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Dimension Theory for Multimodal Maps

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Abstract. This paper is devoted to the study of dimension theory, in particular multifractal analysis, for multimodal maps. We describe the Lyapunov spectrum and study the multifractal spectrum of pointwise dimension. The lack of regularity of the thermodynamic formalism for this class of maps is reflected in the phase transitions of the spectra.

1. Introduction

The dimension theory of dynamical systems has flourished remarkably over the last 15 years. The main goal of the field is to compute the *size* of dynamically relevant subsets of the phase space. For example, sets where the complicated dynamics is concentrated (*repellers* or *attractors*). Usually, the geometry of these sets is rather complicated. That is why, there are several notions of *size* that can be used. One could say that a set is *large* if it contains a great deal of disorder on it. Formally, one would say that the dynamical system restricted to that subset has large entropy. Another way of measuring the size of a set is using geometric tools, namely Hausdorff dimension. There are usually two conditions required on the dynamical system (X, f) for the dimension theory to be carried out. First, a certain amount of hyperbolicity enables us to use Markov partitions and the thermodynamic formalism machinery associated with the Markov setting. Second, the geometric nature of Hausdorff dimension means that it is convenient to assume that the map f is conformal. In this case, the elements of a Markov partition are *almost* balls and hence can be used to compute Hausdorff dimension (see [2,35] and references therein).

In this paper, we consider smooth one-dimensional maps. This implies that the map is conformal. Nevertheless, we study dynamical systems for which the hyperbolicity is rather weak (these maps have critical points and so have regions of strong contraction). The class of maps we will consider is defined

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as follows. Let $\mathcal{F}$ be the collection of C^3 multimodal interval maps $f : I \to I$, where $I = [0, 1]$, satisfying:

a) the critical set $\mathcal{C}r = \mathcal{C}r(f)$ consists of finitely many critical points c with critical order $1 < \ell_c < \infty$, i.e., there exists a neighbourhood U_c of c and a C^3 diffeomorphism $g_c : U_c \to g_c(U_c)$ with $g_c(c) = 0$ $f(x) = f(c) \pm |g_c(x)|^{\ell_c}$;
b) f has no parabolic cycles;
c) f is topologically transitive on I;
d) $f^n(\mathcal{C}r) \cap f^m(\mathcal{C}r) = \emptyset$ for $m \neq n$.

Note that by [46, Theorem C], given condition a), condition b) then allows us to apply the Koebe distortion theorem. Alternatively, we could assume that maps in $\mathcal{F}$ have negative Schwarzian derivative since this added to the transitivity assumption implies that there are no parabolic cycles. We refer to [22, Remarks 1.1 and 1.2] for more information on this type of family of maps. The thermodynamic formalism for these maps was studied in [22]. We proved that in an interval of the form $(-\infty, t^+)$ for some $t^+ > 0$, the pressure function $t \to P(-t \log |Df|)$ is strictly convex, C^1 and the 'natural/geometric' potential $x \mapsto -t \log |Df(x)|$ has a unique equilibrium state (see Sect. 3.1 for precise definitions and statements). In particular, in the interval $(-\infty, t^+)$, the thermodynamic formalism has similar properties to the uniformly hyperbolic case. In the interval (t^+, ∞), the pressure function is linear. Therefore, at the point $t = t^+$ it exhibits a so-called *first-order phase transition*, that is a point where the pressure is not smooth. This lack of regularity is closely related to the different modes of recurrence of the system (see [42,43]).

We will be interested in a particular class of maps belonging to $\mathcal{F}$. Indeed, let $\mathcal{F}_g \subset \mathcal{F}$ be the collection of maps $f : I \to I$ satisfying the growth condition

$$\lim_{n\to\infty} |Df^n(f(c))| \to \infty \tag{1}$$

for all critical points c. By [8], maps satisfying this condition have an absolutely continuous invariant probability measure (acip) $\mu \ll m$ where m denotes Lebesgue measure; we will often denote this measure by μ_{ac}.

This paper is devoted to the study of the dimension theory for maps in $\mathcal{F}_g$. In particular, we are interested in its multifractal analysis (see Sect. 2 for precise definitions). Our first goal was to describe the *Lyapunov spectrum* (see Sect. 2 for precise definitions). Making use of the thermodynamic formalism, we are able to describe the size (Hausdorff dimension) of the level sets determined by the Lyapunov exponent of these maps. We denote by $J(\lambda)$ the set of points having λ as Lyapunov exponent. Dynamical and geometric features are captured in this decomposition. We next state our first main theorem: the set A is defined in Sect. 4.

Theorem A. *Suppose that $f \in \mathcal{F}_g$. Then for all $\lambda \in \mathbb{R}\setminus A$,*

$$L(\lambda) := \dim_H(J(\lambda)) = \frac{1}{\lambda} \inf_{t\in\mathbb{R}} \left(P(-t \log |Df|) + t\lambda\right).$$

As we will see later, the definition of the set A means that if in addition, the map $f \in \mathcal{F}_g$ is unimodal, then the above formula holds for every

$\lambda \neq 0$. Theorem A implies that the function $\lambda \mapsto \lambda \dim_H(J(\lambda))$ is the Legendre-Fenchel transform of the pressure function $t \mapsto P(-t \log |Df|)$. The lack of hyperbolicity of $f \in \mathcal{F}_g$ is reflected in the lack of regularity of the pressure functions (i.e., the presence of *phase transitions*). Therefore, the Lyapunov spectrum keeps track of all the changes in the recurrence modes of the system (see [43]).

Theorem A has been proved in different settings with different assumptions on the hyperbolicity of the system. For example, for the Gauss map it was proved by Kesseböhmer and Stratmann [27]; for maps with parabolic fixed points, related results were shown in [3,15,23,27,32,37]; and for maps with countably many branches and parabolic fixed points, this was shown by Iommi [20]. For rational maps on the complex plane, a similar result was recently shown by Gelfert et al. [14].

We also study the multifractal spectrum of the pointwise dimension of equilibrium measures for Hölder potentials. The first thing that needs to be proved is that in this non-uniformly hyperbolic setting, Hölder potentials have unique equilibrium states. We study the pointwise dimension for equilibrium states for φ through the analysis of potentials of the form $-t \log |Df| + s\varphi$. See [35] for a general account of this approach. So for example, as shown in Sect. 6, using [22], we obtain:

Theorem 1.1. *Suppose $f \in \mathcal{F}_g$ and $\varphi : I \to \mathbb{R}$ is a Hölder potential with $\varphi < P(\varphi)$. Then there exists $\varepsilon > 0$ such that for each $t \in (-\varepsilon, \varepsilon)$ there is a unique equilibrium state $\mu_{\varphi+t\log|Df|}$ for $(I, f, \varphi + t \log |Df|)$. Moreover $h(\mu_{\varphi+t\log|Df|}) > 0$.*

This theorem was proved by Bruin and Todd [9] for a narrower range of potentials φ: potentials not too far from the constant function. Therefore, in some ways, the above theorem is an improvement on Bruin and Todd's results. However, we note that the statistical properties of the equilibrium states in [9] and the relevant properties of the pressure function are stronger.

Remark 1.1. Existence of a 'conformal measure' for the Hölder potential φ as in Theorem 1.1 follows from [12] as well as [24], and uniqueness of both the conformal measure and its accompanying equilibrium state follow as in [13, Theorem 8]; see also [22]. For a discussion of the different classes of smoothness of potentials required to guarantee the existence of equilibrium states see [9, Sect. 1].

We describe the multifractal decomposition induced by the pointwise dimension of equilibrium measures for Hölder potentials (see Sect. 2 for precise definitions). When considering uniformly hyperbolic dynamical systems and Hölder potentials, the multifractal spectrum of pointwise dimension is very regular, indeed it has bounded domain, it is strictly concave and real analytic (see [35, Chapter 7]). In our setting, the multifractal spectrum can exhibit different behaviour. Not only can it have unbounded domain, but it can also have points where it is not analytic and sub-intervals where it is not strictly concave. This is a consequence of the lack of hyperbolicity of our dynamical systems and of the following result (see Sect. 7). Given (I, f, φ) we let μ_φ be the equilibrium state and denote

$$T_\varphi(q) := \inf\{t \in \mathbb{R} : P(-t \log |Df| + q\varphi) = 0\}.$$

We show in Sect. 6 that this function is C^1 and strictly convex in an interval $(q_\varphi^-, q_\varphi^+)$. The size of this interval is discussed in Sect. 6 and in Remark 7.1. We let $\mathfrak{D}_{\mu_\varphi}(\alpha)$ be the multifractal spectrum of pointwise dimension (see Sect. 5 for definitions). Note that in [47, Theorem A] it was necessary to restrict the results to points with positive pointwise upper Lyapunov exponent. In the next result, we are able to remove this restriction by results of [41]. For further discussion of the condition that μ_φ is not absolutely continuous with respect to the Lebesgue measure see Sect. 7.

Theorem B. *Suppose that $f \in \mathcal{F}_g$ and $\varphi : I \to \mathbb{R}$ is a Hölder potential with $\varphi < P(\varphi) = 0$. If μ_φ is not absolutely continuous with respect to the Lebesgue measure, then the dimension spectrum satisfies*

$$\mathfrak{D}_{\mu_\varphi}(\alpha) = \inf_{q \in \mathbb{R}} (T_\varphi(q) + q\alpha)$$

for all $\alpha \in (-DT_\varphi(q_\varphi^+), -D^+T_\varphi(q_\varphi^-))$.

This formula for the dimension spectrum was first rigorously proved by Olsen [34] and by Pesin and Weiss [36] for uniformly hyperbolic maps and for Gibbs measures. The case of the Manneville Pomeau map (non-uniformly hyperbolic map) was studied by Nakaishi [32], Pollicott and Wiess [37], and Jordan and Rams [23]. The case of Horseshoes with a parabolic fixed point was considered in Barreira and Iommi [3]. Multifractal analysis of pointwise dimension was also considered in the countable Markov shift setting by Hanus et al. [16] and Iommi [19]. For general piecewise continuous maps, analysis of this type was addressed in [18]. For multimodal maps, the multifractal analysis of pointwise dimension study began with the work of Todd [47].

As in [47], the main tool we use to prove our results is a family of so-called *inducing schemes*, which are explained in Sect. 3.2 and in greater detail in the Appendix. These are dynamical systems associated with f which on the one hand have better expansion and hyperbolicity properties but on the other, are defined on a non-compact space. We translate our problems to this setting, solve it there and then push the results back into the original system. We use the fact that $f \in \mathcal{F}_g$ to ensure that this process does not miss too many points.

The structure of the paper. In Sect. 2, we define the notions we will use from dimension theory. In Sect. 3, we define the ideas we need from thermodynamic formalism, introduce our inducing schemes and then discuss thermodynamic formalism for inducing schemes. In Sect. 4, we prove Theorem A. We give some basic ideas for the dimension spectrum in Sect. 5. We set up the proof of Theorem B in Sect. 6 and then prove the theorem in Sect. 7. In the Appendix, we give the necessary results from [47].

2. Preliminaries: Dimension Theory

Here, we recall basic definitions and results from dimension theory (see [35,40] for details). A countable collection of sets $\{U_i\}_{i\in\mathbb{N}}$ is called a δ-cover of $F \subset \mathbb{R}$ if $F \subset \bigcup_{i\in\mathbb{N}} U_i$, and U_i has diameter $|U_i|$ at most δ for every $i \in \mathbb{N}$. Let $s > 0$, we define

$$\mathcal{H}^s(F) := \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_i \text{ a } \delta\text{-cover of } F \right\}.$$

The *Hausdorff dimension* of the set F is defined by

$$\dim_H(F) := \inf \{s > 0 : \mathcal{H}^s(F) = 0\}.$$

Given a finite Borel measure μ in F, the *pointwise dimension* of μ at the point x is defined by

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

whenever the limit exists, where $B(x,r)$ is the ball at x of radius r. This function describes the power law behaviour of $\mu(B(x,r))$ as $r \to 0$, i.e.

$$\mu(B(x,r)) \sim r^{d_\mu(x)}.$$

The pointwise dimension quantifies how concentrated a measure is around a point: the larger it is the less concentrated the measure is around that point. Note that if μ is an atomic measure supported at the point x_0 then $d_\mu(x_0) = 0$ and if $x_1 \neq x_0$ then $d_\mu(x_1) = \infty$.

The following propositions relating the pointwise dimension with the Hausdorff dimension can be found in [35, Chapter 2, p. 42].

Proposition 2.1. *Given a finite Borel measure μ, if $d_\mu(x) \leq d$ for every $x \in F$, then $\dim_H(F) \leq d$.*

The *Hausdorff dimension* of the measure μ is defined by

$$\dim_H(\mu) := \inf \{\dim_H(Z) : \mu(Z) = 1\}.$$

Proposition 2.2. *Given a finite Borel measure μ, if $d_\mu(x) = d$ for μ-almost every $x \in F$, then $\dim_H(\mu) = d$.*

In this paper, we will be interested in several types of multifractal spectra. In order to give a unified definition of the objects and of the problem, we will present the general concept of multifractal analysis as developed by Barreira et al. [4] (see also [2, Chapter 7]).

Consider a function $g : Y \to [-\infty, +\infty]$, where Y is a subset of the space X. The level sets induced by the function g are defined by

$$K_g(\alpha) = \{x \in Y : g(x) = \alpha\}.$$

Since they are pairwise disjoint they induce the *multifractal decomposition*

$$X = (X \backslash Y) \cup \bigcup_{\alpha\in[-\infty,+\infty]} K_g(\alpha).$$

Let G be a real function defined on the set of subsets of X. The *multifractal spectrum* $\mathcal{S} : [-\infty, +\infty] \to \mathbb{R}$ is the function that encodes the decomposition given by g by means of the function G, that is

$$\mathcal{S}(\alpha) = G(K_g(\alpha)).$$

We stress that in this definition no dynamical system is involved. The functions g that we will consider are related to the dynamics of a certain systems and are, in general, only measurable functions. Hence, the multifractal decomposition is rather complicated. Given a multimodal map $f : I \to I$ (our dynamical system) the functions g that we will consider in this paper are:

1. The *Lyapunov exponent*, that is the function defined by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)|,$$

whenever the limit exits.

2. The *pointwise dimension* of an equilibrium state μ.

The function G we will consider here is the Hausdorff dimension. Note that we could also use entropy as a way of measuring the size of sets.

3. Preliminaries: Thermodynamic Formalism and Inducing Schemes

In this section, we will introduce some ideas from thermodynamic formalism. Then we will discuss inducing schemes, and finally, we bring these together in thermodynamic formalism for countable Markov shifts.

3.1. Thermodynamic Formalism

Let f be a map of a metric space (I, d), denote by $\mathcal{M}_f$ the set of f-invariant probability measures. Let $\varphi : I \to [-\infty, \infty]$ be a *potential*. The *topological pressure* of φ with respect to f is defined by

$$P_f(\varphi) = P(\varphi) = \sup\left\{h(\mu) + \int \varphi \, \mathrm{d}\mu : \mu \in \mathcal{M}_f \text{ and } -\int \varphi \, \mathrm{d}\mu < \infty\right\},$$

where $h(\mu)$ denotes the measure theoretic entropy of f with respect to μ. The pressure function $t \to P(t\varphi)$ is convex, being the supremum of convex functions (see [48, Chapter 9] and [26] for other properties of the pressure).

A measure $\mu_\varphi \in \mathcal{M}_f$ is called an *equilibrium state* for φ if it satisfies:

$$P(\varphi) = h(\mu_\varphi) + \int \varphi \, \mathrm{d}\mu_\varphi.$$

For $\mu \in \mathcal{M}_f$, we define the *Lyapunov exponent* of μ as

$$\lambda(\mu) = \lambda_f(\mu) := \int \log |Df| \, \mathrm{d}\mu.$$

We say that a measure m is *φ-conformal* if for any Borel set A such that $f : A \to f(A)$ is injective,

$$m(f(A)) = \int_A e^{-\varphi} \, \mathrm{d}m.$$

The following two theorems regarding existence and uniqueness of equilibrium states and the regularity of the pressure function are Theorems A and B of [22].

Theorem 3.1. *Let $f \in \mathcal{F}$. Then there exists $t^+ \in (0, +\infty]$ such that if $t \in (-\infty, t^+)$ there exists a unique equilibrium measure μ_t for the potential $-t \log |Df|$. Moreover, the measure μ_t has positive Lyapunov exponent.*

We define the pressure function $p(t) := P(-t \log |Df|)$.

Theorem 3.2. *Let $f \in \mathcal{F}$. Then for t^+ as in Theorem 3.1, if $t \in (-\infty, t^+)$ then the pressure function $t \mapsto p(t)$ is strictly convex, decreasing and C^1.*

Remark 3.1. The constant t^+ can be defined as

$$t^+ := \sup\{t : p(t) > -\lambda_m t\},$$

where λ_m is the minimal Lyapunov exponent of measures defined below in Eq. (4). As in [22, Sect. 9], $f \in \mathcal{F}_g$ implies $t^+ \geq 1$.

3.2. Inducing Schemes

A strategy used to study multimodal maps $f \in \mathcal{F}$, considering that they lack Markov structure and expansiveness, is to consider a generalisation of the first return map. These maps are expanding and are Markov (although over a countable alphabet). The price one has to pay is to lose compactness. The idea is to study the inducing scheme and then to translate the results into the original system.

We say that (X, F, τ) is an *inducing scheme* for (I, f) if

- X is an interval containing a finite or countable collection of disjoint intervals X_i such that F maps each X_i diffeomorphically onto X, with bounded distortion (i.e. there exists $K > 0$ so that for all i and $x, y \in X_i, 1/K \leq DF(x)/DF(y) \leq K$);
- $\tau|_{X_i} = \tau_i$ for some $\tau_i \in \mathbb{N}$ and $F|_{X_i} = f^{\tau_i}$. If $x \notin \cup_i X_i$ then $\tau(x) = \infty$.

The function $\tau : \cup_i X_i \to \mathbb{N}$ is called the *inducing time*. It may happen that $\tau(x)$ is the first return time of x to X, but that is certainly not the general case. For ease of notation, we will frequently write $(X, F) = (X, F, \tau)$. We denote the set of points $x \in I$ for which there exists $k \in \mathbb{N}$ such that $\tau(F^n(f^k(x))) < \infty$ for all $n \in \mathbb{N}$ by $(X, F)^\infty$.

Given (I, f) and a potential φ, the next definition gives us the relevant potentials for an inducing scheme for f.

Definition 3.1. Let (X, F, τ) be an inducing scheme for the map f. Then for a potential $\varphi : I \to \mathbb{R}$, the induced potential Φ for (X, F, τ) is given by

$$\Phi(x) = \Phi^F(x) := \varphi(x) + \cdots + \varphi \circ f^{\tau(x)-1}(x).$$

For example, for the geometric potential $\log|Df|$, the induced potential for a scheme (X, F) is $\log|DF|$.

Given an inducing scheme (X, F, τ), we say that a measure μ_F is a *lift* of μ if for all μ-measurable subsets $A \subset I$,

$$\mu(A) = \frac{1}{\int_X \tau \, \mathrm{d}\mu_F} \sum_i \sum_{k=0}^{\tau_i - 1} \mu_F \left(X_i \cap f^{-k}(A) \right). \tag{2}$$

Conversely, given a measure μ_F for (X, F, τ), we say that μ_F *projects* to μ if (2) holds. We call a measure μ *compatible to* the inducing scheme (X, F, τ) if

- $\mu(X) > 0$ and $\mu\left(X \backslash (X, F)^\infty\right) = 0$ and
- there exists a measure μ_F which projects to μ by (2): in particular $\int_X \tau \, \mathrm{d}\mu_F < \infty$.

The following result can be proved using [47] (see also [10]). We provide a proof in the Appendix for completeness.

Theorem 3.3. *Let $f \in \mathcal{F}$. There exist a countable collection $\{(X^n, F_n)\}_n$ of inducing schemes with $\partial X^n \notin (X^n, F_n)^\infty$ such that:*

a) *any ergodic invariant probability measure μ with $\lambda(\mu) > 0$ is compatible with one of the inducing schemes (X^n, F_n). In particular there exists an ergodic F_n-invariant probability measure μ_{F_n} which projects to μ as in (2);*
b) *any equilibrium state for $-t\log|Df|$ where $t \in \mathbb{R}$ with $\lambda(\mu) > 0$, or for a Hölder continuous potential $\varphi : I \to \mathbb{R}$ with $\varphi < P(\varphi)$, is compatible with all inducing schemes (X^n, F_n).*
c) *if $f \in \mathcal{F}_g$ then*

$$dim_H \left(I \backslash \left(\cup_{n=1}^\infty (X^n, F_n)^\infty\right)\right) = 0.$$

If (X, F, τ) is an inducing scheme for the map f with $\partial X \notin (X, F)^\infty$, then the system $F : (X, F)^\infty \to (X, F)^\infty$ is topologically conjugated to the full-shift on a countable alphabet. Hence, we can transfer our study to those shifts. We explain this in the next subsection.

3.3. Countable Markov Shifts

Let $\sigma\colon \Sigma \to \Sigma$ be a one-sided Markov shift with a countable alphabet S. We equip Σ with the topology generated by the cylinder sets

$$C_{i_0 \cdots i_n} = \{x \in \Sigma : x_j = i_j \text{ for } 0 \le j \le n\}.$$

Given a function $\varphi\colon \Sigma \to \mathbb{R}$, for each $n \ge 1$ we set

$$V_n(\varphi) = \sup\left\{|\varphi(x) - \varphi(y)| : x, y \in \Sigma,\ x_i = y_i \text{ for } 0 \le i \le n-1\right\}.$$

We say that φ has *summable variation* if $\sum_{n=2}^\infty V_n(\varphi) < \infty$. Clearly, if φ has summable variation then it is continuous. The so-called *Gurevich pressure* of φ was defined by Sarig [42] as

$$P_G(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x:\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right) \chi_{C_{i_0}}(x),$$

where $\chi_{C_{i_0}}(x)$ is the characteristic function of the cylinder $C_{i_0} \subset \Sigma$. We consider a special class of invariant measures. We say that $\mu \in \mathcal{M}_\sigma$ is a *Gibbs measure* for the function $\varphi \colon \Sigma \to \mathbb{R}$ if for some constants $P, C > 0$ and every $n \in \mathbb{N}$ and $x \in C_{i_0 \cdots i_n}$ we have

$$\frac{1}{C} \leq \frac{\mu(C_{i_0 \cdots i_n})}{\exp(-nP + \sum_{i=0}^{n} \varphi(\sigma^k x))} \leq C. \tag{3}$$

It was proved by Mauldin and Urbański [30] and by Sarig in [44] that if (Σ, σ) is a full-shift and the function φ is of summable variations with finite Gurevich pressure, then it has a unique φ-conformal Gibbs measure m_Φ along with a unique invariant Gibbs measure μ_Φ, where $\frac{\mathrm{d}\mu_\Phi}{\mathrm{d}m_\Phi}$ is uniformly bounded away from 0 and ∞. Moreover, if φ is weakly Hölder (see [44] for precise definition) then the function

$$t \mapsto P_G(t\varphi)$$

is real analytic for every $t \geq 1$ (see [43]) whenever $P_G(t\varphi)$ is finite.

Remark 3.2. Since the system $F : (X, F)^\infty \to (X, F)^\infty$ is topologically conjugated to the full-shift on a countable alphabet. In particular, every potential $\Phi : X \to \mathbb{R}$ has a symbolic version, $\overline{\Phi} : \Sigma \to \mathbb{R}$. In all the cases of induced systems we consider in this paper we have, by the Variational Principle [42, Theorem 3], $P(\Phi) = P_G(\overline{\Phi})$. Therefore, in order to simplify the notation, we will denote the pressure by $P(\Phi)$ when the underlying system is the induced system and when it is the full-shift on a countable alphabet.

4. The Lyapunov Spectrum

In this section, we consider the multifractal decomposition of the interval obtained by studying the level sets associated with the Lyapunov exponent for maps $f \in \mathcal{F}$. In recent years, a great deal of attention has been paid to this decomposition. This is partly due to the fact that the Lyapunov exponent is a dynamical characteristic that captures important features of the dynamics. It is closely related to the existence of absolutely continuous (with respect to Lebesgue) invariant measures.

The *lower/upper pointwise Lyapunov exponent* at $x \in I$ is defined by

$$\underline{\lambda}_f(x) := \liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| Df\left(f^j(x)\right)\right|,$$

$$\text{and} \quad \overline{\lambda}_f(x) := \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| Df\left(f^j(x)\right)\right|,$$

respectively. If $\underline{\lambda}_f(x) = \overline{\lambda}_f(x)$, then the *Lyapunov exponent* of the map f at x is defined by $\lambda(x) = \lambda_f(x) = \underline{\lambda}_f(x) = \overline{\lambda}_f(x)$.

The associated level sets for $\alpha \geq 0$ are defined by,

$$J(\alpha) = \left\{ x \in I : \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(x)| = \alpha \right\}.$$

Note that for some values of α we have $J(\alpha) = \emptyset$, a trivial example being for $\alpha > \log(\sup_{x\in I} |Df(x)|)$. Let

$$J' = \left\{ x \in I : \text{the limit } \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \text{ does not exist} \right\}.$$

The unit interval can be decomposed in the following way (the *multifractal decomposition*),

$$[0,1] = J' \cup (\cup_\alpha J(\alpha)).$$

The function that encodes this decomposition is called *multifractal spectrum of the Lyapunov exponents* and is defined by

$$L(\alpha) := \dim_H(J(\alpha)).$$

This function was first studied by Weiss [49] in the context of axiom A maps. The study of the multifractal spectrum of the Lyapunov exponent for multimodal maps began with the work of Todd [47].

We define

$$\lambda_m := \inf \{\lambda(\mu) : \mu \in \mathcal{M}\} \quad \text{and} \quad \lambda_M := \sup \{\lambda(\mu) : \mu \in \mathcal{M}\}. \tag{4}$$

We next show that the range of values that the Lyapunov exponent can attain is an interval contained in $[\lambda_m, \lambda_M]$.

We define

$$\lambda_{\inf} := \inf\{\lambda(x) : x \in I \text{ and this value is defined}\}$$

and

$$\lambda_{\sup} := \sup\{\lambda(x) : x \in I \text{ and this value is defined}\}.$$

Lemma 4.1. *$\lambda_m \geq \lambda_{\inf}$ and $\lambda_M = \lambda_{\sup}$.*

Proof. The fact that $\lambda_m \geq \lambda_{\inf}$ follows from the fact that for an ergodic measure $\mu \in \mathcal{M}$ we have $\lambda(x) = \lambda(\mu)$ for μ-a.e. $x \in I$. Similarly $\lambda_{\sup} \geq \lambda_M$. To show that $\lambda_{\sup} \leq \lambda_M$, suppose $x \in I$ is such that $\lambda(x)$ is well defined. Then let

$$\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$$

where δ_y is the Dirac mass on y. Let μ be a weak* limit of this sequence. Since $\log |Df|$ is upper semicontinuous,

$$\lambda_M \geq \lambda(\mu) \geq \lim_{n\to\infty} \lambda(\mu_n) = \lambda(x)$$

as required. □

Remark 4.1. We expect that $\lambda_m = \lambda_{\inf}$, which is proved in the complex case in [14, Lemma 6]. The argument there is that if there is a point x such that $\lambda(x)$ exists and is in $(-\infty, \lambda_m)$ then this point can be closely approximated by a periodic point with Lyapunov exponent arbitrarily close to $\lambda(x)$. The Dirac measure on this periodic cycle must have Lyapunov exponent larger than λ_m, so taking limits they obtain $\lambda(x) \geq \lambda_m$. The main issue in this argument is to show that points in the orbit of x can be pulled back with bounded distortion by suitable inverse branches of the complex map. In [41, Theorem B], it is shown that for C^3 maps in $\mathcal{F}$ the points which cannot be suitably pulled back have zero Hausdorff dimension; therefore, putting these arguments together, we can conclude that the set $\{x : \lambda(x) < \lambda_m\}$ has zero Hausdorff dimension.

Note that in the case $\lambda_m = 0$ then $\lambda_{\inf}$ is also equal to zero by [38].

We define

$$\tilde{\lambda}_m := \inf \{\lambda(\mu) : \mu \in \mathcal{M} \text{ ergodic and } \lambda(\mu) > 0\}$$

and

$$A := \begin{cases} [\lambda_{\inf}, -D^-p(t^+)) & \text{if } \tilde{\lambda}_m > 0, \\ \{0\} & \text{if } \tilde{\lambda}_m = 0. \end{cases}$$

Remark 4.2. Clearly $\lambda_m > 0$ implies $\tilde{\lambda}_m = \lambda_m$. Moreover, the case that $f \in \mathcal{F}_g$ is unimodal, $\lambda_m = 0$ implies $\tilde{\lambda}_m = 0$ by [33]. For complex maps, this is proved in [39]. We believe that this should hold in the real multimodal case also.

It can be shown by Theorem 3.1 that for every $\lambda \in (-D^-p(t^+), \lambda_M]$ there exists a unique parameter $t_\lambda \in \mathbb{R}$ such that for μ_λ, the unique equilibrium measure μ_λ corresponding to $-t_\lambda \log |Df|$ has

$$\lambda(\mu_\lambda) = \lambda.$$

However, as in [7, Lemma 5.5] there are maps $f \in \mathcal{F}$ with no measure $\mu \in \mathcal{M}_f$ with $\lambda(\mu) = \lambda_m = 0$.

Theorem 4.1. *Suppose that $f \in \mathcal{F}_g$. Let $\lambda \in \mathbb{R}\backslash A$. The Lyapunov spectrum satisfies the following relation*

$$L(\lambda) = \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda). \tag{5}$$

If $\lambda \in (-D^-p(t^+), \lambda_M)$ then we also have

$$L(\lambda) = \frac{1}{\lambda} (p(t_\lambda) + t_\lambda \lambda) = \frac{h(\mu_\lambda)}{\lambda}. \tag{6}$$

If $\lambda_m > 0$ and $\lambda \in A$ then

$$L(\lambda) \geq \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda).$$

Moreover, the irregular set J' has full Hausdorff dimension.

Theorem A follows immediately from this.

Remark 4.3. Theorem 4.1 along with Remark 4.2 implies that if $f \in \mathcal{F}_g$ is unimodal with $\lambda_m = 0$ then for every $\lambda \in (0, -D^-p(1))$ we have that $L(\lambda) = 1$ (see also Lemma 4.2 for a detailed proof of this fact). In the unimodal case, as in [33], $\lambda_m = 0$ implies that the Collet-Eckmann condition fails. In the multimodal case, as well as in the unimodal case where $\lambda_m > 0$, we expect that the formula $L(\lambda) = \frac{1}{\lambda} \inf_{t \in \mathbb{R}} (p(t) + t\lambda)$ still holds for $\lambda \in [\lambda_m, -D^-p(t^+))$, but we do not find an upper bound on this value in this paper.

Remark 4.4. The above formula (5) for $L(\lambda)$ does not imply that the Lyapunov spectrum is concave. For a discussion on that issue see the work of Iommi and Kiwi [21].

Proof of the lower bound for Theorem 4.1. Let $\lambda \in (-D^-p(t^+), \lambda_M)$. In order to prove the lower bound on the formula (5), consider the equilibrium measure μ_λ corresponding to $-t_\lambda \log |Df|$ such that $\lambda(\mu_\lambda) = \lambda$. We have

1. $\mu_\lambda(I \backslash J(\lambda)) = 0$;
2. the measure μ_λ is ergodic;
3. by [17], the pointwise dimension is μ_λ- almost everywhere equal to

$$\lim_{r \to 0} \frac{\log \mu_\lambda(B(x,r))}{\log r} = \frac{h(\mu_\lambda)}{\lambda},$$

where $B(x,r)$ is the ball of radius $r > 0$ centred at $x \in [0,1]$.

Therefore, Proposition 2.2 implies

$$\dim_H(J(\lambda)) \geq \frac{h(\mu_\lambda)}{\lambda}.$$

We next consider $\lambda \in (\lambda_m, -D^-p(t^+))$. The following lemma applies when $\lambda_m = 0$.

Lemma 4.2. *Suppose that $f \in \mathcal{F}_g$ has $\tilde{\lambda}_m = 0$. Then for any $\alpha \in (0, -D^-p(1))$ and $\varepsilon > 0$ there exists an ergodic measure $\mu \in \mathcal{M}$ with $\lambda(\mu) = \alpha$ and $\dim_H(\mu) \geq 1 - \varepsilon$.*

The proof follows by approximating (I, f) by hyperbolic sets on which we have equilibrium states with small Lyapunov exponent and large Hausdorff dimension. The hyperbolic sets are invariant sets for truncated inducing schemes.

Proof. We may assume that $D^-p(t^+) < 0$, otherwise there is nothing to prove. Let $\varepsilon' \in (0, \alpha)$. Since in this case $\tilde{\lambda}_m = 0$, we can choose $\alpha' \in (0, \varepsilon'/(1+\varepsilon')]$ and an ergodic measure $\mu \in \mathcal{M}$ with $\lambda(\mu) \in (0, \alpha']$. We can then choose an inducing scheme (X', F', τ') as in Theorem 3.3 compatible with μ and with distortion sufficiently low that on one of the domains $X_i, \log |DF(x)| \leq \alpha' \tau_i$ for all $x \in X_i'$. In particular, there is a fixed point of F in X_i' with this property. Let $p = \tau_i$ and call this fixed point x_p. Note that this is a periodic point for f with period $\leq p$ and is such that $\lambda(x) \leq \alpha'$. Now take the first return map by F' to X_i' as our inducing scheme (X, F, τ). Note that $\tau(x) \geq p$ for all $x \in X$. We can truncate (X, F, τ) to a scheme with N branches $(\tilde{X}^N, \tilde{F}_N, \tilde{\tau}_N)$ and define

$$p_N(t) := \sup \left\{ h(\mu) - t\lambda(\mu) - p(t) : \mu \in \mathcal{M} \text{ and } \mu \text{ is compatible with } (\tilde{X}^N, \tilde{F}_N) \right\}.$$

Claim 1. There exist $\delta(N) > 0$ where $\delta(N) \to 0$ as $N \to \infty$ such that for $t \in (1 - \varepsilon', 1], p_N(t) \geq -\delta(N)$ and for $t \in (1, 1 + \varepsilon'), p_N(t) \geq -t\alpha'$.

Proof. By Theorem 3.3, $P(-t\log|DF| - \tau p(t)) = 0$ and indeed $p_N(t) \to p(t)$ for all $t \leq 1$, so the first part of the claim follows.

We now suppose that $t \geq 1$. For any $N \geq 1$ the Dirac measure μ_{x_p} on the orbit of x_p lifts to $(\tilde{X}^N, \tilde{F}_N, \tilde{\tau}_N)$ so

$$p_N(t) \geq h(\mu_{x_p}) - t\int \log|Df| \,\mathrm{d}\mu_{x_p} \geq -t\alpha',$$

as required. □

The claim implies that $Dp_N(t) \to Dp(t)$ for $t < 1$ also. Since for $t < 1$ we have $-Dp(t) = \lambda(\mu) > \alpha$ for μ the equilibrium state for $-t\log|Df|$, the claim also implies that for $\varepsilon' > 0$ as above there exists N such that there is $t \in [1, 1+\varepsilon')$ with $-Dp_N(t) = \alpha$ and also $p_N(t) > -\varepsilon'$. Therefore, there is an equilibrium state μ_α which is a projection of $(X, F, -t\log|D\tilde{F}_N|)$ such that

$$h(\mu_\alpha) - \alpha \geq -\varepsilon'.$$

Hence $L(\alpha) \geq \dim_H(\mu_\alpha) = h(\mu_\alpha)/\alpha \geq 1 - \varepsilon'/\alpha$. The proof of the lemma concludes by setting $\varepsilon' := \varepsilon\alpha$. □

For the case where $\lambda_m > 0$, and so $t^+ \in (1, \infty)$, and p is not C^1 at $t = t^+$ we can apply a similar argument. We showed in [22, Remark 9.2] that $t^+ \in (1, \infty)$ implies that $\lambda(\mu) \neq \lambda_m$ for all $\mu \in \mathcal{M}$. Therefore, we can use the fact that for any $\varepsilon' > 0$ there exists $\mu \in \mathcal{M}$ such that $\lambda(\mu) \in (\lambda_m, \lambda_m + \varepsilon')$. In this case, we obtain the lower bound:

$$L(\alpha) \geq t^+ + \frac{p(t^+)}{\alpha},$$

as required. □

Proof of the upper bound for Theorem 4.1. In the case $\lambda_m = 0$ and $\alpha \in (0, -D^-p(1))$ we showed $L(\alpha) \geq 1$, so in fact $L(\alpha) = 1$. Therefore, to complete the proof of Theorem 4.1, we will prove the upper bound for $L(\alpha)$ when $\alpha \in [-D^-p(1), \lambda_M]$ and λ_m is any value.

Let (X, F, τ) be an inducing scheme for the map f. Note that the (X, F) is topologically conjugated to the full-shift on a countable alphabet. Recall that (see Sect. 3.3) every potential $\varphi : X \to \mathbb{R}$ of summable variations and finite pressure has a Gibbs measure [44].

Remark 4.5. Note that if μ_t is the equilibrium measure for $-t\log|Df|$ then the lifted measure $\mu_{F,t}$ is the Gibbs measure corresponding to the potential $\Phi_t = -t\log|DF| - P(-t\log|Df|)\tau$. Note that Φ_t has summable variations by, for example, [10, Lemma 8].

For an inducing scheme (X^n, F_n, τ_n) constructed as in the proof of Theorem 3.3, consider the level set

$$J_n(\lambda) := \left\{ x \in X^n : \lim_{k\to\infty} \frac{\sum_{j=0}^{k-1} \log\left|DF_n\left(F_n^j(x)\right)\right|}{\sum_{j=0}^{k-1} \tau_n\left(F_n^j x\right)} = \lambda \right\}.$$

Note that if $y \in I$ has $\lambda(y) = \lambda$ and $f^j(y) \in (X^n, F_n)^\infty$ for some $j \geq 0$, then $f^j(y) \in J_n(\lambda)$.

Remark 4.6. If μ_t is the equilibrium measure for $-t\log|Df|$ and $\lambda(\mu_t) = \lambda$ then the lifted measure $\mu_{F_n,t}$ has

$$\mu_{F_n,t}\left(I \backslash J_n(\lambda)\right) = 0.$$

Denote by $I_k^n(x)$ the cylinder (with respect to the Markov dynamical system (X^n, F_n)) of length k that contains the point $x \in X$ and by $|I_k^n(x)|$ its Euclidean length. By definition, there exists a positive constant $K > 0$ such that for every $x \in X$ which is not the preimage of a boundary point and every $k \in \mathbb{N}$ we have

$$\frac{1}{K} \leq \frac{|I_k^n(x)|}{|DF_n^k(x)|} \leq K.$$

Definition 4.1. For an inducing scheme (X^n, F_n) and a point $x \in X^n$ not a preimage of a boundary point of X^n, we define the *Markov pointwise dimension of $\mu_{F_n,t}$ at the point x* as

$$\delta_{\mu_{F_n,t}}(x) := \lim_{k\to\infty} \frac{\log \mu_{F_n,t}\left(I_k^n(x)\right)}{\log|I_k^n(x)|}$$

if this limit exists.

Lemma 4.3. *The Hausdorff dimension of $J_n(\lambda)$ is given by*

$$\dim_H(J_n(\lambda)) = \frac{h(\mu_t)}{\lambda} = \delta_{\mu_{F_n,t}}(x)$$

for $\mu_{F_n,t}$-a.e. $x \in X^n$.

Proof. Let $x \in J_n(\lambda)$ and $\mu_{F_n,t}$ be the Gibbs measure with respect to $\Phi_{t,n} := -t\log|DF_n| - P(-t\log|Df|)\tau_n$. Since we have bounded distortion, the Markov pointwise dimension of $\mu_{F_n,t}$ at the point $x \in X^n$, if it exists, is

$$\begin{aligned}
\delta_{\mu_{F_n,t}}(x) &= \lim_{k\to\infty} \frac{\log \mu_{F_n,t}\left(I_k^n(x)\right)}{\log|I_k^n(x)|} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1}\Phi_{t,n}\left(F_n^i(x)\right)}{-\log|DF_n^k(x)|} \\
&= \lim_{k\to\infty} \frac{-t\log\left|DF_n^k(x)\right| - P\left(-t\log|Df|\right)\sum_{i=0}^{k-1}\tau_n\left(F_n^i(x)\right)}{-\log|DF_n^k(x)|} \\
&= t + P\left(-t\log|Df|\right)\lim_{k\to\infty}\frac{\sum_{i=0}^{k-1}\tau_n\left(F_n^i(x)\right)}{\log|DF_n^k(x)|} \\
&= t + \left(h(\mu_t) - t\int \log|Df|\mathrm{d}\mu_t\right)\lim_{k\to\infty}\frac{\sum_{i=0}^{k-1}\tau_n\left(F_n^i(x)\right)}{\log|DF_n^k(x)|}.
\end{aligned}$$

But since $x \in J_n(\lambda)$ we have that

$$\lim_{k\to\infty}\frac{\sum_{i=0}^{k-1}\tau_n(F_n^i(x))}{\log|DF_n^k(x)|} = \frac{1}{\lambda}.$$

Therefore,

$$\delta_{\mu_{F_n,t}}(x) = t + \frac{h(\mu_t) - t\lambda}{\lambda} = \frac{h(\mu_t)}{\lambda}.$$

The following result was proved by Pollicott and Weiss [37, Proposition 3]. Suppose that $\delta_{\mu_{F_n,t}}(x)$ and $\lambda(x)$ exist, then

$$d_{\mu_{F_n,t}}(x) = \delta_{\mu_{F_n,t}}(x).$$

Therefore, we have that for every point $x \in J_n(\lambda)$ the pointwise dimension is given by

$$d_{\mu_{F_n,t}}(x) = \frac{h(\mu_t)}{\lambda}.$$

Since $\mu_{F_n,t}(X^n \backslash J_n(\lambda)) = 1$ we have that

$$\dim_H(J_n(\lambda)) = \frac{h(\mu_t)}{\lambda},$$

as required. □

Note that the projection map $\pi_n : X^n \to I$ from each inducing scheme (X^n, F_n) into the interval I is a bilipschitz map. Therefore,

$$\dim_H(\pi_n(J_n(\lambda))) = \frac{h(\mu_t)}{\lambda}.$$

Similarly, $\dim_H \left(\cup_{k\geq 0} f^{-k}\left(\pi_n(J_n(\lambda))\right)\right) = \frac{h(\mu_t)}{\lambda}$. By the above arguments, plus Theorem 3.3 c), $J(\lambda)$ is contained in $\cup_n \cup_{k\geq 0} f^{-k}\left(\pi_n(J_n(\lambda))\right)$ up to a set of Hausdorff dimension 0. Hence, we obtain the desired upper bound,

$$\begin{aligned}\dim_H(J(\lambda)) &\leq \dim_H \left(\cup_n \cup_{k\geq 0} f^{-k}\left(\pi_n(J_n(\lambda))\right)\right)\\ &= \sup_n \{\dim_H \left(\pi_n(J_n(\lambda))\right)\} = \frac{h(\mu_t)}{\lambda}.\end{aligned}$$

□

Remark 4.7. Note that, as we did in Lemma 4.2, we can truncate (X, F, τ) to a scheme with N branches $(\tilde{X}^N, \tilde{F}_N, \tilde{\tau}_N)$. The Hausdorff dimension of X is approximated by those of $\tilde{X}^N$ (see for example [29, Theorem 3.15] or [19, Theorem 3.1]). It is then a direct consequence of the results of Barreira and Schmeling [5] that the irregular set for X has full Hasudorff dimension. It follows then that the set J' has full Hausdorff dimension.

5. The Pointwise Dimension Spectrum

In this section, we explain the multifractal spectrum of the pointwise dimension of equilibrium states. As in [47], this can be seen as a generalisation of the results on the Lyapunov spectrum. As in Sect. 2, the *pointwise dimension* of the measure μ at the point $x \in I$ is defined by

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu((x-r, x+r))}{\log r},$$

provided the limit exists. This function describes the power law behaviour of the measure of an interval,

$$\mu((x-r, x+r)) \sim r^{d_\mu(x)}.$$

The pointwise dimension induces a decomposition of the space into level sets:

$$K(\alpha) = \{x \in \Sigma : d_\mu(x) = \alpha\}, K' = \{x \in \Sigma : \text{the limit } d_\mu(x) \text{ does not exist}\}.$$

The set K' is called the *irregular set*. The decomposition:

$$I = \left(\bigcup_\alpha K(\alpha)\right) \bigcup K'$$

is called the *multifractal decomposition*. The *multifractal spectrum of pointwise dimension* is defined by

$$\mathfrak{D}_\mu(\alpha) = \dim_H(K(\alpha)).$$

Note that for maps $f \in \mathcal{F}$, points which are not 'seen' by inducing scheme (i.e. not in any set $(X^n, F_n)^\infty$ for an inducing scheme (X^n, F_n) from Theorem 3.3) are beyond our analysis. However, as in Theorem 3.3 c), our inducing schemes capture all sets of positive Hausdorff dimension.

In order to describe the function $\mathfrak{D}_\mu$ we will study an auxiliary function: the so-called *temperature function* is defined in terms of the thermodynamic formalism and shown to be the Legendre-Fenchel transform of the multifractal spectrum.

6. The Temperature Function

In this section, we study the temperature function which allows us to describe the multifractal spectrum. First, we need to establish the existence of the measures that we are going to analyse. The measures we will study will be equilibrium states. The class of potentials that we consider is

$$\mathcal{P} := \{\varphi : I \to [\varphi_{\min}, \varphi_{\max}] \text{ for some } \varphi_{\min}, \varphi_{\max} \in (-\infty, 0) \text{ and } P(\varphi) = 0\}.$$

Note that any bounded potential φ' with $\varphi' < P(\varphi')$ can be translated into this class by setting $\varphi := \varphi' - P(\varphi')$. Any equilibrium state for φ' is an equilibrium state for φ.

We let $\mathcal{P}_H \subset \mathcal{P}$ be the set of Hölder potentials on I. It is well known (see for example [26, Section 4]) that potentials in $\mathcal{P}$ have (potentially many) equilibrium states with positive entropy. Theorem 1.1 shows this.

Theorem 1.1 is a corollary of Theorem 6.1 below. It follows using the inducing techniques as in [22, Section 5]. Note that the Hölder condition on φ guarantees the summable variations for the inducing schemes; see the Appendix.

As in the introduction, the *temperature function with respect to* φ is the function $T_\varphi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ implicitly defined, for $q \in \mathbb{R}$, by the equation

$$T_\varphi(q) = \inf\{t \in \mathbb{R} : P(-t \log |Df| + q\varphi) = 0\}.$$

If for a fixed q and for every $t \in \mathbb{R}$ we have that $P(-t \log |Df| + q\varphi) > 0$ then $T_\varphi(q) = \infty$. If there exists a finite number

$$q_\infty := \sup\{q \in \mathbb{R} : T_\varphi(q) = \infty\},$$

then we say that T_φ has an *infinite phase transition* at q_∞.

Remark 6.1. Note that for $\varphi \in \mathcal{P}$ we have $T_\varphi(1) = 0$, since by definition $P(\varphi) = 0$. Moreover, $T_\varphi(0)$ is the smallest root of the Bowen equation $P(-t \log |Df|) = 0$. It follows from the statement of [1, Theorem 10.5] that there are unimodal maps in $\mathcal{F}$ with critical order $\ell_c > 2$ for which $T_\varphi(0) < 1$. This phenomenon is associated with the presence of a 'wild attractor'. For any unimodal map with quadratic critical point (i.e. $\ell_c = 2$), there is no wild attractor and we have $T_\varphi(0) = 1$. This is also true for any map in $\mathcal{F}_g$.

Remark 6.2. Note that $q' \leq q$ implies $T_\varphi(q') \geq T_\varphi(q)$. Therefore if T_φ has an infinite phase transition at q_∞ then $T(q) = \infty$ for all $q < q_\infty$.

Example 6.1. (*Regular*) Let $f : I \to I$ be a Collet-Eckmann unimodal map. Then the pressure function $t \to p(t)$ is strictly decreasing as in Theorem 3.2. Moreover, p is C^1 in an interval $(-\infty, t^+) \supset [0, 1]$. Consider the potential $\varphi = -h_{\text{top}}(f)$ (i.e. minus the topological entropy of the map f). In this case, the function $T_\varphi(q)$ is obtained by the equation in the variable $t \in \mathbb{R}$ given by

$$P(-t \log |Df|) = q h_{\text{top}}(f).$$

For every $q \in \mathbb{R}$ this equation has a unique solution. Moreover, for q in a neighbourhood of $[0, 1]$, by Theorem 3.1 there exists a unique equilibrium state μ_{φ_q} corresponding to the potential $\varphi_q = -T_\varphi(q) \log |Df| - q h_{\text{top}}(f)$.

Example 6.2. (*Infinite phase transition*) If $f \in \mathcal{F}$ is a unimodal map which is not Collet-Eckmann, then results in [33] imply

$$P(-t \log |Df|) = \begin{cases} \text{positive} & \text{if } t < 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

If we consider the constant potential $\varphi := -h_{\text{top}}(f)$ then for every $q < 0$ we have that

$$P(-t \log |Df| - q h_{\text{top}}(f)) = P(-t \log |Df|) + |q| h_{\text{top}}(f) \geq |q| h_{\text{top}}(f) > 0.$$

i.e.

$$T_\varphi(q) = \begin{cases} \text{infinite} & \text{if } q < 0, \\ \text{finite} & \text{if } q \geq 0. \end{cases}$$

In this case, the function $T_\varphi(q)$ has an infinite phase transition.

We stress that infinite phase transitions can only occur at $q = 0$. This is contained in the following proposition where we collect some basic properties of T_φ.

Proposition 6.1. *Suppose that $f \in \mathcal{F}$ and $\varphi \in \mathcal{P}$. Then*

a) *$T_\varphi(q) \in \mathbb{R}$ for all $q \geq 0$;*
b) *the function $T_\varphi(q)$ can only have an infinite phase transition at $q_\infty = 0$;*
c) *T_φ, when finite, is strictly decreasing.*

We will use the following two Lemmas.

Lemma 6.1. *Suppose that $f \in \mathcal{F}$ and $\varphi \in \mathcal{P}$. If $q \geq 0$ then the function $T_\varphi(q)$ is finite.*

Proof. If $q \geq 0$ then $q\varphi \leq 0$. Therefore,

$$P(-t\log|Df| + q\varphi) \leq P(-t\log|Df|).$$

Since $P(-\log|Df|) \leq 0$ and $t \mapsto P(-t\log|Df|)$ is decreasing, this implies that $T_\varphi(q) \leq 1$. It remains to check $T_\varphi(q) \neq -\infty$.

We have

$$P(-t\log|Df| + q\varphi) \geq P(-t\log|Df| + q\varphi_{\min}) = P(-t\log|Df|) + q\varphi_{\min}.$$

Since

$$\lim_{t\to-\infty} P(-t\log|Df|) = \infty,$$

there exists $t_0 < 0$ such that

$$P(-t_0\log|Df|) - q\varphi_{\min} > 0.$$

i.e.

$$P(-t_0\log|Df| + q\varphi) > 0.$$

Since the function $t \to P(-t\log|Df| + q\varphi)$ is continuous, the Intermediate Value Theorem implies that there exists $T_\varphi(q) \in (t_0, 1]$ such that

$$T_\varphi(q) = \inf\{t \in \mathbb{R} : P(-t\log|Df| + q\varphi) = 0\},$$

as required. □

Lemma 6.2. *Suppose that $f \in \mathcal{F}$ and $\varphi \in \mathcal{P}$. If*

$$\lim_{t\to+\infty} P(-t\log|Df|) = -\infty$$

then $T_\varphi(q)$ is finite for every $q \in \mathbb{R}$.

Proof. We will show that $T_\varphi(q)$ must lie in a finite interval. First note that if $q < 0$ then

$$P(-t\log|Df| + q\varphi) \leq P(-t\log|Df|) + q\varphi_{\min}.$$

Therefore, by assumption if we take $t_1 > 0$ large enough we have that

$$P(-t_1\log|Df| + q\varphi) \leq 0.$$

From the other side, as in the proof of Lemma 6.1 we can find $t_0 \in \mathbb{R}$ such that

$$P(-t_0\log|Df| + q\varphi) > 0.$$

Hence $T_\varphi(q)$ lies in the finite interval $(t_0, t_1]$.

The case of positive q is handled by Lemma 6.1. □

Proof of Proposition 6.1. *Part* a): This follows immediately from Lemma 6.1.

Part b): Lemma 6.2 implies that if $\lim_{t\to+\infty} P(-t\log|Df|) = -\infty$ then we cannot have an infinite phase transition. Therefore, adding this to Lemma 6.1, to prove part 1 of the proposition we only need to examine the case when the limit is finite: $\lim_{t\to+\infty} P(-t\log|Df|) > -\infty$ and $q < 0$.

By definition, $Dp(t) \leq -\lambda_m t$; therefore, the only way that we can have $\lim_{t\to+\infty} P(-t\log|Df|) > -\infty$ is if $\lambda_m = 0$ (note that $\lambda(\mu) \geq 0$ for all $\mu \in \mathcal{M}$

by [38]). This implies $P(-t\log|Df|) \geq 0$ for all $t \in \mathbb{R}$. Now suppose that $q < 0$. Then

$$P(-T(q)\log|Df| + q\varphi) \geq P(-T(q)\log|Df|) + q\varphi_{\max} \geq q\varphi_{\max} > 0.$$

Hence $T(q) = \infty$. Since this holds for all negative q, the infinite phase transition must occur at 0.

Part c): Let $q \in \mathbb{R}$ and $\delta > 0$. Then

$$\begin{aligned} P(-T_\varphi(q)\log|Df| + (q+\delta)\varphi) &\leq P(-T_\varphi(q)\log|Df| + q\varphi) + \delta\varphi_{\max} \\ &< P(-T_\varphi(q)\log|Df| + q\varphi). \end{aligned}$$

Hence, there is no way that $T_\varphi(q)$ can be $T_\varphi(q+\delta)$, proving part c). □

In the next theorem, we establish the existence of equilibrium measures for the potential

$$\varphi_q := -T(q)\log|Df| + q\varphi$$

for a maximal range of values of the parameter $q \in \mathbb{R}$. The strategy of the proof follows the arguments developed in [22] to prove the existence and uniqueness of equilibrium measures for the geometric potential $-t\log|Df|$.

We define the constants $q_\varphi^- \leq q_\varphi^+$ as follows:

- q_φ^+ is defined, if possible, to be the infimum of $q \geq 1$ such that there exists $\varepsilon_q > 0$ such that for all $\varepsilon \in (0, \varepsilon_q)$ there exists $\delta > 0$ such that for any $\mu \in \mathcal{M}$,
$$\left| h(\mu) + \int \varphi_q \, \mathrm{d}\mu \right| < \delta \quad \text{implies} \quad h(\mu) > \varepsilon.$$
If there is no such value, then $q_\varphi^+ := \infty$.
- If T_φ has an infinite phase transition then $q_\varphi^- := 0$. If not then, if possible, it is defined as being the supremum of $q \leq 1$ such that there exists $\varepsilon_q > 0$ such that for all $\varepsilon \in (0, \varepsilon_q)$ there exists $\delta > 0$ such that for any $\mu \in \mathcal{M}$,
$$\left| h(\mu) + \int \varphi_q \, \mathrm{d}\mu \right| < \delta \quad \text{implies} \quad h(\mu) > \varepsilon.$$
If there is no such value then $q_\varphi^- := -\infty$.

Lemma 6.3. *If $f \in \mathcal{F}$ and $\varphi \in \mathcal{P}$ then $q_\varphi^- \leq 0$ and $q_\varphi^+ \geq 1$.*

Proof. Suppose $q \in (0,1)$. Then $T_\varphi(q) \geq 0$. Suppose that there is an equilibrium state $\mu_{\varphi_q} \in \mathcal{M}$ for φ_q. Then by definition

$$T_\varphi(q)\lambda(\mu_{\varphi_q}) = h(\mu_{\varphi_q}) + q\int \varphi \, \mathrm{d}\mu_{\varphi_q} \geq 0$$

since by [38], $\lambda(\mu_{\varphi_q}) \geq 0$. Since $q\int \varphi \, \mathrm{d}\mu_{\varphi_q} < 0$, we must have $h(\mu_{\varphi_q}) > 0$. The lemma then follows by extending this argument to the case of measures μ with $h(\mu) + \int \varphi_q \, \mathrm{d}\mu$ close to 0. □

For here on, we assume that $f \in \mathcal{F}_g$ to ensure that Hölder potentials φ yield induced potentials Φ for our inducing schemes which are locally Hölder continuous; see the Appendix.

Remark 6.3. By [9, Theorem 6], if $f \in \mathcal{F}_g$ and $\varphi \in \mathcal{P}_H$ and $\varphi_{\max} - \varphi_{\min} < h_{\mathrm{top}}(f)$ then $q_\varphi^+ > 1$.

Theorem 6.1. *Suppose that $f \in \mathcal{F}_g$ and $\varphi \in \mathcal{P}_H$. Then for every $q \in (q_\varphi^-, q_\varphi^+)$ the potential φ_q has a unique equilibrium measure μ_{φ_q}. Moreover, it is a measure of positive entropy.*

Since the proof of this theorem goes along the same lines as the proof of Theorem A given in [22], we only sketch it here. Note that Theorem 1.1 is a corollary of this.

Proof. Proposition 6.1 implies that there exists $\tilde{q}_\varphi^- \in [-\infty, 0]$ such that for every $q \in (\tilde{q}_\varphi^-, \infty)$ there exists a unique root $T(q) \in \mathbb{R}$ of the equation $P(-t \log |Df| + q\varphi) = 0$.

Lemma 6.3 implies that for $q \in (q_\varphi^-, q_\varphi^+)$ and any measure $\mu \in \mathcal{M}$ with $h(\mu) + \int \varphi_q \, \mathrm{d}\mu$ close to 0 must have strictly positive entropy.

The rest of the proof follows as in [22, Sect. 5]. The steps are as follows:

Approximation of the pressure with compatible measures. The first step in the proof is to construct an inducing scheme, such that there exists a sequence of measures that approximate the pressure and are all compatible with it. More precisely:

Proposition 6.2. *Suppose that $f \in \mathcal{F}_g$ and $\varphi \in \mathcal{P}_H$. Let $q \in (q_\varphi^-, q_\varphi^+)$, then there exists an inducing scheme (X, F) and a sequence of measures $(\mu_n)_n \subset \mathcal{M}$ all compatible with (X, F) such that*

$$h(\mu_n) - T(q) \int \log |Df| \; d\mu_n + q \int \varphi \; d\mu_n \to 0 \textit{ and } \inf_n h(\mu_n) > 0.$$

Moreover, if Φ_q denotes the induced potential of φ_q then $P(\Phi_q) = 0$.

The proof of this results follows from two observations: the first is that by definition there exist $\varepsilon, \delta > 0$ such that any measure μ with

$$\left| h(\mu) + \int \varphi_q \, \mathrm{d}\mu \right| < \delta$$

is such that $h(\mu) > \varepsilon$. The other result used in the proof is that, given $\varepsilon > 0$ there exists a finite number of inducing schemes, such that any ergodic measure with $h(\mu) > \varepsilon$ is compatible with one of these schemes and has integrable return time (this was first proved in [10, Remark 6]; see also [22, Lemma 4.1]). Combining the previous two observations we obtain that $P(\Phi_q) \geq 0$. The fact that $P(\Phi_q) \leq 0$ follows from an approximation argument (see [22, Lemma 3.1]). We note here that the potential Φ_q has summable variations by combining [9, Lemma 4] and [10, Lemma 8]; see the Appendix for further details.

Since the inducing system (X, F) can be coded by a full-shift on a countable alphabet, as in Sect. 3.3 we have a Gibbs measure μ_{Φ_q} corresponding to Φ_q.

The Gibbs measure has integrable inducing time. The next step is to show that the inducing time is integrable with respect to the Gibbs measure μ_{Φ_q}. This follows as in [22, Proposition 5.2].

Uniqueness of the equilibrium measure. This follows as in [22, Proposition 6.1]. □

A detailed study of the temperature function will allow us to describe the multifractal spectrum. In order to study the regularity properties of the function $T_\varphi(q)$, we need to understand the thermodynamic formalism for the potential φ_q.

Theorem 6.2. *Suppose that $f \in \mathcal{F}_g$ and $\varphi \in \mathcal{P}_H$. If $q \in (q_\varphi^-, q_\varphi^+)$ then*

a) *the temperature function, $q \mapsto T_\varphi(q)$ is differentiable;*
b) $DT_\varphi(q) = \frac{\int \varphi \; d\mu_{\varphi_q}}{\int \log |Df| \; d\mu_{\varphi_q}}$;
c) $T_\varphi(q) = \dim_H(\mu_{\varphi_q}) + qDT_\varphi(q)$;
d) *T_φ is convex;*
e) *if $f \in \mathcal{F}_g$ and $\mu_{ac} \neq \mu_\varphi$ then T_φ is strictly convex;*
f) *T_φ is linear in $(-\infty, q_\varphi^-)$ and (q_φ^+, ∞);*
g) *T_φ is C^1 at q_φ^+.*

Proof. Part a). It is a consequence of Theorem 6.1 and [22, Proposition 8.1] that given $q \in (q_\varphi^-, q_\varphi^+)$ there exists $\epsilon > 0$ such that if $t \in (T_\varphi(q) - \epsilon, T_\varphi(q) + \epsilon)$ the pressure function

$$(t, q) \mapsto P(t, q) = P(-t \log |Df| + q\varphi)$$

is differentiable in each variable. Therefore, by the implicit function theorem we obtain that $T_\varphi(q)$ is differentiable.

Part b). This has been proved in several settings (see [35, Proposition 21.2]). Consider the pressure function on two variables

$$(t, q) \to P(t, q) = P(-t \log |Df| + q\varphi).$$

There exists $\epsilon > 0$ such that $P(t, q)$ is differentiable on each variable in the range $t \in (T_\varphi(q) - \epsilon, T_\varphi(q) + \epsilon)$ and $q \in \mathbb{R}$ satisfying the hypothesis of the theorem. As in for example [40, Chapter 8], [36, Section II] or [35, Chapter 7, p.211],

$$DT_\varphi(q) = \left.\frac{\partial P(q,t)}{\partial t}\right|_{t=T_\varphi(q)} \left(\left.\frac{\partial P(q,t)}{\partial q}\right|_{t=T_\varphi(q)} \right)^{-1}.$$

Furthermore, formulas for the derivative of the pressure (recall that it is differentiable in this range) give

$$DT_\varphi(q) = \frac{\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\int \log |Df| \, \mathrm{d}\mu_{\varphi_q}}$$

as required. Note that in the above references the analogues of the remaining parts of the proof of this theorem would be proved using higher derivatives of T_φ. However, we do not have information on these; hence, we have to use other methods in the rest of this proof to get convexity, etc.

Part c). Using b) and [17],

$$T_\varphi(q) = \frac{h(\mu_{\varphi_q})}{\int \log|Df| \, \mathrm{d}\mu_{\varphi_q}} + q\frac{\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\log|Df|\mathrm{d}\mu_{\varphi_q}} = \dim_H(\mu_{\varphi_q}) + q\frac{\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\log|Df|\mathrm{d}\mu_{\varphi_q}}$$
$$= \dim_H(\mu_{\varphi_q}) + qDT_\varphi(q).$$

Part d). Given $q \in (q_\varphi^-, q_\varphi^+)$ there is an equilibrium state μ_{φ_q} for φ_q. We can write

$$T_\varphi(q) = \frac{h(\mu_{\varphi_q}) + q\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\lambda(\mu_{\varphi_q})}.$$

By the definitions of T_φ and pressure, for $\kappa \in \mathbb{R}$,

$$\begin{aligned} T_\varphi(q+\kappa) &\geq \frac{h(\mu_{\varphi_q}) + (q+\kappa)\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\lambda(\mu_{\varphi_q})} \\ &= T_\varphi(q) + \frac{\kappa\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\lambda(\mu_{\varphi_q})} = T_\varphi(q) + \kappa DT_\varphi(q). \end{aligned}$$

Whence T_φ is convex in $(q_\varphi^-, q_\varphi^+)$.

Part e). To show strict convexity, we use an improved version of the argument in [47, Lemma 6]. There it is shown that if the graph of T_φ is not strictly convex then it must be affine. Similarly, in this case suppose that DT_φ has slope $-\gamma$ in the interval $[q_1, q_2] \subset [q_\varphi^-, q_\varphi^+]$. It can be derived from the above computations that the equilibrium state for φ_q is the same for all $q \in [q_1, q_2]$ (see also, for example, the proof of [47, Lemma 6]).

We will show that if T_φ is not strictly convex then $\gamma = 1$ and μ_φ is equivalent to the acip. Let (X, F) be an inducing scheme as in Theorem 3.3 to which μ_{φ_q} is compatible. By the Gibbs property of μ_{Φ_q} for $q, q+\delta \in [q_1, q_2]$, and for '$\asymp_{\text{dis}}$' meaning 'equal up to a distortion constant' we must have

$$|X_i|^{T_\varphi(q)} e^{q\Phi_i} \asymp_{\text{dis}} |X_i|^{T_\varphi(q+\delta)} e^{(q+\delta)\Phi_i} = |X_i|^{T_\varphi(q)-\delta\gamma} e^{(q+\delta)\Phi_i}$$

where $\Phi_i := \sup_{x\in X_i} \Phi(x)$. This implies $|X_i|^\gamma \asymp_{\text{dis}} e^{\Phi_i}$. We can extend this argument from 1-cylinders to any k-cylinder. This implies that we have a Gibbs measure $\mu_{\Phi/\gamma}$ for the potential Φ/γ, and indeed that $\mu_{-\log|DF|} \equiv \mu_{\Phi/\gamma}$. This also shows that $P(\Phi/\gamma) = 0$. Since f has an acip, $\int \tau \, \mathrm{d}\mu_{\Phi/\gamma} < \infty$ and $\mu_{\Phi/\gamma}$ projects to a measure $\mu_{\varphi/\gamma}$. By Theorem 3.3(b), $\mu_{\varphi/\gamma}$ must be an equilibrium state for φ/γ as well as for $-\log|Df|$, i.e. $\mu_{\varphi/\gamma} = \mu_{ac}$. Moreover, $P(\varphi/\gamma) = 0$. Since $\varphi < 0$,

$$\gamma > 1 \text{ implies } P(\varphi) < P(\varphi/\gamma) \quad \text{and} \quad \gamma < 1 \text{ implies } P(\varphi) > P(\varphi/\gamma).$$

Since $P(\varphi) = P(\varphi/\gamma) = 0$, we must have $\gamma = 1$, so $\mu_\varphi = \mu_{ac}$ contradicting our assumption.

Part f). We may assume that $q_\varphi^- < 0$. Since the entropy of measures around q_φ^- is vanishingly small, we must have

$$T_\varphi(q) = \lim_{q\searrow q_\varphi^-} \frac{q_\varphi^- \int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\lambda(\mu_{\varphi_q})}.$$

If T_φ was not linear in $(-\infty, q_\varphi^-)$, we must have measures μ with $\frac{\int \varphi \, d\mu}{\lambda(\mu)} > \frac{\int \varphi \, d\mu_{\varphi_q}}{\lambda(\mu_{\varphi_q})}$. This contradicts the definition of the value of $T_\varphi(q_\varphi^-)$. A similar argument follows for q_φ^+.

Part g). For $q \geq q_\varphi^+$, we have $T_\varphi(q) < 0$ and so φ_q is upper semicontinuous and there is an equilibrium state for φ_q. Using part f) we can show any equilibrium state μ_{φ^+} for φ_q for $q \geq q_\varphi^+$ is an equilibrium state for φ_q for any other $q \geq q_\varphi^+$. Since $-T_\varphi(q)\lambda(\mu_{\varphi^-}) + q\varphi = 0$, and $\varphi < 0$, we have $\lambda(\mu_{\varphi^-}) > 0$. If T_φ was not C^1 at q_φ^- then we could take a limit μ of the measures μ_{φ_q} where $q \to q_\varphi^-$. As in the proof of [22, Theorem B], μ must be an equilibrium state for $\varphi_{q_\varphi^-}$ with $\lambda(\mu) > 0$, and not equal to μ_{φ^-}. As in [22, Proposition 6.1], there can be at most one equilibrium state for $\varphi_{q_\varphi^-}$ of positive Lyapunov exponent. Hence T_φ is C^1 at q_φ^-, as required. □

7. Multifractal Spectrum of Pointwise Dimension

In this section, we prove that the dimension spectrum of pointwise dimension $\tilde{\mathfrak{D}}_\mu$ is the Legendre-Fenchel transform of the temperature function T_φ. The following is a slightly embellished version of Theorem B.

Theorem 7.1. *Suppose that $f \in \mathcal{F}_g$ and $\varphi \in \mathcal{P}_H$. If $\mu_\varphi \neq \mu_{ac}$ then the dimension spectrum satisfies the following equations*

$$\mathfrak{D}_{\mu_\varphi}(\alpha) = \inf_{q \in \mathbb{R}} (T_\varphi(q) + q\alpha)$$

for all $\alpha \in (-DT_\varphi(q_\varphi^+), -D^+T_\varphi(q_\varphi^-))$. Or equivalently,

$$\mathfrak{D}_{\mu_\varphi}(-DT_\varphi(q)) = T_\varphi(q) - qDT_\varphi(q)$$

for $q \in (q_\varphi^-, q_\varphi^+]$.

Remark 7.1. As in Remark 4.3, we expect that $\mathfrak{D}_\mu(\alpha) = \inf_{q \in \mathbb{R}} (T_\varphi(q) + q\alpha)$ for $\alpha \in [-D^+T_\varphi(q_\varphi^-), -D^-T_\varphi(q_\varphi^-)]$. Similarly, if $q \notin [q_\varphi^-, q_\varphi^+]$ then any equilibrium state μ for φ_q must have $h(\mu) = 0$. In this case $\dim_H(\mu) = 0$. This suggests that $\mathfrak{D}_\mu(\alpha) = 0$ for $\alpha \notin (-DT_\varphi(q_\varphi^+), -D^-T_\varphi(q_\varphi^-)]$.

Proof of Theorem B. We begin by getting information for both the upper and lower bounds from our inducing schemes.

As in Lemma 4.2, for an inducing scheme (X^n, F_n), for $x \in (X^n, F_n)^\infty$, we can define $\delta_{\mu_{F_n,\Phi}}(x)$ and show that for $x \in K_{\Phi_n}(\alpha)$, we have $\delta_{\mu_{F_n,\Phi}}(x) = d_{\mu_\Phi}(x)$. By Propositions B.2 and B.1, for any $y \in (X^n, F_n)^\infty$, there is some $k \geq 0$ such that $x := f^k(y) \in X^n$ and $d_{\mu_\Phi}(x) = d_{\mu_\varphi}(x) = d_{\mu_\varphi}(y)$. Hence, for $x \in K_\varphi(\alpha) \cap (X^n, F_n)^\infty$, we have

$$d_{\mu_\Phi}(x) = d_{\mu_\varphi}(x) = \delta_{\mu_{F_n,\Phi}}(x).$$

We first prove the lower bound on $\mathfrak{D}_{\mu_\varphi}$. The above argument along with that of Lemma 4.3 implies that for μ_{φ_q}-a.e. $x \in X$ and any inducing scheme with $x \in (X,F)^\infty$,

$$d_{\mu_{\varphi_q}}(x) = d_{\mu_{\Phi_q}}(x) = \frac{-\int \Phi \, \mathrm{d}\mu_{\Phi_q}}{\lambda(\mu_{\Phi_q})} = \frac{-\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\lambda(\mu_{\varphi_q})}.$$

Therefore,

$$\mu_{\varphi_q}\left(I \Big\backslash K_\varphi\left(\frac{-\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\int \log|Df| \mathrm{d}\mu_{\varphi_q}}\right)\right) = 0.$$

Hence by Theorem 6.2 c),

$$\mathfrak{D}_{\mu_\varphi}\left(K_\varphi\left(\frac{-\int \varphi \, \mathrm{d}\mu_{\varphi_q}}{\int \log|Df| \mathrm{d}\mu_{\varphi_q}}\right)\right) \geq \dim_H(\mu_{\varphi_q}) = T_\varphi(q) - qDT_\varphi(q).$$

By Theorem 6.2 b) we obtain the lower bounds for $\mathfrak{D}_{\mu_\varphi}$ for any α in the range of the derivative of T_φ.

Similarly for the upper bound, as in Lemma 4.2 we obtain

$$\dim_H(K_\varphi(\alpha)) \leq \max\left\{\sup_n\{\dim_H(K_{\Phi_n}(\alpha))\}, \dim_H\left(I \backslash \left(\cup_n (X^n, F_n)^\infty\right)\right)\right\}.$$

By [19], $\dim_H(K_{\Phi_n}(\alpha)) = T_{\Phi_n}(q) - qDT_{\Phi_n}(q)$. By the final part of Proposition 6.2, $T_{\Phi_n}(\alpha) = T_\varphi(\alpha)$ for $q \in (q_\varphi^-, q_\varphi^+)$. Since $\dim_H\left(I \backslash \left(\cup_n (X^n, F_n)^\infty\right)\right) = 0$, we thus obtain $\dim_H(K_\varphi(\alpha)) \leq T_\varphi(\alpha) - qDT_\varphi$ as required. □

The following result is a consequence of the Legendre-Fenchel relation between the temperature function and the dimension spectrum. Let us stress that there is strong contrast between the behaviour of the dimension spectrum described in Theorem 7.2 and the dimension spectrum for equilibrium states in hyperbolic systems (see for example [35, Chapter 7]). The lack of hyperbolicity of the map f is reflected in the regularity properties of the spectrum.

Theorem 7.2. *Suppose that $f \in \mathcal{F}_g, \varphi \in \mathcal{P}_H$ and $\mu_{ac} \neq \mu_\varphi$. Assume that the temperature function is such that*

$$T_\varphi(q) = \begin{cases} \text{infinite} & \text{if } q < 0, \\ \text{finite} & \text{if } q \geq 0. \end{cases}$$

Then the domain of $\mathfrak{D}_{\mu_\varphi}$ is unbounded. Moreover, $D^+T_\varphi(0) = \frac{\int \varphi \, d\mu_{ac}}{\lambda(\mu_{ac})}$ and for every $\alpha \geq -D^+T_\varphi(0)$ we have that $\mathfrak{D}_{\mu_\varphi}(\alpha) = T_\varphi(0) = 1$.

Proof. The usual derivative formulas imply that if there exists a measure μ_{φ_0} for the potential φ_0 then $D^+T_\varphi(0) = \frac{\int \varphi \, \mathrm{d}\mu_{\varphi_0}}{\lambda(\mu_{\varphi_0})}$. Since $\varphi_0 := -\log|Df|$, as in [28], $\mu_{\varphi_0} = \mu_{ac}$ the acip. The fact that $\mathfrak{D}_{\mu_\varphi}(\alpha) = T_\varphi(0)$ for $\alpha \geq -D^+T_\varphi(0)$ follows as in Lemma 4.2. □

We finish this section by giving a proposition which gives further information on the condition $\mu_\varphi \neq \mu_{ac}$ imposed in the above theorems. One way that μ_φ can be equal to μ_{ac} is if φ is *cohomologous* to $-\log|Df|$, i.e. if there exists a solution $\psi : I \to \mathbb{R}$ to the equation

$$\varphi = -\log|Df| + \psi \circ f - \psi. \tag{7}$$

It is unknown if this is the only way that μ_φ can be equal to μ_{ac}. The study of such equations, and their smoothness is part of Livšic theory, studied for interval maps with critical points in [6].

Let $\mathcal{F}_g' \supset \mathcal{F}_g$ be the class of maps as above, but allowing preperiodic critical points. The following result is proved using ideas from [6].

Proposition 7.1. *Let $f \in \mathcal{F}_g'$ be a unimodal map. If $\varphi : I \mapsto \mathbb{R}$ is a Hölder function then the only way (7) can have a solution is if the critical point is preperiodic.*

Proof. Theorem 6 of [6] holds for $f \in \mathcal{F}_g'$. Therefore, the potential $\varphi' := \varphi + \log|Df|$ satisfies the conditions in [6, Theorem 6]: in particular, it satisfies condition (2) of [6, Section 3.1] for example. By that theorem, any solution ψ to the equation $\varphi' = \psi \circ f - \psi$ must be Hölder continuous. Letting c be the critical point, we may assume that $f(c)$ is a maximum for f. As in [6, Corollary 3], ψ must be bounded on any interval compactly contained in $[f^2(c), f(c)]$. But by construction, ψ must be unbounded on any element of $\cup_{n\geq 1} f^n(c)$. In the case of transitive unimodal maps, this can only occur when $f^2(c) = 0$ and 0 is a fixed point. □

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Appendix A. The Proof of Theorem 3.3

In this section, we explain the essential parts the proof of Theorem 3.3, which can otherwise be found in [47]. The theorem is the same as [22, Theorem 6.1] except for the inclusion of equilibrium states for Hölder continuous potentials in the second part and, most importantly, the final assertion. The generalisation of the second part of [22, Theorem 6.1] for such equilibrium states follows immediately from the arguments in [22]. To prove the final assertion, we will need to revisit the method of producing the inducing schemes. This involves the Hofbauer extension, sometimes also known as a Hofbauer tower, whose construction we give below.

We first consider the dynamically defined cylinders. We let $\mathcal{P}_0 := I$ and $\mathcal{P}_n$ denote the collection of maximal intervals $\mathbf{C}_n$ so that $f^n : \mathbf{C}_n \to f^n(\mathbf{C}_n)$ is a homeomorphism. We let $\mathbf{C}_n[x]$ denote the member of $\mathcal{P}_n$ containing x. If $x \in \cup_{n\geq 0} f^{-n}(\mathcal{C}r)$ there may be more than one such interval, but this ambiguity will not cause us any problems here.

The *Hofbauer extension* is defined as

$$\hat{I} := \bigsqcup_{k \geq 0} \bigsqcup_{\mathbf{C}_k \in \mathcal{P}_k} f^k(\mathbf{C}_k)/\sim$$

where $f^k(\mathbf{C}_k) \sim f^{k'}(\mathbf{C}_{k'})$ as components of the disjoint union $\hat{I}$ if $f^k(\mathbf{C}_k) = f^{k'}(\mathbf{C}_{k'})$ as subsets in I. Let $\mathcal{D}$ be the collection of domains of $\hat{I}$ and $\pi : \hat{I} \to I$ be the natural inclusion map. A point $\hat{x} \in \hat{I}$ can be represented by (x, D) where $\hat{x} \in D$ for $D \in \mathcal{D}$ and $x = \pi(\hat{x})$. Given $\hat{x} \in \hat{I}$, we can denote the domain $D \in \mathcal{D}$ it belongs to by $D_{\hat{x}}$.

The map $\hat{f} : \hat{I} \to \hat{I}$ is defined by

$$\hat{f}(\hat{x}) = \hat{f}(x, D) = (f(x), D')$$

if there are cylinder sets $\mathbf{C}_k \supset \mathbf{C}_{k+1}$ such that $x \in f^k(\mathbf{C}_{k+1}) \subset f^k(\mathbf{C}_k) = D$ and $D' = f^{k+1}(\mathbf{C}_{k+1})$. In this case, we write $D \to D'$, giving $(\mathcal{D}, \to)$ the structure of a directed graph. Therefore, the map π acts as a semiconjugacy between $\hat{f}$ and f:

$$\pi \circ \hat{f} = f \circ \pi.$$

We denote the 'base' of $\hat{I}$, the copy of I in $\hat{I}$ by D_0. For $D \in \mathcal{D}$, we define $\mathrm{lev}(D)$ to be the length of the shortest path $D_0 \to \cdots \to D$ starting at the base D_0. For each $R \in \mathbb{N}$, let $\hat{I}_R$ be the compact part of the Hofbauer extension defined by the disjoint union

$$\hat{I}_R := \sqcup\{D \in \mathcal{D} : \mathrm{lev}(D) \leq R\}.$$

For maps in $\mathcal{F}$, we can say more about the graph structure of $(\mathcal{D}, \to)$ since Lemma 1 of [10] implies that if $f \in \mathcal{F}$ then there is a closed primitive subgraph $\mathcal{D}_{\mathcal{T}}$ of $\mathcal{D}$, i.e. for any $D, D' \in \mathcal{D}_{\mathcal{T}}$ there is a path $D \to \cdots \to D'$; and for any $D \in \mathcal{D}_{\mathcal{T}}$, if there is a path $D \to D'$ then $D' \in \mathcal{D}_{\mathcal{T}}$ too. We can denote the disjoint union of these domains by $\hat{I}_{\mathcal{T}}$. The same lemma says that if $f \in \mathcal{F}$ then $\pi(\hat{I}_{\mathcal{T}}) = \Omega$ and $\hat{f}$ is transitive on $\hat{I}_{\mathcal{T}}$.

Given $\mu \in \mathcal{M}_{\mathrm{erg}}$, we say that μ *lifts to* $\hat{I}$ if there exists an ergodic $\hat{f}$-invariant probability measure $\hat{\mu}$ on $\hat{I}$ such that $\hat{\mu} \circ \pi^{-1} = \mu$. For $f \in \mathcal{F}$, if $\mu \in \mathcal{M}_{\mathrm{erg}}$ and $\lambda(\mu) > 0$ then μ lifts to $\hat{I}$; see [7,25].

We let $\iota := \pi|_{D_0}^{-1}$. Note that there is a natural distance function $d_{\hat{I}}$ within domains $D \in \mathcal{D}$ (but not between them) induced from the Euclidean metric on I.

We obtain our inducing scheme as a first return map in the Hofbauer extension, i.e. we choose $\hat{X} \subset \hat{I}_{\mathcal{T}}$ and use a first return map to $\hat{X}$ to give the inducing scheme on $X := \pi(\hat{X})$. We will always choose X to be a cylinder in $\mathcal{P}_n$, for various values of $n \in \mathbb{N}$.

The set $\hat{X}$ is an interval in a single domain $D \in \mathcal{D}_{\mathcal{T}}$. Then for $x \in X$ there exists a unique $\hat{x} \in \hat{X}$ so that $\pi(\hat{x}) = x$. Then $\tau(x)$ is defined as the first return time of $\hat{x}$ to $\hat{X}$. We choose $\hat{X}$ so that $X \in \mathcal{P}_n$ for some n, and $\hat{X}$ is compactly contained in D. These properties mean that (X, F, τ) is an inducing scheme which is *extendible*. That is to say, letting $X' = \pi(D)$, for any domain X_i of (X,F) there is an extension of f^{τ_i} to $X_i' \supset X_i$ so that $f^{\tau_i} : X_i' \to X'$ is a homeomorphism. By [46, Theorem C(2)], this means that (X,F) has uniformly bounded distortion, with distortion constant depending on $\delta := d_{\hat{I}}(\hat{X}, \partial D)$. In this way, we can cover $\hat{I} \backslash \partial \hat{I}$ with a countable number of sets $\hat{X}$. Since any ergodic measure with positive Lyapunov exponent lifts to $\hat{I}$, this means that there is some $\hat{X}$ chosen in this way for which $\hat{\mu}(\hat{X}) > 0$ and so μ is compatible with the corresponding inducing scheme. The fact that our equilibrium states are compatible to all inducing schemes follows from [22, Section 6].

The main difference between Theorem 3.3 and Theorem 6.1 of [22] is that we are able to show that the following type of points are in $\cup_n (X^n, F_n)^\infty$.

Definition A.1. Let $f \in \mathcal{F}$ and $\varepsilon > 0$. We say that $x \in I$ *goes to ε-large scale* at time n if the interval $(f^n(x) - \varepsilon, f^n(x) + \varepsilon)$ can be pulled back diffeomorphically by the branch of f^{-n} corresponding to the orbit of x. We say that *x goes to large scale infinitely often* if there exists $\varepsilon > 0$ such that x goes to ε-large scale for infinitely many times $n \in \mathbb{N}$.

The argument of [25, Theorem 5] implies that for any $\varepsilon > 0$ there exists $R \in \mathbb{N}$ such that if x goes to ε-large scale infinitely often then ιx maps into $\hat{I}_R$ by $\hat{f}$ infinitely often. Therefore, such an x is contained in $\cup_{n=1}^\infty (X^n, F_n)^\infty$: indeed there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\iota x \in \cup_{n=1}^N (X^n, F_n)^\infty$. The final part of the argument for c) is provided by Rivera-Letelier and Shen [41, Corollary 6.3]. This implies that for a map in $\mathcal{F}_g$ for all $\eta > 0$ there exists $\varepsilon > 0$ such that the set of points which do not go to ε-large scale infinitely often has Hausdorff dimension less than η. Note that c) also follows for any $f \in \mathcal{F}$ which is ‘backward contracting’; see [41] for the definition (in [8] it is shown that all $f \in \mathcal{F}_g$ satisfy this condition).

Appendix B. The Dimension of Induced Measures

In this section, we give the relation between pointwise dimension of a measure and its induced version, which can also be found in [47]. As in [9, Lemma 4], if φ is Hölder then the induced potential Φ for any of our inducing schemes has summable variations. Note that there was an error in that proof, pointed out to the authors by W. Shen and J. Rivera-Letelier. The proof is corrected by using [41, Theorem A].

The following proposition is proved in [24]. A more general version is proved in [47]. Note that the uniqueness of the measures here is shown in [22, Section 6]; see also for example [13, Theorem 8].

Proposition B.1. *Given $f \in \mathcal{F}$ and a Hölder potential $\varphi \in \mathcal{P}$, then there exists an equilibrium state μ_φ and a φ-conformal measure m_φ and $C_\varphi > 0$ so that $\frac{d\mu_\varphi}{dm_\varphi}$ is uniformly bounded away from 0 and ∞.*

Notice that this implies that $d_{m_\varphi} = d_{\mu_\varphi}$ and, by the conformality of $m_\varphi, d_{\mu_\varphi}(x) = d_{\mu_\varphi}(f^n(x))$ for all $n \in \mathbb{N}$.

Proposition B.2. *Suppose that $f \in \mathcal{F}$ satisfies (1) and $\varphi \in \mathcal{P}$ is a Hölder potential. For any inducing scheme (X, F) as in Theorem 3.3 with induced potential $\Phi : X \to \mathbb{R}$, for the equilibrium states μ_φ for (I, f, φ) and μ_Φ for (X, F, Φ), there exists $C_\Phi > 0$ so that*

$$\frac{1}{C_\Phi} \leq \frac{d\mu_\Phi}{d\mu_\varphi} \leq C_\Phi.$$

Proof of Proposition B.2. Suppose that (X, F) is an inducing scheme as in the statement, with induced potential Φ. Since m_φ is φ-conformal, the measure $\frac{m_\varphi|_X}{m_\varphi(X)}$ is Φ-conformal. Since the Φ-conformal measure is unique, $m_\Phi = \frac{m_\varphi|_X}{m_\varphi(X)}$. Since by Proposition B.1, $\frac{\mathrm{d}\mu_\varphi}{\mathrm{d}m_\varphi}$ is bounded above and below, and since $\frac{\mathrm{d}\mu_\Phi}{\mathrm{d}m_\Phi}$ is uniformly bounded above and below, this implies that $\frac{\mathrm{d}\mu_\Phi}{\mathrm{d}\mu_\varphi}$ is also uniformly bounded above and below. □

We use the above proposition in the proof of Theorem B to show that $d_{\mu_\varphi}(x) = d_{\mu_\Phi}(x)$ for a full Hausdorff dimension set of points in $K_\varphi(\alpha)$.

References

[1] Avila, A., Lyubich, M.: Hausdorff dimension and conformal measures of Feigenbaum Julia sets. J. Am. Math. Soc. **21**, 305–363 (2008)

[2] Barreira, L.: Dimension and Recurrence in Hyperbolic Dynamics, Progress in Mathematics, vol. 272. Birkhäuser Verlag, Basel (2008)

[3] Barreira, L., Iommi, G.: Phase transitions and multifractal analysis for parabolic horseshoes. Israel J. Math. (to appear)

[4] Barreira, L., Pesin, Y., Schmeling, J.: On a general concept of multifractality: multifractal spectra for dimensions, entropies, and Lyapunov exponents, multifractal rigidity. Chaos **7**, 27–38 (1997)

[5] Barreira, L., Schmeling, J.: Sets of "non-typical" points have full topological entropy and full Hausdorff dimension. Israel J. Math. **116**, 29–70 (2000)

[6] Bruin, H., Holland, M., Nicol, M.: Livšic regularity for Markov systems. Ergod. Theory Dyn. Syst. **25**, 1739–1765 (2005)

[7] Bruin, H., Keller, G.: Equilibrium states for S-unimodal maps. Ergod. Theory Dyn. Syst. **18**, 765–789 (1998)

[8] Bruin, H., Rivera-Letelier, J., Shen, W., van Strien, S.: Large derivatives, backward contraction and invariant densities for interval maps. Invent. Math. **172**, 509–593 (2008)

[9] Bruin, H., Todd, M.: Equilibrium states for potentials with $\sup \varphi - \inf \varphi < h_{top}(f)$. Commun. Math. Phys. **283**, 579–611 (2008)

[10] Bruin, H., Todd, M.: Equilibrium states for interval maps: the potential $-t \log |Df|$. Ann. Sci. École Norm. Sup. (4) **42**, 559–600 (2009)

[11] Cedervall, S.: Invariant measures and decay of correlations for S-multimodal interval maps. Thesis, Imperial College, London (2006)

[12] Denker, M., Urbański, M.: On the existence of conformal measures. Trans. Am. Math. Soc. **328**, 563–587 (1991)

[13] Dobbs, N.: Measures with positive Lyapunov exponent and conformal measures in rational dynamics. Trans. Am. Math. Soc. (arXiv:0804.3753) (to appear)

[14] Gelfert, K., Przytycki, F., Rams, M.: Lyapunov spectrum for rational maps. Math. Ann. **348**, 965–1004 (2010)

[15] Gelfert, K., Rams, M.: The Lyapunov spectrum of some parabolic systems. Ergod. Theory Dyn. Syst. **29**, 919–940 (2009)

[16] Hanus, P., Mauldin, R.D., Urbański, M.: Thermodynamic formalism and mutifractal analysis of conformal infinite iterated function systems. Acta Math. Hung. **96**, 27–98 (2002)

[17] Hofbauer, F.: Local dimension for piecewise monotonic maps on the interval. Ergod. Theory Dyn. Syst. **15**, 1119–1142 (1995)

[18] Hofbauer, F., Raith, P., Steinberger, T.: Multifractal dimensions for invariant subsets of piecewise monotonic interval maps. Fund. Math. **176**, 209–232 (2003)

[19] Iommi, G.: Multifractal analysis for countable Markov shifts. Ergod. Theory Dyn. Syst. **25**, 1881–1907 (2005)

[20] Iommi, G.: Multifractal analysis of Lyapunov exponent for the backward continued fraction map. Ergod. Theory Dyn. Syst. **30**, 211–232 (2010)

[21] Iommi, G., Kiwi, J.: The Lyapunov spectrum is not always concave. J. Stat. Phys. **135**, 535–546 (2009)

[22] Iommi, G., Todd, M.: Natural equilibrium states for multimodal maps. Commun. Math. Phys. **300**, 65–94 (2010)

[23] Jordan, T., Rams, M.: Multifractal analysis of weak Gibbs measures for non-uniformly expanding C^1 maps. Ergod. Theory Dyn. Syst. **31**, 143–164 (2011)

[24] Keller, G.: Generalized bounded variation and applications to piecewise monotonic transformations. Z. Wahrsch. Verw. Gebiete **69**, 461–478 (1985)

[25] Keller, G.: Lifting measures to Markov extensions. Monatsh. Math. **108**, 83–200 (1989)

[26] Keller, G.: Equilibrium States in Ergodic Theory. London Mathematical Society Student Texts, vol. 42. Cambridge University Press, Cambridge (1998)

[27] Kesseböhmer, M., Stratmann, B.: A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates. J. Für Reine Angewandte Math. **605**, 133–163 (2007)

[28] Ledrappier, F.: Some properties of absolutely continuous invariant measures on an interval. Ergod. Theory Dyn. Syst. **1**, 77–93 (1981)

[29] Mauldin, R., Urbański, M.: Dimensions and measures in infinite iterated function systems. Proc. Lond. Math. Soc. (3) **73**(1), 105–154 (1996)

[30] Mauldin, R., Urbański, M.: Gibbs states on the symbolic space over an infinite alphabet. Israel J. Math. **125**, 93–130 (2001)

[31] de Melo, W., van Strien, S.: One Dimensional Dynamics, Ergebnisse Series, vol. 25. Springer, Berlin (1993)

[32] Nakaishi, K.: Multifractal formalism for some parabolic maps. Ergod. Theory Dyn. Syst. **24**, 843–857 (2000)

[33] Nowicki, T., Sands, D.: Non-uniform hyperbolicity and universal bounds for S-unimodal maps. Invent. Math. **132**, 633–680 (1998)

[34] Olsen, L.: A multifractal formalism. Adv. Math. **116**, 82–196 (1995)

[35] Pesin, Y.: Dimension Theory in Dynamical Systems. CUP (1997)

[36] Pesin, Y., Weiss, H.: A multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions. J. Stat. Phys. **86**, 233–275 (1997)

[37] Pollicott, M., Weiss, H.: Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation. Commun. Math. Phys. **207**, 145–171 (1999)

[38] Przytycki, F.: Lyapunov characteristic exponents are nonnegative. Proc. Am. Math. Soc. **119**, 309–317 (1993)

[39] Przytycki, F., Rivera-Letelier, J., Smirnov, S.: Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps. Invent. Math. **151**, 29–63 (2003)

[40] Przytycki, F., Urbański, M.: Fractals in the Plane, Ergodic Theory Methods. Cambridge University Press, Cambridge (2010)

[41] Rivera-Letelier, J., Shen, W.: Statistical properties of one-dimensional maps under weak hyperbolicity assumptions. Preprint, arXiv:1004.0230

[42] Sarig, O.: Thermodynamic formalism for countable Markov shifts. Ergod. Theory Dyn. Syst. **19**, 1565–1593 (1999)

[43] Sarig, O.: Phase transitions for countable Markov shifts. Commun. Math. Phys. **217**, 555–577 (2001)

[44] Sarig, O.: Existence of Gibbs measures for countable Markov shifts. Proc. Am. Math. Soc. **131**, 1751–1758 (2003)

[45] Schmeling, J.: On the completeness of multifractal spectra. Ergod. Theory Dyn. Syst. **19**, 1595–1616 (1999)

[46] van Strien, S., Vargas, E.: Real bounds, ergodicity and negative Schwarzian for multimodal maps. J. Am. Math. Soc. **17**, 749–782 (2004)

[47] Todd, M.: Multifractal analysis for multimodal maps. Preprint, arXiv: 0809.1074 (2008)

[48] Walters, P.: An Introduction to Ergodic Theory, Graduate Texts in Mathematics 79. Springer, Berlin (1981)

[49] Weiss, H.: The Lyapunov spectrum for conformal expanding maps and Axiom a surface diffeomorphisms. J. Stat. Phys. **95**, 615–632 (1999)

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