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The Cauchy Problem on a Characteristic Cone for the Einstein Equations in Arbitrary Dimensions

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Abstract. We derive explicit formulae for a set of constraints for the Einstein equations on a null hypersurface, in arbitrary space–time dimensions $n+1 \geq 3$. We solve these constraints and show that they provide necessary and sufficient conditions so that a spacetime solution of the Cauchy problem on a characteristic cone for the hyperbolic system of the reduced Einstein equations in wave-map gauge also satisfies the full Einstein equations. We prove a geometric uniqueness theorem for this Cauchy problem in the vacuum case.

1. Introduction

The simplest way to obtain a well-posed system for the vacuum Einstein equations is to suppose that the coordinates satisfy so-called harmonicity conditions, or, more generally, to introduce a preassigned metric $\hat{g}$, called target metric, which permits to write the Ricci tensor as the sum of two tensorial operators, one of which is a hyperbolic operator acting on g, called the reduced Ricci tensor, and the other a homogeneous first-order differential operator acting on a vector H, called wave-map gauge vector, which vanishes when the identity map is a wave map from (V, g) onto $(V, \hat{g})$. When the initial manifold M_0 is spacelike, classical theorems of analysis show existence and uniqueness of solutions of so-reduced Einstein equations. The case where the initial manifold is null has analogies with the spacelike case but also important differences: First, the induced metric is degenerate, and unconstrained in the regions where τ, the divergence of the light-cone (see (4.25) below), has no zeroes. Next, the second fundamental forms defined on a spacelike and on a null manifold, for which the normal is also tangent, have very different properties. Finally, null initial data on a light cone, or on two-intersecting null hypersurfaces, determine the solution in one time direction only, past or future.

A complete understanding of this problem is still lacking, even in space–time dimension four. The most exhaustive studies are for the case of two intersecting null surfaces [4,5,16–18,23,24,44,49,51]; compare [2,3,31]. The most complete construction of equations satisfied by initial data has been given by Damour and Schmidt [17], and the most satisfactory treatment of the local evolution by Rendall [47]. The problem with data on a characteristic cone presents new mathematical difficulties due to its singularity at the vertex, and only partial results have been obtained before in [9,20,24,27,48].

The object of this work is to present a treatment of the Einstein equations with data on a characteristic cone in all dimensions $n+1\geq 3$. We proceed as follows:

Though the equations are geometric and the final results coordinate independent, it is useful to introduce adapted coordinates to carry-out the analysis. We take a C^∞ manifold V diffeomorphic to $\mathbf{R}^{n+1}$, and we consider a cone C_O in V with vertex $O\in V$ and equation, in coordinates y^α compatible with the C^∞ structure of V,

$$y^0=r,\qquad r:=\left\{\sum_{i=1,\dots,n}(y^i)^2\right\}^{\frac{1}{2}}.$$

We consider the Cauchy problem with data on C_O for the Einstein equations with unknown a Lorentzian metric g, assuming that C_O will be a characteristic cone of the metric g and the lines $y^0=r$, $\frac{y^i}{r}=c^i$, where the c^i are constants, its null rays. It is well known[1] that the characteristic cone of a $C^{1,1}$ Lorentzian metric admits always such a representation in a neighbourhood of its vertex. We review in Sect. 3 an existence theorem which applies to the reduced Einstein equations in wave-map gauge with Minkowski target reading in these coordinates

$$\hat g=-(\mathrm{d}y^0)^2+\sum_{i=1,\dots,n}(\mathrm{d}y^i)^2. \tag{1.1}$$

We introduce in Sect. 4 what we call adapted null coordinates, singular on the line $r=0$, in particular at the vertex O of C_O, but C^∞ elsewhere, by setting

$$x^0:=r-y^0,\qquad x^1:=r,$$

and defining x^A, $A=2,\dots n$, to be local coordinates on the sphere S^{n-1}. In coordinates x^α the trace $\overline{g}$ on C_O of the metric g we are looking for has the form

$$\overline{g}=\overline{g}_{00}(\mathrm{d}x^0)^2+2\nu_0\mathrm{d}x^0\mathrm{d}x^1+2\nu_A\mathrm{d}x^0\mathrm{d}x^A+\underbrace{\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B}_{=:\tilde g}. \tag{1.2}$$

[1] This is guaranteed to hold only in a neighbourhood of the vertex, as there can be caustics.

Remark that the question, whether x^1 is an affine parameter on the null rays $x^A = c^A$, depends on derivatives transversal to C_O of the spacetime metric g, which are usually not considered as part of the initial data for characteristic Cauchy problems.

The adapted null, but singular at the vertex, coordinates x^α are used to solve "wave-map gauge constraints" satisfied by $\overline{g}$.

In Sect. 5 we review the standard argument that the Bianchi identities imply that if g satisfies the reduced Einstein equations with source a divergence-free stress energy tensor, then the vector H satisfies a homogeneous hyperbolic system; it vanishes in the future of C_O if its trace $\overline{H}$ vanishes on C_O.

We show in Sects. 6 to 11 that $\overline{H} = 0$ if the initial data $\overline{g}$ satisfy a set of $n+1$ equations which we call the wave-map gauge constraints. These constraints read as a hierarchical system of ordinary differential equations along the light rays, singular at the vertex O, if one uses the adapted null coordinates x^α. We write this complete system for a general $\hat{g}$ and generalized wave gauge, in arbitrary dimensions $n+1 \geq 3$. We integrate them successively under natural limit conditions on the unknowns at O. We study briefly in Sect. 6.2 the case when the degenerate metric $\tilde{g}$ induced on C_O (i.e. the x^1 dependent quadratic $\overline{g}_{AB}$, see (1.2)) is prescribed.

In Sect. 7, in order to have an evolutionary equation for the divergence τ we set, as many authors before us, $\overline{g}_{AB} = \Omega^2 \gamma_{AB}$, with γ an arbitrarily given x^1 dependent metric on S^{n-1}. The first wave-map gauge constraint can be written in a form which involves the two unknowns ν_0 and Ω. Its general solution is obtained by the introduction of an arbitrary function κ. We study in particular the case $\kappa = 0$ which leads to the Raychaudhuri equation for τ for which we prove global existence for a small $|\sigma|$ which depends only on the given γ. A simple integration determines then Ω; hence $\overline{g}_{AB}$ and we are back to the equations for $\nu_0, \nu_A, \overline{g}_{00}$ with given $\tilde{g}$. We remark that the equation for ν_0 (for $\kappa = 0$) implies that the vector ℓ is parallelly transported along the null ray by the connection of a spacetime metric in wave-map gauge satisfying the Einstein equations. In Sects. 8, 9, 10 and 11 we establish, and integrate, the other constraints determining ν_A and $\overline{g}_{00}$. A theorem in Sect. 12 summarizes our analysis of the wave-gauge constraint equations. A uniqueness theorem is proved in Sect. 13.

A major question left open by our work is the description of the largest class of unconstrained initial data which lead to solutions of the wave-map gauge constraints such that the components in y^α coordinates of the trace $\overline{g}$ satisfy the (non trivial) initial conditions given in Sect. 3.1 for the existence theorem for quasilinear wave equations. The problem is that the wave-gauge constraint equations determine the components of $\overline{g}$ in the x^α coordinates, and these components are linked with the components in the y^α coordinates by linear relations which are singular at the vertex. We simply note here that initial data which are Minkowskian in a neighbourhood of the vertex are easily seen to be in the class where the existence theorem holds; see also [13] for a more general family of data. We plan to return to this problem in a near future.

2. Definitions

2.1. Ricci Tensor and Harmonicity Functions

The Ricci tensor of any pseudo Riemannian metric is given in local coordinates by

$$R_{\alpha\beta} := \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda}, \quad \partial_\lambda := \frac{\partial}{\partial x^\lambda}, \tag{2.1}$$

with $\Gamma^\lambda_{\alpha\beta}$ the Christoffel symbols

$$\Gamma^\lambda_{\alpha\beta} := g^{\lambda\mu}[\mu, \alpha\beta], \quad [\mu, \alpha\beta] := \frac{1}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}). \tag{2.2}$$

The Ricci tensor satisfies the identity

$$R_{\alpha\beta} \equiv R^{(h)}_{\alpha\beta} + \frac{1}{2}(g_{\alpha\lambda}\partial_\beta \Gamma^\lambda + g_{\beta\lambda}\partial_\alpha \Gamma^\lambda), \tag{2.3}$$

where $\mathrm{Ricc}^{(h)}(g)$, the reduced Ricci tensor, is a quasi-linear, quasi-diagonal operator on the components of g,

$$R^{(h)}_{\alpha\beta} \equiv -\frac{1}{2} g^{\lambda\mu}\partial_\lambda\partial_\mu g_{\alpha\beta} + f[g, \partial g]_{\alpha\beta}, \tag{2.4}$$

and $f[g, \partial g]_{\alpha\beta}$ is a quadratic form in the first derivatives ∂g of g with coefficients polynomial in g and its controvariant associate.

The Γ^λ's, called *harmonicity functions*, are defined as

$$\Gamma^\alpha := g^{\lambda\mu}\Gamma^\alpha_{\lambda\mu}. \tag{2.5}$$

The condition $\Gamma^\alpha = 0$ expresses that the coordinate function x^α satisfies the wave equation in the metric g.

2.2. Wave-Map Gauge

The harmonicity functions are coordinate dependent and only defined locally in general, whether in space, or time, or both. The wave-map gauge, which we are about to define, provides conditions which are tensorial. A metric g on a manifold V will be said to be *in $\hat{g}$-wave-map gauge* if the identity map $V \to V$ is a harmonic diffeomorphism from the spacetime (V, g) onto the pseudo-Riemannian manifold $(V, \hat{g})$, with $\hat{g}$ a given metric on V. Recall that a mapping $f : (V, g) \to (V, \hat{g})$ is a harmonic map if it satisfies the equation, in abstract index notation,

$$\hat{\Box} f^\alpha := g^{\lambda\mu}(\partial^2_{\lambda\mu} f^\alpha - \Gamma^\sigma_{\lambda\mu}\partial_\sigma f^\alpha + \partial_\lambda f^\sigma \partial_\mu f^\rho \hat{\Gamma}^\alpha_{\sigma\rho}) = 0. \tag{2.6}$$

In a subset in which f is the identity map defined by $f^\alpha(x) = x^\alpha$, the above equation reduces to $H = 0$, where the *wave-gauge vector* H is given in arbitrary coordinates by the formula

$$H^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} - W^\lambda, \text{ with } \quad W^\lambda := g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta}, \tag{2.7}$$

where $\hat{\Gamma}^\lambda_{\alpha\beta}$ are the Christoffel symbols of the *target* metric $\hat{g}$. See [10] for a more complete discussion of the concepts and results in this section.

The following identity has been proved to hold, with $\hat{D}$ the Riemannian covariant derivative in the metric $\hat{g}$ [10, p. 163],

$$R_{\alpha\beta} \equiv R^{(H)}_{\alpha\beta} + \frac{1}{2}(g_{\alpha\lambda}\hat{D}_\beta H^\lambda + g_{\beta\lambda}\hat{D}_\alpha H^\lambda), \tag{2.8}$$

where $R^{(H)}_{\alpha\beta}(g)$, called the reduced Ricci tensor of the metric g in $\hat{g}$-wave-map gauge, is a quasi-linear, quasi-diagonal operator on g, tensor-valued, depending on $\hat{g}$:

$$R^{(H)}_{\alpha\beta} \equiv -\frac{1}{2}g^{\lambda\mu}\hat{D}_\lambda\hat{D}_\mu g_{\alpha\beta} + \hat{f}[g,\hat{D}g]_{\alpha\beta}, \tag{2.9}$$

where $\hat{f}[g,\hat{D}g]_{\alpha\beta}$, independent of the second derivatives of g, is a tensor quadratic in $\hat{D}g$ with coefficients depending upon g and $\hat{g}$, of the form (see formula (7.7) in chapter VI of [10])

$$P^{\rho\sigma\gamma\delta\lambda\mu}_{\alpha\beta}(g)\hat{D}_\rho g_{\gamma\delta}\hat{D}_\sigma g_{\lambda\mu} + \frac{1}{2}g^{\lambda\mu}\{g_{\alpha\rho}\hat{R}_\lambda{}^\rho{}_{\beta\mu} + g_{\beta\rho}\hat{R}_\lambda{}^\rho{}_{\alpha\mu}\}, \tag{2.10}$$

with $\hat{R}$ the Riemann curvature tensor of the covariant derivative $\hat{D}$. We will frequently restrict ourselves to the case in which the target metric is the Minkowski metric η and then denote by D the covariant derivative. In this case, and if using coordinates such that the Minkowski metric takes the canonical form (1.1), the reduced Ricci tensor in wave-map gauge coincides with the one in harmonic coordinates.

We emphasise that, unless explicitly stated, our computations are valid for a general $\hat{g}$.

Our main results below assume that W takes the form (2.7). However, several results apply to a large class of W's of the form[2]

$$W^\lambda := g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta} + \hat{W}^\lambda, \tag{2.11}$$

where $\hat{W}$ is a vector which may depend upon $g, \hat{g}$ and possibly some other fields, but not upon the derivatives of g; the relevant restrictions are pointed out in (7.6)–(7.7), (8.28)–(8.29) and (10.44)–(10.45). The reduced Ricci tensor becomes then

$$R^{(H,\hat{W})}_{\alpha\beta} \equiv R^{(H)}_{\alpha\beta} + \frac{1}{2}\left(g_{\alpha\lambda}\hat{D}_\beta\hat{W}^\lambda + g_{\beta\lambda}\hat{D}_\alpha\hat{W}^\lambda\right). \tag{2.12}$$

However, unless explicitly indicated otherwise we assume that $\hat{W}$ is identically zero.

Another interesting generalization (see, e.g., [45] and references therein) has been inspired by numerical simulations: if one uses the decomposition (2.8), the identities $H^\lambda \equiv 0$ are only obeyed to some finite precision and H^λ shows a generic tendency to deviate from zero. Attempts to cure that have

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[2] Friedrich [25] introduced generalized harmonic coordinates by adding arbitrary functions to the harmonicity conditions.

been made by introducing *constraint damping* terms [30], changing the choice of the reduced Ricci tensor $R^{(H)}_{\alpha\beta}$ to

$$R^{(H)}_{\alpha\beta} + \frac{1}{2}\epsilon(n_\alpha g_{\beta\lambda} + n_\beta g_{\alpha\lambda} - \frac{2\rho}{n-1} n_\lambda g_{\alpha\beta})H^\lambda, \tag{2.13}$$

or, equivalently, the reduced Einstein tensor $S^{(H)}_{\alpha\beta}$ to

$$S^{(H)}_{\alpha\beta} + \frac{1}{2}\epsilon(n_\alpha g_{\beta\lambda} + n_\beta g_{\alpha\lambda} + (\rho-1) n_\lambda g_{\alpha\beta})H^\lambda, \tag{2.14}$$

where n^μ is a vector field and ϵ is a small positive constant which controls the rate of damping of the gauge conditions. (As shown in [30] the constant ρ must also be positive to have damping.) We will show that the damping terms are consistent with our analysis. For definiteness we will assume that n^μ has been prescribed, though certain more general situations can easily be incorporated into our scheme.

3. Characteristic Cauchy Problem

The Einstein equations in wave-map gauge with source a given stress-energy tensor T,

$$R^{(H)}_{\alpha\beta} \equiv -\frac{1}{2} g^{\lambda\mu}\hat{D}_\lambda \hat{D}_\mu g_{\alpha\beta} + \hat{f}[g, \hat{D}g]_{\alpha\beta} = \rho_{\alpha\beta}, \quad \rho_{\alpha\beta} := T_{\alpha\beta} - \frac{\mathrm{tr}_g T}{n-1} g_{\alpha\beta}, \tag{3.1}$$

form a quasi-diagonal, hyperquasi-linear[3] system of wave equations for the Lorentzian metric g. The Cauchy problem for such systems with data on a spacelike n-manifold M_0 is well understood, the Cauchy data are the values of the unknown on M_0 and their first transversal derivatives. When M_0 is not spacelike in the spacetime (V, g) which we are going to construct, the problem is more delicate. It is known since Leray's work (see [37]), [4] that the Cauchy problem for a linear hyperbolic system on a given globally hyperbolic spacetime is well posed if M_0 is "compact towards the past"; that is, is intersected along a compact set by the past of any compact subset of V. However the data depend on the nature of M_0 and the formulation of a theorem requires more care. In the case where M_0 is a null hypersurface, except at some singular subsets (intersection in the case of two null hypersurfaces, vertex in the case of a null cone) the data are only the function, not its transversal derivative, with some hypotheses which need to be made as one approaches the singular set.

In this article we concentrate on the case of the light cone, though most of the calculations of our equations apply to any null hypersurface.

[3] That is, the principal second order terms are diagonal and their coefficients depend on the unknowns but not on their derivatives.

[4] See [32,43] for a treatment of generalized solutions of a linear wave equation with data on a achronal Lipschitz section of a spacetime with compact spacelike sections.

3.1. The Cagnac–Dossa Theorem

To prove the local existence of solutions of Einstein equations with data on a characteristic cone we use a wave-map gauge and an existence theorem for solutions of quasi linear wave equations with such data.

The proof of an existence theorem for such a characteristic quasilinear Cauchy problem is inspired by Leray's idea of the linear case, applied to the characteristic cone and linear wave equations in Cagnac [7] (cf. also Cagnac [6] and Friedlander [22])and extended to the quasilinear case by Cagnac [8]. The most complete results appear in Dossa's thesis, the second part of which is published in abbreviated form in [19]. One considers quasi-diagonal, quasi-linear second-order system for a set v of scalar functions $v^I, I = 1, \ldots, N$, on $\mathbf{R}^{n+1}$ of the form

$$A^{\lambda\mu}(y, v)\partial^2_{\lambda\mu}v + f(y, v, \partial v) = 0,\ y = (y^\lambda) \in \mathbf{R}^{n+1}, \quad \lambda, \mu = 0, 1, \ldots, n \geq 2, \tag{3.2}$$

$$v=(v^I),\quad \partial v=\left(\frac{\partial v^I}{\partial y^\lambda}\right),\quad \partial^2_{\lambda\mu}v=\left(\frac{\partial^2 v^I}{\partial y^\lambda \partial y^\mu}\right),\quad f=(f^I),\quad I=1,\ldots,N. \tag{3.3}$$

The initial data

$$\overline{v} := v|_{C_O} = \phi \tag{3.4}$$

are given on a subset, including its vertex O, of a characteristic cone C_O. Throughout this work a bar over an object denotes the restriction of that object to C_O.

Cagnac and Dossa assume that there is a domain $U \subset \mathbf{R}^{n+1}$ where C_O is represented by the following cone[1] in $\mathbf{R}^{n+1}$ (compare Fig. 1 below)

$$C_O := \{x^0 \equiv r - y^0 = 0\}, \quad r^2 := \sum_{i=1,\ldots,n} (y^i)^2.$$

The initial data ϕ is assumed to be defined on the domain

$$C^T_O := C_O \cap \{0 \leq t := y^0 \leq T\}. \tag{3.5}$$

They denote

$$Y_O := \{t := y^0 > r\}, \quad \text{the interior of } C_O,\ Y^T_O := Y_O \cap \{0 \leq y^0 \leq T\}. \tag{3.6}$$

They also set

$$\Sigma_\tau := C_O \cap \{y^0 = \tau\}, \quad \text{diffeomorphic to } S^{n-1}, \tag{3.7}$$

$$S_\tau := Y_O \cap \{y^0 = \tau\}, \quad \text{diffeomorphic to the ball } B^{n-1}. \tag{3.8}$$

We will use the following theorem given in the first part of Dossa's thesis: it assumes some more differentiability of the data than the theorem in [19], but it is simpler to apply to the Einstein equations whose initial data must satisfy wave-map-gauge constraints and is sufficient for us here.

Observe that these results assume more regularity from the data on the cone than the regularity obtained for the solution, a constant fact in characteristic Cauchy problem already seen in other contexts.

Theorem 3.1. *Consider the problem* (3.2)–(3.4). *Suppose that*

1. *There is an open set* $U \times W \subset \mathbf{R}^{n+1} \times \mathbf{R}^N, Y_O^T \subset U$ *where the functions* $A^{\lambda\mu}$ *are* C^{2m+2} *in* y *and* v. *The function* f *is* C^{2m} *in* $y \in U$ *and* $v \in W$ *and in* $\partial v \in \mathbf{R}^{(n+1)N}$.
2. *For* $(y, v) \in U \times W$ *the quadratic form* $A^{\lambda\mu}$ *has Lorentzian signature; it takes the Minkowskian values for* $y = 0$ *and* $v = 0$.
3. a. *The function* ϕ *takes its values in* W. *The cone* C_O^T *is null for the metric* $A^{\lambda\mu}(y, \phi)$ *and* $\phi(O) = 0$.
   b. ϕ *is the trace on* C_O^T *of a* C^{2m+2} *function in* U.

Then there is a number $0 < T_0 \leq T < +\infty, T_0 = T$ *if* ϕ *is small enough in* C^{2m+2} *norm, such that the problem* (3.2)–(3.4) *has one and only one solution* v *in* $Y_O^{T_0}$, *such that*

1. *If* $m > \frac{n}{2} + 1, v \in K^{m+1}(Y_O^{T_0}) \cap F^{m+1}(Y_O^{T_0})$, *in particular* $|\partial v|$ *is bounded.*
2. *If* $m = \infty, v$ *can be extended by continuity to a* C^∞ *function defined on a neighbourhood of the origin in* $\mathbf{R}^{N+1}$.

The spaces $K^m(Y_O^T)$ and $F^m(Y_O^T)$ are Banach spaces of sets of functions on Y_O^T which together with their time and space derivatives of order less or equal to m admit a square integrable restriction to each S_t and for which, respectively, the following norms are finite:

$$||v||_{K^m(Y_O^T)} := \sum_{I=1,\ldots,N} \left\{ \int_0^T t^{-n} \sum_{0 \leq |k| \leq m} ||\partial^k v^I||^2_{L^2(S_t)} \, dt \right\}^{\frac{1}{2}},$$

$$||v||_{F^m(Y_O^T)} := \sum_{I=1,\ldots,N} \sup_{0 \leq t \leq T} t^{-\frac{n}{2}} \sum_{0 \leq |k| \leq m} ||\partial^k v^I||_{L^2(S_t)}.$$

The Euclidean metric, $e := \sum_i (\mathrm{d}y^i)^2$, is used to define the measure on S_t and as usual k denotes a multi-index, $k := (k_0, k_1, \ldots, k_n), \partial^k$ the derivation of order $|k| := k_0 + k_1 + \cdots + k_n$:

$$\partial^k := (\partial_0)^{k_0} (\partial_1)^{k_1} \ldots (\partial_n)^{k_n}, \quad \text{with } \partial_\alpha := \frac{\partial}{\partial y^\alpha}. \tag{3.9}$$

3.2. Einstein Equations in the Wave-Map Gauge

We know that the wave-map gauge reduced Einstein equations on a manifold V are tensorial equations under coordinate changes, so that any coordinates can be used. Note that the principal part of the wave-map reduced Einstein equations is independent of the target manifold, and so the Einstein equations on $\mathbf{R}^{n+1}$ in wave-map gauge are of the form (3.2) for an unknown h, when we set $g \equiv \eta + h$ and work in the y coordinates where the Minkowski metric takes the standard form

$$\eta \equiv -(\mathrm{d}y^0)^2 + \sum_i (\mathrm{d}y^i)^2.$$

As an application of Theorem 3.1 we obtain (see also [20] in space-dimension $n = 3$):

Theorem 3.2 (Existence for the wave-gauge reduced Einstein equations). *Let $\overline{g} = \overline{\eta} + \overline{h}$ be a quadratic form on C_O^T such that the components $\overline{h}_{\mu\nu}$ in the coordinates y^μ satisfy the hypotheses of the Existence Theorem 3.1. Then, if the source ρ is of class C^{2m} in Y_O^T, there exists $T_0 > 0$ such that the wave-gauge reduced Einstein equations[5] $R_{\alpha\beta}^{(H)} = \rho_{\alpha\beta}$ admit one and only one solution on $Y_O^{T_0}$, a Lorentzian metric $g^{(H)} = \eta + h$, with h satisfying the conclusions of that theorem.*

The following theorem is a straightforward adaptation of a theorem proved long ago by one of us [21] for spacelike Cauchy data.

Theorem 3.3. *Let $g^{(H)}$ be a C^3 Lorentzian metric, solution on Y_O^T of the Einstein equations in wave-map gauge $S_{\alpha\beta}^{(H)} = T_{\alpha\beta}$. Then, $g^{(H)}$ is a solution on Y_O^T of the full Einstein equations $S_{\alpha\beta} = T_{\alpha\beta}$ if the wave-gauge vector vanishes on C_O^T and the source T satisfies the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$.*

Proof. The identity (2.8) implies (indices raised with g)

$$S^{\alpha\beta} \equiv S^{\alpha\beta(H)} + \frac{1}{2}(\hat{D}^\beta H^\alpha + \hat{D}^\alpha H^\beta - g^{\alpha\beta}\hat{D}_\lambda H^\lambda). \tag{3.10}$$

Hence, the equations in wave-map gauge $S_{\alpha\beta}^{(H)} = T_{\alpha\beta}$ and the Bianchi identities imply that H satisfy the quasidiagonal linear homogeneous system of second order equations

$$\nabla_\alpha \hat{D}^\alpha H^\beta + \nabla_\alpha \hat{D}^\beta H^\alpha - \nabla^\beta \hat{D}_\alpha H^\alpha = 0, \tag{3.11}$$

whose principal terms are wave equations in the metric g since $\nabla_\alpha \hat{D}^\beta H^\alpha - \nabla^\beta \hat{D}_\alpha H^\alpha$ is at most first order in H. If g is C^3, H is C^2, and an energy inequality applied to this linear equation implies easily that $H = 0$ in Y_O^T if $\overline{H} := H|_{C_O^T} = 0$. $\square$

When the support of the initial data is a spacelike manifold M_0 the vanishing of H is guaranteed when the constraint equations $(S_{\alpha\beta} - T_{\alpha\beta})n^\beta|_{M_0} = 0$ are satisfied by the initial data, where n^β is the field of unit normals to M_0 in the space–time one seeks to construct. One of the main goals of this work is to present a method to construct initial data on the light-cone which ensures the vanishing of $\overline{H}$.

4. Null Hypersurfaces, Adapted Coordinates

The obtention, and solution, of equations to be satisfied by initial data to ensure the vanishing of $\overline{H}$ is simpler in coordinates adapted to the geometry of the null initial manifold.

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[5] We use abstract index notation when it helps formulate properties of geometric objects.

4.1. Adapted Coordinates

Let M_0 be a hypersurface in $\mathbf{R}^{n+1}$ which will be a null submanifold of the spacetime (V,g) with V some domain of $\mathbf{R}^{n+1}$. M_0 is generated by geodesic null curves, called rays. In a manner classical for null surfaces we choose coordinates x^α so that M_0 is given by the equation $x^0=0$, and on M_0 the coordinate x^1 is a parameter along the rays, denoting by ℓ the tangent vector $\frac{\partial}{\partial x^1}$. We assume that the subspaces $\Sigma_{x^1}:\{x^1{=}\text{constant},\ x^0{=}0\}$ are spacelike and diffeomorphic to the same $n-1$ manifold Σ, except possibly for Σ_0 which reduces to a point in the case of a characteristic cone. We denote by x^A local coordinates on Σ. We have $\ell^0=0,\ell^1=1,\ell^A=0$.

The covariant vector $n:=\operatorname{grad}x^0$, with $x^0=0$ the equation of M_0, is a null vector normal and tangent to M_0 with components $n_0=1,n_1=n_A=0$. By uniqueness of null directions tangent to a light cone we have also $\ell_A=0$ and hence, using that $\ell^\alpha=\delta^\alpha_1,\overline{g}_{1A}=0$. Then, the trace on M_0 of the spacetime metric reduces in the x^α coordinates to (we put an overbar to denote restriction to M_0 of spacetime quantities)

$$\overline{g}:=g|_{x^0=0}\equiv\overline{g}_{00}(\mathrm{d}x^0)^2+2\nu_0\mathrm{d}x^0\mathrm{d}x^1+2\nu_A\mathrm{d}x^0\mathrm{d}x^A+\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B, \qquad (4.1)$$

where

$$\nu_0:=\overline{g}_{01},\quad \nu_A:=\overline{g}_{0A}, \qquad (4.2)$$

We observe that the $\overline{g}_{AB}$ are the non zero components of the quadratic form $\tilde{g}$ induced by g on M_0 by the identity map. They define an x^1-dependent Riemannian metric on Σ

$$\tilde{g}_\Sigma:=\overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B,\quad A,B=2,\dots,n. \qquad (4.3)$$

The following identities hold on M_0, because $\overline{g}_{\alpha\beta}$ and $\overline{g}^{\alpha\beta}$ are inverse matrices.

$$\overline{g}^{00}\equiv\overline{g}^{0A}\equiv 0,\quad \nu^0:=\overline{g}^{01}=\frac{1}{\nu_0}, \qquad (4.4)$$

$$\overline{g}^{AB}\equiv\tilde{g}^{AB},\ \text{ with } \tilde{g}^{AB} \text{ the inverse matrix of } \overline{g}_{AB}. \qquad (4.5)$$

We denote

$$\nu^B:=\overline{g}^{AB}\nu_A \qquad (4.6)$$

then

$$\overline{g}^{A1}\equiv-\nu^0\nu^A,\quad \overline{g}^{11}\equiv-(\nu^0)^2\overline{g}_{00}+(\nu^0)^2\nu^A\nu_A. \qquad (4.7)$$

Remark 4.1. We use coordinates that are adapted to the initial light-cone but, in contradistinction with other authors, we do not assume that those metric functions that vanish on C_O vanish elsewhere.

In Appendix A we collect formulae useful for explicit calculations, such as the trace on M_0 of the Christoffel symbols of g, etc.

See [28,29,34,42] for various useful results concerning null surfaces.

4.2. Characteristic Cones

4.2.1. General Properties. It is no geometric restriction[6] to assume that in a neighbourhood of its vertex the characteristic cone[7] of the spacetime we are looking for is represented in some admissible coordinates $y := (y^\alpha) \equiv (y^0, y^i, i = 1, \ldots, n)$ of $\mathbf{R}^{n+1}$ by the equation of a Minkowskian cone with vertex O,

$$r - y^0 = 0, \quad r := \left\{\sum_i (y^i)^2\right\}^{\frac{1}{2}}. \tag{4.8}$$

Given the coordinates y^α we can define coordinates x^α on $\mathbf{R}^{n+1}$ adapted to the null cone C_O as we did for a general null surface by setting

$$x^0 = r - y^0, \quad x^1 = r, \quad x^A = \mu^A\left(\frac{y^i}{r}\right), \tag{4.9}$$

with x^A local coordinates[8] on S^{n-1}. The null geodesics issued from O have equation $x^0 = 0$, $x^A =$constant, so that $\frac{\partial}{\partial x^1}$ is tangent to those geodesics. On C_O (but not outside of it in general) the spacetime metric g that we are going to construct takes the form (4.1), that is, such that $\overline{g}_{11} = 0$ and $\overline{g}_{1A} = 0$.

We emphasize that there is no loss of generality in writing $\bar{g}$ in the form (4.1). However, $\overline{g}_{00}, \nu_0, \nu_A$ are not invariant under an isometry of space-time which leaves C_O invariant, they are gauge-dependent quantities (see Sects. 4.5 and 13).

We compute the relation between the components of a tensor T in the coordinates y and x using (4.9) and its inverse:

$$y^0 = x^1 - x^0, \quad y^i = x^1\Theta^i(x^A), \quad \text{with} \quad \sum_i (\Theta^i)^2 = 1. \tag{4.10}$$

If the components of a spacetime symmetric tensor T in the coordinates x^α are denoted $T_{\alpha\beta}$ and if in the coordinates y^α they are denoted $\underline{T_{\alpha\beta}}$, then the transformation law for two-covariant tensors, $T_{\lambda\mu} = \underline{T_{\alpha\beta}}\frac{\partial y^\alpha}{\partial x^\lambda}\frac{\partial y^\beta}{\partial x^\mu}$, give the identities

$$T_{00} \equiv \underline{T_{00}}, \quad T_{01} \equiv -\underline{T_{00}} - \underline{T_{0i}}\Theta^i, \quad T_{0A} \equiv -\underline{T_{0i}}r\frac{\partial\Theta^i}{\partial x^A}, \tag{4.11}$$

$$T_{11} \equiv \underline{T_{00}} + 2\underline{T_{0i}}\Theta^i + \underline{T_{ij}}\Theta^i\Theta^j, \quad T_{1A} \equiv \underline{T_{0i}}r\frac{\partial\Theta^i}{\partial x^A} + \underline{T_{ji}}r\Theta^j\frac{\partial\Theta^i}{\partial x^A}, \tag{4.12}$$

$$T_{AB} \equiv \underline{T_{ij}}r^2\frac{\partial\Theta^i}{\partial x^A}\frac{\partial\Theta^j}{\partial x^B}. \tag{4.13}$$

[6] See footnote 1 and details in Sect. 4.5.

[7] A cone is a topological manifold but it is not differentiable at its vertex.

[8] They can be angular coordinates, see e.g. [14, Chapter V, Sect. 4], or stereographic coordinates, as in Christodoulou [15].

Conversely, $\underline{T_{\lambda\mu}} = \frac{\partial x^\alpha}{\partial y^\lambda}\frac{\partial x^\beta}{\partial y^\mu}T_{\alpha\beta}$ gives

$$\underline{T_{00}} \equiv T_{00}, \quad \underline{T_{0i}} \equiv -(T_{00}+T_{01})\Theta^i - T_{0A}\frac{\partial x^A}{\partial y^i}, \tag{4.14}$$

$$\begin{aligned}\underline{T_{ij}} = (T_{00}+2T_{01}+T_{11})\Theta^i\Theta^j + (T_{0A}+T_{1A})\left(\Theta^i\frac{\partial x^A}{\partial y^j}+\Theta^j\frac{\partial x^A}{\partial y^i}\right)\\ + T_{AB}\frac{\partial x^A}{\partial y^i}\frac{\partial x^B}{\partial y^j},\end{aligned} \tag{4.15}$$

with

$$\frac{\partial x^A}{\partial y^i} = r^{-1}\mu_i^A,$$

where the μ_i^A's are C^∞ functions of the x^B on any subset of S^{n-1} where the x^A's are admissible local coordinates.

One checks, using the identities

$$\sum_i(\Theta^i)^2 = 1 \quad \text{and} \quad \sum_i\frac{\partial\Theta^i}{\partial x^A}\frac{\partial\Theta^i}{\partial x^B} \equiv s_{AB},$$

with

$$s_{AB}\mathrm{d}x^A\mathrm{d}x^B \equiv s_{n-1}, \quad \text{the metric of } S^{n-1},$$

that, when $T=\eta$,

$$-(\mathrm{d}y^0)^2 + \sum_i(\mathrm{d}y^i)^2 = -(\mathrm{d}x^0)^2 + 2\mathrm{d}x^0\mathrm{d}x^1 + (x^1)^2 s_{n-1}.$$

4.2.2. Case $\hat{g}=\eta$, the Minkowski Metric. It is natural to take as given metric $\hat{g}$ the metric of a model spacetime such as Minkowski, or de Sitter, or anti-de Sitter. While most our formulae will be completely general, the analysis will mainly be concerned with the case where the metric $\hat{g}$ is a Minkowski metric η given by the formula written above in the introduced coordinates y^α and in the adapted null coordinates x^α. The Riemannian curvature of the Minkowski metric η is zero. The non zero Christoffel symbols of η are in our coordinates x^α, with S^A_{BC} the Christoffel symbols of the metric s,

$$\hat{\Gamma}^B_{1A} \overset{\eta}{\equiv} \frac{1}{x^1}\delta^B_A, \quad \hat{\Gamma}^B_{AC} \overset{\eta}{\equiv} S^B_{AC}, \tag{4.16}$$

$$\hat{\Gamma}^0_{AB} \overset{\eta}{\equiv} -x^1 s_{AB}, \quad \hat{\Gamma}^1_{AB} \overset{\eta}{\equiv} -x^1 s_{AB}. \tag{4.17}$$

Equalities and identities assuming given metric $\hat{g}=\eta$ and $W^\lambda \equiv g^{\alpha\beta}\hat{\Gamma}^\lambda_{\alpha\beta}$ will be denoted with symbols $\overset{\eta}{=}$ and $\overset{\eta}{\equiv}$, respectively, when ambiguous.

We have

$$\overline{W}^0 \overset{\eta}{\equiv} -x^1\overline{g}^{AB}s_{AB} \overset{\eta}{\equiv} \overline{W}^1, \tag{4.18}$$

$$\overline{W}^A \overset{\eta}{\equiv} 2\overline{g}^{1C}\hat{\Gamma}^A_{1C} + \overline{g}^{BC}\hat{\Gamma}^A_{BC} \overset{\eta}{\equiv} -\frac{2}{x^1}\nu^0\nu^A + \overline{g}^{BC}S^A_{BC}. \tag{4.19}$$

4.2.3. Limits at the Vertex. We set $g = \eta + h$. The condition $\underline{\overline{h}_{\alpha\beta}}(O) = 0$ of Theorem 3.2 can always be satisfied by choice of an orthonormal frame for the natural frame of the coordinates y^α at the vertex. Since the coordinates x^μ cease to form a coordinate system at $x^1 = 0$, the behaviour near $x^1 = 0$ of the components $\overline{h}_{\alpha\beta}$ in x coordinates is obtained only by considering limits. As explained above, we can, and will, choose coordinates on C_O such that $\overline{h}_{11} \equiv 0$, i.e. $C_O : x^0 = 0$ is a null cone for g, and $\overline{h}_{1A} = 0$, i.e. the vector $\ell := \frac{\partial}{\partial x^1}$ is on C_O a null vector. Then, the components of $\underline{\overline{h}}$ are

$$\underline{\overline{h}_{00}} \equiv \overline{h}_{00}, \quad \underline{\overline{h}_{0i}} \equiv -(\overline{h}_{00} + \overline{h}_{01})\Theta^i - \overline{h}_{0A}\tfrac{\partial x^A}{\partial y^i}$$
$$\text{with} \quad \overline{h}_{01} := \nu_0 - 1, \quad \overline{h}_{0A} := \nu_A,$$

$$\underline{\overline{h}_{ij}} = (\overline{h}_{00} + 2\overline{h}_{01})\Theta^i\Theta^j + \overline{h}_{0A}\left(\Theta^i \frac{\partial x^A}{\partial y^j} + \Theta^j \frac{\partial x^A}{\partial y^i}\right) + \overline{h}_{AB}\frac{\partial x^A}{\partial y^i}\frac{\partial x^B}{\partial y^j}.$$

We see that the condition $\underline{\overline{h}_{\alpha\beta}}(O) = 0$ is equivalent to the following conditions in the coordinates x^α:

$$\lim_{r\to 0}(1 + \overline{g}_{00}) = \lim_{r\to 0}(\nu_0 - 1) = \lim_{r\to 0}(r^{-1}\nu_A) = \lim_{r\to 0}(r^{-2}\overline{h}_{AB}) = 0. \quad (4.20)$$

4.2.4. A Lemma. For further use we note the following observation:

Lemma 4.2. *If a C^1 spacetime function f is such that on C_O in the coordinates x^α it holds that*

$$\lim_{r\equiv x^1\to 0} \partial_1 \overline{f} = 0$$

then it also holds

$$\lim_{r\equiv x^1\to 0} \overline{\partial_0 f} = 0.$$

Proof. One has the trivial identity

$$\partial_1 f \equiv \underline{\partial_\alpha f}\frac{\partial y^\alpha}{\partial x^1} \equiv \underline{\partial_0 f} + \underline{\partial_i f}\Theta^i.$$

If f is C^1 in a neighbourhood of O, $\underline{\partial_i f}$ tends to a limit, a number a_i at O and hence the above equation implies

$$\lim_{r\to 0} \underline{\overline{\partial_0 f}} + a_i\Theta^i = 0,$$

condition which can be satisfied for all x^A if and only if $a_i = 0$ and $\lim_{r\to 0} \underline{\overline{\partial_0 f}} = 0$. Therefore,

$$\lim_{r\to 0} \overline{\partial_0 f} \equiv -\lim_{r\to 0} \underline{\overline{\partial_0 f}} = 0.$$

□

4.3. The Affine-Parameterisation Condition

The vector field $\ell := \frac{\partial}{\partial x^1}$, tangent to the null rays in M_0, obeys the geodesic property

$$\overline{\ell^\alpha \nabla_\alpha \ell^\beta} = \overline{\Gamma}^\beta_{11}, \tag{4.21}$$

with

$$\overline{\Gamma}^0_{11} \equiv \overline{\Gamma}^A_{11} \equiv 0 \quad \text{and} \quad \overline{\Gamma}^1_{11} \equiv \nu^0 \left(\partial_1 \nu_0 - \frac{1}{2} \overline{\partial_0 g_{11}} \right). \tag{4.22}$$

If we impose the condition $\overline{\Gamma}^1_{11} = 0$, then the vector ℓ is parallelly transported and x^1 is said to be an affine parameter on the rays; this condition gives an equation involving ν_0, a metric coefficient which will appear in our first wave-map gauge constraint. However, we stress that the equation $\overline{\Gamma}^1_{11} = 0$ involves also a derivative transversal to M_0, and thus cannot be made to hold just by a gauge choice of the coordinate x^1 on the cone. We will see later how we can circumvent this problem in the wave-map gauge.

4.4. Null Extrinsic Curvature

4.4.1. General Properties. Let M_0 be a null hypersurface with a field of null tangents ℓ. The null extrinsic curvature at $x \in M_0$ is defined (see, e.g., [28]) as the bilinear form with components $\overline{\nabla_\alpha \ell_\beta}$ acting on the quotient of the tangent space to M_0 at x by the direction defined by ℓ, i.e. equivalence classes of tangent vectors of the form $\overline{X} \equiv \underline{X} + c\ell$ with $\underline{X} \in T_x M_0, c$ an arbitrary number. Indeed, the action of the bilinear form on a pair of such tangent vectors, $\overline{\nabla_\alpha \ell_\beta X^\alpha Y^\beta}$, depends only on the equivalence class, that is in our coordinates on the components X^A and Y^A; hence it is defined by the components $\chi_{AB} := \overline{\nabla_A \ell_B}$ of the bilinear form. Using $\ell^\alpha = \delta^\alpha_1$ and $\ell_\alpha := \overline{g}_{\alpha\beta} \delta^\beta_1 \equiv \nu_0 \delta^0_\alpha$ we have

$$\chi_{AB} \equiv -\overline{\Gamma}^0_{AB} \nu_0 \equiv \frac{1}{2} \partial_1 \overline{g}_{AB}. \tag{4.23}$$

We denote by

$$\chi_A{}^B := \overline{g}^{BC} \chi_{AC} \equiv \overline{\Gamma}^B_{1A} \tag{4.24}$$

the mixed, x^1-dependent, 2-tensor on S^{n-1} deduced from the null second fundamental form. We define its trace

$$\tau := \chi_A{}^A \equiv \overline{g}^{AB} \chi_{AB} \equiv \partial_1 (\log \sqrt{\det \tilde{g}_\Sigma}), \tag{4.25}$$

and its traceless part

$$\sigma_A{}^B := \chi_A{}^B - \frac{1}{n-1} \delta^B_A \tau, \quad \text{and we set } |\sigma|^2 := \sigma_A{}^B \sigma_B{}^A. \tag{4.26}$$

See [28,29] for an analysis of the null second fundamental form through the Weingarten map.

4.4.2. Harmonicity Functions. In adapted coordinates, and using the notation above, the harmonicity functions $\Gamma^\alpha \equiv g^{\lambda\mu}\Gamma^\alpha_{\lambda\mu}$ reduce on M_0 to

$$\overline{\Gamma}^0 \equiv \overline{g}^{\lambda\mu}\overline{\Gamma}^0_{\lambda\mu} \equiv 2\nu^0\overline{\Gamma}^0_{01} + \overline{g}^{AB}\overline{\Gamma}^0_{AB} \equiv \nu^0(\nu^0\overline{\partial_0 g_{11}} - \tau), \tag{4.27}$$

$$\overline{\Gamma}^A \equiv \overline{g}^{\lambda\mu}\overline{\Gamma}^A_{\lambda\mu} \equiv 2\nu^0\overline{\Gamma}^A_{01} + 2\overline{g}^{B1}\overline{\Gamma}^A_{B1} + \overline{g}^{BC}\overline{\Gamma}^A_{BC} \tag{4.28}$$

$$\equiv \nu^0\nu^A(\tau - \nu^0\overline{\partial_0 g_{11}}) + \nu^0\overline{g}^{AB}(\overline{\partial_0 g_{1B}} + \partial_1\nu_B - \partial_B\nu_0)$$

$$-2\nu^0\nu^B{\chi_B}^A + \tilde{\Gamma}^A, \tag{4.29}$$

$$\overline{\Gamma}^1 \equiv \overline{g}^{\lambda\mu}\overline{\Gamma}^1_{\lambda\mu} \equiv \overline{g}^{11}\overline{\Gamma}^1_{11} + 2\nu^0\overline{\Gamma}^1_{01} - 2\nu^0\nu^A\overline{\Gamma}^1_{A1} + \overline{g}^{AB}\overline{\Gamma}^1_{AB} \tag{4.30}$$

$$\equiv (\nu^0)^2\partial_1\overline{g}_{00} + \overline{g}^{11}\nu^0\left(\frac{1}{2}\overline{\partial_0 g_{11}} + \partial_1\nu_0 - \tau\nu_0\right)$$

$$+2(\nu^0)^2\nu^A(-\partial_1\nu_A + \nu^B\chi_{AB}) + \nu^0\overline{g}^{AB}\tilde{\nabla}_B\nu_A - \frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}} \tag{4.31}$$

$$\equiv -\partial_1\overline{g}^{11} + \overline{g}^{11}\nu^0\left(\frac{1}{2}\overline{\partial_0 g_{11}} - \partial_1\nu_0 - \tau\nu_0\right) + \nu^0\overline{g}^{AB}\tilde{\nabla}_B\nu_A - \frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}}. \tag{4.32}$$

We have defined

$$\tilde{\Gamma}^A := \overline{g}^{BC}\tilde{\Gamma}^A_{BC}, \tag{4.33}$$

with $\tilde{\Gamma}^A_{BC}$ being the Christoffel symbols of the metric $\overline{g}_{AB}$. We shall also use

$$\overline{\Gamma}_1 := \overline{g}_{1\mu}\overline{\Gamma}^\mu = \nu_0\overline{\Gamma}^0, \tag{4.34}$$

$$\overline{\Gamma}_A := \overline{g}_{AB}\overline{\Gamma}^B \neq \overline{g}_{A\mu}\overline{\Gamma}^\mu, \tag{4.35}$$

and similarly for components of $\overline{W}$ and $\overline{H}$ with subindices.

4.4.3. Vertex Limits. We set

$$\overline{g}_{AB} \equiv r^2(s_{AB} + \overline{f}_{AB}). \tag{4.36}$$

We have seen in Sect. 4.2.3 that it is no geometric restriction for smooth metrics to assume

$$\lim_{r\to 0}(r^{-2}\overline{g}_{AB} - s_{AB}) = 0, \quad \text{i.e.} \quad \lim_{r\to 0}\overline{f}_{AB} = 0,$$

Then

$$\lim_{r\to 0} r^2\overline{g}^{AB} = s^{AB}. \tag{4.37}$$

Recalling that ${\chi_A}^C \equiv \overline{\Gamma}^C_{A1}$ and using the relation between connections in different frames gives

$${\chi_A}^C \equiv \overline{\Gamma}^C_{A1} \equiv \frac{\partial x^C}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^A}\frac{\partial y^\gamma}{\partial x^1}\underline{\overline{\Gamma}^\alpha_{\beta\gamma}} + \frac{\partial x^C}{\partial y^\alpha}\frac{\partial}{\partial x^A}\frac{\partial y^\alpha}{\partial x^1}.$$

Using

$$\frac{\partial y^0}{\partial x^1} = 1, \quad \frac{\partial y^i}{\partial x^1} = \frac{y^i}{r}, \quad \text{hence} \quad \frac{\partial x^C}{\partial y^\alpha}\frac{\partial}{\partial x^A}\frac{\partial y^\alpha}{\partial x^1} = \frac{1}{r}\frac{\partial x^C}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^A} = \frac{1}{r}\delta^C_A,$$

we find

$$\chi_A{}^C \equiv \frac{1}{r}\delta_A^C + \frac{\partial x^C}{\partial y^i}\frac{\partial y^j}{\partial x^A}\left(\underline{\overline{\Gamma}^i_{j0}} + \frac{y^h}{r}\underline{\overline{\Gamma}^i_{jh}}\right).$$

Therefore, if the coefficients $\underline{\overline{\Gamma}^i_{j0}}$ and $\underline{\overline{\Gamma}^i_{jk}}$ are bounded for $0 \le r \le a$, the same property holds for $\chi_A{}^C - \frac{1}{r}\delta_A^C$, for $\psi := \frac{n-1}{r} - \tau$ and for $\sigma_A{}^C := \chi_A{}^C - \frac{1}{n-1}\delta_A^C\tau$. These quantities are then also continuous on each null ray. However, the limits when r tends to zero are in general angle dependent.

As already said, given a C^2 spacetime metric we can choose normal geodesic coordinates in a neighbourhood of a point O. Then the Christoffel symbols vanish at O and it holds that

$$\lim_{r\to 0}\overline{\Gamma}^C_{A1} = \lim_{r\to 0}\frac{\partial x^C}{\partial y^\alpha}\frac{\partial}{\partial x^A}\frac{\partial y^\alpha}{\partial x^1} \equiv \frac{1}{r}\frac{\partial x^C}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^A} \equiv \frac{1}{r}\delta^C_A.$$

Hence,

$$\lim_{r\to 0}\psi = \lim_{r\to 0}\sigma_A{}^C = 0. \tag{4.38}$$

See further results in the next section.

4.5. Boundary Conditions in Coordinates Normal at the Vertex

In the following sections we will give explicit expressions for the wave-map gauge constraints. To study their solutions we will need to know the behaviour of the unknowns at the tip of the light cone, aiming at finding solutions of the constraints which satisfy the Cagnac–Dossa hypotheses. The purpose of this section is to describe this behaviour in coordinate systems useful for the problem at hand. The analysis here is also useful for proving geometric uniqueness of solutions.

Consider a smooth space–time (V, g). Let $O \in V$ and let C_O be the future light-cone emanating from O. Let T be any unit timelike vector at O, and normalize null vectors ℓ at O by requiring that $g(\ell, T) = -1$. The parallel transport of ℓ defines an affine parameter, denoted by s, on the future null geodesics $s \mapsto \gamma_\ell(s)$ with $\gamma_\ell(0) = O$ and with initial tangent ℓ. Let $(z^\mu), \mu = 0, \ldots, n$ be a normal coordinate system centred at O with $T = \partial_{z^0}$, see, e.g. [36,50], or [10, Chapter 12, Sect. 7]. In those coordinates the future light-cone emanating from O is given by the equation

$$C_O = \{z^0 = |\vec{z}|\}, \quad \text{where } \vec{z} := (z^1, \ldots, z^n), \quad |\vec{z}|^2 := \sum_{i=1}^n (z^i)^2.$$

As is well known, in normal coordinates at $O, z = 0$, the Christoffel symbols vanish at O. Hence, for a $C^{1,1}$ metric we have $\partial_\sigma g_{\mu\nu}(O) = 0$, and so, for small $|z| := |z^0| + |\vec{z}|$,

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |z||\partial_\sigma g_{\mu\nu}| \le C|z|^2, \tag{4.39}$$

for some constant C.

In the coordinate system $(x^\mu) = (x^0 \equiv u, x^1 \equiv r, x^A), A = 2, \ldots, n$, where

$$u = |\vec{z}| - z^0, \; r = |\vec{z}|, \tag{4.40}$$

and where the x^A's are any local coordinates on S^{n-1} parameterizing the unit vector $\vec{z}/|\vec{z}|$, the trace of the metric g on C_O takes the desired form (4.1) as long as the metric, and the light-cone, are smooth; assuming smoothness of g, this will always be the case in some neighbourhood of the tip O.

Equation (4.40) shows that

$$\mathrm{d}z^0 = \mathrm{d}r - \mathrm{d}u, \quad \mathrm{d}z^i = \frac{z^i}{r}\mathrm{d}r + \partial_A z^i \mathrm{d}x^A,$$

which allows us to translate the estimates (4.39) to the asymptotic behaviour of the objects of interest near $r = 0$: From

$$\begin{aligned}
&g_{\mu\nu}\mathrm{d}z^\mu \mathrm{d}z^\nu \\
&\quad = (\eta_{\mu\nu} + O(|z|^2))\mathrm{d}z^\mu \mathrm{d}z^\nu \\
&\quad = (-1 + O(|z|^2))(\mathrm{d}u - \mathrm{d}r)^2 + O(|z|^2)(\mathrm{d}u - \mathrm{d}r)\left(\frac{z^i}{r}\mathrm{d}r + \partial_A z^i \mathrm{d}x^A\right) \\
&\qquad +(\delta_i^j + O(|z|^2))\left(\frac{z^i}{r}\mathrm{d}r + \partial_A z^i \mathrm{d}x^A\right)\left(\frac{z^j}{r}\mathrm{d}r + \partial_B z^j \mathrm{d}x^B\right) \\
&\quad = (-1 + O(|z|^2))(\mathrm{d}u)^2 + (2 + O(|z|^2))\mathrm{d}u\mathrm{d}r + O(r|z|^2)\mathrm{d}u\mathrm{d}x^A \\
&\qquad + O(|z|^2)(\mathrm{d}r)^2 + O(r|z|^2)\mathrm{d}x^A\mathrm{d}r + r^2\left(s_{AB} + O(|z|^2)\right)\mathrm{d}x^A\mathrm{d}x^B
\end{aligned}$$

we obtain, at $u = 0$, for small r,

$$\overline{g}_{00} = -1 + O(r^2), \quad \partial_r\overline{g}_{00} = O(r), \quad \overline{\partial_u g_{00}} = O(r), \quad \partial_A\overline{g}_{00} = O(r^2), \tag{4.41}$$

$$\nu_0 = 1 + O(r^2), \quad \partial_r\nu_0 = O(r), \quad \overline{\partial_u g_{01}} = O(r), \quad \partial_A\nu_0 = O(r^2), \tag{4.42}$$

$$\nu_A = O(r^3), \quad \partial_r\nu_A = O(r^2), \quad \overline{\partial_u g_{0A}} = O(r^2), \quad \partial_A\nu_0 = O(r^3), \tag{4.43}$$

$$\overline{g}_{AB} = r^2\left(s_{AB} + O(r^2)\right), \quad \partial_r(\overline{g}_{AB} - r^2 s_{AB}) = O(r^3), \tag{4.44}$$

$$\overline{\partial_u g_{AB}} = O(r^3), \quad \partial_A(\overline{g}_{AB} - r^2 s_{AB}) = O(r^4), \tag{4.45}$$

$$\overline{g}_{11} = 0, \quad \partial_r\overline{g}_{11} = 0, \quad \overline{\partial_u g_{11}} = O(r), \quad \partial_A\overline{g}_{11} = 0, \tag{4.46}$$

$$\overline{g}_{1A} = 0, \quad \partial_r\overline{g}_{1A} = 0, \quad \overline{\partial_u g_{1A}} = O(r^2), \quad \partial_A\overline{g}_{1A} = 0. \tag{4.47}$$

One also has associated second-derivative estimates,

$$\overline{\partial_u\partial_r g_{AB}} = O(r^2), \quad \partial_r^2(\overline{g}_{AB} - r^2 s_{AB}) = O(r^2), \quad \partial_A\partial_r(\overline{g}_{AB} - r^2 s_{AB}) = O(r^3), \tag{4.48}$$

etc. From (4.44) and (4.48) we obtain

$$\chi_A{}^B = \frac{1}{r}\delta_A^B + O(r), \quad \text{hence } \tau = \frac{n-1}{r} + O(r), \quad \sigma_A{}^B = O(r), \tag{4.49}$$

$$\text{as well as } \partial_r(\tau - \frac{n-1}{r}) = O(1), \quad \partial_A\tau = O(r), \tag{4.50}$$

$$\partial_r\sigma_A{}^B = O(1), \quad \partial_C\sigma_A{}^B = O(r). \tag{4.51}$$

Note that (4.49)–(4.51) will hold in any coordinate system which *coincides with the normal coordinates z^μ on the light-cone*. This is due to the fact that the vectors ∂_r and ∂_A are tangent to the light-cone, which implies that

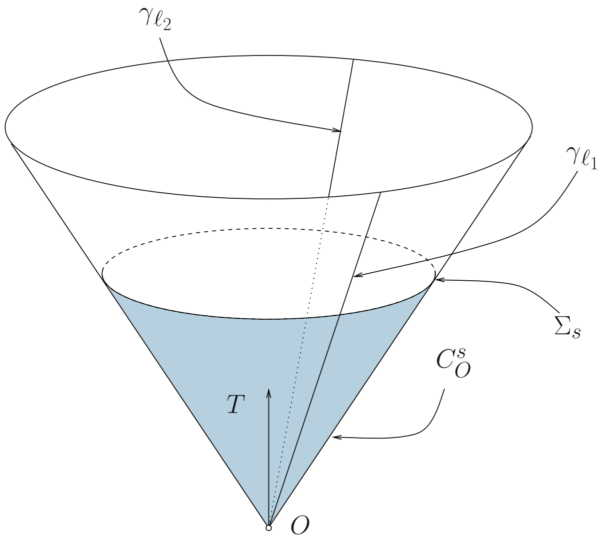

Figure 1. The cross-section Σ_s of the light-cone C_O; C_O^s is the *shaded* region. Two generators γ_{ℓ_1} and γ_{ℓ_2} are also shown

the quadratic form $\tilde{g}_\Sigma = \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B$ is intrinsically defined on the light-cone, independently of how the coordinates are extended away from the light-cone, and from the fact that the matrix $\overline{g}^{AB}$ in (A.2) is the inverse of $\overline{g}_{AB}$.

4.6. The Light-Cone Theorem

A result closely related to our analysis here is the *light-cone theorem*, proved in [12], which reads as follows: Let s be an affine parameter as defined at the beginning of Sect. 4.5. Let Σ_s denote the $(n-1)$-dimensional surface reached by these geodesics after affine time s:

$$\Sigma_s = \{\gamma_\ell(s)\} \subset C_O, \tag{4.52}$$

where the vectors ℓ run over all null future vectors at O normalized as above; see Fig. 1.

We denote by C_O^t the subset of the light-cone covered by all the geodesics up to affine time t:

$$C_O^t = \cup_{0\le s\le t}\Sigma_s. \tag{4.53}$$

Note that $\gamma_\ell(s)$ might not be defined for all s. Further, some of the Σ_s's might not be smooth. However, there exists a maximal $s_0 > 0$ such that Σ_s is defined and smooth for all $0 < s < s_0$. Our considerations only apply to that last region.

It is proved in [12] that, assuming the Einstein equations with a cosmological constant and with sources satisfying the dominant energy condition, the areas of the Σ_s's are less than or equal to the corresponding areas in Minkowski, de Sitter, or anti-de Sitter space–time. Furthermore, if equality holds at some s_2, then on $C_O^{s_2}$ we have

$$\sigma_{AB} = 0 = \overline{T}_{11}, \quad \tau = \frac{n-1}{r}.$$

(This situation will be referred to as that of the Null-Cone Theorem (NCT).) It is further shown in [12] that, under suitably stronger energy conditions, equality implies that the metric is that of the model space on the domain of dependence of $C_O^{s_2}$. The proofs of those facts provide a non-trivial illustration of the formalism developed here, as specialized to the simpler problem treated in [12].

5. Constraints and Gauge Preservation

The obvious analogue on a null submanifold M_0 of the spacelike constraints operator is $\overline{S}_{\alpha\beta}\ell^\beta$, where ℓ denotes the field of null normals to M_0 normalized in some arbitrary way. Derivatives of the metric in $\overline{S}_{\alpha\beta}\ell^\beta$ transversal to the light-cone appear only at first order. Some of them[9] cancel between the various terms contributing to $\overline{S}_{\alpha\beta}\ell^\beta$, and those that remain can be expressed in terms of $\overline{H}$ and $\overline{W}$. So, in the explicit form of $\overline{S}_{\alpha\beta}\ell^\beta$, one can replace every occurrence of $\overline{\partial_0 g_{01}}, \overline{\partial_0 g_{0A}}$ and $\overline{g}^{AB}\overline{\partial_0 g_{AB}}$ by $\overline{H}_\alpha, \overline{W}_\alpha$, and terms containing only derivatives along M_0. We define $n+1$ operators $\mathcal{L}_\alpha(\overline{H}), \alpha = 0, \ldots, n$, by adding all the terms involving $\overline{H}$ in $\overline{S}_{\alpha\beta}\ell^\beta$. One can then define $n+1$ operators $\mathcal{C}_\alpha$ by whatever remains; thus the $\mathcal{C}_\alpha$'s coincide with $\overline{S}_{\alpha\beta}\ell^\beta$ when $\overline{H}_\alpha$ vanishes. Explicit formulae for $\mathcal{C}_\alpha$ are given in (6.13), (8.24) and in (10.41) below, while $\mathcal{L}_\alpha$ can be found in (6.14), (8.22) and (10.38).

We will prove the following theorem, which is the key element of our analysis of the Cauchy problem for the Einstein equations on a light-cone:

Theorem 5.1. 1. *The operator $\overline{S}_{\alpha\beta}\ell^\beta$ on a null submanifold M_0 can be written as a sum,*

$$\overline{S}_{\alpha\beta}\ell^\beta \equiv \mathcal{L}_\alpha + \mathcal{C}_\alpha,$$

where $\mathcal{L}_\alpha$ vanishes when $\overline{H} = 0$, while the operator $\mathcal{C}_\alpha$ depends only on the values $\overline{g}$ on M_0 of the spacetime metric, on the choice of the null vector ℓ, and on $\overline{W}$, which depends on the chosen target space of the wave-map gauge. The operators $\mathcal{C}_\alpha$ will be called Einstein-wave-map gauge constraint operators.

2. *In adapted null coordinates*
   (a) *The operators $\mathcal{C}_\alpha$ lead to a hierarchy of ordinary differential operators for the coefficients of $\overline{g}$ along the generators, all linear when the first constraint $\overline{S}_{\alpha\beta}\ell^\alpha\ell^\beta = \overline{T}_{\alpha\beta}\ell^\alpha\ell^\beta$ has been solved.*
   (b) *The operators $\mathcal{L}_\alpha$ together with the wave-gauge reduced Einstein equations lead to a hierarchy of homogeneous first order ordinary linear differential operators along the generators for the components*

9 Compare [2,17,33,47] in space-dimension three.

$\overline{H}_\alpha$ if the spacetime metric g satisfies on M_0 the reduced Einstein equations.

Theorem 5.1 will be proved in Sects. 6–8 and 10.

A consequence of Theorems 3.3 and 5.1 is the following:

Theorem 5.2. *A C^3 Lorentzian metric $g^{(H)}$, solution of the Einstein equations in wave-map gauge $S^{(H)}_{\alpha\beta} = T_{\alpha\beta}$ in Y^T_O with $\nabla_\alpha T^{\alpha\beta} = 0$, and taking an initial value $\overline{g}$ on C^T_O, is a solution of the full Einstein equations $S_{\alpha\beta} = T_{\alpha\beta}$ if and only if $\overline{g}$ satisfies the constraints $\mathcal{C}_\alpha = \overline{T}_{\alpha\beta}\ell^\beta$.*

Proof. Theorem 5.1 gives the following identities, with $\mathcal{L}_\alpha$ a linear and homogeneous first-order differential operator along the null vector ℓ for the vector $\overline{H}$:

$$\overline{S}_{\alpha\beta}\ell^\beta \equiv \overline{S}^{(H)}_{\alpha\beta}\ell^\beta + \frac{1}{2}(\overline{g}_{\alpha\lambda}\overline{\hat{D}_\beta H^\lambda} + \overline{g}_{\beta\lambda}\overline{\hat{D}_\alpha H^\lambda} - \overline{g}_{\alpha\beta}\overline{\hat{D}_\lambda H^\lambda})\ell^\beta \equiv \mathcal{C}_\alpha + \mathcal{L}_\alpha. \tag{5.1}$$

The "only if" part of the theorem results immediately from the identity (5.1) when the metric g is a solution of the full Einstein equations and is in wave gauge, since then only C_α remains in that identity.

The "if" part will be proved later by showing that $\overline{H}^\alpha = 0$ is the only solution, for metrics which are uniformly C^1 near the tip of the cone, of the equations

$$\frac{1}{2}(\overline{g}_{\alpha\lambda}\overline{\hat{D}_\beta H^\lambda} + \overline{g}_{\beta\lambda}\overline{\hat{D}_\alpha H^\lambda} - \overline{g}_{\alpha\beta}\overline{\hat{D}_\lambda H^\lambda})\ell^\beta = \mathcal{L}_\alpha \tag{5.2}$$

which result from the identity (5.1) when $\mathcal{C}_\alpha = \overline{T}_{\alpha\beta}\ell^\beta$ and $\overline{S}^{(H)}_{\alpha\beta} = \overline{T}_{\alpha\beta}$. □

The question of local geometric uniqueness of solutions is addressed in Sect. 13.

6. The First Constraint

6.1. Computation of $\overline{R}_{11} \equiv \overline{S}_{11} \equiv \overline{S}_{\alpha\beta}\ell^\alpha\ell^\beta$

The component R_{11} can be separated as

$$R_{11} \equiv R^{(1)}_{11} + R^{(2)}_{11},$$

where $R^{(1)}_{11}$ is linear in first derivatives of the Christoffel symbols and $R^{(2)}_{11}$ is quadratic in them. They are given by, after a trivial simplification,

$$R^{(1)}_{11} \equiv \partial_0\Gamma^0_{11} + \partial_A\Gamma^A_{11} - \partial_1\Gamma^0_{10} - \partial_1\Gamma^A_{1A}, \tag{6.1}$$

$$\begin{aligned} R^{(2)}_{11} \equiv\ & \Gamma^0_{11}(\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^A_{0A}) + \Gamma^1_{11}(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^A_{1A}) \\ & + \Gamma^B_{11}(\Gamma^0_{B0} + \Gamma^1_{B1} + \Gamma^A_{BA}) \\ & - \Gamma^0_{10}\Gamma^0_{10} - 2\Gamma^1_{10}\Gamma^0_{11} - \Gamma^1_{11}\Gamma^1_{11} - 2\Gamma^1_{1A}\Gamma^A_{11} - 2\Gamma^0_{1A}\Gamma^A_{10} - \Gamma^A_{1B}\Gamma^B_{1A}. \end{aligned} \tag{6.2}$$

We must take care when taking derivatives transversal to the cone, i.e. ∂_0, that our coordinate conditions are valid only on the cone. We will then use the trivial identity

$$\overline{\partial_\lambda \Gamma^\alpha_{\beta\gamma}} \equiv \overline{g}^{\alpha\mu}(\overline{\partial_\lambda[\mu,\beta\gamma]}) + \overline{(\partial_\lambda g^{\alpha\mu})[\mu,\beta\gamma]}. \tag{6.3}$$

In $\overline{R}^{(1)}_{11}$ only $\overline{\Gamma}^0_{11}$ is differentiated transversally to C_a. We have, since $\overline{g}_{11} = \overline{g}_{1A} = 0$,

$$\overline{\partial_0\Gamma^0_{11}} = \frac{1}{2}\nu^0\partial_1\overline{\partial_0 g_{11}} + \left(\partial_1\nu_0 - \frac{1}{2}\overline{\partial_0 g_{11}}\right)\overline{\partial_0 g^{00}}, \quad \text{with} \quad \overline{\partial_0 g^{00}} = -(\nu^0)^2\overline{\partial_0 g_{11}}, \tag{6.4}$$

$$-\partial_1\overline{\Gamma}^0_{10} \equiv -\partial_1\left(\frac{1}{2}\nu^0\overline{\partial_0 g_{11}}\right). \tag{6.5}$$

By using also

$$\overline{\Gamma}^A_{11} = 0, \quad \overline{\Gamma}^A_{1A} = \frac{1}{2}\overline{g}^{AB}\partial_1\overline{g}_{AB} =: \tau \tag{6.6}$$

and the harmonicity function (4.27), we get

$$\overline{R}^{(1)}_{11} \equiv (\nu^0)^2\frac{1}{2}\overline{\partial_0 g_{11}}\,\overline{\partial_0 g_{11}} + \frac{1}{2}\partial_1\nu^0\overline{\partial_0 g_{11}} - \partial_1\tau \tag{6.7}$$

$$\equiv \frac{1}{2}(\overline{\Gamma}_1+\tau)^2 - \frac{1}{2}\nu^0\partial_1\nu_0(\overline{\Gamma}_1+\tau) - \partial_1\tau. \tag{6.8}$$

The part $\overline{R}^{(2)}_{11}$ depends only on the values of the Christoffel symbols on C_O. Using $\overline{\Gamma}^0_{11} = \overline{\Gamma}^A_{11} = \overline{\Gamma}^0_{1A} = 0$ and trivial simplifications we find that

$$\overline{R}^{(2)}_{11} \equiv \overline{\Gamma}^1_{11}(\overline{\Gamma}^0_{10} + \overline{\Gamma}^A_{1A}) - \overline{\Gamma}^0_{10}\overline{\Gamma}^0_{10} - \overline{\Gamma}^A_{1B}\overline{\Gamma}^B_{1A}.$$

In the chosen coordinates, $\overline{R}^{(2)}_{11}$ reduces to

$$\overline{R}^{(2)}_{11} \equiv -\frac{1}{2}(\overline{\Gamma}_1+\tau)^2 + \frac{1}{2}\nu^0\partial_1\nu_0(\overline{\Gamma}_1+\tau) + \nu^0\partial_1\nu_0\tau - \frac{1}{2}\tau(\overline{\Gamma}_1+\tau) - \chi_A{}^B\chi_B{}^A. \tag{6.9}$$

Adding (6.8) and (6.9) we obtain

$$\overline{R}_{11} \equiv -\partial_1\tau + \nu^0\partial_1\nu_0\tau - \frac{1}{2}\tau(\overline{\Gamma}_1+\tau) - \chi_A{}^B\chi_B{}^A \tag{6.10}$$

$$\equiv -\partial_1\tau + \overline{\Gamma}^1_{11}\tau - \chi_A{}^B\chi_B{}^A, \tag{6.11}$$

6.2. The $\mathcal{C}_1$ Constraint Operator

By definition of the wave-gauge vector H we have $\overline{\Gamma}_1 \equiv \overline{W}_1 + \overline{H}_1$, and hence, (6.10) decomposes as

$$\overline{R}_{11} \equiv \mathcal{C}_1 + \mathcal{L}_1, \tag{6.12}$$

with

$$\mathcal{C}_1 := -\partial_1\tau + \left(\nu^0\partial_1\nu_0 - \frac{1}{2}(\overline{W}_1+\tau)\right)\tau - |\sigma|^2 - \frac{\tau^2}{n-1}, \tag{6.13}$$

where we have separated $\chi_A{}^B$ in trace-free and pure trace parts (see (4.26)), and

$$\mathcal{L}_1 := -\frac{1}{2}\overline{H}_1\tau. \tag{6.14}$$

As announced, (6.13) involves only the values of the metric coefficients on the light-cone; equivalently, no derivatives of the metric transverse to the light-cone occur there:

$$\overline{W}_1 = 2\hat{\Gamma}^0_{01} + \nu_0\overline{g}^{AB}\hat{\Gamma}^0_{AB} \overset{\eta}{=} -\nu_0 x^1\overline{g}^{AB}s_{AB}, \tag{6.15}$$

where we have assumed that the target metric takes the adapted form *in the same coordinate system*, so that $\hat{\Gamma}^0_{11} = 0$ and $\hat{\Gamma}^0_{1A} = 0$. The Einstein equation $R_{11} = T_{11}$ in wave-gauge provides, in this sense, a constraint equation $\mathcal{C}_1 = \overline{T}_{11}$ for the metric components $\overline{g}_{\mu\nu}$.

The constraint equation $\mathcal{C}_1 = \overline{T}_{11}$ contains as unknowns only the components $\overline{g}_{AB}$ and ν_0 if it is so of $\overline{T}_{11}$. A simple strategy is then to prescribe $\overline{g}_{AB}$ (compare Bondi et al. [1]) and use the definition (4.23) to compute χ_{AB}; hence, also $\sigma_A{}^B$ and τ. The first constraint reads then as a differential first-order equation for ν_0, linear if $\overline{T}_{11}$ is independent of ν_0 since $\overline{W}_1$ is linear in ν_0. (Recall that we are assuming $\hat{W}^\mu = 0$ unless explicitly indicated otherwise.) The solution will lead to a Lorentzian metric as long as ν_0 is positive.

However, the equation is singular wherever τ vanishes, as the resulting ODE for ν_0 involves inverse powers of τ. For this reason it is of interest to look for alternatives, where τ is computed from the constraint, rather than prescribed in advance. Following [47] we will prescribe only the conformal class of $\tilde{g}$. The wave-map gauge constraint deduced from (6.13) is then an equation for ν_0 and the conformal factor Ω^2. We can prescribe arbitrarily ν_0 and then determine Ω. We can also, generalizing an idea of Damour and Schmidt, impose on ν_0 to satisfy a well-chosen differential equation containing an arbitrarily given function κ. We treat in detail the case $\kappa = 0$, which implies that for the obtained solution ν_0 the vector ℓ will be parallelly transported, in other words r will be an affine parameter, in the resulting space–time.

7. Solution of the $\mathcal{C}_1$ Constraint for given σ

The operator $\mathcal{C}_1$ relates the three functions τ (which, via Eq. (4.25), essentially describes the evolution of the volume element of the sections Σ), ν_0 and $|\sigma|^2 := \sigma_A{}^B\sigma_B{}^A$. We recall the following well known fact:

Lemma 7.1. *The tensor σ is determined by the conformal class of the induced quadratic form $\tilde{g}$.*

Proof. To see this, let us write

$$\tilde{g} = \Omega^2\gamma,$$

with γ a degenerate quadratic form on C_O such that $\gamma_{11} \equiv \gamma_{1A} \equiv 0$. Then

$$\chi_{AB} \equiv \frac{1}{2}\Omega^2\partial_1\gamma_{AB} + \gamma_{AB}\Omega\,\partial_1\Omega,$$

and thus,

$$\chi_A{}^C \equiv \frac{1}{2}\gamma^{BC}\partial_1\gamma_{AB} + \delta_A^C\,\partial_1 \log\Omega; \tag{7.1}$$

hence, the trace-free part of $\chi_A{}^B$ is

$$\sigma_A{}^C \equiv \frac{1}{2}\gamma^{BC}\partial_1\gamma_{AB} - \delta_A^C\frac{\partial_1(\log\sqrt{\det\gamma_\Sigma})}{n-1}, \tag{7.2}$$

where γ_Σ denotes the positive definite x^1 dependent quadratic form on Σ with components γ_{AB}. We see that the traceless tensor σ is independent of the conformal factor and hence depends only on the conformal class of $\tilde g$. In particular, σ vanishes if $\tilde g$ is conformal to a quadratic form independent of $r := x^1$. $\square$

If γ and its first derivatives satisfy the vertex limits spelled out for $\tilde g$ in Sect. 4.5, then $\lim_{r\to 0} r|\sigma| = 0$; we say that a degenerate quadratic form γ on C_O, with $\gamma_{11}\equiv\gamma_{1A}\equiv 0$, is admissible if it is C^1 on $C_O - O$, i.e. for $r>0$, and such that $|\sigma|^2$ is C^0 for $r\ge 0$ and hence bounded for finite $r\ge 0$. Given σ the constraint $C_{11}=\overline{T}_{11}$ appears as a relation between the functions τ and ν_0. Since it involves radial derivatives of both τ and ν_0 (which can actually be grouped as $\partial_1(\nu^0\tau)$), we could prescribe one of them and integrate for the remaining field, or else provide an additional differential equation for either of τ or ν_0 and integrate simultaneously the coupled system of the constraint and this new equation. In the remainder of this section we show how to solve the constraint by prescribing ν_0, either explicitly (Sect. 7.1) or through a differential condition (rest of Sect. 7).

7.1. Prescribed ν_0

Suppose the function ν_0 is arbitrarily prescribed, then the constraint equation becomes a differential equation for τ. It is convenient to introduce the scalar function (recall that $\tilde g_\Sigma$ denotes the restriction of $\tilde g$ to Σ)

$$\varphi := \left(\frac{\det\tilde g_\Sigma}{\det s_{n-1}}\right)^{1/(2n-2)} = \Omega\left(\frac{\det\gamma_\Sigma}{\det s_{n-1}}\right)^{1/(2n-2)}, \tag{7.3}$$

so that

$$\tau = (n-1)\,\partial_1\log\varphi, \qquad \text{or} \qquad \partial_1\varphi = \frac{\tau}{n-1}\varphi. \tag{7.4}$$

The normalization of φ has been chosen to have $\varphi = r$ for Minkowski. Using this variable the constraint reads

$$-\partial^2_{11}\varphi + \left(\nu^0\partial_1\nu_0 - \frac{1}{2}\overline{W}_1 - \frac{n-1}{2}\frac{\partial_1\varphi}{\varphi}\right)\partial_1\varphi = \frac{|\sigma|^2+\overline{T}_{11}}{n-1}\varphi, \tag{7.5}$$

to be integrated outwards with initial data $\varphi(O)=0$ and $\partial_1\varphi(O)=1$. As already mentioned, $\overline{W}_1$ contains φ nonlinearly, and in principle $\overline{T}_{11}$ could also depend on φ. In general, this scheme could be considered for a larger class of gauges

$$\overline{W}_1 = \overline{W}_1(\gamma_{AB},\varphi,\nu_0,r,x^A), \tag{7.6}$$

and sources of the form

$$\overline{T}_{11} = \overline{T}_{11}(\text{source data}, \gamma_{AB}, \partial_i\gamma_{AB}, \varphi, \partial_1\varphi, \nu_0, \partial_i\nu_0, r, x^A), \tag{7.7}$$

where ∂_i denotes derivatives tangential to the light-cone, and by "source data" we mean non gravitational data, for example fields determined from characteristic initial data for scalar, or Maxwell fields. The wave-map gauge condition (4.18) is clearly of the form (7.6). In Sect. 7.7 we show that both scalar and Maxwell fields lead to a coefficient $\overline{T}_{11}$ of the energy-momentum tensor compatible with (7.7).

7.2. Differential Equation for ν_0

The choice made by Rendall is to assume that x^1 is an affine parameter along the null rays; in other words that the vector $\ell = \frac{\partial}{\partial x^1}$ is parallelly transported along the null rays by the connection of the spacetime he constructs, i.e. $\overline{\Gamma}^1_{11} = 0$; equivalently $\partial_1\nu_0 = (\overline{\Gamma}_1 + \tau)\nu^0/2$. Now, this last equation contains a derivative transversal to the light-cone, which is not part of the characteristic initial data. Extending to the cone an idea of Damour and Schmidt[17] concerning two intersecting surfaces, in anticipation of the fact that our solution will satisfy $\overline{H}_1 = 0$, we could impose on ν_0 to satisfy the equation

$$\partial_1\nu_0 = \frac{1}{2}(\overline{W}_1 + \tau)\nu_0, \tag{7.8}$$

which implies, modulo $\overline{H}_1 = 0$, that $\overline{\Gamma}^1_{11} = 0$. When ν_0 satisfies (7.8) the constraint $\mathcal{C}_1 = \overline{T}_{11}$ reduces to a Raychaudhuri type equation for the only unknown τ

$$\partial_1\tau + \frac{\tau^2}{n-1} + |\sigma|^2 + \overline{T}_{11} = 0. \tag{7.9}$$

More generally, all solutions of (7.9) can be obtained by introducing an arbitrary function κ and solving the pair of equations

$$\partial_1\tau - \kappa\tau + \frac{\tau^2}{n-1} + |\sigma|^2 + \overline{T}_{11} = 0, \tag{7.10}$$

whose only unknown is τ when $|\sigma|^2$ and $\overline{T}_{11}$ are known, and

$$\partial_1\nu_0 = \frac{1}{2}(\overline{W}_1 + \tau)\nu_0 + \kappa\nu_0. \tag{7.11}$$

When ν_0 satisfies this equation and $\overline{H}_1 = 0$, then $\overline{\Gamma}^1_{11} = \kappa$.

Once τ is determined we can use (7.4) to obtain φ and finally (7.11) to compute ν_0.

Remark 7.2. The equations derived here would be dramatically simplified if one simultaneously imposes $\nu_0 = 1$ and $\kappa = 0$; see, e.g., [33]. However, these two conditions together with the wave-gauge condition, which is the cornerstone of our analysis, would impose undesirable geometrical restrictions on the initial data.

Equation (7.10) is, if $\overline{T}_{11}$ does not depend on τ or φ, a Riccati differential equation on each null ray and hence can be rewritten, precisely using the variable φ, as a linear second-order equation

$$-\partial^2_{11}\varphi + \kappa\,\partial_1\varphi = \frac{|\sigma|^2 + \overline{T}_{11}}{n-1}\,\varphi, \tag{7.12}$$

to be integrated outwards with initial values $\varphi(O)$ and $\partial_1\varphi(O)$. We have assumed chosen an admissible γ; hence, $|\sigma|^2$ continuous for $r \geq 0$. We assume the same holds for $\overline{T}_{11}$. Observe that for a continuous tensor T, i.e. with continuous components $\underline{T_{\mu\nu}}$ in the coordinates y^α, we will have $\overline{T}_{11}$ continuous, but $\lim_{r\to 0}\overline{T}_{11}$ a function of angles in general since it holds that

$$\lim_{r\to 0}\overline{T}_{11} = \underline{T_{00}}(0) + 2\underline{T_{i0}}(0)\frac{y^i}{r} + \underline{T_{ij}}(0)\frac{y^i y^j}{r^2}.$$

When $|\sigma|^2 + \overline{T}_{11}$ is continuous for $r \geq 0$ standard ODE theory guarantees that a solution with given initial values exists globally. However, a positive definite metric $\overline{g}_{AB}$ is only obtained from the positive part of the solution. The relevant initial conditions are $\varphi(O) = 0$ and $\partial_1\varphi(O) = 1$, so φ is initially tangent to $\varphi = r$.

We consider the case $\kappa = 0$, that is, x^1 is an affine parameter. Assuming $\overline{T}_{11} \geq 0$, the equation satisfied by φ shows that it is a concave function of x^1 on each null ray wherever positive, and hence there are two possibilities: either φ is a monotone increasing function for all real r, with $0 \leq \varphi \leq r$ and $0 < \tau < (n-1)/r$, or else there is a first local maximum, at which $\partial_1\varphi = 0$ and hence the expansion τ also vanishes there. This is related to the formation of outer-trapped surfaces on the cone C_O. Once a maximum has been reached, φ will necessarily vanish for some larger value of r.

We now turn to a direct analysis in terms of τ, which allows stating results in a more geometric way.

7.3. Solution of the Raychaudhuri Equation

We continue to use a Minkowski target and we make the choice $\kappa = 0$, so that x^1 will be an affine parameter along the null rays. Equation (7.10) then reads as a Raychauduri equation

$$\partial_1\tau + \frac{1}{n-1}\tau^2 + |\sigma|^2 + \overline{T}_{11} = 0. \tag{7.13}$$

This is a first-order ODE for τ when $|\sigma|^2 := \sigma_A{}^B\sigma_B{}^A$ and $\overline{T}_{11}$ are known.

7.3.1. NCT Case. When $|\sigma|^2 + \overline{T}_{11} = 0$, the equation admits the solution corresponding to the Minkowskian cone[10]:

$$\tau_\eta = \frac{n-1}{x^1}. \tag{7.14}$$

The value $|\sigma| = 0$ further imposes

$$\chi_A{}^B = \frac{1}{2x^1}\delta^B_A, \quad \text{i.e.} \quad \partial_1\overline{g}_{AB} = \frac{2}{x^1}\overline{g}_{AB}. \tag{7.15}$$

[10] It is the only solution such that τ^{-1} tends to zero at the vertex of the cone.

With our choice of frame of coordinates at the vertex, the solution is the Minkowskian solution

$$\overline{g}_{AB} = (x^1)^2 s_{AB}, \tag{7.16}$$

as used in the null-cone theorem of [12].

7.3.2. General Case, a Global Existence Theorem. We denote x^1 by r and ∂_1 by a prime. In all that follows we write, and solve, differential equations in r, with constant initial values (mostly zero) for $r = 0$. We do not write explicitly the dependence on the other coordinates x^A, though it occurs in the solutions and in the coefficients.

1. In a neighbourhood of $r = 0$ we define a new function y by

$$y := \frac{n-1}{\tau}.$$

Equation (7.13) becomes

$$y' = 1 + \frac{1}{n-1} f^2 y^2, \quad f^2 := |\sigma|^2 + \overline{T}_{11}. \tag{7.17}$$

In agreement with Sect. 4.4.3, we seek a solution such that $y(0) = 0$. The equation implies that y is increasing and $y \geq r$.

We assume as before that $\frac{1}{n-1} f^2$ is continuous and bounded by a number A^2. Then y exists, is of class C^1, is unique, and is bounded by the solution of the problem

$$z' = 1 + A^2 z^2, \quad z(0) = 0,$$

as long as that solution exists. The solution is

$$z = A^{-1} \tan(Ar). \tag{7.18}$$

Hence, z is defined, C^∞, and bounded, as well as all its derivatives, for $0 \leq r \leq a$, for any $a < A^{-1}\frac{\pi}{2}$.

For $0 \leq r \leq a \leq A^{-1}, z$ is such that

$$r \leq z \leq r + A^2 r^3.$$

We have defined ψ as

$$\psi := \tau_\eta - \tau, \qquad \tau_\eta \equiv \frac{n-1}{r}, \tag{7.19}$$

and hence we have, since $r < y \leq z$,

$$0 \leq \frac{1}{n-1}\psi \equiv \frac{1}{r} - \frac{1}{y} \leq \frac{1}{r} - \frac{1}{z} \leq \frac{1}{r} - \frac{1}{r + A^2 r^3} = \frac{A^2 r}{1 + A^2 r^2} \leq A^2 r. \tag{7.20}$$

That is

$$0 \leq \psi \leq (n-1) A^2 r. \tag{7.21}$$

2. For large r we use the decay of f^2. Using the definition (7.19) we obtain

$$\psi' + \frac{2}{r}\psi = \frac{1}{n-1}\psi^2 + f^2. \tag{7.22}$$

This gives

$$u' = \frac{1}{n-1}\frac{u^2}{r^2} + r^2 f^2 \geq 0, \quad u := r^2\psi. \tag{7.23}$$

Hence, u is an increasing function of r, and both u and ψ are positive as $u(0) = 0$.

For $r \geq a$ we replace the problem to solve by the integral equation

$$u(r) = \frac{1}{n-1}\int_a^r \frac{u^2(\rho)}{\rho^2}\mathrm{d}\rho + \int_a^r \rho^2 f^2(\rho)\mathrm{d}\rho + u_a, \tag{7.24}$$

with $u_a := u(a) \equiv a^2\psi(a)$. By (7.18) and the inequality $y \leq z$ for $r \leq a$ we have

$$\psi(r) \leq \tau_\eta(r) - \frac{n-1}{z(r)} = (n-1)\left(\frac{1}{r} - \frac{A}{\tan(Ar)}\right),$$

hence,

$$u_a \leq (n-1)a\left(1 - \frac{Aa}{\tan(Aa)}\right). \tag{7.25}$$

We assume that $r^2 f^2$ is integrable for $r \in [a, \infty)$, and we set

$$C_a := u_a + B_a, \quad B_a := \int_a^\infty r^2 f^2 \mathrm{d}r.$$

The solution u of the integral equation (7.24) exists and is bounded by a solution v of the equation

$$v(r) = \frac{1}{n-1}\int_a^r \frac{v^2(\rho)}{\rho^2}\mathrm{d}\rho + C_a,$$

as long as such a solution v exists; equivalently, as long as the differential equation

$$v' = \frac{1}{n-1}\frac{v^2}{r^2}$$

admits a solution v with $v(a) = C_a$. The general solution of the above equation is

$$\frac{1}{v} = \frac{1}{(n-1)r} + c, \qquad \text{i.e. } v = \frac{(n-1)r}{1+(n-1)rc}.$$

It takes the value C_a for $r = a$ if and only if

$$\frac{a(n-1)}{1+(n-1)ac} = C_a, \quad \text{i.e. } c = c_a := \frac{1}{C_a} - \frac{1}{(n-1)a}. \tag{7.26}$$

The function v remains positive and bounded if

$$1 + (n-1)rc_a > 0, \tag{7.27}$$

hence, v is defined and bounded for all r if $c_a \geq 0$, i.e. $C_a \leq (n-1)a$; that is, when

$$u_a + B_a \leq (n-1)a. \tag{7.28}$$

It follows from (7.25) that this last inequality will hold when

$$B_a \leq (n-1)\frac{Aa^2}{\tan(Aa)}.$$

In the case where $f^2 \equiv 0$ for $0 \leq r \leq a$ it holds also that $u(r) \equiv 0$ in that interval we have $u_a = 0$ and $C_a = B_a$. Condition (7.28) reduces to

$$B_a \leq (n-1)a. \tag{7.29}$$

3. Assume that $\tau \geq 0$ exists in the interval $r \in (0, b]$ and denote $\tau_b := \tau(b)$. If for $r \geq b$, $f^2 = 0$, the equation for τ reduces, for $r \geq b$, to

$$\tau' + \frac{1}{n-1}\tau^2 = 0, \tag{7.30}$$

with initial value

$$\tau(b) = \tau_b, \quad 0 < \tau_b \leq \tau_0(b) \equiv \frac{n-1}{b}. \tag{7.31}$$

The solution is

$$\frac{1}{\tau} = \frac{r}{n-1} + \frac{1}{\tau_b} - \frac{b}{n-1}. \tag{7.32}$$

Therefore,

$$\tau = \frac{(n-1)\tau_b}{(n-1) + (r-b)\tau_b} = \frac{(n-1)}{r + d_b}, \tag{7.33}$$

with

$$d_b := (n-1)\tau_b^{-1} - b \geq 0. \tag{7.34}$$

Hence, for large r,

$$\tau = \frac{n-1}{r}\left(1 - \frac{d_b}{r} + \cdots\right). \tag{7.35}$$

We have proved the following theorem.

Theorem 7.3. *The equation for τ deduced from the first constraint,*

$$\tau' + \frac{1}{n-1}\tau^2 + f^2 = 0,$$

with $f^2 := |\sigma^2| + \overline{T}_{11}$ continuous and $r^2 f^2$ integrable in r for $r \in [0, \infty)$, has a global solution $\tau(r) > 0$, and the function

$$\psi := \frac{n-1}{r} - \tau$$

is of class C^1 if

1. *We assume that there exists $a \in (0, \infty)$ such that it holds*

$$A < \frac{\pi}{2a}, \quad \text{with} \quad A^2 := \sup_{0 \leq r \leq a} \frac{1}{n-1} f^2. \tag{7.36}$$

In the interval $0 \le r \le a$ *it then follows that*

$$\tau \ge \frac{n-1}{r} z(Ar),$$

with

$$z(x) := \frac{x}{\tan x} \le 1. \tag{7.37}$$

2. *In the interval* $a \le r < \infty$ *we assume that*

$$\int_a^\infty r^2 f^2 \, \mathrm{d}r \le (n-1)a \, z(Aa). \tag{7.38}$$

In this interval we then have

$$\tau \ge \frac{n-1}{r+k_a},$$

with

$$k_a = (n-1)(\tau_a - a^{-2}B_a)^{-1} - a, \quad B_a = \int_a^\infty r^2 f^2 \, dr. \tag{7.39}$$

3. *Regardless of point* 1., *if* $\sigma = 0 = \overline{T}_{11}$ *for* $r \ge b$, *and if* $\tau_b := \tau(b) > 0$, *then the solution exists for all* $r \ge b$ *and it holds that*

$$\tau = \frac{n-1}{r+k_b^{(0)}}, \quad k_b^{(0)} := (n-1)\tau_b^{-1} - b \ge 0.$$

Remark 7.4. If $f^2 := |\sigma|^2 + \overline{T}_{11}$ has compact support $\{a \le r \le b\}$ with $a > 0$, it follows from (7.28) that (7.38) can be replaced by

$$\int_a^b r^2 f^2 \mathrm{d}r \le (n-1)a,$$

which will be satisfied if, e.g.

$$\sup_{a \le r \le b} r^2 f^2 \le \frac{(n-1)a}{b-a}.$$

Remark 7.5. It follows from the equations above (compare [12, Proposition 2.2]) that if there exists $r_2 > 0$ such that

$$\int_0^{r_2} \rho^2 f^2(\rho, x^A) \mathrm{d}\rho \ge (n-1)r_2, \tag{7.40}$$

then the expansion $\tau(r, x^A)$ will become negative at some value of r strictly smaller than r_2. If this happens for all x^A, then one expects existence of an outer trapped surface in the associated space–time. (See [15,35,46] for recent important results concerning formation of trapped surfaces.)

7.4. Determination of $\overline{g}_{AB}$

Recall that we have set

$$\overline{g}_{AB} := \Omega^2 \gamma_{AB} \equiv \varphi^2 \left(\frac{\det s_{n-1}}{\det \gamma_\Sigma}\right)^{1/(n-1)} \gamma_{AB}, \tag{7.41}$$

and that φ satisfies the equation

$$\partial_1 \log \varphi = \frac{\tau}{n-1} = \frac{1}{r} - \frac{\psi}{n-1}, \tag{7.42}$$

with the initial condition $\varphi(0) = 0$. Its integration gives

$$\varphi(r) = r \exp\left(-\int_0^r \frac{\psi(\rho)}{n-1} \mathrm{d}\rho\right). \tag{7.43}$$

We assume that the free data $\overline{T}_{11}$ and γ_{AB} are such that τ exists and satisfies the conclusions of Theorem 7.3, with some $a \in (0, \infty)$. We have then

1. For small r, using the inequality (7.21), valid for $r < a \le A^{-1}$,

$$0 \le \psi \le (n-1)A^2 r$$

we conclude that in such interval we have

$$\exp\left(-\int_0^r \frac{\psi(\rho)}{n-1} \mathrm{d}\rho\right) \ge \exp\left(-\frac{1}{2}A^2 r^2\right),$$

and therefore

$$0 \le r - \varphi(r) \le r\left(1 - \exp\left(-\frac{1}{2}A^2 r^2\right)\right) \le \frac{1}{2}A^2 r^3. \tag{7.44}$$

2. For $r \ge a$, let ψ be as in (7.19), we use

$$\psi \equiv \frac{u}{r^2} \le \frac{v}{r^2} \equiv \frac{(n-1)}{r\{1 + (n-1)rc_a\}} \tag{7.45}$$

to obtain

$$\partial_1 \log \varphi = \frac{1}{r} - \frac{\psi}{n-1} \ge \frac{(n-1)c_a}{1 + (n-1)rc_a}. \tag{7.46}$$

This shows that φ is an increasing C^1 function bounded below by

$$\varphi(a) \frac{1 + (n-1)rc_a}{1 + (n-1)ac_a}. \tag{7.47}$$

3. In the case where one assumes that $f^2 = 0$ for $r > b$ it holds exactly

$$\partial_1 \log \varphi = \frac{1}{r + d_b} > 0, \qquad d_b = (n-1)\tau_b^{-1} - b \ge 0. \tag{7.48}$$

Therefore, using the notation $\varphi_b := \varphi(b)$,

$$\varphi(r) = \varphi_b \frac{r + d_b}{b + d_b} > \varphi_b, \quad \text{if} \ \ r > b. \tag{7.49}$$

In conclusion, if γ_{AB} is admissible and $\overline{T}_{11}$ is known and continuous, we can solve (7.9) for τ on some maximal (possibly angle-dependent) interval of

r's. Subsequently, (7.4) can be solved with initial value $\varphi(0) = 0$. This provides $\overline{g}_{AB}$. The quantity φ and hence also Ω, depend only on the conformal class of γ; the same is true of $\tilde{g}$, defined by (7.41).

7.5. Determination of ν_0

Once $\overline{g}_{AB}$ is known we can integrate equation (7.11) for ν_0, with the initial condition $\nu_0|_{r=0} = 1$,

$$\partial_1\nu_0 = \frac{1}{2}(\overline{W}_1 + \tau)\nu_0 + \kappa\nu_0.$$

(Note that at this stage any "wave-gauge source" $\overline{W}_1$ of the form

$$\overline{W}_1(\tau, \overline{g}_{AB}, \nu_0, r, x^A) \tag{7.50}$$

with an appropriate behaviour near $r = 0$ could be used, though as said before, in this section we assume $\kappa = 0$, and a Minkowski target.) The equation for ν_0 reads

$$\frac{\partial_1\nu_0}{\nu_0^2} = \frac{1}{2}\left\{-\overline{g}^{AB} r s_{AB} + \frac{\tau}{\nu_0}\right\}, \tag{7.51}$$

i.e. since $\nu^0 = \frac{1}{\nu_0}$,

$$\partial_1\nu^0 = -\frac{1}{2}\tau\nu^0 + \frac{1}{2}\overline{g}^{AB} r s_{AB}. \tag{7.52}$$

This is a linear equation for ν^0, with coefficients singular for $r = 0$, but continuous for $r > 0$. Its solution taking a given initial value for $r_0 > 0$ exists, is C^1 and unique for $r \geq r_0$ as long as τ and Ω^{-1} exist and are continuous. Note, however, that ν^0 could go to zero in finite affine time, which would lead to a (geometric or coordinate) singularity.

7.5.1. NCT Case. To study solutions with initial data at $r = 0$, we start with the NCT case. We have then $\tau \equiv \tau_\eta = \frac{n-1}{r}$, and (7.52) reduces to

$$\partial_1\nu^0 = -\frac{n-1}{2r}(\nu^0 - 1). \tag{7.53}$$

The general solution is, for some constant k,

$$\nu^0 - 1 = kr^{-\frac{n-1}{2}}. \tag{7.54}$$

The solution tending to one as r tends to zero corresponds to $k = 0$, and is $\nu^0 = \nu_0 = 1$.

7.5.2. General Case. To construct a solution tending to 1 as r tends to zero we set

$$Y := 1 - \nu^0. \tag{7.55}$$

The Eq. (7.52) for ν^0 becomes the linear non homogeneous equation

$$Y' = -\frac{1}{2}\tau Y + F, \tag{7.56}$$

with F a continuous function (recall the notations $\tau \equiv \tau_\eta - \psi, \tau_\eta = \frac{n-1}{r} = \eta^{AB} r s_{AB}$ and the assumed boundary conditions (4.44) and (4.49))

$$F := \frac{1}{2}(\tau - \overline{g}^{AB} r s_{AB}) \equiv \frac{1}{2}\{(\eta^{AB} - \overline{g}^{AB}) r s_{AB} - \psi\} = O(r), \quad (7.57)$$

where $\psi \geq 0$ and

$$(\eta^{AB} - \overline{g}^{AB}) r s_{AB} = \frac{n-1}{r} - r\Omega^{-2}\gamma^{AB} s_{AB}. \quad (7.58)$$

Incidentally, this implies that $F \leq 0$ for initial data such that

$$r^2\Omega^{-2}\gamma^{AB} s_{AB} \geq n-1. \quad (7.59)$$

Now, in the notation of (7.3), this can be rewritten in the form

$$r^2\Omega^{-2}\gamma^{AB} s_{AB} = r^2\varphi^{-2}\left[(\det \gamma_\Sigma)^{1/(n-1)}\gamma^{AB}\right]\left[(\det s_{n-1})^{-1/(n-1)} s_{AB}\right], \quad (7.60)$$

such that the two expressions in square brackets have unit determinants. Using $\varphi \leq r$, hence $r^2\varphi^{-2} \geq 1$, the last equation allows one to deduce (7.59) from a condition involving only the conformal metric γ_{AB}.

We want to find a solution Y which tends to zero with r, but this solution will lead to data for a Lorentzian metric only if ν^0 remains bounded and non zero; that is, if $Y < 1$.

The homogeneous equation associated with (7.56) is

$$Y' = \left(-\frac{n-1}{2r} + \frac{1}{2}\psi\right) Y. \quad (7.61)$$

Setting $Y = \exp Z$, this equation reads

$$Z' = -\frac{n-1}{2r} + \frac{1}{2}\psi. \quad (7.62)$$

The general solution of (7.56) is of the form

$$Y = w \exp Z, \quad \text{with} \quad w' = \exp(-Z)F. \quad (7.63)$$

1. Case $0 \leq r \leq a$.

Without loss of generality we can choose

$$Z = -\frac{n-1}{2}\log r + \frac{1}{2}\int_0^r \psi(\rho)\mathrm{d}\rho; \quad (7.64)$$

hence,

$$Y(r) = w r^{-\frac{n-1}{2}} \exp\left(\frac{1}{2}\int_0^r \psi(\rho)\mathrm{d}\rho\right),$$

$$\text{with} \quad w' = r^{\frac{n-1}{2}} \exp\left(-\frac{1}{2}\int_0^r \psi(\rho)\mathrm{d}\rho\right) F(r). \quad (7.65)$$

We find the solution $Y := 1 - \nu^0$ tending to zero with r (compare (4.42)) by integrating w' between 0 and r,

$$Y(r) = r^{-\frac{n-1}{2}} \exp\left(\frac{1}{2}\int_0^r \psi(\rho)\mathrm{d}\rho\right) \int_0^r \rho^{\frac{n-1}{2}} \exp\left(-\frac{1}{2}\int_0^\rho \psi(\chi)\mathrm{d}\chi\right) F(\rho)\mathrm{d}\rho. \tag{7.66}$$

Keeping in mind that there exist numbers C_a, a=1,2, such that

$$0 \le \psi \le C_1 r, \quad |F(r)| \le C_2 r,$$

we see that for $0 \le r \le a$ we have

$$\exp\left(\frac{1}{2}\int_0^r \psi(\rho)\mathrm{d}\rho\right) \le \exp\left(\frac{1}{4}C_1 r^2\right),$$

$$\int_0^r \rho^{\frac{n-1}{2}} \exp\left(-\frac{1}{2}\int_0^\rho \psi(\chi)\mathrm{d}\chi\right) |F(\rho)|\mathrm{d}\rho \le C_2 \int_0^r \rho^{\frac{n+1}{2}}\mathrm{d}\rho = \frac{2}{n+3}C_2\, r^{\frac{n+3}{2}},$$

leading to the bound, still for $r \le a$,

$$|Y| \le \frac{2C_2 r^2}{n+3} \exp\left(\frac{1}{4}C_1 r^2\right).$$

Since $Y := 1 - \nu^0$ the function ν^0 is bounded. From (A.7) the metric will have Lorentzian signature, $\overline{g}_{AB}$ being Riemannian, if and only if ν^0 remains bounded and non zero (hence positive since equal to 1 for $r = 0$). This will hold if $Y < 1$, which will be true for any C_2 if a is small enough. In vacuum C_2 is determined by $|\sigma|^2$, so for any a it will hold that $Y < 1$ for $r \in [0, a]$ if σ is small enough.

Note that if $F \le 0$, then $Y \le 0$; hence $\nu^0 \ge 1$ without restriction on the size of a or $|\sigma|$.

2. $a \le r < \infty$.

By the same reasoning as for $r \le a$, the solution Y taking the value $Y(a)$ for $r = a$ is

$$Y(r) = Y(a) + r^{-\frac{n-1}{2}} \exp\left(\frac{1}{2}\int_a^r \psi(\rho)\mathrm{d}\rho\right)$$
$$\times \int_a^r \rho^{\frac{n-1}{2}} \exp\left(-\frac{1}{2}\int_a^\rho \psi(\chi)\mathrm{d}\chi\right) F(\rho)\mathrm{d}\rho.$$

3. Suppose that for $r \ge b$ we have $\overline{g}^{AB} = \varphi^{-2} s^{AB}$, hence

$$F(r) \equiv \frac{1}{2}\{(s^{AB}(r^{-2} - \varphi^{-2}) r s_{AB} - \psi\} \equiv \frac{1}{2}\left\{(1 - r^2\varphi^{-2})\frac{n-1}{r} - \psi\right\}.$$

We have seen that $\varphi \le r$, that is $r^2\varphi^{-2} \ge 1$ and hence $F(r) \le 0$ and $\nu^0(r) \ge \nu^0(b)$.

7.6. Vanishing of $\overline{H}_1$

Consider a solution of the wave-map-reduced Einstein equations $\overline{R}^{(H)}_{11} = \overline{T}_{11}$ with initial data on C_O, and with Minkowski target. Suppose that the data there satisfy the constraint $\mathcal{C}_1 = \overline{T}_{11}$. The identity (see (2.8))

$$\overline{R}_{11} \overset{\eta}{\equiv} \overline{R}^{(H)}_{11} + \nu_0 D_1 \overline{H}^0$$

shows that $\overline{H}^0$ satisfies a linear homogeneous differential equation on C_O, namely,

$$D_1 \overline{H}^0 + \frac{1}{2}\overline{H}^0 \tau \overset{\eta}{=} 0. \tag{7.67}$$

Keeping in mind that D is the covariant derivative of the Minkowski metric, in our adapted coordinate system we have

$$D_1 \overline{H}^0 \overset{\eta}{\equiv} \partial_1 \overline{H}^0.$$

For all solutions which satisfy uniform C^1 bounds near the vertex in the (y^μ) coordinate system, the y^μ-components of the wave-gauge vector are bounded near the vertex. It follows that $\overline{H}^0$ is bounded near the vertex. But every solution of (7.67) which is not identically zero behaves, for small r, as $r^{-(n-1)/2}$ along some generators. So, in the uniformly C^1 case, we can deduce from (7.67) that

$$\overline{H}^0 = 0, \quad \text{hence also} \quad \overline{H}_1 \equiv \nu_0 \overline{H}^0 = 0.$$

Remark 7.6. If we add constraint damping terms as in (2.13), we obtain instead

$$\mathcal{L}_1 = \left(-\frac{1}{2}\tau + \epsilon n_1\right)\overline{H}_1. \tag{7.68}$$

No term proportional to $\overline{H}^A$ or $\overline{H}^1$ appears, and hence the damping term is compatible with this first step of the wave-map-gauge constraint hierarchy. The new term does not change the terms which are singular in r in (7.67), and hence $\overline{H}_1 = 0$ is still the only solution with the required behaviour.

7.7. Scalar and Maxwell Fields

We wish to check that scalar fields lead to equations compatible with the required hierarchical structure of the equations. For this, consider a scalar field ϕ coupled with the gravitational field through an energy-momentum tensor of the form

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right)g_{\alpha\beta}.$$

In the adapted coordinate system the components relevant for our argument are

$$\overline{T}_{11} = (\partial_1\overline{\phi})^2, \tag{7.69}$$

$$\overline{T}_{A1} = \partial_A\overline{\phi}\,\partial_1\overline{\phi}, \tag{7.70}$$

$$\begin{aligned}\overline{T}_{01} &= -\nu_0\frac{1}{2}\overline{g}^{AB}\partial_A\overline{\phi}\,\partial_B\overline{\phi} + \nu^A\partial_A\overline{\phi}\,\partial_1\overline{\phi} \\ &\quad -\nu_0\frac{1}{2}\overline{g}^{11}(\partial_1\overline{\phi})^2 - V(\phi)\nu_0.\end{aligned} \tag{7.71}$$

Keeping in mind that the initial data for the scalar field on C_O are provided by $\overline{\phi} := \phi|_{C_O}$, we see that prescribing $\overline{\phi}$ provides a $\overline{T}_{11}$ which can be used in (7.5) or in (7.10) (compare (7.7)).

Next, the relevant components of the stress-energy tensor for the Maxwell field $F_{\mu\nu}$ are

$$\overline{T}_{11} = \overline{g}^{AB}\overline{F}_{A1}\overline{F}_{B1}, \tag{7.72}$$

$$\overline{T}_{A1} = -\nu^0\overline{F}_{A1}\overline{F}_{01} - \overline{g}^{BC}\overline{F}_{AB}\overline{F}_{C1} + \nu^0\nu^B\overline{F}_{A1}\overline{F}_{B1}, \tag{7.73}$$

$$\begin{aligned}\overline{T}_{01} &= -\frac{1}{4}\nu_0\overline{g}^{AC}\overline{g}^{BD}\overline{F}_{AB}\overline{F}_{CD} - \overline{g}^{BC}\nu^A\overline{F}_{AB}\overline{F}_{C1} + \frac{1}{2}\nu^0\nu^A\nu^B\overline{F}_{A1}\overline{F}_{B1} \\ &\quad -\frac{1}{2}\nu_0\overline{g}^{AB}\overline{g}^{11}\overline{F}_{A1}\overline{F}_{B1} - \frac{1}{2}\nu^0(\overline{F}_{01})^2.\end{aligned} \tag{7.74}$$

We defer a complete discussion of the Cauchy problem for the Einstein-Maxwell equations to separate work. Here we note that if F_{1A} is given on the null cone, then (7.72) is *not* of the right form for viewing (7.10) as a first-order equation for τ: Instead, (7.10) should be considered as a second-order equation for φ, using (7.4). On the other hand, (7.72) is of the form (7.7), needed for the analysis of the problem when ν_0 has been given. The remainder of our analysis of the $\mathcal{C}_1$ constraint goes through as before.

For further reference, we note that the combination of stress-energy components appearing in the final constraint $\mathcal{C}_0$ is

$$\overline{g}^{11}\overline{T}_{11} + 2\overline{g}^{A1}\overline{T}_{A1} + 2\overline{g}^{01}\overline{T}_{01} = -\frac{1}{2}\overline{g}^{AC}\overline{g}^{BD}\overline{F}_{AB}\overline{F}_{CD} - (\overline{g}^{01}\overline{F}_{01} + \overline{g}^{A1}\overline{F}_{A1})^2 \tag{7.75}$$

for the Maxwell field, and

$$\overline{g}^{11}\overline{T}_{11} + 2\overline{g}^{A1}\overline{T}_{A1} + 2\overline{g}^{01}\overline{T}_{01} = -\overline{g}^{AB}\partial_A\overline{\phi}\,\partial_B\overline{\phi} \tag{7.76}$$

for the scalar field.

8. The $\mathcal{C}_A$ Constraint

The $\mathcal{C}_A$ wave-map-gauge constraint operator will be obtained from an analysis of

$$\overline{S}_{1A} \equiv \overline{R}_{1A} \equiv \overline{R}^{(1)}_{1A} + \overline{R}^{(2)}_{1A}, \tag{8.1}$$

where we have again separated terms including derivatives of Christoffels in $\overline{R}^{(1)}_{1A}$ from the rest in $\overline{R}^{(2)}_{1A}$. Trivial simplification gives

$$\overline{R}^{(1)}_{1A} \equiv \overline{\partial_0 \Gamma^0_{1A}} + \partial_B \overline{\Gamma}^B_{1A} - \partial_1 \overline{\Gamma}^0_{A0} - \partial_1 \overline{\Gamma}^B_{AB}. \tag{8.2}$$

We have by the choice of coordinates $\overline{\Gamma}^0_{1A} \equiv \overline{\Gamma}^B_{11} \equiv \overline{\Gamma}^0_{11} \equiv 0$, and therefore

$$\begin{aligned}\overline{R}^{(2)}_{1A} \equiv{}& \overline{\Gamma}^1_{1A}(\overline{\Gamma}^0_{10} + \overline{\Gamma}^B_{1B}) + \overline{\Gamma}^B_{1A}(\overline{\Gamma}^0_{B0} + \overline{\Gamma}^1_{B1} + \overline{\Gamma}^C_{BC}) \\ &- \overline{\Gamma}^0_{10}\overline{\Gamma}^0_{A0} - \overline{\Gamma}^B_{10}\overline{\Gamma}^0_{AB} - \overline{\Gamma}^1_{1B}\overline{\Gamma}^B_{A1} - \overline{\Gamma}^B_{1C}\overline{\Gamma}^C_{AB}.\end{aligned} \tag{8.3}$$

We find, for the terms in $\overline{R}^{(1)}_{1A}$,

$$\overline{\partial_0 \Gamma^0_{1A}} = \overline{\partial_0 g^{00}[0,1A]} + \overline{\partial_0 g^{0B}[B,1A]} + \frac{1}{2}\nu^0 \overline{\partial_0 \partial_A g_{11}} \tag{8.4}$$

$$\begin{aligned}&= \frac{1}{2}\partial_A(\nu^0 \overline{\partial_0 g_{11}}) - \nu^0 \chi_A{}^B \overline{\partial_0 g_{1B}} \\ &\quad + \frac{1}{2}(\nu^0)^2(\overline{\partial_0 g_{1A}} - \partial_1 \nu_A + 2\nu_B \chi_A{}^B)\overline{\partial_0 g_{11}},\end{aligned} \tag{8.5}$$

$$-\partial_1 \overline{\Gamma}^0_{A0} \equiv -\frac{1}{2}\partial_1\{\nu^0(\partial_A \nu_0 + \overline{\partial_0 g_{1A}} - \partial_1 \nu_A)\}, \tag{8.6}$$

$$\partial_B \overline{\Gamma}^B_{1A} - \partial_1 \overline{\Gamma}^B_{AB} \equiv \partial_B \chi_A{}^B - \partial_1(\nu^0 \nu_B \chi_A{}^B) - \partial_A \tau. \tag{8.7}$$

And for the terms in $\overline{R}^{(2)}_{1A}$ we find

$$\overline{\Gamma}^1_{1A}\overline{\Gamma}^B_{1B} \equiv \left\{\frac{1}{2}\nu^0(\partial_1 \nu_A + \partial_A \nu_0 - \overline{\partial_0 g_{1A}}) - \nu^0 \nu_B \chi_A{}^B\right\}\tau, \tag{8.8}$$

$$\overline{\Gamma}^B_{1A}(\overline{\Gamma}^0_{B0} + \overline{\Gamma}^1_{B1} + \overline{\Gamma}^C_{BC}) \equiv \chi_A{}^B(\nu^0 \partial_B \nu_0 + \tilde{\Gamma}^C_{BC}), \tag{8.9}$$

$$-\overline{\Gamma}^0_{10}\overline{\Gamma}^0_{A0} \equiv \frac{1}{4}(\nu^0)^2(\partial_1 \nu_A - \partial_A \nu_0 - \overline{\partial_0 g_{1A}})\overline{\partial_0 g_{11}}, \tag{8.10}$$

$$\overline{\Gamma}^1_{A1}\overline{\Gamma}^0_{10} \equiv \frac{1}{4}(\nu^0)^2(\partial_1 \nu_A + \partial_A \nu_0 - \overline{\partial_0 g_{1A}} - 2\nu_B \chi_A{}^B)\overline{\partial_0 g_{11}}, \tag{8.11}$$

$$-\overline{\Gamma}^B_{10}\overline{\Gamma}^0_{AB} \equiv \frac{1}{2}\nu^0 \chi_A{}^B(\overline{\partial_0 g_{1B}} + \partial_1 \nu_B - \partial_B \nu_0 - \nu^0 \nu_B \overline{\partial_0 g_{11}}), \tag{8.12}$$

$$-\overline{\Gamma}^1_{1B}\overline{\Gamma}^B_{A1} - \overline{\Gamma}^B_{1C}\overline{\Gamma}^C_{AB} \equiv \frac{1}{2}\nu^0 \chi_A{}^B(\overline{\partial_0 g_{1B}} - \partial_1 \nu_B - \partial_B \nu_0) - \chi_B{}^C \tilde{\Gamma}^B_{AC}. \tag{8.13}$$

All terms in these formulae can be computed on C_O, except for those that contain $\overline{\partial_0 g_{1B}}$ or $\overline{\partial_0 g_{11}}$, and whose sum simplifies to

$$\overline{R}_{1A,\partial_0} = -\frac{1}{2}\partial_1(\nu^0 \overline{\partial_0 g_{1A}}) - \frac{1}{2}\tau\,\nu^0 \overline{\partial_0 g_{1A}} + \frac{1}{2}\partial_A(\nu^0 \overline{\partial_0 g_{11}}). \tag{8.14}$$

(We see that all terms quadratic in ∂_0 derivatives cancel out.) The rest is given by

$$\begin{aligned}\overline{R}_{1A} - \overline{R}_{1A,\partial_0} \equiv{}& \frac{1}{2}\nu^0 \partial_1(\partial_1 \nu_A - \partial_A \nu_0 - 2\nu_B \chi_A{}^B) + \tilde{\nabla}_B \chi_A{}^B - \nu_0 \partial_A(\tau \nu^0) \\ &+ \frac{1}{2}(\partial_1 \nu^0 + \tau \nu^0)(\partial_1 \nu_A - \partial_A \nu_0 - 2\nu_B \chi_A{}^B).\end{aligned} \tag{8.15}$$

8.1. Use of Harmonicity Functions

From the identities (4.27) and (4.29) we get

$$\overline{\Gamma}_A := \overline{g}_{AB}\overline{\Gamma}^B \equiv -\nu_A\overline{\Gamma}^0 + \nu^0(\overline{\partial_0 g_{1A}} + \partial_1\nu_A - \partial_A\nu_0) - 2\nu^0\nu_B\chi_A{}^B + \tilde{\Gamma}_A. \tag{8.16}$$

Hence,

$$\overline{\partial_0 g_{1A}} \equiv -\partial_1\nu_A + \partial_A\nu_0 + 2\nu_B\chi_A{}^B + \nu_A\overline{\Gamma}_1 + \nu_0(\overline{\Gamma}_A - \tilde{\Gamma}_A). \tag{8.17}$$

On the other hand,

$$\overline{\Gamma}_A \equiv \overline{H}_A + \overline{W}_A, \quad \text{with} \quad \overline{H}_A := \overline{g}_{AB}\overline{H}^B, \tag{8.18}$$

similarly for $\overline{\Gamma}_A$ and $\overline{W}_A$. Therefore, we have

$$\overline{\partial_0 g_{1A}} \equiv -\partial_1\nu_A + \partial_A\nu_0 + 2\nu_B\chi_A{}^B + \nu_0 f_A + \nu_0\overline{H}_A + \nu_A\overline{H}_1, \tag{8.19}$$

with

$$f_A := \nu^0\nu_A\overline{W}_1 + \overline{W}_A - \tilde{\Gamma}_A, \tag{8.20}$$

For a Minkowski target, using (4.18) and (4.19), this is

$$f_A \overset{\eta}{=} -\left(x^1\overline{g}^{CD}s_{CD} + \frac{2\nu^0}{x^1}\right)\nu_A + \overline{g}_{AB}\overline{g}^{CD}(S^B_{CD} - \tilde{\Gamma}^B_{CD}). \tag{8.21}$$

8.2. Computation of $\mathcal{L}_A$ and $\mathcal{C}_A$

We see from the identities obtained that $\overline{R}_{1A}$ is the sum of a linear homogeneous operator $\mathcal{L}_A$ on $\overline{H}_A := \overline{g}_{AB}\overline{H}^B$ and a second-order linear operator $\mathcal{C}_A$ on ν_A, both with coefficients depending only on the x^1-dependent metric $\tilde{g}$ and scalar ν_0 previously determined. (Strictly speaking, ν_A also appears in $\mathcal{L}_A$, but multiplied by $\overline{H}^0$ which, with appropriate boundary conditions and a Minkowski target, can be shown to be zero at this stage of the argument, as explained above.)

$$\mathcal{L}_A \equiv -\frac{1}{2}\partial_1(\overline{H}_A + \nu_A\overline{H}^0) - \frac{1}{2}\tau(\overline{H}_A + \nu_A\overline{H}^0) + \frac{1}{2}\partial_A(\nu_0\overline{H}^0). \tag{8.22}$$

From (8.14),

$$\overline{R}_{1A,\partial_0} = -\frac{1}{2}\nu^0\partial_1\overline{\partial_0 g_{1A}} - \frac{1}{2}(\partial_1\nu^0 + \tau\nu^0)\overline{\partial_0 g_{1A}} + \frac{1}{2}\partial_A(\nu^0\overline{\partial_0 g_{11}}),$$

and using the formula (8.19) we find that

$$\begin{aligned}\overline{R}_{1A,\partial_0} - \mathcal{L}_A \equiv & -\frac{1}{2}\nu^0\partial_1\{-\partial_1\nu_A + \partial_A\nu_0 + 2\nu_B\chi_A{}^B + \nu_0 f_A\} \\ & -\frac{1}{2}(\partial_1\nu^0 + \tau\nu^0)\{-\partial_1\nu_A + \partial_A\nu_0 + 2\nu_B\chi_A{}^B + \nu_0 f_A\} \\ & +\frac{1}{2}\partial_A(\overline{W}_1 + \tau).\end{aligned} \tag{8.23}$$

Finally, assembling results (8.15) and (8.23) gives

$$\begin{aligned}\mathcal{C}_A \equiv\ & \overline{R}_{1A} - \mathcal{L}_A \equiv\ \frac{1}{2}\nu^0\partial_1\{2\partial_1\nu_A - 4\nu_B{\chi_A}^B - \nu_0 f_A\}\\ &+\frac{1}{2}(\partial_1\nu^0 + \tau\nu^0)\{2\partial_1\nu_A - 4\nu_B{\chi_A}^B - \nu_0 f_A\}\\ &+\tilde{\nabla}_B{\chi_A}^B - \frac{1}{2}\partial_A(\tau - \overline{W}_1 + 2\nu^0\partial_1\nu_0).\end{aligned}\tag{8.24}$$

It turns out that there is a simple way of rewriting (8.24) in terms of

$$\xi_A := -2\nu^0\partial_1\nu_A + 4\nu^0\nu_B{\chi_A}^B + \underbrace{\nu^0\nu_A\overline{W}_1 + \overline{W}_A - \tilde{\Gamma}_A}_{\equiv f_A},\tag{8.25}$$

where $\tilde{\Gamma}_A := \overline{g}_{AD}\overline{g}^{BC}\tilde{\Gamma}^D_{BC}$ (compare (4.33)). The vector ξ_A equals $-2\overline{\Gamma}^1_{1A}$ after using the harmonicity conditions $\overline{\Gamma} = \overline{W}$. Note that ξ_A vanishes when $\nu_A = 0$ and $\overline{g}_{AB} = \hat{g}_{AB}$. The wave-map-gauge constraint operator $\mathcal{C}_A$ can be expressed in terms of ξ_A as

$$\mathcal{C}_A = -\frac{1}{2}(\partial_1\xi_A + \tau\xi_A) + \tilde{\nabla}_B{\chi_A}^B - \frac{1}{2}\partial_A(\tau - \overline{W}_1 + 2\nu^0\partial_1\nu_0).\tag{8.26}$$

Separating different orders of ∂_1 derivatives we get, for a Minkowski target,

$$\begin{aligned}\mathcal{C}_A \overset{\eta}{=}\ & \nu^0\partial^2_{11}\nu_A - 2\nu^0{\chi_A}^B\partial_1\nu_B + \nu^0\left(\tau + \frac{1}{x^1} - \frac{1}{2}\overline{W}_1 - \nu^0\partial_1\nu_0\right)\partial_1\nu_A\\ &-\nu^0\left(2\partial_1{\chi_A}^B + 2(\nu_0\partial_1\nu^0 + \tau){\chi_A}^B\right)\nu_B + \tilde{\nabla}_B{\chi_A}^B\\ &-\left(\frac{1}{2}\partial_1\overline{W}_1 + \frac{1}{2}(\nu_0\partial_1\nu^0 + \tau)\left(\overline{W}_1 - \frac{2}{x^1}\right) + \frac{1}{(x^1)^2}\right)\nu^0\nu_A\\ &-\partial_A\left(\frac{1}{2}\tau + \nu^0\partial_1\nu_0 - \frac{1}{2}\overline{W}_1\right) - \frac{1}{2}(\partial_1 + \tau)(\overline{g}_{AB}\overline{g}^{CD}S^B_{CD} - \tilde{\Gamma}_A).\end{aligned}\tag{8.27}$$

In the general case, in addition to (7.6)–(7.7) one can assume that

$$\overline{W}_A = \overline{W}_A(\gamma_{AB}, \varphi, \nu_0, \nu_A, r, x^A),\tag{8.28}$$

$$\overline{T}_{1A} = \overline{T}_{1A}(\text{source data}, \gamma_{AB}, \partial_i\gamma_{AB}, \varphi, \partial_i\varphi, \nu_0, \partial_i\nu_0, \nu_A, \partial_1\nu_A, \overline{\partial_0 g_{11}}, r, x^A),\tag{8.29}$$

where as before ∂_i denotes derivatives tangential to the light-cone. This is clearly compatible with the wave-map gauge (4.19), and with scalar fields or Maxwell fields as sources (compare Sect. 7.7).

9. Solution of the $\mathcal{C}_A$ Constraint

9.1. NCT Case

In the vacuum case with Minkowski target and when ${\sigma_A}^B = 0$ we have ${\chi_A}^B = r^{-1}\delta^B_A, \overline{g}_{AB} = r^2 s_{AB}$ (therefore $\tilde{\Gamma}^A_{BC} = S^A_{BC}$), $\nu_0 = 1$ and $\tau = -\overline{W}_1 = \frac{n-1}{r}$. It has been shown in [12] that the $\mathcal{C}_A$ wave-map-gauge constraint reduces then to

$$\mathcal{C}_A \equiv \overline{R}_{1A} - \mathcal{L}_A \equiv \partial^2_{11}\nu_A + \frac{3n-5}{2r}\partial_1\nu_A + \frac{1}{2}\frac{(n-2)(n-3)}{r^2}\nu_A = 0. \quad (9.1)$$

This is a Fuchsian type linear equation, with Fuchsian exponents $p = \frac{3-n}{2}$ and $2-n$. Thus, the only solution satisfying (4.20), i.e. $\lim_{r\to 0}(r^{-1}\nu_A) = 0$, is $\nu_A \equiv 0$. (In fact, the only solution $\nu_A = o(r^{\frac{3-n}{2}})$ is zero.)

9.2. General Case

From the identity (8.26) we then find

Lemma 9.1. *Assuming (7.6)–(7.7) and (8.28)–(8.29), the wave-map-gauge constraint operator $\mathcal{C}_A \equiv \overline{S}_{1A} - \mathcal{L}_A$ is a first-order linear ordinary differential operator for the field ξ_A, with κ as in (7.11)*

$$\mathcal{C}_A \equiv -\frac{1}{2}(\partial_1\xi_A + \tau\xi_A) + \tilde{\nabla}_B\sigma_A{}^B - \frac{n-2}{n-1}\partial_A\tau - \partial_A\kappa, \quad (9.2)$$

where ξ_A is defined as (8.25), which particularizes for a Minkowski target as

$$\xi_A \overset{\eta}{:=} -2\nu^0\partial_1\nu_A + 4\nu^0\nu_B\chi_A{}^B + \left(\overline{W}^0 - \frac{2}{r}\nu^0\right)\nu_A + \gamma_{AB}\gamma^{CD}(S^B_{CD} - \tilde{\Gamma}^B_{CD}). \quad (9.3)$$

Anticipating, we note that ν_A will also appear in the last wave-map-gauge constraint $\mathcal{C}_0$ through ξ_A only.

If one assumes that $\overline{T}_{1A}$ is known (e.g., in vacuum, or for scalar fields, compare (7.70)), the homogeneous part of the equation $\mathcal{C}_A = \overline{T}_{1A}$ reads

$$-\frac{1}{2}(\partial_1\xi_A + \tau\xi_A) = 0,$$

and admits as general solution, keeping in mind that $\tau \equiv (n-1)\partial_1 \log\varphi$,

$$\xi_A = \check{\xi}_A\varphi^{-(n-1)}, \quad (9.4)$$

for some vector field on the sphere $\check{\xi}_A$. The final solution ξ_A is of the form (9.4), with $\check{\xi}_A$ obtained by integrating the following equation for $\check{\xi}_A$

$$\partial_1\check{\xi}_A = 2\varphi^{n-1}\{\tilde{\nabla}_B\sigma_A{}^B - \frac{n-2}{n-1}\partial_A\tau - \partial_A\kappa - \overline{T}_{1A}\}, \quad (9.5)$$

with the boundary condition $\check{\xi}_A = 0$, deduced from the finiteness of

$$\lim_{r\to 0}\xi_A = \lim_{r\to 0} r^{-n+1}\check{\xi}_A$$

(compare (4.42)–(4.45), and(4.49)). The field ν_A is then obtained by integrating (9.3), with the boundary condition $\nu_A = 0$ at $r = 0$. These equations constitute a first-order linear system of ODEs with coefficients singular for $r = 0$.

In the NCT case we have $\xi_A \equiv 0$ and $\overline{W}^0 \overset{\eta}{\equiv} -x^1\overline{g}^{AB}s_{AB}$, and hence

$$2\partial_1\nu_A + \frac{n-3}{r}\nu_A = 0, \quad (9.6)$$

whose general solution is

$$\nu_A = k_A r^{-\frac{n-3}{2}}$$

with k_A independent of $x^1 \equiv r$. The solution tending to zero with r, and compatible with the boundary condition (4.43) for $n \geq 2$, is $\nu_A = 0$. In the general case, but with Minkowski target, the Eq. (9.3) reads

$$2\partial_1 \nu_A + \frac{n-3}{r}\nu_A - 4\nu_C \sigma_A{}^C + \left\{ r\nu_0 \overline{g}^{AB} s_{AB} - \frac{n-1}{r} + \frac{4\psi}{n-1} \right\} \nu_A \\ \stackrel{\eta}{=} \nu_0 \gamma_{AB} \gamma^{CD} (S^B_{CD} - \tilde{\Gamma}^B_{CD}) - \nu_0 \xi_A. \tag{9.7}$$

Setting

$$\nu_A = k_A r^{-\frac{n-3}{2}}$$

gives for k_A, with k_A which must tend to zero with r, a differential system with coefficients continuous and right-hand side tending to zero like $r^{\frac{n-3}{2}}$,

$$\partial_1 k_A \stackrel{\eta}{=} 2k_C \sigma_A{}^C + \lambda k_A + \mu_A$$

with

$$\lambda := -\frac{1}{2}\left\{ r\nu_0 \overline{g}^{AB} s_{AB} - \frac{n-1}{r} + \frac{4\psi}{n-1} \right\},$$

$$\mu_A := \frac{1}{2} r^{\frac{n-3}{2}} \{\nu_0 \gamma_{AB} \gamma^{CD} (S^B_{CD} - \tilde{\Gamma}^B_{CD}) - \nu_0 \xi_A\}.$$

Such a system can be solved by iterated integration starting from $k_A^{(0)} = 0$,

$$k_A^{(p)} = \int_0^r \{2k_C^{(p-1)} \sigma_A{}^C + \lambda k_A^{(p-1)} + \mu_A\}(\rho) \mathrm{d}\rho.$$

Convergence, and the bound $|k_A| \leq C r^{\frac{n+1}{2}}$, result from the bounds of σ, λ and μ. In conclusion, in vacuum, and in the wave-map gauge, the solution of (9.5) exists as long as τ does, with $\nu_A \in C^1$ and $|\nu_A| \leq C r^2$.

9.3. Vanishing of $\overline{H}_A$

The general identity (2.8) gives in our coordinates

$$\mathcal{C}_A + \mathcal{L}_A \equiv \overline{R}_{1A} \equiv \overline{R}^{(H)}_{1A} + \frac{1}{2}(\nu_0 \overline{\hat{D}_A H^0} + \nu_A \overline{\hat{D}_1 H^0} + \overline{g}_{AB} \overline{\hat{D}_1 H^B}), \tag{9.8}$$

with $\hat{D}$ the covariant derivative of the target metric, which in this subsection will be chosen to be the Minkowski metric, and hence $\hat{D} = D$. Therefore, if a metric solves $R^{(H)}_{1A} = T_{1A}$ and $\mathcal{C}_A = \overline{T}_{1A}$, we will have, taking $\overline{H}^0 = 0$ (which, for sufficiently regular solutions, and for a Minkowski target, has been justified in Sect. 7.6) on the left-hand side,

$$\mathcal{L}_A = -\frac{1}{2}\{\partial_1 \overline{H}_A + \tau \overline{H}_A\} \stackrel{\eta}{=} \frac{1}{2}(\nu_0 \overline{D_A H^0} + \nu_A \overline{D_1 H^0} + \overline{g}_{AB} \overline{D_1 H^B}), \tag{9.9}$$

For Minkowski target these derivatives are, still for $\overline{H}^0 = 0$,

$$\overline{D_A H^0} \stackrel{\eta}{=} \hat{\Gamma}^0_{AB}\overline{H}^B \stackrel{\eta}{=} -rs_{AB}\overline{H}^B, \qquad \overline{D_1 H^0} \stackrel{\eta}{=} 0, \tag{9.10}$$

$$\overline{D_1 H^B} \stackrel{\eta}{=} \partial_1 \overline{H}^B + \hat{\Gamma}^B_{1C}\overline{H}^C \stackrel{\eta}{=} \overline{g}^{BC}\partial_1\overline{H}_C + \overline{H}_C\overline{g}^{CD}\left(\frac{1}{r}\delta^B_D - 2\chi_D{}^B\right). \tag{9.11}$$

Therefore, (9.9) reads

$$2\partial_1\overline{H}_A + \tau\overline{H}_A \stackrel{\eta}{=} \left\{\nu_0 r s_{AB}\overline{g}^{BC} - \frac{1}{r}\delta^C_A + 2\chi_A{}^C\right\}\overline{H}_C. \tag{9.12}$$

Taking leading orders in r near the vertex, as given in Sect. 4.5, we find

$$\partial_1\overline{H}_A + \left(\frac{n-3}{2r}\delta^B_A + O^B_A\right)\overline{H}_B \stackrel{\eta}{=} 0, \tag{9.13}$$

where the O^B_A are $O(r)$ functions. Hence,

$$\overline{H}_A = r^{-\frac{n-3}{2}}k_A, \qquad \text{with} \qquad \partial_1 k_A + O^B_A k_B = 0. \tag{9.14}$$

Standard ODE arguments show that $\overline{H}_A = 0$ is the only solution of (9.12) such that $\overline{H}_A = O(r)$, which is the case for metrics having uniform C^1 estimates at the vertex.

Remark 9.2. If we add constraint damping terms as in (2.13) we obtain instead, using again $\overline{H}^0 = 0$,

$$\mathcal{L}_A = -\frac{1}{2}\partial_1\overline{H}_A + \frac{1}{2}(-\tau + \epsilon n_1)\overline{H}_A. \tag{9.15}$$

No term proportional to $\overline{H}^1$ appears, and hence the constraint damping term is compatible with this second step of the constraint hierarchy. The new term does not change the leading orders in r of Eq. (9.13) and hence $\overline{H}_A = 0$ is still the only regular solution.

10. The $\mathcal{C}_0$ Constraint

We compute $S_{01} \equiv S_{0\alpha}\ell^\alpha$ on C_O. We have

$$S_{01} := R_{01} - \frac{1}{2}g_{01}R, \tag{10.1}$$

hence, in our coordinates

$$\overline{S}_{01} \equiv -\frac{1}{2}\nu_0\overline{g}^{AB}\overline{R}_{AB} + \overline{R}_{1A}\nu^A - \frac{1}{2}\nu_0\overline{g}^{11}\overline{R}_{11}. \tag{10.2}$$

We write

$$\overline{R}_{AB} := \overline{R}^{(1)}_{AB} + \overline{R}^{(2)}_{AB}, \tag{10.3}$$

with

$$\overline{R}^{(1)}_{AB} := \overline{\partial_\alpha\Gamma^\alpha_{AB}} - \partial_A\overline{\Gamma}^\alpha_{B\alpha}, \qquad \overline{R}^{(2)}_{AB} := \overline{\Gamma}^\alpha_{AB}\overline{\Gamma}^\lambda_{\alpha\lambda} - \overline{\Gamma}^\alpha_{A\beta}\overline{\Gamma}^\beta_{B\alpha}. \tag{10.4}$$

We will see that the $\mathcal{C}_0$ wave-map-gauge constraint is obtained, like the other constraints, by decomposing the term in $\overline{S}_{01}$ which has not already been computed, $\overline{g}^{AB}\overline{R}_{AB}$, into terms defined by data of the degenerate metric on the cone and terms which vanish when harmonicity conditions are satisfied on the cone.

Equations (4.27) and (4.31) allow us to express the transversal derivative $\overline{g}^{AB}\overline{\partial_0 g_{AB}}$ in terms of harmonicity functions,

$$\begin{aligned}\frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}} &\equiv -\overline{\Gamma}^1 + (\nu^0)^2\partial_1\overline{g}_{00} + \overline{g}^{11}\left(\nu^0\partial_1\nu_0 - \frac{1}{2}\tau + \frac{1}{2}\overline{\Gamma}_1\right)\\ &\quad + 2(\nu^0)^2\nu^A(-\partial_1\nu_A + \nu_B{\chi_A}^B) + \nu^0\overline{g}^{AB}\tilde{\nabla}_B\nu_A \qquad (10.5)\\ &\equiv -\overline{\Gamma}^1 - \partial_1\overline{g}^{11} + \overline{g}^{11}\left(-\nu^0\partial_1\nu_0 - \frac{1}{2}\tau + \frac{1}{2}\overline{\Gamma}_1\right) + \nu^0\overline{g}^{AB}\tilde{\nabla}_B\nu_A. \qquad (10.6)\end{aligned}$$

10.1. Computation of $\overline{g}^{AB}\overline{R}^{(1)}_{AB}$

We have

$$\overline{g}^{AB}\overline{R}^{(1)}_{AB} := \overline{g}^{AB}\left\{\overline{\partial_0\Gamma^0_{AB}} + \partial_1\overline{\Gamma}^1_{AB} + \partial_C\overline{\Gamma}^C_{AB} - \partial_A\overline{\Gamma}^\alpha_{B\alpha}\right\}. \qquad (10.7)$$

To compute we proceed in a straightforward way, using the values of the Christoffel symbols of the first kind and elementary algebraic relations in our coordinates on the cone. Equations (A.36) and (A.37) of Appendix A are useful for the calculations that follow.

We set

$$\overline{g}^{AB}\overline{\partial_0\Gamma^0_{AB}} \equiv I_1 + II_1, \qquad (10.8)$$

with

$$I_1 := \overline{g}^{AB}\nu^0\overline{\partial_0[1,AB]} \equiv -\frac{1}{2}\overline{g}^{AB}\nu^0\partial_1\overline{\partial_0 g_{AB}} + \overline{g}^{AB}\nu^0\partial_A\overline{\partial_0 g_{1B}}, \qquad (10.9)$$

$$\begin{aligned}II_1 &:= \overline{g}^{AB}\overline{\partial_0 g^{0\alpha}[\alpha,AB]} \equiv (\nu^0)^2\overline{\partial_0 g_{11}}\overline{g}^{AB}\left(\frac{1}{2}\overline{\partial_0 g_{AB}} - \partial_A\nu_B\right)\\ &\quad -\overline{\partial_0 g^{01}}\tau + \overline{\partial_0 g^{0C}}\tilde{\Gamma}_C, \qquad (10.10)\end{aligned}$$

and

$$\begin{aligned}\overline{\partial_0 g^{01}} &= -(\nu^0)^2\overline{\partial_0 g_{01}} - \nu^0\overline{g}^{11}\overline{\partial_0 g_{11}} + (\nu^0)^2\nu^C\overline{\partial_0 g_{1C}}, \qquad (10.11)\\ \overline{\partial_0 g^{0C}} &= (\nu^0)^2\nu^C\overline{\partial_0 g_{11}} - \nu^0\overline{g}^{CA}\overline{\partial_0 g_{1A}}. \qquad (10.12)\end{aligned}$$

Grouping terms gives

$$\begin{aligned}\overline{g}^{AB}\overline{\partial_0\Gamma^0_{AB}} &\equiv -\frac{1}{2}\nu^0\overline{g}^{AB}\partial_1\overline{\partial_0 g_{AB}} + \nu^0\overline{g}^{AB}\tilde{\nabla}_A\overline{\partial_0 g_{1B}}\\ &\quad +(\nu^0)^2\overline{\partial_0 g_{11}}\left\{\frac{1}{2}\overline{g}^{AB}\overline{\partial_0 g_{AB}} - \tilde{\nabla}_A\nu^A + \nu_0\tau\overline{g}^{11}\right\}\\ &\quad +(\nu^0)^2\overline{\partial_0 g_{01}}\tau - (\nu^0)^2\nu^A\overline{\partial_0 g_{1A}}\tau. \qquad (10.13)\end{aligned}$$

We now separate

$$\overline{g}^{AB}\overline{\partial_1\Gamma^1_{AB}} \equiv III_1 + IV_1, \qquad (10.14)$$

with

$$
\begin{aligned}
III_1 &\equiv \overline{g}^{AB}\left\{\nu^0\partial_1\overline{[0,AB]}+\overline{g}^{11}\partial_1\overline{[1,AB]}+\overline{g}^{1C}\partial_1\overline{[C,AB]}\right\}\\
&= -\frac{1}{2}\overline{g}^{AB}\nu^0\partial_1\overline{\partial_0 g_{AB}}+\overline{g}^{AB}\nu^0\partial_1\partial_A\nu_B-\overline{g}^{11}\overline{g}^{AB}\partial_1\chi_{AB}\\
&\quad -\nu^0\nu^C\overline{g}^{AB}\partial_1\overline{[C,AB]}, \qquad (10.15)\\
IV_1 &:= \overline{g}^{AB}\partial_1\overline{g}^{1\alpha}\overline{[\alpha,AB]}\\
&\equiv \overline{g}^{AB}\partial_1\nu^0(\partial_A\nu_B-\frac{1}{2}\overline{\partial_0 g_{AB}})-\tau\partial_1\overline{g}^{11}-\overline{g}^{AB}\partial_1(\nu^0\nu^C)\overline{[C,AB]}. \qquad (10.16)
\end{aligned}
$$

Grouping terms gives

$$
\begin{aligned}
\overline{g}^{AB}\overline{\partial_1\Gamma^1_{AB}} \equiv{}& -\frac{1}{2}\partial_1(\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}})-\nu^0\chi^{AB}\overline{\partial_0 g_{AB}}+\overline{g}^{AB}\partial_1(\nu^0\tilde{\nabla}_A\nu_B)\\
&-\overline{g}^{11}\overline{g}^{AB}\partial_1\chi_{AB}-\tau\partial_1 g^{11}. \qquad (10.17)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\overline{g}^{AB}\partial_C\overline{\Gamma}^C_{AB} &\equiv \overline{g}^{AB}\nu^C\nu^0\partial_C\chi_{AB}+\tau\partial_C(\nu^C\nu^0)+\overline{g}^{AB}\partial_C\tilde{\Gamma}^C_{AB}, \qquad (10.18)\\
-\overline{g}^{AB}\partial_A\overline{\Gamma}^\alpha_{B\alpha} &\equiv -\overline{g}^{AB}\partial^2_{AB}(\log\sqrt{\det g}) \equiv -\overline{g}^{AB}\partial^2_{AB}\{\log(\nu_0\sqrt{\det\tilde{g}})\}\\
&\equiv -\overline{g}^{AB}\{\partial_A\nu^0\partial_B\nu_0+\nu^0\partial^2_{AB}\nu_0+\partial_A\tilde{\Gamma}^C_{BC}\}. \qquad (10.19)
\end{aligned}
$$

10.2. Computation of $\overline{g}^{AB}\overline{R}^{(2)}_{AB}$

We set

$$
g^{AB}\overline{R}^{(2)}_{AB} := \overline{g}^{AB}\{\overline{\Gamma}^\alpha_{AB}\overline{\Gamma}^\beta_{\alpha\beta}-\overline{\Gamma}^\alpha_{A\beta}\overline{\Gamma}^\beta_{B\alpha}\} \equiv (I_2+II_2+III_2+IV_2+V_2+VI_2), \qquad (10.20)
$$

with

$$
I_2 := \overline{g}^{AB}\overline{\Gamma}^0_{AB}\overline{\Gamma}^\beta_{0\beta}, \qquad II_2 := \overline{g}^{AB}\overline{\Gamma}^1_{AB}\overline{\Gamma}^\beta_{1\beta}. \qquad (10.21)
$$

We find by straightforward computation

$$
I_2 \equiv -\nu^0\tau\left\{\nu^0\overline{\partial_0 g_{01}}+\frac{1}{2}\overline{g}^{11}\overline{\partial_0 g_{11}}-\nu^0\nu^A\overline{\partial_0 g_{1A}}+\frac{1}{2}\overline{g}^{AB}\overline{\partial_0 g_{AB}}\right\}, \qquad (10.22)
$$

$$
II_2 \equiv \left\{\nu^0\tilde{\nabla}_A\nu^A-\frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}}-\overline{g}^{11}\tau\right\}(\nu^0\partial_1\nu_0+\tau). \qquad (10.23)
$$

Then we have, recalling that $\tilde{\Gamma}$ denotes Christoffel symbols of the metric $\tilde{g}$,

$$
III_2 := \overline{g}^{AB}\overline{\Gamma}^C_{AB}\overline{\Gamma}^\beta_{C\beta} \equiv (\nu^0\nu^C\tau+\overline{g}^{AB}\tilde{\Gamma}^C_{AB})(\tilde{\Gamma}^D_{CD}+\nu^0\partial_C\nu_0), \qquad (10.24)
$$

Next,

$$
IV_2 := -\overline{g}^{AB}\{\overline{\Gamma}^0_{A0}\overline{\Gamma}^0_{B0}+\overline{\Gamma}^1_{A1}\overline{\Gamma}^1_{B1}\}. \qquad (10.25)
$$

Furthermore,

$$
\overline{\Gamma}^1_{1A} \equiv -\overline{\Gamma}^0_{0A}-\nu^0\nu_B\chi_A{}^B+\nu^0\partial_A\nu_0, \qquad (10.26)
$$

with (ζ_A is sometimes called the torsion form)

$$
\overline{\Gamma}^0_{0A} \equiv \frac{1}{2}\nu^0\zeta_A, \qquad \zeta_A := \overline{\partial_0 g_{1A}}+\partial_A\nu_0-\partial_1\nu_A. \qquad (10.27)
$$

Hence, using (8.19),

$$\zeta_A = 2\partial_A\nu_0 - 2\partial_1\nu_A + 2\nu_B\chi_A{}^B + \nu_0(f_A + \overline{H}_A + \nu_A\overline{H}^0). \tag{10.28}$$

In terms of this object we have

$$IV_2 \equiv -\overline{g}^{AB}(\nu^0)^2 \times \left\{\frac{1}{2}\zeta_A\zeta_B + \zeta_A(\nu_C\chi_B{}^C - \partial_B\nu_0) + (\nu_C\chi_A{}^C - \partial_A\nu_0)(\nu_D\chi_B{}^D - \partial_B\nu_0)\right\}. \tag{10.29}$$

We set

$$\begin{aligned} V_2 &:= -2\overline{g}^{AB}(\overline{\Gamma}^0_{AC}\overline{\Gamma}^C_{B0} + \overline{\Gamma}^1_{AC}\overline{\Gamma}^C_{B1}) \\ &= \chi^{AB}\{-(\nu^0)^2\nu_A\zeta_B + 2\nu^0\overline{\partial_0 g_{AB}} - 2\nu^0\tilde{\nabla}_A\nu_B + 2\overline{g}^{11}\chi_{AB}\}. \end{aligned} \tag{10.30}$$

Finally,

$$\begin{aligned} VI_2 &\equiv -\overline{g}^{AB}\overline{\Gamma}^C_{AD}\overline{\Gamma}^D_{BC} \equiv -\overline{g}^{AB}(\nu^0\nu^C\chi_{AD} + \tilde{\Gamma}^C_{AD})(\nu^0\nu^D\chi_{BC} + \tilde{\Gamma}^D_{BC}) \\ &= -(\nu^0)^2\nu^C\nu^D\chi_C{}^A\chi_{AD} - 2\tilde{\Gamma}^C_{AD}\nu^0\nu^D\chi_C{}^A - \overline{g}^{AB}\tilde{\Gamma}^C_{AD}\tilde{\Gamma}^D_{BC}. \end{aligned} \tag{10.31}$$

10.3. Final Computation of $\overline{g}^{AB}\overline{R}_{AB}$

Adding the results of Sects. 10.1 and 10.2, we get the final result

$$\begin{aligned} \overline{g}^{AB}\overline{R}_{AB} &\equiv 2(\partial_1 + \overline{\Gamma}^1_{11})^2\overline{g}^{11} + 3\tau(\partial_1 + \overline{\Gamma}^1_{11})\overline{g}^{11} + (\partial_1\tau + \tau^2)\overline{g}^{11} \\ &\quad + 2(\partial_1 + \overline{\Gamma}^1_{11} + \tau)\overline{\Gamma}^1 + \tilde{R} - 2\overline{g}^{AB}\overline{\Gamma}^1_{1A}\overline{\Gamma}^1_{1B} - 2\overline{g}^{AB}\tilde{\nabla}_A\overline{\Gamma}^1_{1B} \end{aligned} \tag{10.32}$$

$$\begin{aligned} &\equiv 2(\partial_1 + \overline{\Gamma}^1_{11} + \tau)\left[(\partial_1 + \overline{\Gamma}^1_{11} + \frac{\tau}{2})\overline{g}^{11} + \overline{\Gamma}^1\right] \\ &\quad + \tilde{R} - 2\overline{g}^{AB}\overline{\Gamma}^1_{1A}\overline{\Gamma}^1_{1B} - 2\overline{g}^{AB}\tilde{\nabla}_A\overline{\Gamma}^1_{1B} \end{aligned} \tag{10.33}$$

(an explicit expression for $\overline{\Gamma}^1_{11}$ can be found in (4.22)), where

$$-2\overline{\Gamma}^1_{1A} = \nu^0\overline{\partial_0 g_{1A}} - \nu^0\partial_1\nu_A + 2\nu^0\nu_B\chi_A{}^B - \nu^0\partial_A\nu_0. \tag{10.34}$$

In this way we have isolated the transversal derivatives in $\overline{\Gamma}^1_{11}, \overline{\Gamma}^1$ and the vector $\overline{\Gamma}^1_{1A}$. Decomposing $\overline{\Gamma}_A = \overline{W}_A + \overline{H}_A$ we find the relation

$$-2\overline{\Gamma}^1_{1A} = \xi_A + \overline{H}_A + \nu^0\nu_A\overline{H}_1, \tag{10.35}$$

with ξ_A defined in (8.25).

We note the interesting fact that both the second and third constraints naturally break into two first-order equations, with the intermediate variable being a Christoffel, respectively, $\overline{\Gamma}^1_{1A}$ and $\overline{g}^{AB}\overline{\Gamma}^1_{AB}$.

Remark 10.1. The expression in square brackets in (10.33) can be rewritten as

$$\left(\partial_1 + \overline{\Gamma}^1_{11} + \frac{\tau}{2}\right)\overline{g}^{11} + \overline{\Gamma}^1 = -\frac{\tau}{2}\overline{g}^{11} + \nu^0\overline{g}^{AB}\left(\tilde{\nabla}_B\nu_A - \frac{1}{2}\overline{\partial_0 g_{AB}}\right) = \overline{g}^{AB}\overline{\Gamma}^1_{AB}. \tag{10.36}$$

This shows that $\overline{g}^{AB}\overline{R}_{AB}$ originally contains only a first-order radial derivative of $\overline{g}^{11}$, if we keep the radial derivative of $\overline{g}^{AB}\overline{\partial_0 g_{AB}}$. It is precisely the elimination of the latter object using the harmonicity condition (10.6) that introduces the second-order radial derivative of $\overline{g}^{11}$. Hence, our $\mathcal{C}_0$ constraint operator below will contain such second-order derivative.

This leads to the following lemmata:

Lemma 10.2. *All terms in $\overline{g}^{AB}\overline{R}_{AB}$ involving the derivatives $\overline{\partial_0 g_{01}}$ and $\chi^{AB}\overline{\partial_0 g_{AB}}$ cancel out. The only new remaining transversal derivative is $\overline{g}^{AB}\overline{\partial_0 g_{AB}}$, which can be eliminated using $\overline{\Gamma}^1 \equiv \overline{H}^1 + \overline{W}^1$.*

Lemma 10.3. *It holds that*

$$\overline{S}_{01} \equiv -\frac{1}{2}\nu_0\overline{g}^{AB}\overline{R}_{AB} + \overline{R}_{1A}\nu^A - \frac{1}{2}\nu_0\overline{g}^{11}\overline{R}_{11} \equiv \mathcal{C}_0 + \mathcal{L}_0, \tag{10.37}$$

where $\mathcal{C}_0$ depends only on the quadratic form $\tilde{g}$ on the cone and the $\overline{W}^\alpha$, while $\mathcal{L}_0$ is obtained by replacing $\overline{\Gamma}^\alpha$ by $\overline{H}^\alpha$.

The explicit formula for $\mathcal{L}_0$ reads

$$\begin{aligned} 2\nu^0\mathcal{L}_0 = & -2\overline{g}^{1A}\mathcal{L}_A - \overline{g}^{11}\mathcal{L}_1 - 2\partial_1\overline{H}^1 - \left(\tau + 2\nu^0\partial_1\nu_0 - \nu_0\overline{W}^0\right)\overline{H}^1 \\ & -\tilde{\nabla}_A\overline{H}^A + \xi_A\overline{H}^A + \nu_0\overline{g}^{11}\partial_1\overline{H}^0 - \nu^A\partial_A\overline{H}^0 \\ & +\Bigg(\overline{g}_{00}\overline{W}^0 + \nu_0\overline{W}^1 + \nu_A\overline{W}^A + \frac{1}{2}\nu_0\tau\overline{g}^{11} \\ & -\overline{g}^{11}\partial_1\nu_0 - 2\nu^0\partial_1\overline{g}_{00} - \overline{g}^{AB}\partial_A\nu_B + 2\nu^0\nu^A\partial_1\nu_A\Bigg)\overline{H}^0 \\ & +\frac{1}{2}\left\{\overline{g}_{00}(\overline{H}^0)^2 + 2\nu_0\overline{H}^0\overline{H}^1 + 2\nu_A\overline{H}^0\overline{H}^A + \overline{g}_{AB}\overline{H}^A\overline{H}^B\right\}. \end{aligned} \tag{10.38}$$

Note that the last line in the previous equation is quadratic in the wave-gauge vector $\overline{H}$, and equals

$$\frac{1}{2}\overline{g}_{\mu\nu}\overline{H}^\mu\overline{H}^\nu. \tag{10.39}$$

All other terms are linear in $\overline{H}$.

10.4. Constraint

To write the wave-map-gauge constraint $\mathcal{C}_0 - \overline{T}_{01} = 0$ as an equation for $\overline{g}^{11}$, we use the other constraints, which have been satisfied since $\mathcal{L}_1 = \mathcal{L}_A = 0 = \overline{H}_1 = \overline{H}_A$,

$$\overline{R}_{1A} = \overline{T}_{1A}, \quad \overline{R}_{11} = \overline{T}_{11}. \tag{10.40}$$

We find

$$\begin{aligned} -2\nu^0(\mathcal{C}_0 - \overline{T}_{01}) \equiv & \ 2(\partial_1 + \kappa)^2\overline{g}^{11} + 3\tau(\partial_1 + \kappa)\overline{g}^{11} + (\partial_1\tau + \tau^2)\overline{g}^{11} \\ & +2(\partial_1 + \kappa + \tau)\overline{W}^1 + \tilde{R} - \frac{1}{2}\overline{g}^{AB}\xi_A\xi_B + \overline{g}^{AB}\tilde{\nabla}_A\xi_B \\ & +\overline{g}^{11}\overline{T}_{11} + 2\overline{g}^{1A}\overline{T}_{1A} + 2\overline{g}^{01}\overline{T}_{01} \\ = & \ 0, \end{aligned} \tag{10.41}$$

where ξ_A is the vector (8.25) and recall that

$$\kappa \equiv \nu^0\partial_1\nu_0 - \frac{1}{2}(\overline{W}_1 + \tau).$$

To avoid ambiguities, we emphasise that the right-hand-side of (10.41) vanishes identically in wave-map gauge when $T_{\mu\nu}$ there is replaced by the Einstein tensor $S_{\mu\nu}$. This fact reflects the identity, valid for any dimension with our choice of coordinates,

$$\overline{g}^{AB}\overline{R}_{AB} + \overline{g}^{11}\overline{S}_{11} + 2\overline{g}^{1A}\overline{S}_{1A} + 2\overline{g}^{01}\overline{S}_{01} = 0. \tag{10.42}$$

A slightly simplified form of the differential part of the constraint is, using (10.33),

$$\begin{aligned} -2\nu^0(\mathcal{C}_0 - \overline{T}_{01}) \equiv\ & 2(\partial_1 + \kappa + \tau)\left[(\partial_1 + \kappa + \frac{\tau}{2})\overline{g}^{11} + \overline{W}^1\right] \\ & +\tilde{R} - \frac{1}{2}\overline{g}^{AB}\xi_A\xi_B + \overline{g}^{AB}\tilde{\nabla}_A\xi_B \\ & +\overline{g}^{11}\overline{T}_{11} + 2\overline{g}^{1A}\overline{T}_{1A} + 2\overline{g}^{01}\overline{T}_{01} \\ =\ & 0. \end{aligned} \tag{10.43}$$

Suppose that in addition to (7.6)–(7.7) and (8.28)–(8.29) it holds that

$$\overline{W}_0 = \overline{W}_0(\gamma_{AB}, \varphi, \nu_0, \nu_A, \overline{g}^{11}, r, x^A), \tag{10.44}$$

$$\overline{T}_{11} = \overline{T}_{11}(\ldots, \overline{\partial_0 g_{1A}}, \overline{g}^{11}, \partial_1\overline{g}^{11}), \tag{10.45}$$

where ... in (10.45) denotes the collection of fields already occurring in (8.29). This is clearly compatible with the wave-map gauge (4.18), and with scalar fields or Maxwell fields as sources (compare Sect. 7.7). Then, (10.41) becomes a second-order ODE for $\overline{g}^{11}$, linear when the vacuum Einstein equations and the wave-map gauge have been assumed.

11. Solution of the $\mathcal{C}_0$ Constraint

Throughout this section we assume that the target metric is Minkowski, $\kappa = 0$ and that the relevant components of the tensor $\overline{T}$ are known (e.g., zero). Using the $\mathcal{C}_1$ constraint,

$$\overline{T}_{11} = -\left(\partial_1\tau + \frac{1}{n-1}\tau^2 + |\sigma|^2\right), \tag{11.1}$$

we find that the $\mathcal{C}_0$ wave-map-gauge constraint operator can be written as

$$\begin{aligned} \nu^0\mathcal{C}_0 \equiv & -\partial^2_{11}\overline{g}^{11} - \frac{3}{2}\tau\partial_1\overline{g}^{11} - \frac{1}{2}\left(\frac{n-2}{n-1}\tau^2 - |\sigma|^2\right)\overline{g}^{11} \\ & -\partial_1\overline{W}^1 - \tau\overline{W}^1 - \frac{1}{2}\tilde{R} + \frac{1}{4}\overline{g}^{AB}\xi_A\xi_B - \frac{1}{2}\overline{g}^{AB}\tilde{\nabla}_A\xi_B + \nu^0\overline{T}_{1A}\nu^A. \end{aligned} \tag{11.2}$$

Hence, setting $\overline{g}^{11} \equiv 1 - \alpha$ and using previous notations, the equation for the $\mathcal{C}_0$ wave-map-gauge constraint, $\mathcal{C}_0 - \overline{T}_{01} = 0$, reads as the linear second-order

ODE for α

$$L(\alpha) \equiv \alpha'' + \frac{3}{2}\tau\alpha' + \frac{1}{2}\left(\frac{n-2}{n-1}\tau^2 - |\sigma|^2\right)\alpha = \Phi, \tag{11.3}$$

with Φ the function known from the previous sections

$$\begin{aligned}\Phi :=& \frac{1}{2}\left(\frac{n-2}{n-1}\tau^2 - |\sigma|^2\right) + \partial_1\overline{W}^1 + \tau\overline{W}^1\\ &+\frac{1}{2}\tilde{R} - \frac{1}{4}\overline{g}^{AB}\xi_A\xi_B + \frac{1}{2}\overline{g}^{AB}\tilde{\nabla}_A\xi_B - \nu^0\overline{T}_{1A}\nu^A + \nu^0\overline{T}_{01}.\end{aligned} \tag{11.4}$$

$L(\alpha)$ simplifies to

$$\begin{aligned}L(\alpha) \equiv& \alpha'' + \frac{3}{2}\left(\frac{n-1}{r} - \psi\right)\alpha'\\ &+\frac{1}{2}\left(\frac{(n-1)(n-2)}{r^2} - 2\frac{(n-2)}{r}\psi + \frac{n-2}{n-1}\psi^2 - |\sigma|^2\right)\alpha.\end{aligned} \tag{11.5}$$

This linear equation has smooth coefficients for $r > 0$; it has a global solution with initial data given for $r = a > 0$.

We proceed to the study of solutions starting from $r = 0$.

11.1. NCT Case

In the NCT case it holds that $\nu^0 = 1, f_A = 0, \nu_A = 0, \tau = \frac{n-1}{r} = -\overline{W}_1, \partial_1\overline{W}^1 + \tau\overline{W}^1 = -(n-1)(n-2)/r^2 = -\tilde{R}$ and $\overline{T}_{\alpha\beta} = 0$. Hence $\Phi = 0$. The $\mathcal{C}_0$ wave-map-gauge constraint for $\alpha = 1 - \overline{g}^{11}$ reduces to

$$2\alpha'' + \frac{3(n-1)}{r}\alpha' + \frac{(n-1)(n-2)}{r^2}\alpha = 0, \tag{11.6}$$

it is a Fuchsian equation with characteristic polynomial

$$2p(p-1) + 3(n-1)p + (n-1)(n-2).$$

The zeroes of this polynomial are

$$p_+ = \frac{1-n}{2}, \quad p_- = 2-n;$$

both negative or zero for $n \geq 2$. The general solution of (11.6) is

$$\alpha := a_+ r^{(1-n)/2} + a_- r^{2-n},$$

with $a_\pm$ independent of r. The only member of this general solution where α tends to zero as r tends to zero is $\alpha \equiv 0$.

11.2. General Case

We look for a solution starting from $r = 0$ and such that

$$\lim_{r\to 0}\alpha = \lim_{r\to 0}(r\partial_1\alpha) = 0.$$

We set $\partial_1\alpha = \alpha'$ and decompose L as follows:

$$L(\alpha) \equiv L_0(\alpha) + L_1(\alpha), \tag{11.7}$$

where L_0 is the Fuchsian operator appearing in the NCT case,

$$L_0(\alpha) \equiv \alpha'' + \frac{a_0}{r}\alpha' + \frac{b_0}{r^2}\alpha := \alpha'' + \frac{3(n-1)}{2r}\alpha' + \frac{(n-1)(n-2)}{2r^2}\alpha, \qquad (11.8)$$

and

$$L_1(\alpha) \equiv a_1\alpha' + b_1\alpha := -\frac{3}{2}\psi\alpha' + \left\{-\frac{(n-2)}{r}\psi + \frac{1}{2}\left(\frac{n-2}{n-1}\psi^2 - |\sigma|^2\right)\right\}\alpha. \qquad (11.9)$$

In order to use the idea of the Fuchs theorem,[11] we write the second-order equation (11.3) as a first-order system for a pair of unknowns $v := (v_1, v_2)$ by setting $v_1 := \alpha, v_2 \equiv r\alpha'$; hence $r\alpha'' = -r^{-1}v_2 + v_2'$. The system reads

$$rv_1' - v_2 = 0,$$
$$rv_2' + (a_0 - 1)v_2 + b_0 v_1 + r(a_1 v_2 + b_1 r v_1 - r\Phi) = 0.$$

It is of the form

$$rv' + Av = r\{F_1(r)v + F_0(r)\}, \qquad (11.10)$$

with A the constant linear operator

$$A \equiv \begin{pmatrix} 0 & -1 \\ b_0 & a_0 - 1 \end{pmatrix}$$

whose eigenvalues $\mu_\pm$ are found by solving the equation

$$\det\begin{pmatrix} 0-\mu & -1 \\ b_0 & a_0 - 1 - \mu \end{pmatrix} \equiv \mu^2 + \mu(1-a_0) + b_0 = 0.$$

The solutions are the opposites, $-p_\pm$, of the characteristic indices computed in the NCT case; hence nonnegative. Further

$$F_1(r)v \equiv \begin{pmatrix} 0 \\ -a_1 v_2 - r b_1 v_1 \end{pmatrix}, \quad F_0(r) \equiv \begin{pmatrix} 0 \\ r\Phi \end{pmatrix},$$

where

$$a_1 \equiv -\frac{3}{2}\psi, \quad rb_1 \equiv -(n-2)\psi + \frac{r}{2}\left(\frac{n-2}{n-1}\psi^2 - |\sigma|^2\right)$$

are bounded functions smooth away from $r = 0$, as well as $r\Phi$. What has been said shows that F_0 and F_1 are continuous at $r = 0$ for admissible γ_{AB}.

Lemma 11.1. *Let*

$$rv' + Av = r\{F_1(r)v + F_0(r)\}, \qquad (11.11)$$

be a linear differential system with A a constant linear operator with non-negative eigenvalues. Let F_1 be a continuous linear map and F_2 a continuous function, for $0 \le r \le r_0$. The system admits one and only one solution in $C^1([0, r_0])$ which vanishes at $r = 0$.

[11] See e.g. [10, Appendix V].

Proof. We set $v = Mw$, with M a 2×2 matrix satisfying the homogeneous equation

$$rM' + AM = 0.$$

We choose for M the matrix

$$M = r^{-A} \equiv e^{-A \log r}.$$

The Eq. (11.10) reads then

$$w' = r^A \{F_1(r)v + F_0(r)\}. \tag{11.12}$$

Hence, the Eq. (11.11) together with the condition $v|_{r=0} = 0$ is equivalent to the integral equation

$$v(r) = \int_0^r (r^{-1}\rho)^A \{F_1(\rho)v(\rho) + F_0(\rho)\} \mathrm{d}\rho. \tag{11.13}$$

We have

$$\sup_{0 \le \rho \le r} |(r^{-1}\rho)^A| \le 1. \tag{11.14}$$

We set, with a an arbitrary positive number,

$$C_1 := \sup_{0 \le r \le a} |F_1(r)|, \qquad C_0 := \sup_{0 \le r \le a} |F_0(r)|. \tag{11.15}$$

The integral equation (11.13) can then be solved by iteration, setting

$$v_0(r) = \int_0^r (r^{-1}\rho)^A F_0(\rho) \mathrm{d}\rho.$$

Hence, for $r \le a$

$$|v_0(r)| \le rC_0,$$
$$v_1(r) := \int_0^r (r^{-1}\rho)^A F_1(\rho) v_0(\rho) \mathrm{d}\rho + v_0(r),$$

and so

$$|v_1(r) - v_0(r)| \le \frac{1}{2} r^2 \, C_1 C_0,$$
$$v_{n+1}(r) := \int_0^r (r^{-1}\rho)^A F_1(\rho) v_n(\rho) \mathrm{d}\rho + v_0(r),$$
$$|v_{n+1}(r) - v_n(r)| \le \int_0^r C_1 |v_n(\rho) - v_{n-1}(\rho)| \mathrm{d}\rho.$$

Assume that, as satisfied for $n = 1$,

$$|v_n(\rho) - v_{n-1}(\rho)| \le \frac{1}{(n+1)!} r^{n+1} C_1^n C_0,$$

Then the same inequality is satisfied when replacing n by $n+1$. The sequence v_n converges therefore in uniform norm to a limit v, solution of the integral equation (11.13); hence of the differential equation (11.11). This solution $v := (\alpha, r\alpha'), \alpha(0) = (r\alpha')(0) = 0$, is defined, continuous and bounded for any finite r. □

We deduce from this lemma the following theorem:

Theorem 11.2. *In the interval of $r \geq 0$, possibly angle dependent, where the $\mathcal{C}_1$ constraint has a global solution and $\nu_0 > 0$, the $\mathcal{C}_0$ wave-map-gauge constraint with coefficients deduced from the solutions of the $\mathcal{C}_1$ and $\mathcal{C}_A$ constraints admits a solution $\overline{g}^{11} \equiv 1 - \alpha$ with $\alpha(0) = (r\alpha')(0) = 0, \alpha$ and $r\alpha'$ which are C^1 in r. The solution is global when it is so of the solution of the previous constraints, since the system is linear.*

11.3. Vanishing of $\overline{H}_0$

In previous sections we have seen how to achieve $\overline{H}^0 = \overline{H}^A = 0$, and hence $\mathcal{L}_1 = \mathcal{L}_A = 0$. Specializing equation (10.38) to this case we get (with $\overline{H}_0 := \nu_0\overline{H}^1$)

$$\mathcal{L}_0 \equiv -\partial_1\overline{H}_0 + \frac{1}{2}(\overline{W}_1 - \tau)\overline{H}_0. \tag{11.16}$$

On the other hand, the identity

$$S_{01} \equiv S_{01}^{(H)} + \frac{1}{2}(g_{0\alpha}\hat{D}_1 H^\alpha + g_{1\alpha}\hat{D}_0 H^\alpha - g_{01}\hat{D}_\alpha H^\alpha) \tag{11.17}$$

reduces on C_O to

$$\overline{S}_{01} \equiv \overline{S}_{01}^{(H)} + \frac{1}{2}(\overline{g}_{00}\overline{\hat{D}_1 H^0} + \nu_A\overline{\hat{D}_1 H^A} - \nu_0\overline{\hat{D}_A H^A}). \tag{11.18}$$

Using again the conditions $\overline{H}^0 = \overline{H}^A = 0$ we have, for an arbitrary target space in adapted coordinates,

$$\overline{\hat{D}_1 H^0} = 0, \qquad \overline{\hat{D}_1 H^A} = 0, \qquad \overline{\hat{D}_A H^A} = \hat{\Gamma}^A_{A1}\overline{H}^1, \tag{11.19}$$

and hence (11.18) further reduces to

$$\overline{S}_{01} \equiv \overline{S}_{01}^{(H)} - \frac{1}{2}\hat{\Gamma}^A_{A1}\overline{H}_0. \tag{11.20}$$

For a solution of the Einstein equations in wave-map gauge it holds then that

$$\overline{S}_{01} \equiv \mathcal{C}_0 + \mathcal{L}_0 = \overline{T}_{01} - \frac{1}{2}\hat{\Gamma}^A_{A1}\overline{H}_0. \tag{11.21}$$

Therefore, when $\overline{H}^0 = \overline{H}^A = 0$ and the initial data satisfy the wave-map-gauge constraint

$$\mathcal{C}_0 - \overline{T}_{01} = 0, \tag{11.22}$$

then $\overline{H}_0$ satisfies the equation

$$\partial_1\overline{H}_0 = \frac{1}{2}(\overline{W}_1 - \tau + \hat{\Gamma}^A_{A1})\overline{H}_0. \tag{11.23}$$

For a Minkowski target, and using the boundary conditions (4.20), we have

$$\hat{\Gamma}^A_{A1} \overset{\eta}{=} \frac{n-1}{r}, \quad \lim_{r\to 0}(r\overline{W}_1) \overset{\eta}{\equiv} \lim_{r\to 0}(-r^2\overline{g}^{AB}s_{AB}) = -(n-1),$$
$$\lim_{r\to 0}(r\tau) = n-1, \tag{11.24}$$

and hence Eq. (11.23) takes a Fuchsian form

$$r\partial_1\overline{H}_0 + \frac{n-1}{2}\overline{H}_0 + rM\overline{H}_0 \overset{\eta}{=} 0. \tag{11.25}$$

with M a continuous function up to $r=0$.

We want to prove that $H_0 = 0$ when the spacetime metric is a C^2 solution of the Einstein equations in Minkowski-wave-ap gauge; in this case the wave gauge vector H is C^1; then $\overline{H}_0$ tends to a finite limit at the vertex. The Eq. (11.25) implies that this limit is zero and hence that the only solution is zero.

Remark 11.3. If we add constraint damping terms as in (2.13) we obtain instead, using again $\overline{H}^0 = \overline{H}^A = 0$,

$$\mathcal{L}_0 = -\partial_1\overline{H}_0 + \frac{1}{2}(\overline{W}_1 - \tau + \epsilon\rho n_1)\overline{H}_0. \tag{11.26}$$

This new term does not change the leading orders in r of Eq. (11.25) and hence $\overline{H}_0 = 0$ is still the only regular solution. We conclude that the addition of constraint damping terms is fully compatible with the wave-map-gauge constraint hierarchy.

12. Wave-Map Gauge Constraints: A Summary

We have defined C_O to be the cone represented in $\mathbf{R}^{n+1}$ by the Minkowskian cone

$$y^0 = r, \quad r^2 := \sum_{i=1}(y^i)^2, \tag{12.1}$$

equivalently

$$x^0 = 0, \quad x^1 = r, \quad \Theta^i(x^A) = \frac{y^i}{r}. \tag{12.2}$$

We have considered on C_O a non degenerate quadratic form given in x^α coordinates by

$$\overline{g}_{00}(\mathrm{d}x^0)^2 + 2\nu_0\mathrm{d}x^0\mathrm{d}x^1 + 2\nu_A\mathrm{d}x^0\mathrm{d}x^A + \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B.$$

We have proved (recall that admissible means hypotheses on smoothness and limits at the vertex spelled out in various sections)

Theorem 12.1. 1. *Let $\tilde{g}$ be a given admissible degenerate quadratic form on $C_O^{T_0}$,*

$$\tilde{g} = \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B.$$

There exists on C_O^T, for some T with $0 < T \leq T_0$, coefficients $\nu_0, \nu_A, \overline{g}_{00}$ satisfying the vacuum Einstein wave-map gauge constraints, unique when admissible vertex limits are imposed.

2. *An admissible degenerate quadratic form $\tilde{g}$ together with a non vanishing ν_0 can be determined on C_O^T, for some T with $0 < T \leq T_0$, from the first vacuum Einstein wave-map gauge constraints, an admissible quadratic form γ and a scalar function κ being arbitrarily given. Then, $\tilde{g}$ is conformal to γ and depends only on its conformal class and ν_0 is linked to κ by the differential equation (7.11). They are unique under admissible vertex limits.*
   When $\tilde{g}$ is known, ν_A and $\overline{g}_{00}$ are determined as in point 1. by the second and third wave-map gauge constraint and admissible vertex limits.

13. Local Geometric Uniqueness for the Vacuum Einstein Equations

In this section only the vacuum Einstein equations will be considered.

Recall that two spacetimes (V_a, g_a) and (V_b, g_b) are considered as (both geometrically and physically) the same if there exists a diffeomorphism $\phi : V_a \to V_b$ such that on V_a it holds that $g_a = \phi_* g_b$. We have said before that given a $C^{1,1}$ metric g_a on a manifold V_a and $O_a \in V_a$ there are in some neighbourhood of O_a normal coordinates y_a^α centred at O_a, where the characteristic cone C_{O_a} is represented, for $0 \leq y_a^0 \leq T_a$ by the equation of a Minkowskian cone in R^{n+1}

$$y_a^0 = r_a, \quad r_a^2 := \sum_{i=1}^n (y_a^i)^2.$$

The null rays issued from O_a are represented by the generators of this cone. We have defined adapted null coordinates by setting

$$x_a^0 := r_a - y_a^0, \quad x_a^1 = r_a, \quad \text{with} \quad r_a^2 = \sum_{i=1}^n (y_a^i)^2 \tag{13.1}$$

and x_a^A local coordinates on the sphere S^{n-1}. In the coordinates x_a^α the metric g_a reads on the cone C_{O_a}

$$\overline{g}_{a,00}(\mathrm{d}x_a^0)^2 + 2\nu_{a,0}\mathrm{d}x_a^0\mathrm{d}x_a^1 + 2\nu_{a,A}\mathrm{d}x_{a,}^0\mathrm{d}x_a^A + \overline{g}_{a,AB}\mathrm{d}x_a^A\mathrm{d}x_a^B. \tag{13.2}$$

We have shown moreover (see Sect. 4.5) that it is possible to choose the coordinates y_a^α so that it holds

$$\lim_{r\to 0} r_a^{-3}(\overline{g}_{a,AB} - r_a^2 s_{AB}) = 0, \quad \lim_{r\to 0} r_a^{-2}\partial_1(\overline{g}_{a,AB} - r_a^2 s_{AB}) = 0, \tag{13.3}$$

$$\lim_{r\to 0} r_a^{-1}(\nu_{a,0} - 1) = 0, \quad \lim_{r\to 0} r_a^{-2}\nu_{a,A} = \lim_{r\to 0} r_a^{-1}\partial_1\nu_{a,A} = 0, \tag{13.4}$$

and even

$$\lim_{r\to 0} r_a^{-2}\overline{\partial_0(g_{a,AB} - r_a^2 s_{AB})} = 0, \tag{13.5}$$

$$\lim_{r\to 0} r_a^{-1}\overline{\partial_0 g_{a,1A}} = \lim_{r\to 0} r_a^{-1}\overline{\partial_0 g_{a,0A}} = 0, \tag{13.6}$$

while

$$\lim_{r\to 0} r_a^{-1}(\overline{g}_{a,00}+1)=0, \quad \lim_{r\to 0}\partial_1\overline{g}_{a,00}=0=\lim_{r\to 0}\overline{\partial_0 g_{a,00}}. \tag{13.7}$$

Having chosen such coordinates y_a^μ, respectively, y_b^μ, for the metrics g_a and g_b, we obtain a diffeomorphism ϕ_N by $y_b^\alpha(y_a^\alpha):=y_a^\alpha$, defined in the subset $y_a^0\le T:=\min(T_a,T_b)$. Such a diffeomorphism will be called *canonical.* We remark that canonical diffeomorphisms are *not* unique, and that above we have not required r to be an affine parameter.

The metrics g_b and $\phi_{N,*}g_b$ are geometrically equivalent, and one has equality of components $(\phi_{N,*}g_b)^{\lambda\mu}(y_a)=g_b^{\lambda\mu}(y_b)$ for $y_b^\alpha=y_a^\alpha$. The coordinates y^α are normal for both metrics and they satisfy in the coordinates x^α the vertex limits (13.3–13.7) recalled above.

To study the geometric uniqueness of our characteristic Cauchy problem we first consider two metrics g_a and g_b on the same manifold which satisfy the characteristic Cauchy problem on the same cone C_O. We will prove the following theorem, using the notations given in previous sections for C_O and Y_O (note that we are not assuming an affine parameterisation of the cone generators here):

Theorem 13.1. *Consider two smooth solutions g_a and g_b in Y_O^T of the Cauchy problem for the vacuum Einstein equations Ricci$(g)=0$ with data on the cone C_O^T, characteristic for both metrics. There exists $T'\le T$ so that g_a is equivalent to g_b in $Y_O^{T'}$ if and only if they induce on C_O^T the same degenerate quadratic form satisfying in the coordinates x^α the vertex limits* (13.3–13.7).

Proof. We put the metric g_a in Minkowski wave-map gauge by constructing a wave map f_a, that is a solution of the semilinear, tensorial, partial differential equations which read in abstract index notation

$$\Box_{g_a,\hat{g}}f_a^\alpha\equiv g_a^{\lambda\mu}(\partial^2_{\lambda\mu}f_a^\alpha-\Gamma_{a,}{}^{\sigma}_{\lambda\mu}\,\partial_\sigma f_a^\alpha+\partial_\lambda f_a^\sigma\partial_\mu f_a^\rho\hat{\Gamma}^\alpha_{\sigma\rho})=0, \tag{13.8}$$

which on C_O^T is the trace of the identity mapping I of $\mathbf{R}^{n+1}$. To simplify the writing we suppress the index a in the following computations, valid for any metric g with normal coordinates y^α and adapted null coordinates x^α, we will reestablish a and b in the conclusions.

The components $\underline{f^\alpha}$ and f^α of the image point are linked by the same relations as the coordinates y and x. They take in coordinates x^α the initial data

$$\overline{f}^0=0,\quad \overline{f}^1=x^1,\quad \overline{f}^A=x^A,\quad \text{for } x^0=0; \tag{13.9}$$

and in the coordinates y^α the initial data

$$\overline{\underline{f}}^i=y^i,\quad \overline{\underline{f}}^0=r; \tag{13.10}$$

we see that in the y coordinates the initial data are the trace on C_O of the set of C^∞ functions on $\mathbf{R}^{n+1}$

$$\underline{I}^i=y^i,\quad \underline{I}^0=y^0.$$

The existence of a C^2 wave map f in some $Y_O^{T_a}$ taking these initial data can therefore be proved using the Cagnac–Dossa theorem. In fact, since the equations are linear in a coordinate system where the $\hat{\Gamma}$'s vanish, the usual linear theory [22] suffices to obtain the result. The resulting wave map extends to a C^2 mapping (though not in general to a C^2 wave map).

To prove that it is a diffeomorphism at least in a neighbourhood of the vertex we first remark that our definitions imply

$$\overline{\underline{\partial_i f^j}} = \delta_i^j. \tag{13.11}$$

To study the derivatives $\underline{\partial_0}$ we return to the x coordinates and consider the set of functions

$$f^0 - x^0, \quad f^1 - x^1, \quad f^A - x^A. \tag{13.12}$$

They vanish on C_O and so do therefore their tangential derivatives on C_O; hence by application of the Lemma 4.2

$$\lim_{r\to 0} \overline{\partial_0 f^1} = 0, \quad \lim_{r\to 0} \overline{\partial_A f^0} = 0, \qquad \lim_{r\to 0} \overline{\partial_0 f^0} = 1. \tag{13.13}$$

By definition of the coordinates x and y we have

$$\underline{f}^0 \equiv (\Sigma(f^i)^2)^{\frac{1}{2}} - f^0;$$

hence,

$$\underline{\partial_0 f^0} := \frac{\partial}{\partial y^0}\underline{f}^0 \equiv -\frac{\partial}{\partial y^0} f^0 = \frac{\partial}{\partial x^0} f^0 := \partial_0 f^0 \tag{13.14}$$

while $\underline{f}^i$ depends only on f^1 and f^A. Therefore,

$$\lim_{r\to 0} \overline{\underline{\partial_0 f^i}} = 0, \quad \lim_{r\to 0} \overline{\underline{\partial_0 f^0}} = 1. \tag{13.15}$$

Since the Jacobian of the C^1 mapping f tends to 1 at O, it is a diffeomorphism, between at least small neighbourhoods of O.

The initial data, trace $\overline{g}^{(H)}$ of the metric $g^{(H)}$ in wave gauge are linked with the original $\overline{g}$ by the classical relation

$$\overline{g}_{\alpha\beta} \equiv \overline{\partial_\alpha f^\lambda}\,\overline{\partial_\beta f^\mu}\,\overline{g}^{(H)}_{\lambda\mu}. \tag{13.16}$$

The values of $\partial_i \overline{f}$ in the coordinates x^α show the equality of quadratic forms $\tilde{g}^{(H)} \equiv \tilde{g}$, indeed in these coordinates:

$$\overline{g}^{(H)}_{11} = \overline{g}_{11} = 0, \quad \overline{g}^{(H)}_{1A} = \overline{g}_{1A} = 0, \quad \overline{g}^{(H)}_{AB} = \overline{g}_{AB}. \tag{13.17}$$

Since $g^{(H)}$ is in wave gauge, and satisfies the vacuum Einstein equations, its trace $\overline{g}^{(H)}$ satisfies the wave-map gauge constraint $\mathcal{C}_1 = 0$, and $\nu_0^{(H)}$ satisfies the same equation as ν_0,

$$\partial_1 \nu_0^{(H)} = \nu_0^{(H)} \left\{ \frac{\partial_1 \tau}{\tau} + \frac{1}{2}(\nu_0^{(H)} \overline{W}^0 + \tau) + \frac{|\sigma|^2}{\tau} + \frac{\tau}{n-1} \right\}, \tag{13.18}$$

since the coefficients depend only on $\tilde{g}$, to show that $\nu_0^{(H)}$ tends to 1 at the vertex like ν_0 we use the identity

$$\nu_0 \equiv \overline{\partial_0 f^\lambda}\,\overline{\partial_1 f^\mu}\,\overline{g}^{(H)}_{\lambda\mu}. \tag{13.19}$$

and the limits (13.13)–(13.15) give

$$1 = \lim_{r\to 0} \nu_0 = \lim_{r\to 0} \nu_0^{(H)}.$$

Uniqueness of solutions of (13.18) (with non zero τ) with this limit at O shows that the function $\nu_0^{(H)}$ depends only on $\tilde{g}$, i.e. $\nu_{a,0}^{(H)} = \nu_{b,0}^{(H)}$ since $\tilde{g}_a = \tilde{g}_b$, that is $\overline{g}_{a,AB} = \overline{g}_{b,AB}$.

As a consequence, the wave-map-gauge constraints $C_{a,A} = 0$ written with $\overline{g}_{AB} \equiv \overline{g}_{AB}^{(H)}$ and $\nu_{a,0}^{(H)}$ are the same equation for $\nu_{a,A}^{(H)}$, $a = 1$ or 2. The vertex limit of $\nu_A^{(H)}$ will be deduced from the definition

$$\nu_A := \overline{g}_{0A} \equiv \overline{\partial_0 f^\lambda}\,\overline{\partial_A f^\mu}\,\overline{g}_{\lambda\mu}^{(H)} \equiv \overline{\partial_0 f^\lambda}\,\overline{g}_{A\lambda}^{(H)} \tag{13.20}$$

which implies using (13.9) (compare (4.43))

$$0 = \lim_{r\to 0} r^{-2}\nu_A = \lim_{r\to 0} r^{-2}\nu_A^{(H)}, \quad \lim_{r\to 0} \nu^A = \lim_{r\to 0} \overline{g}^{AB}\nu_B = \lim_{r\to 0} r^{-2}s^{AB}\nu_B = 0. \tag{13.21}$$

Differentiating (13.20) gives

$$\partial_1\nu_A \equiv \overline{\partial_1\partial_0 f^\lambda}\,\overline{g}_{A\lambda}^{(H)} + \overline{\partial_0 f^\lambda}\,\partial_1\overline{g}_{A\lambda}^{(H)}.$$

We have

$$\lim_{r\to 0} r^{-1}\partial_1\nu_A \equiv \lim_{r\to 0} (r^{-1}\overline{\partial_1\partial_0 f^\lambda}\,\overline{g}_{A\lambda}^{(H)}) + \lim_{r\to 0} (r^{-1}\overline{\partial_0 f^\lambda}\,\partial_1\overline{g}_{A\lambda}^{(H)})$$

with, by (13.9) and (13.13),

$$\lim_{r\to 0} (\overline{\partial_0 f^\lambda}\, r^{-1}\partial_1\overline{g}_{A\lambda}^{(H)}) = \lim_{r\to 0} r^{-1}\partial_1\overline{g}_{A0}^{(H)},$$

and

$$\lim_{r\to 0} (r^{-1}\overline{\partial_1\partial_0 f^\lambda}\,\overline{g}_{A\lambda}^{(H)}) = \lim_{r\to 0} (r\overline{\partial_1\partial_0 f^B}) s_{AB}.$$

Taking the trace on the cone of the wave map equation, with Minkowski target, gives

$$\begin{aligned} &2\overline{g}^{01}(\partial_1\overline{\partial_0 f^A} - \overline{\Gamma}^0_{10}\overline{\partial_0 f^A} - \overline{\Gamma}^A_{10} + \hat{\Gamma}^A_{1B}\overline{\partial_0 f^B}) \\ &\quad - 2\overline{g}^{1B}(\overline{\Gamma}^A_{1B} - \hat{\Gamma}^A_{1B}) - \overline{g}^{BC}(\overline{\Gamma}^0_{BC}\overline{\partial_0 f^A} + \overline{\Gamma}^A_{BC} - \hat{\Gamma}^A_{BC}) = 0. \end{aligned} \tag{13.22}$$

We have

$$\begin{aligned} &\lim_{r\to 0} \overline{\Gamma}^0_{10} = 0 \quad \text{and} \quad \lim_{r\to 0} \overline{\partial_0 f^A} = 0, \\ &\lim_{r\to 0} r\overline{\Gamma}^A_{10} = \lim_{r\to 0} \frac{1}{2}\{-r\nu^A\nu^0\overline{\partial_0 g_{11}} + r\overline{g}^{AB}(\overline{\partial_0 g_{1B}} + \partial_1\nu_B - \partial_B\nu_0)\} = 0. \end{aligned}$$

Finally, for a wave map f

$$\lim_{r\to 0} (r\partial_1\overline{\partial_0 f^A}) = 0;$$

hence, since $\lim_{r\to 0} r^{-1}\partial_1\nu_A = 0$, we obtain

$$\lim_{r\to 0} r^{-1}\partial_1\nu_A^{(H)} = 0.$$

Since $\nu_{a,A}$ and $\nu_{b,A}$ satisfy the same equation and the same boundary conditions, they are equal.

It remains to analyse the boundary conditions for the functions $\overline{g}_{a,00}$, which again satisfy the same equation for $a = 1$ or 2. We have

$$g_{00} = \partial_0 f^\lambda \partial_0 f^\mu g^{(H)}_{\lambda\mu}.$$

It implies

$$\lim_{r\to 0} \overline{g}_{00} \equiv \lim_{r\to 0} \left(\overline{g}^{(H)}_{00} \overline{\partial_0 f^0 \partial_0 f^0} + 2\overline{\partial_0 f^0 \partial_0 f^1} \nu^{(H)}_0 + \overline{g}^{(H)}_{AB} \overline{\partial_0 f^A \partial_0 f^B} \right).$$

The previous limits imply then

$$\lim_{r\to 0} \overline{g}_{00} = \lim_{r\to 0} \overline{g}^{(H)}_{00} = -1.$$

Also,

$$\partial_1 g_{00} = \partial_0 f^\lambda \partial_0 f^\mu \partial_1 g^{(H)}_{\lambda\mu} + 2\partial_1 \partial_0 f^\lambda \partial_0 f^\mu g^{(H)}_{\lambda\mu};$$

hence using previous limits

$$\lim_{r\to 0} r\partial_1 \overline{g}_{00} = \lim_{r\to 0} \left\{ r\partial_1 \overline{g}^{(H)}_{00} + 2r\left(\overline{\partial_1 \partial_0 f^0} \overline{g}^{(H)}_{00} + \overline{\partial_1 \partial_0 f^1} \nu^{(H)}_0 \right) \right\}.$$

We have, by definition of a wave map with Minkowskian target,

$$2\nu^0 (\partial_1 \overline{\partial_0 f^0} - \overline{\Gamma}^0_{10} \overline{\partial_0 f^0}) + \nu^0 \tau \overline{\partial_0 f^0} + \overline{g}^{AB} \hat{\Gamma}^0_{AB} = 0.$$

Hence,

$$\lim_{r\to 0} r \left\{ 2(\partial_1 \overline{\partial_0 f^0} - \lim \overline{\Gamma}^0_{10}) + \frac{n-1}{r} - \psi - \frac{n-1}{r} \right\} = 0,$$

which gives

$$\lim_{r\to 0} r\partial_1 \overline{\partial_0 f^0} = 0.$$

One finds also

$$\lim_{r\to 0} r\partial_1 \overline{\partial_0 f^1} = 0,$$

hence,

$$\lim_{r\to 0} r\partial_1 \overline{g}_{00} = \lim_{r\to 0} r\partial_1 \overline{g}^{(H)}_{00} = 0.$$

We have proved that $\tilde{g}_a = \tilde{g}_b$ on C^T_O implies $\overline{g}^{(H)}_a = \overline{g}^{(H)}_b$ on C^T_O and hence, by uniqueness for the hyperbolic system of the Einstein equations in wave gauge $g^{(H)}_a = g^{(H)}_b$ in Y^T_O. The metrics g_a and g_b are geometrically equivalent.

The reverse implication is trivial. □

Our next result, one of the main results of this paper, is a straightforward corollary of Theorem 13.1:

Theorem 13.2. *Given points* $O_a \in V_a$ *and* $O_b \in V_b$ *denote by* C_{O_a} *and* C_{O_b} *the characteristic (null) cones of smooth Lorentzian metrics* g_a *on* V_a *and* g_b *on* V_b. *Denote by* J_a^+ *the future of the point* O_a *in the metric* g_a. *There are neighbourhoods* U_a *of* O_a *and* U_b *of* O_b *such that the spacetimes* $(U_a \cap J_a^+, g_a)$ *and* $(U_b \cap J_b^+, g_b)$ *are locally geometrically the same if and only if the pull back* $\phi_N^* \tilde{g}_b$, *where* ϕ_N *is a canonical diffeomorphism of* U_a *onto* U_b, *equals* $\tilde{g}_a$.

Proof. The spacetimes $(U_b \cap J_b^+, g_b)$ and $(\phi_N^{-1}(U_b \cap J_b^+) \subset U_a \cap J_a^+), \phi_N^* g_b)$ are geometrically equivalent. Theorem 13.1 shows that the second one is locally geometrically equivalent to $(U_a \cap J_a^+), g_A)$; the conclusion follows from the fact that $\phi_N^* \tilde{g}_b = \widetilde{\phi_N^* g_b}$ and satisfies the required vertex limits. □

From the Uniqueness Theorem 12.1 for the constraints one deduces straightforwardly a formulation of geometric local uniqueness starting from data γ and κ.

14. Conclusions, and Open Problems

We have shown that the trace $\overline{g}$ on a characteristic cone of a solution of Einstein equations which is also a solution of the reduced Einstein equations in wave-map gauge satisfies necessarily a set of $n+1$ equations which we have called wave-map gauge constraints, written out explicitly and solved. We have shown that, conversely a solution of the reduced Einstein equations in wave-map gauge with trace satisfying these wave-map gauge constraints satisfies the original Einstein equations. Finally, we have shown that every solution of the vacuum Einstein equations is locally (i.e. in a neighbourhood of the vertex) isometric to a solution in wave map gauge, uniquely determined by the degenerate quadratic form induced on the characteristic cone by the spacetime metric.

There remain many interesting open problems:

- Determine the minimum regularity, in particular at the vertex, under which the initial data lead to a local solution (see also [13]).
- Extend our analysis to a characteristic cone with vertex at i^- (cf. [24]).
- Study the asymptotic behaviour of the solutions of the wave-map gauge constraint equations at future null infinity.
- Prove global existence for small initial data of solutions of the Einstein equations in higher dimensions by a conformal method, as was done for the spacelike Cauchy problem with data identically Schwarzschild outside of a bounded region [11].
- Prove global existence using the approach of Lindblad–Rodnianski [38–40] (compare [2,26]).

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Appendix A: Collected Formulae

The metric on $C_O = \{x^0 = 0\}$ is written as

$$g = \overline{g}_{00}(\mathrm{d}x^0)^2 + 2\overline{g}_{0A}\mathrm{d}x^0\mathrm{d}x^A + 2\overline{g}_{01}\mathrm{d}x^0\mathrm{d}x^1 + \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{A.1}$$

and recall that we *do not* assume that this form of the metric is preserved under differentiation in the x^0-direction. Here and elsewhere we put overbars on the relevant quantities whenever the formulae hold only on C_O. The inverse is

$$g^\sharp = \overline{g}^{11}\partial_1^2 + 2\overline{g}^{1A}\partial_1\partial_A + 2\overline{g}^{01}\partial_0\partial_1 + \overline{g}^{AB}\partial_A\partial_B, \tag{A.2}$$

with

$$\overline{g}^{01} = \frac{1}{\overline{g}_{01}}, \quad \overline{g}^{1A} = -\overline{g}^{01}\overline{g}^{AB}\overline{g}_{0B}, \quad \overline{g}^{11} = (\overline{g}^{01})^2(-\overline{g}_{00} + \overline{g}^{AB}\overline{g}_{0A}\overline{g}_{0B}). \tag{A.3}$$

We introduce the special notations

$$\nu_0 := \overline{g}_{01}, \qquad \nu_A := \overline{g}_{0A}, \qquad \tilde{g} := \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B, \tag{A.4}$$

$$\nu^0 := \overline{g}^{01} = \frac{1}{\nu_0}, \qquad \nu^A := -\overline{g}_{01}\overline{g}^{1A} = \overline{g}^{AB}\nu_B. \tag{A.5}$$

Then,

$$\overline{g}^{1A} = -\nu^0\nu^A, \qquad \overline{g}^{11} = (\nu^0)^2(-\overline{g}_{00} + \nu^A\nu_A). \tag{A.6}$$

The determinant reads

$$\sqrt{|\det \overline{g}|} = \nu_0\sqrt{\det \tilde{g}_\Sigma}. \tag{A.7}$$

The Levi–Civita connection of the metric $\overline{g}_{AB}$ will be denoted as $\tilde{\nabla}_A$, with corresponding Christoffel symbols $\tilde{\Gamma}^C_{AB}$ with respect to the derivative ∂_A.

We have the following Christoffel symbols on the null hypersurface:

$$\overline{\Gamma}^0_{00} = \frac{1}{2}\nu^0(-\partial_1\overline{g}_{00} + 2\overline{\partial_0 g_{01}}), \tag{A.8}$$

$$\overline{\Gamma}^0_{01} = \frac{1}{2}\nu^0\overline{\partial_0 g_{11}}, \tag{A.9}$$

$$\overline{\Gamma}^0_{11} = 0, \tag{A.10}$$

$$
\begin{aligned}
\overline{\Gamma}^1_{00} &= \frac{1}{2}\nu^0\nu^A(\partial_A\overline{g}_{00} - 2\overline{\partial_0 g_{0A}})\\
&\quad +\frac{1}{2}\overline{g}^{11}(-\partial_1\overline{g}_{00} + 2\overline{\partial_0 g_{01}}) + \frac{1}{2}\nu^0\overline{\partial_0 g_{00}}, && \text{(A.11)}\\
\overline{\Gamma}^1_{01} &= \frac{1}{2}\left(\nu^0\partial_1\overline{g}_{00} + \nu^0\nu^A(\partial_A\nu_0 - \partial_1\nu_A - \overline{\partial_0 g_{1A}}) + \overline{g}^{11}\overline{\partial_0 g_{11}}\right), && \text{(A.12)}\\
\overline{\Gamma}^1_{11} &= \nu^0\partial_1\nu_0 - \frac{1}{2}\nu^0\overline{\partial_0 g_{11}}, && \text{(A.13)}\\
\overline{\Gamma}^0_{A0} &= \frac{1}{2}\nu^0(\partial_A\nu_0 + \overline{\partial_0 g_{1A}} - \partial_1\nu_A), && \text{(A.14)}\\
\overline{\Gamma}^0_{A1} &= 0, && \text{(A.15)}\\
\overline{\Gamma}^1_{A0} &= \frac{1}{2}\nu^0(\partial_A\overline{g}_{00} - \nu^B(\tilde{\nabla}_A\nu_B - \tilde{\nabla}_B\nu_A + \overline{\partial_0 g_{AB}}))\\
&\quad +\frac{1}{2}\overline{g}^{11}(\partial_A\nu_0 + \overline{\partial_0 g_{1A}} - \partial_1\nu_A), && \text{(A.16)}\\
\overline{\Gamma}^1_{A1} &= \frac{1}{2}\nu^0(\partial_A\nu_0 - \overline{\partial_0 g_{1A}} + \partial_1\nu_A - \nu^B\partial_1\overline{g}_{AB}), && \text{(A.17)}\\
\overline{\Gamma}^0_{AB} &= -\frac{1}{2}\nu^0\partial_1\overline{g}_{AB}, && \text{(A.18)}\\
\overline{\Gamma}^1_{AB} &= \frac{1}{2}\nu^0(\tilde{\nabla}_A\nu_B + \tilde{\nabla}_B\nu_A - \overline{\partial_0 g_{AB}}) - \frac{1}{2}\overline{g}^{11}\partial_1\overline{g}_{AB}, && \text{(A.19)}\\
\overline{\Gamma}^C_{00} &= -\frac{1}{2}\overline{g}^{CA}\partial_A\overline{g}_{00} + \frac{1}{2}\nu^0\nu^C\partial_1\overline{g}_{00} + \overline{g}^{CA}\overline{\partial_0 g_{0A}} - \nu^0\nu^C\overline{\partial_0 g_{01}}, && \text{(A.20)}\\
\overline{\Gamma}^C_{01} &= \frac{1}{2}\overline{g}^{CA}(\overline{\partial_0 g_{1A}} + \partial_1\nu_A - \partial_A\nu_0) - \frac{1}{2}\nu^0\nu^C\overline{\partial_0 g_{11}}, && \text{(A.21)}\\
\overline{\Gamma}^C_{11} &= 0, && \text{(A.22)}\\
\overline{\Gamma}^C_{A0} &= -\frac{1}{2}\nu^0\nu^C(\overline{\partial_0 g_{1A}} + \partial_A\nu_0 - \partial_1\nu_A)\\
&\quad +\frac{1}{2}\overline{g}^{BC}(\tilde{\nabla}_A\nu_B - \tilde{\nabla}_B\nu_A + \overline{\partial_0 g_{AB}}), && \text{(A.23)}\\
\overline{\Gamma}^C_{A1} &= \frac{1}{2}\overline{g}^{BC}\partial_1\overline{g}_{AB}, && \text{(A.24)}\\
\overline{\Gamma}^C_{AB} &= \tilde{\Gamma}^C_{AB} + \frac{1}{2}\nu^0\nu^C\partial_1\overline{g}_{AB}. && \text{(A.25)}
\end{aligned}
$$

The remaining ones are obtainable by symmetry. Note that in spite of having $\overline{g}_{AB} = \tilde{g}_{AB}$, the Christoffel symbols $\overline{\Gamma}^C_{AB}$ (a part of $\overline{\Gamma}^\lambda_{\mu\nu}$) and $\tilde{\Gamma}^C_{AB}$ (the Christoffel symbols of $\tilde{g}_{AB}$) do not coincide in general.

We note the following traces of the Christoffel symbols:

$$
\begin{aligned}
\overline{\Gamma}^\mu_{0\mu} &= \nu^0\overline{\partial_0 g_{01}} + \frac{1}{2}\overline{g}^{11}\overline{\partial_0 g_{11}} - \nu^0\nu^A\overline{\partial_0 g_{1A}} + \frac{1}{2}\overline{g}^{AB}\overline{\partial_0 g_{AB}}, && \text{(A.26)}\\
\overline{\Gamma}^\mu_{1\mu} &= \nu^0\partial_1\nu_0 + \frac{1}{2}\overline{g}^{AB}\partial_1\overline{g}_{AB}, && \text{(A.27)}\\
\overline{\Gamma}^\mu_{A\mu} &= \nu^0\partial_A\nu_0 + \frac{1}{2}\overline{g}^{BC}\partial_A\overline{g}_{BC}. && \text{(A.28)}
\end{aligned}
$$

The harmonicity vector on the null surface reads

$$\overline{\Gamma}^0 = (\nu^0)^2\overline{\partial_0 g_{11}} - \frac{1}{2}\nu^0\overline{g}^{AB}\partial_1\overline{g}_{AB}, \tag{A.29}$$

$$\begin{aligned}\overline{\Gamma}^1 &= \nu^0\overline{g}^{AB}\tilde{\nabla}_B\nu_A + \overline{g}^{11}\nu^0\partial_1\nu_0 - \frac{1}{2}\overline{g}^{11}\overline{g}^{AB}\partial_1\overline{g}_{AB} + (\nu^0)^2\nu^A\nu^B\partial_1\overline{g}_{AB} \\ &\quad +(\nu^0)^2\partial_1\overline{g}_{00} - 2(\nu^0)^2\nu^A\partial_1\nu_A - \frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}} + \frac{1}{2}\nu^0\overline{g}^{11}\overline{\partial_0 g_{11}} \qquad (A.30) \\ &= \nu^0\overline{g}^{AB}\tilde{\nabla}_B\nu_A - \frac{\partial_1(\nu_0\overline{g}^{11}\sqrt{\det\tilde{g}_\Sigma})}{\nu_0\sqrt{\det\tilde{g}_\Sigma}} - \frac{1}{2}\nu^0\overline{g}^{AB}\overline{\partial_0 g_{AB}} + \frac{1}{2}\nu^0\overline{g}^{11}\overline{\partial_0 g_{11}}, \qquad (A.31)\end{aligned}$$

$$\begin{aligned}\overline{\Gamma}^A &= -\overline{g}^{AB}\nu^0\partial_B\nu_0 + \overline{g}^{CD}\tilde{\Gamma}^A_{CD} + \frac{1}{2}\overline{g}^{BC}\nu^0\nu^A\partial_1\overline{g}_{BC} - \overline{g}^{AC}\nu^0\nu^B\partial_1\overline{g}_{BC} \\ &\quad +\nu^0(\overline{g}^{AB}\partial_1\nu_B + \overline{g}^{AB}\overline{\partial_0 g_{1B}} - \nu^0\nu^A\overline{\partial_0 g_{11}}),\end{aligned} \tag{A.32}$$

$$\begin{aligned}\overline{g}_{0\mu}\overline{\Gamma}^\mu &= -\nu^A\nu^0\partial_A\nu_0 + \overline{g}^{AB}\partial_B\nu_A + \nu^0\partial_1\overline{g}_{00} + \overline{g}^{11}\partial_1\nu_0 - \nu^0\nu^A\partial_1\nu_A \\ &\quad -\frac{1}{2}\overline{g}^{AB}\overline{\partial_0 g_{AB}} + \nu^0\nu^A\overline{\partial_0 g_{1A}} - \frac{1}{2}\overline{g}^{11}\overline{\partial_0 g_{11}},\end{aligned} \tag{A.33}$$

$$\overline{g}_{1\mu}\overline{\Gamma}^\mu = -\frac{1}{2}\overline{g}^{AB}\partial_1\overline{g}_{AB} + \nu^0\overline{\partial_0 g_{11}}, \tag{A.34}$$

$$\overline{g}_{A\mu}\overline{\Gamma}^\mu = -\nu^0(\partial_A\nu_0 - \partial_1\nu_A - \overline{\partial_0 g_{1A}} + \nu^B\partial_1\overline{g}_{AB}) + \overline{g}^{BC}\overline{g}_{AD}\tilde{\Gamma}^D_{BC}. \tag{A.35}$$

(In the main body of the paper we also use $\overline{\Gamma}_A := \overline{g}_{AB}\overline{\Gamma}^B$, see (4.35).)

The following formulae are often used in our calculations:

$$\overline{\partial_0 g^{00}} \equiv -(\nu^0)^2\overline{\partial_0 g_{11}}, \quad \overline{\partial_0 g^{0B}} \equiv -\nu^0(-\nu^0\nu^B\overline{\partial_0 g_{11}} + \overline{g}^{BC}\overline{\partial_0 g_{1C}}), \tag{A.36}$$

$$\overline{\partial_0 g^{10}} \equiv -\{(\nu^0)^2\overline{\partial_0 g_{01}} + \nu^0\overline{g}^{11}\overline{\partial_0 g_{11}} - (\nu^0)^2\nu^C\overline{\partial_0 g_{1C}})\}. \tag{A.37}$$

The scalar wave operator acting on a function f reads

$$\begin{aligned}\overline{\Box_g f} &= \frac{1}{\sqrt{|\det\overline{g}|}}\overline{\partial_\mu(\sqrt{|\det g|}g^{\mu\nu}\partial_\nu f)} \\ &= -\overline{\Gamma}^\nu\overline{\partial_\nu f} + \overline{g}^{11}\partial_1^2\overline{f} - 2\nu^0\nu^A\partial_1\partial_A\overline{f} + 2\nu^0\partial_1\overline{\partial_0 f} \\ &\quad +\overline{g}^{AB}\partial_A\partial_B\overline{f}.\end{aligned} \tag{A.38}$$

The tensor computations in this article have been checked with the computer algebra framework *xAct* [41].

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