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Generalized Second Bargmann Transforms Associated with the Hyperbolic Landau Levels on the Poincaré Disk

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Abstract. We deal with a family of generalized coherent states associated to the hyperbolic Landau levels of the Schrödinger operator with uniform magnetic field on the Poincaré disk. Their associated coherent state transforms constitute a class of generalized second Bargmann transforms.

1. Introduction

The classical Bargmann transform made the space of square integrable functions f on the real line isometric to the space of entire functions that are $e^{-|z|^2}d\mu$-square integrable, $d\mu$ being the Lebesgue measure on the complex plane. It is defined in [2] as

$$\mathscr{B}[f](z) := \pi^{-\frac{1}{4}} \int\limits_{\mathbb{R}} \exp\left(-\frac{\xi^2}{2} + \sqrt{2}\xi z - \frac{z^2}{2}\right) f(\xi)d\xi; \quad z \in \mathbb{C}, \tag{1.1}$$

where the involved kernel function is related to the generating function of the Hermite polynomials. In the same paper [2, p. 203], Bargmann has also introduced a second transform $\mathscr{T}_\nu$; $\nu > 1/2$, as a unitary integral operator whose kernel function corresponds to the generating function of the Laguerre polynomials $L_m^{(\alpha)}(\cdot)$ [20,22]. It maps isometrically the space of $\xi^{2\nu}d\xi/\Gamma(2\nu)\xi$-square integrable functions on the positive real half-line onto the weighted Bergman space

$$\mathcal{A}^{2,\nu}(\mathbb{D}) := \left\{ f \quad \text{holomorphic on } \mathbb{D}; \int\limits_{\mathbb{D}} |f(z)|^2 \left(1-|z|^2\right)^{2\nu-2} d\mu(z) < \infty \right\} \tag{1.2}$$

on the unit disk $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$. Explicitly, we have

$$\mathscr{T}_\nu[\psi](z) = \left(\frac{2\nu-1}{\pi}\right)^{1/2}(1-z)^{-2\nu}\int_0^\infty \exp\left(-\frac{\xi}{2}\left(\frac{1+z}{1-z}\right)\right)\psi(\xi)\xi^{2\nu}(\Gamma(2\nu))^{-1}\frac{\mathrm{d}\xi}{\xi}, \tag{1.3}$$

where the positive real number $2\nu - 1$ represents the parameter γ in [2, p. 203]. Thus, using the canonical isometry from $L^2(\mathbb{R}_+^*, \mathrm{d}\xi/\xi)$ onto $L^2\left(\mathbb{R}_+^*, \xi^{2\nu}\mathrm{d}\xi/\Gamma(2\nu)\xi\right)$, one extends $\mathscr{T}_\nu$ to the transform

$$\mathscr{W}_\nu[\varphi](z) := \left(\frac{2\nu-1}{\pi\Gamma(2\nu)}\right)^{1/2}(1-z)^{-2\nu}\int_0^{+\infty}\xi^\nu \exp\left(-\frac{\xi}{2}\left(\frac{1+z}{1-z}\right)\right)\varphi(\xi)\frac{\mathrm{d}\xi}{\xi} \tag{1.4}$$

mapping isometrically the Hilbert space $L^2(\mathbb{R}_+^*, \mathrm{d}\xi/\xi)$ onto $\mathcal{A}^{2,\nu}(\mathbb{D})$. Note that the space $\mathcal{A}^{2,\nu}(\mathbb{D})$ in (1.2) can also be realized as the null space of the second order differential operator

$$H_\nu = -4(1-|z|^2)\left((1-|z|^2)\frac{\partial^2}{\partial z\partial\bar{z}} - 2\nu\bar{z}\frac{\partial}{\partial\bar{z}}\right). \tag{1.5}$$

The latter one is acting on the Hilbert space $L^{2,\nu}(\mathbb{D}) := L^2(\mathbb{D}, (1-|z|^2)^{2\nu-2}\mathrm{d}\mu)$ and can be unitarily intertwined to represent a Hamiltonian of uniform magnetic field on the unit disk.

In this paper, we will be concerned with the L^2-eigenspaces

$$\mathcal{A}_m^{2,\nu}(\mathbb{D}) := \left\{F \in L^{2,\nu}(\mathbb{D});\ H_\nu F = \epsilon_m^\nu F\right\} \tag{1.6}$$

associated to the discrete spectrum of H_ν consisting of the eigenvalues (*hyperbolic Landau levels*):

$$\epsilon_m^\nu = 4m(2\nu - m - 1); \quad m = 0, 1, 2, \ldots, [\nu - (1/2)], \tag{1.7}$$

where $[x]$ denotes the greatest integer less than x. The transform (1.4) can be viewed as resulting from a coherent state which turns out to be a superposition of the basis elements of $L^2(\mathbb{R}_+, x^{-1}\mathrm{d}x)$ considered as a states Hilbert space. In this superposition, the coefficients are the analytic functions constituting the basis of the Bergman space $\mathcal{A}^{2,\nu}(\mathbb{D})$. Here our aim is to keep the same states Hilbert space $L^2(\mathbb{R}_+, x^{-1}\mathrm{d}x)$ and to replace the coefficients by the basis elements of the eigenspace $\mathcal{A}_m^{2,\nu}(\mathbb{D})$, for a fixed positive integer m, in order to construct a family of integral transforms generalizing (1.4). The method used is based on the coherent states analysis [12] together with the concrete description of the L^2-spectral theory of the operator H_ν [5–8, 23]. Precisely, we establish the following

Theorem 1.1. *Let ν be a real number such that $\nu > 1/2$ and $m = 0, 1, 2, \ldots,$ $[\nu - (1/2)]$. Then, the mapping*

$$\mathscr{W}_{\nu,m}[\varphi](z) = \left(\frac{(2(\nu-m)-1)m!}{\pi\Gamma(2\nu-m)}\right)^{\frac{1}{2}} \left(\frac{|1-z|}{1-|z|^2}\right)^{2m} (1-z)^{-2\nu}$$
$$\times \int_0^{+\infty} \xi^{\nu-m} \exp\left(-\frac{\xi}{2}\left(\frac{1+z}{1-z}\right)\right) L_m^{2(\nu-m)-1}\left(\xi\frac{(1-|z|^2)}{|1-z|^2}\right)\varphi(\xi)\frac{\mathrm{d}\xi}{\xi} \tag{1.8}$$

defines a unitary isomorphism from $L^2(\mathbb{R}_+^*, d\xi/\xi)$ onto the Hilbert space $\mathcal{A}_m^{2,\nu}(\mathbb{D})$. The particular case of $m = 0$ reproduces the second Bargmann transform $\mathscr{W}_\nu$ in (1.4).

The paper is organized as follows. In Sect. 2, we review some needed facts on the L^2-spectral theory of the operator H_ν on the unit disk. In Sect. 3, we recall the coherent states formalism we will be using. In Sect. 4, we define a family of generalized coherent states attached to hyperbolic Landau levels. The associated coherent state transforms (CST) constitute a family of generalized second Bargmann transforms.

2. Spectral Analysis of H_ν; $\nu > 0$

The second order differential operator H_ν in (1.5) appears as the Laplace–Beltrami operator on the unit disk perturbed by a first order differential operator. It can be interpreted as the Hamiltonian of a charged particle moving in an external uniform magnetic field. In fact, H_ν is unitary equivalent to the magnetic Schrödinger operator [13]:

$$\mathscr{L}_\nu := (d + \sqrt{-1}\nu\theta)^*(d + \sqrt{-1}\nu\theta), \tag{2.1}$$

associated to the gauge potential vector $\theta(z) = -\sqrt{-1}(\partial - \bar{\partial})\log(1-|z|^2)$, acting on $L^2(\mathbb{D}) = L^2(\mathbb{D}, (1-|z|^2)^{-2}\mathrm{d}\mu)$. Indeed, we have $(1-|z|^2)^\nu$ $H_\nu(1-|z|^2)^{-\nu} = \mathscr{L}_\nu$. Different aspects of its spectral analysis have been studied by many authors, e.g. [5–8,23]. For instance, note that H_ν is an elliptic densely defined operator on the Hilbert space $L^{2,\nu}(\mathbb{D})$ and admits a unique self-adjoint realization that we denote also by H_ν. Note also that such operator commutes with the action of the group $SU(1,1)$ defined on $L^{2,\nu}(\mathbb{D})$ by

$$(T_g^\nu f)(z) := (\det(g')(z))^\nu f(g.z);$$
$$g.z = (az+b)(cz+d)^{-1},$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1),$$

where g' is the complex Jacobian, and its needed principal properties are listed as follows:

- The discrete part of the spectrum of H_ν is not empty if and only if that $2\nu > 1$. It consists of the eigenvalues ϵ_m^ν given through (1.7) and occurring with infinite multiplicities.
- Let ν be such that $2\nu > 1$. Then, for every fixed $m = 0, 1, 2, \ldots, [\nu - (1/2)]$, we have
  - (i) The family of functions given explicitly in terms of the Jacobi polynomials $P_j^{(\alpha,\beta)}(\cdot)$ as

$$\phi_k^{\nu,m}(z) = (-1)^{\min(m,k)}(1-|z|^2)^{-m} \times |z|^{|m-k|} e^{-i(m-k)\arg z} P_{\min(m,k)}^{(|m-k|,2(\nu-m)-1)}(1-2|z|^2) \tag{2.2}$$

    constitutes an orthogonal basis of the L^2-eigenspace in (1.6).
  - (ii) The square norm of $\phi_k^{\nu,m}$ in $L^{2,\nu}(\mathbb{D})$ is given by

$$\rho_k^{\nu,m} = \left(\frac{\pi}{2(\nu-m)-1}\right) \frac{(\max(m,k))!\Gamma(2(\nu-m)+\min(m,k))}{(\min(m,k))!\Gamma(2(\nu-m)+\max(m,k))}. \tag{2.3}$$

- The set of functions

$$\Phi_k^{\nu,m} := \frac{\phi_k^{\nu,m}}{\sqrt{\rho_k^{\nu,m}}}; \quad k = 0, 1, 2, \ldots, \tag{2.4}$$

  is an orthonormal basis of $\mathcal{A}_m^{2,\nu}(\mathbb{D})$ and can be rewritten as

$$\Phi_k^{\nu,m}(z) = (-1)^k \left(\frac{2(\nu-m)-1}{\pi}\right)^{1/2} \left(\frac{k!\Gamma(2(\nu-m)+m)}{m!\Gamma(2(\nu-m)+k)}\right)^{1/2} \times (1-|z|^2)^{-m} \bar{z}^{m-k} P_k^{(m-k,2(\nu-m)-1)}(1-2|z|^2), \tag{2.5}$$

  by making use of the identity [22, p. 63]:

$$\frac{\Gamma(m+1)}{\Gamma(m-s+1)} P_m^{(-s,\alpha)}(t) = \frac{\Gamma(m+\alpha+1)}{\Gamma(m-s+\alpha+1)} \left(\frac{t-1}{2}\right)^s P_{m-s}^{(s,\alpha)}(t), \quad 1 \le s \le m, \tag{2.6}$$

  for $s = m-k$, $t = 1-2|z|^2$ and $\alpha = 2(\nu-m)-1$.
- The reproducing kernel of the Hilbert space $\mathcal{A}_m^{2,\nu}(\mathbb{D})$ is given by

$$K_m^\nu(z,w) = \left(\frac{2(\nu-m)-1}{\pi}\right)(1-z\bar{w})^{-2\nu}\left(\frac{|1-z\bar{w}|^2}{(1-|z|^2)(1-|w|^2)}\right)^m \times P_m^{(0,2(\nu-m)-1)}\left(2\frac{(1-|z|^2)(1-|w|^2)}{|1-z\bar{w}|^2}-1\right), \tag{2.7}$$

  with the diagonal function

$$K_m^\nu(z,z) = \left(\frac{2(\nu-m)-1}{\pi}\right)(1-|z|^2)^{-2\nu}, \quad z \in \mathbb{D}. \tag{2.8}$$

Remark 2.1. In view of (2.2), the L^2-eigenspace $\mathcal{A}_0^{2,\nu}(\mathbb{D})$, corresponding to $m = 0$ in (1.6) and associated to the bottom energy $\epsilon_0^\nu = 0$, reduces further to the weighted Bergman space $\mathcal{A}^{2,\nu}(\mathbb{D})$ consisting of complex-valued holomorphic functions F on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |F(z)|^2 (1 - |z|^2)^{2\nu - 2} \mathrm{d}\mu(z) < +\infty. \tag{2.9}$$

Remark 2.2. The condition $2\nu > 1$ ensuring the existence of the eigenvalues (1.7) should implies that the magnetic field $B = \mathrm{d}\theta_\nu = 2\nu\Omega(z)$, where Ω stands for the Khäler 2-form on $\mathbb{D}$, has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the orbit of the particle will intercept the disk boundary whose points stand for "points at infinity" (see [3, p. 189]).

3. Generalized Coherent States

The first model of coherent states was the 'nonspreading wavepacket' of the harmonic oscillator, which have been constructed by Schrödinger [21]. In suitable units, wave functions of these states can be written as

$$\Phi_{\mathfrak{z}}(\xi) := \langle \xi \mid \mathfrak{z} \rangle = \pi^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\xi^2 + \sqrt{2}\xi\mathfrak{z} - \frac{1}{2}\mathfrak{z}^2 - \frac{1}{2}|\mathfrak{z}|^2 \right), \quad \xi \in \mathbb{R}, \tag{3.1}$$

where $\mathfrak{z} \in \mathbb{C}$ determines the mean values of coordinate $\widehat{x}$ and momentum $\widehat{p}$ according to $\langle \widehat{x} \rangle := \langle \Phi_{\mathfrak{z}}, x\Phi_{\mathfrak{z}} \rangle = \sqrt{2}\Re\mathfrak{z}$ and $\langle \widehat{p} \rangle := \langle \Phi_{\mathfrak{z}}, p\Phi_{\mathfrak{z}} \rangle = \sqrt{2}\Im\mathfrak{z}$. The variances $\sigma_x = \langle \widehat{x}^2 \rangle - \langle \widehat{x} \rangle^2 = \frac{1}{2}$ and $\sigma_p = \langle \widehat{p}^2 \rangle - \langle \widehat{p} \rangle^2 = \frac{1}{2}$ have equal values, so their product assumes the minimal value permitted by the Heisenberg uncertainty relation. The coherent states $\Phi_{\mathfrak{z}}$ have been also obtained by Glauber [14] from the vacuum state $\mid 0\rangle$ by means of the unitary displacement operator $\exp(\mathfrak{z}A^* - \bar{\mathfrak{z}}A)$ as

$$\Phi_{\mathfrak{z}} = \exp(\mathfrak{z}A^* - \bar{\mathfrak{z}}A) \mid 0\rangle, \tag{3.2}$$

where A and A^* are, respectively, the annihilation and the creation operators defined by

$$A = \frac{1}{\sqrt{2}}(\widehat{x} + i\widehat{p}), \quad A^* = \frac{1}{\sqrt{2}}(\widehat{x} - i\widehat{p}). \tag{3.3}$$

Following [4, p. 2], it was Iwata [16] who used the well known expansion over the Fock basis $|n\rangle$ to give an expression of $\Phi_{\mathfrak{z}}$ as

$$\Phi_{\mathfrak{z}} = e^{-\frac{1}{2}|\mathfrak{z}|^2} \sum_{n=0}^{+\infty} \frac{\mathfrak{z}^n}{\sqrt{n!}} |n\rangle. \tag{3.4}$$

Actually, various generalizations of coherent states are proposed. Here, we shall focus on a generalization of (2.1), according to a construction starting from a measure space X "as a set of data" which was presented for the first time in [11] by Gazeau et al. (see also [10]). Precisely, let $(X, \mathrm{d}\lambda)$ be a measure space and $\mathcal{A}^2 \subset L^2(X, \mathrm{d}\lambda)$ be a closed subspace of infinite dimension. Let $\{f_k\}_{k=1}^{\infty}$ be an orthogonal basis of $\mathcal{A}^2$ satisfying

$$\omega(u) := \sum_{k=1}^{\infty} \rho_k^{-1} |f_k(u)|^2 < +\infty \tag{3.5}$$

for every $u \in X$, where $\rho_k := \|f_k\|^2_{L^2(X)}$. Therefore, the function

$$K(u,v) := \sum_{k=1}^{\infty} \rho_k^{-1} f_k(u) \overline{f_k(v)}, \tag{3.6}$$

defined on $X \times X$, is a reproducing kernel of the Hilbert space $\mathcal{A}^2$ so that we have $\omega(u) = K(u,u)$; $u \in X$.

Definition 3.1. For given infinite dimensional Hilbert space $\mathcal{H}$ with $\{\psi_k\}_{k=1}^{\infty}$ as an orthonormal basis, the vectors $(\Psi_u)_{u \in X}$ defined by

$$\Psi_u := (\omega(u))^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{f_k(u)}{\sqrt{\rho_k}} \psi_k \tag{3.7}$$

will be called generalized coherent states for the data of $(X; \mathcal{A}^2; \{f_k\})$ and $(\mathcal{H}; \{\psi_k\})$.

The choice of the Hilbert space $\mathcal{H}$ defines in fact a quantization of $X = \{u\}$ by the coherent states Ψ_u, via the inclusion map $u \to \Psi_u$ from X into $\mathcal{H}$. Moreover, according to the fact that $\langle \Psi_u, \Psi_u \rangle_{\mathcal{H}} = 1$, one can show that the transform given by

$$\mathscr{W}[f](u) := (\omega(u))^{\frac{1}{2}} \langle \Psi_u, f \rangle_{\mathcal{H}} \tag{3.8}$$

defines an isometry from $\mathcal{H}$ into $\mathcal{A}^2$. Thereby we have a resolution of the identity, i.e., the following integral representation holds:

$$f(\cdot) = \int_X \langle \Psi_u, f \rangle_{\mathcal{H}} \Psi_u(\cdot) \omega(u) \mathrm{d}\lambda(u) \tag{3.9}$$

for every $f \in \mathcal{H}$.

Definition 3.2. The transform $\mathscr{W} : \mathcal{H} \to \mathcal{A}^2 \subset L^2(X, \mathrm{d}\lambda)$ in (3.8) will be called the CST associated to the set of coherent states Ψ_u; $u \in X$.

For introductory papers on coherent states, we refer to [15] by Glauber for the radiation field and [1] by Arecchi et al. for atomic states. For an overview of all aspect of the theory of coherent states and their genesis, we refer to the early papers by Klauder and particularly [17], the survey [4] (with a list of 451 references) by Dodonov or also the recent book [12] by Gazeau.

4. Generalized Coherent States Attached to Landau Levels ϵ_m^ν

Now, we are in position to attach to each hyperbolic Landau level ϵ_m^ν in (1.7) a set of generalized coherent states according to formula (3.7). Namely, we have

$$\Psi_{\nu,m;z} := (K_m^\nu(z,z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{\phi_k^{\nu,m}(z)}{\sqrt{\rho_k^{\nu,m}}} \psi_{\nu,m;k} \tag{4.1}$$

with the following specifications:

- $(X, \mathrm{d}\lambda) = (\mathbb{D}, (1-|z|^2)^{2\nu-2}\mathrm{d}\mu)$.
- $\mathcal{A}^2 = \mathcal{A}_m^{2,\nu}(\mathbb{D})$ is the eigenspace in (1.6).
- $K_m^\nu(z,z) = \pi^{-1}(2(\nu-m)-1)(1-|z|^2)^{-2\nu}$ as in (2.8).
- $f_k = \phi_k^{\nu,m}$ are the eigenfunctions given by (2.2).
- $\rho_k^{\nu,m}$ being the square norm of $\Phi_k^{B,m}$ given in (2.3).
- $\mathcal{H} = L^2(\mathbb{R}_+^*, \xi^{-1}\mathrm{d}\xi)$ is the Hilbert space carrying the coherent states (4.1).
- $\psi_k = \psi_{\nu,m;k}$, $k = 0, 1, 2, \ldots$, the basis of $\mathcal{H}$ given by

$$\psi_{\nu,m;k}(\xi) := \left(\frac{k!}{\Gamma(2(\nu-m)+k)}\right)^{\frac{1}{2}} \xi^{\nu-m} \mathrm{e}^{-\frac{1}{2}\xi} L_k^{(2(\nu-m)-1)}(\xi), \quad \xi > 0. \tag{4.2}$$

In view of (2.5) and (4.1), the coherent states belonging to the Hilbert space $\mathcal{H}$ and corresponding to the eigenspace in (1.6) are defined by their wave functions through the series expansion

$$\Psi_{\nu,m;z}(\xi) := (1-|z|^2)^{\nu-m} \sum_{k=0}^{+\infty} (-1)^k \left(\frac{k!\Gamma(2(\nu-m)+m)}{m!\Gamma(2(\nu-m)+k)}\right)^{1/2} \times \bar{z}^{m-k} P_k^{(m-k,2(\nu-m)-1)}(1-2|z|^2)\psi_{\nu,m;k}(\xi). \tag{4.3}$$

A closed form for (4.3) can be obtained in terms of Laguerre polynomials as follows.

Proposition 4.1. *Let $2\nu > 1$ and $m = 0, 1, 2, \ldots, [\nu - (1/2)]$. Then, the wave functions of the states in (4.3) can be expressed as*

$$\Psi_{\nu,m;z}(\xi) = (-1)^m \left(\frac{m!}{\Gamma(2\nu-m)}\right)^{\frac{1}{2}} \frac{|1-z|^{2m}}{(1-z)^{2\nu}} (1-|z|^2)^{\nu-m} \xi^{\nu-m} \times \exp\left(-\frac{\xi}{2}\frac{1+z}{1-z}\right) L_m^{2(\nu-m)-1}\left(\xi \frac{1-|z|^2}{|1-z|^2}\right). \tag{4.4}$$

Proof. Set $\alpha = 2(\nu-m)-1$ and $t = 1-2|z|^2$. Then, the expression in (4.3) reads

$$\Psi_{\nu,m;z}(\xi) := (1-|z|^2)^{\nu-m} \sum_{k=0}^{+\infty} (-1)^k \left(\frac{k!\Gamma(\alpha+1+m)}{m!\Gamma(\alpha+1+k)}\right)^{\frac{1}{2}} \bar{z}^{m-k} P_k^{(m-k,\alpha)}(t)\psi_{\nu,m;k}(\xi). \tag{4.5}$$

By inserting the expression of $\psi_{\nu,m;k}(\xi)$ given by (4.2) in (4.5), we infers

$$\Psi_{\nu,m;z}(\xi) = \left(\frac{\Gamma(\alpha+1+m)}{m!}\right)^{\frac{1}{2}} (1-|z|^2)^{\nu-m}\xi^{\nu-m}\mathrm{e}^{-\frac{1}{2}\xi} \times \sum_{k=0}^{+\infty} \frac{(-1)^k k!}{\Gamma(\alpha+1+k)} \overline{z}^{m-k} P_k^{(m-k,\alpha)}(t) L_k^{(\alpha)}(\xi) \tag{4.6}$$

$$= \left(\frac{\Gamma(\alpha+1+m)}{m!}\right)^{\frac{1}{2}} (1-|z|^2)^{\nu-m}\xi^{\nu-m}\mathrm{e}^{-\frac{1}{2}\xi} \times \sum_{k=0}^{+\infty} \frac{k!}{\Gamma(\alpha+1+k)} \overline{z}^{m-k} P_k^{(\alpha,m-k)}(-t)\, L_k^{(\alpha)}(\xi). \tag{4.7}$$

The last equality is readily derived by means of the symmetry relation [20, p. 256]:

$$P_k^{(a,b)}(t) = (-1)^k P_k^{(b,a)}(-t). \tag{4.8}$$

In order to use the bilateral generating function ([20, p. 213]):

$$\sum_{k=0}^{+\infty} \lambda^k {}_2F_1(-k, b; 1+\alpha; y) L_k^{(\alpha)}(\xi) = \frac{(1-\lambda)^{b-1-\alpha}}{(1-\lambda+y\lambda)^b} \exp\left(\frac{-\xi\lambda}{1-\lambda}\right) \times {}_1F_1\left(b; 1+\alpha; \frac{\xi y\lambda}{(1-\lambda)(1-\lambda+y\lambda)}\right), \tag{4.9}$$

involving a Laguerre polynomial and a terminating Gauss hypergeometric ${}_2F_1$-sum, we make appeal to the fact [20, p. 254]:

$$P_k^{(\alpha,\eta)}(x) = \frac{\Gamma(1+\alpha+k)}{k!\Gamma(1+\alpha)} \left(\frac{1+x}{2}\right)^k {}_2F_1\left(-k, -(\eta+k), 1+\alpha; \frac{x-1}{x+1}\right) \tag{4.10}$$

with $\eta = m-k$ and $x = -t = -1+2|z|^2$. Hence, we obtain

$$\Psi_{\nu,m;z}(\xi) = \left(\frac{\Gamma(\alpha+1+m)}{m!}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\alpha+1)} (1-|z|^2)^{\nu-m}\xi^{\nu-m} e^{-\frac{1}{2}\xi} \times \sum_{k=0}^{+\infty} \overline{z}^m z^k {}_2F_1\left(-k, -m; 1+\alpha; \frac{t+1}{t-1}\right) L_k^{(\alpha)}(\xi). \tag{4.11}$$

Thus, by applying (4.9) with $\lambda = z$, $b = -m$ and $y = \frac{t+1}{t-1} = -\frac{1-|z|^2}{|z|^2}$, we check that

$$\Psi_{\nu,m;z}(\xi) = \left(\frac{\Gamma(\alpha+1+m)}{m!}\right)^{\frac{1}{2}} \frac{(-1)^m}{\Gamma(\alpha+1)} \frac{|1-z|^{2m}}{(1-z)^{2\nu}} (1-|z|^2)^{\nu-m} \times \xi^{\nu-m} \exp\left(-\frac{\xi}{2}\left(\frac{1+z}{1-z}\right)\right) {}_1F_1\left(-m; \alpha+1; \frac{\xi(1-|z|^2)}{|1-z|^2}\right). \tag{4.12}$$

Finally, making use of [22, p. 103]

$$ {}_1F_1(-m;1+\alpha;x) = \frac{m!\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)} L_m^{(\alpha)}(x), \tag{4.13} $$

with $x = \xi(1-|z|^2)/|1-z|^2$ yields

$$ \Psi_{\nu,m;z}(\xi) = (-1)^m \left(\frac{m!}{\Gamma(2\nu-m)}\right)^{\frac{1}{2}} \frac{|1-z|^{2m}}{(1-z)^{2\nu}} (1-|z|^2)^{\nu-m} \times \xi^{\nu-m} \exp\left(-\frac{\xi}{2}\left(\frac{1+z}{1-z}\right)\right) L_m^{(2(\nu-m)-1)}\left(\frac{\xi(1-|z|^2)}{|1-z|^2}\right). \tag{4.14} $$

This completes the proof. □

According to Definition 3.2, the CST associated with the coherent states in (4.14) is the unitary map:

$$ \mathscr{W}_{\nu,m} : L^2(\mathbb{R}_+^*, \mathrm{d}\xi/\xi) \longrightarrow \mathcal{A}_m^{2,\nu}(\mathbb{D}) \tag{4.15} $$

$$ \phi \longmapsto \mathscr{W}_{\nu,m}\left[\phi\right](z) := \left(K_m^\nu(z,z)\right)^{\frac{1}{2}} \left\langle \Psi_{\nu,m;z}, \phi \right\rangle_{L^2(\mathbb{R}_+^*, \xi^{-1}\mathrm{d}\xi)}. \tag{4.16} $$

Explicitly, we have

$$ \mathscr{W}_{\nu,m}\left[\phi\right](z) = \left(\frac{m!(2(\nu-m)-1)}{\pi\Gamma(2\nu-m)}\right)^{\frac{1}{2}} \left(\frac{1-|z|^2}{|1-z|^2}\right)^{-m} (1-z)^{-2\nu} \times \int_0^{+\infty} \xi^{\nu-m} \exp\left(-\frac{\xi}{2}\left(\frac{1+z}{1-z}\right)\right) L_m^{2(\nu-m)-1}\left(\xi\frac{1-|z|^2}{|1-z|^2}\right) \phi(\xi) \frac{\mathrm{d}\xi}{\xi} \tag{4.17} $$

thanks to Proposition 4.1. The assertion of Theorem 1.1 follows then from the fact that the CST in (3.8) is an isometry. Note that for $m = 0$, the above transform in (4.17) reduces to the second Bargmann transform in (1.4).

Definition 4.2. The CST $\mathscr{W}_{\nu,m}$ in (4.17) will be called a generalized second Bargmann transform of index $m = 0, 1, 2, \ldots, [\nu - (1/2)]$.

We end this section with the following remarks.

Remark 4.3. In [18, pp. 5–6] a class of coherent states attached to Landau levels have been constructed using the similar procedure as in the present work. In fact, the functions $\Psi_{\nu,m;k}$, in the right hand side of (4.1), have been taken as elements of an orthonormal basis of the weighted Bergman space $\mathcal{H}_\nu := \mathcal{A}^{2,\nu}(\mathbb{D}) = \mathcal{A}_0^{2,\nu}(\mathbb{D})$ in (1.2) with the form

$$ \Psi_{\nu,m;k}(\xi) = \left(\frac{\Gamma(2\nu+m-k)}{\Gamma(2\nu)\Gamma(1+m-k)}\right)^{\frac{1}{2}} \xi^{m-k}; \quad -\infty \le k \le m, \ z \in \mathbb{D}. \tag{4.18} $$

The obtained CST maps isometrically eigenstates of the first hyperbolic Landau level ϵ_0^ν into eigenstates corresponding to mth level ϵ_m^ν as an integral transform $\mathcal{A}_0^{2,\nu}(\mathbb{D}) \to \mathcal{A}_m^{2,\nu}(\mathbb{D})$.

Remark 4.4. In [19] Perelomov's coherent states have been attached to Landau levels on the Poincaré upper half-plane by means of the affine group without connection with the second Bargmann transform. Now, this connection appears to us when dealing with the Landau problem in the disk model and constructing coherent states as series expansions by the formalism stated in [12, pp. 72–76].

Remark 4.5. It would also be of interest to consider a set of Gazeau–Klauder coherent states ([9, pp. 124–127]) for the total discrete dynamics of the Hamiltonian in (1.5) by performing a finite sum over all discrete eigenvalues ϵ_m^ν in (1.7) as

$$|J,\gamma,\nu,k> \, = N(J)^{-1} \sum_{m=0}^{[\nu-(1/2)]} \frac{J^{m/2}\exp(i\epsilon_m^\nu \gamma)}{\sqrt{\rho_m}} |\phi_k^{\nu,m}>, \tag{4.19}$$

where $J \geq 0$, $-\infty < \gamma < +\infty$, $2\nu > 1$, k a fixed integer, $N(J)$ a normalization factor and the quantity ρ_m is defined by $\rho_0 = 1$ and $\rho_m = \epsilon_1^\nu \epsilon_2^\nu \cdots \epsilon_m^\nu$. Then establishing properties for such states should add new results for the Landau problem on the Poincaré disk.

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