

Local and Global Analytic Solutions for a Class of Characteristic Problems of the Einstein Vacuum Equations in the “Double Null Foliation Gauge”

Giulio Caciotta and Francesco Nicolò

Abstract. The main goal of this work consists in showing that the analytic solutions for a class of characteristic problems for the Einstein vacuum equations have an existence region much larger than the one provided by the Cauchy–Kowalevski theorem due to the intrinsic hyperbolicity of the Einstein equations. To prove this result we first describe a geometric way of writing the vacuum Einstein equations for the characteristic problems we are considering, in a gauge characterized by the introduction of a double null cone foliation of the spacetime. Then we prove that the existence region for the analytic solutions can be extended to a larger region which depends only on the validity of the a priori estimates for the Weyl equations, associated with the “Bel-Robinson norms”. In particular, if the initial data are sufficiently small we show that the analytic solution is global. Before showing how to extend the existence region we describe the same result in the case of the Burger equation, which, even if much simpler, nevertheless requires analogous logical steps required for the general proof. Due to length of this work, in this paper we mainly concentrate on the definition of the gauge we use and on writing in a “geometric” way the Einstein equations, then we show how the Cauchy–Kowalevski theorem is adapted to the characteristic problem for the Einstein equations and we describe how the existence region can be extended in the case of the Burger equation. Finally, we describe the structure of the extension proof in the case of the Einstein equations. The technical parts of this last result is the content of a second paper.

1. Introduction

In this paper, we prove a result about the existence region of the analytic solution of a class of characteristic problems, namely those whose “initial data”

are given on a null hypersurface consisting of the union of a truncated outgoing null cone and of a truncated incoming cone intersecting the previous one along a surface diffeomorphic to S^2 . This class of characteristic problems has been studied by different authors, for instance Muller Zum Hagen [19], Muller Zum Hagen and Seifert [18], Christodoulou and Muller Zum Hagen [7], Dossa [9, and references therein] in a series of papers, but, in particular, we recall the anticipating work by Rendall [24], where a thorough examination has been done to show how to obtain initial data satisfying the constraint equations and the harmonic gauge conditions and, subsequently, a way of obtaining a local existence result is presented. Recently following, but largely improving the Rendall result, we suggest to the reader attention the paper by Y. Choquet-Bruhat, P.Y. Chrusciel, J.M. Martin-Garcia, “The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions”, to appear in *Annales Henri Poincaré*. In that paper the authors prove a local existence result for the characteristic problem with initial data on a null cone, using again the harmonic gauge and proving in a very detailed way how the initial data constraints have to be satisfied and how, relying on the Dossa results, the local existence result can be proved. Moreover, the nature of the characteristic problem, initial data on the null cone, adds the extra problems of the “tip of the cone” they solve completely.

In the present paper, our goal is to show that the real analytic solutions of the class of characteristic problems we are considering have, due to the hyperbolicity of the Einstein equations, a larger existence region than the one we can obtain by the application of the Cauchy–Kowalevski theorem. More precisely we prove that the extension of the analyticity existence region depends only on a finite number of derivatives of the initial data, namely on some appropriate Sobolev norms; moreover, if we assume the initial data small in these norms we can prove the global existence of the analytic solutions. Some analogous results have been obtained in the past by Alinhac and Metivier [1, references therein], where they proved the propagation of the analyticity for hyperbolic systems of p.d.e.; more recent results can be found in Spagnolo [25], and references therein. To prove this result, we present a strategy analogous to that used by Klainerman and one of the authors (F.N.) [14], for the Burger equation in a non-characteristic problem. Clearly, as the Einstein equations are much more complicated, the extension to the present case is significantly more difficult. Nevertheless, some general aspects are borrowed from that toy model and suggests the nature of the different technical problems we have to deal.

As it will appear clearly in the rest of this introduction, in the bulk of this paper and of the following one, preliminary to the existence proof of the analytic solutions a detailed examination of the gauge used and how the constraints are satisfied is needed. This part, crucial to the development of the existence proof, is, in our opinion, of interest in itself and new in many aspects. To it a greater part of the present paper is devoted. Let us summarize the various steps of our approach.

- (a) As discussed in detail in Sect. 4.1 and stated before the crucial ingredient to extend the analyticity region is the hyperbolicity of the p.d.e. equations we are considering. The region where the analytical solution exists is the region where the a priori energy estimates can be proved. Therefore, the control of energy norms is the key step. To achieve these norms and their a priori estimates is trivial in the Burger case, but much more delicate in the Einstein equations, even more if we look for energy type estimates valid (for small initial data) everywhere. It is, in fact, well known that the standard energy norms associated with the vacuum Einstein equation in the harmonic gauge are very difficult to use to obtain a global solution even in the non-characteristic problem, see [17], and no results are at our disposal for the characteristic case. It turns out that an efficient strategy to achieve a global existence result is the one introduced first by Christoudoulou and Klainerman [8] and subsequently modified by Klainerman and Nicolò [15]. In this approach, the “energy type” norms to bound are those associated with the Bel-Robinson tensor, quadratic in the Riemann tensor. The control of these norms is strictly tied to the control of the connection coefficients¹ of the spacetime and the equations which control these last quantities are the so-called “structure equations”, see [26], vol. 2, which have the form of elliptic Hodge systems or of transport equations along the null directions.
- (b) This strongly suggests the use of a foliation already introduced in [15], Chapter 3, the “double null cone foliation”. The main differences are that first, the structure equations were used there to obtain good estimates for the various L^p integral norms while here to show how the Einstein equations can be expressed as a subset of these structure equations. More precisely, as in the non-characteristic case, one can foliate the spacetime with a family of spacelike hypersurfaces and write the Einstein equations as a system of first order equations for the (Riemannian) metric and the second fundamental form adapted to this foliation, here we assume the spacetime foliated by a family of outgoing cones and incoming (truncated) cones and write the Einstein equations as a set of first order equations involving the metric adapted to these cones and the connection coefficients, basically, the first derivatives of this metric. The complete detailed description of the procedure to write the Einstein equations in the way we are sketching here is given in Sects. 2.3 and 2.4.

The second difference is that in [15] a local solution was already assumed to exist and proved in the more standard harmonic gauge, here even the local existence is proved in the “double null cone foliation gauge”. Therefore, in this approach, we will never use any foliation made by spacelike hypersurfaces, the derivatives of the various unknown functions of our equations are always done with respect to the angular variables and to the \underline{u} and u variables, the affine parameters of the null geodesics generating the outgoing or incoming cones.

¹ Sometimes called Ricci coefficients.

- (c) Exactly as in the non-characteristic case the choice of the spacelike hypersurface foliation is basically equivalent to the choice of a gauge, in the present case specifying the “double null cone foliation” is just the choice of the gauge and the quantities Ω and X which appear in the expression of our metric in the adapted coordinates,

$$\mathbf{g} = -2\Omega^2(d\underline{u}d\underline{u} + d\underline{u}d\underline{u}) + \gamma_{ab}(X^a d\underline{u} + d\underline{\omega}^a)(X^b d\underline{u} + d\underline{\omega}^b), \quad (1.1)$$

play the role of the lapse function and the shift vector. Analogously to them they will have to satisfy some differential equations. In Sect. 2.3 a very extended discussion about this gauge is given.

- (d) As we are dealing with a characteristic problem it is expected that the initial data cannot be given in a complete free way, but they have to satisfy some constraints. In the case of the Einstein equations the situation is more complicated as even in the non-characteristic case the initial data cannot be given in a free way.² Therefore, our initial data have to satisfy two different kinds of constraints, those due to the nature of the Einstein equations, the analogous of the constraints equations for the second fundamental form k_{ij} and those associated with the gauge choice namely, in the present case, the equations for Ω and X . In our presentation, more geometric than the one using, for instance, the harmonic gauge, the distinction between these two kinds of constraints is completely clear and it is natural that in these equations no transverse (to the cones) derivatives appear.

Finally, to show that the solutions of our equations are in fact solutions of the Einstein equations we have to prove, exactly as in the non-characteristic case, that the (Einstein) constraint equations once satisfied by the initial data are satisfied everywhere. This is proved in Sect. 2.4.

- (e) As we want to prove that our analytic solutions can be extended to the whole spacetime³ we have first to provide a local analytic solution of the characteristic problem. This is discussed in Sect. 3, where we adapt the Cauchy–Kowalevski theorem to the characteristic problem following Duff [10] and Friedrich [11].
- (f) In Sect. 4 the central part of our program is described and partially proved. In Sect. 4.1 an analogous result is proved for the Burger equation using, and somewhat extending, a previous result of Klainerman and one of the present authors (F.N.) [14]. Although the problem, in that case, is much simpler some of the basic ideas can be borrowed and transported. In the Burger equation case we prove that, due to the hyperbolicity of the equation, some a priori estimates hold for the Sobolev energy norms (with $s = 2$) up to a time T , depending only on these norms. Then it is proved that in the region of analyticity provided by the Cauchy–Kowalevski theorem it is possible to show that all the derivatives satisfy some appropriate estimates such that the series describing the analytic solutions have a convergence radius depending on the initial analytic data and

² See, for instance the detailed discussion in [12].

³ A slightly imprecise statement which, nevertheless, should be clear.

on the Sobolev norms associated with the first derivatives, but independent from the point (of the analyticity region) around which we perform the series expansion. This is proved controlling, with a delicate inductive argument, the norms of all order derivatives. Once this result is achieved the analyticity region can be extended and as, again, the series convergence radius depends only on the initial data and on the Sobolev norms (bounded, via the a priori estimates, in terms of the Sobolev initial data norms) it can be proved that the procedure can be repeated in the whole time interval where the a priori estimates hold, obtaining the final result.

In the second part of the section, Sect. 4.2, we discuss how this approach can be implemented in the case of the Einstein equations. This requires a lot of technical work to which a subsequent paper is devoted. There we discuss in a very detailed way the problems one encounters and how we have solved them. We recall here some of these problems and give in Sect. 4.2 an extended discussion of how they are faced.

- (1) The first problem is the position of the initial data, in Sect. 2.4 it was already discussed the constraints they have to satisfy, here we have to show how they can be given on the whole initial null hypersurface and not only in a portion of it, a generalization of what has been done in [4], where we were interested only on Sobolev initial data.
- (2) The local existence result has been proved in Sect. 3, the strategy to extend the analyticity region is to repeat the inductive mechanism which allows to control the norms of all derivatives again in an uniform way. This is the more complicated technical part; to achieve it we have to use the transport equations for the connection coefficients and the hyperbolicity of the Einstein equations, more precisely the a priori estimates for the integral norms of the Bel-Robinson tensor. The main lemma needed to prove our result, Lemma 4.7, is stated in Sect. 4.2 while its proof and the subsequent steps to prove our result are written in the subsequent paper.

To summarize this discussion, we are convinced that to satisfy our goal of extending as much as possible the analyticity region for the Einstein equations our gauge choice is the most convenient, even more as it seems so naturally intertwined to the characteristic problem. Moreover, this formalism is perfectly suited to control the integral norms of the Bel-Robinson tensor and prove that, for small initial data, they can be bounded in the whole spacetime.

2. The Characteristic Problem for the Vacuum Einstein Equations, Assuming the Spacetime Foliated by Outgoing and Incoming Null Cones

In this section and in the following one we present a way of writing the Einstein equations suited to study and solve the class of characteristic problems we are investigating. The basic idea is to assume that the spacetime we are constructing is foliated by a family of outgoing and incoming null cones, a foliation used

in [15] which we believe very appropriate to study the characteristic problems for the Einstein equations.

We will see that, with obvious differences, our approach is similar to the one used for the non-characteristic problem when the spacetime is foliated by three-dimensional hypersurfaces and the evolution part of the Einstein equations,⁴ are first order equations in terms of the Riemannian metric of the hypersurfaces, g_{ij} , and their extrinsic curvature k_{ij} .

Therefore, the various steps required to accomplish our goal are in order

- (a) Define the class of characteristic problems we are considering.
- (b) Define the gauge we use.
- (c) Identify the evolution equations in the coordinates associated with the gauge.
- (d) Identify the constraints equations in this formalism.
- (e) the conservation of the constraints: Once steps (a)–(d) are clearly done, we show how the analytic solutions can be obtained in this characteristic case with the appropriate Cauchy–Kowalewski approach, that exactly as in non-characteristic case we can define our analytic solution as a solution of a “reduced problem” and subsequently that, once the constraints are satisfied from the initial data, they are satisfied in all the existence region so that the analytic solution is really a solution of the (vacuum) Einstein equations.

In the following subsections and in Sect. 3 we will concentrate on steps (a) to (e).

2.1. The Class of Characteristic Problems

As we mentioned in Sect. 1 we are considering the case of initial data on a null hypersurface consisting of the union of a truncated outgoing null cone $C_0 \equiv C(\lambda_0)$, see later for the “cone” definition and also [4], and of a truncated incoming cone $\underline{C}_0 \equiv \underline{C}(\nu_0)$ intersecting the previous one along an S^2 surface. Moreover, we expect that analogous results can be easily obtained when the initial hypersurface is made by the intersection of two null hyperplanes. More delicate is when looking for the solutions of the Einstein equations inside an outgoing cone where we give the initial data. We believe that this problem, solved for the local existence in the previous work by Y. Choquet-Bruhat, P.Y. Chrusciel, J.M. Martin-Garcia, quoted in Sect. 1, can also be faced with our technique, in our double null cone gauge.

2.2. The “Double Null Foliation Gauge”

Let us recall what is the meaning of a “geometric” gauge choice in the non-characteristic problem associated with a spacetime foliation. In that case, see also [15], Lemma 1.3.2, we can write the metric in the following way

$$\mathbf{g}(\cdot, \cdot) = -\Phi^2 dt^2 + g_{ij}(X^i dt + dx^j)(X^j dt + dx^j), \quad (2.2)$$

⁴ Therefore apart from the constraint equations.

where g_{ij} are the components of the Riemannian metric induced on the generic spacelike hypersurface of the foliation. The unit normal to the generic hypersurface is

$$N = \frac{1}{\sqrt{\Phi^2 + |X|^2}} \left(\frac{\partial}{\partial t} - X^j \frac{\partial}{\partial x^j} \right). \tag{2.3}$$

The coordinates $\{x^i\}, i \in \{1, 2, 3\}$ define a point p of a generic hypersurface labeled by a parameter $\tau, p \in \Sigma_\tau$, and the diffeomorphism $\Psi_N(\delta)$ associated with the vector field N send p to a point $q \in \Sigma_{\tau+\delta}$ with the same $\{x^i\}$ coordinates.

Up to now, although we have defined the coordinates $\{t, x^i\}$, the “gauge” is not yet defined as we have not specified the lapse and the shift functions Φ and X . They are defined completely once that we define the hypersurfaces which foliate the spacetime. For instance in the choice used in [8], the hypersurfaces are assumed labeled by the time coordinate $\tau = t$ and moreover they are assumed “maximal” which means that the trace of their second fundamental form is identically zero. The first statement implies that $X = 0$ and the second one that the lapse function Φ satisfies the elliptic equation

$$\Delta\Phi = |k|^2\Phi.$$

Similar considerations can be done for the “gauge” associated with the “CMC foliation” used by Andersson and Moncrief [2].

Following these ideas the “geometric” gauge we are choosing is associated with a double null cone foliation, see [15] and later on for its definition. The spacetime we are going to construct is foliated by a family of null outgoing cones $C(\lambda)$ and a family of null incoming cones $\underline{C}(\nu)$, more precisely a portion of null cones, see later on for details. We denote $S(\lambda, \nu)$ their intersections,

$$S(\lambda, \nu) = C(\lambda) \cap \underline{C}(\nu) \tag{2.4}$$

which are two-dimensional surfaces diffeomorphic⁵ to S^2 and with N and \underline{N} the equivariant vector fields whose associated diffeomorphisms Φ and $\underline{\Phi}$ send the $S(\lambda, \nu)$ surfaces to the analogous surfaces on the outgoing or the incoming cones respectively,

$$\begin{aligned} \Phi(\delta)[S(\lambda, \nu)] &= S(\lambda, \nu + \delta) \\ \underline{\Phi}(\delta)[S(\lambda, \nu)] &= S(\lambda + \delta, \nu). \end{aligned} \tag{2.5}$$

Let us also define the map $\Psi(\lambda, \nu)$ we will use extensively later on,

$$\begin{aligned} \Psi(\lambda, \nu) : S_0 \ni p_0 &\rightarrow q = \Psi(\lambda, \nu)(p_0) \\ &= \Phi(\nu - \nu_0)(\underline{\Phi}(\lambda - \lambda_0)(p_0) \in S(\lambda, \nu). \end{aligned} \tag{2.6}$$

Once we have specified the foliation of the spacetime we can define two coordinates adapted to our gauge, namely the parameter λ and ν which determine

⁵ Here this is part of the assumptions we are doing about the foliation, later on we prove that this is true where the solution exists.

the incoming or the outgoing cones. More precisely we denote these coordinates u and \underline{u} and the outgoing and incoming cones $C(\lambda)$ and $\underline{C}(\nu)$ are, respectively,

$$C(\lambda) = \{p \in \mathcal{M} | u(p) = \lambda\}, \quad Cb(\nu) = \{p \in \mathcal{M} | \underline{u}(p) = \nu\} \tag{2.7}$$

where \mathcal{M} denotes the spacetime.

As the coordinates u, \underline{u} allow to define the null cones it is important to know how to connect them to an arbitrary set of coordinates. This is provided by the Eikonal equation,

$$g^{\mu\nu} \partial_\mu w \partial_\nu w = 0, \tag{2.8}$$

where $g^{\mu\nu}$ are the components of the inverse of the metric tensor written in an arbitrary set of coordinates x^μ . Then $u = u(x)$ and $\underline{u} = \underline{u}(x)$ are solution of the Eikonal equations with appropriate initial data⁶ such that their level hypersurfaces are, at least locally, the null cones of our foliation. It is well known, see for instance [15], that the vector fields

$$L = -g^{\mu\nu} \partial_\nu u \frac{\partial}{\partial x^\mu}, \quad \underline{L} = -g^{\mu\nu} \partial_\nu \underline{u} \frac{\partial}{\partial x^\mu}$$

are the tangent vector field of the null geodesics generating the outgoing and the incoming cones, satisfying

$$D_L L = 0, \quad D_{\underline{L}} \underline{L} = 0.$$

Their scalar product defines the scalar function Ω ,

$$g(L, \underline{L}) = -(2\Omega)^{-2} \tag{2.9}$$

and the vector fields e_3, e_4 ,

$$e_3 = 2\Omega \underline{L}, \quad e_4 = 2\Omega L, \tag{2.10}$$

satisfy the relation $g(e_3, e_4) = -2$. Finally the equivariant vector fields previously introduced, N, \underline{N} ,⁷ have the following expressions

$$N = \Omega e_4, \quad \underline{N} = \Omega e_3. \tag{2.11}$$

Once introduced the coordinates u, \underline{u} we are left with defining the remaining two coordinates, we have to interpret as angular coordinates, which allow to specify a point on each surface $S(\lambda, \nu)$. The procedure is similar to the one envisaged previously in the non-characteristic case, to define them we have to define a map which sends a point p of “angular coordinates” $\{\omega^a\}$, $a \in \{1, 2\}$ belonging to the intersection of the “initial cones” $S(\lambda_0, \nu_0) = C(\lambda_0) \cap \underline{C}(\nu_0)$ to a point $q \in S(\lambda, \nu)$ to which the same coordinates are assigned. The map we choose is made in the following way: first we move, starting from a point $p \in S(\lambda_0, \nu_0)$ of angular coordinates $\{\omega^a\}$, along the $\underline{C}(\nu_0)$ cone using the integral curves of the vector field \underline{N} up to a point $q' \in S(\lambda, \nu_0)$, then applying the diffeomorphism generated by the vector field N we move “inside” the region

⁶ Different choice of the “initial data” for (2.8) on the external outgoing and incoming cones give rise to different null cone foliations.

⁷ It is immediately to check that they satisfy Eq. (2.5).

of the spacetime up to the point $q \in (\lambda, \nu)$ and assign to this point the same angular coordinates $\{\omega^a\}$; formally, for any a ,

$$\omega^a(\Psi(\lambda, \nu)(p)) \equiv \omega^a(\Phi(\nu - \nu_0)\underline{\Phi}(\lambda - \lambda_0)(p)) = \omega^a(p). \tag{2.12}$$

Once we have introduced the coordinates relative to our gauge we have still to write, in these coordinates, an explicit expression for the metric tensor. This can be done defining a null moving frame, $\{e_4, e_3, e_A\}, A \in \{1, 2\}$ adapted to this double null foliation where

$$e_A = e_A^a \frac{\partial}{\partial \omega^a}, \quad A \in \{1, 2\}, \tag{2.13}$$

are orthonormal vector fields tangent at each point $p \in \mathcal{M}$ to the surface $S(\lambda, \nu)$ containing p and e_3, e_4 are null vector fields orthogonal to the e_A 's, outgoing and incoming, respectively, which, basically, means “tangent” to the null hypersurfaces $C(\lambda)$ and $\underline{C}(\nu)$.⁸ Moreover, recalling the meaning of the vector fields N and \underline{N} , it follows that

$$N = \frac{\partial}{\partial u} \quad \text{and} \quad e_4 = \frac{1}{\Omega} \frac{\partial}{\partial \underline{u}}. \tag{2.14}$$

The explicit expression of \underline{N} , and therefore of e_3 is somewhat different as in a curved spacetime these two vector fields do not commute; it can be proved, see [15] that \underline{N} must have the following expression

$$\underline{N} = \frac{\partial}{\partial u} + X^a \frac{\partial}{\partial \omega^a} \quad \text{and therefore} \quad e_3 = \frac{1}{\Omega} \left(\frac{\partial}{\partial u} + X^a \frac{\partial}{\partial \omega^a} \right). \tag{2.15}$$

where the property and the equation that the vector field X has to satisfy will be discussed later on.⁹

Once we have the explicit expression of the null frame in the adapted¹⁰ coordinates we can write the metric tensor in these coordinates obtaining

$$\mathbf{g} = -2\Omega^2(dud\underline{u} + d\underline{u}du) + \gamma_{ab}(X^a du + d\omega^a)(X^b du + d\omega^b), \tag{2.16}$$

where γ_{ab} are the components of the induced metric on $S(\lambda, \nu)$.

Remark. Observe that this is not the more general metric we can write “adapted” to the foliation, in fact there are only seven metric components different from zero. This follows as we have chosen the coordinate \underline{u} such that the N vector field was $\frac{\partial}{\partial \underline{u}}$. We could nevertheless define the coordinates in such a way that $N = \frac{\partial}{\partial \underline{u}} + Y^a \frac{\partial}{\partial \omega^a}$, in this case repeating the previous argument the metric turns out to be, written in the same u, \underline{u} coordinates,

$$\mathbf{g} = -2\Omega^2(dud\underline{u} + d\underline{u}du) + \gamma_{ab}(X^a du + Y^a d\underline{u} + d\omega^a)(X^b du + Y^b d\underline{u} + d\omega^b)$$

with ten components different from zero.¹¹ This last expression is the one analogous to the metric (2.2) written for the non-characteristic problem

$$\mathbf{g}(\cdot, \cdot) = -\Phi^2 dt^2 + g_{ij}(X^i dt + dx^i)(X^j dt + dx^j)$$

⁸ e_4 is, at same time, tangent and normal to $C(\lambda)$ and analogously e_3 with respect to $\underline{C}(\nu)$.

⁹ The definition of the coordinates ω^a , Eq. (2.12), implies that $X = 0$ on \underline{C}_0 .

¹⁰ Here with adapted we mean both adapted to the “gauge” we are defining, both to the leaves $S(\lambda, \nu)$ of the null cones.

¹¹ Nevertheless not all independent due to the nature of the double null cone foliation.

while the previous expression (2.16) corresponds to the one used for instance in [8] associated with the “maximal” foliation, namely

$$\mathbf{g}(\cdot, \cdot) = -\Phi^2 dt^2 + g_{ij} dx^i dx^j.$$

Comparing the metric expression (2.16) with the one in (2.2) we see that Ω plays basically the role of the lapse function and X of the shift vector. Exactly as in that case we expect that the gauge choice is completed once we are able to determine these functions. In the non-characteristic case this could be done in different ways, for instance in [15] the shift vector was imposed equal to zero and the lapse function had to satisfy an elliptic equation. In other cases, see for instance [3, 8] or [2, 3],¹² both quantities had to satisfy some differential equations. In the present characteristic case we will see that both Ω and X^a have to satisfy some first order differential equations. We will obtain them in the next sections after we write the Einstein equations in these coordinates. To do it we use in a systematic way the structure equations for a Lorentzian manifold.

2.3. The Einstein Equations in the $\{\lambda, \nu, \theta, \phi\}$ Coordinates

In this section and in the following section we fulfill the first goal of this paper, namely we write the Einstein equations in a way suited to the class of characteristic problems we are considering and connected to the gauge choice discussed in the previous subsection. We recall shortly the main properties of our approach.

- (a) As we said before we assume the spacetime foliated by null cones, outgoing and incoming as prescribed by the chosen gauge. Therefore, the Einstein equations become evolution equations for the metric components γ_{ab} of the two-dimensional leaves $S(\lambda, \nu)$ which foliate the null cones. Moreover, as in the non-characteristic case, the choice of the gauge will imply that the lapse function, Ω , and the shift vector X have to satisfy some equations.
- (b) As in the analogous formulation of the non-characteristic case we express the Einstein equations as a system of first order equations; we define the family of first order equations which express the Einstein equations and separate them into two groups, in the first one we collect those equations which can be interpreted as the evolution part of the Einstein equations¹³ and in the second one those which can be interpreted as constraint equations. This is the central goal of this subsection.
- (c) The third aspect to remark is that we write our equations in terms of the coordinates $u \equiv \lambda, \underline{u} \equiv \nu, \omega^a$, where ν and λ are, as defined in the previous subsection, affine parameters for the null geodesics along the null outgoing and incoming cones and we do not use the more standard

¹² In the case of [2] the spacetime is spatially compact and the reduced equations are the evolution equations (2.5a), (2.5b) together with the elliptic equations (2.8a), (2.8b) relative to N and X which specify the gauge; given a solution of this set, the constraint equations are proved to hold everywhere in the spacetime once they are assumed for the initial data.

¹³ Together with the equations for Ω and X .

time coordinate, t .¹⁴ With this choice our equations have a more geometric “flavour” as all the quantities we introduce are connected in a direct way to the geometric properties of our foliation and the equations are transport equations along the null cones. This approach and this formalism turn out to be very appropriate, as already discussed in [15], to obtain a priori estimates for “energy-type” quantities. This, moreover, will allow us to prove the main goal of this work, namely, as discussed in Sect. 1, that the analytic solution of the characteristic problem has a much larger existence region depending on the “hyperbolic” a priori estimates, a region which can be unbounded if the initial data have H^s norms, with appropriate s , sufficiently small.

To write the Einstein equations in a way satisfying (a)–(c), we use intensively the structure equations, see for instance [26], vol 2, adapted to a Lorentzian manifold.

2.3.1. The Structure Equations. We recall some general aspects of the structure equations. We denote the null orthonormal frame in the following way:

$$\{e_{(\alpha)}\} = \{e_\alpha\} = \{e_{(1)}, e_{(2)}, e_{(3)}, e_{(4)}\}, \tag{2.17}$$

where

$$e_{(3)} = 2\Omega\underline{L}, \quad e_{(4)} = 2\Omega L, \quad e_{(1)} = e_\theta, \quad e_{(2)} = e_\phi \tag{2.18}$$

and

$$\{\theta^{(\alpha)}\} = \{\theta^\alpha\} = \{\theta^{(1)}, \theta^{(2)}, \theta^{(3)}, \theta^{(4)}\} \tag{2.19}$$

are the corresponding forms satisfying

$$\theta^{(\alpha)}(e_{(\beta)}) = \delta_\beta^\alpha \tag{2.20}$$

and it follows that

$$\theta_\mu^{(3)} = -\frac{1}{2}g_{\mu\nu}e_4^\nu, \quad \theta_\mu^{(4)} = -\frac{1}{2}g_{\mu\nu}e_3^\nu. \tag{2.21}$$

We define

$$\begin{aligned} \mathbf{D}_{e_\alpha} e_\beta &\equiv \Gamma_{\alpha\beta}^\gamma e_\gamma \\ \mathbf{R}(e_\alpha e_\beta) e_\gamma &\equiv \mathbf{R}_{\gamma\alpha\beta}^\delta e_\delta, \end{aligned} \tag{2.22}$$

where \mathbf{D} is the connection of the spacetime associated with the Lorentz metric g , $\mathbf{D}_{e_\alpha} e_\beta$ is the covariant derivative of the vector field e_β in the direction e_α

¹⁴ The simplest analogy is solving the two-dimensional homogeneous linear wave equation written as:

$$\frac{\partial^2}{\partial\lambda\nu} u = 0.$$

and \mathbf{R} is the Riemann tensor (here the first Greek letters are “names” and do not denote components),¹⁵

$$\mathbf{R}(e_\alpha e_\beta)e_\gamma = \mathbf{D}_{e_\alpha}(\mathbf{D}_{e_\beta}e_\gamma) - \mathbf{D}_{e_\beta}(\mathbf{D}_{e_\alpha}e_\gamma) - \mathbf{D}_{[e_\alpha, e_\beta]}e_\gamma.$$

Defining the following one and two forms:

$$\begin{aligned} \omega_\beta^\alpha &\equiv \Gamma_{\gamma\beta}^\alpha \theta^\gamma \\ \Omega_\beta^\alpha &\equiv \frac{1}{2} \mathbf{R}_{\beta\gamma\delta}^\alpha \theta^\gamma \wedge \theta^\delta, \end{aligned} \tag{2.23}$$

we have the following result, see [26], vol. 2, whose proof is in Appendix to this section.

Proposition 2.1. ω_β^α and Ω_β^α satisfy the following structure equations

$$d\theta^\alpha = -\omega_\gamma^\alpha \wedge \theta^\gamma \tag{2.24}$$

$$d\omega_\gamma^\delta = -\omega_\sigma^\delta \wedge \omega_\gamma^\sigma + \Omega_\gamma^\delta, \tag{2.25}$$

called, respectively, the first and the second structure equations.

The knowledge of a null orthonormal frame in a whole region is equivalent to knowing the metric in that region, therefore the first set of structure equations can be thought as “first order equations” for the metric components. Viceversa the one forms ω_β^α are connected to the first derivatives of the (components of the) moving frame and, therefore play the role of the first derivatives of the metric components; the second set of structure equations represent first order equations for these first derivatives.¹⁶

To have an explicit expression for the structure equations we recall the definition of the “connection coefficients” (sometimes called “Ricci coefficients”). In the defined null orthonormal frame they have the following expressions:

$$\begin{aligned} \chi_{AB} &= \mathbf{g}(\mathbf{D}_{e_A}e_4, e_B), & \chi_{ab} &= \mathbf{g}(\mathbf{D}_{e_A}e_3, e_B) \\ \xi_{\underline{A}} &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_3}e_3, e_A), & \xi_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_4}e_4, e_A) \\ \underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_{e_3}e_3, e_4), & \omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_{e_4}e_4, e_3) \\ \underline{\eta}_{\underline{A}} &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_4}e_3, e_A), & \eta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_3}e_4, e_A) \\ \zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_A}e_4, e_3). \end{aligned}$$

¹⁵ Obviously choosing a coordinate basis $\{\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$ the $\Gamma_{\alpha\beta}^\gamma$ defined in (2.22) are just the usual Christoffel symbols, $\Gamma_{\nu\rho}^\mu, \mathbf{D}_{\frac{\partial}{\partial x^\mu}} \equiv D_\mu$ is the covariant derivative with respect to ∂_μ and $\mathbf{R}_{\gamma\alpha\beta}^\delta = R_{\nu\rho\sigma}^\mu e_\alpha^\nu e_\beta^\rho e_\mu^\sigma \theta^\delta$.

¹⁶ Observe that the second group of structure equations depends, through Ω_β^α , also on the Riemann tensor components. This could suggest that expressing these equations as partial differential equations for the various components, to have a closed system of equations one should also consider the Bianchi equations for the Riemann tensor and the Riemann components as independent variables. Although this could be done, see for instance [11], this is not what we do, as we discuss in great detail later on.

They are 2-covariant tensors, vectors and scalar functions defined on the two-dimensional surfaces $S(\lambda, \nu)$.

In terms of these quantities the one forms, ω^β_α , have the following expressions:

$$\begin{aligned} \omega_3^4 &= 0, \quad \omega_4^4 = -2\omega\theta^4 + 2\underline{\omega}\theta^3 - \zeta_a\theta^A, \quad \omega_A^4 = \underline{\eta}_A\theta^4 + \frac{1}{2}\underline{\chi}_{BA}\theta^B \\ \omega_3^3 &= 2\omega\theta^4 - 2\underline{\omega}\theta^3 + \zeta_a\theta^A, \quad \omega_4^3 = 0, \quad \omega_A^3 = \eta_A\theta^3 + \frac{1}{2}\chi_{BA}\theta^B \\ \omega_4^A &= 2\eta_A\theta^3 + \chi_{BA}\theta^B, \quad \omega_3^A = 2\underline{\eta}_A\theta^4 + \underline{\chi}_{BA}\theta^B. \end{aligned} \tag{2.26}$$

The one forms ω_B^A have a different expression which do not depend on the connection coefficients we have introduced,

$$\omega_B^A = \mathbf{g}(\mathcal{D}_4 e_B, e_A)\theta^4 + \mathbf{g}(\mathcal{D}_3 e_B, e_A)\theta^3 + \mathbf{g}(\mathbf{D}_C e_B, e_A)\theta^C. \tag{2.27}$$

It is a long, but simple task to write the structure equations in terms of the metric components and the connection coefficients. The first set of structure equations

$$d\theta^\alpha(e_\beta, e_\gamma) = -\omega_\delta^\alpha \wedge \theta^\delta(e_\beta, e_\gamma)$$

becomes:

$$\begin{aligned} \{\alpha, \beta, \gamma\} = \{A, B, 4\} &: \frac{\partial}{\partial \nu} \gamma_{ab} = 2\Omega \chi_{ab} \\ \{\alpha, \beta, \gamma\} = \{A, B, 3\} &: \frac{\partial}{\partial \lambda} \gamma_{ab} + \mathcal{L}_X \gamma_{ab} = 2\Omega \underline{\chi}_{ab} \\ \{\alpha, \beta, \gamma\} = \{A, 4, 3\} &: \frac{\partial}{\partial \nu} X^c = 4\Omega^2 \zeta_C e^C \\ \{\alpha, \beta, \gamma\} = \{4, 4, 3\} &: \frac{\partial}{\partial \nu} \log \Omega = -2\Omega \omega \\ \{\alpha, \beta, \gamma\} = \{3, 3, 4\} &: \left(\frac{\partial}{\partial \lambda} + X^a \frac{\partial}{\partial \omega^a} \right) \log \Omega = -2\Omega \underline{\omega} \\ \{\alpha, \beta, \gamma\} = \{4, A, 4\} &: \nabla_A \log \Omega = \underline{\eta}_A + \zeta_A \\ \{\alpha, \beta, \gamma\} = \{3, A, 3\} &: \nabla_A \log \Omega = \eta_A - \zeta_A \\ \{\alpha, \beta, \gamma\} = \{4, A, 3\} &: \underline{\xi}_A = 0 \\ \{\alpha, \beta, \gamma\} = \{3, A, 4\} &: \xi_A = 0. \end{aligned} \tag{2.28}$$

Remark. To obtain Eqs. (2.28) and (2.29) we have used only the relation $e_3 = 2\Omega \underline{L}$, $e_4 = 2\Omega L$. Recall that the structure equations for a moving (null orthonormal) frame do not imply that $\{e_3, e_A\}$, $A \in \{1, 2\}$ or $\{e_4, e_A\}$, $A \in \{1, 2\}$ are integrable distributions. In our present case, with the previous definition of e_3, e_4 , this is true.

As our purpose is to write the Einstein equations as a system of first order equations for the component of the metric and their derivatives, Eq. (2.28) of the first set of structure equations tell us exactly that we have to look for first order partial differential equations for the connection coefficients $\{\chi, \underline{\chi}, \zeta, \omega, \underline{\omega}\}$. They correspond in fact to the derivatives along the inward

(or outward) direction with respect to the null cones outgoing or incoming, of the metric components γ_{ab} , of the lapse function and of the shift vector. Once we have written the appropriate partial differential equations for these connection coefficients we will show that we reduce to a closed set of equations in the metric components and in these connection coefficients. To find the appropriate equations we rely on the second set of the structure equations,

$$(d\omega_\gamma^\delta + \omega_\sigma^\delta \wedge \omega_\gamma^\sigma)(e_\alpha, e_\beta) = \Omega_\gamma^\delta(e_\alpha, e_\beta).$$

These equations can be written in a more explicit way in terms of the connection coefficients. One has, nevertheless, to remember that the structure equations are identities, valid in a generic manifold with a (Riemannian or Lorentzian) metric. They can also be seen as “integrability conditions” for the existence of a moving frame in the whole manifold. One has to remark that in their right hand side the terms $\Omega_\gamma^\delta(e_\alpha, e_\beta)$ appear which can be written in terms of the various components of the Riemann tensor. Therefore, these equations can be seen as identities defining the Riemann tensor in terms of first derivatives of the connection coefficients, but if we impose some condition on the Riemann tensor they become partial differential equations to solve with respect to the connection coefficients. The condition to impose to the Riemann tensor, or more precisely to Ω_γ^δ , is that the vacuum Einstein equations have to be satisfied, namely that the corresponding Ricci tensor is identically zero. Therefore, we have to look to the explicit expression of these equations under this condition.

It is a long, but standard and certainly not new¹⁷ to realize that the structure equations have the expressions written in the following where we indicate also the Ricci components (=0) to which they are associated.¹⁸

$$[\mathbf{R}(e_4, e_4) = 0 :]$$

$$\mathbf{D}_4 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 + 2\omega \text{tr}\chi + |\hat{\chi}|^2 = 0$$

$$[\mathbf{R}(e_4, e_A) = 0 :]$$

$$\mathbf{D}_4 \zeta + \zeta \chi + \text{tr}\chi \zeta - \mathcal{D}iv \chi + \nabla \text{tr}\chi + \mathbf{D}_4 \nabla \log \Omega = 0$$

$$[\delta_{AB} \mathbf{R}(e_A, e_B) = 0 :]$$

$$\mathbf{D}_4 \text{tr}\underline{\chi} + \frac{1}{2} \text{tr}\chi \text{tr}\underline{\chi} - 2\omega \text{tr}\underline{\chi} + \hat{\chi} \cdot \underline{\hat{\chi}} + 2\mathcal{D}iv(\zeta - \nabla \log \Omega) - 2|\zeta - \nabla \log \Omega|^2 = 2\rho$$

$$[\widehat{\mathbf{R}}(e_A, e_B) = 0 :]$$

$$\mathbf{D}_4 \hat{\chi} + \frac{1}{2} \text{tr}\chi \hat{\chi} + \frac{1}{2} \text{tr}\underline{\chi} \hat{\chi} - 2\omega \hat{\chi} + \nabla \hat{\otimes} (\zeta - \nabla \log \Omega)$$

$$-(\zeta - \nabla \log \Omega) \hat{\otimes} (\zeta - \nabla \log \Omega) = 0$$

¹⁷ What is certainly more uncommon is that we do not use a subset of these equations to get some norm estimates for various terms assuming we already have a solution, but to solve them, which requires a delicate choice of the subset.

¹⁸ Remind that we used the relations between $\eta, \underline{\eta}$ and ζ , the expression of ω and $\underline{\omega}$ in terms of Ω and the fact that $\xi = \underline{\xi} = 0$, obtained from the first set of structure equations, see for instance [15], Chapter 3.

$$\begin{aligned}
 & [\mathbf{R}(e_3, e_4) = 0 :] \\
 & \mathbf{D}_4 \underline{\omega} - 2\omega \underline{\omega} - \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 \\
 & \quad + \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}) = 0 \\
 & [\mathbf{R}(e_3, e_3) = 0 :] \\
 & \mathbf{D}_3 \text{tr} \underline{\chi} + \frac{1}{2} (\text{tr} \underline{\chi})^2 + 2\omega \text{tr} \underline{\chi} + |\hat{\chi}|^2 = 0 \\
 & [\mathbf{R}(e_3, e_A) = 0 :] \\
 & \mathfrak{D}_3 \zeta + \text{tr} \underline{\chi} \zeta + \zeta \underline{\chi} + \mathfrak{d}iv \underline{\chi} - \nabla \text{tr} \underline{\chi} - \mathfrak{D}_3 \nabla \log \Omega = 0 \tag{2.30} \\
 & [\delta_{AB} \mathbf{R}(e_A, e_B) = 0 :] \\
 & \mathfrak{D}_3 \text{tr} \chi + \text{tr} \underline{\chi} \text{tr} \chi - 2\omega \text{tr} \chi - 2\mathfrak{d}iv (\zeta + \nabla \log \Omega) - 2|\zeta + \nabla \log \Omega|^2 + 2\mathbf{K} = 0 \\
 & [\widehat{\mathbf{R}(e_A, e_B)} = 0 :] \\
 & \mathfrak{D}_3 \hat{\chi} + \frac{1}{2} \text{tr} \underline{\chi} \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} - 2\omega \hat{\chi} \\
 & \quad - \nabla \widehat{\otimes} (\zeta + \nabla \log \Omega) - (\zeta + \nabla \log \Omega) \widehat{\otimes} (\zeta + \nabla \log \Omega) = 0 \\
 & \mathbf{R}(e_4, e_3) = 0 :] \\
 & \mathbf{D}_3 \omega - 2\omega \omega + \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 \\
 & \quad + \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \underline{\chi} \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}) = 0
 \end{aligned}$$

where $\mathbf{D}_3 = \mathbf{D}_{e_3}, \mathbf{D}_4 = \mathbf{D}_{e_4}, \mathfrak{D}_3, \mathfrak{D}_4$ their projection on the tangent spaces $TS(\lambda, \nu), \nabla$ the covariant derivatives associated with the metric γ induced by g on the surfaces $S(\lambda, \nu), \mathbf{K}$ is the curvature of these $S(\lambda, \nu)$ surfaces and

$$\widehat{\mathbf{R}(e_A, e_B)} = \mathbf{R}(e_A, e_B) - 2^{-1} \delta_{AB} \mathbf{R}(e_A, e_B).$$

In Appendix, we write the general form of the second set of structure equations and show how from them, the first set and our gauge choice equations (2.30) follow.

We have now the explicit expression for the first set and the second set of the structure equations, namely (2.28) and (2.30). They are 24 equations, ten from the first set and fourteen from the second one while we have only sixteen unknown functions $\Omega, X, \gamma, \chi, \underline{\chi}, \zeta, \omega, \underline{\omega}$. The fact that there are more equations than unknown functions is not, in the present case, a real difficulty as the structure equations are automatically satisfied in a Lorentzian manifold and, therefore, also in a vacuum Einstein manifold. Therefore, we have to choose a subset between them which forms a complete set of equations for the 16 unknown functions and then prove, as expected, that the remaining equations play the analogous role of the standard constraint equations and are automatically satisfied once they are imposed on the initial data. The nature of the constraint equations is discussed in detail in Sect. 2.4. The 16 equations we choose are:

$$\begin{aligned}
 & \frac{\partial \gamma}{\partial \lambda} - 2\Omega \underline{\chi} + \mathcal{L}_X \gamma = 0 \\
 & \frac{\partial \log \Omega}{\partial \lambda} + \nabla_X \log \Omega + 2\Omega \underline{\omega} = 0 \\
 & \mathcal{D}_3 \text{tr} \underline{\chi} + \text{tr} \underline{\chi} \text{tr} \underline{\chi} - 2\underline{\omega} \text{tr} \underline{\chi} - 2d\text{iv}(\zeta + \nabla \log \Omega) - 2|\zeta + \nabla \log \Omega|^2 + 2\mathbf{K} = 0 \\
 & \mathcal{D}_3 \hat{\chi} + \frac{1}{2} \text{tr} \underline{\chi} \hat{\chi} + \frac{1}{2} \text{tr} \underline{\chi} \hat{\chi} - 2\underline{\omega} \hat{\chi} - \nabla \widehat{\otimes}(\zeta + \nabla \log \Omega) \\
 & \quad - (\zeta + \nabla \log \Omega) \widehat{\otimes}(\zeta + \nabla \log \Omega) = 0 \tag{2.31} \\
 & \mathcal{D}_3 \zeta + \text{tr} \underline{\chi} \zeta + \zeta \underline{\chi} + d\text{iv} \underline{\chi} - \nabla \text{tr} \underline{\chi} - \mathcal{D}_3 \nabla \log \Omega = 0 \\
 & \mathbf{D}_3 \omega - 2\underline{\omega} \omega + \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 \\
 & \quad + \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \underline{\chi} \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}) = 0 \\
 & \frac{\partial X}{\partial \nu} + 4\Omega^2 Z = 0
 \end{aligned}$$

$$\begin{aligned}
 & L\mathcal{D}_4 \text{tr} \underline{\chi} + \text{tr} \underline{\chi} \text{tr} \underline{\chi} - 2\omega \text{tr} \underline{\chi} + 2d\text{iv}(\zeta - \nabla \log \Omega) - 2|\zeta - \nabla \log \Omega|^2 + 2\mathbf{K} = 0 \\
 & \mathcal{D}_4 \hat{\chi} + \frac{1}{2} \text{tr} \underline{\chi} \hat{\chi} + \frac{1}{2} \text{tr} \underline{\chi} \hat{\chi} - 2\omega \hat{\chi} + \nabla \widehat{\otimes}(\zeta - \nabla \log \Omega) \\
 & \quad - (\zeta - \nabla \log \Omega) \widehat{\otimes}(\zeta - \nabla \log \Omega) = 0 \tag{2.32} \\
 & \mathbf{D}_4 \underline{\omega} - 2\omega \underline{\omega} - \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 \\
 & \quad + \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \underline{\chi} \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}) = 0.
 \end{aligned}$$

Remarks. (i) The Eqs. (2.31) and (2.32) are appropriate, as we will see in the following, to apply Cauchy–Kowalevski theorem and find analytic solutions. To write them as first order p.d.e. equations for the tensor components requires still some more work due to the presence of the Gauss curvature K which depends on the second angular variables of the metric γ . This will be discussed in the next subsection.

(ii) The second important remark is that once we have solved Eqs. (2.31) and (2.32), looking at (2.30) our analytic solution is such that

$$\mathbf{R}(e_A, e_B) = 0, \quad \mathbf{R}(e_A, e_4) = 0, \quad \mathbf{R}(e_3, e_4) = 0$$

and we have still to prove that the remaining Ricci equations are satisfied. This is discussed in detail in Sect. 2.4 where we show that the remaining equations to be satisfied have to be considered as “constraint equations”.

(iii) Observe that in this approach there are ten independent connection coefficients, $\chi, \underline{\chi}, \zeta, \omega, \underline{\omega}$. They, basically, correspond to the second fundamental form k_{ij} of the “maximal foliation gauge” or of the “CMC foliation gauge”. The difference is that k has only six components. This is due to the fact that in those cases the foliation is made by only one family of hypersurfaces Σ_t while here there are both the $C(\lambda)$ and the $\underline{C}(\nu)$ null hypersurfaces. If we consider only the $\{C(\lambda)\}$ foliation, $\underline{\chi}, \zeta, \underline{\omega}$ are

the S -tensors corresponding to k (χ, ζ, ω in the opposite case). In both situations there are six components, as expected.

Equations (2.31), (2.32) are perfectly defined as tensorial equations, but, to consider them as p.d.e. equations, they have to be written as equations for the tensor components. In this case they do not maintain exactly the same expressions. There are many ways to rewrite these equations as standard partial differential equations whose unknown are the components of the various tensors involved, for instance one could choose a Fermi transported null orthonormal frame¹⁹ as was done in [15]; here, we present a more general approach using the diffeomorphism $\Psi(\lambda, \nu)$ previously introduced to map, via the pull-back associated with $\Psi(\lambda, \nu)$, these equations on a manifold $S_0 \times R^2$ where the equations become equations for the various components in the angular variables and in the variables λ and ν , which are just the parameters of the diffeomorphism $\Psi(\lambda, \nu)$. Here, we state the result and its detailed proof is in Appendix to this section.

Proposition 2.2. *In the coordinates $\{\lambda, \nu, \theta, \phi\}$ associated with the “double null foliation gauge”, Eqs. (2.31) and (2.32) written for the various of metric and connection coefficients components have the following expression:*

$$\begin{aligned}
 & \frac{\partial \gamma}{\partial \lambda} - 2\Omega \underline{\chi} + \mathcal{L}_X \gamma = 0 \\
 & \frac{\partial \log \Omega}{\partial \lambda} + \nabla_X \log \Omega + 2\Omega \underline{\omega} = 0 \\
 & \frac{\partial \text{tr} \chi}{\partial \lambda} + \Omega \text{tr} \underline{\chi} \text{tr} \chi - 2\Omega \underline{\omega} \text{tr} \chi + \nabla_X \text{tr} \chi - 2\Omega \text{div}(\zeta + \nabla \log \Omega) \\
 & \quad - 2\Omega |\zeta + \nabla \log \Omega|^2 + 2\Omega \mathbf{K} = 0 \\
 & \frac{\partial \hat{\chi}}{\partial \lambda} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \chi}{2} \underline{\hat{\chi}} + \frac{\partial \log \Omega}{\partial \lambda} \hat{\chi} + (\nabla_X \log \Omega) \hat{\chi} - \Omega (\hat{\chi} \cdot \underline{\hat{\chi}}) \gamma \\
 & - \Omega \nabla \hat{\otimes} (\zeta + \nabla \log \Omega) - \Omega (\zeta + \nabla \log \Omega) \hat{\otimes} (\zeta + \nabla \log \Omega) + \mathcal{L}_X \hat{\chi} = 0 \\
 & \frac{\partial \zeta}{\partial \lambda} + \Omega \text{tr} \underline{\chi} \zeta + \Omega \text{div} \underline{\chi} - \Omega \nabla \text{tr} \underline{\chi} - \frac{\partial \nabla \log \Omega}{\partial \lambda} + \Omega \nabla \log \Omega \cdot \underline{\chi} \\
 & \quad + \mathcal{L}_X \zeta - \mathcal{L}_X \nabla \log \Omega = 0 \\
 & \frac{\partial \omega}{\partial \lambda} + \nabla_X \omega - 2\Omega \underline{\omega} \omega - \frac{3}{2} \Omega |\zeta|^2 + \Omega \zeta \cdot \nabla \log \Omega + \frac{1}{2} \Omega |\nabla \log \Omega|^2 \\
 & \quad + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} \right) = 0 \tag{2.33} \\
 & \frac{\partial X}{\partial \nu} + 4\Omega^2 Z = 0 \\
 & \frac{\partial \text{tr} \chi}{\partial \nu} + \Omega \text{tr} \chi \text{tr} \chi - 2\Omega \omega \text{tr} \chi - 2\Omega \text{div}(-\zeta + \nabla \log \Omega) - 2\Omega | \\
 & \quad -\zeta + \nabla \log \Omega|^2 + 2\Omega \mathbf{K} = 0
 \end{aligned}$$

¹⁹ Nevertheless this is possible only with respect to a null direction, but not simultaneously to both.

$$\begin{aligned}
 & \frac{\partial \hat{\chi}}{\partial \nu} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \underline{\chi}}{2} \hat{\chi} + \frac{\partial \log \Omega}{\partial \nu} \hat{\chi} - \Omega(\hat{\chi} \cdot \hat{\chi})\gamma - \Omega \nabla \hat{\otimes}(-\zeta + \nabla \log \Omega) \\
 & - \Omega(-\zeta + \nabla \log \Omega) \hat{\otimes}(-\zeta + \nabla \log \Omega) = 0 \\
 & \frac{\partial \underline{\omega}}{\partial \nu} - 2\Omega \underline{\omega} \underline{\omega} - \frac{3}{2}\Omega |\zeta|^2 - \Omega \zeta \cdot \nabla \log \Omega + \frac{1}{2}\Omega |\nabla \log \Omega|^2 \\
 & + \frac{1}{2}\Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0.
 \end{aligned} \tag{2.34}$$

Equations (2.33), (2.34) are not yet a system of first order equations. In fact \mathbf{K} , the curvature of the two-dimensional surfaces $S(\lambda, \nu)$, depends on second tangential derivatives of γ . Moreover, in the (transport) evolution equations along $\underline{C}(\nu)$ for $\text{tr} \chi, \hat{\chi}, \zeta$ and in the (transport) evolution equations along $C(\lambda)$ for $\text{tr} \underline{\chi}$ and $\hat{\chi}$ the second derivatives of $\log \Omega$ with respect to the angular variables are present.

To have a real first order system we define some new independent variables and their evolution equations, namely:

$$v_{\dots} = \not\partial . \gamma(\cdot, \cdot), \quad w_{\dots} = \not\partial . X(\cdot), \quad \psi_{\dots} = \not\partial . \log \Omega. \tag{2.35}$$

Their evolution equations are obtained deriving the evolution equations of $\gamma(\cdot, \cdot), X(\cdot)$ and $\log \Omega$, the second one in the outgoing direction, the other two in the incoming one. The unknown function w_{\dots} is introduced as in the evolution equation for v the second tangential derivatives of X appear. These equations do not contain more than first derivatives of the previous unknown variables and this transforms the system of equations into a larger system of first order equations. It is a matter of computation, which we report in the appendix to this section, to obtain the following evolution equations for ψ_a, v_{adb} and for w_{ab} :

$$\begin{aligned}
 \frac{\partial}{\partial \lambda} \psi_a &= -2\Omega \partial_a \underline{\omega} - 2\Omega \underline{\omega} \psi_a - (\nabla_a X)^c \psi_c - X^c \nabla_c \psi_a \\
 \frac{\partial}{\partial \lambda} v_{cba} &= -(\partial_c X^d) v_{dab} - \partial_X v_{cab} + \partial_c w_{ab} + \partial_c w_{ba} + 2\Omega \partial_c \underline{\chi}_{ab} + 2\Omega \psi_c \underline{\chi}_{ab} \\
 \frac{\partial w_{ab}}{\partial \nu} &= -8\Omega^2 \psi_a \zeta_b - 4\Omega^2 \partial_a \zeta_b + 2\Omega \psi_a \chi_{bc} X^c + 2\Omega (\partial_a \chi_{bc}) X^c.
 \end{aligned} \tag{2.36}$$

We write now the final system of first order equations for the various tensors components omitting the indices to simplify the notations,

$$\begin{aligned}
 \frac{\partial \gamma}{\partial \omega} - v = 0, \quad \frac{\partial \log \Omega}{\partial \omega} - \psi = 0, \quad \frac{\partial \hat{X}}{\partial \omega} - w = 0 \\
 \frac{\partial \gamma}{\partial \lambda} - 2\Omega \underline{\chi} + \mathcal{L}_X \gamma = 0 \\
 \frac{\partial \log \Omega}{\partial \lambda} + \psi(X) + 2\Omega \underline{\omega} = 0 \\
 \frac{\partial v}{\partial \lambda} + \nabla_X v + (\not\partial X) \cdot v - S(\not\partial \otimes w) - 2\Omega \not\partial \otimes \underline{\chi} - 2\Omega \psi \otimes \underline{\chi} = 0 \\
 \frac{\partial \psi}{\partial \lambda} + \nabla_X \psi + 2\Omega \underline{\omega} \psi + \psi(\nabla X) + 2\Omega \not\partial \underline{\omega} = 0
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\partial \text{tr}\chi}{\partial \lambda} + \Omega \text{tr}\chi \text{tr}\chi - 2\Omega \omega \text{tr}\chi + \nabla_X \text{tr}\chi - 2\Omega \text{div}(\zeta + \psi) - 2\Omega |\zeta + \psi|^2 + 2\Omega \mathbf{K} = 0 \\
 & \frac{\partial \hat{\chi}}{\partial \lambda} + \mathcal{L}_X \hat{\chi} - \frac{\Omega \text{tr}\chi}{2} \hat{\chi} + \frac{\Omega \text{tr}\chi}{2} \hat{\chi} - 2\Omega \omega \hat{\chi} - \Omega(\hat{\chi} \cdot \hat{\chi})\gamma - \Omega \nabla \widehat{\otimes}(\zeta + \psi) \\
 & \quad - \Omega(\zeta + \psi) \widehat{\otimes}(\zeta + \psi) = 0 \\
 & \frac{\partial \zeta}{\partial \lambda} + \mathcal{L}_X \zeta + \Omega \text{tr}\chi \zeta + \Omega \text{div} \hat{\chi} - \frac{1}{2} \Omega \not\partial \text{tr}\chi + 2\Omega \omega \psi + 2\Omega \not\partial \omega + \Omega \psi \cdot \chi = 0 \\
 & \quad \frac{\partial \omega}{\partial \lambda} + \not\partial_X \omega - 2\Omega \omega \omega - \frac{3}{2} \Omega |\zeta|^2 + \frac{1}{4} \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 \\
 & \quad + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr}\chi \text{tr}\chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0 \tag{2.37} \\
 & \quad \frac{\partial \hat{X}}{\partial \nu} + 4\Omega^2 \zeta = 0
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\partial w}{\partial \nu} + 8\Omega^2 \psi \otimes \zeta + 4\Omega^2 \not\partial \otimes \zeta - 2\Omega \psi \otimes (\chi \cdot X) - 2\Omega (\not\partial \otimes \chi) \cdot X = 0 \\
 & \frac{\partial \text{tr}\chi}{\partial \nu} + \Omega \text{tr}\chi \text{tr}\chi - 2\Omega \omega \text{tr}\chi + 2\Omega \text{div} \zeta - 2\Omega \text{div} \psi - 2\Omega |\zeta - \psi|^2 + 2\Omega \mathbf{K} = 0 \\
 & \frac{\partial \hat{\chi}}{\partial \nu} - \frac{\Omega \text{tr}\chi}{2} \hat{\chi} + \frac{\Omega \text{tr}\chi}{2} \hat{\chi} - 2\Omega \omega \hat{\chi} - \Omega(\hat{\chi} \cdot \hat{\chi})\gamma + \Omega \nabla \widehat{\otimes}(\zeta - \psi) \\
 & \quad - \Omega(\zeta - \psi) \widehat{\otimes}(\zeta - \psi) = 0 \\
 & \quad \frac{\partial \omega}{\partial \nu} - 2\Omega \omega \omega - \frac{3}{2} \Omega |\zeta|^2 - \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 \\
 & \quad + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr}\chi \text{tr}\chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0 \tag{2.38}
 \end{aligned}$$

where \hat{X} is the covariant vector $\hat{X}_a = \gamma_{ab} X^b$, S means symmetrization, $V \widehat{\otimes} W$ is twice the traceless part of the symmetric tensorial product $S(V \otimes W)$, \mathbf{K} has to be thought as a function of γ, v and $\not\partial v$.²⁰

2.4. The First Order System of Equations as Solutions of the Vacuum Einstein Characteristic Problem, the Constraint Problem

The first order system of p.d.e. equations (2.37), (2.38) describes a characteristic problem which can be solved via the Cauchy–Kowalevski theorem (its characteristic version, as discussed later on in Sect. 3), giving the initial data on the two null hypersurfaces C_0 and \underline{C}_0 .²¹ As we said before, see remark (ii) after equations (2.32), the equations we want to solve are not all the equations associated with $\mathbf{R}_{\mu\nu} = 0$. Therefore, we have to determine under which conditions a solution of the Eqs. (2.37), (2.38) is a solution of the Einstein equations. Observe that, looking at the structure equations, apart from Eqs. (2.37), (2.38), the following equations have to be satisfied by the vacuum Einstein equations:

²⁰ $\not\partial$ is the ordinary partial derivative with respect to the angular variables, ω^a , and ∇ is the Levi-Civita connection with respect to γ .

²¹ Which initial data can be given freely and which constrained is a delicate point we discuss in detail in Sect. 4.2.

$$\begin{aligned}
 \mathbf{R}(e_4, e_A) = 0 : & \quad \frac{\partial \gamma}{\partial \nu} - 2\Omega\chi = 0, \quad \frac{\partial \log \Omega}{\partial \nu} + 2\Omega\omega = 0 \\
 & \quad \frac{\partial \zeta}{\partial \nu} + \Omega \operatorname{tr}\chi\zeta - \Omega \not{d}\nu\chi + \Omega \not{\nabla}\operatorname{tr}\chi - 2\Omega\omega\psi \\
 & \quad \quad - 2\Omega \not{\nabla}\omega - \Omega\psi \cdot \chi = 0 \tag{2.39} \\
 \mathbf{R}(e_4, e_4) = 0 : & \quad \frac{\partial \operatorname{tr}\chi}{\partial \nu} + \frac{\Omega \operatorname{tr}\chi}{2} \operatorname{tr}\chi + 2\Omega\omega \operatorname{tr}\chi + \Omega|\hat{\chi}|^2 = 0 \\
 \mathbf{R}(e_3, e_3) = 0 : & \quad \frac{\partial \operatorname{tr}\chi}{\partial \lambda} + \frac{\Omega \operatorname{tr}\chi}{2} \operatorname{tr}\chi + \not{\partial}_X \operatorname{tr}\chi + 2\Omega\omega \operatorname{tr}\chi + \Omega|\hat{\chi}|^2 = 0.
 \end{aligned}$$

The first two equations of (2.39) are at the “level” of equations for the metric components and follow from the first set of the structure equations. This is somewhat analogous to what happens in the maximal foliation gauge used by Christodoulou and Klainerman [8], where, once we impose $\operatorname{tr}k = 0$ on Σ_0 , one has to prove that $\operatorname{tr}k$ remains equal zero on any t -constant hypersurface, justifying the definition of the maximal foliation gauge. In other words, proving that the first two equations, once satisfied on C_0 , are satisfied on any $C(\lambda)$ shows that we are in the double null foliation gauge.

The remaining three equations have to be satisfied to make the components of the Ricci tensor, $\mathbf{R}(e_4, e_A)$, $\mathbf{R}(e_4, e_4)$ and $\mathbf{R}(e_3, e_3)$, identically zero. Here, it is appropriate to introduce the notion of “signature” for the various functions involved.²²

Definition 2.1. We call “signature” of the various connection coefficients the number of times the null vector e_4 appears in their definition minus the number of times e_3 is present. Each derivative along C_0 increases the signature by one and viceversa for each derivative along \underline{C}_0 .

Observe that the last three equations in (2.39) are at the level of connection coefficients and have signature $+1, +2$ and -2 . This can be interpreted as the indication that in these equations there are no “derivatives” with respect to the transverse directions, e_3 for $C(\lambda)$ and e_4 for $\underline{C}(\nu)$. These equations have, therefore, to be seen as constraint equations and we have to prove that, if satisfied from the initial data on C_0 and \underline{C}_0 , they are satisfied on each outgoing or incoming cone, respectively.²³ This is the content of the following lemma which connects the solutions of (2.37), (2.38) to the solutions of the Einstein equations,

Lemma 2.1. *Let $\Psi = (\gamma_{ab}, \log \Omega, X_a, v_{c,ab}, w_{ba}, \chi_{ab}, \zeta_a, \omega, \underline{\chi}_{ab}, \underline{\omega})$ be a solution of the first order system made by Eqs. (2.37), (2.38).*

If $\{\gamma_{ab}, \log \Omega, \operatorname{tr}\chi, \zeta\}$ are a solution of the first four equations of (2.39) on C_0 and $\operatorname{tr}\chi$ is a solution of the last one on \underline{C}_0 , it follows that they are solutions of the same equations on any cone $C(\lambda)$ and $\underline{C}(\nu)$, respectively.

²² This was introduced by Christodoulou and Klainerman [8], for the null components of the Riemann tensor.

²³ Observe that the effect of a coordinate choice and of the choice of the system of equations make the set of equations (2.39) asymmetric with respect to the λ, ν interchange. It is also easy to see that we have a certain arbitrariness in choosing the first order system, for instance one could interchange the role of the “ ν ” and “ λ ” directions.

Proof. We show that there exist first order transport equations along λ for the left hand sides of the first four equations of (2.39) and a transport equation along ν for the left hand side of the last equation. Therefore, if these expressions are zero on C_0 or on \underline{C}_0 , they are identically zero for all λ, ν values. A way to obtain these transport equations is just a long computation using Eqs. (2.37), (2.38). We write only the proof for the second equation of the first line of (2.39).

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial \log \Omega}{\partial \nu} + 2\Omega \omega \right) = \frac{\partial}{\partial \nu} \left(\frac{\partial \log \Omega}{\partial \lambda} \right) + 2 \left(\frac{\partial \log \Omega}{\partial \lambda} \right) \Omega \omega + 2\Omega \frac{\partial \omega}{\partial \lambda}. \tag{2.40}$$

Using again Eqs. (2.37), (2.38), the terms in the right hand side become

$$\begin{aligned} \frac{\partial}{\partial \nu} \left(\frac{\partial \log \Omega}{\partial \lambda} \right) &= \frac{\partial}{\partial \nu} (-2\Omega \omega - \partial_X \log \Omega) \\ &= -2 \left(\frac{\partial}{\partial \nu} \log \Omega \right) \Omega \omega - 2\Omega \frac{\partial \omega}{\partial \nu} - \frac{\partial X^a}{\partial \nu} \partial_a \log \Omega - \partial_X \left(\frac{\partial \log \Omega}{\partial \nu} \right) \\ 2 \left(\frac{\partial \log \Omega}{\partial \lambda} \right) \Omega \omega &= 2\Omega \omega (-2\Omega \omega - \partial_X \log \Omega) = -4\Omega^2 \omega \omega - 2\Omega \omega \partial_X \log \Omega. \end{aligned} \tag{2.41}$$

Substituting in (2.40) and denoting $\mathcal{I} = \left(\frac{\partial \log \Omega}{\partial \nu} + 2\Omega \omega \right)$ we obtain

$$\frac{\partial \mathcal{I}}{\partial \lambda} = -\nabla_X \mathcal{I} - 2\Omega \omega \mathcal{I} + 2\Omega \left[\left(\frac{\partial \omega}{\partial \lambda} + \partial_X \omega \right) - \frac{\partial \omega}{\partial \nu} - \frac{1}{2\Omega} \frac{\partial X^a}{\partial \nu} \partial_a \log \Omega \right] \tag{2.42}$$

and, using again Eqs. (2.37), (2.38), the term in the square bracket is identically zero. Therefore \mathcal{I} satisfies the following equation

$$\frac{\partial \mathcal{I}}{\partial \lambda} + \nabla_X \mathcal{I} + 2\Omega \omega \mathcal{I} = 0 \tag{2.43}$$

which implies that \mathcal{I} is equal to zero on every outgoing cone $C(\lambda)$ provided it is set equal zero on C_0 .

To complete the proof of the lemma at the level of the metric components, let us consider the first equation of (2.39). Proceeding as before and denoting

$$\mathcal{I}_{ab} = \frac{\partial \gamma_{ab}}{\partial \nu} - 2\Omega \chi_{ab},$$

a long, but straightforward computation shows that \mathcal{I}_{ab} satisfies the equation:

$$\frac{\partial \mathcal{I}_{ab}}{\partial \lambda} - \Omega \text{tr} \chi \mathcal{I}_{ab} + \mathcal{L}_X \mathcal{I}_{ab} = 0 \tag{2.44}$$

which implies again that if $\frac{\partial \gamma_{ab}}{\partial \nu} - 2\Omega \chi_{ab} = 0$ on C_0 then this relation holds on any $C(\lambda)$. To complete Lemma 2.1 the same result has to be proved for the remaining equations in (2.39). If we proceed as before the computation will turn out very long and laborious. This can be avoided observing that this

result follows by a straightforward application of the Bianchi equations. In fact let us consider the Lorentzian manifold with metric

$$g(\cdot, \cdot) = |X|^2 d\lambda^2 - 2\Omega^2(d\lambda d\nu + d\nu d\lambda) - X_a(d\lambda d\omega^a + d\omega^a d\lambda) + \gamma_{ab}d\omega^a d\omega^b,$$

where X, Ω, γ satisfies Eqs. (2.37), (2.38). From these equations and the result just stated it follows that the components of $\Psi, \{\chi_{ab}, \zeta_a, \omega, \underline{\chi}_{ab}, \underline{\omega}\}$, can be interpreted as the connection coefficients associated with this metric. Therefore, as they satisfy Eqs. (2.37), (2.38) it follows that, see Eq. (2.30), the null components $\mathbf{R}(e_A, e_B), \mathbf{R}(e_3, e_4)$ and $\mathbf{R}(e_3, e_A)$ of the Ricci tensor are identically zero. To prove the remaining part of Lemma 2.1 amounts to prove that also $\mathbf{R}(e_4, e_4), \mathbf{R}(e_A, e_4)$ and $\mathbf{R}(e_3, e_3)$ are identically zero, provided they are equal to zero on the initial hypersurface. To prove this result we use the contracted Bianchi equations. In fact from them one deduces the following identities

$$D^\mu R_{\mu\nu} - \frac{1}{2}D_\nu R = 0. \tag{2.45}$$

Denoting $\{e_3, e_4, e_A\}, A \in \{1, 2\}$, a null orthonormal frame and writing

$$g^{\mu\nu} = -\frac{1}{2}(e_3^\mu e_4^\nu + e_4^\mu e_3^\nu) + \sum_A e_A^\mu e_A^\nu,$$

Equation (2.45) can be written, multiplying it with e_4 and e_B respectively,

$$\begin{aligned} &-\frac{1}{2}(D_3 R_{\mu\nu})e_4^\mu e_4^\nu + \sum_A (D_A R_{\mu\nu})e_A^\mu e_4^\nu - \frac{1}{2}(D_4 R_{\mu\nu})e_A^\mu e_A^\nu = 0 \\ &-\frac{1}{2}(D_4 R_{\mu\nu})e_3^\mu e_B^\nu - \frac{1}{2}(D_3 R_{\mu\nu})e_4^\mu e_B^\nu + \sum_A (D_A R_{\mu\nu})e_A^\mu e_B^\nu + \frac{1}{2}(D_B R_{\mu\nu})e_4^\mu e_3^\nu \\ &-\sum_A \frac{1}{2}(D_B R_{\mu\nu})e_A^\mu e_A^\nu = 0. \end{aligned} \tag{2.46}$$

Rewriting these equations as transport equations for the various null Ricci components, from the first set of structure equations for the null frame, see for instance [15], Chapter 3:

$$\begin{aligned} \mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2}\chi_{AB}e_3 + \frac{1}{2}\underline{\chi}_{AB}e_4 \\ \mathbf{D}_A e_3 &= \underline{\chi}_{AB}e_B + \zeta_A e_3, & \mathbf{D}_A e_4 &= \chi_{AB}e_B - \zeta_A e_4 \\ \mathbf{D}_3 e_A &= \mathfrak{D}_3 e_A + \eta_A e_3, & \mathbf{D}_4 e_A &= \mathfrak{D}_4 e_A + \underline{\eta}_A e_4 \\ \mathbf{D}_3 e_3 &= (\mathbf{D}_3 \log \Omega)e_3, & \mathbf{D}_3 e_4 &= -(\mathbf{D}_3 \log \Omega)e_4 + 2\eta_B e_B \\ \mathbf{D}_4 e_4 &= (\mathbf{D}_4 \log \Omega)e_4, & \mathbf{D}_4 e_3 &= -(\mathbf{D}_4 \log \Omega)e_3 + 2\underline{\eta}_B e_B \end{aligned}$$

and recalling that all the null Ricci components with signature -1 and 0 are already equal to zero, Eq. (2.46) become:

$$\begin{aligned}
 & - \left(\frac{\partial}{\partial \lambda} + \partial_X \right) \mathbf{R}(e_4, e_4) - \text{tr} \underline{\chi} \mathbf{R}(e_4, e_4) + 4 \underline{\omega} \mathbf{R}(e_4, e_4) + 2 \nabla_A \mathbf{R}(e_A, e_4) \\
 & \quad - 2(\eta_B - g(\nabla_A e_A, e_B) + \nabla_B \log \Omega) \mathbf{R}(e_B, e_4) = 0 \tag{2.47} \\
 & - \left(\frac{\partial}{\partial \lambda} + \partial_X \right) \mathbf{R}(e_B, e_4) - \text{tr} \underline{\chi} \mathbf{R}(e_A, e_4) - \underline{\chi}(e_A, e_B) \mathbf{R}(e_B, e_4) - 4 \underline{\omega} \mathbf{R}(e_B, e_4) \\
 & \quad + g(\mathcal{D}_3 e_A, e_B) \mathbf{R}(e_B, e_4) = 0.
 \end{aligned}$$

From the second equation and the assumed initial conditions it follows that $\mathbf{R}(e_B, e_4) = 0$, which, substituted in the first equation, implies that $\mathbf{R}(e_4, e_4) = 0$. The proof that $\mathbf{R}(e_3, e_3) = 0$ goes in the same way, is somewhat simpler and we do not report here. Once Lemma 2.1 is proved it follows that Ψ is a solution of the vacuum Einstein equations.

Remark. The previous discussion makes crystal clear, in the Einstein equations characteristic problem, which are the equations we have to consider as evolution equations and which have to be interpreted as constraint equations, which is enough to satisfy on the “initial data” to have them satisfied everywhere, The first ones are Eqs. (2.37), (2.38), while the “constraint equations” are Eq. (2.39) which do not involve inward (outward) derivatives of the initial data.

One also has to remark that if we consider only the Eqs. (2.37), (2.38) and we do not care about initial data satisfying (2.39), we are still considering a well-defined characteristic problem whose solutions nevertheless do not define an Einstein vacuum spacetime. Nevertheless, as this is a characteristic problem in itself, even in this more general case the initial data have to satisfy some constraints, namely the initial data associated with $\{\gamma, \Omega, v, \psi, \chi, \zeta, \omega\}$ have to satisfy the constraints prescribed by Eq. (2.37) on $\underline{C}_0 \equiv \underline{C}(\nu_0)$, while they are given in a free way on $C_0 = C(\lambda_0)$ and the opposite has to be imposed for the initial data of $\{\hat{X}, w, \underline{\chi}, \underline{\omega}\}$.

In conclusion one has to recognize that, in some sense, the characteristic problem for the Einstein equations has two kind of constraint equations that the initial data have to satisfy, the first one connected to the more general problem (2.37), (2.38) and the second one to the requirement that also equations (2.39) have to be satisfied. We summarize this discussion in the following theorem,

Theorem 2.1. *Let $\Psi = (\gamma_{ab}, \log \Omega, X_a, v_{c,ab}, \psi_a, w_{ba}, \chi_{ab}, \zeta_a, \omega, \underline{\chi}_{ab}, \underline{\omega})$ be a solution, in a region, $\{(\lambda, \nu) | (\lambda, \nu) \in [\lambda_0, \bar{\lambda}] \times [\nu_0, \bar{\nu}]\}$, of the characteristic first order Cauchy problem made by Eqs. (2.37), (2.38) with the initial data on the null hypersurface $\mathcal{S} = C_0 \cup \underline{C}_0$ satisfying on \mathcal{S} , beside Eqs. (2.37), (2.38), considered as equations on \underline{C}_0 and C_0 respectively, the constraint equations, see (2.39),*

$$\begin{aligned}
\text{On } C_0 : \quad & \frac{\partial \gamma}{\partial \nu} - 2\Omega \chi = 0, \quad \frac{\partial \log \Omega}{\partial \nu} + 2\Omega \omega = 0 \\
& \frac{\partial \text{tr} \chi}{\partial \nu} + \frac{\Omega \text{tr} \chi}{2} \text{tr} \chi + 2\Omega \omega \text{tr} \chi + \Omega |\hat{\chi}|^2 = 0 \\
& \frac{\partial \zeta}{\partial \nu} + \Omega \text{tr} \chi \zeta - \Omega \text{div} \chi + \Omega \not\partial \text{tr} \chi + \frac{\partial \not\partial \log \Omega}{\partial \nu} - \Omega \not\partial \log \Omega \cdot \chi = 0 \\
\text{On } \underline{C}_0 : \quad & \frac{\partial \text{tr} \underline{\chi}}{\partial \lambda} + \frac{\Omega \text{tr} \underline{\chi}}{2} \text{tr} \underline{\chi} + \partial_X \text{tr} \underline{\chi} + 2\Omega \underline{\omega} \text{tr} \underline{\chi} + \Omega |\hat{\underline{\chi}}|^2 = 0, \tag{2.48}
\end{aligned}$$

then in the same region the metric tensor

$$\mathbf{g}(\cdot, \cdot) = |X|^2 d\lambda^2 - 2\Omega^2 (d\lambda d\nu + \nu d\lambda) - X_a (d\lambda d\omega^a + d\omega^a d\lambda) + \gamma_{ab} d\omega^a d\omega^b \tag{2.49}$$

is a solution of the Einstein vacuum equations.

Summarizing we collect here all the constraint equations the initial data have to satisfy, namely Eqs. (2.37), (2.38) and (2.39),

On C_0 :

$$\begin{aligned}
& \not\partial \gamma - \nu = 0, \quad \not\partial X - w = 0, \quad \not\partial \log \Omega - \psi = 0 \\
& \frac{\partial \gamma}{\partial \nu} - 2\Omega \chi = 0, \quad \frac{\partial X}{\partial \nu} + 4\Omega^2 Z = 0, \quad \frac{\partial \log \Omega}{\partial \nu} + 2\Omega \omega = 0 \\
& \frac{\partial \text{tr} \chi}{\partial \nu} + \frac{\Omega \text{tr} \chi}{2} \text{tr} \chi + 2\Omega \omega \text{tr} \chi + \Omega |\hat{\chi}|^2 = 0 \\
& \frac{\partial \zeta}{\partial \nu} + \Omega \text{tr} \chi \zeta - \Omega \text{div} \chi + \Omega \not\partial \text{tr} \chi + \frac{\partial \not\partial \log \Omega}{\partial \nu} - \Omega \not\partial \log \Omega \cdot \chi = 0 \\
& \frac{\partial \text{tr} \underline{\chi}}{\partial \nu} + \Omega \text{tr} \chi \text{tr} \underline{\chi} - 2\Omega \omega \text{tr} \underline{\chi} + 2\Omega \text{div} \zeta - 2\Omega \not\Delta \log \Omega \\
& \quad - 2\Omega |\zeta - \not\partial \log \Omega|^2 + 2\Omega \mathbf{K} = 0 \\
& \frac{\partial \hat{\chi}}{\partial \nu} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \chi}{2} \hat{\chi} - 2\Omega \omega \hat{\chi} - \Omega (\hat{\chi} \cdot \hat{\chi}) \gamma + \Omega \nabla \hat{\otimes} \zeta - \Omega \nabla \hat{\otimes} \not\partial \log \Omega \\
& \quad - \Omega (-\zeta + \not\partial \log \Omega) \hat{\otimes} (-\zeta + \not\partial \log \Omega) = 0 \\
& \frac{\partial \underline{\omega}}{\partial \nu} - 2\Omega \omega \underline{\omega} - \frac{3}{2} \Omega |\zeta|^2 + \Omega \zeta \cdot \not\partial \log \Omega \\
& \quad + \frac{1}{2} \Omega |\not\partial \log \Omega|^2 + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0
\end{aligned} \tag{2.50}$$

On \underline{C}_0 :

$$\begin{aligned}
& \frac{\partial \gamma}{\partial \lambda} - 2\Omega \underline{\chi} + \mathcal{L}_X \gamma = 0, \quad \frac{\partial \log \Omega}{\partial \lambda} + \partial_X \log \Omega + 2\Omega \underline{\omega} = 0 \\
& \frac{\partial \text{tr} \underline{\chi}}{\partial \lambda} + \frac{\Omega \text{tr} \underline{\chi}}{2} \text{tr} \underline{\chi} + \partial_X \text{tr} \underline{\chi} + 2\Omega \underline{\omega} \text{tr} \underline{\chi} + \Omega |\hat{\underline{\chi}}|^2 = 0 \\
& \frac{\partial \zeta}{\partial \lambda} + \Omega \text{tr} \chi \zeta + \Omega \text{div} \underline{\chi} - \Omega \not\partial \text{tr} \underline{\chi} - \frac{\partial (\not\partial \log \Omega)}{\partial \lambda} + \Omega \not\partial \log \Omega \cdot \underline{\chi} \\
& \quad + \mathcal{L}_X \zeta - \mathcal{L}_X \not\partial \log \Omega = 0
\end{aligned}$$

$$\begin{aligned}
 & \frac{\partial \text{tr} \chi}{\partial \lambda} + \Omega \text{tr} \underline{\chi} \text{tr} \chi - 2\Omega \underline{\omega} \text{tr} \chi + \partial_X \text{tr} \chi - 2\Omega \text{div} \zeta - 2\Omega \Delta \log \Omega \\
 & - 2\Omega |\zeta + \nabla \Omega|^2 + 2\Omega \mathbf{K} = 0 \\
 & \frac{\partial \hat{\chi}}{\partial \lambda} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\partial \log \Omega}{\partial \lambda} \hat{\chi} + (\partial_X \log \Omega) \hat{\chi} - \Omega (\hat{\chi} \cdot \hat{\chi}) \gamma - \Omega \nabla \hat{\otimes} \zeta \\
 & - \Omega \nabla \hat{\otimes} \not\partial \log \Omega - \Omega (\zeta + \not\partial \log \Omega) \hat{\otimes} (\zeta + \not\partial \log \Omega) + \mathcal{L}_X \hat{\chi} = 0 \tag{2.51} \\
 & \frac{\partial \omega}{\partial \lambda} + \nabla_X \omega - 2\Omega \underline{\omega} \omega - \frac{3}{8} \Omega |\zeta|^2 - \Omega \zeta \cdot \not\partial \log \Omega + \frac{1}{2} \Omega |\not\partial \log \Omega|^2 \\
 & + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \underline{\chi} \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0.
 \end{aligned}$$

Remark. The implementation of the initial conditions, namely the way of obtaining initial data satisfying the constraint equations (2.50) and (2.51) with appropriate (Sobolev) regularity and asymptotic behaviour has been discussed in [4]. Here we will have basically to repeat the same argument, but imposing the analyticity, this makes this problem more complicated and how we solve it will be discussed later on. Next section, Sect. 3, is devoted to find a local analytic solution for the system of Eqs. (2.37) and (2.38). To do it we rewrite them in a more compact notation and show how, following Duff [10] and Friedrich [11], we can apply the Cauchy–Kowalevski theorem to this characteristic case.

3. The Analytic Solution of the Characteristic Problem via the Cauchy-Kowalevski Theorem

The method we use to obtain a real analytic solution of the characteristic problem defined by the system of equations (2.37) and (2.38) with initial data satisfying Eqs. (2.50) and (2.51) is a variant of the Cauchy–Kowalevski method. The adaptation of the Cauchy–Kowalevski theorem to characteristic problems has been developed by Duff [10], for the linear case and, subsequently, by Friedrich [11], for the non-linear problem. Friedrich result is suited to the present case, therefore we just recall the main lines of the proof, a straightforward adaptation of his result.

The system of Eqs. (2.37) and (2.38) can be written in a much more compact form in the following way:

$$\begin{aligned}
 \frac{\partial \mathbf{V}}{\partial \lambda} &= \mathbf{F}(\mathbf{V}, \mathbf{W}, \not\partial \mathbf{V}, \not\partial \mathbf{W}) \\
 \frac{\partial \mathbf{W}}{\partial \nu} &= \mathbf{G}(\mathbf{V}, \mathbf{W}, \not\partial \mathbf{V}),
 \end{aligned} \tag{3.52}$$

where \mathbf{V} and \mathbf{W} are vector functions valued in R^{18} and R^{10} , respectively, defined by

$$\begin{aligned}
 \mathbf{V} = \{V^s\} &= \{\gamma, \Omega, v, \psi; \omega, \zeta, \chi\}, & \mathbf{W} = \{W^t\} &= \{X, w, \underline{\omega}, \underline{\chi}\} \\
 s \in \{1, \dots, 18\}, & & t \in \{1, \dots, 10\}.
 \end{aligned} \tag{3.53}$$

The initial data are assigned on the union of the null hypersurfaces C_0 and \underline{C}_0 . They have to be analytic functions satisfying the constraint equations (2.50) and (2.51); this is possible as it is proved later on and has been proved in [4] for initial data belonging to a suitable Sobolev space.²⁴ Let us denote the initial data $\mathbf{V}_0 = \mathbf{V}_0(\nu, \omega^a)$, $\mathbf{W}_0 = \mathbf{W}_0(\lambda, \omega^a)$ on C_0 and \underline{C}_0 , respectively.²⁵

The existence of a local real analytic solution of the system (3.52) with initial data $\mathbf{V}_0 = \mathbf{V}_0(\nu, \omega^a)$, $\mathbf{W}_0 = \mathbf{W}_0(\lambda, \omega^a)$ is proved in the following theorem:²⁶

Theorem 3.1. *The system of equations (3.52) with initial data $\mathbf{V}_0 = \mathbf{V}_0(\nu, \omega^a)$, $\mathbf{W}_0 = \mathbf{W}_0(\lambda, \omega^a)$ admits a unique real analytic solution in a region $(\lambda, \nu) \in [0, \bar{\lambda}] \times [0, \bar{\nu}]$ whose size is determined by the initial data.*

Proof. System (3.52) can be rewritten for the new unknown functions

$$(\mathbf{V} - \mathbf{V}_0)(\lambda, \nu, \omega^a) \quad \text{and} \quad (\mathbf{W} - \mathbf{W}_0)(\lambda, \nu, \omega^a)$$

we denote again \mathbf{V} and \mathbf{W} . It is easy to show that it has the form:

$$\frac{\partial V^s}{\partial \lambda} = F_{s'}^{s,a} \frac{\partial V^{s'}}{\partial \omega^a} + \tilde{F}_{t'}^{s,a} \frac{\partial W^{t'}}{\partial \omega^a} + f^s \tag{3.54}$$

$$\frac{\partial W^t}{\partial \nu} = \tilde{G}_{s'}^{t,a} \frac{\partial V^{s'}}{\partial \omega^a} + g^t, \tag{3.55}$$

with $\omega^a \in \{\theta, \phi\}$ and we sum over repeated indices. With these new \mathbf{V}, \mathbf{W} , the initial data are

$$\mathbf{V}_0 = (0, \dots, 0), \quad \mathbf{W}_0 = (0, \dots, 0) \tag{3.56}$$

and the coefficients $F_{s'}^{s,a}, \tilde{F}_{t'}^{s,a}, f^s, \tilde{G}_{s'}^{t,a}, g^t$ depend, besides \mathbf{V} and \mathbf{W} , on the original data \mathbf{V}_0 and \mathbf{W}_0 and through them on the coordinates $\{x^\mu\}$.

Following Friedrich [11], the proof is basically made by two main steps. The first one provides a recursive mechanism to get all the derivatives in ν, λ, ω^a for V and W observing that, as initial data, we have all the derivatives in ν, ω^a for V and in λ, ω^a for W . The remaining mixed derivatives are obtained through the equations (3.54) and (3.55). In the second and more delicate step we prove the convergence of the formal power series we have obtained.

(i) **The recursive determination of the derivatives:** From Eq. (3.55) we control $\partial_\nu W$ and $\partial_\nu \nabla^q W$ for any $q \geq 0$. From Eq. (3.54) we control $\partial_\lambda V$ and $\partial_\lambda \nabla^q V$ for any $q \geq 0$. Deriving Eq. (3.54) with respect to ν we control $\partial_\nu \partial_\lambda \nabla^q V$, deriving Eq. (3.55) with respect to λ we control $\partial_\nu \partial_\lambda \nabla^q W$. Deriving with respect to λ equation (3.54) we control $\partial_\lambda^2 \nabla^q V$ and deriving with respect to ν

²⁴ The situation is somewhat simpler here with respect to [4] due to the fact that we are considering a local problem and we do not have to worry about the asymptotic behaviour of the initial data, but only require that the initial data be analytic.

²⁵ The initial data defined here, $\mathbf{V}_0 = \mathbf{V}_0(\nu, \omega^a)$, $\mathbf{W}_0 = \mathbf{W}_0(\lambda, \omega^a)$, satisfy the constraint equations (2.39), the remaining ones are automatically obtained using the Cauchy-Kowalevski method.

²⁶ For notational simplicity we have redefined λ, ν as $\lambda - \lambda_0, \nu - \nu_0$, see the proof in Lemma 3.1.

equation (3.55) we control $\partial_\nu^2 \nabla^q W$. Iterating the procedure we obtain all the mixed derivatives.

Remark. It should be clear that this procedure which allows to obtain formal power series for V and W both on C_0 and on \underline{C}_0 satisfies also the constraint equations for V on \underline{C}_0 and W on C_0 as discussed in remark before Theorem 2.1.

(ii) **The convergence of the formal series:** The functions $F_{s'}^{s,a}, \tilde{F}_{t'}^{s,a}, \tilde{G}_{s'}^{t,a}, f^s, g^t$ depend on the analytic initial data \mathbf{V}_0 and \mathbf{W}_0 and on the unknown functions \mathbf{V} and \mathbf{W} . More specifically, looking at the explicit expression of system (2.37) and (2.38), they are polynomials in \mathbf{V} and \mathbf{W} and can be written as

$$\begin{aligned}
 F_{s'}^{s,a} &= \sum_{\alpha\beta} F_{s';\alpha\beta}^{s,a}(x^\mu)W^\alpha V^\beta, & \tilde{F}_{t'}^{s,a} &= \sum_{\alpha\beta} \tilde{F}_{t';\alpha\beta}^{s,a}(x^\mu)W^\alpha V^\beta \\
 \tilde{G}_{s'}^{t,a} &= \sum_{\alpha\beta} \tilde{G}_{s';\alpha\beta}^{t,a}(x^\mu)W^\alpha V^\beta \\
 f^s &= \sum_{\alpha\beta} f_{\alpha\beta}^s(x^\mu)W^\alpha V^\beta, & g^t &= \sum_{\alpha\beta} g_{\alpha\beta}^t(x^\mu)W^\alpha V^\beta,
 \end{aligned}
 \tag{3.57}$$

with

$$\begin{aligned}
 |\alpha| = 0, \quad |\beta| \leq 2 \text{ for } F, \quad |\alpha| = 0, \quad |\beta| \leq 1 \text{ for } \tilde{F} \\
 |\alpha| = 0, \quad |\beta| \leq 2 \text{ for } \tilde{G} \\
 |\alpha| \leq 1, \quad |\beta| \leq 3 \text{ for } f, \quad |\alpha| \leq 1, \quad |\beta| \leq 3 \text{ for } g.
 \end{aligned}
 \tag{3.58}$$

As we assumed that the initial data $\mathbf{V}_0, \mathbf{W}_0$ are real analytic, the functions $F_{s';\alpha\beta}^{s,a}(x^\mu), \tilde{F}_{t';\alpha\beta}^{s,a}(x^\mu), \tilde{G}_{s';\alpha\beta}^{t,a}(x^\mu), f_{\alpha\beta}^s(x^\mu), g_{\alpha\beta}^t(x^\mu)$ are real analytic²⁷ and can be written as power series in $\{x^\mu\}$,

$$\begin{aligned}
 F_{s';\alpha\beta}^{s,a}(x) &= \sum_\gamma F_{s';\alpha\beta\gamma}^{s,a} x^\gamma, & \tilde{F}_{t';\alpha\beta}^{s,a}(x) &= \sum_\gamma \tilde{F}_{t';\alpha\beta\gamma}^{s,a} x^\gamma \\
 \tilde{G}_{s';\alpha\beta}^{t,a}(x) &= \sum_\gamma \tilde{G}_{s';\alpha\beta\gamma}^{t,a} x^\gamma \\
 f_{\alpha\beta}^s(x) &= \sum_\gamma f_{\alpha\beta\gamma}^s x^\gamma, & g_{\alpha\beta}^t(x) &= \sum_\gamma g_{\alpha\beta\gamma}^t x^\gamma.
 \end{aligned}
 \tag{3.59}$$

Due to the real analyticity we can assume that their convergence radius be $> R$,²⁸ for a given $R > 0$ and that in $B_R(0) \subset R^4$ the coefficients of the expansions (3.59) satisfy

$$|F_{s';\alpha\beta\gamma}^{s,a}|, |\tilde{F}_{t';\alpha\beta\gamma}^{s,a}|, |\tilde{G}_{s';\alpha\beta\gamma}^{t,a}|, |f_{\alpha\beta\gamma}^s|, |g_{\alpha\beta\gamma}^t| \leq \frac{M}{R^{|\gamma|}}.
 \tag{3.60}$$

²⁷ $\{x^\mu\} = \{x^1, x^2, x^3, x^4\} = \{\lambda, \nu, \omega^a\}$.

²⁸ As we are concerned about a local solution we consider a compact portion of $C_0 \cup \underline{C}_0$.

Let us define the function $H(\theta_\mu x^\mu) = H(y)$, with $\theta_1 > 1, \theta_2 = \theta_3 = \theta_4 = 1$:

$$\begin{aligned}
 H(\theta_\mu x^\mu) &\equiv \frac{M}{1 - \frac{\theta_\mu x^\mu}{R}} = \frac{MR}{R - \theta_\mu x^\mu} \\
 &= \sum_\gamma \frac{M}{R^{|\gamma|}} \frac{|\gamma|!}{\gamma_1! \gamma_2! \gamma_3! \gamma_4!} \theta_1^{\gamma_1} x^\gamma = \sum_\gamma H_\gamma x^\gamma. \tag{3.61}
 \end{aligned}$$

It follows from the previous definitions that

$$|F_{s';\alpha\beta\gamma}^{s,a}|, |\tilde{F}_{t';\alpha\beta\gamma}^{s,a}|, |\tilde{G}_{s';\alpha\beta\gamma}^{t,a}|, |f_{\alpha\beta\gamma}^s|, |g_{\alpha\beta\gamma}^t| \leq H_\gamma \tag{3.62}$$

which means that the function $H(\theta_\mu x^\mu)$ majorizes the functions $F_{s';\alpha\beta}^{s,a}(x)$, $\tilde{F}_{t';\alpha\beta}^{s,a}(x)$, $\tilde{G}_{s';\alpha\beta}^{t,a}(x)$, $f_{\alpha\beta}^s(x)$, $g_{\alpha\beta}^t(x)$. We define now the functions $\hat{F}_{s'}^{s,a}$, $\hat{F}_{t'}^{s,a}$, $\hat{G}_{s'}^{t,a}$, \hat{f}^s, \hat{g}^t as the power series (3.57) with the coefficient functions $F_{s';\alpha\beta}^{s,a}(x^\mu) \dots$ substituted by the function $H(\theta_\mu x^\mu)$ times a matrix $P_{s';\alpha\beta}^{s,a} \dots$ with non-negative real coefficients and which defines the same polynomial $r(\mathbf{V}, \mathbf{W}) = \sum_{\alpha\beta} \tilde{p}_{\alpha\beta} W^\alpha V^\beta, \tilde{p}_{\alpha\beta} > 0, |\alpha| \leq 1, |\beta| \leq 3$, for all indices s, s', t, a . Therefore²⁹

$$\begin{aligned}
 \hat{F}_{s'}^{s,a} &= H(\theta_\mu x^\mu) \sum_{\alpha\beta} P_{s';\alpha\beta}^{s,a} W^\alpha V^\beta = H(\theta_\mu x^\mu) r(\mathbf{V}, \mathbf{W}) P_{s'}^{s,a} \\
 \hat{F}_{t'}^{s,a} &= H(\theta_\mu x^\mu) \sum_{\alpha\beta} \tilde{P}_{t';\alpha\beta}^{s,a} W^\alpha V^\beta = H(\theta_\mu x^\mu) r(\mathbf{V}, \mathbf{W}) \tilde{P}_{t'}^{s,a} \\
 \hat{G}_{s'}^{t,a} &= H(\theta_\mu x^\mu) \sum_{\alpha\beta} \tilde{Q}_{s';\alpha\beta}^{t,a} W^\alpha V^\beta = H(\theta_\mu x^\mu) r(\mathbf{V}, \mathbf{W}) \tilde{Q}_{s'}^{t,a} \tag{3.63} \\
 \hat{f}^s &= H(\theta_\mu x^\mu) \sum_{\alpha\beta} p_{\alpha\beta}^s W^\alpha V^\beta = H(\theta_\mu x^\mu) r(\mathbf{V}, \mathbf{W}) p^s \\
 \hat{g}^t &= H(\theta_\mu x^\mu) \sum_{\alpha\beta} q_{\alpha\beta}^t W^\alpha V^\beta = H(\theta_\mu x^\mu) r(\mathbf{V}, \mathbf{W}) q^t.
 \end{aligned}$$

Let us consider the following system of equations:

$$\begin{aligned}
 \frac{\partial \hat{V}^s}{\partial \lambda} &= \hat{F}_{s'}^{s,a} \frac{\partial \hat{V}^{s'}}{\partial \omega^a} + \hat{F}_{t'}^{s,a} \frac{\partial \hat{W}^{t'}}{\partial \omega^a} + \hat{f}^s(x, \hat{V}, \hat{W}) \\
 \frac{\partial \hat{W}^t}{\partial \nu} &= \hat{G}_{s'}^{t,a} \frac{\partial \hat{V}^{s'}}{\partial \omega^a} + \hat{g}^t(x, \hat{V}, \hat{W}), \tag{3.64}
 \end{aligned}$$

the following lemma holds:

Lemma 3.1. *The solution $\hat{\mathbf{V}}, \hat{\mathbf{W}}$ of system (3.64) with initial conditions $\hat{\mathbf{V}}_0 = \hat{\mathbf{W}}_0$ majorizing the initial conditions $\mathbf{V}_0 = \mathbf{W}_0 = 0$, is majorizing \mathbf{V} and \mathbf{W} , solution of system (3.54) (3.55) with initial conditions $\mathbf{V}_0 = \mathbf{W}_0 = 0$.*

Proof. The first step is to write formal power expansions for \mathbf{V} and \mathbf{W} ; this is a formal expansion around a generic point of $S_0 \equiv S(\lambda_0, \nu_0)$; redefining λ and ν as $\lambda - \lambda_0, \nu - \nu_0$ we consider it as an expansion around $(\lambda, \nu) = (0, 0)$. The convergence of the formal expansion can be repeated for all the angular

²⁹ The matrix elements $P_{s'}^{s,a}, \tilde{P}_{t'}^{s,a}, \tilde{Q}_{s'}^{t,a}, p^s, q^t$ are all zero or one.

coordinates $\{\omega^a\}$ of the points of S_0 so that, finally, we obtain an analytic solution in a neighbourhood of S_0 . Therefore, redefining also ω^a as $(\omega^a - \omega_0^a)$ we have:

$$V^s(x) = \sum_{j,k,\underline{a}} v_{jk\underline{a}}^s \lambda^j \nu^k \omega_1^{a_1} \omega_2^{a_2}, \quad W^t(x) = \sum_{j,k,\underline{a}} w_{jk\underline{a}}^t \lambda^j \nu^k \omega_1^{a_1} \omega_2^{a_2}. \quad (3.65)$$

Observe that the initial conditions imply $v_{0k\underline{a}}^s = w_{j0\underline{a}}^t = 0$ and that, plugging the formal expansions in (3.54) and (3.55), we obtain recursively, as explained before, all the coefficients $v_{jk\underline{a}}^s$ and $w_{jk\underline{a}}^t$. We proceed in the same way expanding the solution $\hat{\mathbf{V}}, \hat{\mathbf{W}}$ of system (3.64),

$$\hat{V}^s(x) = \sum_{j,k,\underline{a}} \hat{v}_{jk\underline{a}}^s \lambda^j \nu^k \omega_1^{a_1} \omega_2^{a_2}, \quad \hat{W}^t(x) = \sum_{j,k,\underline{a}} \hat{w}_{jk\underline{a}}^t \lambda^j \nu^k \omega_1^{a_1} \omega_2^{a_2} \quad (3.66)$$

and we observe that, due to the definition of the functions $\hat{F}_{s'}^{s,a}, \hat{F}_{t'}^{s,a}, \hat{G}_{s'}^{t,a}, \hat{f}^s, \hat{g}^t$, (3.63), it follows that $\hat{v}_{jk\underline{a}}^s, \hat{w}_{jk\underline{a}}^t \geq 0$ and that the formal solution $(\hat{\mathbf{V}}, \hat{\mathbf{W}})$, (3.66), majorizes the (formal) solution (\mathbf{V}, \mathbf{W}) , (3.65), namely the following inequalities hold, for all index values,

$$\hat{v}_{jk\underline{a}}^s \geq |v_{jk\underline{a}}^s|, \quad \hat{w}_{jk\underline{a}}^t \geq |w_{jk\underline{a}}^t|. \quad (3.67)$$

The proof is achieved if we can find a real analytic solution of the system (3.64) with initial conditions such that the coefficients of its power expansion satisfy $\hat{v}_{0k\underline{a}}^s \geq 0, \hat{w}_{j0\underline{a}}^t \geq 0$; in this case, for an appropriate radius R , the expansions (3.65) describe a real analytic solution of system (3.55) with initial conditions $\mathbf{V}_0 = \mathbf{W}_0 = 0$.

To find a solution of the system (3.64) with initial conditions $\hat{\mathbf{V}}_0 = \hat{\mathbf{W}}_0$ such that

$$\hat{v}_{0k\underline{a}}^s \geq 0, \hat{w}_{j0\underline{a}}^t \geq 0, \quad (3.68)$$

we make the following ansatz:

$$\begin{aligned} \hat{V}^s(x^\mu) &= \tilde{V}^s(y = \theta_\mu x^\mu) = \tilde{V}(y) \\ \hat{W}^t(x^\mu) &= \tilde{W}^t(y = \theta_\mu x^\mu) = \tilde{W}(y). \end{aligned} \quad (3.69)$$

Each equation for \hat{V}^s and \hat{W}^t of system (3.64) becomes in terms of \tilde{V} and \tilde{W} , denoting

$$\begin{aligned} \tilde{V}' &= \frac{\partial \tilde{V}}{\partial y}, \quad \tilde{W}' = \frac{\partial \tilde{W}}{\partial y}, \\ \theta_1 \tilde{V}' &= H(y)q(\tilde{V}, \tilde{W})(36\tilde{V}' + 20\tilde{W}' + c) \\ \tilde{W}' &= H(y)q(\tilde{V}, \tilde{W})(36\tilde{V}' + 1) \end{aligned} \quad (3.70)$$

where $q(\tilde{V}, \tilde{W})$ is a polynomial in \tilde{V}, \tilde{W} with positive coefficients; therefore

$$\begin{aligned}
 (\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)\tilde{V}' &= Hq(20Hq + 1) \\
 \tilde{W}' &= Hq + 36Hq \frac{Hq(20Hq + c1)}{(\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)} \\
 &= \frac{1}{(\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)} [\theta_1 Hq]
 \end{aligned}
 \tag{3.71}$$

and the two equations can be rewritten as

$$\begin{aligned}
 (\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)\tilde{V}' &= Hq(20Hq + 1) \\
 (\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)\tilde{W}' &= \theta_1 Hq.
 \end{aligned}
 \tag{3.72}$$

Initial conditions for $\hat{\mathbf{V}}_0 = \hat{\mathbf{W}}_0$ such that (3.68) holds are satisfied if we require as initial conditions for \tilde{V} and \tilde{W}

$$\tilde{V}(0) = 0, \quad \tilde{W}(0) = 0,
 \tag{3.73}$$

therefore we are reduced to prove that the system of equations

$$\begin{aligned}
 (\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)\tilde{V}' &= Hq(20Hq + 1) \\
 (\theta_1 - 36Hq - 20 \cdot 36(Hq)^2)\tilde{W}' &= \theta_1 Hq
 \end{aligned}
 \tag{3.74}$$

with initial conditions (3.73) has a real analytic solution in a neighborhood of the origin. Choosing θ_1 such that

$$\theta_1 - 36H(0)q(0, 0) - 20 \cdot 36(H(0)q(0, 0))^2 > 0
 \tag{3.75}$$

equations (3.74) can be written in the form

$$\begin{aligned}
 \tilde{V}' &= H(y, \tilde{V}, \tilde{W}) \\
 \tilde{W}' &= J(y, \tilde{V}, \tilde{W}),
 \end{aligned}
 \tag{3.76}$$

where H and J are real analytic functions having power series expansions at $y = 0, \tilde{V} = 0, \tilde{W} = 0$ with non-negative expansion coefficients. Therefore a real analytic solution of the system (3.64) exists and there is a neighborhood of $S_0 = S(\lambda_0, \nu_0)$ where the real analytic solution of the system (3.55) does exist. This completes the analytic part of our result giving a concrete meaning to Theorem 2.1.

Remark. Recall that the local analytic solution whose proof has been sketched now is a solution for $\mathbf{V} - \mathbf{V}_0, \mathbf{W} - \mathbf{W}_0$. Then \mathbf{V}, \mathbf{W} are a local solution of the Einstein equations provided \mathbf{V}_0 and \mathbf{W}_0 are analytic functions satisfying the initial data constraints (2.50), (2.51).

4. A Common Region of Existence for the Real Analytic Solutions of the Non-Linear Characteristic Problem

We look for an “Alhincac type” result [1,13] proving that, if some Sobolev norms, H^s , of a real analytic solution are controlled up to a certain s ,³⁰ then its “Cauchy–Kowalevski existence region” can be extended to a larger region whose size depends only on these Sobolev norms. This kind of results have been started by Lax [16], extended by Nirenberg [23] and proved, in the main lines, for the Burger equation by Klainerman and one of the authors (F.N.) [14]. These results requires the hyperbolicity of the partial differential equations we are considering which is evident in the case of the Burger equation, but more hidden in the case of the Einstein equations.³¹

It is appropriate, before discussing our result for the Einstein equations, to look in a detailed way what has been obtained in the case of the Burger equation [14].

4.1. A Summary of the Analytic Extended Solution Approach to the Burger Equation in [14]

We give a complete survey of the various steps of this approach in the case of the Burger equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \tag{4.77}$$

Step 1: We introduce a Banach space $B_{\alpha,\rho}$ defined by the norm

$$\|f(\cdot, t)\|_{B_{\alpha,\rho}} = \sum_{n=0}^{\infty} \frac{\|D^n f(\cdot, t)\|_{L^2}}{n!} n^\alpha \rho^n, \tag{4.78}$$

where $L^2 = L^2(R^{n_0})$ and, defining the multiindex $\beta = (\beta_1, \beta_2, \dots, \beta_{n_0})$,

$$\frac{\|D^n f(\cdot, t)\|_{L^2}}{n!} \equiv \sum_{\beta; |\beta|=n} \frac{\|D^\beta f(\cdot, t)\|_{L^2}}{\beta!}. \tag{4.79}$$

In the Burger equation case, $n_0 = 1$, but we keep $n_0 \geq 1$ as we are interested to extend this result to a more general case. We assume that the initial data of the hyperbolic equation we are considering are given by an analytic function $u^{(0)}$ which belongs to B_{α,ρ_0} for a certain $\alpha > 2$ and $\rho_0 > 0$.

The next two lemmas prove that if $f(\cdot, t) \in B_{\alpha,\rho}$ then, as a function of the “spatial” variables $\{x^i\}$, $f(\cdot, t)$ is real analytic in R^{n_0} . and that also the viceversa holds.

Lemma 4.1. *Let f be a function belonging to the Banach space $B_{\alpha,\rho}$ with*

$$\alpha \geq \left\lceil \frac{n_0}{2} \right\rceil,$$

then $f(\cdot, t)$ is real analytic in $B(x)_\rho$.

³⁰ For our problem they are explicitly defined in Sect. 4.2.

³¹ The hyperbolicity for the Einstein equations is explicit in the harmonic gauge, in a more general setting, for instance in the geometric gauge we are considering here, it expresses itself in the existence of a priori estimates for some energy-type norms, see also [8,15].

We do not report its simple proof here. The opposite statement follows in next lemma and its proof is immediate,

Lemma 4.2. *Let $f(t, x)$ be a real analytic function in a strip around the real axis of uniform height 2ρ , then for any $T_a < \rho f(t, \cdot)$ belongs to the Banach space $B_{\alpha, \rho'}$ with appropriate $\rho' < \rho$ and α .*

Step 2: The following theorem holds:³²

Theorem 4.1. *Let us consider a p.d.e. whose coefficients are real analytic functions, let $u^{(0)} \in B_{\alpha, \rho_0}$ be the initial data defined on R^{n_0} ; given $\rho \leq \rho_0$ there exists a time $T_a(\rho, \rho_0, u^{(0)})$ and a solution $u(t, x) \in C^1([0, T_a], B_{\alpha, \rho})$ satisfying the initial conditions.*

This result follows more or less immediately recalling how Cauchy–Kowalevski theorem works; given (real) analytic initial data, using the p.d.e. equation it is possible to control all the derivatives in space and time of the (formal) series which defines the function $u(t, x)$, analytic solution of our equation with initial data $u^{(0)}$ provided the series is convergent. Once this is proved the solution $u(t, x)$ is real analytic with a convergence radius ρ , a priori depending on the point $x \in R^{n_0}$; if $u^{(0)}$ belongs to B_{α, ρ_0} this implies that the convergence radius of its power series is uniform in x . If the real analytic coefficients of the p.d.e. are defined, for instance, in the whole R^{n_0+1} with a common convergence radius, it follows that $u(t, x)$ can be defined in a strip around the real axis of uniform width $2\Delta_1$. Therefore the solution exists for all $|t| < \Delta_1$. This does not prevent the possibility that choosing as initial data on the hypersurface $\Sigma_{\bar{t}}$ with $|\bar{t}| = \Delta_1 - \epsilon$ the function $u(\bar{t}, x)$ it could be possible, again using the Cauchy–Kowalevski theorem, to extend the solution to a larger strip and then iterating this procedure forever or up to a moment when this has to end. The first case can happen, for instance, if the p.d.e. equation is linear as it follows that the strip width does not depend on t , while the second case takes place if in the iteration steps the width becomes smaller and smaller so that the strip where the analyticity is proved cannot be extended anymore.

Nevertheless, it has to be pointed out that in the statement of the theorem the time $T_a(\rho, \rho_0, u^{(0)})$ does not define the larger possible existence region, but only the region where the function $u(t, x)$, thought as a function of the space variables only is real analytic, with convergence radius ρ . In other words, T_a finite does not prevent the possibility that $u(t, x)$ can be extended as real analytic function in a larger region, but only that for $t > T_a$ the function $u(t, \cdot)$ does not belong to $B_{\alpha, \rho}$ but, possibly, to a $B_{\alpha, \rho'}$ with $\rho' < \rho$. To consider the (strip) largest possible analyticity region, possible infinite, we define

$$\bar{T} \equiv \sup\{T_a > 0 \mid \rho(T_a, \rho_0, u^{(0)}) > 0\}. \tag{4.80}$$

If $\bar{T} < \infty$, it must happen that

$$\lim_{T_a \rightarrow \bar{T}} \rho(T_a, \rho_0) = 0 \tag{4.81}$$

³² Although this theorem is valid for a larger class of partial differential equations, we look at its result applied to an evolution equation of hyperbolic type.

and therefore in the limit $t \rightarrow \bar{T}$ the function $\lim_{t \rightarrow \bar{T}} u(t, \cdot)$ is not real analytic anymore in the space variables.

The main conclusion from this discussion is that this approach does not provide a real control of the largest existence region. This can be a problem, for instance, if we are considering a sequence of real analytic solutions $u^{(n)}(t, x)$ associated with a sequence of real analytic initial data converging to a function belonging to a Sobolev space. In fact, based on the previous theorem we cannot make a statement on the existence region of the limit solution as the $T_a^{(n)}$ can shrink to zero as $n \rightarrow \infty$.

In the case of the hyperbolic equation the situation is different. While in the general case, as discussed, the control of the time \bar{T} is based, via the Cauchy-Kowalevski theorem, on the estimates of the coefficients of the p.d.e. equation and on the initial data $u^{(0)}$, when the equation we are considering is hyperbolic one can have a better control of the solution existence time, namely one can prove that the analytic solution can be extended beyond \bar{T} up to a time T which depends only on the suitable Sobolev H^s norm of the initial data. In the following of this section we discuss in some detail the Klainerman and Nicolò [14] result for the Burger equation.

We start considering the Burger equation solution $u(t, \cdot) \in C^1([0, T_a], B_{\alpha, \rho})$ with initial data $u^{(0)} \in B_{\alpha, \rho_0}$ and with, as discussed before,

$$\rho = \rho(T_a, \rho_0, u^{(0)}) < \rho_0.$$

Energy conservation and Sobolev inequalities imply that

$$\|u(\cdot, t)\|_{H^2} \leq \|u^{(0)}\|_{H^2} \exp c \int_0^t \|u(\cdot, s)\|_{H^2} ds \tag{4.82}$$

and

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^2} \leq \|u^{(0)}\|_{H^2} \exp c \int_0^T \|u(\cdot, s)\|_{H^2} ds \tag{4.83}$$

which implies

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^2} \leq \|u^{(0)}\|_{H^2} \exp (cT \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^2}). \tag{4.84}$$

Therefore, we have the following lemma.

Lemma 4.3. *Let $u(t, \cdot) \in C^1([0, T_a], B_{\alpha, \rho})$ be solution of the Burger equation with initial data $u^{(0)} \in B_{\alpha, \rho_0}$, then there exists a time $T > 0$, depending only on $\|u^{(0)}\|_{H^2}$ such that for any $T' \leq \min\{T, T_a\}$*

$$\sup_{t \in [0, T']} \|u(\cdot, t)\|_{H^2} \leq c_0 \|u^{(0)}\|_{H^2}. \tag{4.85}$$

Inequality (4.85) is satisfied if T verifies,

$$T \leq \frac{\log c_0}{c_0 c \|u^{(0)}\|_{H^2}}. \tag{4.86}$$

Remark. In the non-linear hyperbolic equations it is possible that the a priori estimates do not hold beyond a certain time value due to the fact that the equations do not have global solutions. In this case there will be a time value $T^* > T$ where some norm blows up. It could also happen that with more refined a priori estimates inequality like (4.85) can be extended for all t values. In the case of the Burger equation we are in the first situation.³³

Step 3: Next goal is to prove the following fundamental theorem:

Theorem 4.2. *Let $u^{(0)} \in B_{\alpha, \rho_0}$ be the initial data, then for $t \in [0, T']$ there exists an analytic solution, $u(\cdot, t) \in B_{\alpha, \rho''}$ where $\rho'' < \rho_0$ and, moreover, ρ'' does not depend on T_a , but only on $\rho_0, \|u^{(0)}\|_{B_{\alpha, \rho_0}}$. Finally in the time interval $[0, T']$, the following inequality holds with a constant c_4 ,*

$$\sup_{t \in [0, T']} \|u(\cdot, t)\|_{B_{\alpha, \rho''}} \leq c_4 \|u^{(0)}\|_{B_{\alpha, \rho_0}}. \tag{4.87}$$

Remark. The central part of this result is that ρ'' appearing in the analytic solution

$$u \in C^1([0, T'], B_{\alpha, \rho''}),$$

does not depend on T_a ,

$$\rho'' = \rho''(\rho_0, \|u^{(0)}\|_{B_{\alpha, \rho_0}}),$$

while before, from the Cauchy–Kowalevski theorem, we had

$$u \in C^1([0, T_a], B_{\alpha, \rho})$$

with $\rho = \rho(T_a, \rho_0, u^{(0)})$. It is this fact that allows to extend the solution beyond the time \bar{T} defined in (4.80), see the next theorem.

Proof. The proof of the theorem is based on the following lemma,

Lemma 4.4. *Let $u(t, \cdot) \in C^1([0, T_a], B_{\alpha, \rho})$ be the analytic solution of the Burger equation with initial data $u^{(0)} \in B_{\alpha, \rho_0}$, with $\alpha > 2$. Then, for any J , the following estimates hold,³⁴*

$$\|D^J u(t)\|_{L^2} \leq C_0 e^{(J-2)\gamma t} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J}, \tag{4.88}$$

where C_0 is a constant satisfying

$$c_0 \|u^{(0)}\|_{B_{\alpha, \rho_0}} \leq C_0 \tag{4.89}$$

and c_0 is defined in the inequality (4.85). $\gamma > \gamma_0$, where

$$\gamma_0 = \sup_{t \in [0, T']} \|u(\cdot, t)\|_{H^2} \tag{4.90}$$

and $\gamma - \gamma_0$ depends on C_0 .

³³ For the vacuum Einstein equations with small data we are in the second case, see [8].

³⁴ We use the simplified notation $\|D^J u(t)\|_{L^2} \equiv \|D^J u(\cdot, t)\|_{L^2}$. moreover with D^J we indicate the spatial derivatives, in the Burger case $D^J = \frac{d^J}{dx^J}$.

Sketch of the proof of the lemma. The proof of Lemma 4.4, see [14], is inductive, therefore one starts proving that inequality (4.88) is true for $J = 1, 2$.³⁵ This is obtained from the a priori estimate (4.85), in fact, for $J = 1, 2$ and $t \leq T'$

$$\|D^{J=1,2}u(t)\|_{L^2} \leq \|u(\cdot, t)\|_{H^2} \leq c_0\|u(0)\|_{H^2} = c_0 \sum_{J=0}^2 \|D^J u^{(0)}\|_{L^2}. \tag{4.91}$$

Therefore it is enough to choose C_0 such that

$$\begin{aligned} \|D^{J=1}u(t)\|_{L^2} &\leq c_0\|u(0)\|_{H^2} \leq C_0 \frac{1}{\rho_0} \\ \|D^{J=2}u(t)\|_{L^2} &\leq c_0\|u(0)\|_{H^2} \leq C_0 \frac{1}{2^\alpha} \frac{1}{\rho_0^2}. \end{aligned} \tag{4.92}$$

Both inequalities are satisfied if, with a constant $c_1 \geq c_0(2^\alpha \rho_0^2)$, C_0 satisfies:

$$C_0 \geq c_1 \|u^{(0)}\|_{H^2}. \tag{4.93}$$

We do not repeat here the inductive estimates, see pages 97, 98 of [14], but we recall that in the case of the Burger equation the structure of the estimate is

$$\|D^N u(t)\|_{L^2} \leq e^{(N+2)\gamma_0 t} \|D^N u(0)\|_{L^2} + \int_0^t dt F(\{D^{J < N} u(t)\}), \tag{4.94}$$

where F is a complicated expression, see Eqs. (7.20)–(7.23) of [14], which, nevertheless, depends only on the partial derivatives of u of order lower than N and is estimated using the inductive assumption. To estimate $\|D^N u(0)\|_{L^2}$ for an arbitrary $N > 2$ we can only use the fact that $u(0, \cdot) = u^{(0)}$ belongs to B_{α, ρ_0} and therefore satisfies

$$\|D^N u(0)\|_{L^2} \leq \|u^{(0)}\|_{B_{\alpha, \rho_0}} \frac{N!}{N^\alpha} \frac{1}{\rho_0^N}. \tag{4.95}$$

Therefore, inequality (4.94) becomes, for $N > 2$ and $\gamma > 5\gamma_0$,

$$\begin{aligned} \|D^N u(t)\|_{L^2} &\leq e^{(N+2)\gamma_0 t} \|u^{(0)}\|_{B_{\alpha, \rho_0}} \frac{N!}{N^\alpha} \frac{1}{\rho_0^N} + \int_0^t dt F(\{D^{J < N} u(t)\}) \\ &\leq e^{(N-2)\gamma t} \|u^{(0)}\|_{B_{\alpha, \rho_0}} \frac{N!}{N^\alpha} \frac{1}{\rho_0^N} + \int_0^t dt F(\{D^{J < N} u(t)\}) \end{aligned} \tag{4.96}$$

$$\leq e^{(N-2)\gamma t} \frac{C_0}{2} \frac{N!}{N^\alpha} \frac{1}{\rho_0^N} + \int_0^t dt F(\{D^{J < N} u(t)\}) \tag{4.97}$$

provided we choose C_0 satisfying, beside (4.93),

$$C_0 \geq 2 \|u^{(0)}\|_{B_{\alpha, \rho_0}}. \tag{4.98}$$

³⁵ In the General Relativity case the induction will start at a different value of J .

Using the inductive assumption the integration part in (4.96) can be estimated, see [14], as

$$\int_0^t dt F(\{D^{J < N} u(t)\}) \leq e^{(N-2)\gamma t} \frac{C_0}{2} \frac{N!}{N^\alpha} \frac{1}{\rho_0^N} \left[\frac{2C_0 c_1 c_2 k_1 k_2}{\rho_0(\gamma - \gamma_0)} \right], \tag{4.99}$$

where c_1, c_2, k_1, k_2 are constants which do not depend on the solution norms. It follows that the lemma is proved if

$$\left[\frac{2C_0 c_1 c_2 k_1 k_2}{\rho_0(\gamma - \gamma_0)} \right] \leq 1. \tag{4.100}$$

Inequality (4.100) gives a lower bound on $\gamma - \gamma_0$. Therefore, Lemma 4.4 is proved provided $C_0, (\gamma - \gamma_0), \alpha$ satisfy

$$C_0 \geq (c_1 + 2) \|u^{(0)}\|_{B_{\alpha, \rho_0}}, \quad (\gamma - \gamma_0) \geq \max \left(\frac{2c_1 c_2 k_1 k_2}{\rho_0} C_0, 4\gamma_0 \right), \quad \alpha > 2. \tag{4.101}$$

Defining

$$\rho' = \rho_0 e^{-\gamma T'} \tag{4.102}$$

it follows that the previous estimates for the J derivative become

$$\sup_{t \in [0, T']} \|D^J u(t)\|_{L^2} \leq C_0 \frac{J!}{J^\alpha} \frac{1}{\rho'^J} \tag{4.103}$$

and defining c_3 such that

$$c_3 \|u^{(0)}\|_{B_{\alpha, \rho_0}} \geq C_0 \geq c_1 \|u^{(0)}\|_{B_{\alpha, \rho_0}} \tag{4.104}$$

we obtain, with $\rho'' < \rho' < \rho_0$,

$$\sup_{t \in [0, T']} \|u(t)\|_{B_{\alpha, \rho''}} \leq C_0 \sum_{N=0}^{\infty} \left(\frac{\rho''}{\rho'} \right)^N \leq c_3 \frac{\rho'}{\rho' - \rho''} \|u^{(0)}\|_{B_{\alpha, \rho_0}} \leq c_4 \|u^{(0)}\|_{B_{\alpha, \rho_0}}, \tag{4.105}$$

proving Theorem 4.2.

Step 4: We are left to prove that we can extend the solution beyond T_a , this is the content of the following theorem:

Theorem 4.3. *Let u be an analytic solution of the Burger equation with initial data $u(\cdot, 0) = u^{(0)} \in B_{\alpha, \rho_0}$ with $\alpha > 2$ then the solution can be extended to an analytic solution $\in B_{\alpha, \rho'''}$ with*

$$\rho''' < \rho_1 = \rho_0 e^{-\gamma T}, \tag{4.106}$$

up to a time T which depends only on the Sobolev norm of the initial data $\|u^{(0)}\|_{H^2}$, see (4.86). Moreover

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{B_{\alpha, \rho'''}} \leq c_5 \|u^{(0)}\|_{B_{\alpha, \rho_0}}. \tag{4.107}$$

Proof. Given the initial data $u^{(0)} \in B_{\alpha, \rho_0}$ we know that given T_a there exists an analytic solution u such that for any $t \in [0, T_a]$

$$u(\cdot, t) \in C^1([0, T_a]; B_{\alpha, \rho})$$

with $\rho = \rho(T_a, \rho_0, u^{(0)})$. The previous theorem, Theorem 4.2 says something much stronger, namely that there exists $\rho_1 = \rho_0 e^{-\gamma T}$ such that for any $T_a < T$,³⁶

$$u \in C^1([0, T_a]; B_{\alpha, \rho_1}), \tag{4.108}$$

where γ and T depend only on $\|u^{(0)}\|_{B_{\alpha, \rho_0}}$ and $\|u^{(0)}\|_{H^2(\mathbb{R}^{n_0})}$, respectively. Let us define T^* in the following way:

$$T^* = \sup\{T_a \in [0, T] \mid \text{for any } t \in [0, T_a] \ u(T_a, \cdot) \in B_{\alpha, \rho_1}\}. \tag{4.109}$$

As in the interval $[0, T_a]$, u is an analytic solution in all the variables, it follows that taking the limit $t \rightarrow T^*$ also $u(\cdot, T^*)$ is an analytic solution $\in B_{\alpha, \rho''}$ with $\rho'' < \rho_1$. In fact, as discussed before, T_a does not define the largest region of the Cauchy Kowalevski solution, but the largest region where $u(\cdot, t) \in B_{\alpha, \rho_1}$.

If $T^* = T$ we have proved our result. Let us, therefore, assume that $T^* < T$, we can apply again the Cauchy–Kowalevski theorem defining the real analytic function $u^{(1)} \equiv u(\cdot, T^*)$ as the initial data on Σ_{T^*} and proving that there exists an interval $[T^*, T^* + \Delta]$ with $T \geq T^* + \Delta$ such that, for t in this interval, there is a function, solution of the Burger equation with this initial data,

$$v(t, \cdot) \in B_{\alpha, \tilde{\rho}},$$

with $\tilde{\rho} > 0$.³⁷

Observe that Δ depends only on $\tilde{\rho}, \rho_1$ and $\|u^{(1)}\|_{B_{\alpha, \rho_1}}$ or, in other words, only $\tilde{\rho}, \rho_0, T, \|u^{(0)}\|_{B_{\alpha, \rho_0}}$. Let us consider now the function w defined in the following way, for an arbitrary $k \geq 1$,

$$\begin{aligned} w(t, \cdot) &= u(t, \cdot) \in C^k([0, T^*], B_{\alpha, \rho''}), \quad \text{for } t \in [0, T^*] \\ w(t, \cdot) &= v(t, \cdot) \in C^k([T^*, T^* + \Delta], B_{\alpha, \tilde{\rho}}), \quad \text{for } t \in [T^*, T^* + \Delta]. \end{aligned} \tag{4.110}$$

It is clear that $w(t, \cdot)$ is C^k in the time variable for $t \in [0, T^* + \Delta]$ and we can conclude that ($\tilde{\rho} < \rho'''$),

$$w(t, \cdot) \in C^k([0, T^* + \Delta], B_{\alpha, \tilde{\rho}})$$

and is a solution of the Burger equation in this time interval with initial data $w(0, \cdot) = u^{(0)}(\cdot)$.

Let Δ be such that $T^* + \Delta \leq T$, we can apply again Theorem 4.2 to this function, prove that exactly the same inductive estimates hold³⁸ and conclude that

³⁶ Recall that in the previous theorem $\rho'' < \rho' = \rho_0 e^{-\gamma T'}$ and $T' = \min\{T_a, T\}$, therefore if $T_a < T$ we can choose $\rho'' > \rho_1 = \rho_0 e^{-\gamma T}$.

³⁷ It follows simply applying Lemma 4.2.

³⁸ In this case $T_a = T^* + \Delta$.

$$w(t, \cdot) \in C^k([0, T^* + \Delta], B_{\alpha, \rho''}) \subset C^k([0, T^* + \Delta], B_{\alpha, \rho'''}).$$

As $T^* + \Delta > T^*$, it follows that T^* cannot be the sup as defined in (4.109) unless $T^* = T$. Therefore, we conclude that with initial data $u^{(0)} \in B_{\alpha, \rho_0}$ there exists a solution u in $[0, T]$ such that

$$u(t, \cdot) \in C^k([0, T], B_{\alpha, \rho'''}),$$

which means that the size of the spacetime region where the Cauchy-Kowalevski solution can be extended, depends only on the $H^{s=2}$ norm of $u^{(0)}$.

Remarks. (1) Observe that all the previous results are valid if the initial data belong to a different Banach space B_{α, ρ'_0} with $\rho'_0 < \rho_0$, the time T and the fact that the solution can be extended at least up to T does not change.

(2) The fact that C_0 is lower bounded by $\|u^{(0)}\|_{B_{\alpha, \rho_0}}$ and not only on $\|u^{(0)}\|_{H^2}$ is not harmful and, on the other side, is what has to be expected. In fact, at the end of the whole proof, the various real analytic solutions u_n (associated with the sequence of real analytic initial data $\{u_n^{(0)}\}$) will have the same existence time T , but their higher derivatives will have increasing bounds as $n \rightarrow \infty$. Therefore denoting $C_0^{(n)}$ the constant C_0 “associated” to $u_n^{(0)}$ it will follow that $C_0^{(n)} \rightarrow \infty$ as $n \rightarrow \infty$.

Step 5: The construction of a Sobolev solution. In this case we have a sequence of real analytic functions $\{u_n^{(0)}\}$ converging in the H^2 norms to $u^{(0)} \in H^2$. It is clear that due to this convergence the H^2 norms of all the $u_n^{(0)}$ functions are bounded by $c\|u^{(0)}\|_{H^2}$. On the other side the L^2 norms of the higher tangential derivatives will, in general, diverge as $n \rightarrow \infty$, therefore we also have

$$\lim_{n \rightarrow \infty} \|u_n^{(0)}(\cdot)\|_{B_{\alpha, \rho_0^{(n)}}} = \infty. \tag{4.111}$$

Then we have a sequence of real analytic solutions, $\{u_n(t, x)\}$ defined in the same interval of time $[0, T]$ and the L^2 norms of the higher, > 2 , derivatives satisfy

$$\|D^J u_n(t)\|_{L^2} \leq C_0^{(n)} \frac{j!}{j^\alpha \rho(n)^j} \tag{4.112}$$

and as $C_0^{(n)} \geq c' \|u_n^{(0)}(\cdot)\|_{B_{\alpha, \rho}}$,

$$\lim_{n \rightarrow \infty} C_0^{(n)} = \infty. \tag{4.113}$$

This is exactly what we expect as the sequence of real analytic functions converge to a function which is not real analytic. The fact that this sequence has a radius ρ depending on $n, \rho^{(n)}$ follows immediately repeating the procedure to obtain the estimate (4.112) and looking at the needed estimate (4.100) which now reads

$$\left[\frac{2C_0^{(n)} c_1 c_2 k_1 k_2}{\rho_0(\gamma^{(n)} - \gamma_0)} \right] \leq 1. \tag{4.114}$$

As $\gamma^{(n)}$ has to increase it follows that³⁹

$$\rho_{(n)} \leq \rho_1^{(n)} = \rho_0 e^{-\gamma^{(n)} T}. \tag{4.115}$$

Therefore, as already said, although the radius of convergence of the sequence of the analytic solutions of the Burger equation, with initial data approximating Sobolev data, tends to shrink, nevertheless the time interval $[0, T]$ where the analytic solutions do exist does not change.

4.2. The Extension of the Previous Result to the Einstein Vacuum Equations in the Characteristic Case

The extension of the previous result to the Einstein vacuum equations in the characteristic case requires a detailed discussion, let us indicate all the differences, with respect to the Burger case, we have to deal with.

(1) The initial data: The analogous of data on Σ_0 for the Burger equation are the data given on $\mathcal{C} = C_0 \cup \underline{C}_0$. The discussion on how to obtain real analytic data satisfying the constraints on this null hypersurface requires again the use of a ‘‘Cauchy–Kowalevski-type’’ argument for the local part and a ‘‘Burger-type’’ argument for the global one. More precisely, let us consider first the initial data on C_0 , first we have to specify on S_0 all the tangential derivatives for the \mathcal{O} and the $\underline{\mathcal{O}}$ quantities, where only some constraints have to be fulfilled, see [4]. Then we use the transport equations, on (a portion of) C_0 , for the $\nabla^N \mathcal{O}$ variables for any N . Next, once we have these estimates, we can use the transport equations for the $\nabla^N \underline{\mathcal{O}}$ variables for any N . Finally we use the transport equations just as relations expressing derivatives with respect to ν in terms of lower order ν derivatives (in the case of the \mathcal{O} quantities) or lower order ν derivatives and tangential derivatives for the $\underline{\mathcal{O}}$ quantities. Once all these estimates have been done we know that the \mathcal{O} and $\underline{\mathcal{O}}$ quantities satisfy the transport equations along (a portion of) C_0 which are exactly the ‘‘initial condition equations’’ of Theorem 2.1, Eqs. (2.50) and (2.51).

Exactly the same has to be done on (a portion of) \underline{C}_0 inverting the role of the \mathcal{O} and $\underline{\mathcal{O}}$ quantities. This completes the local construction of the analytic initial data.

Applying now a ‘‘Burger type argument’’, we obtain estimates for all the derivatives in $\{\nu, \omega^b\}$ of the \mathcal{O} and $\underline{\mathcal{O}}$ quantities proving that, being these quantities in a $\mathcal{B}_{\alpha, \rho_0}$ -type Banach space, they can be extended as real analytic functions on the whole $C_0 \cup \underline{C}_0$. The details of this construction are given in the following paper.⁴⁰

³⁹ In the same way also in the construction of the initial data approximating Sobolev data we expect that ρ_0 depends on $n, \rho_0 = \rho_0^{(n)}$.

⁴⁰ The Burger argument developed before requires a bootstrap argument and, before, an a priori estimate. In the case of the initial data the mechanism is slightly different and although also in the initial data case, we need a recursive mechanism and a bootstrap argument, the

(2) The local existence for real analytic solutions: Next step is to prove a local existence for our problem. This is done using the initial data on a local neighborhood of S_0 and the version of the Cauchy–Kowalevski theorem adapted to this non-linear characteristic problem as discussed in Sect. 3. This is the preliminary step needed to prove a larger existence region as discussed in Point 3.

(3) The largest existence region: This is the central part of the proof. One has to show, as in the Burger case, that the existence region of an analytic solution depends only on some appropriate Sobolev norms of the analytic initial data. Arguing as in the previous discussion of the Burger equation this implies that we can have a sequence of real analytic data $\{\Psi_n^{(0)}\}$ converging in some Sobolev norms to a “Sobolev” initial data whose corresponding solutions have a common region of analyticity. This would allow to construct H^s solutions as a limit of a sequence of analytic solutions.

The proof that the existence region of an analytic solution depends only on some appropriate Sobolev norms is obtained, as in the case of the Burger equation, using a contradiction argument, namely we show that we can define a Banach space, $\mathcal{B}_{\alpha,\rho}$, where the real analytic solution belongs (as function of space variables), when restricted to a certain region \mathcal{K}_a , and prove that the solution remains in this Banach space even when the region is extended to a larger one, \mathcal{K} , whose size depends only on some Sobolev norms of the initial data. Of course, again, the choice of the Banach space must be such that if a function belongs to it and solve the (“hyperbolic”) Einstein equations then it is real analytic and can be used as a real analytic initial data on the upper boundary of the largest \mathcal{K}_a region to extend the solution to \mathcal{K} . Moreover, this argument requires that the set of regions \mathcal{K}_a is not empty which requires preliminary the local existence result discussed in (2). Therefore, next problem is that of defining the Banach spaces $\mathcal{B}_{\alpha,\rho}$.

(4) The Banach spaces: To state precisely the analogy with the Burger equation, the Banach spaces $\mathcal{B}_{\alpha,\rho}$ has to be the analogous ones of $B_{\alpha,\rho}$, the region \mathcal{K}_a the analogous of the region $R \times [0, T_a]$, the region \mathcal{K} of the region $R \times [0, T]$.

Let us recall the definition of the Banach space norm in the case of the Burger equation:

$$\|u(t, \cdot)\|_{\mathcal{B}_{\alpha,\rho}} = \sum_{n=0}^{\infty} \frac{\|D^n f(t, \cdot)\|_{L^2}}{n!} n^\alpha \rho^n. \tag{4.116}$$

In the present case let $\Psi = \Psi_0 + \Psi_1$ be a real analytic solution, where

$$\begin{aligned} \Psi &= (V; W) = (\gamma, \Omega, v, \psi, \omega, \zeta, \chi; X, w, \underline{\omega}, \underline{\chi}) \equiv (\gamma, \Omega, v, \psi, \mathcal{O}; X, w, \underline{\mathcal{O}}) \\ \Psi_0 &= (\gamma, \Omega, v, \psi; X, w), \quad \Psi_1 = (\mathcal{O}; \underline{\mathcal{O}}), \end{aligned} \tag{4.117}$$

Footnote 40 continued

fact that we can assign in a free way some connection coefficients makes the control of the initial data along the whole $C_0 \cup \underline{C}_0$ easier.

in a region, $\{(\lambda, \nu) | (\lambda, \nu) \in [0, \Lambda_a] \times [0, \Pi_a]\}$, of the characteristic problem for the Einstein vacuum equations. Recalling that V denotes the part of the Cauchy–Kowalevski solution whose equations are in the incoming λ -direction and W the opposite one, we define

$$\begin{aligned} \|\mathcal{O}(\lambda, \cdot)\|_{B_{\alpha, \rho}} &= \sum_{n=0}^{\infty} \frac{\left(\sup_{p \in \{2,4\}} \sup_{\nu \in C_0} \|r^{n+\psi(n)-\frac{2}{p}} \nabla^n \mathcal{O}(\lambda, \cdot)\|_{L^p(S_0)}\right)}{n!} n^\alpha \rho^n \\ \|\underline{\mathcal{O}}(\nu, \cdot)\|_{B_{\alpha, \rho}} &= \sum_{n=0}^{\infty} \frac{\left(\sup_{p \in \{2,4\}} \sup_{\lambda \in \underline{C}_0} \|f_n(r, \lambda; p) \nabla^n \underline{\mathcal{O}}(\nu, \cdot)\|_{L^p(S_0)}\right)}{n!} n^\alpha \rho^n, \end{aligned} \tag{4.118}$$

where $\|\nabla^n \mathcal{O}(\lambda, \cdot)\|_{L^p(S_0)}, \|\nabla^n \underline{\mathcal{O}}(\nu, \cdot)\|_{L^p(S_0)}$ are slightly symbolic expressions for

$$\begin{aligned} \|\nabla^n \mathcal{O}(\nu, \cdot)\|_{L^p(S_0)} &:= \sum_{p,q;p+q=n} \|\mathcal{D}_4^p \nabla^q \mathcal{O}(\nu, \cdot)\|_{L^p(S_0)} \\ \|\nabla^n \underline{\mathcal{O}}(\lambda, \cdot)\|_{L^p(S_0)} &:= \sum_{p,q;p+q=n} \|\mathcal{D}_3^p \nabla^q \underline{\mathcal{O}}(\lambda, \cdot)\|_{L^p(S_0)}, \end{aligned} \tag{4.119}$$

$\psi(n)$ depends on which element of \mathcal{O} is considered, $f_n(r, \lambda)$ depends on which element of $\underline{\mathcal{O}}$ is considered and has the structure $f_n(r, \lambda) = r^{\alpha(n)-\frac{2}{p}} |\lambda|^{\beta(n)}$, see for instance the analogous definitions for the first derivatives in [15]. Finally

$$\|f(\lambda; \cdot)\|_{L^p(S_0)} = \left(\int_{S_0} d\mu_{\hat{\gamma}_0} |f(\lambda; \nu, \omega^a)|^p \right)^{\frac{1}{p}}, \tag{4.120}$$

where the measure is done with respect to a given metric tensor $\hat{\gamma}_0$, analogous to the metric γ_0 on S_0 , but with a different radial factor

$$\hat{r}_0(\lambda, \nu) \equiv r_0 + \frac{1}{2}(\nu - \lambda), \tag{4.121}$$

$$\hat{\gamma}_{0(\lambda, \nu)}(\cdot, \cdot) = \frac{\hat{r}_0(\lambda, \nu)}{r_0} \gamma_0(\cdot, \cdot). \tag{4.122}$$

Finally, if q is an h -covariant tensor on S_0 its pointwise norm $|q|$, defined with respect to the metric tensor $\hat{\gamma}_0$, is⁴¹

$$|q|^2 = q_{a_1 a_2 \dots a_s} q_{b_1 b_2 \dots b_s} \hat{\gamma}_0^{a_1 b_1} \hat{\gamma}_0^{a_2 b_2} \dots \hat{\gamma}_0^{a_s b_s} \equiv |q|_{\hat{\gamma}_0}^2. \tag{4.123}$$

The Banach space $\mathcal{B}_{\alpha, \rho}$ is defined through the norm

$$\|\Psi_{(1)}(\lambda, \nu; \cdot)\|_{\mathcal{B}_{\alpha, \rho}} = \|\mathcal{O}(\lambda, \cdot)\|_{B_{\alpha, \rho}} + \|\underline{\mathcal{O}}(\nu, \cdot)\|_{B_{\alpha, \rho}}. \tag{4.124}$$

⁴¹ We use the metric $\hat{\gamma}_0$ instead of the metric γ_0 as we want a “background” metric “near” to $\gamma(\lambda, \nu)$ for all the values of λ, ν .

Remarks. (i) The norms

$$\begin{aligned} & \sup_{p \in \{2,4\}} \sup_{\nu \in C_0} \|r^{n+\psi(n)-\frac{2}{p}} \nabla^n \mathcal{O}(\lambda, \cdot)\|_{L^p(S_0)}, \\ & \sup_{p \in \{2,4\}} \sup_{\lambda \in \underline{C}_0} \|f_n(r, \lambda; p) \nabla^n \underline{\mathcal{Q}}(\nu, \cdot)\|_{L^p(S_0)} \end{aligned}$$

play the role of the $\|D^n u(t, \cdot)\|_{L^2(R^n)}$ norm in the Burger case. The main difference is that, in that case, $D = \frac{\partial}{\partial x}$ involves all the variables in R^n (more specifically the only existing one, but in the case of R^n with $n > 1$ all the derivatives $\{\partial_{x^i}\}$ would be present). In the Einstein case the derivatives involved are those “tangent” to C_0 and \underline{C}_0 respectively, namely $\{\mathcal{D}_4, \nabla\}$ on C_0 , $\{\mathcal{D}_3, \nabla\}$ on \underline{C}_0 .⁴²

(ii) The previous norms are relative to the connection coefficients \mathcal{O} and $\underline{\mathcal{Q}}$ while the Cauchy–Kowalevski solution Ψ involves also the functions $\gamma, \Omega, \nu, \psi; X, w$. Also for these quantities we have to define analogous norms; nevertheless once Ψ_1 belongs to the Banach space $\mathcal{B}_{\alpha;\rho}$ we can easily prove that the analogous norms for these quantities are bounded.

(5) The main result: The core result we have proved and we present in the subsequent paper is that our solution belongs to a $\mathcal{B}_{\alpha;\rho}$ Banach space, with appropriate ρ and α , with (λ, ν) (coordinates of points) in a region \mathcal{K} whose size is determined only from some Sobolev norms.⁴³ From this it follows immediately that in the same region the solution is real analytic.⁴⁴ Looking at the discussion of the similar result for the Burger equation we state, first of all, Lemma 4.5 and Theorem 4.4 the analogous of Lemma 4.1 and Theorem 4.1 respectively.

Lemma 4.5. *Let $\mathcal{O}(\lambda, \cdot), \underline{\mathcal{Q}}(\nu, \cdot)$ be functions belonging to the Banach space $\mathcal{B}_{\alpha;\rho}$ with*

$$\alpha \geq \left[\frac{3}{2} \right],$$

then, for any $x = \{\nu, \omega^a\}, \mathcal{O}(\lambda, \cdot)$ and, for any $y = \{\lambda, \omega^a\}, \underline{\mathcal{Q}}(\nu, \cdot)$, are real analytic in $B(x)_\rho, B(y)_\rho$ respectively.

Proof. The proof goes exactly as the proof of Lemma 4.1 with the obvious modifications and we do not repeat it here. Next result, the analogous of Theorem 4.1, specifies the existence region of the analytic solution of the characteristic problem solved via the characteristic Cauchy–Kowalevski theorem as discussed in Sect. 3.

Theorem 4.4. *Let the “initial data” $\Psi^{(0)} \in \mathcal{B}_{\alpha;\rho_0}$ then, given $\rho \leq \rho_0$, there exists a solution of the system of equations (2.37) and (2.38), $\Psi(\lambda, \nu, \omega^a) \in \mathcal{B}_{\alpha;\rho}$ satisfying the initial conditions in a region,*

$$\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [0, \Lambda_a] \times [0, \Pi_a]\},$$

⁴² Applying $\mathcal{D}_4, \mathcal{D}_3$ implies deriving with $\mathbf{D}_4, \mathbf{D}_3$ and projecting on the S -tangent space.

⁴³ The region \mathcal{K} is a region of the spacetime, but is completely defined once we give (λ, ν) .

⁴⁴ In fact in a slightly larger region.

where Λ_a, Π_a depend on $\rho, \rho_0, \Psi^{(0)}$ and are such that

$$\lim_{\rho \rightarrow \rho_0} \Lambda_a(\rho, \rho_0, \Psi^{(0)}) = \lim_{\rho \rightarrow \rho_0} \Pi_a(\rho, \rho_0, \Psi^{(0)}) = 0.$$

Proof. The proof goes exactly as the proof of Theorem 4.1 with the obvious modifications and we do not repeat it here.

Theorem 4.4 tells us, exactly as in the Burger case, that if we define $\bar{\Lambda}, \bar{\Pi}$ such that

$$\mathcal{K}(\bar{\Lambda}, \bar{\Pi}) = \sup\{\mathcal{K}(\Lambda_a, \Pi_a) | \Psi(\lambda, \nu, \omega^a) \in B_{\alpha, \rho} \text{ with } \rho > 0\} \quad (4.125)$$

then

$$\lim_{\Lambda_a, \Pi_a \rightarrow \bar{\Lambda}} \rho(\Lambda_a, \Pi_a; \rho_0, \Psi^{(0)}) = \lim_{\Pi_a \rightarrow \bar{\Pi}} \rho(\Lambda_a, \Pi_a; \rho_0, \Psi^{(0)}) = 0. \quad (4.126)$$

Next step is to prove Lemma 4.6 and Theorem 4.5 which correspond to Lemma 4.3 and Theorem 4.2. We first define the analogous of the H^s norm for the Burger equation; we denote

$$\mathbf{g} = (\gamma, \Omega, X), \quad \mathcal{O} = (\chi, \eta, \omega), \quad \underline{\mathcal{O}} = (\underline{\chi}, \underline{\eta}, \underline{\omega})$$

and define, ⁴⁵

$$\begin{aligned} \|\Psi(\lambda, \nu; \cdot)\|_{H_p^s(S_0)} &= \|\mathbf{g}\|_{H_p^s(S_0)}(\lambda, \nu) + \|\mathcal{O}\|_{H_p^s(S_0)}(\lambda, \nu) + \|\underline{\mathcal{O}}\|_{H_p^s(S_0)}(\lambda, \nu) \\ \|\mathbf{g}\|_{H_p^s(S_0)} &\equiv \|\gamma\|_{H_p^s(S_0)}(\lambda, \nu) + \|\Omega - \frac{1}{2}\|_{H_p^s(S_0)}(\lambda, \nu) + \|X\|_{H_p^s(S_0)}(\lambda, \nu). \end{aligned}$$

For the initial data we define analogous norms,

$$\begin{aligned} \|\Psi^{(0)}\|_{s,p} &\equiv \sup_{\nu \in \mathcal{C}_0} \left(\|\mathbf{g}\|_{H_p^s(S_0)}(0, \nu) + \|\mathcal{O}\|_{H_p^s(S_0)}(0, \nu) + \|\underline{\mathcal{O}}\|_{H_p^s(S_0)}(0, \nu) \right) \\ &+ \sup_{\lambda \in \mathcal{C}_0} \left(\|\mathbf{g}\|_{H_p^s(S_0)}(\lambda, 0) + \|\mathcal{O}\|_{H_p^s(S_0)}(\lambda, 0) + \|\underline{\mathcal{O}}\|_{H_p^s(S_0)}(\lambda, 0) \right). \end{aligned}$$

Lemma 4.6. *Let $\Psi = (\mathbf{g}; \mathcal{O}, \underline{\mathcal{O}}) = (\gamma, \Omega, X; \omega, \zeta, \chi, \underline{\omega}, \underline{\chi})$ be a solution of the characteristic problem for the Einstein vacuum equations, in a region,*

$$\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [0, \Lambda_a] \times [0, \Pi_a]\},$$

with initial data satisfying $\|\Psi^{(0)}\|_{s,p} \leq c$, with $p \in \{2, 4\}$. Then there exists a region $\mathcal{K}(\Lambda, \Pi)$, whose size depends only on the norm $\|\Psi^{(0)}\|_{s,p}$ such that for $s \geq 7, p \in \{2, 4\}$, there exists a constant c_0 such that

45

$$\begin{aligned} \|\mathcal{O}\|_{H_p^s} &= \left(\sum_{k=0}^s \int_{S_0} d\mu_{\gamma_0} |r^{k+\psi(k)-\frac{2}{p}} \nabla^k f(\lambda; \nu, \omega^a)|^p \right)^{\frac{1}{p}}. \\ \|\underline{\mathcal{O}}\|_{H_p^s} &= \left(\sum_{k=0}^s \int_{S_0} d\mu_{\gamma_0} |f_k(r, \lambda; p) \nabla^k f(\lambda; \nu, \omega^a)|^p \right)^{\frac{1}{p}}. \end{aligned}$$

$$\sup_{(\lambda, \nu) \in \mathcal{K}(\Lambda', \Pi')} \|\Psi(\lambda, \nu; \cdot)\|_{H_p^s(S_0)} \leq c_0 \|\Psi^{(0)}\|_{s,p} \tag{4.127}$$

for any region $\mathcal{K}(\Lambda', \Pi') \subset \min(\mathcal{K}(\Lambda, \Pi), \mathcal{K}(\Lambda_a, \Pi_a))$.

Remark. Lemma 4.6 states, basically, the a priori estimates for the Einstein equations. Differently from the simple Burger equation these estimates are much more involved as the hyperbolicity of the Einstein equations is more difficult to exploit.

Next theorem, the analogous of Theorem 4.2, is the basic ingredient to prove our result.

Theorem 4.5. *Let $\Psi = (\mathbf{g}; \mathcal{O}, \underline{\mathcal{O}}) = (\gamma, \Omega, v, \psi, w, X; \omega, \zeta, \chi, \underline{\omega}, \underline{\chi})$ be a real analytic solution in the region,*

$$\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [0, \Lambda_a] \times [0, \Pi_a]\},$$

of the characteristic problem for the Einstein vacuum equations with analytic initial data on $C_0 \cup \underline{C}_0$, belonging to the Banach space $\mathcal{B}_{\alpha; \rho_0}$.

Then in any region $\mathcal{K} \subset \mathcal{K}(\Lambda', \Pi') \subset \mathcal{K}(\Lambda_a, \Pi_a)$ the following relation holds

$$\sup_{(\lambda, \nu) \in \mathcal{K}} \|\Psi_1(\lambda, \nu; \cdot)\|_{\mathcal{B}_{\alpha; \rho''}} \leq c_1 \|\Psi_1^{(0)}\|_{\mathcal{B}_{\alpha; \rho_0}} \tag{4.128}$$

for some ρ'' which depends only on ρ_0 and on the $\|\Psi_1^{(0)}\|_{\mathcal{B}_{\alpha; \rho_0}}$ norm, but does not depend on the region $\mathcal{K}(\Lambda_a, \Pi_a)$.

Remark. Observe that ρ'' and ρ_0 are the analogous of the same quantities defined in Theorem 4.2 in the Burger’s equation case. Moreover (Λ_a, Π_a) are the analogous of T_a , (Λ', Π') are the analogous of T' and (Λ, Π) are the analogous of T and, exactly in the same way, they depend only on the a priori estimates which means that the size of the region $\mathcal{K}(\Lambda, \Pi)$ depends only on $\|\Psi^{(0)}\|_{H_p^{s'}(S_0)}$.

Lemma 4.6 and Theorem 4.5 are the core of the result we want to obtain, exactly as the proof of Theorem 4.2 is based on Lemma 4.4, the proof of Theorem 4.5 is based on the following lemma.

Lemma 4.7. *Let $\Psi = (\mathbf{g}; \mathcal{O}, \underline{\mathcal{O}}) = (\gamma, \Omega, v, \psi, w, X; \omega, \zeta, \chi, \underline{\omega}, \underline{\chi})$ be a real analytic solution in the region,*

$$\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [0, \Lambda_a] \times [0, \Pi_a]\},$$

of the characteristic problem for the Einstein vacuum equations with analytic initial data on $C_0 \cup \underline{C}_0$, sufficiently small and belonging to the Banach space $\mathcal{B}_{\alpha; \rho_0}$, with $\alpha > 3$. Then, for the generic connection coefficient we indicate with

\mathcal{U} , the following estimates hold for any J ⁴⁶

$$|r^{2+J-\frac{2}{p}} \nabla^J \mathcal{U}|_{p,S}(\lambda, \nu) \leq C_0 \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta} e^{J\Gamma(\lambda, \nu)}}{\rho^J}, \tag{4.129}$$

where $\delta > \delta_0 > 0$, C_0 is a constant satisfying

$$C_0 \geq c_1 \|\Psi^{(0)}\|_{B_{\alpha, \rho_0}} \tag{4.130}$$

and

$$\Gamma(\lambda, \nu) = \Gamma(\nu) + \underline{\Gamma}(\lambda) \leq \gamma \tag{4.131}$$

where, with an appropriate $\hat{C} > 0$,

$$\Gamma(\nu) = -\hat{C} \frac{(\nu - \nu_0)}{\nu\nu_0}, \quad \underline{\Gamma}(\lambda) = \hat{C} \frac{(\lambda - \lambda_0)}{\lambda\lambda_0}. \tag{4.132}$$

$\gamma > \gamma_0$ where $\gamma - \gamma_0$ depends on C_0 and

$$\Gamma_0(\nu) + \underline{\Gamma}_0(\lambda) \leq \gamma_0 \tag{4.133}$$

where, with an appropriate $\tilde{C}_0 > 0$,

$$\Gamma_0(\nu) = \tilde{C}_0 \frac{(\nu - \nu_0)}{\nu\nu_0}, \quad \underline{\Gamma}_0(\lambda) = \tilde{C}_0 \frac{(\lambda - \lambda_0)}{\lambda\lambda_0}. \tag{4.134}$$

$\Gamma_0(\nu) + \underline{\Gamma}_0(\lambda)$, γ_0, δ_0 are relative to the initial data which are assumed to belong to $\mathcal{B}_{\alpha; \rho_0}$ and, moreover their angular covariant derivatives satisfy analogous bounds to (4.129), with $\gamma_0, \delta_0, \Gamma_0(\nu) + \underline{\Gamma}_0(\lambda)$ instead of $\gamma, \delta, \Gamma(\nu) + \underline{\Gamma}(\lambda)$.

Remarks. (a) The crucial fact in this lemma is that the estimates (4.129) do not depend on Λ_a, Π_a . Therefore the limit (4.126) is not anymore true and this basically allows to extend the region.

(b) Differently from Burger, when initial data are small a factor $|\nu| + |\lambda|$ in the exponential factor (the analogous of T in the Burger case) is not needed (if we do not assume initial data “small” we expect nevertheless that the $|\nu| + |\lambda|$ has to be present).

(c) The goal of this lemma is to provide the appropriate estimates for the covariant derivatives of Ψ to conclude that $\Psi \in B_{\alpha, \rho'}$. Nevertheless, to obtain this result we need to control *the* $|\cdot|_{p,s}$ norms not only for the angular derivatives but also for the \mathcal{D}_4 derivatives for the V components and the \mathcal{D}_3 derivatives for the W ones and also for the mixed derivatives. Nevertheless, we first prove Lemma 4.7 and obtain the appropriate estimates for the angular derivatives, then using again the structure equations we control all the remaining mixed derivatives.

(d) A technical remark is appropriate here, although it will be discussed at length in the subsequent paper where the technical details of Sect. 4.2 are exploited. In the proof of global existence the double null cone foliation, the gauge we are using, have to be further specified. Namely

⁴⁶ The notation of inequality (4.129) is a bit symbolic, J is an integer except that in ∇^J where it has to be considered a multiindex with $|J| = J$, moreover if with \mathcal{U} we denote a \mathcal{Q} connection coefficient then the weight factor $r^{2+J-\frac{2}{p}}$ has to be modified, see equations (4.118).

between these foliations one has to choose a specific one, we call “double null cone canonical foliation”, previously introduced in [15]. The reason for this choice, although technical, can be simply justified claiming that asymptotically, in the null outgoing directions, we expect the cone structure be very similar to the one in the Minkowski case. We do not discuss it at length here as it will be done carefully in the subsequent paper and also because its choice and the need of it is not new, see for instance [15, 22].

Once Lemma 4.6 and Theorem 4.5 are proved, next step is the proof of the final theorem, Theorem 4.6, which is the analogous of Theorem 4.3 for the Burger equation and which we state here. Nevertheless, it has to be remarked that its proof is significantly different as we are considering now a characteristic problem.

Theorem 4.6. *Let $\Psi = (\mathbf{g}; \mathcal{O}, \underline{\mathcal{Q}}) = (\gamma, \Omega, v, \psi, w, X; \omega, \zeta, \chi, \underline{\omega}, \underline{\chi})$ be a real analytic solution problem for the Einstein vacuum equations with analytic initial data on $C_0 \cup \underline{C}_0$, belonging to the Banach space $\mathcal{B}_{\alpha; \rho_0}$. Then this solution can be extended to an analytic solution $\in \mathcal{B}_{\alpha; \rho''}$ in the region $\mathcal{K}(\Lambda, \Pi)$ with*

$$\rho''' < \rho_1 := \rho_0 e^{-\gamma(|\Lambda|+|\Pi|)}, \tag{4.135}$$

where $\gamma > 0$ depends on the $\mathcal{B}_{\alpha; \rho_0}$ norms and Λ, Π depend only on the Sobolev norms of the initial data. Moreover if the initial data are “sufficiently small” then the region where the analytic solution exists is unbounded (in the ν variable, $\mathcal{K}(\Lambda, \infty)$).

Proof. The first step to prove the theorem is the proof of the following lemma:

Lemma 4.8. *If the results of Theorem 4.4 and Lemma 4.6 are true then it is possible to extend this solution to a new solution $\tilde{\Psi}$ real analytic in the region $\mathcal{K}(\Lambda_a + \Delta, \Pi_a + \Delta)$ and such that, for each λ, ν in this region, it belongs to the Banach space $\mathcal{B}_{\alpha; \rho''}$ with, depending on the initial conditions,*

$$\rho''' < \rho_0 e^{-\gamma(|\Lambda|+|\Pi|)}. \tag{4.136}$$

Proof of Lemma 4.8. First we prove, using these propositions, that assuming as initial data those on $C_0 \cup \underline{C}_0$ and the real analytic solution on $\underline{C}(\Pi_a; [0, \Lambda_a])$ and on $C(\Lambda_a; [0, \Pi_a])$ we can extend the solution to a real analytic solution in $\mathcal{K}(\Lambda_a, \Pi_a + \Delta) \cup \mathcal{K}(\Lambda_a + \Delta, \Pi_a)$ and also on the diamond region

$$\mathcal{K}(\Lambda_a + \Delta, \Pi_a + \Delta) / \mathcal{K}(\Lambda_a, \Pi_a + \Delta) \cup \mathcal{K}(\Lambda_a + \Delta, \Pi_a).$$

The proof of the existence of a real analytic solution in the two strips $\mathcal{K}(\Lambda_a, \Pi_a + \Delta)$ and $\mathcal{K}(\Lambda_a + \Delta, \Pi_a)$ mimics the analogous proof made in [5], with the difference that, there, we were building an H^s solution while here we prove the existence of a real analytic solution. The problem is nevertheless of the same type: as these stripes in one direction cannot have an uniformly bounded length we have to use a “sub-bootstrap” mechanism to prove the existence of the solution there. In the real analytic case the existence of the solution in a strip of a width which does not tend to zero is provided again by the “a

priori estimates” which allow to show that the solution exists along the whole strip.

Once Lemma 4.8 has been proved, the remaining steps to prove Theorem 4.6 are basically identical to those used for the Burger equation, namely Theorem 4.3. Observe that instead of $[0, T^*]$ we have $\mathcal{K}(\Lambda^*, \Pi^*)$ and the definition of T^*

$$T^* = \sup\{T_a \in [0, T] \mid \text{for any } t \in [0, T_a] \ u(t, \cdot) \in B_{\alpha, \rho_1}\}.$$

is substituted by

$$\begin{aligned} &\mathcal{K}(\Lambda^*, \Pi^*) \\ &= \sup \{ \mathcal{K}(\Lambda_a, \Pi_a) \subseteq \mathcal{K}(\Lambda, \Pi) \mid \forall (\lambda, \nu) \in \mathcal{K}(\Lambda_a, \Pi_a); \Psi(\lambda, \nu; \cdot) \in \mathcal{B}_{\alpha; \rho_1} \}. \end{aligned}$$

Lemma 4.8 shows that, unless

$$\mathcal{K}(\Lambda^*, \Pi^*) = \mathcal{K}(\Lambda, \Pi), \tag{4.137}$$

the region $\mathcal{K}(\Lambda^*, \Pi^*)$ can be extended contradicting its definition. Therefore, (4.137) is proved.

5. Conclusions

The results presented in this paper concern the existence region of the real analytic solutions of a class of characteristic problems for the vacuum Einstein equations. As a byproduct to obtain it, we have discussed and developed a gauge associated with the double null cone foliation, first introduced in [15]. We consider this gauge very appropriate to this kind of problems and, in particular, we believe that to obtain the same global result proved here in a gauge like the harmonic gauge could be much more difficult, even if possible.

The choice of the “double null cone gauge” also has the advantage, in our opinion, that it allows writing the Einstein equations as first order equations in a very natural and geometric way⁴⁷ because this foliation, differently from any spacelike hypersurfaces foliation, is more intrinsic due to the physical meaning of the null cones. Moreover in this approach the distinction, in the characteristic case, between those Einstein equations which can be considered “evolution equations” and those which have to be interpreted as “constraint equations” is completely clear. We also believe that our approach could be used to deal with the characteristic problem with “initial data” on a null outgoing cone hypersurface, but this has not yet been completely worked. Nevertheless in Appendix 7, we give a short sketch on how, in our opinion the global existence for the cone problem can be faced and solved.

⁴⁷ This is certainly not new, see for instance [20, 21], but we believe it is presented here in a more complete way.

Acknowledgements

The idea of trying to propagate as much as possible the analyticity of the solutions of the Einstein equations was suggested, years ago, to one of the author (F.N.) by S. Klainerman and was implemented together on the “toy model” of the Burger equation in [14]; moreover, both authors are pleased to thank S. Klainerman one for the long and rich collaboration and (G.C.) for the useful period spent in the Math. Department of Princeton University and for the long and profitable conversations he had with him.

6. Appendix

6.1. General Aspects of the Structure Equations

Proof of Proposition 2.1. We have

$$(\theta^\gamma \wedge \omega_\gamma^\alpha)(e_\beta, e_\delta) = \omega_\beta^\alpha(e_\delta) - \omega_\delta^\alpha(e_\beta) \tag{6.138}$$

on the other side, recalling how d operates,

$$\begin{aligned} d\theta^\alpha(e_\beta, e_\delta) &= e_\beta(\theta^\alpha(e_\delta)) - e_\delta(\theta^\alpha(e_\beta)) - \theta^\alpha([e_\beta, e_\delta]) = -\theta^\alpha([e_\beta, e_\delta]) \\ &= -\theta^\alpha(\mathbf{D}_{e_\beta} e_\delta) + \theta^\alpha(\mathbf{D}_{e_\delta} e_\beta) \\ &= -\Gamma_{\beta\delta}^\gamma \theta^\alpha(e_\gamma) + \Gamma_{\delta\beta}^\gamma \theta^\alpha(e_\gamma) = -\Gamma_{\beta\delta}^\alpha + \Gamma_{\delta\beta}^\alpha \\ &= -\omega_\delta^\alpha(e_\beta) + \omega_\beta^\alpha(e_\delta) = \theta^\gamma \wedge \omega_\gamma^\alpha(e_\beta, e_\delta), \end{aligned} \tag{6.139}$$

which proves the first structure equation. To prove the second structure equation we write

$$\begin{aligned} \mathbf{R}_{\gamma\alpha\beta}^\delta e_\delta &= \mathbf{R}(e_\alpha, e_\beta)e_\gamma = \mathbf{D}_{e_\alpha}(\mathbf{D}_{e_\beta} e_\gamma) - \mathbf{D}_{e_\beta}(\mathbf{D}_{e_\alpha} e_\gamma) - \mathbf{D}_{[e_\alpha, e_\beta]} e_\gamma \\ &= \mathbf{D}_{e_\alpha}(\Gamma_{\beta\gamma}^\lambda e_\lambda) - \mathbf{D}_{e_\beta}(\Gamma_{\alpha\gamma}^\lambda e_\lambda) - \mathbf{D}_{[e_\alpha, e_\beta]} e_\gamma \\ &= e_\alpha(\Gamma_{\beta\gamma}^\lambda) e_\lambda - e_\beta(\Gamma_{\alpha\gamma}^\lambda) e_\lambda + \Gamma_{\beta\gamma}^\lambda \Gamma_{\alpha\lambda}^\epsilon e_\epsilon - \Gamma_{\alpha\gamma}^\lambda \Gamma_{\beta\lambda}^\epsilon e_\epsilon - \mathbf{D}_{[e_\alpha, e_\beta]} e_\gamma \end{aligned} \tag{6.140}$$

therefore

$$\begin{aligned} \mathbf{R}_{\gamma\alpha\beta}^\delta &= e_\alpha(\Gamma_{\beta\gamma}^\delta) - e_\beta(\Gamma_{\alpha\gamma}^\delta) + \Gamma_{\beta\gamma}^\lambda \Gamma_{\alpha\lambda}^\delta - \Gamma_{\alpha\gamma}^\lambda \Gamma_{\beta\lambda}^\delta - \theta^\delta(\mathbf{D}_{[e_\alpha, e_\beta]} e_\gamma) \\ &= e_\alpha(\omega_\gamma^\delta(e_\beta)) - e_\beta(\omega_\gamma^\delta(e_\alpha)) + \omega_\gamma^\lambda(e_\beta)\omega_\lambda^\delta(e_\alpha) \\ &\quad - \omega_\gamma^\lambda(e_\alpha)\omega_\lambda^\delta(e_\beta) - \theta^\delta(\mathbf{D}_{[e_\alpha, e_\beta]} e_\gamma). \end{aligned}$$

On the other side

$$\begin{aligned} (d\omega_\gamma^\delta + \omega_\sigma^\delta \wedge \omega_\gamma^\sigma)(e_\alpha, e_\beta) &= d\omega_\gamma^\delta(e_\alpha, e_\beta) + \omega_\sigma^\delta(e_\alpha)\omega_\gamma^\sigma(e_\beta) - \omega_\sigma^\delta(e_\beta)\omega_\gamma^\sigma(e_\alpha) \\ &= e_\alpha(\omega_\gamma^\delta(e_\beta)) - e_\beta(\omega_\gamma^\delta(e_\alpha)) - \omega_\gamma^\delta([e_\alpha, e_\beta]) \\ &\quad + \omega_\sigma^\delta(e_\alpha)\omega_\gamma^\sigma(e_\beta) - \omega_\sigma^\delta(e_\beta)\omega_\gamma^\sigma(e_\alpha) \end{aligned} \tag{6.141}$$

and

$$\begin{aligned} \omega_\gamma^\delta([e_\alpha, e_\beta]) &= \omega_\gamma^\delta(c_{\alpha\beta}^\sigma e_\sigma) = c_{\alpha\beta}^\sigma \omega_\gamma^\delta(e_\sigma) = c_{\alpha\beta}^\sigma \Gamma_{\sigma\gamma}^\delta \\ &= c_{\alpha\beta}^\sigma \theta^\delta(\mathbf{D}_{e_\sigma} e_\gamma) = \theta^\delta(\mathbf{D}_{c_{\alpha\beta}^\sigma e_\sigma} e_\gamma) = \theta^\delta(\mathbf{D}_{[e_\alpha, e_\beta]} e_\gamma) \end{aligned} \tag{6.142}$$

and the thesis follows.

6.1.1. The General Structure Equations: $d\omega_\gamma^\delta = -\omega_\sigma^\delta \wedge \omega_\gamma^\sigma + \Omega_\gamma^\delta$.

$$\left\{ \begin{array}{l}
 \mathcal{D}_3 \underline{\chi} + tr \chi \underline{\hat{\chi}} + 2\omega \underline{\hat{\chi}} - \nabla \widehat{\otimes} \underline{\xi} + (2\zeta - \underline{\eta} - \eta) \widehat{\otimes} \underline{\xi} = -\underline{\alpha} \\
 \mathcal{D}_3 tr \underline{\chi} + \frac{1}{2}(tr \underline{\chi})^2 + 2\underline{\omega} tr \underline{\chi} + |\underline{\hat{\chi}}|^2 - 2d\text{iv} \underline{\xi} - 2\underline{\xi} \cdot (\underline{\eta} + \underline{\eta} - 2\zeta) = 0 \\
 -\text{curl} \underline{\xi} + (2\zeta + \underline{\eta} - \eta) \wedge \underline{\xi} = 0 \\
 \\
 \mathcal{D}_4 \underline{\hat{\chi}} + \frac{1}{2} tr \chi \underline{\hat{\chi}} + \frac{1}{2} tr \chi \underline{\hat{\chi}} - 2\omega \underline{\hat{\chi}} - \nabla \widehat{\otimes} \underline{\eta} - \underline{\eta} \widehat{\otimes} \underline{\eta} - \underline{\xi} \widehat{\otimes} \underline{\xi} = 0 \\
 \mathcal{D}_4 tr \underline{\chi} + \frac{1}{2} tr \chi tr \underline{\chi} - 2\underline{\omega} tr \underline{\chi} + \underline{\hat{\chi}} \cdot \underline{\hat{\chi}} - 2d\text{iv} \underline{\eta} - 2|\underline{\eta}|^2 - 2\underline{\xi} \cdot \underline{\xi} = 2\rho \\
 \text{curl} \underline{\eta} + \underline{\xi} \wedge \underline{\xi} - \frac{1}{2}(\underline{\hat{\chi}} \wedge \underline{\hat{\chi}}) = -\sigma \\
 \\
 \mathcal{D}_3 \underline{\hat{\chi}} + \frac{1}{2} tr \chi \underline{\hat{\chi}} + \frac{1}{2} tr \chi \underline{\hat{\chi}} - 2\omega \underline{\hat{\chi}} - \nabla \widehat{\otimes} \underline{\eta} - \underline{\eta} \widehat{\otimes} \underline{\eta} - \underline{\xi} \widehat{\otimes} \underline{\xi} = 0 \\
 \mathcal{D}_3 tr \chi + \frac{1}{2} tr \chi tr \chi - 2\underline{\omega} tr \chi + \underline{\hat{\chi}} \cdot \underline{\hat{\chi}} - 2d\text{iv} \underline{\eta} - 2|\underline{\eta}|^2 - 2\underline{\xi} \cdot \underline{\xi} = 2\rho \\
 \text{curl} \underline{\eta} - \frac{1}{2} \underline{\hat{\chi}} \wedge \underline{\hat{\chi}} + \underline{\xi} \wedge \underline{\xi} = \sigma \\
 \\
 \mathcal{D}_4 \underline{\hat{\chi}} + tr \chi \underline{\hat{\chi}} + 2\omega \underline{\hat{\chi}} - \nabla \widehat{\otimes} \underline{\xi} - (2\zeta + \underline{\eta} + \eta) \widehat{\otimes} \underline{\xi} = -\underline{\alpha} \\
 \mathcal{D}_4 tr \chi + \frac{1}{2}(tr \chi)^2 + 2\underline{\omega} tr \chi + |\underline{\hat{\chi}}|^2 - 2d\text{iv} \underline{\xi} - 2\underline{\xi} \cdot (\underline{\eta} + \underline{\eta} + 2\zeta) = 0 \\
 \text{curl} \underline{\xi} - (-2\zeta + \underline{\eta} - \underline{\eta}) \wedge \underline{\xi} = 0 \\
 \\
 \text{curl} \underline{\chi} - \zeta \wedge \underline{\chi} = * \underline{\beta} \\
 \nabla tr \underline{\chi} - d\text{iv} \underline{\chi} + \zeta \cdot \underline{\chi} - \zeta tr \underline{\chi} = -\underline{\beta} \\
 \\
 \text{curl} \chi + \zeta \wedge \chi = - * \beta \\
 \nabla tr \chi - d\text{iv} \chi - \zeta \cdot \chi + \zeta tr \chi = \beta \\
 \\
 \mathcal{D}_4 \underline{\xi} - \mathcal{D}_3 \underline{\eta} - (\underline{\eta} - \eta) \cdot \underline{\chi} - 4\underline{\omega} \underline{\xi} = -\underline{\beta} \\
 \mathcal{D}_3 \underline{\xi} - \mathcal{D}_4 \underline{\eta} - (\underline{\eta} - \eta) \cdot \chi - 4\underline{\omega} \underline{\xi} = \underline{\beta} \\
 \\
 \mathcal{D}_3 \zeta + 2\nabla \underline{\omega} + \underline{\chi} \cdot (\underline{\eta} + \zeta) - \chi \cdot \underline{\xi} + 2\underline{\omega}(\underline{\eta} - \zeta) - 2\underline{\omega} \underline{\xi} = -\underline{\beta} \\
 \mathcal{D}_4 \zeta - 2\nabla \underline{\omega} - \chi(\underline{\eta} - \zeta) + \underline{\chi} \underline{\xi} - 2\underline{\omega}(\underline{\eta} + \zeta) + 2\underline{\omega} \underline{\xi} = -\underline{\beta} \\
 \\
 \text{curl} \zeta - \frac{1}{2} \underline{\hat{\chi}} \wedge \underline{\hat{\chi}} = \sigma \\
 \mathcal{D}_3 \underline{\omega} + \mathcal{D}_4 \underline{\omega} - 4\underline{\omega} \underline{\omega} + (\underline{\eta} \underline{\eta} - \underline{\xi} \cdot \underline{\xi} + \underline{\eta} \cdot \zeta - \eta \cdot \zeta) = \rho \\
 \frac{1}{2} {}^{(2)}\mathbf{R}_{dca b} \delta_{da} \delta_{cb} + \frac{1}{4} tr \chi tr \chi - \frac{1}{2} \underline{\hat{\chi}} \cdot \underline{\hat{\chi}} = -\rho
 \end{array} \right.$$

6.2. The Connection Coefficients as Functions on $S_0 \times R \times R$

To express the tensorial equations (2.31) and (2.32) as a system of p.d.e. equations we have to transform the covariant derivatives relative to the null directions into partial derivatives. This can be done expressing everything in a generic set of coordinates. We do it projecting all the connection coefficients on the two-dimensional surface $S_0 \equiv S(0, 0)$ through the pullback associated with the diffeomorphism (2.12) which specifies a coordinate set adapted to the double null foliation.

More precisely assuming the spacetime $(\mathcal{M}, \mathbf{g})$ foliated by a double null foliation,⁴⁸ whose leaves we call, with a slight abuse of notation, outgoing or incoming cones and denote by $C(\lambda)$ and $\underline{C}(\nu)$ respectively, the two-dimensional surfaces $S(\lambda, \nu) = C(\lambda) \cap \underline{C}(\nu)$ produce a foliation of each outgoing (incoming) cone, $C(\bar{\lambda})$, for instance, being foliated by the leaves $\{S(\bar{\lambda}, \nu)\}$. We recall that the vector fields

$$N = 2\Omega^2 L, \quad \underline{N} = 2\Omega^2 \underline{L}, \tag{6.143}$$

are equivariant vector fields with respect to the leaves $S(\lambda, \nu)$.⁴⁹ This means that the diffeomorphism Φ_ν , generated by the vector field N , sends $S(\lambda, 0)$ onto $\Phi_\nu[S(\lambda, 0)] = S(\lambda, \nu)$, and $\underline{\Phi}_\lambda$, the diffeomorphism generated by the equivariant vector field \underline{N} , sends $S(0, \nu)$ onto $\underline{\Phi}_\lambda[S(0, \nu)] = S(\lambda, \nu)$. We have previously defined, see (2.12), the diffeomorphism,

$$\Psi(\lambda, \nu) : S_0 \ni p_0 \rightarrow q = \Psi(\lambda, \nu)(p_0) \in S(\lambda, \nu) \tag{6.144}$$

where $\Psi(\lambda, \nu)(p_0) = \Phi_\nu \circ \underline{\Phi}_\lambda(p_0) = \Phi_\nu(\underline{\Phi}_\lambda(p_0)) = \Phi_\nu(p)$, $p = \underline{\Phi}_\lambda(p_0)$.

$$\tag{6.145}$$

From $\Psi(\lambda, \nu)$, we derive the pullback $\Psi^*(\lambda, \nu)$ which sends the metric components and the various connection coefficients, tensors belonging to $T^*S(\lambda, \nu) \otimes T^*S(\lambda, \nu) \otimes \dots \otimes T^*S(\lambda, \nu)$, to tensors belonging to $T^*S_0 \otimes T^*S_0 \otimes \dots \otimes T^*S_0$ and depending on the parameters λ, ν . We define:

$$\begin{aligned} \tilde{\zeta}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\zeta)(p_0) = \Psi^*(\lambda, \nu)(\zeta \circ \Psi(\lambda, \nu)(p_0)) \\ &= \underline{\Phi}_\lambda^* \Phi_\nu^* \zeta(\Phi_\nu \circ \underline{\Phi}_\lambda(p_0)) = \underline{\Phi}_\lambda^* \Phi_\nu^* \zeta(q) \\ \tilde{\chi}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\chi)(p_0) = \Psi^*(\lambda, \nu)(\chi \circ \Psi(\lambda, \nu)(p_0)) \\ &= \underline{\Phi}_\lambda^* \Phi_\nu^* \chi(\Phi_\nu \circ \underline{\Phi}_\lambda(p_0)) = \underline{\Phi}_\lambda^* \Phi_\nu^* \chi(q) \\ \tilde{\underline{\chi}}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\underline{\chi})(p_0) = \Psi^*(\lambda, \nu)(\underline{\chi} \circ \Psi(\lambda, \nu)(p_0)) \\ &= \underline{\Phi}_\lambda^* \Phi_\nu^* \underline{\chi}(\Phi_\nu \circ \underline{\Phi}_\lambda(p_0)) = \underline{\Phi}_\lambda^* \Phi_\nu^* \underline{\chi}(q) \end{aligned} \tag{6.146}$$

$$\begin{aligned} \tilde{\omega}(\lambda, \nu; p_0) &= (\omega \circ \Psi(\lambda, \nu))(p_0) = (\omega \circ \Phi_\nu \circ \underline{\Phi}_\lambda)(p_0) = \omega(q) \\ \tilde{\underline{\omega}}(\lambda, \nu; p_0) &= (\underline{\omega} \circ \Psi(\lambda, \nu))(p_0) = (\underline{\omega} \circ \Phi_\nu \circ \underline{\Phi}_\lambda)(p_0) = \underline{\omega}(q) \end{aligned}$$

where $p_0 \in S_0 \equiv S(0, 0)$, is specified by its coordinates (θ, ϕ) . Therefore, $\tilde{\chi}, \tilde{\underline{\chi}}, \tilde{\zeta}, \tilde{\omega}, \tilde{\underline{\omega}}$ are covariant tensors defined on S_0 and depending on the parameters λ, ν ,

$$\tilde{\chi}(\lambda, \nu; \theta, \phi), \quad \tilde{\underline{\chi}}(\lambda, \nu; \theta, \phi), \quad \tilde{\zeta}(\lambda, \nu; \theta, \phi), \quad \tilde{\omega}(\lambda, \nu; \theta, \phi), \quad \tilde{\underline{\omega}}(\lambda, \nu; \theta, \phi).$$

The Einstein equations will take the form of a system of first order partial differential equations, with respect to the variable $\{\lambda, \nu, \theta, \phi\}$, for the components of these covariant tensors and of the metric tensor. Defining on S_0 an

⁴⁸ In [15] the double null foliation had to be ‘‘canonical’’, see its definition in Chapter 3. This is not needed here.

⁴⁹ Obviously they do not commute and their commutator is $[N, \underline{N}] = -4\Omega^2 \zeta(e_A)e_A$.

orthonormal basis $\{\tilde{e}_\theta, \tilde{e}_\phi\}$ with respect to a fixed metric γ_0 assigned on S_0 , the explicit expression for ζ is:

$$\tilde{\zeta}(\lambda, \nu; \theta, \phi)(\cdot) = \sum_A \tilde{\zeta}_A(\lambda, \nu; \theta, \phi) \tilde{\theta}_A(\cdot) = \tilde{\zeta}_\theta(\lambda, \nu; \theta, \phi) d\theta(\cdot) + \tilde{\zeta}_\phi(\lambda, \nu; \theta, \phi) d\phi(\cdot) \tag{6.147}$$

where

$$\tilde{\zeta}_A(\lambda, \nu; \theta, \phi) \equiv \tilde{\zeta}(\lambda, \nu; \theta, \phi)(\tilde{e}_A) \tag{6.148}$$

and exactly analogous expressions hold for $\chi, \underline{\chi}, \omega, \underline{\omega}$. Finally writing in \mathcal{M} the metric $\mathbf{g}(\cdot, \cdot)$ as

$$\mathbf{g} = |X|^2 d\lambda^2 - 2\Omega^2(d\lambda d\nu + d\nu d\lambda) - X_a(d\lambda d\omega^a + d\omega^a d\lambda) + \gamma_{ab} d\omega^a d\omega^b, \tag{6.149}$$

we denote the pull back of its components, $\tilde{X}, \tilde{\Omega}, \tilde{\gamma}$. In conclusion we have defined a diffeomorphism from $(R^2 \times S_0, \tilde{\mathbf{g}})$ to $(\mathcal{M}, \mathbf{g})$

$$R^2 \times S_0 \ni p = (\lambda, \nu, \theta, \phi) \rightarrow q = \Psi(\lambda, \nu)(p_0) \tag{6.150}$$

where $p_0 = (\theta, \phi) \in S_0$ and

$$\tilde{\mathbf{g}} = |\tilde{X}|^2 d\lambda^2 - 2\tilde{\Omega}^2(d\lambda d\nu + d\nu d\lambda) - \tilde{X}_a(d\lambda d\omega^a + d\omega^a d\lambda) + \tilde{\gamma}_{ab} d\omega^a d\omega^b. \tag{6.151}$$

The goal of this appendix is to rewrite equations (2.30), as a set of first order partial differential equations in the variables $\lambda, \nu, \theta, \phi$. Following the discussion of Sect. 3 choosing an appropriate subset of equations (2.30), supplemented with some other equations for the metric components we obtain Eqs. (2.31) and (2.32) written (for the various components) in the $(\lambda, \nu, \theta, \phi)$ coordinates. These are the Einstein equations written in the “double null foliation” gauge and Proposition 2.2 is, therefore, proved.

6.3. The Structure Equations for the Connection Coefficients in a Vacuum Einstein Manifold in the $\{\lambda, \nu, \theta, \phi\}$ Coordinates

To write the previous equations in terms of the pulled back quantities we look first at those satisfied by ζ , corresponding to $\mathbf{R}(e_4, e_A) = 0$ and $\mathbf{R}(e_3, e_A) = 0$,

$$\begin{aligned} \mathcal{D}_4 \zeta + \frac{3}{2} \text{tr} \chi \zeta + \zeta \hat{\chi} - \text{div} \hat{\chi} + \frac{1}{2} \nabla \text{tr} \chi + \mathcal{D}_4 \nabla \log \Omega &= 0 \\ \mathcal{D}_3 \zeta + \frac{3}{2} \text{tr} \underline{\chi} \zeta + \zeta \hat{\underline{\chi}} + \text{div} \hat{\underline{\chi}} - \frac{1}{2} \nabla \text{tr} \underline{\chi} - \mathcal{D}_3 \nabla \log \Omega &= 0. \end{aligned} \tag{6.152}$$

The derivative with respect to the parameter ν of $\tilde{\zeta}$ is:

$$\frac{\partial \tilde{\zeta}}{\partial \nu}(\lambda, \nu; p_0) = (\Phi_\lambda^* \Phi_\nu^* \mathcal{L}_N \zeta)(q). \tag{6.153}$$

In fact

$$\begin{aligned} \frac{\partial \tilde{\zeta}}{\partial \nu}(\lambda, \nu; p_0) &= \underline{\Phi}_\lambda^* \lim_{h \rightarrow 0} \frac{1}{h} [\Phi_{\nu+h}^* \zeta(\Phi_{\nu+h} \circ \underline{\Phi}_\lambda(p_0)) - (\Phi_\nu^* \zeta)(\Phi_\nu \circ \underline{\Phi}_\lambda(p_0))] \\ &= \underline{\Phi}_\lambda^* \Phi_\nu^* \lim_{h \rightarrow 0} \frac{1}{h} [\Phi_h^* \zeta(\Phi_h(q)) - \zeta(q)] = (\underline{\Phi}_\lambda^* \Phi_\nu^* \mathcal{L}_N \zeta)(\Psi(\lambda, \nu)(p_0)) \\ &= (\underline{\Phi}_\lambda^* \Phi_\nu^* \mathcal{L}_N \zeta)(q). \end{aligned} \tag{6.154}$$

A simple computation gives

$$\begin{aligned} (\mathcal{L}_N \zeta)(\cdot) &= \Omega[\mathcal{D}_4 \zeta(\cdot) + (\zeta \cdot \chi)(\cdot)] \\ &= \Omega[-\text{tr} \chi \zeta(\cdot) + \text{d}\text{iv} \chi(\cdot) - \nabla \text{tr} \chi(\cdot) - \mathcal{D}_4 \nabla \log \Omega(\cdot)] \end{aligned}$$

which implies

$$\frac{\partial \tilde{\zeta}}{\partial \nu}(\lambda, \nu; \theta, \phi)(\cdot) = (\underline{\Phi}_\lambda^* \Phi_\nu^* \Omega [-\text{tr} \chi \zeta + \text{d}\text{iv} \chi - \nabla \text{tr} \chi - \mathcal{D}_4 \nabla \log \Omega])(\lambda, \nu; \theta, \phi)(\cdot). \tag{6.155}$$

Looking at the r.h.s. of (6.155) it is easy to recognize that, denoting p_0 a point of S_0 the following relationships hold, whose proof is given later on:

$$\begin{aligned} (\underline{\Phi}_\lambda^* \Phi_\nu^* \Omega \text{tr} \chi \zeta)(p_0) &= \tilde{\Omega}(\lambda, \nu; p_0) \widetilde{\text{tr} \chi}(\lambda, \nu; p_0) \tilde{\zeta}(\lambda, \nu; p_0) \\ (\underline{\Phi}_\lambda^* \Phi_\nu^* \Omega \zeta \cdot \chi)(p_0) &= \tilde{\Omega}(\lambda, \nu; p_0) \tilde{\zeta}(\lambda, \nu; p_0) \cdot \tilde{\chi}(\lambda, \nu; p_0) \\ (\underline{\Phi}_\lambda^* \Phi_\nu^* \Omega \text{d}\text{iv} \chi)(p_0) &= \tilde{\Omega}(\lambda, \nu; p_0) \widetilde{\text{d}\text{iv} \chi}(\lambda, \nu; p_0) \\ (\underline{\Phi}_\lambda^* \Phi_\nu^* \Omega \nabla \text{tr} \chi)(p_0) &= \tilde{\Omega}(\lambda, \nu; p_0) \widetilde{\nabla \text{tr} \chi}(\lambda, \nu; p_0) \\ \underline{\Phi}_\lambda^* \Phi_\nu^* \Omega \mathcal{D}_4 \nabla \log \Omega(p_0) &= \frac{\partial \widetilde{\nabla \log \Omega}}{\partial \nu}(\lambda, \nu; p_0) - \tilde{\Omega} \widetilde{\nabla \log \Omega} \cdot \tilde{\chi}(\lambda, \nu; p_0). \end{aligned} \tag{6.156}$$

In conclusion the pullback on S_0 of the first equation of (6.152) is

$$\frac{\partial \tilde{\zeta}}{\partial \nu} + \tilde{\Omega} \widetilde{\text{tr} \chi} \tilde{\zeta} - \tilde{\Omega} \widetilde{\text{d}\text{iv} \chi} + \tilde{\Omega} \widetilde{\nabla \text{tr} \chi} + \frac{\partial \widetilde{\nabla \log \Omega}}{\partial \nu} - \tilde{\Omega} \widetilde{\nabla \log \Omega} \cdot \tilde{\chi} = 0. \tag{6.157}$$

Equation (6.157) has to be compared with the second equation of (2.30). The remaining equations along the outgoing cone are obtained in the same way. We recall that to write the pull back of the evolution equation along $C(\lambda)$ for $\hat{\chi}$ the following relation has been used⁵⁰

$$(\mathcal{L}_N \hat{\chi})(\cdot, \cdot) = \Omega[\mathcal{D}_4 \hat{\chi} + \hat{\chi} \cdot \chi + \chi \cdot \hat{\chi}](\cdot, \cdot) = \Omega[\mathcal{D}_4 \hat{\chi} + (\hat{\chi} \cdot \hat{\chi})\gamma + \text{tr} \chi \hat{\chi}](\cdot, \cdot).$$

⁵⁰ The notation here can be misleading. With $\hat{\chi} \cdot \chi$ we mean $(\hat{\chi} \cdot \chi)_{ab} = \hat{\chi}_{ac} \gamma^{cd} \chi_{db}$, while with $(\hat{\chi} \cdot \hat{\chi})$ we indicate $(\hat{\chi} \cdot \hat{\chi}) = \hat{\chi}_{ab} \gamma^{ac} \gamma^{bd} \hat{\chi}_{cd}$.

Finally the remaining “outgoing equations” have the following expressions:

$$\begin{aligned} & \frac{\partial \widetilde{\text{tr}}\chi}{\partial \nu} + \frac{\widetilde{\Omega} \widetilde{\text{tr}}\chi}{2} \widetilde{\text{tr}}\chi + 2\widetilde{\Omega} \widetilde{\omega} \widetilde{\text{tr}}\chi + \widetilde{\Omega} |\widehat{\chi}|^2 = 0 \\ & \frac{\partial \widetilde{\text{tr}}\chi}{\partial \nu} + \widetilde{\Omega} \widetilde{\text{tr}}\chi \widetilde{\text{tr}}\chi - 2\widetilde{\Omega} \widetilde{\omega} \widetilde{\text{tr}}\chi - 2\widetilde{\Omega} \widetilde{d}\widehat{\nu} \widetilde{\eta} - 2\widetilde{\Omega} |\widetilde{\eta}|^2 + 2\widetilde{\Omega} \widetilde{\mathbf{K}} = 0 \\ & \frac{\partial \widehat{\chi}}{\partial \nu} - \frac{\widetilde{\Omega} \widetilde{\text{tr}}\chi}{2} \widehat{\chi} + \frac{\widetilde{\Omega} \widetilde{\text{tr}}\chi}{2} \widetilde{\chi} + \frac{\partial \log \widetilde{\Omega}}{\partial \nu} \widehat{\chi} - \widetilde{\Omega} (\widehat{\chi} \cdot \widehat{\chi}) \gamma - \widetilde{\Omega} \widetilde{\nabla} \widehat{\eta} - \widetilde{\Omega} (\widehat{\eta} \widehat{\eta}) = 0 \\ & \frac{\partial \widetilde{\omega}}{\partial \nu} - 2\widetilde{\Omega} \widetilde{\omega} \widetilde{\omega} - \frac{3}{2} \widetilde{\Omega} |\widetilde{\zeta}|^2 + \widetilde{\Omega} \widetilde{\zeta} \cdot \widetilde{\nabla} \log \widetilde{\Omega} + \frac{1}{2} \widetilde{\Omega} |\widetilde{\nabla} \log \widetilde{\Omega}|^2 \\ & + \frac{1}{2} \widetilde{\Omega} \left(\widetilde{\mathbf{K}} + \frac{1}{4} \widetilde{\text{tr}}\chi \widetilde{\text{tr}}\chi - \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi} \right) = 0. \end{aligned}$$

As expected from the fact that the vector fields N and \underline{N} do not commute, the projection on S_0 produces non-equivalent expressions when applied to the equations along the incoming cones.⁵¹ Let us consider again the equation for ζ . The derivative with respect to λ of $\widetilde{\zeta}$ is:

$$\frac{\partial \widetilde{\zeta}}{\partial \lambda}(\lambda, \nu; p_0) = \Phi_\lambda^* \Phi_\nu^*(\mathcal{L}_V \zeta)(q), \tag{6.158}$$

where

$$V = \Phi_{*\nu} \underline{N} \tag{6.159}$$

is the vector field generating the diffeomorphism $\Phi_\nu \circ \Phi_\lambda \circ \Phi_\nu^{-1}$. In fact,

$$\begin{aligned} \frac{\partial \widetilde{\zeta}}{\partial \lambda}(\lambda, \nu; p_0) &= \Phi_\lambda^* \lim_{h \rightarrow 0} \frac{1}{h} [\Phi_h^* \Phi_\nu^* \zeta(\Phi_\nu \circ \Phi_{\lambda+h}(p_0)) - \Phi_\nu^* \zeta(\Phi_\nu \circ \Phi_\lambda(p_0))] \\ &= \Phi_\lambda^* \Phi_\nu^* \lim_{h \rightarrow 0} \frac{1}{h} [\Phi_\nu^{-1} \Phi_h^* \Phi_\nu^* \zeta(\Phi_\nu \circ \Phi_{\lambda+h}(p_0)) - \zeta(\Phi_\nu \circ \Phi_\lambda(p_0))] \\ &= \Phi_\lambda^* \Phi_\nu^* \lim_{h \rightarrow 0} \frac{1}{h} [(\Phi_\nu^{-1} \Phi_h^* \Phi_\nu^* \zeta)(q) - \zeta(q)] \\ &= \Phi_\lambda^* \Phi_\nu^*(\mathcal{L}_V \zeta)(\Psi(\lambda, \nu)(p_0)) \\ &= \Phi_\lambda^* \Phi_\nu^*(\mathcal{L}_V \zeta)(q). \end{aligned} \tag{6.160}$$

The following relation holds:

$$(\mathcal{L}_V \zeta)(q) = (\mathcal{L}_{\underline{N}} \zeta)(q) - (\mathcal{L}_X \zeta)(q) \tag{6.161}$$

and $\widetilde{\zeta}$ satisfies the following equation:

$$\begin{aligned} & \frac{\partial \widetilde{\zeta}}{\partial \lambda} - \frac{\partial \widetilde{\nabla} \log \widetilde{\Omega}}{\partial \lambda} + \widetilde{\Omega} \widetilde{\text{tr}}\chi \widetilde{\zeta} + \widetilde{\Omega} \widetilde{d}\widehat{\nu} \widetilde{\chi} - \widetilde{\Omega} \widetilde{\nabla} \widetilde{\text{tr}}\chi + \widetilde{\nabla} \log \widetilde{\Omega} \cdot \widetilde{\chi} + \mathcal{L}_{\widetilde{X}} \widetilde{\zeta} \\ & - \mathcal{L}_{\widetilde{X}} \widetilde{\nabla} \log \widetilde{\Omega} = 0. \end{aligned} \tag{6.162}$$

To express all the remaining equations on the incoming cones pulled back to S_0 we use also the following equation, whose proof follows in the next subsection:

$$\Phi_\lambda^* \Phi_\nu^*(\Omega \mathcal{D}_3 f)(q) = \frac{\partial \widetilde{f}}{\partial \lambda}(p_0) + \widetilde{X}|_{p_0}(\widetilde{f}). \tag{6.163}$$

⁵¹ Apart from the obvious substitution of λ with ν .

Using also Eq. (6.163) we obtain

$$\begin{aligned}
 & \frac{\partial \tilde{\omega}}{\partial \lambda} + \tilde{\nabla}_{\tilde{x}} \tilde{\omega} - 2\tilde{\Omega} \tilde{\omega} - \frac{3}{2}\tilde{\Omega}|\tilde{\zeta}|^2 - \tilde{\Omega}\tilde{\zeta} \cdot \tilde{\nabla} \log \tilde{\Omega} + \frac{1}{2}\tilde{\Omega}|\tilde{\nabla} \log \tilde{\Omega}|^2 \\
 & \quad + \frac{1}{2}\tilde{\Omega} \left(\tilde{\mathbf{K}} + \frac{1}{4}\widetilde{\text{tr}}\chi\widetilde{\text{tr}}\chi - \frac{1}{2}\hat{\chi} \cdot \hat{\chi} \right) = 0 \\
 & \frac{\partial \widetilde{\text{tr}}\chi}{\partial \lambda} + \tilde{\Omega}\widetilde{\text{tr}}\chi\widetilde{\text{tr}}\chi - 2\tilde{\Omega}\tilde{\omega}\widetilde{\text{tr}}\chi + \tilde{\nabla}_{\tilde{x}}\widetilde{\text{tr}}\chi - 2\tilde{\Omega}\widetilde{\text{tr}}\chi\tilde{\eta} - 2\tilde{\Omega}|\tilde{\eta}|^2 + 2\tilde{\Omega}\tilde{\mathbf{K}} = 0 \\
 & \frac{\partial \hat{\chi}}{\partial \lambda} - \frac{\tilde{\Omega}\widetilde{\text{tr}}\chi}{2}\hat{\chi} + \frac{\tilde{\Omega}\widetilde{\text{tr}}\chi}{2}\hat{\chi} + \frac{\partial \log \tilde{\Omega}}{\partial \lambda}\hat{\chi} + (\tilde{\nabla}_{\tilde{x}} \log \tilde{\Omega})\hat{\chi} - \tilde{\Omega}(\hat{\chi} \cdot \hat{\chi})\gamma - \tilde{\Omega} \tilde{\nabla} \otimes \tilde{\eta} \\
 & \quad - \tilde{\Omega}(\tilde{\eta} \otimes \tilde{\eta}) + \mathcal{L}_{\tilde{x}}\hat{\chi} = 0 \tag{6.164} \\
 & \frac{\partial \tilde{\zeta}}{\partial \lambda} + \tilde{\Omega} \widetilde{\text{tr}}\chi\tilde{\zeta} + \tilde{\Omega}\widetilde{\text{tr}}\chi\tilde{\zeta} - \tilde{\Omega}\tilde{\nabla}\widetilde{\text{tr}}\chi - \frac{\partial \tilde{\nabla} \log \tilde{\Omega}}{\partial \lambda} + \tilde{\Omega}\tilde{\nabla} \log \tilde{\Omega} \cdot \tilde{\chi} \\
 & \quad + \mathcal{L}_{\tilde{x}}\tilde{\zeta} - \mathcal{L}_{\tilde{x}}\tilde{\nabla} \log \tilde{\Omega} = 0. \\
 & \frac{\partial \widetilde{\text{tr}}\chi}{\partial \lambda} + \frac{\tilde{\Omega}\widetilde{\text{tr}}\chi}{2}\widetilde{\text{tr}}\chi + \tilde{\nabla}_{\tilde{x}}\widetilde{\text{tr}}\chi + 2\tilde{\Omega}\tilde{\omega}\widetilde{\text{tr}}\chi + \tilde{\Omega}|\hat{\chi}|^2 = 0.
 \end{aligned}$$

6.4. The Structure Equations in a vacuum Einstein Spacetime for the Metric Components in the $\{\lambda, \nu, \theta, \phi\}$ Coordinates

To complete the set of p.d.e. equations representing the Einstein equations in this “double null foliation gauge” we have still to add some other equations. The previous equations are the analogue of the first order equations for the second fundamental form k_{ij} associated with the spacelike foliation of the Einstein spacetime. We need the analogue of the equations for the three dimensional metric g_{ij} . Observe that in this foliation the metric written in the “adapted” coordinates is given in (6.149) and the six quantity associated with the metric are Ω, X_a, γ_{ab} . The corresponding quantities, pulledback to TS_0 , are

$$\begin{aligned}
 \tilde{\gamma}(\lambda, \nu; p_0) &= \Phi_\lambda^* \Phi_\nu^* \gamma(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Phi_\lambda^* \Phi_\nu^* \gamma(q) \\
 \tilde{X}(\lambda, \nu; p_0) &= \Phi_\lambda^* \Phi_\nu^* X(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Phi_\lambda^* \Phi_\nu^* X(q) \\
 \tilde{\Omega}(\lambda, \nu; p_0) &= \Omega(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Omega(q),
 \end{aligned}$$

and, proceeding as before, their partial derivatives with respect to ν and λ are

$$\begin{aligned}
 \frac{\partial \tilde{\gamma}}{\partial \nu}(\cdot, \cdot)(\lambda, \nu; \theta, \phi) &= \Phi_\lambda^* \Phi_\nu^* (\mathcal{L}_N \gamma)(\cdot, \cdot)(q) = 2\tilde{\Omega} \tilde{\chi}(\cdot, \cdot)(\lambda, \nu; \theta, \phi) \\
 \frac{\partial \tilde{\gamma}}{\partial \lambda}(\cdot, \cdot)(\lambda, \nu; \theta, \phi) &= \Phi_\lambda^* \Phi_\nu^* (\mathcal{L}_V \gamma)(\cdot, \cdot)(q) = 2\tilde{\Omega} \tilde{\chi}(\cdot, \cdot)(\lambda, \nu; \theta, \phi) \\
 & \quad - (\mathcal{L}_{\tilde{x}} \tilde{\gamma})(\cdot, \cdot)(\lambda, \nu; \theta, \phi). \tag{6.165}
 \end{aligned}$$

Beside these equations we need, in these coordinates, the equation connecting X to the connection coefficients and analogous equations for Ω . From the

definition of ω and $\underline{\omega}$, in Sect. 2.3, it follows

$$\begin{aligned} \tilde{\omega}(\lambda, \nu; p_0) &= -\frac{1}{2\tilde{\Omega}} \frac{\partial \log \tilde{\Omega}}{\partial \nu}(\lambda, \nu; p_0) \\ \underline{\tilde{\omega}}(\lambda, \nu; p_0) &= -\frac{1}{2\tilde{\Omega}} \left(\frac{\partial \log \tilde{\Omega}}{\partial \lambda} + \tilde{\nabla}_{\tilde{X}} \log \tilde{\Omega} \right)(\lambda, \nu; p_0). \end{aligned} \tag{6.166}$$

The equation for X is derived from the commutation relation of N with \underline{N} ,

$$\begin{aligned} [N, \underline{N}] &= -4\Omega^2 Z, \\ [N, \underline{N}] &= \mathcal{L}_N \left(\frac{\partial}{\partial \lambda} + X \right) = \mathcal{L}_{\frac{\partial}{\partial \nu}} \left(\frac{\partial}{\partial \lambda} + X \right) = \mathcal{L}_N X. \end{aligned} \tag{6.167}$$

It follows, in the $\{\lambda, \nu, \omega^a\}$ coordinates,

$$\frac{\partial \tilde{X}}{\partial \nu}(\lambda, \nu; p_0) = -\Phi_{*\lambda}^{-1} \Phi_{*\nu}^{-1} 4\Omega^2 Z = 4\tilde{\Omega}^2 \tilde{Z}(\lambda, \nu; p_0), \tag{6.168}$$

where $\tilde{Z}(\cdot) = \sum_{A \in \{1,2\}} \tilde{\zeta}(\tilde{e}_A) \tilde{e}_A(\cdot)$.

6.5. The Equations (2.30) in the $\{\lambda, \nu, \theta, \phi\}$ Coordinates

We collect all the equations (2.30), written now as p.d.e. equations for covariant tensors on S_0 , depending on the parameters λ, ν .⁵² They are, indicating also the Ricci component to which they are associated,

$$\begin{aligned} \mathbf{R}(e_4, e_4) &= 0: \\ \frac{\partial \widetilde{\text{tr}} \chi}{\partial \nu} + \frac{\tilde{\Omega} \widetilde{\text{tr}} \chi}{2} + 2\tilde{\Omega} \tilde{\omega} \widetilde{\text{tr}} \chi + \tilde{\Omega} |\hat{\chi}|^2 &= 0 \\ \mathbf{R}(e_4, e_A) &= 0: \\ \frac{\partial \tilde{\zeta}}{\partial \nu} + \tilde{\Omega} \widetilde{\text{tr}} \chi \tilde{\zeta} - \tilde{\Omega} \widehat{d}iv \tilde{\chi} + \tilde{\Omega} \tilde{\nabla} \widetilde{\text{tr}} \chi + \frac{\partial \tilde{\nabla} \log \tilde{\Omega}}{\partial \nu} - \tilde{\Omega} \tilde{\nabla} \log \tilde{\Omega} \cdot \tilde{\chi} &= 0 \\ \delta_{AB} \mathbf{R}(e_A, e_B) &= 0: \\ \frac{\partial \widetilde{\text{tr}} \chi}{\partial \nu} + \tilde{\Omega} \widetilde{\text{tr}} \chi \widetilde{\text{tr}} \chi - 2\tilde{\Omega} \tilde{\omega} \widetilde{\text{tr}} \chi - 2\tilde{\Omega} \widehat{d}iv \tilde{\eta} - 2\tilde{\Omega} |\tilde{\eta}|^2 + 2\tilde{\Omega} \tilde{\mathbf{K}} &= 0 \\ \mathbf{R}(\widehat{e_A}, \widehat{e_B}) &= 0: \\ \frac{\partial \hat{\chi}}{\partial \nu} - \frac{\tilde{\Omega} \widetilde{\text{tr}} \chi}{2} \hat{\chi} + \frac{\tilde{\Omega} \widetilde{\text{tr}} \chi}{2} \tilde{\chi} + \frac{\partial \log \tilde{\Omega}}{\partial \nu} \hat{\chi} - \tilde{\Omega} (\hat{\chi} \cdot \hat{\chi}) \gamma - \tilde{\Omega} \tilde{\nabla} \hat{\eta} - \tilde{\Omega} (\hat{\eta} \hat{\otimes} \hat{\eta}) &= 0 \\ \mathbf{R}(\widehat{e_3}, \widehat{e_4}) &= 0: \\ \frac{\partial \underline{\tilde{\omega}}}{\partial \nu} - 2\tilde{\Omega} \tilde{\omega} \underline{\tilde{\omega}} - \frac{3}{2} \tilde{\Omega} |\tilde{\zeta}|^2 + \tilde{\Omega} \tilde{\zeta} \cdot \tilde{\nabla} \log \tilde{\Omega} + \frac{1}{2} \tilde{\Omega} |\tilde{\nabla} \log \tilde{\Omega}|^2 \\ + \frac{1}{2} \tilde{\Omega} \left(\tilde{\mathbf{K}} + \frac{1}{4} \widetilde{\text{tr}} \chi \widetilde{\text{tr}} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) &= 0 \\ \mathbf{R}(\widehat{e_3}, \widehat{e_4}) &= 0: \end{aligned}$$

⁵² It is clear that this equations can be immediately written as equations for the various components of these tensors. In fact the orthonormal frame introduced in S_0 does not depend on λ, ν .

$$\begin{aligned} & \frac{\partial \tilde{\omega}}{\partial \lambda} + \tilde{\nabla}_{\tilde{X}} \tilde{\omega} - 2\tilde{\Omega} \tilde{\omega} - \frac{3}{2}\tilde{\Omega}|\tilde{\zeta}|^2 - \tilde{\Omega}\tilde{\zeta} \cdot \tilde{\nabla} \log \tilde{\Omega} \\ & + \frac{1}{2}\tilde{\Omega}|\tilde{\nabla} \log \tilde{\Omega}|^2 + \frac{1}{2}\tilde{\Omega} \left(\tilde{\mathbf{K}} + \frac{1}{4}\text{tr}\tilde{\chi}\text{tr}\tilde{\chi} - \frac{1}{2}\hat{\chi} \cdot \hat{\chi} \right) = 0 \\ & \delta_{AB}\mathbf{R}(e_A, e_B) = 0: \\ & \frac{\partial \text{tr}\tilde{\chi}}{\partial \lambda} + \tilde{\Omega}\text{tr}\tilde{\chi}\text{tr}\tilde{\chi} - 2\tilde{\Omega}\tilde{\omega}\text{tr}\tilde{\chi} + \tilde{\nabla}_{\tilde{X}}\text{tr}\tilde{\chi} - 2\tilde{\Omega}\text{d}\tilde{v}\tilde{\eta} - 2\tilde{\Omega}|\tilde{\eta}|^2 + 2\tilde{\Omega}\tilde{\mathbf{K}} = 0 \\ & \mathbf{R}(\widehat{e_A, e_B}) = 0: \\ & \frac{\partial \hat{\chi}}{\partial \lambda} - \frac{\tilde{\Omega}\text{tr}\tilde{\chi}}{2}\hat{\chi} + \frac{\tilde{\Omega}\text{tr}\tilde{\chi}}{2}\hat{\chi} + \frac{\partial \log \tilde{\Omega}}{\partial \lambda}\hat{\chi} + (\tilde{\nabla}_{\tilde{X}} \log \tilde{\Omega})\hat{\chi} - \tilde{\Omega}(\hat{\chi} \cdot \hat{\chi})\gamma - \tilde{\Omega} \tilde{\nabla} \hat{\eta} \\ & - \tilde{\Omega}(\hat{\eta} \hat{\otimes} \hat{\eta}) + \mathcal{L}_{\tilde{X}}\hat{\chi} = 0 \end{aligned} \tag{6.169}$$

$$\begin{aligned} & \mathbf{R}(e_3, e_A) = 0: \\ & \frac{\partial \tilde{\zeta}}{\partial \lambda} + \tilde{\Omega} \text{tr}\tilde{\chi}\tilde{\zeta} + \tilde{\Omega}\text{d}\tilde{v}\tilde{\chi} - \tilde{\Omega}\tilde{\nabla}\text{tr}\tilde{\chi} - \frac{\partial \tilde{\nabla} \log \tilde{\Omega}}{\partial \lambda} + \tilde{\Omega}\tilde{\nabla} \log \tilde{\Omega} \cdot \tilde{\chi} \\ & + \mathcal{L}_{\tilde{X}}\tilde{\zeta} - \mathcal{L}_{\tilde{X}}\tilde{\nabla} \log \tilde{\Omega} = 0. \\ & \mathbf{R}(\widehat{e_3, e_3}) = 0: \\ & \frac{\partial \text{tr}\tilde{\chi}}{\partial \lambda} + \frac{\tilde{\Omega}\text{tr}\tilde{\chi}}{2}\text{tr}\tilde{\chi} + \tilde{\nabla}_{\tilde{X}}\text{tr}\tilde{\chi} + 2\tilde{\Omega}\tilde{\omega}\text{tr}\tilde{\chi} + \tilde{\Omega}|\hat{\chi}|^2 = 0, \end{aligned}$$

where $\eta = \zeta + \tilde{\nabla} \log \Omega$ and $\underline{\eta} = -\zeta + \tilde{\nabla} \log \Omega$ and $\mathbf{R}(\cdot, \cdot)$ is the traceless part of the Ricci tensor. To have a closed set of equations we have to add the equations at the level of the metric components (6.165), (6.166) and (6.168)

$$\begin{aligned} & \frac{\partial \tilde{\gamma}}{\partial \nu} - 2\tilde{\Omega} \tilde{\chi} = 0, \quad \frac{\partial \tilde{\gamma}}{\partial \lambda} - 2\tilde{\Omega} \tilde{\chi} + \mathcal{L}_{\tilde{X}}\tilde{\gamma} = 0 \\ & \frac{\partial \log \tilde{\Omega}}{\partial \nu} + 2\tilde{\Omega} \tilde{\omega} = 0, \quad \frac{\partial \log \tilde{\Omega}}{\partial \lambda} + \tilde{\nabla}_X \log \tilde{\Omega} + 2\tilde{\Omega} \tilde{\omega} = 0 \\ & \frac{\partial \tilde{X}}{\partial \nu} - 4\tilde{\Omega}^2 \tilde{Z} = 0. \end{aligned} \tag{6.170}$$

In the next subsection we prove equations (2.36) which complete the construction of a system of first order partial differential equations from the structure equations (2.30).

6.6. Proof of Equations (2.36)

Observe that $\gamma(\lambda, \nu)(\cdot, \cdot)$ is a symmetric covariant two tensor defined on S_0 . We can consider it as a metric tensor, but the natural metric tensor on S_0 is $\gamma_0(\cdot, \cdot) \equiv \gamma(0, 0)(\cdot, \cdot)$.⁵³ On the other side, as discussed before, $\gamma(\lambda, \nu)(\cdot, \cdot)$ is the pull back via $\Psi^*(\lambda, \nu)$ of a metric tensor induced on $S(\lambda, \nu)$. The relationship,

⁵³ With respect to the metric $\gamma_0(\cdot, \cdot) \equiv \gamma(0, 0)(\cdot, \cdot)$, we define the area of S_0 as $|S_0|_{\gamma_0}$ and its radius $\sqrt{4\pi r_0} \equiv |S_0|_{\gamma_0}^{\frac{1}{2}}$. As S_0 is diffeomorphic to S^2 , if the initial data on S_0 are “small”, see for more details [4], we can conclude that the metric $\gamma_0(\cdot, \cdot)$ is “near” to $r_0(d\theta^2 + \sin^2\theta d\phi^2)$.

reintroducing the tildas temporarily, is (6.165),

$$\frac{\partial}{\partial \nu} \tilde{\gamma}(\lambda, \nu; p_0)(\cdot, \cdot) = (\Psi^*(\lambda, \nu) \mathcal{L}_N \gamma)(\Psi(\lambda, \nu)(p_0))(\cdot, \cdot).$$

Denoting $\{e_A^{(0)}\}$ an orthonormal frame on S_0 with respect to the metric γ_0 it is clear that, in general, $\tilde{\gamma}(e_A^{(0)}, e_B^{(0)}) \neq \delta_{AB}$. The variables $v_{cba}(\lambda, \nu, \omega)$ are the (non-covariant) derivatives with respect to the angular variables of $\tilde{\gamma}(\lambda, \nu; p_0) = \tilde{\gamma}(\lambda, \nu; \omega)$ and do not have to be thought as the transport of partial derivatives of $\gamma(\lambda, \nu; p_0)$ on $S(\lambda, \nu)$. The pullback is defined for the covariant tensors on $S(\lambda, \nu)$. This is true also for the covariant derivatives in the sense that denoted W a tensor on $S(\lambda, \nu)$ it follows that $\widetilde{\nabla} W = \tilde{\nabla} \tilde{W}$ and $\tilde{\nabla}$ is the covariant derivative done with respect to the metric $\tilde{\gamma}$. Therefore if, for instance, W is a one form we have

$$\widetilde{\nabla} W_{ab} = \tilde{\nabla}_a \tilde{W}_b = \frac{\partial}{\partial \omega^a} \tilde{W}_b - \tilde{\Gamma}_{ab}^c \tilde{W}_c \tag{6.171}$$

$$\text{and } \tilde{\Gamma}_{ab}^c = \frac{1}{2} \tilde{\gamma}^{cd} (v_{adb} + v_{bad} - v_{dab}), \tag{6.172}$$

where v_{abc} are the unknown functions of the first order system defined in (2.35). Omitting the tildas, we have:

$$\begin{aligned} \frac{\partial}{\partial \lambda} \psi_a &= \frac{\partial}{\partial \lambda} \nabla_a \log \Omega = \nabla_a \frac{\partial}{\partial \lambda} \log \Omega = \nabla_a (-2\Omega \omega - \nabla_X \log \Omega) \\ &= -2\Omega \nabla_a \omega - 2\Omega \omega \psi_a - (\nabla_a X)^c \psi_c - X^c \nabla_c \psi_a. \end{aligned} \tag{6.173}$$

$$\begin{aligned} \frac{\partial}{\partial \lambda} v_{cba} &= \frac{\partial}{\partial \lambda} (\partial \gamma)_{cba} = \partial_c \frac{\partial}{\partial \lambda} \gamma_{ba} = -\partial_c (\mathcal{L}_X \gamma)_{ab} + \partial_c (2\Omega \chi)_{ab} \\ &= -(\partial_c X^d) \partial_d \gamma_{ab} - \partial_X \partial_c \gamma_{ab} + \partial_c (\partial_a X^d \gamma_{db} + \partial_b X^d \gamma_{ad}) + 2\Omega \partial_c \chi_{ab} \\ &\quad + 2\Omega (\nabla_c \log \Omega) \chi_{ab} \\ &= -(\partial_c X^d) v_{dab} - \partial_X v_{cab} + \partial_c \partial_a X_b + \partial_c \partial_b X_a + 2\Omega \partial_c \chi_{ab} + 2\Omega \psi_c \chi_{ab} \\ &= -(\partial_c X^d) v_{dab} - \partial_X v_{cab} + \partial_c w_{ab} + \partial_c w_{ba} + 2\Omega \partial_c \chi_{ab} + 2\Omega \psi_c \chi_{ab}. \end{aligned}$$

To obtain the equation for w_{ab} we write:

$$\begin{aligned} \frac{\partial w_{ab}}{\partial \nu} &= \frac{\partial \partial_a X_b}{\partial \nu} = \partial_a \frac{\partial \gamma_{bc} X^c}{\partial \nu} = -4\partial_a (\Omega^2 \zeta_b) + \left(\partial_a \frac{\partial \gamma_{bc}}{\partial \nu} \right) X^c \\ &= -4\partial_a (\Omega^2 \zeta_b) + \partial_a (2\Omega \chi_{bc}) X^c \\ &= -8\Omega^2 (\nabla_a \log \Omega) \zeta_b - 4\Omega^2 \partial_a \zeta_b + 2\Omega (\nabla_a \log \Omega) \chi_{bc} X^c + 2\Omega (\partial_a \chi_{bc}) X^c \\ &= -8\Omega^2 \psi_a \zeta_b - 4\Omega^2 \partial_a \zeta_b + 2\Omega \psi_a \chi_{bc} X^c + 2\Omega (\partial_a \chi_{bc}) X^c. \end{aligned}$$

We collect the three equations obtained,

$$\begin{aligned} \frac{\partial}{\partial \lambda} \psi_a &= -2\Omega \partial_a \omega - 2\Omega \omega \psi_a - (\nabla_a X)^c \psi_c - X^c \nabla_c \psi_a \\ \frac{\partial}{\partial \lambda} v_{cba} &= -(\partial_c X^d) v_{dab} - \partial_X v_{cab} + \partial_c w_{ab} + \partial_c w_{ba} + 2\Omega \partial_c \chi_{ab} + 2\Omega \psi_c \chi_{ab} \\ \frac{\partial w_{ab}}{\partial \nu} &= -8\Omega^2 \psi_a \zeta_b - 4\Omega^2 \partial_a \zeta_b + 2\Omega \psi_a \chi_{bc} X^c + 2\Omega (\partial_a \chi_{bc}) X^c. \end{aligned} \tag{6.174}$$

6.7. The Null Components of the Riemann Tensor on $S_0 \times R \times R$ and their Bianchi Equations

In our notation the ten independent components of the (conformal part of the) Riemann tensor are denoted $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$, respectively, two covariant two-tensors, two covariant vectors and two scalar functions, tangential at each point to $S(\lambda, \nu)$, the two-dimensional surface containing it. All these components written in terms of the connection coefficients and their derivatives are, see [8, 15]:

$$\begin{aligned}
 \alpha &= -\mathcal{D}_4 \hat{\chi} - \text{tr} \chi \hat{\chi} + (\mathbf{D}_4 \log \Omega) \hat{\chi} \\
 \beta &= \nabla \text{tr} \chi - \text{div} \chi - \zeta \cdot \chi + \zeta \text{tr} \chi \\
 \rho &= -\mathbf{K} - \frac{1}{4} \text{tr} \chi \text{tr} \chi + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \\
 \sigma &= \text{curl} \zeta - \frac{1}{2} \hat{\chi} \wedge \hat{\chi} \\
 \underline{\beta} &= -\nabla \text{tr} \underline{\chi} + \text{div} \underline{\chi} - \zeta \cdot \underline{\chi} + \zeta \text{tr} \underline{\chi} \\
 \underline{\alpha} &= -\mathcal{D}_3 \hat{\chi} - \text{tr} \underline{\chi} \hat{\chi} + (\mathbf{D}_3 \log \Omega) \hat{\chi}.
 \end{aligned} \tag{6.175}$$

We will consider these quantities pulled back on the two-dimensional surface S_0 , as we did before for the connection coefficients. Therefore, we define

$$\begin{aligned}
 \tilde{\alpha}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\alpha)(p_0) = \Phi_\lambda^* \Phi_\nu^* \alpha(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Phi_\lambda^* \Phi_\nu^* \alpha(q) \\
 \tilde{\beta}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\beta)(p_0) = \Phi_\lambda^* \Phi_\nu^* \beta(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Phi_\lambda^* \Phi_\nu^* \beta(q) \\
 \tilde{\underline{\beta}}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\underline{\beta})(p_0) = \Phi_\lambda^* \Phi_\nu^* \underline{\beta}(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Phi_\lambda^* \Phi_\nu^* \underline{\beta}(q) \\
 \tilde{\underline{\alpha}}(\lambda, \nu; p_0) &= (\Psi^*(\lambda, \nu)\underline{\alpha})(p_0) = \Phi_\lambda^* \Phi_\nu^* \underline{\alpha}(\Phi_\nu \circ \Phi_\lambda(p_0)) = \Phi_\lambda^* \Phi_\nu^* \underline{\alpha}(q) \\
 \tilde{\rho}(\lambda, \nu; p_0) &= (\rho \circ \Psi(\lambda, \nu))(p_0) = (\rho \circ \Phi_\nu \circ \Phi_\lambda)(p_0) = \rho(q) \\
 \tilde{\sigma}(\lambda, \nu; p_0) &= (\sigma \circ \Psi(\lambda, \nu))(p_0) = (\sigma \circ \Phi_\nu \circ \Phi_\lambda)(p_0) = \sigma(q)
 \end{aligned} \tag{6.176}$$

where $p_0 \in S_0$ and is specified by its coordinates (θ, ϕ) . In conclusion, we have the following covariant tensors defined on S_0 ,

$$\begin{aligned}
 &\tilde{\alpha}(\lambda, \nu; \theta, \phi), \quad \tilde{\underline{\alpha}}(\lambda, \nu; \theta, \phi), \quad \tilde{\beta}(\lambda, \nu; \theta, \phi), \\
 &\tilde{\underline{\beta}}(\lambda, \nu; \theta, \phi), \quad \tilde{\rho}(\lambda, \nu; \theta, \phi), \quad \tilde{\sigma}(\lambda, \nu; \theta, \phi).
 \end{aligned}$$

Lemma 6.1. *The explicit expressions for the pullback of the various null Riemann components are*

$$\begin{aligned}
 \tilde{\alpha} &= -\frac{1}{\tilde{\Omega}} \frac{\partial \hat{\chi}}{\partial \nu} + |\hat{\chi}|^2 \gamma - 2\tilde{\omega} \hat{\chi} \\
 \tilde{\beta} &= \frac{1}{2} \nabla \text{tr} \tilde{\chi} - \tilde{\text{div}} \tilde{\chi} - \tilde{\zeta} \cdot \tilde{\chi} + \frac{1}{2} \tilde{\zeta} \text{tr} \tilde{\chi} \\
 \tilde{\rho} &= -\tilde{\mathbf{K}} - \frac{1}{4} \text{tr} \tilde{\chi} \text{tr} \tilde{\chi} + \frac{1}{2} \tilde{\chi} \cdot \tilde{\chi} \\
 \tilde{\sigma} &= \tilde{\text{curl}} \tilde{\zeta} - \frac{1}{2} \tilde{\chi} \wedge \tilde{\chi}
 \end{aligned} \tag{6.177}$$

$$\begin{aligned} \underline{\tilde{\beta}} &= -\frac{1}{2}\tilde{\nabla}\widetilde{\text{tr}\chi} + \widetilde{\text{div}}\hat{\chi} - \zeta \cdot \hat{\chi} + \frac{1}{2}\tilde{\zeta}\widetilde{\text{tr}\chi} \\ \underline{\tilde{\alpha}} &= -\frac{1}{\tilde{\Omega}}\left(\frac{\partial\hat{\chi}}{\partial\lambda} + \mathcal{L}_{\hat{\chi}}\hat{\chi}\right) + |\hat{\chi}|^2\gamma - 2\tilde{\omega}\hat{\chi}. \end{aligned}$$

Proof. It follows immediately from the explicit expressions of the various Riemann null components (6.175) and the pullback of the various connection coefficients and their derivatives.

The Bianchi equations take the form of a system of transport equations with respect to the variables λ and ν for these covariant tensors. Defining on S_0 an orthonormal basis $\{\tilde{e}_\theta, \tilde{e}_\phi\}$, each of them can be written in the following way:

$$\tilde{\alpha}(\cdot) = \sum_A \tilde{\alpha}_{AB}(\lambda, \nu; \theta, \phi)\tilde{\theta}_A \otimes \tilde{\theta}_B(\cdot, \cdot), \tag{6.178}$$

where

$$\tilde{\alpha}_{AB}(\lambda, \nu; \theta, \phi) \equiv \tilde{\alpha}(\lambda, \nu; \theta, \phi)(\tilde{e}_A, \tilde{e}_B).$$

We start writing the derivative with respect to ν of $\tilde{\alpha}$ obtaining:

$$\frac{\partial\tilde{\alpha}}{\partial\nu}(\lambda, \nu; p_0) = \Phi_\lambda^*\Phi_\nu^*(\mathcal{L}_{N\alpha})(q). \tag{6.179}$$

The proof goes over exactly as the one for $\frac{\partial\tilde{\zeta}}{\partial\nu}(\lambda, \nu; p_0)$ and we do not repeat it. A simple computation gives

$$\begin{aligned} (\mathcal{L}_{N\alpha})(\cdot) &= \Omega\mathcal{D}_4\alpha(\cdot) + \Omega(\alpha \cdot \chi + \chi \cdot \alpha)(\cdot) = \Omega[\mathcal{D}_4\alpha + \text{tr}\chi\alpha + (\alpha \cdot \hat{\chi})\gamma](\cdot) \\ &= \Omega\left[\frac{\text{tr}\chi}{2}\alpha + (\alpha \cdot \hat{\chi})\gamma - \nabla\hat{\otimes}\underline{\beta} + (4\omega\alpha - 3(\hat{\chi}\rho - \hat{\chi}\sigma) + (\zeta - 4\underline{\eta})\hat{\otimes}\underline{\beta})\right] \end{aligned} \tag{6.180}$$

which implies

$$\begin{aligned} \frac{\partial\tilde{\alpha}}{\partial\nu}(\lambda, \nu; \theta, \phi)(\cdot) &= \left(\Phi_\lambda^*\Phi_\nu^*\Omega\left[\frac{\text{tr}\chi}{2}\alpha + (\alpha \cdot \hat{\chi})\gamma - \nabla\hat{\otimes}\underline{\beta} + (4\omega\alpha - 3(\hat{\chi}\rho - \hat{\chi}\sigma) \right. \right. \\ &\quad \left. \left. + (\zeta - 4\underline{\eta})\hat{\otimes}\underline{\beta})\right]\right)(\lambda, \nu; \theta, \phi)(\cdot). \end{aligned} \tag{6.181}$$

Applying, as before, the pull-back on the right hand side, the final expression of the Bianchi equations, for this component, is

$$\begin{aligned} \frac{\partial\tilde{\alpha}}{\partial\nu}(\lambda, \nu; \theta, \phi)(\cdot) &= \tilde{\Omega}\left[\frac{\text{tr}\chi}{2}\tilde{\alpha} + (\tilde{\alpha} \cdot \hat{\chi})\tilde{\gamma} - \nabla\hat{\otimes}\tilde{\beta} + (4\tilde{\omega}\tilde{\alpha} - 3(\hat{\chi}\tilde{\rho} - \hat{\chi}\tilde{\sigma}) \right. \\ &\quad \left. + (\tilde{\zeta} - 4\tilde{\eta})\hat{\otimes}\tilde{\beta})\right](\lambda, \nu; \theta, \phi)(\cdot). \end{aligned} \tag{6.182}$$

It is easy now to project the remaining Bianchi equations on S_0 .

$$\begin{aligned}
 \frac{\partial \underline{\tilde{\beta}}}{\partial \nu} &= \underline{\Phi}_\lambda^* \Phi_\nu^* (\Omega [\mathcal{D}_4 \underline{\beta} + \chi \cdot \underline{\beta}]) = \underline{\Phi}_\lambda^* \Phi_\nu^* \left[\Omega \left(-\frac{1}{2} \text{tr} \chi \underline{\beta} + \hat{\chi} \cdot \underline{\beta} - \nabla \rho + [2\omega \underline{\beta} + 2\hat{\chi} \cdot \beta \right. \right. \\
 &\quad \left. \left. + {}^* \nabla \sigma - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma) \right) \right] \\
 &= \tilde{\Omega} \left(-\frac{1}{2} \widetilde{\text{tr}} \chi \tilde{\beta} + \hat{\chi} \cdot \tilde{\beta} - \tilde{\nabla} \tilde{\rho} + [2\tilde{\omega} \tilde{\beta} + 2\hat{\chi} \cdot \tilde{\beta} + {}^* \tilde{\nabla} \tilde{\sigma} - 3(\tilde{\eta} \tilde{\rho} - {}^* \tilde{\eta} \tilde{\sigma})] \right) \\
 \frac{\partial \tilde{\rho}}{\partial \nu} &= \underline{\Phi}_\lambda^* \Phi_\nu^* (\Omega \mathcal{D}_4 \rho) \\
 &= \underline{\Phi}_\lambda^* \Phi_\nu^* \left[\Omega \left(-\frac{3}{2} \text{tr} \chi \rho + \mathcal{d} \! \! \! / \nu \beta - \left[\frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta - 2 \nabla \log \Omega \cdot \beta \right] \right) \right] \\
 &= \tilde{\Omega} \left(-\frac{3}{2} \widetilde{\text{tr}} \chi \tilde{\rho} + \widetilde{\mathcal{d} \! \! \! / \nu} \tilde{\beta} - \left[\frac{1}{2} \hat{\chi} \cdot \tilde{\alpha} + \tilde{\zeta} \cdot \tilde{\beta} - 2 \nabla \log \Omega \cdot \tilde{\beta} \right] \right) \\
 \frac{\partial \tilde{\sigma}}{\partial \nu} &= \underline{\Phi}_\lambda^* \Phi_\nu^* (\Omega \mathcal{D}_4 \sigma) \\
 &= \underline{\Phi}_\lambda^* \Phi_\nu^* \left[\Omega \left(-\frac{3}{2} \text{tr} \chi \sigma + \mathcal{d} \! \! \! / \nu {}^* \beta - \left[\frac{1}{2} \hat{\chi} \cdot {}^* \alpha + \zeta \cdot {}^* \beta - 2 \nabla \log \Omega \cdot {}^* \beta \right] \right) \right] \\
 &= \tilde{\Omega} \left(-\frac{3}{2} \widetilde{\text{tr}} \chi \tilde{\sigma} + \widetilde{\mathcal{d} \! \! \! / \nu} {}^* \tilde{\beta} - \left[\frac{1}{2} \hat{\chi} \cdot {}^* \tilde{\alpha} + \tilde{\zeta} \cdot {}^* \tilde{\beta} - 2 \nabla \log \Omega \cdot {}^* \tilde{\beta} \right] \right) \\
 \frac{\partial \tilde{\beta}}{\partial \nu} &= \underline{\Phi}_\lambda^* \Phi_\nu^* (\Omega [\mathcal{D}_4 \beta + \chi \cdot \beta]) \\
 &= \underline{\Phi}_\lambda^* \Phi_\nu^* \left[\Omega \left(-\frac{3}{2} \text{tr} \chi + \mathcal{d} \! \! \! / \nu \alpha - 2\omega \beta + \hat{\chi} \cdot \beta + (\zeta + \nabla \log \Omega) \cdot \alpha \right) \right] \\
 &= \tilde{\Omega} \left(-\frac{3}{2} \widetilde{\text{tr}} \chi \tilde{\beta} + \widetilde{\mathcal{d} \! \! \! / \nu} \tilde{\alpha} - 2\omega \beta + [\hat{\chi} \cdot \beta + (\zeta + \nabla \log \Omega) \cdot \alpha] \right). \tag{6.183}
 \end{aligned}$$

The Bianchi equations which are transport equations along the incoming cones can be obtained exactly in the same way as done for the connection coefficients. In this case we have to use the relation, see (6.161), for a generic Riemann null component w ,

$$\frac{\partial \tilde{w}}{\partial \lambda} (\lambda, \nu; \theta, \phi) (\cdot) = \underline{\Phi}_\lambda^* \Phi_\nu^* [(\mathcal{L}_N w)(q) - (\mathcal{L}_X w)(q)]. \tag{6.184}$$

$$\begin{aligned}
 \frac{\partial \tilde{\beta}}{\partial \lambda} &= \tilde{\Omega} \left(-\frac{3}{2} \widetilde{\text{tr}} \chi \tilde{\beta} + \hat{\chi} \cdot \tilde{\beta} - \mathcal{d} \! \! \! / \nu \tilde{\alpha} - [2\tilde{\omega} \tilde{\beta} + (-2\tilde{\zeta} + \tilde{\eta}) \cdot \tilde{\alpha}] \right) - \mathcal{L}_{\tilde{X}} \tilde{\beta} \\
 \frac{\partial \tilde{\rho}}{\partial \lambda} &= \tilde{\Omega} \left(-\frac{3}{2} \widetilde{\text{tr}} \chi \tilde{\rho} - \widetilde{\mathcal{d} \! \! \! / \nu} \tilde{\beta} - \left[\frac{1}{2} \hat{\chi} \cdot \tilde{\alpha} - \tilde{\zeta} \cdot \tilde{\beta} + 2\tilde{\eta} \cdot \tilde{\beta} \right] \right) - \tilde{\nabla}_{\tilde{X}} \tilde{\rho} \\
 \frac{\partial \tilde{\sigma}}{\partial \lambda} &= \tilde{\Omega} \left(-\frac{3}{2} \widetilde{\text{tr}} \chi \tilde{\sigma} - \widetilde{\mathcal{d} \! \! \! / \nu} {}^* \tilde{\beta} + \left[\frac{1}{2} \hat{\chi} \cdot {}^* \tilde{\alpha} - \tilde{\zeta} \cdot {}^* \tilde{\beta} - 2\tilde{\eta} \cdot {}^* \tilde{\beta} \right] \right) - \tilde{\nabla}_{\tilde{X}} \tilde{\sigma} \tag{6.185} \\
 \frac{\partial \tilde{\beta}}{\partial \lambda} &= \tilde{\Omega} \left(-\frac{1}{2} \widetilde{\text{tr}} \chi \tilde{\beta} + \hat{\chi} \cdot \tilde{\beta} + \tilde{\nabla} \tilde{\rho} + [2\tilde{\omega} \tilde{\beta} + 2\hat{\chi} \cdot \underline{\beta} + {}^* \tilde{\nabla} \tilde{\sigma} + 3(\tilde{\eta} \tilde{\rho} + {}^* \tilde{\eta} \tilde{\sigma})] \right) \\
 &\quad - \mathcal{L}_{\tilde{X}} \tilde{\beta}
 \end{aligned}$$

$$\frac{\partial \tilde{\alpha}}{\partial \lambda} = \tilde{\Omega} \left[\frac{\text{tr} \tilde{\chi}}{2} \tilde{\alpha} + (\tilde{\alpha} \cdot \hat{\underline{\chi}}) \gamma + \nabla \hat{\otimes} \tilde{\beta} + \left(4 \underline{\tilde{\omega}} \tilde{\alpha} - 3(\hat{\underline{\chi}} \tilde{\rho} + \hat{\underline{\chi}} \tilde{\sigma}) + (\tilde{\zeta} + 4 \tilde{\eta}) \hat{\otimes} \tilde{\beta} \right) \right] - \mathcal{L}_{\hat{\underline{x}}} \tilde{\alpha}.$$

7. The Cone Problem, a Strategy for its Solution

In this paper and in the following one where various technical details and proofs are provided, we proved that analytic solutions exist in the same region where the a priori estimates for some “energy-type” norms, associated with the first few metric derivatives hold, which, for sufficiently small initial data, means that we have proved global existence for the analytic solutions.

On the other side proving a priori estimates for the first few derivatives amounts to provide the basic step in proving global existence for “Sobolev space solutions”, a result we have proved in [6]. Therefore, to extend the results of this paper to the “cone” situation, meaning with it the “global” region $J^{(+)}(P_0)$, P_0 being the lower vertex we can interpret as the spacetime origin, we need first to prove a less regular global existence result in our “double null foliation” setting. This is still an open problem, whatever gauge we consider, but we expect that, once we prove it, the extension of the existence proof to analytic solutions can be obtained basically in the same way as done in the present paper.

In the following we sketch the structure of a global existence proof in the present geometric gauge without giving any detail, but only pointing out the main problems to face and presenting a possible strategy, in our opinion, to solve them.

- (1) The initial data at the tip of the cone. We have to consider what happens when the initial surface of our present problem $S_0 = C_0 \cup \underline{C}_0$ tends to a point. We only point out two problems: first we need to consider the limit of the connection coefficients when they tend to the tip of the cone. It will be necessary to rescale them as they tend to infinity as r , the radius of S_0 , tends to zero. Second, observing that, as S_0 tends to a point, \underline{C}_0 , the incoming cone of our present work, becomes smaller and smaller up to vanishing, we have to exploit how to recover the initial data defined on it.
- (2) Once carefully solved the initial data problem for the connection coefficients, we have to find the appropriate foliation of C_0 in S^2 surfaces; this basically amounts to define on C_0 an $u|_{C_0}$ function such that the level surfaces of the solution of the Eikonal equation with this initial data are not generic null hypersurfaces, but “incoming cones”, meaning with it that all the incoming null geodesics with tangent vector $\underline{L} = (2\Omega)^{-1} e_3$, starting from an S^2 surface, $S(\lambda_0, \nu)$, just defined on C_0 , meet all at the same point $P(\nu)$, hence generating an incoming cone with $P(\nu)$ as vertex. The family of points $P(\nu)$ must form a timelike geodesic curve $\lambda(\nu)$ which can be interpreted as the timelike geodesic of the “spacetime origin”. This is a delicate problem and one of the main difference with our

present work, where the choice of Ω is free, apart from some norm upper bounds. Incidentally let us notice that there is not only one solution to this problem as it depends on the choice of the geodesic curve $\lambda(\nu)$.⁵⁴

- (3) On the curve $\lambda(\nu)$ where the vertices of the incoming cones lie we have to specify how the null incoming direction $e_3(\theta, \phi)$ is connected to a null outgoing direction $e_4(\theta', \phi')$. The null vector field $e_3(\theta, \phi)$ is continued at the tip of the cone with $\tilde{e}_4(\theta, \phi) = e_4(-\theta, \phi + \pi)$, the antipodal direction to $e_4(\theta, \phi)$, therefore the underlined connection coefficients $\underline{\mathcal{O}}$ has to be continued on the points of the curve $\lambda(\cdot)$ with the not underlined ones \mathcal{O} along the antipodal direction. Therefore, on the points of the curve $\lambda(\cdot)$, their analytic behaviour has to match exactly.
- (4) These previous steps performed, we have a local solution of the Einstein equations with a double cone foliation on it and well defined (rescaled) connection coefficients \mathcal{O} and $\underline{\mathcal{O}}$ satisfying appropriate bounds. Therefore we can prove the global existence via a bootstrap argument similar to the one in [15]. This goes in the following way:

- (a) Once that the local existence of a (small) region $J^{(+)}(P_0) \cap J^{(-)}(P_1)$ endowed with a double cone foliation has been provided, we assume that the largest region $J^{(+)}(P_0) \cap J^{(-)}(P^*)$, ($P^* \in \lambda(\tau)$, with $\tau_* > \tau_1$) is the largest possible region with this foliation and appropriate bounds on the \mathcal{O} , $\underline{\mathcal{O}}$ and \mathcal{R} norms. What remains to prove is that, given small initial data on C_0 these norms can be made smaller and a region $J^{(+)}(P_0) \cap J^{(-)}(P^{**})$ with the same properties does exist, with P^{**} near to P^* ($\tau_{**} > \tau_*$) on the continuation of the geodesic curve $\lambda(\tau)$.
- (b) To prove this “extension step” one has to proceed in the following way: on $C_0(\nu_0, \nu_*)$ the initial data are assigned, but Ω is not free as it has to satisfy a constraint equation. We extend the free initial data $\hat{\chi}$ on C_0 , the other ones have to satisfy the various constraint equations as previously discussed, we assign freely also Ω on $C_0([\nu_*, \nu_* + \delta])$ and we consider the “rectangular” region $\mathcal{V}([\lambda_0, \lambda_*] \times [\nu_*, \nu_* + \delta])$ whose boundaries are:

$$\begin{aligned} \partial\mathcal{V}([\lambda_0, \lambda_*] \times [\nu_*, \nu_* + \delta]) &= C_0([\nu_*, \nu_* + \delta]) \\ &\cup \underline{C}(\nu_*; [\lambda_0, \lambda_* = \nu_*]) \cup C(\lambda_*, [\nu_*, \nu_* + \delta]) \\ &\cup \underline{C}(\nu_* + \delta; [\lambda_0, \lambda_* = \nu_*]). \end{aligned} \tag{7.186}$$

This region has exactly the same structure of the regions we have studied insofar therefore we could easily proceed as in [6]. Moreover

⁵⁴ If we consider a portion sufficiently small of the cone, $J^{(+)}(P_0) \cap J^{(-)}(P_1)$, with P_1 very “near” to P_0 , we can assume this portion contained in a convex norm neighbourhood of the spacetime and therefore diffeomorphically mapped into the region between an incoming and an outgoing cone of a Minkowski spacetime (the tangent space TM_{P_0}). In this case the geodesic curve $\lambda(\nu)$ is the image of a timelike straight line and different geodesic curves $\lambda(\nu)$ correspond to different timelike straight (geodesic) lines which can be transformed one into each other by boosts.

we can redefine the canonical foliation starting now from the new canonical definition of Ω on $\underline{C}(\nu_* + \delta; [\lambda_0, \lambda_* = \nu_*])$.⁵⁵

- (c) We have not yet reached the result as we have still to build the diamond region $J^{(+)}(P^*) \cap J^{(-)}(P^{**})$ where the null coordinates of P^* and P^{**} are $(\nu_*, \lambda_* = \nu_*)$ and $(\nu^{**} = \nu_* + \delta, \lambda^{**} = \nu^{**})$. This region is still the interior of a small portion of a cone and, therefore, its local existence is achieved as at the beginning, recalling that we have estimates for the initial data on $C(\lambda_*, [\nu_*, \nu_* + \delta])$ having proved the existence of the previous region. Again there is a matching between the incoming cones in $J^{(+)}(P^*) \cap J^{(-)}(P^{**})$ and those in $\mathcal{V}([\lambda_0, \lambda_*] \times [\nu_*, \nu_* + \delta])$.

References

- [1] Alinhac, S., Metivier, G.: Propagation de l'analyticité des solutions de systèmes hyperboliques non-linéaires. *Invent. Math.* **75**, 189–204 (1984)
- [2] Andersson, L., Moncrief, V.: Elliptic-hyperbolic systems and the Einstein equation. arXiv:gr-qc/0110111 v3 (2002)
- [3] Andersson, L., Moncrief, V.: Future complete vacuum spacetimes. arXiv:gr-qc/030305 (2003)
- [4] Caciotta, G., Nicolò, F.: Global characteristic problem for Einstein vacuum equations with small initial data: (I) the initial data constraints. *JHDE* **2**(1), 201–277 (2005)
- [5] Caciotta, G., Nicolò, F.: Global characteristic problem for Einstein vacuum equations with small initial data II. arXiv:gr-qc/0608038
- [6] Caciotta, G., Nicolò, F.: On a class of global characteristic problems for the Einstein vacuum equations with small initial data. *JMP* **51**, 1–21 (2010)
- [7] Christodoulou, D., Müller Zum Hagen, H.: Problème de valeur initiale caractéristique pour des systèmes quasi linéaires du second ordre. *C. R. Acad. Sc. Paris Serie I* **293**, 39–42 (1981)
- [8] Christodoulou, D., Klainerman, S.: *The Global Non Linear Stability of the Minkowski Space*, vol. 41. Princeton Mathematical Series. Princeton University Press, Princeton (1993)
- [9] Dossa, M.: Espaces de Sobolev non isotropes, à poids et problèmes de Cauchy quasi-linéaires sur un cône caractéristique. *Ann. Inst. H. Poincaré* **66**(1), 37–107 (1997)
- [10] Duff, G.F.D.: Mixed problems for linear systems of first order equations. *Can. J. Math.* **X**, 127–160 (1958)
- [11] Friedrich, H.: On the existence of analytic null asymptotically flat solutions of Einstein's vacuum field equations. *Proc. Roy. Soc. Lond. A* **381**, 361–371 (1982)

⁵⁵ This requires some care as from one side the Ω defined on $\underline{C}(\nu_* + \delta)$ has to be such to provide a canonical foliation, but on the other side also the outgoing null hypersurfaces have to be outgoing cones starting from $\lambda(\cdot)$ which requires a different Ω on the last slice. The solution requires a careful matching between the outgoing cones starting from $\lambda(\cdot)$ and the outgoing null hypersurfaces starting from the last slice, a problem similar to the one worked in [8].

- [12] Friedrich, H., Rendall, A.: The Cauchy problem for the Einstein equations. arXiv:gr-qc/0002074 (22 Feb 2000)
- [13] Fritz, J.: Partial Differential Equations. Applied Mathematical Sciences, vol.1. Springer, Berlin (1982)
- [14] Klainerman, S., Nicolò, F.: On local and global aspects of the Cauchy problem in general relativity. *Class. Quantum Gravity* **16**, R73–R157 (1999)
- [15] Klainerman, S., Nicolò, F.: The Evolution Problem in General Relativity. *Progress in Mathematical Physics*, vol. 25. Birkhauser, Basel (2002)
- [16] Lax, P.D.: Non linear hyperbolic equations. *Commun. Pure Appl. Math.* **VI**, 231–258 (1953)
- [17] Lindblad, H., Rodnianski, I.: Global existence for the Einstein vacuum equations in wave coordinates. arXiv:math.AP/0312479 V.1, (2003)
- [18] Müller Zum Hagen, H., Seifert, H.J.: On characteristic initial value and mixed problems. *Gen. Rel. Grav.* **8**(4), 259–301 (1977)
- [19] Müller Zum Hagen, H.: Characteristic initial value problem for hyperbolic systems of second order differential equations. *Ann. Inst. H. Poincaré* **53**(2), 159–216 (1990)
- [20] Newman, E.T., Penrose, R.: An approach to gravitational radiation by a method of spin coefficients. *J. Math. Phys.* **3**, 566–578 (1962)
- [21] Newman, E.T., Penrose, R.: New conservation laws for zero rest-mass fields in asymptotically flat space–time. *Proc. Roy. Soc. Lond.* **A305**, 175–204 (1968)
- [22] Nicolò, F.: Canonical foliation on a null hypersurface. *JHDE* **1**(3), 367–427 (2004)
- [23] Nirenberg, L.: An abstract form of the nonlinear Cauchy–Kowalewski theorem. *J. Diff. Geometry* **6**, 561–576 (1972)
- [24] Rendall, A.D.: Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations. *Proc. Roy. Soc. Lond. A* **427**, 221–239 (1990). arXiv: 1012.3949
- [25] Spagnolo, S.: Propagation of analyticity for a class of nonlinear hyperbolic equations. arXiv:1012.3949v1 [math.AP] (2010)
- [26] Spivak, M.: *A Comprehensive Introduction to Differential Geometry*. Publish or Perish, Wilmington (1970)

Giulio Caciotta and Francesco Nicolò
Dipartimento di Matematica
Università degli Studi di Roma “Tor Vergata”
Via della Ricerca Scientifica
00133 Rome, Italy
e-mail: giulioc42@gmail.com;
francesco.nicolo@gmail.com

Communicated by Piotr T. Chrusciel.

Received: March 22, 2011.

Accepted: October 5, 2011.