finite local complexity. These features, however, might be needed for further results, for example, related to the Integrated Density of States (see [21,29–31]).

The present note is organized as follows: in Sect. 2 we state the main results, in Sect. 3 we adapt the Bootstrap MSA to fit our new setting. In Sect. 4 we prove the results on the dynamics in both annealed and quenched regimes. In Sect. 5 we prove uniform Wegner estimates for Delone-Anderson random Schrödinger operators. In Sect. 6, in the lines of [19] we proof the existence of a metal–insulator transition for a Landau Hamiltonian with a Delone-Anderson potential and the existence of almost sure spectrum near the band edges, that has non empty intersection with the localization region.

2. Main Results

For $x \in \mathbb{R}^d$ we denote by $\|x\|$ the usual euclidean norm while the supremum norm is defined as $|x|_\infty = \max_{1\leq i\leq d} |x_i|$, where $|\cdot|$ stands for absolute value.

Given $x \in \mathbb{R}^d$ and $L > 0$ we denote by $B(x, L)$ the ball of center x and radius L in the $\|\cdot\|$-norm, while the set

$$\Lambda_L(x) = \left\{ y \in \mathbb{R}^d : |y - x|_\infty < \frac{L}{2} \right\}$$

defines the cube of side L centered at x, also denoted as $\Lambda_{x,L}$. We denote the volume of a Borel set $\Lambda \subset \mathbb{R}^d$ with respect to the Lebesgue measure as $|\Lambda| = \int_{\mathbb{R}^d} \chi_\Lambda(x) d^d x$, where χ_Λ is the characteristic function of the set Λ. We will often write $\chi_{x,L}$ for $\chi_{\Lambda_L(x)}$ and denote by $\|f\|_{x,L}$ or $\|f\|_{\Lambda_L(x)}$ the norm of f in $\mathrm{L}^2(\Lambda_{x,L})$.

We denote by $\mathcal{C}_c^\infty(\Lambda)$ the vector space of real-valued infinitely differentiable functions with compact support contained in Λ, with $C_{c,+}^\infty(\Lambda)$ being the subclass of nonnegative functions.

We denote by $\mathcal{B}(\mathcal{H})$ the Banach space of bounded linear operators on the Hilbert space $\mathcal{H}$. For a closed, densely defined operator A with adjoint A^*, we denote its domain by $\mathcal{D}(A) \subset \mathrm{L}^2(\Lambda)$ and by $\|A\| = \sup\{\|A\phi\|;\ \|\phi\|_2 = 1\}$ its (uniform) norm if bounded. We define its absolute value by $|A| = \sqrt{A^*A}$ and, for $p > 1$, we define its (Schatten) p-norm in the Banach space $\mathcal{J}_p(\mathrm{L}^2(\Lambda))$ as $\|A\|_p = (\mathrm{tr}|A|^p)^{1/p}$. In particular, $\mathcal{J}_1$ is the space of trace-class operators and $\mathcal{J}_2$, the space of Hilbert–Schmidt operators. We write $\langle x \rangle = \sqrt{(1 + \|x\|^2)}$ and use $\langle X \rangle$ to denote the operator given by multiplication by the function $\langle x \rangle$.

For convenience we denote a constant C depending only on the parameters $a, b, \ldots$ by $C_{a,b,\ldots}$.

We consider a random Schödinger operator of the form

$$H_\omega = H_0 + \lambda V_\omega \quad \text{on } \mathrm{L}^2(\mathbb{R}^d), \tag{2.1}$$

where H_0 is the free Hamiltonian, λ measures the disorder strength which in the following we consider fix, and V_ω, called *random potential*, is the operator multiplication by V_ω, such that $\{V_\omega(x) : x \in \mathbb{R}^d\}$ is a real-valued measurable process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having the following properties:

(R) $V_\omega = V_\omega^+ + V_\omega^-$, where V_ω^+ and V_ω^- are real valued measurable processes on Ω such that for $\mathbb{P}$-a.e. $\omega : 0 \le V_\omega^+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_ω^- is relatively form-bounded with respect to $-\Delta$, with relative bound <1, i.e., there are nonnegative constants $\Theta_1 < 1$ and Θ_2 independent of ω such that for all $\psi \in \mathcal{D}(\nabla)$ we have

$$|\langle \psi, V_\omega^- \psi \rangle| \le \Theta_1 \|\nabla \psi\|^2 + \Theta_2 \|\psi\|^2 \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

(IAD) There exists $\varrho > 0$ such that for any bounded sets $B_1, B_2 \subset \mathbb{R}^d$ with $\mathrm{dist}(B_1, B_2) > \varrho$, the processes $\{V_\omega(x) : x \in B_1\}$ and $\{V_\omega(x) : x \in B_2\}$ are independent.

In the case $H_0 = H_B$, the unperturbed Landau Hamiltonian on $\mathrm{L}^2(\mathbb{R}^2)$

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with } \mathbf{A} = \frac{B}{2}(x_2, -x_1), \tag{2.2}$$

where $\mathbf{A}$ is the vector potential and B is the strength of the magnetic field, we ask $\mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ to satisfy the diamagnetic inequality so we can obtain trace estimates for the Landau Hamiltonian from those of the Laplacian.

It follows that H_ω is a semibounded selfadjoint operator for $\mathbb{P}$-a.e. ω. Moreover, the mapping $\omega \to H_\omega$ is measurable for $\mathbb{P}$-a.e. ω, we denote its spectrum by σ_ω.

In the usual setting for (ergodic) random Hamiltonians, H_ω satisfies a *covariance condition* with respect to the action of a family of unitary (translation) operators U_x, and its associated ergodic group of translations τ_x on the probability space Ω. Throughout this paper we do not make any assumption on the ergodicity of H_ω, so this covariance condition, *a priori*, does not hold, i.e.,

$$H_{\tau_\gamma(\omega)} \neq U_\gamma H_\omega U_\gamma^*, \tag{2.3}$$

which makes H_ω a *non-ergodic* random operator.

For the following assumption we need the notion of a *finite volume operator*, the restriction of H_ω to either an open box $\Lambda_L(x)$ with Dirichlet boundary condition or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary conditions. In this way, we obtain a well defined random operator $H_{\omega,x,L}$ acting on $\mathrm{L}^2(\Lambda_L(x))$ defined by

$$H_{\omega,x,L} = H_{0,x,L} + \lambda V_{\omega,x,L}.$$

We denote its spectrum by $\sigma_{\omega,x,L}$ and by $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ its resolvent operator. We define the spectral projections $P_\omega(J) = \chi_J(H_\omega)$ and $P_{\omega,x,L}(J) = \chi_J(H_{\omega,x,L})$ for $J \subset \mathbb{R}$ a Borel set. When stressing the dependence on λ, it will be added to the subscript.

Definition 2.1. (UWE) We say that H_ω satisfies a uniform Wegner estimate with Hölder exponent s in an open interval $\mathcal{J}$, i.e., for every $E \in \mathcal{J}$ there exists a constant Q_E, bounded on compact subintervals of $\mathcal{J}$ and $0 < s \leq 1$ such that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\{\mathrm{tr}(P_{\omega,x,L}(E-\eta, E+\eta)\} \leq Q_E \eta^s L^d, \tag{2.4}$$

for all $\eta > 0$ and $L \in 2\mathbb{N}$. It satisfies a uniform Wegner estimate at an energy E if it satisfies a uniform Wegner estimate in an open interval $\mathcal{J}$ such that $E \in \mathcal{J}$.

To describe the dynamics, we consider the random moment of order $p \geq 0$ at time t for the time evolution in the Hilbert–Schmidt norm, initially spatially localized in a square of side one around $u \in \mathbb{Z}^2$ and localized in energy by the function $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, i.e.,

$$M_{u,\omega}(p,\mathcal{X},t) = \| \langle X-u\rangle^{p/2} e^{-itH_\omega}\mathcal{X}(H_\omega)\chi_u\|_2^2. \tag{2.5}$$

We next consider its time average,

$$\mathcal{M}_{u,\omega}(p,\mathcal{X},T) = \frac{2}{T}\int_0^\infty e^{-2t/T} M_{u,\omega}(p,\mathcal{X},t)\ \mathrm{d}t. \tag{2.6}$$

Definition 2.2. 1. We say that H_ω exhibits strong Hilbert–Schmidt (HS-) dynamical localization in the open interval I if for all $\mathcal{X} \in \mathcal{C}^\infty_{c,+}(I)$ we have

$$\sup_{u\in\mathbb{Z}^2} \mathbb{E}\{\sup_{t\in\mathbb{R}} M_{u,\omega}(p,\mathcal{X},t)\} < \infty \quad \text{for all } p \geq 0.$$

We say that H_ω exhibits strong Hilbert–Schmidt (HS-) dynamical localization at an energy E if there exists an open interval I with $E \in I$, such that there is strong HS-dynamical localization in the open interval.

2. The strong insulator region for H_ω is defined as

$$\Sigma_{\mathrm{SI}} = \{E \in \mathbb{R} : H_\omega \text{exhibits strong HS-dynamical localization at } E\}.$$

Note that if there exists a $\delta > 0$ such that $\mathrm{dist}(E, \sigma_\omega) > \delta$ for almost every ω, then $E \in \Sigma_{\mathrm{SI}}$.

As we shall see, the existence of such a region for random Schrödinger operators is the consequence of the applicability of the Bootstrap MSA adapted to the non ergodic setting Theorem 2.3.

Given $\theta > 0, E \in \mathbb{R}, x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-*suitable* for H_ω if $E \notin \sigma_{\omega,x,L}$ and

$$\|\Gamma_{x,L} R_{\omega,x,L}(E)\chi_{x,L/3}\|_{x,L} \leq \frac{1}{L^\theta},$$

where $\Gamma_{x,L} = \chi_{\bar{\Lambda}_{L-1}(x)\setminus\Lambda_{L-3}(x)}$. If we replace the polynomial decay $1/L^\theta$ by $e^{-mL/2}$ we say that the box $\Lambda_L(x)$ is $(m, E) -$ *regular* for H_ω.

The following theorem is a reformulation of Theorem 3.4 and Corollary 3.10 [15] in a non ergodic setting,

Theorem 2.3. *Let H_ω be a random Schrödinger operator satisfying a uniform Wegner estimate in an open interval $\mathcal{J}$ with Hölder exponent s and assumptions (R), (IAD). Given $\theta > d$, for each $E \in \mathcal{J}$ there exists a finite scale $\mathcal{L}_\theta(E) = \mathcal{L}(\theta, E, Q_E, d, s)$, bounded in compact subintervals of $\mathcal{J}$, such that if for $\mathcal{L} > \mathcal{L}_\theta(E)$ the following holds*

$$\inf_{x\in\mathbb{Z}^d} \mathbb{P}\{\Lambda_{\mathcal{L}}(x) \text{ is } (\theta, E) - \textit{suitable}\} > 1 - \frac{1}{841^d}, \tag{2.7}$$

then there exists $\delta_0 > 0$ and $C_\zeta > 0$ such that

$$\sup_{u\in\mathbb{Z}^d} \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2\right) \leq C_\zeta e^{-|x|^\zeta}, \tag{2.8}$$

for $0 < \zeta < 1$, where $I(\delta_0) = [E - \delta_0, E + \delta_0]$. Moreover, $E \in \Sigma_{\mathrm{SI}}$ and we have the following properties,

(SUDEC) Summable uniform decay of eigenfunction correlations: for a.e. $\omega \in \Omega$, the Hamiltonian H_ω has pure point spectrum in $I \subset \Sigma_{\mathrm{SI}}$ with finite multiplicity. Let $\{\epsilon_{n,\omega}\}_{n\in\mathbb{N}}$ be an enumeration of the distinct eigenvalues of H_ω in I. Then for each $\zeta \in]0,1[$ and $\epsilon > 0$ we have, for every $x, u \in \mathbb{Z}^d$,

$$\|\chi_{x+u}\phi\|\|\chi_u\varphi\| \leq C_{I,\zeta,\epsilon,\omega}\|T_u^{-1}\phi\|\|T_u^{-1}\varphi\|\langle x+u\rangle^{\frac{d+\epsilon}{2}}\langle u\rangle^{\frac{d+\epsilon}{2}} e^{-|x|^\zeta}, \tag{2.9}$$

for all $\phi, \varphi \in$ Ran $P_\omega(\{\epsilon_{n,\omega}\})$ (see Sect. 3).

(*DFP*) *Decay of the Fermi projections: for $E \in \Sigma_{\mathrm{SI}}$ and for any $\zeta \in]0,1[$ we have*

$$\sup_{u\in\mathbb{Z}^d} \mathbb{E}\{\|\chi_{x+u} P_\omega((-\infty,E])\chi_u\|_2^2\} \le C_{\zeta,\lambda,E} e^{-|x|^\zeta}, \tag{2.10}$$

where the constant $C_{\zeta,E}$ is locally bounded in E.

Remark 2.4. The condition (2.7) is called the initial length scale estimate (ILSE) of the Bootstrap MSA. In practice is often useful to prove the equivalent estimate [17, Theorem 4.2]: For some $\theta > d$, we have

$$\limsup_{L\to\infty} \inf_{x\in\mathbb{Z}^d} \mathbb{P}\{\Lambda_{\mathcal{L}}(x) \text{ is } (\theta,E)\text{-suitable}\} = 1. \tag{2.11}$$

Definition 2.5. The multiscale analysis region for H_ω is defined as the set of energies where we can perform the bootstrap MSA, i.e.,

$$\Sigma_{\mathrm{MSA}} = \{E \in \mathbb{R} : H_\omega \text{ satisfies a uniform Wegner estimate at } E \text{ and (ILSE) holds for some } \mathcal{L} > \mathcal{L}_\theta(E)\}.$$

By Theorem 2.3, we have $\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}}$.

We introduce the (lower) transport exponent in the annealed regime:

$$\beta(p,\mathcal{X}) = \liminf_{T\to\infty} \frac{\log_+ \sup_u \mathbb{E}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T))}{p\log T}, \tag{2.12}$$

for $p \ge 0, \mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, where $\log_+ t = \max\{0,\log t\}$, and define the p-th local transport exponent at the energy E, by

$$\beta(p,E) = \inf_{I\ni E} \sup_{\mathcal{X}\in C^\infty_{c,+}(I)} \beta(p,\mathcal{X}), \tag{2.13}$$

where I denotes an open interval. The exponents $\beta(p,E)$ provide a measure of the rate of transport in wave packets with spectral support near E. Since they are increasing in p, we define the local (lower) transport exponent $\beta(E)$ by

$$\beta(E) = \lim_{p\to\infty} \beta(p,E) = \sup_{p>0} \beta(p,E). \tag{2.14}$$

With the help of this transport rate we can define two complementary sets in the energy axis for fixed $B>0, \lambda>0$, the region of *dynamical localization*

$$\Xi^{\mathrm{DL}} = \{E \in \mathbb{R}: \ \beta(E) = 0\}, \tag{2.15}$$

also called the trivial transport region (TT) in [17] and the region of *dynamical delocalization*

$$\Xi^{\mathrm{DD}} = \{E \in \mathbb{R}: \ \beta(E) > 0\}, \tag{2.16}$$

also called the weak metallic transport region (WMT), in [17]. Recalling Theorem 2.3 we have that $\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}} \subset \Xi^{\mathrm{DL}}$.

The following result is an improvement of [17, Theorem 2.11] for the non ergodic setting,

Theorem 2.6. *Let H_ω be a Schrödinger operator satisfying a uniform Wegner estimate with Hölder exponent s in an open interval $\mathcal{J}$ and assumptions (R), (IAD). Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{J}, \alpha \geq 0$ and $p > p(\alpha, s) := 12\frac{d}{s} + 2\alpha\frac{d}{s}$. If*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} \frac{1}{T^\alpha}\mathbb{E}\left(\mathcal{M}_{u,\omega}(p,\mathcal{X},T)\right) < \infty, \tag{2.17}$$

then $J \subset \Sigma_{\mathrm{MSA}}$. In particular, it follows that (2.17) holds for any $p \geq 0$.

Moreover, we can extend this result to a quenched regime, a new feature in both ergodic and non-ergodic situations:

Theorem 2.7. *Let H_ω be a Schrödinger operator satisfying a uniform Wegner estimate with Hölder exponent s in an open interval $\mathcal{J}$ and assumptions (R), (IAD). Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{J}, \alpha \geq 0$ and $p > p(\alpha, s) := 15\frac{d}{s} + 2\alpha\frac{d}{s}$. If*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} T^{\frac{s}{d}}\mathbb{P}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T) > T^\alpha) = 0, \tag{2.18}$$

then $J \subset \Sigma_{\mathrm{MSA}}$. In particular, it follows that (2.18) holds for any $p \geq 0$.

Remark 2.8. If the moment increases almost surely at any other rate less than polynomial, this implies in particular condition (2.18) for some $\alpha > 0$, and the result follows.

Moreover, if condition (2.28) in [17, Theorem 2.11] holds for $\alpha > 0$ and $p > p(\alpha, s) + d$, then condition (2.18) holds for $\alpha' = \alpha + \delta$ and the same p, where $0 < s/2 < \delta < \frac{s(p-p(\alpha,s))}{2d}$ and $p > p(\alpha', s)$, since by Chebyshev's inequality we have, for all $T > 0$

$$T^{\frac{s}{d}} \sup_u \mathbb{P}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T) > T^{\alpha'}) \leq \frac{1}{T^{\alpha+\delta-s/2}} \sup_u \mathbb{E}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T)). \tag{2.19}$$

This also shows that (2.18) is indeed a weaker condition than (2.17).

By Theorem 2.6 we have that $\Xi^{\mathrm{DL}} \subset \Sigma_{\mathrm{MSA}}$, so Theorems 2.8 and 2.10 of [17] hold in our setting. Thus, the local transport exponent $\beta(E)$ gives a characterization of the metal–insulator transport transition for non ergodic models as for the usual ergodic setting. Moreover, if we consider only the random moments in a quenched regime to behave asymptotically slow, we see the same behavior for the ergodic and non ergodic setting, in agreement with the annealed regime.

As an application of these results, we can study a Landau Hamiltonian with constant magnetic field B perturbed by a Delone-Anderson potential and show an Anderson metal–insulator transition.

Definition 2.9. A subset D of $\mathbb{R}^d$ is called a *(r,R)-Delone set* if there exist reals r and R such that for any cubes Λ_r, Λ_R of sides r and R respectively, we have $\sharp(D \cap \Lambda_r) \leq 1$ and $\sharp(D \cap \Lambda_R) \geq 1$, where $\sharp$ stands for cardinality.

Remark 2.10. Note that in a (r, R)-Delone set there exists a minimal distance between any two points, $r/2$, and a maximal distance between neighbors, $\sqrt{d}R$. Such a set is said to be *uniformly discrete* and *relatively dense*. Lattices and the set of vertices in a Penrose tiling are particular cases of Delone sets.

We consider the case where the free Hamiltonian in (2.1) is H_B, the unperturbed Landau Hamiltonian on $\mathrm{L}^2(\mathbb{R}^2)$ defined by

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with } \mathbf{A} = \frac{B}{2}(x_2, -x_1), \tag{2.20}$$

where $\mathbf{A}$ is the vector potential and B is the strength of the magnetic field, and the random potential represents impurities placed in a Delone set, that is,

$$V_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u(x - \gamma), \tag{2.21}$$

where ω_γ are independent identically distributed random variables of bounded probability density ρ, D is a (r, R)-Delone set (for the case $H_0 = -\Delta$ see [13]), and the single-site potential u has small support such that supports of the translations $u_i = u(\cdot - i)$ for $i \in \mathbb{R}^2$ do not overlap. We assume furthermore that $u(0) = \|u\|_\infty = 1$.

The spectrum of H_B is pure point and consists of a sequence of infinitely degenerate eigenvalues, the Landau levels $\{B_n = (2n+1)|B|;\ n = 0, 1, \ldots\}$, with associated orthogonal projection operators Π_n. As the spectrum is independent of the sign of B, we will always assume $B > 0$.

We denote the spectrum of the operator $H_{B,\lambda,\omega}$ by $\sigma_{B,\lambda,\omega}$. By perturbation theory [23, Theorem V.4.10] we know that for each $\omega \in \Omega$

$$\sigma_{B,\lambda,\omega} \subset \bigcup_{n=0}^{\infty} \mathcal{B}_n(B, \lambda),$$

where $\mathcal{B}_n(B, \lambda) = [B_n - \lambda m_0, B_n + \lambda M_0]$ is called the *n-th Landau band*. Moreover, by a Borel–Cantelli argument, for almost every $\omega \in \Omega$,

$$\sigma_B \subset \sigma_{B,\lambda,\omega}, \tag{2.22}$$

where σ_B is the spectrum of the free Landau operator. We also show that there exists almost surely spectrum near the band edges so our results are not empty (see Sect. 6.4).

For B fixed λ is small enough such that

$$\lambda(m_0 + M_0) < 2B, \tag{2.23}$$

i.e., the Landau bands $\mathcal{B}_n(B, \lambda)$ are disjoint and hence the open intervals

$$\mathcal{G}_n(B, \lambda) =]B_n + \lambda M_0, B_{n+1} - \lambda m_0[, \quad n = 0, 1, 2, \ldots, \tag{2.24}$$

are nonempty spectral gaps for $H_{B,\lambda,\omega}$.

We aim to prove for this model the existence of complementary regions of dynamical localization and delocalization in the spectrum and therefore, the existence of a dynamical transition energy. By doing this we extend known results for ergodic random Landau Hamiltonians [8, 16, 19, 20] to non-ergodic ones.

Theorem 2.11. *Let $H_{B,\lambda,\omega}$ be the Delone-Landau Hamiltonian, satisfying in particular condition (2.23). Then for any $n = 0, 1, 2, \ldots$ there exists a positive constant $B(n)$, depending on parameters of the model, such that for any $B > B(n)$, $H_{B,\lambda,\omega}$ exhibits almost surely an Anderson metal–insulator transport transition in the n-th Landau band.*

As shown in Sect. 6, Theorem 2.11 is a consequence of Theorems 6.1 and 6.4, which show dynamical localization an delocalization in each Landau band, plus Theorem 6.5 which takes care of the lack of the existence of an almost-sure spectrum and states that these results are non-empty.

3. Proof of Theorem 2.3

3.1. Generalized Eigenfunction Expansion

We have to construct a generalized eigenfunction expansion adapted to the non ergodic case. Compared to [15, Sect. 2.3] we shall use a family of weighted spaces rather than just one in particular, using translations in $u \in \mathbb{Z}^2$ of the operator T defined there and thus without using translation invariance in the proofs.

Let T_u be the operator in $\mathcal{H}$ given by multiplication by the function $(1 + |x - u|^2)^\nu$, where $\nu > d/4, u \in \mathbb{Z}^2$. We define the weighted spaces $\mathcal{H}^u_\pm$ as

$$\mathcal{H}^u_\pm = L^2(\mathbb{R}^d, (1 + |x - u|^2)^{\pm 2\nu}\, \mathrm{d}x; \mathbb{C}). \tag{3.1}$$

The sesquilinear form

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}^u_+, \mathcal{H}^u_-} = \int \overline{\phi}_1 \phi_2(x)\, \mathrm{d}x \quad \text{for } \phi_1 \in \mathcal{H}^u_+, \phi_2 \in \mathcal{H}^u_-$$

makes $\mathcal{H}^u_+$ and $\mathcal{H}^u_-$ conjugates dual to each other and we denote by $\dagger$ the conjugation with respect to this duality. The natural injections $\iota^u_+ : \mathcal{H}^u_+ \to \mathcal{H}$ and $\iota^u_- : \mathcal{H} \to \mathcal{H}^u_-$ are continuous with dense range, with $(\iota^u_+)^\dagger = \iota^u_-$. The operators $T_{u,+} : \mathcal{H}^u_+ \to \mathcal{H}$ and $T_{u,-} : \mathcal{H} \to \mathcal{H}^u_-$ defined by $T_{u,+} = T_u \iota^u_+, T_{u,-} = \iota^u_- T_u$ on $\mathcal{D}(T_u)$ are unitary with $T_{u,-} = T^\dagger_{u,+}$. Note that

$$\|\chi_{x,L}\|_{\mathcal{H},\mathcal{H}^u_+} = \|\chi_{x,L}\|_{\mathcal{H}^u_-,\mathcal{H}} \le C_{L,d,\nu}(1 + |x - u|^2)^\nu, \tag{3.2}$$

for all $x \in \mathbb{R}^d$ and $L > 0$.

With this redefinition we can follow [15], restating assumption GEE for non ergodic operators. We consider a fixed open interval $\mathcal{I}$ and we recall that $P_\omega(J) = \chi_J(H_\omega)$ is the spectral projection of the operator H_ω on a Borel set $J \subset \mathbb{R}$.

(UGEE) *For some $\nu > d/4$, the set $\mathcal{D}^{u,\omega}_+ = \{\phi \in \mathcal{D}(H_\omega) \cap \mathcal{H}^u_+ : H_\omega \phi \in \mathcal{H}^u_+\}$ is dense in $\mathcal{H}_+$ and an operator core for H_ω for $\mathbb{P}$-a.e. ω and all u. There exists a bounded function f, strictly positive on the spectrum of H_ω such that,}*

$$\sup_u \operatorname{tr}_{\mathcal{H}}(T_u^{-1} f(H_\omega) P_\omega(\mathcal{I}) T_u^{-1}) < \infty, \tag{3.3}$$

for $\mathbb{P}$-a.e. ω.

If UGEE holds, for almost every ω and all u we have

$$\mathrm{tr}_{\mathcal{H}}(T_u^{-1} P_\omega(J \cap \mathcal{I}) T_u^{-1}) < \infty, \tag{3.4}$$

for all bounded sets J. Thus with probability one, for all u

$$\mu_{u,\omega}(J) = \mathrm{tr}_{\mathcal{H}}(T_u^{-1} P_\omega(J \cap \mathcal{I}) T_u^{-1}) \tag{3.5}$$

is a spectral measure for the restriction of H_ω to the Hilbert space $P_\omega(\mathcal{I})\mathcal{H}$, and for every bounded set J,

$$\mu_{u,\omega}(J) < \infty. \tag{3.6}$$

Then, we have a generalized eigenfunction expansion as in [15, Sect. 2]: for every u, there exists a $\mu_{u,\omega}$-locally integrable function $\mathbf{P}_{u,\omega}(\tilde{\lambda})$ from $\mathbb{R}$ into $\mathcal{T}_1(\mathcal{H}_+^u, \mathcal{H}_-^u)$, the space of trace class operators from $\mathcal{H}_+^u$ to $\mathcal{H}_-^u$, with

$$\mathbf{P}_{u,\omega}(\tilde{\lambda}) = \mathbf{P}_{u,\omega}(\tilde{\lambda})^\dagger \tag{3.7}$$

and

$$\mathrm{tr}_{\mathcal{H}}\left(T_{u,-}^{-1} \mathbf{P}_{u,\omega}(\tilde{\lambda}) T_{u,+}^{-1}\right) = 1 \quad \text{for } \mu_{u,\omega}\text{-a.e. } \tilde{\lambda}, \tag{3.8}$$

such that

$$\iota_-^u P_\omega(J \cap \mathcal{I}) \iota_+^u = \int_J \mathbf{P}_{u,\omega}(\tilde{\lambda})\, \mathrm{d}\mu_{u,\omega}(\tilde{\lambda}) \quad \text{for bounded Borel sets } J, \tag{3.9}$$

where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+^u, \mathcal{H}_-^u)$-valued functions.

The following (a restatement of assumption SGEE), is a stronger version of UGEE:

(USGEE) *We have that UGEE holds with*

$$\sup_u \mathbb{E}([\mathrm{tr}_{\mathcal{H}}(T_u^{-1} f(H_\omega) P_\omega(\mathcal{I}) T_u^{-1})]^2) < \infty. \tag{3.10}$$

So for every bounded set J,

$$\sup_u \mathbb{E}(\mu_{u,\omega}(J)^2) < \infty. \tag{3.11}$$

3.2. Kernel Decay and Dynamical Localization

Following the arguments in [15] for ergodic operators, we can show that HS-strong dynamical localization is a consequence of the applicability of the Bootstrap MSA for the non ergodic setting ([15, Theorem 3.4] with the stronger initial ILSE (2.7) instead of the original one).

We can restate Lemma 2.5 and Lemma 4.1 [15] as follows, extending the proofs to our new definitions,

Lemma 3.1. *Let H_ω be a random operator satisfying assumption GEE. We have with probability one, for all u, that for $\mu_{u,\omega}$-almost every $\tilde{\lambda}$,*

$$\|\chi_x \mathbf{P}_{u,\omega}(\tilde{\lambda}) \chi_y\|_1 \le C(1 + |x-u|^2)^\nu (1 + |y-u|^2)^\nu, \tag{3.12}$$

for all $x, y \in \mathbb{R}^d$, with C a finite constant independent of $\tilde{\lambda}, \omega$ and u.

Suppose, moreover, that assumption EDI in [15] *is satisfied in some compact interval $I_0 \subset \mathcal{I}$. Given $I \subset I_0, m > 0, L \in 6\mathbb{N}$ and $x, y \in \mathbb{Z}^d$, if $\omega \in R(m, L, I, x, y)$, with $R(m, L, I, x, y)$ defined as in* (3.18)*, then*

$$\|\chi_x \mathbf{P}_{u,\omega}(\tilde{\lambda})\chi_y\|_2 \leq Ce^{-mL/4}(1 + |x-u|^2)^{\nu}(1 + |y-u|^2)^{\nu}, \tag{3.13}$$

for $\mu_{u,\omega}$-almost all $\tilde{\lambda} \in I$, with $C = C(m, d, \nu, \tilde{\gamma}_{I_0})$, where $\tilde{\gamma}_{I_0}$ is the constant on assumption EDI.

Proof of Theorem 2.3. To apply the MSA in the non ergodic case we first need to verify for an operator satisfying only properties R, IAD and UWE, the standard assumptions SLI, EDI [15], plus UNE and USGEE, which are stronger assumptions than those stated in the mentioned article.

As for SLI and EDI, these are deterministic assumptions that hold for each $\omega \in \Omega$ and their proof, done in [17, Appendix A], relies on property R, with no use of ergodicity. In the same appendix we see that assumption NE is uniform on cubes centered in $x \in \mathbb{R}^d$ and relies on property R so it holds in our more general setting. The same is true for [17, Lemma A.3], and can be extended in an analog way to the case $H_0 = H_B$ [5, Sect. 2.1], proving the first part of USGEE (and UGEE).

As for the trace estimate (3.10) , for the case $H_0 = -\Delta$ it follows from [17, Lemma A.4] and [24, Theorem 1.1], taking $V = \langle X - u\rangle^{-2\nu}$ there, the result being uniform in u. It can be extended to the case $H_0 = H_B$ as in [5, Proposition 2.1].

To obtain the basic result of MSA [15, Theorem 3.4] we need conditions IAD, SLI, UNE and UWE to follow an analog iteration procedure. Recall that in their article, Germinet and Klein take two versions of MSA by Figotin and Klein, improve their estimates yielding other two MSA and then bootstrapping them to obtain the strongest result out of the weakest hypothesis, so in order to extend this results to the non ergodic setting we reformulate this methods. Each step consists of a purely geometric deterministic part where we use SLI, and therefore it does not depend on the placement of the boxes were we perform the procedure, and a probabilistic part, where we use UWE instead of WE to obtain an estimate on the probability of having bad events, in a stronger sense than the usual, that is, uniform with respect to the placement of the box in space.

We begin with the single energy multiscale analyses, Theorems 5.1 and 5.6 [15], which in our non-ergodic setting consists in estimating the decay of

$$p_L = \sup_{x \in \mathbb{Z}^d} p_{x,L}, \tag{3.14}$$

where

$$p_{x,L} = \mathbb{P}\{\Lambda_L(x) \text{ is bad}\} \tag{3.15}$$

(here a box is bad if it is not (θ, E)-*suitable* for H_ω). In the ergodic case we need only to consider $p_{0,L}$. Hypothesis (2.7) ensures we can follow the same iteration procedure in all boxes centered in $x \in \mathbb{Z}^d$, where $p_{0,L}$ is thus replaced by p_L. We use properties SLI and UWE instead of WE, and the deterministic arguments remain the same, since they do not depend on the location of the box. Considering a Hölder exponent s in WE implies that the choice of the initial length scale will also depend on s.

Next we consider the energy interval multiscale analyses, Theorems 5.2 and 5.7 [15], which in our general setting consists in estimating

$$\tilde{p}_L = \sup_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|>L+\varrho}} \tilde{p}_{x,y,L}, \tag{3.16}$$

with

$$\tilde{p}_{x,y,L} = \mathbb{P}\{R(m,L,I(\delta_0),x,y)^c\}, \tag{3.17}$$

where $I(\delta_0) = [E-\delta_0, E+\delta_0]$, for some $\delta_0 > 0$ and

$$R(m,L,I(\delta_0),x,y) = \{\omega : \text{ for every } E \in I(\delta_0), \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is good}\} \tag{3.18}$$

(here a box is good if it is (m,E)-*regular* for H_ω, with m to be specified later). In the ergodic case it suffices to consider $\tilde{p}_{x,y,L}$. We can thus follow the original iteration procedure on this estimate, replacing $\tilde{p}_{x,y,L}$ by $\tilde{p}_L$, obtaining an analog of [15, Eq. 3.4], i.e., there exists $\delta_0 > 0$ such that given any $\zeta, 0<\zeta<1$ there is a length scale $L_0 < \infty$ and a mass $m_\zeta = m(\zeta, L_0) > 0$ such that if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}, 0<\alpha<\zeta^{-1}, k=0,1,2,\ldots$ we have

$$\inf_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|>L+\varrho}} \mathbb{P}\{R(m_\zeta, L_k, I(\delta_0), x, y)\} \geq 1 - e^{-L_k^\zeta}. \tag{3.19}$$

To derive results on the spectrum and the dynamics of the operator from this estimate we need to consider also conditions EDI and USGEE. Thus, with Lemma 3.1 in hand, (3.19) and USGEE we can follow the proof of [15, Theorem 3.8] with minor modifications. We want to show that if (3.19) holds we have that for any $0<\zeta<1$, there is a finite constant C_ζ such that

$$\sup_u \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2\right) \leq C_\zeta e^{-|x|^\zeta}, \tag{3.20}$$

For this, we consider the pair of points x, y as the pair $x+u, u$, and fix $x \in \mathbb{Z}^d$ and k such that $L_{k+1} + \varrho > |x| > L_k + \varrho$. We split the expectation in (3.20) in two parts: the first one over the set $R(m_\zeta, L_k, I(\delta_0), x+u, u)$ and the second one over its complement, which has probability less than $e^{-L_k^\zeta}$, uniformly in u, by (3.19). We follow the arguments in [15, Eq. 4.8-4.13]. By (3.9) and Lemma 3.1 we can write, for a positive constant C_1,

$$\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2 \leq C_1 e^{-L_k^\zeta} \mu_{u,\omega}(I). \tag{3.21}$$

This implies,

$$\sup_u \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2; R(m_\zeta, L_k, I(\delta_0), x+u, u)\right)$$
$$\leq C_1^2 \sup_u \mathbb{E}\{(\mu_{u,\omega}(I(\delta_0)))^2\} e^{-2L_k^\zeta}. \tag{3.22}$$

As for the expectation over $R(m_\zeta, L_k, I(\delta_0), x+u, u)^c$, (3.19) implies that

$$\sup_u \mathbb{P}(R(m_\zeta, L_k, I(\delta_0), x+u, u)^c) < e^{-L_k^\zeta},$$

this yields,

$$\sup_u \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2; R(m_\zeta, L_k, I(\delta_0), x+u, u)^c\right)$$
$$\leq 4^\nu \sup_u \mathbb{E}\{(\mu_{u,\omega}(I(\delta_0)))^2\}^{\frac{1}{2}} e^{-\frac{1}{2}L_k^\zeta}, \tag{3.23}$$

where we use the fact that by (3.5) we can write

$$\|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2 \leq \|f\|^2 \|P_\omega(I(\delta_0))\chi_u\|_2^2 \leq C\|f\|\mu_{u,\omega}(I(\delta_0)). \tag{3.24}$$

Combining (3.22) and (3.23), using USGEE we obtain the desired decay, namely (3.20).

Now we can prove a strong version of dynamical localization as in [15, Corollary 3.10]. Notice that, if $p > 2$

$$\langle X-u\rangle^p = \sum_{x\in\mathbb{Z}^d} (1+\|y-u\|^2)^{p/2}\chi_x(y) \leq C_d \sum_{x\in\mathbb{Z}^d} (1+\|x-u\|^2)^{p/2}\chi_x(y)$$
$$= C_d \sum_{x\in\mathbb{Z}^d} (1+\|x\|^2)^{p/2}\chi_{x+u}(y), \tag{3.25}$$

so we have,

$$\|\langle X-u\rangle^{p/2} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2$$
$$= \mathrm{tr}[\chi_u f(H_\omega) P_\omega(I(\delta_0))\langle X-u\rangle^p P_\omega(I(\delta_0)) f(H_\omega)\chi_u]$$
$$\leq C_d \sum_{x\in\mathbb{Z}^d} (1+\|x\|^2)^{p/2} \mathrm{tr}[\chi_u f(H_\omega) P_\omega(I(\delta_0))\chi_{x+u} P_\omega(I(\delta_0)) f(H_\omega)\chi_u]$$
$$= C_d \sum_{x\in\mathbb{Z}^d} (1+\|x\|^2)^{p/2} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2. \tag{3.26}$$

Taking the expectation and then the supremum over $u \in \mathbb{Z}^2$, by (3.20) we obtain strong HS-dynamical localization in the energy interval $I(\delta_0)$.

Following the proof of [18, Corollary 3], after adapting [18, Theorem 1] to our setting we obtain the summable uniform decay of eigenfunction correlations SUDEC. As for property DFP, it is a consequence of (3.20) combined with [4, Theorem 1.4], which is a deterministic result also valid in our setting, in the lines of [18, Theorem 3]. □

4. Proofs of Theorems 2.6 and 2.7

Here we can proceed as in [17]. First we state the following Lemma, which is an intermediate result in the proof of [17, Lemma 6.4], adapted to the UWE with Hölder exponent s. We consider a cube $\Lambda_L(x)$ with arbitrary x so we omit it from the notation.

Lemma 4.1. *Let H_ω be a random Schrödinger operator satisfying a uniform Wegner estimate in an open interval $\mathcal{I}$, with Wegner constant Q_E and Hölder exponent s. Let $p_0 > 0$ and $\gamma > d$. For each $E \in \mathcal{I}$, there exists $\mathcal{L} = \mathcal{L}(d, E, Q_E, \gamma, p_0, s)$ bounded on compact subsets of $\mathcal{I}$, such that, given $L \in 2\mathbb{N}$ with $L \geq \mathcal{L}$, and subsets B_1 and B_2 of Λ_L (not necessarily disjoint) with $B_1 \subset \Lambda_{L-5/2}$ and $\bar{\Lambda}_{L-1} \setminus \Lambda_{L-3} \subset B_2$, then for each $a > 0$ and $0 < \epsilon \leq 1$ we have*

$$\mathbb{P}\left(\|\chi_2 R_{\omega,L}(E+i\epsilon)\chi_1\|_L > \frac{a}{4}\right) \leq \mathbb{P}\left(\|\chi_2 R_\omega(E+i\epsilon)\chi_1\| > \frac{a}{L^\gamma}\right) + \frac{p_0}{10}, \quad (4.1)$$

and

$$\mathbb{P}\left(\|\chi_2 R_{\omega,L}(E)\chi_1\|_L > \frac{a}{2}\right) \leq \mathbb{P}\left(\|\chi_2 R_\omega(E+i\epsilon)\chi_1\| > \frac{a}{L^\gamma}\right) + Q_E\left(\frac{4\epsilon}{a}\right)^{s/2} L^d + \frac{p_0}{10}, \quad (4.2)$$

where χ_i stands for $\chi_{B_i}, i = 1, 2$.

Proof of Theorem 2.6. By the same arguments used in [17, Theorem 4.2], it suffices to show that, under condition (2.18), for each $E \in J$ there is some $\theta > d/s$ such that

$$\limsup_{L\to\infty} \inf_{y\in\mathbb{Z}^d} \mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,y,L}(E)\chi_{y,L/3}\|_{y,L} \leq \frac{1}{L^\theta}\right) = 1, \quad (4.3)$$

i.e., the starting condition for the bootstrap MSA, (2.7), in its strong version, holds at some finite scale $L > \mathcal{L}_\theta(E)$.

Let $E \in J, \theta > d/s$ and $L \in 6\mathbb{N}$. We start by estimating

$$P_{E,L} := \sup_y \mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,y,L}(E)\chi_{y,L/3}\|_{y,L} > \frac{1}{L^\theta}\right). \quad (4.4)$$

We decompose as in [17, Eq. 6.26–6.28], using

$$\chi_{y,L} = \chi_{y,2L/3} + \chi_{y,L\setminus 2L/3}, \quad \text{where } \chi_{y,L\setminus 2L/3} = \chi_{y,\Lambda_L\setminus\Lambda_{2L/3}},$$

so (for simplicity we omit the subscript y from the norm)

$$P_{E,L} \leq \sup_y \mathbb{P}\left(\frac{1}{4L^\theta} < \|\Gamma_{y,L} R_{\omega,L}(E+i\epsilon)\chi_{y,L/3}\|_L\right) \quad (4.5)$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^\theta} < \epsilon\|R_{\omega,L}(E+i\epsilon)\|_L \|\Gamma_{y,L} R_{\omega,L}(E)\chi_{y,2L/3}\|_L\right) \quad (4.6)$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{4L^\theta} < \epsilon\|R_{\omega,L}(E)\|_L \|\chi_{y,L\setminus 2L/3} R_{\omega,L}(E+i\epsilon)\chi_{y,L/3}\|_L\right). \quad (4.7)$$

To estimate the first term we use (4.1) with $a = L^{-\theta}$. As for the rest, we use (4.2) and (4.1), respectively, with $a = 1$, plus the uniform Wegner estimate. We obtain

$$P_{E,L} \leq \sup_y \mathbb{P}\left(\frac{1}{L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\chi_{y,L/3}\|\right) \tag{4.8}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{L^{\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\chi_{y,2L/3}\|\right) \tag{4.9}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{L^{\gamma}} < \|\chi_{y,L\setminus 2L/3} R_\omega(E+i\epsilon)\chi_{y,L/3}\|\right) \tag{4.10}$$

$$+ Q_I(4\epsilon)^{s/2}L^d + 2Q_I\epsilon^s L^{\theta s+d} + \frac{3p_0}{10}, \tag{4.11}$$

for $L > \mathcal{L}$, with $\mathcal{L}$ as in Lemma 4.1, where $\gamma > d/s, 0 < \epsilon \leq 1, 0 < p_0 < 1$ and $Q_I = \sup_{E\in I} Q_E < \infty$. Set

$$L = L(I,\epsilon) := \left[\left(\frac{p_0}{20Q_I\epsilon^s}\right)^{1/(\theta s+d)}\right]_{6\mathbb{N}}, \tag{4.12}$$

so that

$$Q_I(4\epsilon)^{s/2}L^d \leq \frac{p_0}{10} \quad \text{and} \quad 2Q_I\epsilon^s L^{\theta s+d} \leq \frac{p_0}{10}.$$

We first estimate,

$$\sup_y \mathbb{P}\left(\frac{1}{L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\chi_{y,L/3}\|\right). \tag{4.13}$$

To do this, we decompose the norm using the function $\mathcal{X}(H_\omega)$ that localizes in energy, yielding

$$\sup_y \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right) \tag{4.14}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)(1-\mathcal{X}(H_\omega))\chi_{y,L/3}\|\right). \tag{4.15}$$

For the second term we use Chebyshev's inequality and follow [17, Eq. 6.32–6.34], so we can bound it by $p_0/12$.

Estimating in the same way the terms (4.9) and (4.10) we obtain that for L big enough,

$$P_{E,L} \leq \sup_y \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right) \tag{4.16}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^{\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,2L/3}\|\right) \tag{4.17}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^{\gamma}} < \|\chi_{y,L\setminus 2L/3} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right) + \frac{3p_0}{4}. \tag{4.18}$$

As for the first term,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right)\\
&\quad\leq 2L^{\theta+\gamma}\mathbb{E}\left(\|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right) &(4.19)\\
&\quad\leq 2L^{\theta+\gamma}\sum_{u\in\tilde{\Lambda}_{L/3}(y)}\mathbb{E}\left(\|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right). &(4.20)
\end{aligned}$$

For any u fixed, given a compact subinterval $I\subset J$ and $M>0$ we set:

$$A_{u,M,I,\epsilon}=\left\{E\in I:\ \mathbb{E}\left(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2\right)\leq M\epsilon^{-(\alpha+1)}\right\}.$$

We have, taking $T=\epsilon^{-1}$ and using [17, Lemma 6.3]

$$\begin{aligned}
|I\backslash A_{u,M,I,\epsilon}| &\leq \frac{1}{M\epsilon^{-(\alpha+1)}}\int_{\mathbb{R}}\mathbb{E}(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2)\,\mathrm{d}E\\
&=\frac{2\pi}{MT^{\alpha+1}}\int_0^\infty e^{-2t/T}\mathbb{E}(\|\langle X-u\rangle^{p/2}e^{-itH_\omega}\mathcal{X}(H_\omega)\chi_u\|_2^2)\,\mathrm{d}t\\
&\leq\frac{\pi}{MT^{\alpha}}\sup_u\mathbb{E}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T)). &(4.21)
\end{aligned}$$

Remark 4.2. Notice that the analogous sets $A_{k,I,M}$ in the proof [17, Theorem 2.11] do not work in the non ergodic setting, so we need to consider a family of sets $A_{u,M,I,\epsilon}$, indexed by u.

By hypothesis (2.17) we can pick a sequence $T_k\to\infty$ such that for k big enough, we have $\sup_u\mathbb{E}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T_k))<CT_k^\alpha$, then for the corresponding sequence $\epsilon_k\to 0^+$ we have

$$|I\backslash A_{u,M,I,\epsilon_k}|\leq\frac{C}{M}. \tag{4.22}$$

Notice that this bound is uniform in u.

Thus, for an $E\in I$ fixed and $\epsilon_k=T_k^{-1}$, either $E\in A_{u,I,M,\epsilon_k}$ in which case we have,

$$\begin{aligned}
&\mathbb{E}\left(\|\Gamma_{y,L_k}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|\right)\\
&\quad\leq C_{p,d}L_k^{-p/2}\mathbb{E}(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|_2)\\
&\quad\leq C_{p,d}L_k^{-p/2}\mathbb{E}(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|_2^2)^{1/2}\\
&\quad\leq C_{p,d}L_k^{-p/2}M^{1/2}\epsilon_k^{-(\alpha+1)/2}, &(4.23)
\end{aligned}$$

where we write $L_k=L(I,\epsilon_k)$, or else, $E\in I\backslash A_{u,M,I,\epsilon_k}$, so by (4.22) there exists $E_u\in A_{u,I,M,\epsilon_k}$ such that

$$|E-E_u|\leq\frac{C}{M}$$

and so, by the resolvent identity and the definition of $A_{u,M,I,\epsilon}$,

$$\begin{aligned}&\mathbb{E}(\|\Gamma_{y,L_k}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|)\\&\quad\le\mathbb{E}(\|\Gamma_{y,L_k}R_\omega(E_u+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|)\\&\qquad+|E-E_u|\mathbb{E}(\|R_\omega(E+i\epsilon_k)\|\|R_\omega(E_u+i\epsilon_k)\|)\\&\quad\le C_{p,d}L_k^{-p/2}M^{1/2}\epsilon_k^{-(\alpha+1)/2}+\frac{C}{M\epsilon_k^2}.\end{aligned}\tag{4.24}$$

Therefore,

$$\begin{aligned}&\mathbb{P}\left(\frac{1}{2L_k^{\theta+\gamma}}<\|\Gamma_{y,L_k}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_{y,L_k/3}\|\right)\\&\quad\le C'_{p,d}L_k^{\theta+\gamma-p/2+d}M^{1/2}\epsilon_k^{-(\alpha+1)}+C''_{p,d}\frac{L_k^{\theta+\gamma+d}}{M\epsilon_k^2}.\end{aligned}\tag{4.25}$$

The remaining terms (4.17) and (4.18) are estimated in the same way, using the fact that $\mathrm{dist}(\bar\Lambda_{L-1}\setminus\Lambda_{L-3},\Lambda_{\frac{2L}{3}})\ge\frac{L}{3}-\frac{3}{2}$ and $\mathrm{dist}(\Lambda_{L\setminus\frac{2L}{3}},\Lambda_{\frac{2L}{3}})\ge\frac{L}{6}$. For these terms we obtain an estimate as (4.25) with constants $C'^{(2)}_{p,d}, C''^{(2)}_{p,d}$ and $C'^{(3)}_{p,d}, C''^{(3)}_{p,d}$, respectively, and with no θ in the exponent of L. Denote by $C_{p,d}$ the maximal constant, and since $L^\theta<L^{\theta+\gamma}$, the estimate on (4.25) using $C_{p,d}$ will imply the same estimate on (4.17) and on (4.18).

Now, for p such that $p>p'(\alpha,s)=\alpha\frac{2d}{s}+12\frac{d}{s}$, we can find $\theta,\gamma>d/s$ for which

$$p>5\theta+3\gamma+2d+(\alpha+1)(\theta s+d)/s,\tag{4.26}$$

so if we set

$$M=L_k^{3\theta+\gamma},\tag{4.27}$$

and recall

$$\epsilon_k^{-(\alpha+1)/2}=C_{p_0,Q_I}L_k^{(\alpha+1)(\theta s+d)/2s},\quad \epsilon_k^{-2}=C'_{p_0,Q_I}L_k^{-2(\theta s+d)/s}.\tag{4.28}$$

we obtain, for k big enough depending on $d,I,p,\alpha,\theta,\gamma,s,p_0,Q_I$,

$$C'_{p,d}L_k^{\theta+\gamma-p/2+d}M^{1/2}\epsilon_k^{-(\alpha+1)}<p_0/24\tag{4.29}$$

and

$$C''_{p,d}\frac{L_k^{\theta+\gamma+d}}{M\epsilon_k^2}<p_0/24,\tag{4.30}$$

so there exists a sequence $L_k\to\infty$ such that for k big enough,

$$\mathbb{P}\left(\frac{1}{2L_k^{\theta+\gamma}}<\|\Gamma_{y,L_k}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_{y,L_k/3}\|\right)<\frac{p_0}{12}.\tag{4.31}$$

The same argument shows that the terms (4.17) and (4.18) are smaller than $p_0/12$, for k big enough.

Inserting this in (4.16)–(4.18) we see that

$$\limsup_{k\to\infty}\sup_y\mathbb{P}\left(\frac{1}{L_k^\theta}<\|\Gamma_{y,L_k}R_{\omega,y,L_k}(E)\chi_{y,L_k/3}\|_{L_k}\right)\le p_0,\tag{4.32}$$

Since $0<p_0<1$ is arbitrary, we conclude that (4.3) holds for each $E\in I$. □

Proof of Theorem 2.7. From Eq. (4.3) to (4.18) the previous proof remains valid in the current setting. We will only estimate (4.16), since the remaining terms (4.17) and (4.18) can be estimated in the same way. Notice that

$$\begin{aligned}\mathbb{P}&\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right)\\ &\leq \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \sum_{u\in\tilde{\Lambda}_{L/3}(y)} \|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right)\\ &\leq \sum_{u\in\tilde{\Lambda}_{L/3}(y)} \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma+d}} < \|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right). \qquad (4.33)\end{aligned}$$

To estimate the r.h.s of the last inequality, the following following lemma is crucial,

Lemma 4.3. *There exists $\mathcal{L} = \mathcal{L}(I, p, \theta, \gamma, d, \alpha, s, p_0, Q_I)$ such that for any $u \in \tilde{\Lambda}_{L/3}(y)$ with $L = L(I, \epsilon)$ as in (4.12), $L \geq \mathcal{L}$ and $E \in I$ fixed, if*

$$p > p(\theta,\gamma,d,\alpha,s) := \alpha\frac{(\theta s+d)}{s} + 9\theta + 3\gamma + 2d + \frac{d}{s}, \qquad (4.34)$$

then, for $T = \epsilon^{-1}$,

$$\begin{aligned}&\left\{\omega : \|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\| > \frac{1}{2L^{\theta+\gamma+d}}\right\}\\ &\quad\subset \{\omega : \mathcal{M}_{u,\omega}(p,\mathcal{X},T) > T^\alpha\}. \qquad (4.35)\end{aligned}$$

Now, if $p > p(\alpha, s) := 15\frac{d}{s} + 2\alpha\frac{d}{s}$, then there exist $\theta, \gamma > d/s$ such that $p > p(\theta, \gamma, d, \alpha, s) > p(\alpha, s)$ so Lemma 4.3 holds yielding, for $L = L(I, \epsilon)$ as in (4.12) big enough,

$$\begin{aligned}\mathbb{P}&\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right)\\ &\leq C_{p_0,Q_I}T^{\frac{s}{2}}\sup_u \mathbb{P}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T) > T^\alpha), \qquad (4.36)\end{aligned}$$

where C_{p_0,Q_I} comes from $L^d = C_{p_0,Q_I}T^{\frac{s}{2}}$, by (4.12).

By hypothesis (2.18), we can pick a sequence $T_k \to \infty$ such that for k big enough

$$T_k^{\frac{s}{2}}\sup_u \mathbb{P}(\mathcal{M}_{u,\omega}(p,\mathcal{X},T_k) > T_k^\alpha) < p_0/12. \qquad (4.37)$$

In an analogous way we can estimate (4.17) and (4.18). It follows that for all $E \in I$ we have

$$\limsup_{k\to\infty}\sup_y \mathbb{P}\left(\frac{1}{L_k^\theta} < \|\Gamma_{y,L_k}R_{\omega,y,L_k}(E)\chi_{y,L_k/3}\|_{L_k}\right) < p_0. \qquad (4.38)$$

Since $0 < p_0 < 1$ is arbitrary, we conclude that (4.3) holds for each $E \in I$. □

Proof of Lemma 4.3. Let $\omega \in \{\omega : \mathcal{M}_{u,\omega}(p, \mathcal{X}, T) \leq T^\alpha\}$. For a given compact subinterval $I \subset J, M > 0$ and $L = L(\epsilon, I)$ as in (4.12), we set

$$A_{u,\omega,M,I} = \{E \in I : \|\langle X - u\rangle^{p/2} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2 \leq M\epsilon^{-(\alpha+1)}\}.$$

We have, using [17, Lemma 6.3]

$$\begin{aligned} |I\backslash A_{u,\omega,M,I}| &\leq \frac{1}{M\epsilon^{-(\alpha+1)}} \int_{\mathbb{R}} \|\langle X - u\rangle^{p/2} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2 \, \mathrm{d}E \\ &= \frac{2\pi}{MT^{\alpha+1}} \int_0^\infty e^{-2t/T} \|\langle X - u\rangle^{p/2} e^{-itH_\omega}\mathcal{X}(H_\omega)\chi_u\|_2^2 \, \mathrm{d}t \\ &= \frac{\pi}{MT^\alpha}\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) \\ &\leq \frac{\pi}{M}, \end{aligned} \tag{4.39}$$

where the last bound is uniform on u and ω.

Thus, for an $E \in I$ fixed either $E \in A_{u,\omega,M,I}$ in which case we have

$$\begin{aligned} \|\Gamma_{y,L} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\| &\leq C_{p,d} L^{-p/2} \|\langle X - u\rangle^{p/2} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2 \\ &\leq C_{p,d} L^{-p/2} M^{1/2} \epsilon^{-(\alpha+1)/2} \end{aligned} \tag{4.40}$$

or else, $E \in I\backslash A_{u,\omega,M,I}$, so by (4.39) there exists $E_{u,\omega} \in A_{u,\omega,M,I}$ such that

$$|E - E_{u,\omega}| \leq \frac{\pi}{M}$$

and therefore, by the resolvent identity and the definition of $A_{u,\omega,M,I}$,

$$\begin{aligned} \|\Gamma_{y,L} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\| &\leq \|\Gamma_{y,L} R_\omega(E_{u,\omega} + i\epsilon)\mathcal{X}(H_\omega)\chi_u\| \\ &\quad + |E - E_{u,\omega}| \|R_\omega(E + i\epsilon)\| \|R_\omega(E_{u,\omega} + i\epsilon)\| \\ &\leq C_{p,d} L^{-p/2} M^{1/2} \epsilon^{-(\alpha+1)/2} + \frac{\pi}{M\epsilon^2}. \end{aligned} \tag{4.41}$$

Now, for p such that $p > p(\theta, \gamma, d, \alpha, s)$ we have

$$2(\theta + \gamma + d) < p - 6\theta - \gamma - (1 + \alpha)(\theta s + d)/s, \tag{4.42}$$

so if we set

$$M = L^{6\theta+\gamma}, \tag{4.43}$$

and recall

$$\epsilon^{-(1+\alpha)/2} = C_{p_0,Q_I} L^{(1+\alpha)(\theta s+d)/2s}, \tag{4.44}$$

we obtain, for L big enough depending on $d, I, p, \alpha, \theta, \gamma, s, p_0, Q_I$,

$$\begin{aligned} C_{p,d} L^{-p/2} M^{1/2} \epsilon^{-(\alpha+1)/2} &= C_{p,d,Q_I,p_0} L^{-(p/2-(6\theta+\gamma)/2-(1+\alpha)(\theta s+d)/2s)} \\ &< \frac{1}{4L^{(\theta+\gamma+d)}} \end{aligned} \tag{4.45}$$

and

$$\frac{\pi}{M\epsilon^2} = C'_{p_0,Q_I} L^{6\theta+2\gamma-2(\theta s+d)/s} < \frac{1}{4L^{(\theta+\gamma+d)}}. \tag{4.46}$$

Inserting this in (4.41) proves the lemma. □

5. Uniform Wegner Estimates for Delone-Anderson Type Potentials

Take $0 < r < R < \infty$ and consider the operator $H_\omega = H_0 + \lambda V_\omega$ with random potential given by

$$V_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u(x - \gamma), \tag{5.1}$$

where D is a (r, R)-Delone set, as in Definition 2.9. The measurable function u, called *single-site potential*, is such that $\| \sum_{\gamma \in D} u(\cdot - \gamma)\|_\infty = 1$, it has compact support and satisfies

$$u^- \chi_{0,\epsilon_u} \leq u \leq u^+ \chi_{0,\delta_u}, \tag{5.2}$$

for some constants $0 < \epsilon_u \leq \delta_u < r < \infty$ and $0 < u^- \leq u^+ < \infty$.

Here, $(\omega_\gamma)_{\gamma \in D}$ is a family of independent random variables, with probability distributions μ_γ of bounded and continuous densities ρ_γ such that

$$\rho_+ := \sup_{\gamma \in D} \|\rho_\gamma\|_\infty < \infty, \tag{5.3}$$

$$0 \in \operatorname{supp} \rho_\gamma \subset [-m_0, M_0], \tag{5.4}$$

where $0 \leq m_0 < \infty, 0 < M_0 < \infty$.

Under these assumptions V_ω is a bounded scalar potential jointly measurable in both $\omega \in \Omega$ and $x \in \mathbb{R}^d$, and so the mapping $\omega \mapsto H_\omega$ is measurable.

Denote by $H_{\lambda,\omega,x,L}$ and $H_{0,x,L}$ the restriction of H_ω and H_0 to the cube $\Lambda_L(x)$ with periodic boundary conditions, respectively (in the particular case of the Landau Hamiltonian, details on the finite volume operator $H_{B,L}$ are stated in Sect. 6), with λ fixed and $V_{\omega,x,L}$ being the restriction of V_ω to $\Lambda_L(x)$, defined by

$$V_{\omega,x,L}(\cdot) = \sum_{\gamma \in D \cap \Lambda_{L-\delta_u}(x)} \omega_\gamma u(\cdot - \gamma). \tag{5.5}$$

and denote by $\tilde{V}_{x,L}$ the potential defined by

$$\tilde{V}_{x,L}(\cdot) = \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} u(\cdot - \gamma), \tag{5.6}$$

where $\tilde{\Lambda}_L(x) = D \cap \Lambda_L(x)$.

We denote by $P_{\lambda,\omega,x,L}, P_{0,x,L}$ the spectral projector associated to the finite volume operators $H_{\lambda,\omega,x,L}, H_{0,x,L}$, respectively. In the particular case of the finite volume random Landau Hamiltonian and free Landau Hamiltonian, we write $H_{B,\lambda,\omega,x,L}$ and $H_{B,x,L}$, respectively, and we use the notation $\Pi_{n,x,L}$ for the spectral projector associated to the n-th Landau level, and $\Pi^\perp_{n,x,L}$ for its orthogonal projector (see Sect. 6.1). Define $s(\epsilon) = \sup_{\gamma \in D} \sup_{E \in \mathbb{R}} \mu_\gamma$ $([E, E + \epsilon])$.

We prove several Wegner estimates that we summarize in the following theorem,

Theorem 5.1. (*i*) *For $d = 2$, let H_0 be the Landau Hamiltonian with constant magnetic field $B > 0$ fixed. For any bounded interval $I \in \mathbb{R}$ there exist constants $Q_W = Q_W(B, \lambda, R, r, I, u, m_0, M_0), \eta_{B,\lambda,J} \in]0, 1]$ and a finite scale $\mathcal{L}_*(B, \lambda, I, R)$ such that for every compact subinterval $J \subset I$, with $|J| < \eta_{B,\lambda,J}$ and $L > \mathcal{L}_*$, we have*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\{\mathrm{tr}P_{\lambda,\omega,x,L}(J)\} \leq Q_W \rho_+ s(|J|)L^d. \tag{5.7}$$

(*ii*) *Let $E_0 \in \mathbb{R}\backslash\sigma(H_0)$ for $H_0 = -\Delta + V_0$, where V_0 is $\mathbb{Z}^d$-periodic. For any bounded interval $I \subset \mathbb{R}\backslash\sigma(H_0)$ there exist a constant $Q_W = Q_W(\lambda, R, r, I, u)$ and a finite scale $\mathcal{L}_*(R)$ such that for every compact subinterval $J \subset I$, (5.7) holds.*

(*iii*) *Assume the IDS of H_0 is Hölder continuous with exponent $\delta > 0$ in some open interval I and no further assumption on $s(\epsilon)$. Then there exists a constant $Q'_W = Q'_W(B, \lambda, I, u, R, r, d) > 0$ such that for all compact subintervals $J \subset I$ with $|J|$ small enough, and $0 < \gamma < 1$,*

$$\mathbb{E}\{\mathrm{tr}P_{\lambda,\omega,x,L}(J)\} \leq Q'_W \max\{|J|^{\delta\gamma}, |J|^{-2\gamma}s(|J|)\}L^d. \tag{5.8}$$

In particular, if $s(\epsilon) \leq C\epsilon^\zeta$, for some $\zeta \in [0, 1]$, then

$$\mathbb{E}\{\mathrm{tr}P_{\lambda,\omega,x,L}(J)\} \leq Q'_W |J|^{\frac{\zeta\delta}{\delta+2}} L^d. \tag{5.9}$$

Remark 5.2. We point out that Theorem 5.1 (*i*) provides a Wegner estimate at all energies, in particular, at Landau levels. Compare this result to Theorem 6.2. Both these results are necessary to prove the transport transition of Theorem 2.11. Theorem 6.2 is used to prove localization at the band edges, while Theorem 5.1 (*i*) is needed to apply Theorem 2.6.

Since the results are uniform in x, we state them for $x = 0, \lambda$ fixed and for simplicity we omit these subscripts from the notation.

For the proof we follow [10], based on [9], plus [11,19] in the case of the Landau Hamiltonian. In all cases we need to estimate $\mathbb{E}\{\mathrm{tr}P_{\omega,L}(J)\}$. We decompose it with respect to the free spectral projector of an interval $\tilde{J}$, such that $J \subset \tilde{J}$ and $d_J = \mathrm{dist}(J, \tilde{J}^c) > 0$, that is

$$\mathrm{tr}P_{\omega,L}(J) = \mathrm{tr}P_{\omega,L}(J)P_{0,L}(\tilde{J}) + \mathrm{tr}P_{\omega,L}(J)P_{0,L}(\tilde{J}^c). \tag{5.10}$$

The key step in estimating the first term of the r.h.s is to prove a positivity estimate as in [10, Theorem 2.1]. In order to obtain this estimate in the case of the Landau Hamiltonian, we need some preliminary lemmas.

Lemma 5.3. *Using the notations above, there exists a positive finite constant $C_n(B, u, R)$, so that*

$$\Pi_{n,L}\tilde{V}_{x,L}\Pi_{n,L} \geq C_n(B, u, R)\Pi_{n,L}. \tag{5.11}$$

Proof. From [11] we have that for $n \in \mathbb{N}, \tilde{R} > 0$, for each $0 < \epsilon < \tilde{R}, \kappa > 1$ and $\eta > 0$ there exists a constant $C_0 = C_{0,n,\epsilon,\tilde{R},\eta} > 0$ such that

$$\Pi_n\chi_{0,\epsilon}\Pi_n \geq C_0(\Pi_n\chi_{0,\tilde{R}}\Pi_n - \eta\Pi_n\chi_{0,\kappa\tilde{R}}\Pi_n). \tag{5.12}$$

Because of the invariance of H_B under the magnetic translations (6.1) we have that the projections Π_n commute with these unitary operators, which in turn gives, for an arbitrary $x \in \mathbb{R}^2$,

$$U_x \Pi_n \chi_{0,\epsilon} \Pi_n U_x^* \geq C_0 U_x (\Pi_n \chi_{0,\tilde{R}} \Pi_n - \eta \Pi_n \chi_{0,\kappa\tilde{R}} \Pi_n) U_x^* \tag{5.13}$$
$$\Pi_n U_x \chi_{0,\epsilon} U_x^* \Pi_n \geq C_0 (\Pi_n U_x \chi_{0,\tilde{R}} U_x^* \Pi_n - \eta \Pi_n U_x \chi_{0,\kappa\tilde{R}} U_x^* \Pi_n) \tag{5.14}$$
$$\Pi_n \chi_{x,\epsilon} \Pi_n \geq C_0 (\Pi_n \chi_{x,\tilde{R}} \Pi_n - \eta \Pi_n \chi_{x,\kappa\tilde{R}} \Pi_n), \tag{5.15}$$

since conjugation by unitary operators is a positivity preserving operation.

Now, we recall [19, Lemma 5.3] (which is independent of V and, therefore, D).

Lemma 5.4. *Fix $B > 0, n \in \mathbb{N}, \tilde{R} > 0, 0 < \epsilon < \tilde{R}$ and $\eta > 0$. If $\kappa > 1$ and $L \in \mathbb{N}_B$ (defined as in (6.3)) are such that $L > 2(L_B + \kappa\tilde{R})$ then for all $\tilde{x} \in \Lambda_L(x)$, we have*

$$\Pi_{n,L} \hat{\chi}_{\tilde{x},\epsilon} \Pi_{n,L} \geq C_0 \Pi_{n,L} (\hat{\chi}_{\tilde{x},\tilde{R}} - \eta \hat{\chi}_{\tilde{x},\kappa\tilde{R}}) \Pi_{n,L} + \Pi_{n,L} \mathcal{E}_{n,\tilde{x},L} \Pi_{n,L}, \tag{5.16}$$

where $C_0 = C_{0;n,B,\epsilon,\tilde{R},\eta} > 0$ is a constant as before and the error operator $\mathcal{E}_{n,\tilde{x},L}$ satisfies

$$\|\mathcal{E}_{n,\tilde{x},L}\| \leq C_{n,B,\epsilon,R,\eta} e^{-m_{n,B} L}, \tag{5.17}$$

for some positive constant $m_{n,B}$.

Now, by (5.2) we have

$$\tilde{V}_{x,L}(\cdot) = \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} u(\cdot - \gamma) \geq u^- \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \hat{\chi}_{\gamma,\epsilon_u}. \tag{5.18}$$

We fix $\tilde{R} > 2R + \delta_u$, in which case

$$\sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \hat{\chi}_{\gamma,\tilde{R}} \geq \chi_{x,L}. \tag{5.19}$$

Now fix $\kappa > 1$ and pick $\eta > 0$ such that

$$\eta \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \hat{\chi}_{\gamma,\kappa\tilde{R}} \leq \frac{1}{2} \chi_{x,L}. \tag{5.20}$$

It follows from Lemma 5.4, (5.19) and (5.20) that

$$\Pi_{n,L} \tilde{V}_{x,L} \Pi_{n,L} \geq u^- C_0 \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \Pi_{n,L} (\hat{\chi}_{\gamma,\tilde{R}} - \eta \hat{\chi}_{\gamma,\kappa\tilde{R}}) \Pi_{n,L} + \Pi_{n,L} \mathcal{E}_{n,L} \Pi_{n,L} \tag{5.21}$$
$$\geq \frac{u^- C_0}{2} \Pi_{n,L} + \Pi_{n,L} \mathcal{E}_{n,L} \Pi_{n,L} \tag{5.22}$$
$$\geq C_1 \Pi_{n,L}, \tag{5.23}$$

for $L \geq L^*$ for some $L^* = L^*_{n,B,\epsilon,R,\kappa,\eta} < \infty$ and $C_1 = \frac{u^- C_0}{4}$, since the error operator

$$\Pi_{n,L} \mathcal{E}_{n,L} \Pi_{n,L} = \Pi_{n,L} \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \mathcal{E}_{n,\gamma,L} \Pi_{n,L}$$

by (5.17), satisfies

$$\|\mathcal{E}_{n,L}\| \leq L^2 C_{n,B,\epsilon,R,\eta} e^{-m_{n,B}L}.$$

□

Finally we recall,

Lemma 5.5 [10, Lemma 2.1]. *Suppose that T is a trace class operator independent of ω and u, the single site potential (5.2). We then have*

$$\mathbb{E}\{\mathrm{tr} P_{\omega,L}(J) u_i T u_j\} \leq 8s(|J|)\|u_i T u_j\|_1. \tag{5.24}$$

where we use the notation $u_i = u(x-i), i \in \mathbb{R}^2$.

Proof of Theorem 5.1. To prove (i), using the preliminary lemmas we can follow the proof in [10, Theorem 4.3]. Notice that the spatial homogeneity of the Delone set in the sense that points do not accumulate neither are too far away, so the sums over indexes of elements of D preserves the properties of the sums over indexes of elements of the lattice $\mathbb{Z}^2$ as the original proofs.

Recall that we need to estimate $\mathbb{E}\{\mathrm{tr} P_{\omega,L}(J)\}$ as in (5.10), that is, for an arbitrary $E_0 \in \mathbb{R}$, with J and $\tilde{J}$ closed bounded intervals centered on E_0 such that $J \subset \tilde{J}, |J| < 1, d_J > 0$, we need to estimate

$$\mathrm{tr} P_{\omega,L}(J) = \mathrm{tr} P_{\omega,L}(J)\Pi_{n,L} + \mathrm{tr} P_{\omega,L}(J)\Pi_{n,L}^{\perp}. \tag{5.25}$$

a. Estimate on $\mathbb{E}\{\mathrm{tr} P_{\omega,L}(J)\Pi_{n,L}^{\perp}\}$.

The analysis in [10, Eq. 2.6–2.10] for the n-th Landau band remains valid taking, for the constants defined therein, $M = 1$ and the operator K defined by

$$K \equiv \left(\frac{H_{B,L}+1}{H_{B,L}-E_m}\right)^2, \quad \|K\| \leq K_n \equiv \left(1 + \frac{1+J_+}{d_n}\right)^2, \tag{5.26}$$

where E_m is an eigenvalue of $H_{B,\lambda,\omega,L}, d_n \equiv \min\{\mathrm{dist}(I, B_{n-1}), \mathrm{dist}(I, B_{n+1}\}$ and $J = [J_-, J_+]$.

Then we can obtain the analog of [10, Eq. 4.4],

$$\mathrm{tr} P_{\omega,L}(J)\Pi_{n,L}^{\perp} \leq K_n \lambda^2 \max\{m_0, M_0\}^2 \sum_{i,j\in\tilde{\Lambda}} |\mathrm{tr}\ u_j P_{\omega,L}(J) u_i K_{ij}|, \tag{5.27}$$

where $K_{ij} \equiv \chi_i (H_{B,L}+1)^{-2}\chi_j$, for $\chi \geq 0$ a smooth function of compact support slightly larger than the support of u such that $\chi u = u$. Note that due to the spatial homogeneity of D and the fact that supp u is contained in a cube of side r, the translated supports of u do not overlap.

Now, denote by $\tilde{\Lambda}_0 = \{i, j \in \tilde{\Lambda}/\chi_i\chi_j = 0\}$ and by $\tilde{\Lambda}_0^c = \{i, j \in \tilde{\Lambda}/\chi_i\chi_j \neq 0\}$. For $i, j \in \tilde{\Lambda}_0$, the operator K_{ij} is trace class [5, Lemma 2.2], [10, Lemma 5.1] and it satisfies the Combes–Thomas estimate,

$$\|K_{ij}\|_1 = \|\chi_i (H_{B,L}+1)^{-2}\chi_j\|_1 \leq C_0' e^{-\tilde{C}_0\|i-j\|}, \tag{5.28}$$

where C_0' and $\tilde{C}_0$ are positive constants. So we can use Lemma 5.5 to obtain

$$\mathbb{E}\{|\sum_{i,j\in\tilde{\Lambda}_0} \text{tr } u_j P_{\omega,L}(J) u_i K_{ij}|\} \leq \mathbb{E}\{\sum_{i,j\in\tilde{\Lambda}_0} |\text{tr } u_j P_{\omega,L}(J) u_i K_{ij}|\} \quad (5.29)$$

$$\leq C_0 8 s(|J|) \sum_{i,j\in\tilde{\Lambda}_0} e^{-\tilde{C}_0\|i-j\|} \quad (5.30)$$

$$\leq C_1 s(|J|)|\Lambda|. \quad (5.31)$$

where C_1 also depends on r, since $\sharp(\tilde{\Lambda}_L) \leq C_{r,d} L^d$ for $L > R$, see Eq. (6.21).

On the other hand, for $i, j \in \tilde{\Lambda}_0^c, K_{ij}$ is also trace class [5, Lemma 2.2] so we can apply Lemma 5.5 again, obtaining

$$\mathbb{E}\{\text{tr} P_{\omega,L}(J)\Pi_{n,L}^{\perp}\} \leq C_2 s(|J|)|\Lambda|, \quad (5.32)$$

where $C_2 > 0$ depends on u, I, λ, r and $M = \max\{m_0, M_0\}$.

b. Estimate on $\mathbb{E}\{\text{tr}P_{\omega,L}(J)\Pi_{n,L}\}$.

We use the spectral projector $\Pi_{n,L}$ in order to control the trace. Here the key ingredient is the positivity estimate (5.11) and the fact that, under our hypotheses on u, there exists a finite constant C_u, depending on u only, such that

$$0 < \tilde{V}_L^2 \leq C_u \tilde{V}_L.$$

Now,

$$\text{tr}P_{\omega,L}(J)\Pi_{n,L} \leq \frac{1}{C_n(B,u,R)} \text{tr}P_{\omega,L}(J)\Pi_{n,L}\tilde{V}_L\Pi_{n,L} \quad (5.33)$$

$$\leq \frac{1}{C_n(B,u,R)}\{\text{tr}P_{\omega,L}(J)\tilde{V}_L\Pi_{n,L} - \text{tr}P_{\omega,L}(J)\Pi_{n,L}^{\perp}\tilde{V}_L\Pi_{n,L}\}. \quad (5.34)$$

Then we can proceed as in parts (2) and (3) of the proof of [10, Theorem 4.3], and we finally arrive to the desired result,

$$\mathbb{E}\{\text{tr}P_{\omega,L}(J)\} \leq Q_W s(|J|)|\Lambda|. \quad (5.35)$$

where the constant $Q_W > 0$ depends on B, u, R, r, I, λ and M.

As for (ii), note that in this case $\text{tr}P_{0,L}(\tilde{J}) = 0$ if $\tilde{J} \subset \mathbb{R}\backslash\sigma(H_0)$, so we only need to estimate the second term in the r.h.s. of (5.10), where we do not need the positivity estimate (5.11) for $P_{0,L}$. The proof mimics (i)-a.

In case (iii) we can estimate the first term in the r.h.s. of (5.10) without using the analog of (5.11) for $P_{0,L}$. Instead, the Hölder continuity of the IDS of the non perturbed operator implies that there exists a constant $C > 0$ such that

$$\text{tr}P_{0,L}(\tilde{J}) \leq C|\tilde{J}|^{\delta}|\Lambda|,$$

and so, for $0 < \gamma < 1$

$$\text{tr}P_{\omega,L}(J)P_{0,L}(\tilde{J}) \leq C|J|^{\gamma\delta}|\Lambda|. \quad (5.36)$$

Since, as in the previous case (writing explicitly the dependence on d_J) we have

$$\mathbb{E}\{\mathrm{tr} P_{\omega,L}(J) P_{0,L}(\tilde{J}^c)\} \leq \frac{Q'_W}{d_J^2} s(|J|)|\Lambda|,$$

by taking $d_J = |J|^\gamma$ we obtain the desired result. Furthermore, if $s(\epsilon)$ is ζ-Hölder continuous, we get, taking γ such that $\gamma\delta = \zeta - 2\gamma$,

$$\mathbb{E}\{\mathrm{tr} P_{\omega,L}(J)\} \leq Q'_W \max\{|J|^{\gamma\delta}, |J|^{\zeta-2\gamma}\} L^2 \tag{5.37}$$

$$\leq Q'_W |J|^{\frac{\zeta\delta}{\delta+2}} L^2, \tag{5.38}$$

where Q'_W depends on u, I, λ, R, r and M. □

6. Applications to Non Ergodic Random Landau Operators: Proof of Theorem 2.11

6.1. The Model

Recall the operator $H_{\lambda,B,\omega} = H_B + \lambda V_\omega$, where H_B is given by (2.20), V_ω is the Delone-Anderson potential (2.21), satisfying conditions (5.2)–(5.4) with $\rho_\gamma = \rho$, for all $\gamma \in D$. We assume the random variables are identically distributed and that the single-site potential u satisfies the additional conditions:

(*uc.*) $\delta_u < r/10$, i.e., u has compact support contained in $B(0, r/10)$. This implies that for $i, j \in D$ with $i \neq j$, $\mathrm{supp}\ u_i \cap \mathrm{supp}\ u_j = \emptyset$, where we use the notation $u_i = u(\cdot - i)$ for $i \in \mathbb{R}^2$.

(*u0.*) $\|u\|_\infty = 1$ and $u(0) = 1$.

We define the magnetic translations U_a for $a \in \mathbb{R}^2$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, by

$$U_a \varphi(x) = e^{-i\frac{B}{2}(x_2 a_1 - x_1 a_2)} \varphi(x - a), \tag{6.1}$$

obtaining a projective unitary representation of $\mathbb{R}^2$ on $\mathrm{L}^2(\mathbb{R}^2)$:

$$U_a U_b = e^{i\frac{B}{2}(a_2 b_1 - a_1 b_2)} U_{a+b} = e^{iB(a_2 b_1 - a_1 b_2)} U_b U_a, \quad a, b \in \mathbb{R}^2. \tag{6.2}$$

We then have $U_a H_B U_a^* = H_B$ for all $a \in \mathbb{R}^2$.

We now define finite volume operators following [19]. For $B > 0$, we set

$$K_B = \min\left\{k \in \mathbb{N} : k \geq \sqrt{\frac{B}{4\pi}}\right\} \quad \text{and} \quad L_B = K_B \sqrt{\frac{B}{4\pi}}. \tag{6.3}$$

We denote $\mathbb{N}_B = L_B \mathbb{N}$, $\tilde{\mathbb{N}}_B = \mathbb{N}_B \cup \{\infty\}$ and $\mathbb{Z}_B^2 = L_B \mathbb{Z}^2$.

We consider squares $\Lambda_L(x)$ with $L \in \mathbb{N}_B$ and $x \in \mathbb{R}^2$, and identify them with the torii $\mathbb{T}_{L,x} := \mathbb{R}^2/(L\mathbb{Z}^2 + x)$. We denote by $\chi_{x,L}$ the characteristic function of the cube $\Lambda_L(x)$ and for $\tilde{x} \in \Lambda_L(x)$ and $\tilde{L} < L$ we denote by $\hat{\Lambda}_{\tilde{L}}(\tilde{x})$ and $\hat{\chi}_{\tilde{x},\tilde{L}}$ the cube and characteristic function in $\mathbb{T}_{L,x}$.

For the first order differential operator $\mathbf{D}_B = (-i\nabla - \mathbf{A})$ restricted to $\mathcal{C}_c^\infty(\Lambda_L(x))$ we take its closed, densely defined extension $\mathbf{D}_{B,x,L}$ from $\mathrm{L}^2(\Lambda_L(x))$ to $\mathrm{L}^2(\Lambda_L(x); \mathbb{C}^2)$, with periodic boundary conditions and then set $H_{B,x,L} = \mathbf{D}^*_{B,x,L} \mathbf{D}_{B,x,L}$.

We are left with the operator $H_{B,\lambda,\omega,x,L}$ acting on $\mathrm{L}^2(\Lambda_L(x))$ defined by

$$H_{B,\lambda,\omega,x,L} = H_{B,x,L} + \lambda V_{\omega,x,L}. \tag{6.4}$$

where $V_{\omega,x,L}$ is defined as in (5.5).

We write $R_L(z) = (H_{B,\lambda,\omega,x,L} - z)^{-1}$ for the resolvent operator of $H_{B,\lambda,\omega,x,L}$.

Since $H_{B,x,L}$ has a compact resolvent, its spectrum consists of the Landau Levels but now with finite multiplicity. We denote by $\Pi_{n,L}$ the orthogonal projection associated to the n-th Landau level and define $P_{B,\lambda,\omega,x,L}(J) = \chi_J(H_{B,\lambda,\omega,x,L})$ for $J \subset \mathbb{R}$ a Borel set.

This operator satisfies the compatibility conditions [19, Eq. 4.2]: If $\varphi \in \mathcal{D}(\mathbf{D}_{B,x,L})$ with supp $\varphi \subset \Lambda_{L-\delta_u}(x)$, then $\mathcal{I}_{x,L}\varphi \in \mathcal{D}(\mathbf{D}_B)$ and

$$\begin{aligned} \mathcal{I}_{x,L}\mathbf{D}_{B,x,L}\varphi &= \mathbf{D}_B\mathcal{I}_{x,L}\varphi, \\ \mathcal{I}_{x,L}\chi_{x,L-\delta_u}V_{\omega,x,L} &= \chi_{x,L-\delta_u}V_\omega, \end{aligned} \tag{6.5}$$

where $\mathcal{I}_{x,L} : \mathrm{L}^2(\Lambda_L(x)) \to \mathrm{L}^2(\mathbb{R}^2)$ is the canonical injection

$$\mathcal{I}_{x,L}\varphi(y) = \begin{cases} \varphi(y) & \text{if } y \in \Lambda_L(x) \\ 0 & \text{otherwise.} \end{cases}$$

From this we have

$$\mathcal{I}_{x,L}H_{B,\lambda,\omega,x,L}\varphi = H_{B,\lambda,\omega}\mathcal{I}_{x,L}\varphi,$$

that is, the finite volume operators $H_{B,\lambda,\omega,x,L}$ agree with $H_{B,\lambda,\omega}$ inside the square $\Lambda_L(x)$.

However, $H_{B,\lambda,\omega,x,L}$ does not satisfy the covariance condition (2.3) so we have a priori

$$H_{B,\lambda,\omega,x,L} \neq U_x H_{B,\lambda,\tau_{-x}(\omega),0,L} U_x^*,$$

where U_x is the magnetic translation (6.1) seen as a unitary map from $\mathrm{L}^2(\Lambda_L(0))$ to $\mathrm{L}^2(\Lambda_L(x))$ and τ_x is the translation defined as $\tau_x(\omega_\gamma) = \omega_{\gamma-x}$ for $x \in \mathbb{R}^2$.

6.2. Dynamical Localization in Landau Bands

In this section we prove the following

Theorem 6.1. *Let H_ω be as before. For any $n = 0, 1, 2, \ldots$ there exist finite positive constants $\mathbf{B}(n)$ and $K_n(\lambda)$ depending only on n, M, u and ρ such that for all $B \geq \mathbf{B}(n)$ we can perform MSA in the intervals*

$$\Sigma_{B,n,\lambda,\omega} = \sigma_{B,\lambda,\omega} \cap \left\{ E \in \mathcal{B}_n : |E - B_n| \geq K_n(\lambda)\frac{\log B}{B} \right\}, \tag{6.6}$$

We have strong HS-dynamical localization at energy levels up to a distance $K_n(\lambda)\frac{\log B}{B}$ from the Landau levels for large B.

For the proof we need to verify the conditions to start the modified Multiscale Analysis, Theorem 2.3. As mentioned in the proof of Theorem 2.3, this model satisfies properties IAD, R, EDI, SLI and UNE. What is left to prove is the existence of a suitable length scale L_0 that satisfies (2.7) and UWE. The latter comes from the following improvement in the Wegner estimate of the previous section and it follows [8, Theorem 3.1].

Theorem 6.2. *There exists* $\tilde{B} > 0$ *and a constant* $Q_n = \tilde{Q}_{n,\lambda,u}\|\rho\|_\infty$ *such that for all* $B > \tilde{B}$ *and for any closed interval* $I \subset \mathcal{B}_n \backslash \sigma(H_B)$

$$\mathbb{E}\{\mathrm{tr} P_{B,\lambda,\omega,x,L}(I)\} \leq Q_n \frac{B}{2(\mathrm{dist}(I, B_n))^2} |I| L^2. \tag{6.7}$$

In particular, for $E_0 \notin \sigma(H_B)$ *and all* $0 < \epsilon < |E_0 - B_n|$,

$$\mathbb{P}\{\mathrm{dist}(\sigma(H_{B,\lambda,\omega,x,L}), E_0) \leq \epsilon\} \leq Q_n \frac{B}{(|E_0 - B_n| - \epsilon)^2} \epsilon L^2. \tag{6.8}$$

Proof. Without loss of generality we work within the first Landau band $\mathcal{B}_0$, containing the Landau level B_0. Set $M = \|V_\omega\|_\infty = \max\{m_0, M_0\}$. Let I be an interval such that $I \subset \mathcal{B}_0 \backslash \{B_0\}$ and $\inf I > B_0$, so $\mathrm{dist}(I, B_0) > 0$.

Following the same arguments in [8, Eq. 3.4–3.11], we get

$$\mathbb{E}\{\mathrm{tr} P_L(I)\} < \mathrm{dist}(I, B_0)^{-2} M^2 \|\rho\|_\infty |I| \sum_{i,j \in D} \|\Pi_{0,L}^{ij}\|_1, \tag{6.9}$$

where $P_L(I)$ stands for $P_{B,\lambda,\omega,x,L}(I)$ and we use the notation $A^{ij} = u_i^{1/2} A u_j^{1/2}$ for any bounded operator A.

To evaluate the sum we consider separately the indices i, j for which $\|i - j\| < 4\delta_u$ and those for which $\|i - j\| \geq 4\delta_u$, with δ_u as in (5.2).

Let χ_{ij} be the characteristic function of $\mathrm{supp}(u_i + u_j)$. Again, as in Theorem 5.1, the translated supports of u behave in a similar way as in the lattice. Then we follow the same arguments therein and obtain, using [8, Lemma 2.1],

$$\sum_{|i-j|<4\delta_u} \|\Pi_{0,L}^{ij}\|_1 \leq \|u\|_\infty^2 \sum_{|i-j|<4\delta_u} \|\chi_{ij} \Pi_{0,L} \chi_{ij}\|_1 \leq C_0 B |\Lambda| |\mathrm{supp}\ u|, \tag{6.10}$$

where the constant C_0 actually depends on the index n of the Landau level, which in this case is 0.

Define χ_{ij}^+ to be the characteristic function of the set $\{x \in \mathbb{R}^2 : \|x - i\| < \|x - j\|\}$ and denote $\chi_{ij}^- = 1 - \chi_{ij}^+$. Then we obtain

$$\|\Pi_{0,L}^{ij}\|_1 \leq \|u_j^{1/2} \Pi_{0,L} \chi_{ij}^+\|_2 \|\chi_{ij}^+ \Pi_{0,L} u_i^{1/2}\|_2 + \|u_j^{1/2} \Pi_{0,L} \chi_{ij}^-\|_2 \|\chi_{ij}^- \Pi_{0,L} u_i^{1/2}\|_2.$$

Now, if $|i - j| \geq 4\delta_u$, condition (5.2) implies that

$$\mathrm{dist}(\mathrm{supp}\ \chi_{ij}^+, \mathrm{supp}\ u_j) \geq \frac{\|i - j\|}{2} - \delta_u \geq k\|i - j\|$$

for some $k > 0$. Similarly for $\mathrm{dist}(\mathrm{supp}\ \chi_{ij}^-, \mathrm{supp}\ u_i)$. We then obtain

$$\sum_{|i-j|\geq 4\delta_u} \|\Pi_{0,L}^{ij}\|_1 \leq C_1 |\mathrm{supp}\ u| |\Lambda|. \tag{6.11}$$

Combining (6.9), (6.10) and (6.11) we obtain

$$\mathbb{E}\{\mathrm{tr} P_L(I)\} \leq Q_0 (\mathrm{dist}(I, B_0))^{-2} \|\rho\|_\infty \epsilon B |\Lambda|,$$

where the constant Q_0 depends on $\lambda, M, \|u\|_\infty$ and supp u. Taking $I = [E_0 - \epsilon, E + \epsilon]$ for small $\epsilon > 0$ and applying Chebyshev's inequality we obtain (6.8). □

As for the initial length scale estimate (2.7) to start the multiscale analysis, we need to verify that for some $L_0 \in 6\mathbb{N}$ sufficiently large (as specified in [16]), given $\theta > 0, E \in \mathbb{R}\backslash\sigma(H_{B,L})$,

$$\mathbb{P}\left\{\|\Gamma_{x,L_0} R_{B,\omega,x,L_0}(E)\chi_{x,L_0/3}\| \leq \frac{1}{L_0^\theta}\right\} > 1 - \frac{1}{L_0^p}, \tag{6.12}$$

for a suitable choice of p, where $\Gamma_{x,L} = \chi_{\bar{\Lambda}_{L-1}(x)\backslash\Lambda_{L-3}(x)}$.

To do so we follow the approach [8] to obtain estimates that we will later state as in [16]. We need to show that in the annular region between a box of side $L/3$ and L, there exists a closed, connected ribbon where the potential V satisfies the condition $|V(x) + B_n - E| > a > 0$, for $E \neq B_n$ with a good probability ([8, Eq. 4.2]). To prove this, Combes and Hislop used bond percolation theory, defining occupied bonds of the lattice as those bonds where the potential satisfies this property. However, in our case there is no need to use percolation theory since this fact is assured by the assumption (uc) on the single-site potential. More precisely, we will show that there exist ribbons where the potential is zero almost surely.

Let us consider the *Voronoi diagram* associated to D [32]. Since $\tilde{\Lambda}_L = D \cap \Lambda_L$ is a discrete bounded set, we can write $\tilde{\Lambda}_L = \{p_1, \ldots, p_n\}, n \in \mathbb{N}$. For each site p_i we consider its *Voronoi cell*, defined as

$$\mathcal{V}(p_i) = \{x \in \mathbb{R}^2 : \|x - p_i\| \leq \|x - p_j\|, j \neq i, 1 \leq j \leq n\},$$

i.e., the set of points that are closer to p_i than to any other site in $\tilde{\Lambda}_L$. The Voronoi diagram associated to $\tilde{\Lambda}_L$, denoted by $\mathcal{V}or(\tilde{\Lambda}_L)$ is a subdivision of Λ_L into Voronoi cells,

$$\mathcal{V}or(\tilde{\Lambda}_L) = \bigcup_{1\leq i\leq n} \mathcal{V}(p_i).$$

The edges and vertices of $\mathcal{V}or(\tilde{\Lambda}_L)$ are polygonal connected lines with the property that the minimal and maximal distances from any site p_i to an edge or vertex are $r/4$ and $R/\sqrt{2}$, respectively.

Now, take a covering of $\Lambda_{L/3}$ by a finite collection of Voronoi cells, $\mathcal{V}_\Lambda$, which is a convex polygon. Its perimeter is a polygonal line $\mathcal{C}$ that encloses $\Lambda_{L/3}$ such that $\mathcal{C} \cap D = \emptyset$. Taking L big enough with respect to R we have $\mathcal{C} \subset \Lambda_{L-3}\backslash\Lambda_{L/3}$. Moreover, assumption ($uc$) implies that we can always find a ribbon $\mathcal{R}$ associated to $\mathcal{C}$, i.e., a set

$$\mathcal{R} = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathcal{C}) < \frac{r}{4} - \frac{r}{10}\},$$

such that $V(x) = 0$ for all $x \in \mathcal{R}$ (see Fig. 1)

Then, condition [8, Eq. (4.2)] holds almost surely, therefore [8, Corollary 4.1] holds almost surely, and this implies (see [8, Proposition 5.1], [16, Theorem 4.3])

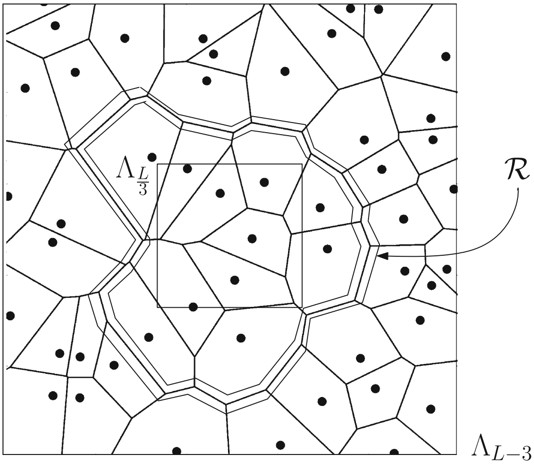

FIGURE 1. Ribbon $\mathcal{R}$ in the Voronoi diagram associated to D. *Points* represent the support of the Delone-Anderson potential

Theorem 6.3. *Let $E = B_n \pm 2a$ for some $n = 0, 1, 2 \ldots$ with $0 < 2a < B$. There exists constants $Y_n, \beta_n > 0$ depending only on n, M, u, δ_u such that for any $0 < \epsilon \leq a, L \in 6\mathbb{N}$ and Q_n as in the previous theorem,*

$$\mathbb{P}\left\{\|\Gamma_{x,L} R_{B,\omega,x,L}(E)\chi_{x,L/3}\| \leq Y_n \frac{B}{a\epsilon^2} e^{-\beta_n min\{aB,\sqrt{B}\}}\right\} > 1 - Q_n \frac{B\epsilon}{a^2} L^2. \quad (6.13)$$

Therefore, to satisfy (6.12) we need only to verify the conditions

$$Y_n \frac{B}{a\epsilon^2} e^{-\beta_n \min\{aB,\sqrt{B}\}} \leq \frac{1}{L_0^{\theta}}, \quad (6.14)$$

$$Q_n \frac{B\epsilon}{a^2} L_0^2 \leq \frac{1}{L_0^{p}}, \quad (6.15)$$

which can be done in the same way as in the proof of [16, Theorem 4.1], yielding Theorem 6.1.

6.3. Dynamical Delocalization in Landau Bands

Theorem 6.4. *Under the disjoint bands condition (2.23) the random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits dynamical delocalization in each Landau band $\mathcal{B}_n(B, \lambda)$, i.e., for all $n = 1, 2, \ldots$,*

$$\Xi^{\mathrm{DD}} \cap \sigma_{B,\lambda,\omega} \cap \mathcal{B}_n(B, \lambda) \neq \emptyset. \quad (6.16)$$

In particular, there exists at least one energy $E_{n,\omega}(B, \lambda) \in \mathcal{B}_n(B, \lambda)$ such that for every $\mathcal{X} \in C^{\infty}_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \ni E_{n,\omega}(B, \lambda)$ and $p > 0$, we have

$$\mathcal{M}_{B,\lambda}(p, \mathcal{X}, T) \geq C_{p,\mathcal{X}} T^{\frac{p}{4}-6}, \quad (6.17)$$

for all $T \geq 0$ with $C_{p,\mathcal{X}} > 0$.

This is a consequence of the quantization of the Hall conductance in each Landau band and the fact that in regions of dynamical localization, the Hall conductance is constant, as proven in [19, Section 3]. We recall the main lines of their strategy.

Consider the switch function $h(t) = \chi_{[\frac{1}{2},\infty)}(t)$ and let h_j denote the multiplication by the function $h(x_j), j = 1, 2$. The Hall conductance is defined as

$$\begin{aligned}\sigma_{H_\omega}(B, \lambda, E) &= -2\pi i \Theta(P_{B,\lambda,\omega,E}) \\ &:= \operatorname{tr}\{P_{B,\lambda,\omega,E}[[P_{B,\lambda,\omega,E}, h_1], [P_{B,\lambda,\omega,E}, h_2]]\}, \qquad (6.18)\end{aligned}$$

where $P_{B,\lambda,\omega,E} := P_{B,\lambda,\omega}((-\infty, E])$.

Following the proof of [19, Lemma 3.2] we see that the Hall conductance is constant in connected components of the dynamical localization region, where property SUDEC is valid, as consequence of Theorem 2.3. On the other hand, it is well known that for $\lambda = 0, \sigma_{H_\omega}(B, \lambda, E) = n$ if $E \in (B_n, B_{n+1})$ for all $n = 0, 1, 2, \ldots$. Under the disjoint bands condition (2.23), if $E \in \mathcal{G}_n(B, \lambda_*)$ for λ_* and some $n \in \{0, 1, 2, \ldots\}$, we can find some $\lambda_E > \lambda_*$ such that $E \in \mathcal{G}_n(B, \lambda)$ for all $\lambda \in [0, \lambda_E]$. That is, the spectral gaps stay open as λ increases. Then we prove along the lines of [19, Lemma 3.3] that $\sigma_{H_\omega}(B, \lambda, E) = n$ if $E \in \mathcal{G}_n(B, \lambda)$, for all $[0, \lambda_E]$. As the spectral gaps $\mathcal{G}_n(B, \lambda)$ are by definition part of the localization region, this implies that the Hall conductance has the same value in different gaps, which is a contradiction. Therefore, we must have $\Xi^{\mathrm{DD}} \cap \sigma_{B,\lambda,\omega} \cap \mathcal{B}_n(B, \lambda) \neq \emptyset$ for every $\omega \in \Omega$.

By Theorems 6.1 and 6.4 we conclude that there exists a dynamical transition energy in each Landau band as stated in Theorem 6.4.

6.4. Almost Sure Existence of Spectrum Near Band Edges

Since we deal with a non ergodic random operator, previous results on the nature of the spectrum do not hold in this setting. In particular, we cannot use the characterization of the spectra as a union of spectra of periodic operators as in [19]. We need a more constructive approach and thus, to go back to the argument used in [8]. We extend [8, Theorem 7.1] to a Delone-Anderson potential to make sure that, although the spectrum $\sigma_{B,\lambda,\omega}$ is random, there exists almost surely some part of $\sigma_{B,\lambda,\omega}$ in the region were we can prove dynamical localization, that is, in the spectral band edges.

Consider the operator acting on $\mathrm{L}^2(\mathbb{R}^2), H_\omega^D = H_B + \lambda V_\omega^D$ where $\lambda > 0$ and V_ω^D is defined as in (2.21). Recall that

$$V_\omega^D(x) = \sum_{\gamma \in D} \omega_\gamma u_\gamma, \qquad (6.19)$$

where D is a (r, R)-Delone set, the random variables ω_γ are i.i.d. with absolute continuous probability density $\rho, \operatorname{supp} \rho = [-M, M]$ and $u_\gamma = u(x - \gamma)$. Assume moreover $u \in \mathcal{C}^2, \|u\|_\infty = 1, \operatorname{supp} u \subset \Lambda_r(0)$ and $u(0) = 1$.

Theorem 6.5. *Under the disjoint bands conditions, for a random Landau Hamiltonian as stated before and any* $n = 0, 1, 2, \ldots$ *there exists a finite positive constant* $B(n)$ *depending on* n, M, u, λ *and* $K_n(\lambda)$ *such that for all* $B > B(n)$, *the intervals* $\Sigma_{B,n,\lambda,\omega}$ *in Theorem 6.1 are almost surely non empty. More precisely, we prove that there exist finite positive constants* $C_n, B(n)$ *depending on* n, M, u *such that for every* $B > B(n)$*, we have for all* $E \in \mathcal{B}_n$*,*

$$\sigma(H_\omega) \cap [E - \lambda C_n B^{-1/2}, E + \lambda C_n B^{-1/2}] \neq \emptyset. \tag{6.20}$$

For a set $A \in \mathbb{R}^2$ we denote by $\tilde{A}$ the intersection $A \cap D$. Recall that we have, for an arbitrary box $\Lambda_L(x)$ of side $L \in \mathbb{N}$ centered in x:

$$C_{R,d} L^d \leq \sharp(\tilde{\Lambda}_L) = \sharp(D \cap \Lambda_L) \leq C_{r,d} L^d, \tag{6.21}$$

where $C_{R,d} = R^{-d}$ and $C_{r,d} = \lceil r^{-d} \rceil$.

Take a sequence $\{x_n\}$ such that $|x_n - x_m| > L$ for every n, m and consider the following sets in the probability space Ω:

$$\Omega_\epsilon^L(x_n) = \{\omega : |\omega_\gamma - \eta| \leq \epsilon \quad \forall \gamma \in \tilde{\Lambda}_L(x_n)\}$$

and

$$\Omega_\epsilon^L = \bigcap_N \bigcup_{n \geq N} \Omega_\epsilon^L(x_n), \tag{6.22}$$

where $\eta \in [-M, M]$. By the choice of $\{x_n\}$, the events $\Omega_\epsilon^L(x_n)$ and $\Omega_\epsilon^L(x_m)$ are independent for $n \neq m$.

Since the random variables are i.i.d. and (6.21) holds for every box $\Lambda_L(x_n)$, we obtain

$$\begin{aligned} \mathbb{P}\left(\Omega_\epsilon^L(x_n)\right) &= \mathbb{P}\left(|\omega_\gamma - \eta| \leq \epsilon, \forall \gamma \in \tilde{\Lambda}_L(x_n)\right) \\ &= \mathbb{P}\left(|\omega_\gamma - \eta| \leq \epsilon\right)^{\sharp(D \cap \Lambda_L(x_n))} &(6.23) \\ &\geq \mathbb{P}\left(|\omega_\gamma - \eta| \leq \epsilon\right)^{C_{r,d} L^d} &(6.24) \\ &= \rho([\eta - \epsilon, \eta + \epsilon])^{C_{r,d} L^d}. &(6.25) \end{aligned}$$

Therefore

$$\sum_n \mathbb{P}\left(\Omega_\epsilon^L(x_n)\right) = \infty, \tag{6.26}$$

which implies that $\mathbb{P}\left(\Omega_\epsilon^L\right) = 1$, by the Borel–Cantelli lemma.

Given $\delta > 0$, take $\epsilon = \delta/(rL)^d$. We have shown that for $\omega \in \Omega_\epsilon^L$, a set of full measure, there exists an infinite sequence $\{x_n\}$ such that for any $\eta \in [-M, M]$,

$$|\omega_\gamma - \eta| < \frac{\delta}{(rL)^d} \quad \text{for all } \gamma \in \tilde{\Lambda}_L(x_n) \tag{6.27}$$

Fix one of these boxes and call it Λ_0 (so Λ_0 depends on ω, but this procedure can be done for all $\omega \in \Omega_0$, the yielding result being uniform in ω).

Without loss of generality, $\tilde{\Lambda}_0$ contains 0. Indeed, if $0 \notin \tilde{\Lambda}_L(x_n)$ for all n, take $L > R$ so that $\tilde{\Lambda}_0 \neq 0$ and take $\gamma_0 \in \tilde{\Lambda}_0$. Consider now the operator

$$H_\omega^{D-\gamma_0} = H_B + \lambda \sum_{\gamma \in D - \gamma_0} \omega_\gamma u_\gamma. \tag{6.28}$$

We have that $\sigma(H_\omega^D) = \sigma(H_\omega^{D-\gamma_0})$, since, taking a translation $\tau_{\gamma_0} : \Omega \times D \to \Omega \times (D-\gamma_0)$ defined by $\tau_{\gamma_0}(\omega_\gamma, \gamma) = (\omega_\gamma, \gamma - \gamma_0)$, that associates the same random variable of a point to its translated, we can see H_ω^D is unitarily equivalent to $H_\omega^{D-\gamma_0}$.

Moreover, by what is known for H_ω^D, with full probability there exists a sequence $\{\tilde{x}_n\} = \{x_n - \gamma_0\}$ such that (6.27) holds. In particular, since the cube Λ_0 is a cube that satisfies (6.27) for H_ω^D, then the cube $\Lambda_{\gamma_0} = \Lambda_0 - \gamma_0$ satisfies (6.27) for $H_\omega^{D-\gamma_0}$.

Define

$$V_{\gamma_0}(x) = \eta \sum_{\gamma \in \tilde{\Lambda}_{\gamma_0}} u_\gamma. \tag{6.29}$$

Since $\gamma_0 \in \tilde{\Lambda}_0 = \Lambda_0 \cap D$ we have that $0 \in \tilde{\Lambda}_{\gamma_0} = (\Lambda_0 - \gamma_0) \cap (D - \gamma_0)$. Moreover, the assumptions on u, namely that $u(0) = 1$ and the supports of u_γ do not overlap, imply that $V_{\gamma_0}(0) = \eta$. Therefore, without loss of generality we can assume $\tilde{\Lambda}_0$ is centered in 0 and so we work from now on with H_ω^D, V_ω^D and V_0 as in (6.29) with $\gamma_0 = 0$.

Remark 6.6. The assumption $u(0) = 1$ is so we can later perform a Taylor expansion around 0.

Proof of Theorem 6.5. From now on L is fixed. For the sake of completeness, we will reproduce the details of [8, Appendix 2] with the corresponding adaptations and work in the *0*-th Landau band. Let Π_0 be the Landau projection in the *0*-th Landau band, around the Landau level B_0. Take the normalized function $\phi_0 \in \Pi_0(\mathcal{H})$,defined by

$$\phi_0(x) = \left(\frac{2B}{\pi}\right)^{1/2} e^{-B|x|^2}. \tag{6.30}$$

Let $E \in [B_0 - \lambda M, B_0 + \lambda M]$, that is, $E = B_0 + \lambda\eta$ for some $\eta \in [-M, M]$. The case $\eta = 0$ is trivial by the previous Borel–Cantelli argument, as $\{B_n\}_{n \geq 0} \subset \sigma(H_\omega)$ almost surely. Since the argument is analog for $\eta < 0$, in the following we consider only $\eta \in (0, M]$, and write

$$\| \left(H_\omega^D - E\right) \phi_0 \| = \| \left(H_\omega^D - B_0 - \lambda\eta\right) \phi_0 \| \tag{6.31}$$

$$\leq \|\Pi_0(\lambda V_\omega^D - \lambda\eta)\phi_0\| + \lambda\|(1 - \Pi_0)V_\omega^D \phi_0\|. \tag{6.32}$$

For simplicity we write V_ω instead of V_ω^D. The deterministic result [8, Lemma A.1] implies that

$$\lambda\|(1 - \Pi_0)V_\omega \phi_0\| \leq \lambda C_1 B^{-1/2}, \tag{6.33}$$

where C_1 is a constant depending only on the single-site potential u. We are left with

$$\|\Pi_0(\lambda V_\omega - \lambda\eta)\phi_0\| \leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma + \sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| \tag{6.34}$$

$$\leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| + \lambda\|\sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \omega_\gamma u_\gamma \phi_0\| \tag{6.35}$$

$$\leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| + \lambda M \sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \|u_\gamma \phi_0\|. \tag{6.36}$$

Recall that

$$\{\gamma \in D : \ \gamma \in D\setminus\tilde{\Lambda}_0\} \subset \{\gamma \in D : \ |\gamma| > r\}. \tag{6.37}$$

The second term in (6.36) can be estimated as in [8, Eq. 7.6], where it is shown that

$$\|u_\gamma\phi_0\|^2 = \int_{\mathbb{R}^2} \phi_0(x)^2 u(x-\gamma)^2 dx \leq \|u\|_\infty^2 e^{-2B|\gamma|^2+4Br|\gamma|}, \tag{6.38}$$

which is summable for γ such that $|\gamma| > r$, yielding that for all $B > B_*$, for a constant B_* big enough,

$$\lambda M \sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \|u_\gamma\phi_0\| \leq \lambda C_2 B^{-1/2}, \tag{6.39}$$

where the constant is uniform in B.

As for the first term in (6.36), recalling the definition of V_0 from (6.29), we write

$$\lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| = \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - V_0 + V_0 - \eta)\phi_0\| \tag{6.40}$$

$$\leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta \sum_{\gamma\in\tilde{\Lambda}_0} u_\gamma)\phi_0\| + \lambda\|(V_0-\eta)\phi_0\| \tag{6.41}$$

$$\leq \lambda\|\sum_{\gamma\in\tilde{\Lambda}_0} (\omega_\gamma - \eta) u_\gamma\phi_0\| + \lambda\|(V_0-\eta)\phi_0\|. \tag{6.42}$$

By the choice of Λ_0 the first term in (6.42) is

$$\lambda\|\sum_{\gamma\in\tilde{\Lambda}_0} (\omega_\gamma - \eta) u_\gamma\phi_0\| \leq \lambda\delta. \tag{6.43}$$

As for the second term in (6.42),

$$\|(V_0-\eta)\phi_0\|^2 = \left(\frac{2}{\pi}\right)\int_{\mathbb{R}^2} |V_0(x)-\eta|^2 e^{-2B\|x\|^2}\ \mathrm{d}x \tag{6.44}$$

$$= \left(\frac{2}{\pi}\right)\int_{\mathbb{R}^2} |V_0(B^{-1/2}x)-\eta|^2 e^{-2\|x\|^2}\ \mathrm{d}x. \tag{6.45}$$

Now, since $V_0(0) = \eta$, we have

$$|V_0(B^{-1/2}x) - \eta| = |V_0(B^{-1/2}x) - V_0(0)| \tag{6.46}$$

and we can perform a Taylor expansion around 0 for V_0, obtaining, since supp $V_0 \subset \Lambda_0$,

$$|V_0(B^{-1/2}x) - V_0(0)| \leq B^{-1/2}\|x\|\|\nabla V_0\|_\infty \leq B^{-1/2}L\|\nabla V_0\|_\infty. \quad (6.47)$$

Notice that $\|\nabla V_0\|_\infty \leq C_3$, for a constant C_3 depending only on u, uniformly with respect to $\eta \in [0, M]$. Replacing this in the integral we obtain

$$\|(V_0 - \eta)\phi_0\|^2 = \left(\frac{C_4}{\pi B}\right) \int e^{-2\|x\|^2} \, \mathrm{d}x. \quad (6.48)$$

So we obtain once more

$$\lambda\|(V_0 - \eta)\phi_0\| \leq \lambda C_5 B^{-1/2}. \quad (6.49)$$

Finally, adding the estimates (6.33), (6.39), (6.43) and (6.49) yields that for all $B > B_*$,

$$\|H_\omega^D - (B_0 + \lambda\eta)\| \leq \lambda C_5 B^{-1/2} + \delta, \quad (6.50)$$

where the bound is uniform in $B, \omega \in \Omega_0$ and in $\eta \in [0, M]$. The same result holds in any Landau band for all B large enough. Therefore, with probability one and for any $E = B_n + \lambda\eta$, we have

$$\sigma(H_\omega^D) \cap [E - \lambda C_5 B^{-1/2} - \delta, E + \lambda C_5 B^{-1/2} + \delta] \neq 0. \quad (6.51)$$

Since $\delta > 0$ is arbitrary,

$$\sigma(H_\omega^D) \cap [E - \lambda C_5 B^{-1/2}, E + \lambda C_5 B^{-1/2}] \neq 0, \quad (6.52)$$

for every $E \in [B_n, B_n + \lambda M]$. This proves that any gap in the spectrum of H_ω^D in the Landau band cannot exceed a length of order $B^{-1/2}$. □

In particular, since we know by perturbation theory that $\sigma(H_\omega^D) \subset [B_n - \lambda M, B_n + \lambda M]$,we have that for $E = B_n + \lambda M$, that is, in the edge of the Landau band,

$$\sigma(H_\omega^D) \cap [B_n + \lambda M - \lambda C_5 B^{-1/2}, B_n + \lambda M] \neq \emptyset. \quad (6.53)$$

On the other hand, by Theorem 6.1 we know the localization region is at a distance $K_n(\lambda)\frac{\ln B}{B}$ from the Landau level B_n. If λ is fixed and B is such that

$$K_n(\lambda)\frac{\ln B}{B} < \lambda M - \frac{\lambda C_n}{\sqrt{B}}, \quad (6.54)$$

then the region of the spectrum that is almost surely near the band edge, that is above $B_n + \lambda M - \lambda C_n B^{-1/2}$, lies in the localization region, that is above $B_n + K_n(\lambda)\frac{\ln B}{B}$. So we have shown Theorem 6.5, that is, for every $n = 0, 1, 2, \ldots$

$$\Sigma_{B,n,\lambda,\omega} \neq \emptyset \quad \text{for a.e. } \omega \in \Omega. \quad (6.55)$$

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